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New univalence criterions for special general integral operators

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Abstract: In this work we consider some integral operators on the special subclasses of the set of analytic functions in the unit disc which are defined by the Hadamard product. Using the univalence criterions, we obtain new sufficient conditions for these operators to be univalent in the open unit disk. We give some applications of the main results.

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1. Introduction

Let $\mathcal{H}$ denote the class of analytic functions in the unit disc $\mathbb{D} = \{z : |z| < 1\}$ on the complex plane $\mathbb{C}$. Let $\mathcal{A}$ denote the class of all functions $f \in \mathcal{H}$ normalized by $f(0) = 0$, $f'(0) = 1$. Let $\mathcal{S}$ be the subclass of $\mathcal{A}$ whose members are univalent in $\mathbb{D}$. Recall that a set $E \subset \mathbb{C}$ is said to be starlike with respect to a point $w_0 \in E$ if and only if the linear segment joining w_0 to every other point $w \in E$ lies entirely in E, while a set E is said to be convex if and only if it is starlike with respect to each of its points, that is if and only if the linear segment joining any two points of E lies entirely in E. The set of all functions $f \in \mathcal{A}$ that are starlike univalent in $\mathbb{D}$ will be denoted by $\mathcal{S}^*$. The set of all functions $f \in \mathcal{A}$ that are convex univalent in $\mathbb{D}$ by $\mathcal{K}$. Robertson introduced in [6] the classes $\mathcal{S}^*(\alpha)$, $\mathcal{K}(\alpha)$ of starlike and convex functions of order $\alpha \le 1$, which are defined by

$$\mathcal{S}^*(\alpha) := \left\{ f \in \mathcal{A} : \ \mathfrak{Re}\left\{\frac{zf'(z)}{f(z)}\right\} > \alpha, \quad \text{for all } z \in \mathbb{D} \right\}, \tag{1.1}$$

$$\mathcal{K}(\alpha) := \left\{ f \in \mathcal{A} : \ \mathfrak{Re}\left\{1 + \frac{zf''(z)}{f'(z)}\right\} > \alpha, \quad \text{for all } z \in \mathbb{D} \right\}. \tag{1.2}$$

If $\alpha \in [0;1)$, then a function in either of these sets is univalent, if $\alpha < 0$ it may fail to be univalent. In particular we denote $\mathcal{S}^*(0) = \mathcal{S}^*$, $\mathcal{K}(0) = \mathcal{K}$. For functions $f, g \in \mathcal{H}$ given by

$$f(z) = \sum_{n=1}^{\infty} a_n z^n, \quad g(z) = \sum_{n=1}^{\infty} b_n z^n; \; (|z| < 1), \tag{1.3}$$

their Hadamard product (or convolution) is defined by:

$$(f * g)(z) = \sum_{n=1}^{\infty} a_n b_n z^n; \; (|z| < 1). \tag{1.4}$$

For a function $g \in \mathcal{H}$ we define the subclass $\mathcal{S}^*(g; a, b, \lambda)$ of functions $f \in \mathcal{H}$ satisfying the condition:

$$\left| \frac{z^{\lambda}(f * g)'(z)}{(f * g)^{\lambda}(z)} - a \right| < b; \; (z \in \mathbb{D}, |a - 1| < b \leq a, \lambda \geq 1), \tag{1.5}$$

such that $(f * g)(z) \neq 0$. If $f \in \mathcal{A}, g(z) = z/(1 - z)$ $(|z| < 1)$ and $\lambda = 1$ we have $(f * g)(z) = f(z)$ and so:

$$\mathcal{S}^*(g; a, b, \lambda) = \mathcal{S}_0^*(a, b) = \left\{ f \in \mathcal{A} : \left| \frac{zf'(z)}{f(z)} - a \right| < b, z \in \mathbb{D}, |a - 1| < b \leq a \right\},$$

where the class $\mathcal{S}_0^*(a, b)$ was introduced and studied by Jakubowski in 1972 (see [4]). Note that

$$\mathcal{S}_0^*(a, b) \subseteq \mathcal{S}_0^*(a - b) \subseteq \mathcal{S}^*(0) = \mathcal{S}^* \subseteq \mathcal{S}.$$

Recently Deniz, Raducanu and Orhan [2] defined the following general integral operator:

$$F_{\alpha,\beta}(z) = \left\{ \beta \int_0^z t^{\beta - 1} \prod_{i=1}^{n} \left(\frac{(f_i * g_i)(t)}{h_i(t)} \right)^{\alpha_i} \mathrm{d}t \right\}^{\frac{1}{\beta}}; \; (\alpha_i, \beta \in \mathbb{C}, z \in \mathbb{D}), \tag{1.6}$$

where $f_i, g_i, h_i \in \mathcal{A}$, $\mathfrak{Re}(\beta) > 0$, $\alpha = (\alpha_1, \ldots, \alpha_n)$ and $(f_i * g_i)(z)/h_i(z) \neq 0$. Note that all powers in (1.6) are principal ones.

Using the convolution, we introduce the following integral operator:

$$G_{\alpha,\beta}(z) = \int_0^z \prod_{i=1}^{n} ((f_i * g_i)'(t))^{\alpha_i} \left(\frac{(f_i * g_i)(t)}{h_i(t)} \right)^{\beta_i} \mathrm{d}t; \; (\alpha_i, \beta_i \in \mathbb{C}, z \in \mathbb{D}), \tag{1.7}$$

where $\alpha = (\alpha_1, \ldots, \alpha_n)$, $\beta = (\beta_1, \ldots, \beta_n)$ and $(f_i * g_i)(z)/h_i(z) \neq 0$.

Remark 1.1 It is useful to see that the integral operators $F_{\alpha,\beta}(z)$ and $G_{\alpha,\beta}(z)$ extend some operators defined by many authors, for example:

1) If $f_1 = \ldots = f_n = f$, $g_1 = \ldots = g_n = z/(1-z)$, $h_1 = \ldots = h_n = z$, $\alpha_1 = \ldots = \alpha_n = \alpha$, $\beta_1 = \ldots = \beta_n = \beta$ and $n = 1$, then we obtain the following integral operators:

$$F_{\alpha,\beta}(z) = \left\{\beta \int_0^z t^{\beta-1} \left(\frac{f(t)}{t}\right)^{\alpha} \mathrm{d}t\right\}^{\frac{1}{\beta}}$$

and

$$G_{\alpha,\beta}(z) = \int_0^z (f'(t))^{\alpha} \left(\frac{f(t)}{t}\right)^{\beta} \mathrm{d}t.$$

2) For $g_1 = \ldots = g_n = \frac{z}{(1-z)^2}$, $h_1 = \ldots = h_n = z$, we obtain the integral operators:

$$F_{\alpha,\beta}(z) = \left\{\beta \int_0^z t^{\beta-1} \prod_{i=1}^{n} (f_i'(t))^{\alpha_i} \mathrm{d}t\right\}^{\frac{1}{\beta}}$$

and

$$G_{\alpha,\beta}(z) = \int_0^z \prod_{i=1}^{n} (f_i'(t) + t f_i''(t))^{\alpha_i} (f_i'(t))^{\beta_i} \mathrm{d}t.$$

In this paper we give new sufficient conditions for the operators $F_{\alpha,\beta}(z)$ and $G_{\alpha,\beta}(z)$ to be univalent in $\mathbb{D}$, where the functions f_i belong to the class $\mathcal{S}^*(g_i; a_i, b_i, \lambda_i)$ for all $i = 1, \ldots, n$. In order to get our main results we will use the following lemmas, so we recall them here.

Lemma 1.1 *[5] Let $\beta \in \mathbb{C}$ with $\mathfrak{Re}(\beta) > 0$. If $f \in \mathcal{A}$ satisfies:*

$$\frac{1 - |z|^{2\mathfrak{Re}(\beta)}}{\mathfrak{Re}(\beta)} \left|\frac{z f''(z)}{f'(z)}\right| \le 1, \quad (z \in \mathbb{D}),$$

then the function

$$F_{\beta}(z) = \left\{\beta \int_0^z t^{\beta-1} f'(t) \mathrm{d}t\right\}^{\frac{1}{\beta}} \tag{1.8}$$

is univalent in $\mathbb{D}$.

2. Main Results

Using the previous lemmas, we state and prove the following:

Theorem 2.1 *Let $f_i \in \mathcal{S}^*(g_i; a_i, b_i, \lambda_i)$, $g_i, h_i \in \mathcal{A}$ for all $i = 1, \ldots, n$ and let $|(f_i * g_i)(z)| < M$ for all $z \in \mathbb{D}$ and $M > 0$. Assume also that $|z h_i'(z)/h_i(z)| \le 1$. If $c \in \mathbb{C} \setminus \{-1\}$ and β with $\mathfrak{Re}(\beta) > 0$*

$$\mathfrak{Re}(\beta) \ge \left(\sum_{i=1}^{n} |\alpha_i| \left(1 + (a_i + b_i) M^{\lambda_i - 1}\right)\right), \ (\alpha_i \in \mathbb{C}),$$

then the function defined by (1.6) is univalent in $\mathbb{D}$.

Proof. Define the function:

$$\phi(z) = \int_0^z \prod_{i=1}^{n} \left(\frac{(f_i * g_i)(t)}{h_i(t)} \right)^{\alpha_i} \mathrm{d}t, \ (\alpha_i \in \mathbb{C}, z \in \mathbb{D}).$$

Then we have $\phi(0) = 0, \phi'(0) = 1$ and:

$$\phi'(z) = \prod_{i=1}^{n} \left(\frac{(f_i * g_i)(z)}{h_i(z)} \right)^{\alpha_i}, \tag{2.1}$$

also $\phi(z)$ is analytic in $\mathbb{D}$. From (2.1) we obtain:

$$\frac{z\phi''(z)}{\phi'(z)} = \sum_{i=1}^{n} \alpha_i \left(\frac{z(f_i * g_i)'(z)}{(f_i * g_i)(z)} - \frac{zh_i'(z)}{h_i(z)} \right), \ (z \in \mathbb{D}). \tag{2.2}$$

Since $f_i \in \mathcal{S}^*(g_i; a_i, b_i, \lambda_i)$ and $|(f_i * g_i)(z)| < M$, by the well-known Schwarz lemma in complex analysis, we see that:

$$\begin{aligned}
\left|\frac{z\phi''(z)}{\phi'(z)}\right| &= \left|\sum_{i=1}^{n} \alpha_i \left(\frac{z(f_i * g_i)'(z)}{(f_i * g_i)(z)} - \frac{zh_i'(z)}{h_i(z)} \right)\right| \\
&\leq \sum_{i=1}^{n} |\alpha_i| \left(\left|\frac{z^{\lambda_i}(f_i * g_i)'(z)}{(f_i * g_i)^{\lambda_i}(z)}\right| \left|\frac{(f_i * g_i)(z)}{z}\right|^{\lambda_i - 1} + \left|\frac{zh_i'(z)}{h_i(z)}\right| \right) \\
&\leq \sum_{i=1}^{n} |\alpha_i| \left(1 + (a_i + b_i)M^{\lambda_i - 1}\right).
\end{aligned}$$

Now the last inequality shows that:

$$\begin{aligned}
\frac{1 - |z|^{2\mathfrak{Re}(\beta)}}{\mathfrak{Re}(\beta)} \left|\frac{z\phi''(z)}{\phi'(z)}\right| &\leq \frac{1}{\mathfrak{Re}(\beta)} \left|\frac{z\phi''(z)}{\phi'(z)}\right| \\
&\leq \frac{1}{\mathfrak{Re}(\beta)} \left(\sum_{i=1}^{n} |\alpha_i| \left(1 + (a_i + b_i)M^{\lambda_i - 1}\right) \right) \\
&\leq 1.
\end{aligned}$$

Applying Lemma 1.1 for the function $\phi(z)$, we conclude that $F_{\alpha,\beta}(z) \in \mathcal{S}$. ■

Letting $a_1 = b_1 = n = \lambda_1, f_1 = f, g_1 = z/(1-z)^2$ and $h_1(z) = z$ in Theorem 2.1, we obtain the following result.

Corollary 2.1 *Let* $f \in \mathcal{A}, |f(z)| < M$ *for all* $z \in \mathbb{D}$ *and* $|zf_i'(z)/f_i(z)| \leq 1$. *If* $\alpha, \beta \in \mathbb{C}$ *with* $\mathfrak{Re}(\beta) \geq 3|\alpha| > 0$, *then the function:*

$$F_{\alpha,\beta}(z) = \left\{ \beta \int_0^z t^{\beta-1} (f'(t))^{\alpha} \mathrm{d}t \right\}^{\frac{1}{\beta}},$$

is univalent in $\mathbb{D}$.

We next give some sufficient conditions for the operator $G_{\alpha,\beta}(z)$ to be univalent in $\mathbb{D}$.

Theorem 2.2 *Let for* $i = 1, \dots, n, g_i, h_i \in \mathcal{A}, f_i \in \mathcal{A} \cap \mathcal{S}^*(g_i; a_i, b_i, \lambda_i)$ *and:*

$$|(f_i * g_i)(z)| < M; \ (z \in \mathbb{D}, M > 0).$$

If $f_i' \in \mathcal{S}^*(g_i(z)/z; a_i', b_i', 1)$ *and* $|zh_i'(z)/h_i(z)| \le 1$ *for all* $z \in \mathbb{D}, i = 1, \dots, n,$ *also:*

$$\sum_{i=1}^{n} |\alpha_i|(a_i' + b_i') + |\beta_i| \left(1 + (a_i + b_i)M^{\lambda_i - 1}\right) \le 1,$$

then the function $G_{\alpha,\beta}(z)$ *defined by* (1.7) *is univalent in* $\mathbb{D}$.

Proof. Define the function $\phi(z)$ by:

$$\phi(z) = G_{\alpha,\beta}(z) = \int_0^z \prod_{i=1}^{n} ((f_i * g_i)'(t))^{\alpha_i} \left(\frac{(f_i * g_i)(t)}{h_i(t)}\right)^{\beta_i} \mathrm{d}t; \ (\alpha_i, \beta_i \in \mathbb{C}, z \in \mathbb{D}),$$

then we have:

$$\phi'(z) = \prod_{i=1}^{n} ((f_i * g_i)'(z))^{\alpha_i} \left(\frac{(f_i * g_i)(z)}{h_i(z)}\right)^{\beta_i},$$

$\phi(0) = 0, \phi'(0) = 1$ and $\phi(z)$ is analytic in $\mathbb{D}$. Differentiating logaritmically from $\phi'(z)$, we obtain:

$$\frac{z\phi''(z)}{\phi'(z)} = \sum_{i=1}^{n} \alpha_i \frac{z(f_i * g_i)''(z)}{(f_i * g_i)'(z)} + \beta_i \left(\frac{z(f_i * g_i)'(z)}{(f_i * g_i)(z)} - \frac{zh_i'(z)}{h_i(z)}\right).$$

Because $f_i, g_i \in \mathcal{A}$, we see that:

$$\frac{(f_i * g_i)''(z)}{(f_i * g_i)'(z)} = \frac{(f_i' * (g_i/z))'(z)}{(f_i' * (g_i/z))(z)}.$$

Now, by suppositions of theorem, we find that:

$$\begin{aligned} \left|\frac{z\phi''(z)}{\phi'(z)}\right| &\le \sum_{i=1}^{n} \left\{ |\alpha_i| \left|\frac{z(f_i' * (g_i/z))'(z)}{(f_i' * (g_i/z))(z)}\right| + \right. \\ &+ \left. |\beta_i| \left(\left|\frac{z^{\lambda_i}(f_i * g_i)'(z)}{(f_i * g_i)^{\lambda_i}(z)}\right| \cdot \left|\frac{(f_i * g_i)(z)}{z}\right|^{\lambda_i - 1} + \left|\frac{zh_i'(z)}{h_i(z)}\right| \right) \right\} \\ &\le \sum_{i=1}^{n} |\alpha_i|(a_i' + b_i') + |\beta_i| \left(1 + (a_i + b_i)M^{\lambda_i - 1}\right). \end{aligned}$$

Using Lemma 1.1 and the last inequality, we conclude that $G_{\alpha,\beta}(z) \in \mathcal{S}$. ■

Taking $a_i = b_i = \lambda_i = a_i' = b_i' = 1, f_i = f, g_i = z/(1-z)^2$ and $h_i(z) = z$ for $i = 1, \dots, n$, in Theorem 2.2, we obtain the following result.

Corollary 2.2 *Let $f \in \mathcal{A}, |zf''(z)/f'(z)| < 1$ and $|zf'(z)| < M$ with $M > 0$ for all $z \in \mathbb{D}$. If:*

$$\left|\frac{z(2f'' + zf''')}{f' + zf''} - 1\right| < 1 \ (z \in \mathbb{D}),$$

and $\sum_{i=1}^{n} 2|\alpha_i| + 3|\beta_i| \le 1$, for $\alpha_i, \beta_i \in \mathbb{C}$, then the function:

$$G_{\alpha,\beta}(z) = \int_0^z (f'(t) + tf''(t))^{n\alpha_i} (f'(t))^{n\beta_i} \mathrm{d}t,$$

is univalent in $\mathbb{D}$.

Proof. Because $g_i(z) = \frac{z}{(1-z)^2} = g(z)$, it is easy to see that, $(f * g)(z) = zf'(z)$, also $f' \in \mathcal{S}^*(g(z)/z; 1, 1, 1)$ if and only if:

$$\left|\frac{z(2f'' + zf''')}{f' + zf''} - 1\right| < 1.$$

This completes the proof. ■

Using the proof of theorem (2.2), we obtain another result:

Corollary 2.3 *Suppose all conditions of Theorem 2.2 are satisfied, then the function $G_{\alpha,\beta}(z)$ defined by (1.7) is in $\mathcal{K}(\gamma)$, where:*

$$\gamma = 1 - \sum_{i=1}^{n} |\alpha_i|(a_i' + b_i') + |\beta_i| \left(1 + (a_i + b_i)M^{\lambda_i - 1}\right).$$

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