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Weak Solutions of Fractional Order Differential Equations via Volterra-Stieltjes Integral Operator

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ABSTRACT: The fractional derivative of the Riemann-Liouville and Caputo types played an important role in the development of the theory of fractional derivatives, integrals and for its applications in pure mathematics ([18], [21]). In this paper, we study the existence of weak solutions for fractional differential equations of Riemann-Liouville and Caputo types. We depend on converting of the mentioned equations to the form of functional integral equations of Volterra-Stieltjes type in reflexive Banach spaces.

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1. Introduction and preliminaries

Let E be a reflexive Banach space with norm $\| \,.\, \|$ and dual E^*. Denote by $C[I, E]$ the Banach space of strongly continuous functions $x : I \to E$ with sup-norm.
Fractional differential equations have received increasing attention due to its applications in physics, chemistry, materials, engineering, biology, finance [15, 16]. Fractional order derivatives have the memory property and can describe many phenomena that integer order derivatives cant characterize. Only a few papers consider fractional differential equations in reflexive Banach spaces with the weak topology [6, 7, 14, 22, 23].
Here we study the existence of weak solutions of the Volterra-Stieltjes integral equation

$$x(t) \;=\; p(t) + \int_0^t f(s, x(s))\; d_s g(t, s), \quad t \in I = [0, T],$$

in the reflexive Banach space E.
Let $\alpha \in (0,1)$. As applications, we study the existence of weak solution for the differential equations of fractional order

$$^{R}D^{\alpha}x(t) = f(t, x(t)),\ t \in (0,T] \tag{1.1}$$

with the initial data

$$x(0) = 0\ , \tag{1.2}$$

where $^{R}D^{\alpha}x(.)$ is a Riemann-Liouville fractional derivative of the function $x : I = [0,T] \to E$.
Also we study the existence of mild solution for the initial value problem

$$^{C}D^{\alpha}x(t) = f(t, x(t)),\ t \in (0,T] \tag{1.3}$$

with the initial data

$$x(0) = x_0\ , \tag{1.4}$$

where $^{C}D^{\alpha}x(.)$ is a Caputo fractional derivative of the function $x : I : [0,T] \to E$.
Functional integral equations of Volterra-Stieltjes type have been studied in the space of continuous functions in many papers for example, (see [1-5] and [8]).
For the properties of the Stieltjes integral (see Banaś [1]).

Definition 1.1. The fractional (arbitrary) order integral of the function $f \in L_1$ of order $\alpha > 0$ is defined as [18, 21]

$$I^{\alpha}f(t) := \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} f(s)\ ds.$$

For the fractional-order derivative we have the following two definitions.

Definition 1.2. The Riemann-Liouville fractional-order derivative of $f(t)$ of order $\alpha \in (0,1)$ is defined as ([18], [21])

$$^{R}D_a^{\alpha}f(t) = \frac{d}{dt}\int_a^t \frac{(t-s)^{-\alpha}}{\Gamma(1-\alpha)} f(s)\ ds$$

or

$$^{R}D_a^{\alpha}f(t) = \frac{d}{dt} I_a^{1-\alpha} f(t).$$

Definition 1.3. The Caputo fractional-order derivative of $g(t)$ of order $\alpha \in (0,1]$ of the absolutely continuous function $g(t)$ is defined as ([9])

$$^{C}D_a^{\alpha}g(t) = \int_a^t \frac{(t-s)^{-\alpha}}{\Gamma(1-\alpha)} \frac{d}{ds} g(s)\ ds$$

or

$$^{C}D_a^{\alpha}g(t) = I_a^{1-\alpha}\frac{d}{dt} g(t).$$

Now, we shall present some auxiliary results that will be need in this work. Let E be a Banach space (need not be reflexive) and let $x : [a,b] \to E$, then

(1-) $x(.)$ is said to be weakly continuous (measurable) at $t_0 \in [a,b]$ if for every $\phi \in E^*$, $\phi(x(.))$ is continuous (measurable) at t_0.

(2-) A function $h : E \to E$ is said to be weakly sequentially continuous if h maps weakly convergent sequences in E to weakly convergent sequences in E.

If x is weakly continuous on I, then x is strongly measurable and hence weakly measurable (see [10] and [13]). It is evident that in reflexive Banach spaces, if x is weakly continuous function on $[a,b]$, then x is weakly Riemann integrable (see [13]).

Definition 1.4. Let $f : I \times E \to E$. Then $f(t,u)$ is said to be weakly-weakly continuous at (t_0, u_0) if given $\epsilon > 0$, $\phi \in E^*$ there exists $\delta > 0$ and a weakly open set U containing u_0 such that

$$| \phi(f(t,u) - f(t_0,u_0)) | < \epsilon$$

whenever

$$| t - t_0 | < \delta \text{ and } u \in U.$$

Now, we have the following fixed point theorem, due to O'Regan, in the reflexive Banach space (see [19]) and some propositions which will be used in the sequel [13, 20].

Theorem 1.5. *Let E be a Banach space and let Q be a nonempty, bounded, closed and convex subset of $C[I,E]$ and let $F : Q \to Q$ be a weakly sequentially continuous and assume that $FQ(t)$ is relatively weakly compact in E for each $t \in I$. Then, F has a fixed point in the set Q.*

Proposition 1.6. *A convex subset of a normed space E is closed if and only if it is weakly closed.*

Proposition 1.7. *A subset of a reflexive Banach space is weakly compact if and only if it is closed in the weak topology and bounded in the norm topology.*

Proposition 1.8. *Let E be a normed space with $y \in E$ and $y \neq 0$. Then there exists a $\phi \in E^*$ with $\| \phi \| = 1$ and $\| y \| = \phi(y)$.*

2. Volterra-Stieltjes integral equation

In this section we prove the existence of weak solutions for the Volterra-Stieltjes integral equation

$$x(t) = p(t) + \int_0^t f(s,x(s))\, d_s g(t,s), \quad t \in I = [0,T], \tag{2.5}$$

in the space $C[I, E]$. To facilitate our discussion, denote Λ by

$$\Lambda = \{(t, s) : 0 \leq s \leq t \leq T\}$$

and let $p : I \to E,\ f : I \times E \to E$ and $g : \Lambda \to R$ be functions such that:

(i) $p \in C[I, E]$.

(ii) The function f is weakly-weakly continuous.

(iii) There exists a constant M such that $\| f(t, x) \| \leq M$.

(iv) The function g is continuous on Λ.

(v) The function $s \to g(t, s)$ is of bounded variation on $[0, t]$ for each fixed $t \in I$.

(vi) For any $\epsilon > 0$ there exists $\delta > 0$ for all $t_1, t_2 \in I$ such that $t_1 < t_2$ and $t_2 - t_1 \leq \delta$ the following inequality holds

$$\bigvee_{0}^{t_1} [g(t_2, s) - g(t_1, s)] \leq \epsilon.$$

(vii) $g(t, 0) = 0$ for any $t \in I$.

Obviously we will assume that g satisfies assumptions (iv)-(vi). For our purposes we will only need the following lemmas.

Lemma 2.1. *[5] The function* $z \to \bigvee_{s=0}^{z} g(t, s)$ *is continuous on* $[0, t]$ *for any fixed* $t \in I$.

Lemma 2.2. *[5] For an arbitrary fixed* $0 < t_2 \in I$ *and for any* $\epsilon > 0$, *there exists* $\delta > 0$ *such that if* $t_1 \in I,\ t_1 < t_2$ *and* $t_2 - t_1 \leq \delta$ *then*

$$\bigvee_{s=t_1}^{t_2} g(t_2, s) \leq \epsilon.$$

Lemma 2.3. *[5] The function* $t \to \bigvee_{s=0}^{t} g(t, s)$ *is continuous on* I. *Then there exists a finite positive constant* K *such that*

$$K = \sup\{\bigvee_{s=0}^{t} g(t, s) : t \in I\}.$$

Definition 2.4. By a weak solution to (2.5) we mean a function $x \in C[I, E]$ which satisfies the integral equation (2.5). This is equivalent to find $x \in C[I, E]$ with

$$\phi(x(t)) \;=\; \phi(p(t) + \int_0^t f(s, x(s))\ d_s g(t, s)),\ t \in I\ \forall\ \phi \in E^*.$$

Now we can prove the following theorem.

Theorem 2.5. *Under the assumptions (i)-(vii), the Volterra-Stieltjes integral equation (2.5) has at least one weak solution* $x \in C[I, E]$.

Proof. Define the nonlinear Volterra-Stieltjes integral operator A by

$$Ax(t) = p(t) + \int_0^t f(s, x(s))\ d_s g(t, s), \quad t \in I.$$

For every $x \in C[I, E]$, $f(., x(.))$ is weakly continuous ([24]). To see this we equip E and $I \times E$ with weak topology and note that $t \mapsto (t, x(t))$ is continuous as a mapping from I into $I \times E$, then $f(., x(.))$ is a composition of this mapping with f and thus for each weakly continuous $x : I \to E$, $f(., x(.)) : I \to E$ is weakly continuous, means that $\phi(f(., x(.)))$ is continuous, for every $\phi \in E^*$, g is of bounded variation. Hence $f(., x(.))$ is weakly Riemann-Stieltjes integrable on I with respect to $s \to g(t, s)$. Thus A makes sense.
For notational purposes $\| x \|_0 = \sup_{t \in I} \| x(t) \|$.
Now, define the set Q by

$$Q = \Big\{ x \in C[I, E] : \| x \|_0 \leq M_0\ ,$$

$$\| x(t_2) - x(t_1) \| \leq \| p(t_2) - p(t_1) \| + MN(\epsilon) + M \bigvee_{s=t_1}^{t_2} g(t_2, s) \Big\}\ .$$

First notice that Q is convex and norm closed. Hence Q is weakly closed by Proposition 1.6.
Note that A is well defined, to see that, Let $t_1,\ t_2 \in I,\ t_2 > t_1$, without loss of generality, assume $Ax(t_2) - Ax(t_1) \neq 0$

$$\begin{aligned}
\| Ax(t_2) - Ax(t_1) \| &= \phi(Ax(t_2) - Ax(t_1)) \leq\ | \phi(p(t_2) - p(t_1)) | \\
&+\ | \int_0^{t_2} \phi(f(s, x(s)))\ d_s g(t_2, s) - \int_0^{t_1} \phi(f(s, x(s)))\ d_s g(t_1, s) | \\
&\leq\ \| p(t_2) - p(t_1) \| + | \int_0^{t_1} \phi(f(s, x(s)))\ d_s g(t_2, s) \\
&+\ \int_{t_1}^{t_2} \phi(f(s, x(s)))\ d_s g(t_2, s) - \int_0^{t_1} \phi(f(s, x(s)))\ d_s g(t_1, s) | \\
&\leq\ \| p(t_2) - p(t_1) \| + | \int_0^{t_1} \phi(f(s, x(s)))\ d_s[g(t_2, s) - g(t_1, s)] | \\
&+\ | \int_{t_1}^{t_2} \phi(f(s, x(s)))\ d_s g(t_2, s) |
\end{aligned}$$

$$
\begin{aligned}
&\leq \ \| p(t_2) - p(t_1) \| \\
&+ \ \int_0^{t_1} | \phi(f(s, x(s))) | \ d_s[\bigvee_{z=0}^{s} (g(t_2, z) - g(t_1, z))] \\
&+ \ \int_{t_1}^{t_2} | \phi(f(s, x(s))) | \ d_s[\bigvee_{z=0}^{s} g(t_2, z)] \\
&\leq \ \| p(t_2) - p(t_1) \| + M \int_0^{t_1} d_s[\bigvee_{z=0}^{s} (g(t_2, z) - g(t_1, z))] \\
&+ \ M \int_{t_1}^{t_2} d_s[\bigvee_{z=0}^{s} g(t_2, z)] \\
&\leq \ \| p(t_2) - p(t_1) \| + M \bigvee_{s=0}^{t_1} (g(t_2, s) - g(t_1, s)) \\
&+ \ M[\bigvee_{s=0}^{t_2} g(t_2, s) - \bigvee_{s=0}^{t_1} g(t_2, s)] \\
&\leq \ \| p(t_2) - p(t_1) \| + MN(\epsilon) + M \bigvee_{s=t_1}^{t_2} g(t_2, s) \ ,
\end{aligned}
$$

where

$$
N(\epsilon) = \sup\{ \bigvee_{s=0}^{t_1} (g(t_2, s) - g(t_1, s)) : t_1, t_2 \in I, \ t_1 < t_2, \ t_2 - t_1 \leq \epsilon \}.
$$

Hence

$$
\| Ax(t_2) - Ax(t_1) \| \leq \| p(t_2) - p(t_1) \| + MN(\epsilon) + M \bigvee_{s=t_1}^{t_2} g(t_2, s), \tag{2.6}
$$

and so $Ax \in C[I, E]$. We claim that $A : Q \to Q$ is weakly sequentially continuous and $A(Q)$ is weakly relatively compact. Once the claim is established, Theorem 1.5 guarantees the existence of a fixed point $x \in C[I, E]$ of the operator A and the integral equation (2.5) has a solution $x \in C[I, E]$.
To prove our claim, we start by showing that $A : Q \to Q$. Take $x \in Q$, note that the inequality (2.6) shows that AQ is norm continuous. Then by using Proposition 1.8

we get

$$\begin{aligned}
\| Ax(t) \| &= \phi(Ax(t)) \leq \ | \phi(p(t)) | + | \phi(\int_0^t f(s,x(s))\ d_s g(t,s)) | \\
&\leq \| p(t) \| + \int_0^t | \phi(f(s,x(s))) |\ \ d_s(\bigvee_{z=0}^{s} g(t,z)) \\
&\leq \| p(t) \| + M \int_0^t \ d_s(\bigvee_{z=0}^{s} g(t,z)) \\
&\leq \| p(t) \| + M \bigvee_{s=0}^{t} g(t,s) \\
&\leq \| p \|_0 + M \sup_{t\in I} \bigvee_{s=0}^{t} g(t,s) \\
&\leq \| p \|_0 + MK = M_0 \ .
\end{aligned}$$

Then

$$\| Ax \|_0 = \sup_{t\in I} \| Ax(t) \| \leq M_0.$$

Hence, $Ax \in Q$ and $AQ \subset Q$ which prove that $A : Q \to Q$, and AQ is bounded in $C[I,E]$.

We need to prove now that $A : Q \to Q$ is weakly sequentially continuous. Let $\{x_n(t)\}$ be sequence in Q weakly convergent to $x(t)$ in E, since Q is closed we have $x \in Q$. Fix $t \in I$, since f satisfies (ii), then we have $f(t,x_n(t))$ converges weakly to $f(t,x(t))$. By the Lebesgue dominated convergence theorem (see assumption (iii)) for Pettis integral ([12]), we have for each $\phi \in E^*,\ s \in I$

$$\phi(\int_0^t f(s,x_n(s))\ d_s g(t,s)) = \int_0^t \phi(f(s,x_n(s)))\ d_s g(t,s)$$

$$\to \int_0^t \phi(f(s,x(s)))\ d_s g(t,s),\ \forall \phi \in E^*,\ t \in I,$$

i.e. $\phi(Ax_n(t)) \to \phi(Ax(t)),\ \forall\ t \in I,\ Ax_n(t)$ converging weakly to $Ax(t)$ in E.

Thus, A is weakly sequentially continuous on Q.

Next we show that $AQ(t)$ is relatively weakly compact in E.

Note that Q is nonempty, closed, convex and uniformly bounded subset of $C[I,E]$ and AQ is bounded in norm. According to Propositions 1.6 and 1.7, AQ is relatively weakly compact in $C[I,E]$ implies $AQ(t)$ is relatively weakly compact in E, for each $t \in I$.

Since all conditions of Theorem 1.5 are satisfied, then the operator A has at least one fixed point $x \in Q$ and the nonlinear Stieltjes integral equation (2.5) has at least one weak solution $x \in C[I,E]$. □

3. Volterra integral equation of fractional order

In this section we show that the Volterra integral equation of fractional order

$$x(t) = p(t) + \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} f(s, x(s))\ ds,\ t \in I \tag{3.7}$$

can be considered as a special case of the Volterra-Stieltjes integral equation (2.1), where the integral is in the sense of weakly Riemann.
First, consider, as previously, that the function $g(t, s) = g : \Lambda \to R$. Moreover, we will assume that the function g satisfies the following condition

(vi') For $t_1, t_2 \in I$, $t_1 < t_2$, the function $s \to g(t_2, s) - g(t_1, s)$ is nonincreasing on $[0, t_1]$.

Now, we have the following lemmas which proved by Banaś et al. [5].

Lemma 3.1. *Under assumptions (vi$^{'}$) and (vii), for any fixed $s \in I$, the function $t \to g(t, s)$ is nonincreasing on $[s, 1]$.*

Lemma 3.2. *Under assumptions (iv), (vi$^{'}$) and (vii), the function g satisfies assumption (vi).*

Consider the function g defined by

$$g(t, s) = \frac{t^\alpha - (t-s)^\alpha}{\Gamma(\alpha+1)}. \tag{3.8}$$

Now, we show that the function g satisfies assumptions (iv), (v), (vi$^{'}$) and (vii).
Clearly that the function g satisfies assumptions (iv) and (vii). Also we get

$$d_s g(t, s) = \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} > 0$$

for $0 \le s < t$. This implies that $s \to g(t, s)$ is increasing on $[0, t]$ for any fixed $t \in I$. Thus the function g satisfies assumption (v).
To show that g satisfies assumption (vi$^{'}$), let us fix arbitrary $t_1, t_2 \in [0, T],\ t_1 < t_2$. Then we get

$$G(s) = g(t_2, s) - g(t_1, s) = \frac{t_2^\alpha - t_1^\alpha - (t_2 - s)^\alpha + (t_1 - s)^\alpha}{\Gamma(\alpha+1)},$$

define on $[0, t_1]$. Thus

$$G'(s) = \frac{(t_2 - s)^{\alpha-1} - (t_1 - s)^{\alpha-1}}{\Gamma(\alpha)} = \frac{1}{\Gamma(\alpha)}[\frac{1}{(t_2-s)^{1-\alpha}} - \frac{1}{(t_1-s)^{1-\alpha}}].$$

Hence $G'(s) < 0$ for $s \in [0, t_1)$. This means that g satisfies assumption (vi$^{'}$). And the function g satisfies assumptions (iv)-(vii) in Theorem 2.5.
Hence, the equation (3.7) can be written in the form

$$x(t) = p(t) + \int_0^t f(s, x(s))\ d_s g(t, s).$$

And the equation (3.7) is a special case of the equation (2.5).
Now, we estimate the constants K, $N(\epsilon)$ used in our proof. To see this, since the function $s \to g(t,s)$ is nondecreasing on $[0,t]$ for any fixed $t \in I$. Then we have

$$\bigvee_{s=0}^{t} g(t,s) = g(t,t) - g(t,0) = g(t,t) = \frac{t^\alpha}{\Gamma(\alpha+1)},$$

and

$$\begin{aligned}
\bigvee_{s=0}^{t_1} (g(t_2,s) - g(t_1,s)) &= \sum_{i=1}^{n} \mid [g(t_2,s_i) - g(t_1,s_i)] - [g(t_2,s_{i-1}) - g(t_1,s_{i-1})] \mid \\
&= \sum_{i=1}^{n} \{[g(t_2,s_{i-1}) - g(t_1,s_{i-1})] - [g(t_2,s_i) - g(t_1,s_i)]\} \\
&= g(t_1,t_1) - g(t_2,t_1) \\
&= \frac{1}{\Gamma(\alpha+1)}[t_1^\alpha - t_2^\alpha + (t_2-t_1)^\alpha].
\end{aligned}$$

Thus

$$K = \sup\{\bigvee_{s=0}^{t} g(t,s) : t \in I\} = \frac{T^\alpha}{\Gamma(\alpha+1)}$$

and

$$\begin{aligned}
N(\epsilon) = \sup\Big\{ \bigvee_{s=0}^{t_1} (g(t_2,s) - g(t_1,s)) : t_1, t_2 \in I,\ t_1 < t_2,\ t_2 - t_1 \le \epsilon \Big\} \\
= \frac{1}{\Gamma(\alpha+1)}[t_1^\alpha - t_2^\alpha + (t_2-t_1)^\alpha].
\end{aligned}$$

Since

$$\begin{aligned}
\bigvee_{s=t_1}^{t_2} g(t_2,s) &= g(t_2,t_2) - g(t_2,t_1) \\
&= \frac{1}{\Gamma(\alpha+1)}[t_2^\alpha - (t_2-t_2)^\alpha - t_2^\alpha + (t_2-t_1)^\alpha] \\
&= \frac{(t_2-t_1)^\alpha}{\Gamma(\alpha+1)}.
\end{aligned}$$

Then

$$\begin{aligned}
Q = \{x \in C[I,E] : \parallel x \parallel_0 \le M_0\ , \\
\parallel x(t_2) - x(t_1) \parallel \le \parallel p(t_2) - p(t_1) \parallel + \frac{M}{\Gamma(\alpha+1)}[\mid t_1^\alpha - t_2^\alpha \mid + 2(t_2-t_1)^\alpha]\}.
\end{aligned}$$

Finally, we can formulate the following existence result concerning the fractional integral equation (3.7).

Theorem 3.3. *Under the assumptions (i)-(iii), the fractional integral equation (3.7) has at least one weak solution $x \in C[I,E]$.*

4. Fractional differential equations

In this section we establish existence results for the fractional differential equations (1.1)-(1.2) and (1.3)-(1.4) in the reflexive Banach space E.

4.1. Weak solution

Consider the integral equation

$$x(t) = \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} f(s, x(s))\ ds,\ t \in I\ , \tag{4.9}$$

where the integral is in the sense of weakly Riemann.

Lemma 4.1. *Let $\alpha \in (0,1)$. A function x is a weak solution of the fractional integral equation (4.9) if and only if x is a solution of the problem (1.1)-(1.2).*

Proof. Integrating (1.1)-(1.2) we obtain the integral equation (4.9). Operating by ${}_RD^\alpha$ on (4.9) we obtain the problem (1.1)-(1.2). So the equivalent between (1.1)-(1.2) and the integral equation (4.9) is proved and then the results follows from Theorem 3.3. □

4.2. Mild solution

Consider now the problem (1.3)-(1.4). According to Definitions 1.1 and 1.3, it is suitable to rewrite the problem (1.3)-(1.4) in the integral equation

$$x(t) = x_0 + \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} f(s, x(s))\ ds,\ t \in I\ . \tag{4.10}$$

Definition 4.2. By the mild solution of the problem (1.3)-(1.4), we mean that the function $x \in C[I, E]$ which satisfies the corresponding integral equation of (1.3)-(1.4) which is (4.10).

Theorem 4.3. *If (i)-(iii) are satisfied, then the problem (1.3)-(1.4) has at least one mild solution $x \in C[I, E]$.*

It is often the case that the problem (1.3)-(1.4) does not have a differentiable solution yet does have a solution, in a mild sense.

References

[1] J. Banaś, *Some properties of Urysohn-Stieltjes integral operators*, Internat. J. Math. and Math. Sci. 21 (1998) 79-88.

[2] J. Banaś, K. Sadarangani, *Solvability of Volterra-Stieltjes operator-integral equations and their applications*, Comput. Math. Appl. 41 12 (2001) 1535-1544.

[3] J. Banaś, J.C. Mena, *Some properties of nonlinear Volterra-Stieltjes integral operators*, Comput. Math. Appl. 49 (2005) 1565-1573.

[4] J. Banaś, D. O'Regan, *Volterra-Stieltjes integral operators*, Math. Comput. Modelling. 41 (2005) 335-344.

[5] J. Banaś, T. Zając, *A new approach to the theory of functional integral equations of fractional order*, J. Math. Anal. Appl. 375 (2011) 375-387.

[6] M. Benchohra, F. Mostefai, *Weak solutions for nonlinear fractional differential equations with integral boundary conditions in Banach spaces*, Opuscula Mathematica 32 1 (2012) 31-40.

[7] M. Benchohra, J.R. Graef and F. Mostefai, *Weak solutions for nonlinear fractional differential equations on reflexive Banach spaces*, Electron. J. Qual. Theory Differ. Equ. 54 (2010) 1-10.

[8] C.W. Bitzer, *Stieltjes-Volterra integral equations*, Illinois J. Math. 14 (1970) 434-451.

[9] M. Caputo, *Linear models of dissipation whose Q is almost frequency independent-II*, Geophys. J.R. Astr. Soc. 13 (1967) 529-539.

[10] N. Dunford, J.T. Schwartz, *Linear Operators*, Interscience, Wiley, New York 1958.

[11] A.M.A. El-Sayed, W.G. El-Sayed and A.A.H. Abd El-Mowla, *Volterra-Stieltjes integral equation in reflexive Banach spaces*, Electronic Journal of Mathematical Analysis and Applications 5 1 (2017) 287-293.

[12] R.F. Geitz, *Pettis integration*, Proc. Amer. Math. Soc. 82 (1981) 81-86.

[13] E. Hille, R.S. Phillips, *Functional Analysis and Semi-groups*, Amer. Math. Soc. Colloq. Publ. Providence, R. I. 1957.

[14] H.H.G. Hashem, *Weak solutions of differential equations in Banach spaces*, Journal of Fractional Calculus and Applications 3 1 (2012) 1-9.

[15] T. Margulies, *Wave propagation in viscoelastic horns using a fractional calculus rheology model*, The Journal of the Acoustical Society of America 114 2442 (2003), https://doi.org/10.1121/1.4779280.

[16] B. Mathieu, P. Melchior, A. Oustaloup and Ch. Ceyral, *Fractional differentiation for edge detection*, Fractional Signal Processing and Applications 83 (2003) 2285-2480.

[17] A.R. Mitchell, Ch. Smith, *An existence theorem for weak solutions of differential equations in Banach spaces*, Nonlinear Equations in Abstract Spaces (V. Lakshmikantham, ed.) (1978) 387-404.

[18] I. Podlubny, *Fractional Differential Equations*, Academic Press, New York 1999.

[19] D. O'Regan, *Fixed point theory for weakly sequentially continuous mapping*, Math. Comput. Modeling 27 (1998) 1-14.

[20] A. Szep, *Existence theorem for weak solutions of ordinary differential equations in reflexive Banach spaces*, Studia Sci. Math. Hungar. 6 (1971) 197-203.

[21] S.G. Samko, A.A. Kilbas and O. Marichev, *Integral and Derivatives of Fractional Orders and Some of Their Applications*, Nauka i Teknika, Minsk 1987.

[22] H.A.H. Salem, A.M.A. El-Sayed, *Weak solution for fractional order integral equations in reflexive Banach spaces*, Math. Slovaca 55 (2005) 169-181.

[23] H.A.H. Salem, A.M.A. El-Sayed, *A note on the fractional calculus in Banach spaces*, Studia Sci. Math. Hungar. 42 2 (2005) 115-130.

[24] H.A.H. Salem, *Quadratic integral equations in reflexive Banach space*, Discuss. Math. Differ. Incl. Control Optim. 30 (2010) 61-69.

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