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On the Alternative Structures for a Three-Grade Markov Manpower System

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and Augustine A. Osagiede*

ABSTRACT: This paper considers a manpower system modelled within the Markov chain context under the condition that recruitment is done to replace outgoing flows. The paper takes up the embeddability problem in a three-grade manpower system and examines it from the standpoint of generating function (i.e., the z-transform of stochastic matrices). The method constructs a stochastic matrix that is made up of a limiting-state probability matrix and a partial sum of transient matrices. Examples are provided to illustrate the utility of the method.

AMS Subject Classification: 15A18, 91D35.

Keywords and Phrases: Embeddability problem; Manpower system; Markov chain; Stochastic matrix; Z-transform.

1. Introduction

Mathematical models are often used to describe how changes take place in a manpower system, where individuals move through a network of states which may be defined in terms of ranks or position. One of the widely used approaches to the modeling of manpower systems is the Markov chain framework [1, 7, 9]. The basic Markov chain model for a k -grade manpower system is expressed algebraically using the following recursive relation

$$n_j(t+1) = \sum_{i=1}^k n_i(t)p_{ij} + R(t+1)r_j, \quad j = 1, 2, \dots, k, \quad (1.1)$$

where $n_i(t)$ is the expected number of individuals in state i at time t , p_{ij} is the internal homogeneous transition probability from state i to state j , r_j is the proportion of recruits allocated to state j and $R(t+1)$ is the expected number of recruits to the system at time $t+1$. The manpower accounts for the system are assumed to take place at the end of the time period and recruitment is recorded as if it took place at the beginning of the next time period [1]. The transition probabilities, p_{ij} 's, are estimated based on data from observable variables using the maximum likelihood method [14]. In many practical instances, the transition probability, p_{ij} , satisfies the conditions: $\sum_{j=1}^k p_{ij} \leq 1$, $i \in S$, $p_{ij} \geq 0$, $i, j \in S$, where $S = \{1, 2, \dots, k\}$ is the set of mutually exclusive and collectively exhaustive states of the k -grade manpower system. The shortfall in the sum $\sum_{j=1}^k p_{ij} \leq 1$ is attributed to outgoing flows (wastage) from the system. With w_i as the wastage from the system,

$$\sum_{j=1}^k p_{ij} + w_i = 1, \quad i \in S. \quad (1.2)$$

The recursive relation in equation (1.1) can be rewritten in matrix notation as

$$\mathbf{n}(t+1) = \mathbf{n}(t)\mathbf{P} + R(t+1)\mathbf{r}, \quad (1.3)$$

where $\mathbf{n}(t) = [n_1(t), n_2(t), \dots, n_k(t)]$ is the structure of the system at any given time t , $\mathbf{P} = (p_{ij})$ is the homogeneous transition matrix and $\mathbf{r} = [r_1, r_2, \dots, r_k]$ is the recruitment vector with $\sum_{j=1}^k r_j = 1$. Let $\mathbf{w} = [w_1, w_2, \dots, w_k]$ denote the wastage vector for the system. Since a fixed size manpower system is considered, where wastage is replaced by new recruits, the expected number of recruits to the system at time $t+1$ is

$$R(t+1) = \mathbf{n}(t)\mathbf{w}'. \quad (1.4)$$

Thus, equation (1.3) can be expressed as

$$\mathbf{n}(t+1) = \mathbf{n}(t)(\mathbf{P} + \mathbf{w}'\mathbf{r}), \quad (1.5)$$

where $(\mathbf{P} + \mathbf{w}'\mathbf{r})$ is a stochastic matrix. Equation (1.5) is suitable to predict what the manpower structure will become one-step ahead year after year. If the manpower structure is to be maintained, then $\mathbf{n}(t+1) = \mathbf{n}(t) = \mathbf{n}$ in equation (1.5), cf. [13].

Suppose for motivational reasons, that the manpower structure is to be projected for a semester beyond one-step (that is, one year and six months) or a quarter beyond one-step (that is, one year and three months). Then representation becomes an issue when we have the fractional indicial stochastic matrix, $(\mathbf{P} + \mathbf{w}'\mathbf{r})^{1+1/n}$, for $n = 2$ or 4 . This problem is an embeddability problem. Singer and Spilerman [11] considered the embeddability problem by verifying whether an observed transition matrix could have arisen from the evolution of a stationary continuous-time Markov process. The approach does not give a unique solution. Osagiede and Ekhosuehi [10] solved the embeddability problem for a manpower system with sparse stochastic matrices within the context of determining the nearest Markov generator arising from the continuous-time Markov chain to the higher order observable Markov chain. The resulting Markov

chain was an approximation to the higher order observable Markov chain. In [6], the problem was solved by finding the diagonalizable form of the observable Markov chain.

This study considers a three-grade manpower system, that is, $k = 3$. Markovian manpower systems with three grades arise in many practical situations [1, 3, 4, 7, 8, 13]. Following [12], the study assumes a fixed size manpower system that operates a policy that allows wastage to be replaced by new recruits. In this case, the consequential outflow from state i which goes back to state j as recruitment would be $w_i r_j$, $i, j \in S$. The study is aimed at finding the fractional indicial stochastic matrix, $(\mathbf{P} + \mathbf{w}'\mathbf{r})^{1+1/n}$, arising from a hierarchical manpower system with three grades using the generating function technique (the so called z-transform). This approach that is based on z-transform has been used to model population dynamics within the Leslie matrices framework [2]. The study develops an additive representation for the stochastic matrix describing the evolution of the personnel structure of a Markov manpower system with fixed total size. The assumption of a fixed total size for manpower system is appropriate in practice when an organization is faced with limited personnel availability on the external labour market, facility and budget restrictions [8]. The usefulness of the additive representation is justified when there is a lack of observations regarding the time unit of the Markov chain (that was earlier estimated using historical data in discrete time) owing to a policy change in the short-term on the effective date of promotion. For instance, extending the effective date of promotion from October 1 of the current year to January 1 of the following year for budgetary reasons. This kind of policy change is dealt with in the additive representation.

2. The generating function standpoint

In this section, we prove the following using the z-transform: If $\mathbf{Q} = (\mathbf{P} + \mathbf{w}'\mathbf{r}) \in \mathbb{R}^{3 \times 3}$ is a stochastic matrix that satisfies the axioms that: (i) \mathbf{Q} is irreducible, (ii) the determinant of \mathbf{Q} is non-singular, and (iii) the characteristic polynomial arising from the determinant $\det(\mathbf{I} - \mathbf{Q}z)$ has linear factors, then the fractional indicial stochastic matrix, $\mathbf{\Gamma} = \mathbf{Q}^{1+1/n}$, $n > 0$, can be expressed in the form

$$\mathbf{\Gamma} = \left\{ \mathbf{X} = (x_{ij}) \in \mathbb{R}^{3 \times 3} \mid \mathbf{X} = \mathbf{A}_m + \mathbf{T}_m(1 + 1/n), \sum_{j=1}^3 x_{ij} = 1, x_{ij} \geq 0, \forall i, j \in S, m = 1, 2 \right\}, \quad (2.1)$$

where \mathbf{A}_m is the 3×3 matrix of limiting-state probabilities for case m and

$$\mathbf{T}_m(1+1/n) = \begin{cases} \alpha_1^{-(2+1/n)}\mathbf{B}_1 + \alpha_2^{-(2+1/n)}\mathbf{C}, & m = 1 \text{ if } (tr(\mathbf{Q})-1)^2 > 4 \det(\mathbf{Q}) \\ (2+1/n)\alpha^{-(3+1/n)}\mathbf{B}_2 + \alpha^{-(2+1/n)}\mathbf{D}, & m = 2 \text{ if } (tr(\mathbf{Q})-1)^2 = 4 \det(\mathbf{Q}) \end{cases}$$

provided that $\alpha, \alpha_1, \alpha_2 \in \Psi = \{v \mid v > 1, v \in \mathbb{R}\}$ with $\alpha, \alpha_1, \alpha_2$ being the zeros of the characteristic function $\det(\mathbf{I} - \mathbf{Q}z) = 1 - tr(\mathbf{Q})z + \left(\sum_{i=1}^3 Q_{ii}\right)z^2 - \det(\mathbf{Q})z^3$ with

Q_{ii} being the cofactor of the diagonal entries in \mathbf{Q} , and \mathbf{B}_m , \mathbf{C} , \mathbf{D} are matrices of constant values for each respective case m .

Consider the recurrence relation in equation (1.5): Using the z-transform, the generation function vector $\mathbf{g}(z)$ that is associated with the manpower structure $\mathbf{n}(t)$ is defined by

$$\mathbf{g}(z) = \sum_{t=0}^{\infty} \mathbf{n}(t)z^t. \quad (2.2)$$

Thus,

$$\mathbf{g}(z)\mathbf{Q} = \sum_{t=0}^{\infty} \mathbf{n}(t)\mathbf{Q}z^t = \sum_{t=0}^{\infty} \mathbf{n}(t+1)z^t = \frac{1}{z} \sum_{t=0}^{\infty} \mathbf{n}(t+1)z^{t+1} = \frac{1}{z}(\mathbf{g}(z) - \mathbf{n}(0)),$$

where $\mathbf{n}(0)$ is the initial manpower structure. Further simplifications lead to

$$\mathbf{g}(z) = \mathbf{n}(0) [\mathbf{I} - \mathbf{Q}z]^{-1}.$$

Let

$$\mathbf{G}(z) = [\mathbf{I} - \mathbf{Q}z]^{-1} = \sum_{t=0}^{\infty} \mathbf{Q}^t z^t, \quad \mathbf{Q}^0 = \mathbf{I}, \quad (2.3)$$

where $\mathbf{G}(z)$ is the 3×3 Green function matrix and \mathbf{I} is the 3×3 identity matrix. Since

$$\mathbf{Q} = \begin{bmatrix} p_{11} & p_{12} & p_{13} \\ p_{21} & p_{22} & p_{23} \\ p_{31} & p_{32} & p_{33} \end{bmatrix} + \begin{bmatrix} w_1 \\ w_2 \\ w_3 \end{bmatrix} \begin{bmatrix} r_1 & r_2 & r_3 \end{bmatrix} = (q_{ij}),$$

where $q_{ij} = p_{ij} + w_i r_j$, $i, j \in S$, then

$$\mathbf{I} - \mathbf{Q}z = \begin{bmatrix} 1 - q_{11}z & -q_{12}z & -q_{13}z \\ -q_{21}z & 1 - q_{22}z & -q_{23}z \\ -q_{31}z & -q_{32}z & 1 - q_{33}z \end{bmatrix}.$$

The inverse of $\mathbf{I} - \mathbf{Q}z$ is defined as

$$[\mathbf{I} - \mathbf{Q}z]^{-1} = \frac{\text{adj}(\mathbf{I} - \mathbf{Q}z)}{\det(\mathbf{I} - \mathbf{Q}z)}. \quad (2.4)$$

The determinant, $\det(\mathbf{I} - \mathbf{Q}z)$, is obtained as follows: Factorizing $(1 - q_{11}z)$, $q_{12}z$, $q_{13}z$ from column 1, 2, 3 respectively of $\det(\mathbf{I} - \mathbf{Q}z)$ yields

$$\det(\mathbf{I} - \mathbf{Q}z) = (1 - q_{11}z)q_{12}q_{13}z^2 \begin{vmatrix} 1 & -1 & -1 \\ -\frac{q_{21}z}{(1-q_{11}z)} & \frac{1-q_{22}z}{q_{12}z} & -\frac{q_{23}}{q_{13}} \\ -\frac{q_{31}z}{(1-q_{11}z)} & -\frac{q_{32}}{q_{12}} & \frac{1-q_{33}z}{q_{13}z} \end{vmatrix}.$$

Subtracting column 2 from column 3, we have

$$\det(\mathbf{I} - \mathbf{Q}z) = (1 - q_{11}z)q_{12}q_{13}z^2 \begin{vmatrix} 1 & -1 & 0 \\ -\frac{q_{21}z}{(1-q_{11}z)} & \frac{1-q_{22}z}{q_{12}z} & -\frac{q_{23}}{q_{13}} - \frac{1-q_{22}z}{q_{12}z} \\ -\frac{q_{31}z}{(1-q_{11}z)} & -\frac{q_{32}}{q_{12}} & \frac{1-q_{33}z}{q_{13}z} + \frac{q_{32}}{q_{12}} \end{vmatrix}.$$

Adding column 1 to column 2,

$$\det(\mathbf{I} - \mathbf{Q}z) = (1 - q_{11}z)q_{12}q_{13}z^2 \begin{vmatrix} 1 & 0 & 0 \\ -\frac{q_{21}z}{(1-q_{11}z)} & \frac{1-q_{22}z}{q_{12}z} - \frac{q_{21}z}{(1-q_{11}z)} & -\frac{q_{23}}{q_{13}} - \frac{1-q_{22}z}{q_{12}z} \\ -\frac{q_{31}z}{(1-q_{11}z)} & -\frac{q_{32}}{q_{12}} - \frac{q_{31}z}{(1-q_{11}z)} & \frac{1-q_{33}z}{q_{13}z} + \frac{q_{32}}{q_{12}} \end{vmatrix}.$$

Taking the determinant

$$\det(\mathbf{I} - \mathbf{Q}z) = (1 - q_{11}z)q_{12}q_{13}z^2 \left(\left(\frac{1 - q_{22}z}{q_{12}z} - \frac{q_{21}z}{(1 - q_{11}z)} \right) \left(\frac{1 - q_{33}z}{q_{13}z} + \frac{q_{32}}{q_{12}} \right) - \left(\frac{q_{23}}{q_{13}} - \frac{1 - q_{22}z}{q_{12}z} \right) \left(\frac{q_{32}}{q_{12}} - \frac{q_{31}z}{(1 - q_{11}z)} \right) \right).$$

This simplifies to

$$\det(\mathbf{I} - \mathbf{Q}z) = 1 - (q_{11} + q_{22} + q_{33})z + (q_{11}q_{22} + q_{11}q_{33} + q_{22}q_{33} - q_{21}q_{12} - q_{23}q_{32} - q_{31}q_{13})z^2 - (q_{11}q_{22}q_{33} - q_{21}q_{12}q_{33} + q_{21}q_{32}q_{13} - q_{23}q_{11}q_{32} + q_{23}q_{12}q_{31} - q_{13}q_{22}q_{31})z^3.$$

Thus

$$\det(\mathbf{I} - \mathbf{Q}z) = 1 - \text{tr}(\mathbf{Q})z + \left(\sum_{i=1}^3 Q_{ii} \right) z^2 - \det(\mathbf{Q})z^3. \quad (2.5)$$

Now $(1 - z)$ is a factor of the cubic characteristic function (2.5) since at $z = 1$,

$$1 - \text{tr}(\mathbf{Q}) + \left(\sum_{i=1}^3 Q_{ii} \right) - \det(\mathbf{Q}) = \begin{vmatrix} 1 - q_{11} & -q_{12} & -q_{13} \\ -q_{21} & 1 - q_{22} & -q_{23} \\ -q_{31} & -q_{32} & 1 - q_{33} \end{vmatrix}. \quad (2.6)$$

Equation (2.6) simplifies to

$$\begin{vmatrix} q_{12} + q_{13} & -q_{12} & -q_{13} \\ -q_{21} & q_{21} + q_{23} & -q_{23} \\ -q_{31} & -q_{32} & q_{31} + q_{32} \end{vmatrix} = \begin{vmatrix} q_{13} & -q_{12} & -q_{13} \\ q_{23} & q_{21} + q_{23} & -q_{23} \\ -(q_{31} + q_{32}) & -q_{32} & q_{31} + q_{32} \end{vmatrix} = 0,$$

as column 1 and column 3 are identical. It follows that

$$\det(\mathbf{I} - \mathbf{Q}z) = (1 - z) (1 - (\text{tr}(\mathbf{Q}) - 1)z + \det(\mathbf{Q})z^2). \quad (2.7)$$

Using the fundamental theorem of algebra, equation (2.7) is expressed as

$$\det(\mathbf{I} - \mathbf{Q}z) = \det(\mathbf{Q})(1 - z)(\alpha_1 - z)(\alpha_2 - z), \quad (2.8)$$

where

$$\alpha_1 = \frac{\text{tr}(\mathbf{Q} - 1)}{2 \det(\mathbf{Q})} \left(1 - \left(1 - \frac{4 \det(\mathbf{Q})}{(\text{tr}(\mathbf{Q}) - 1)^2} \right)^{1/2} \right)$$

and

$$\alpha_2 = \frac{\text{tr}(\mathbf{Q} - 1)}{2 \det(\mathbf{Q})} \left(1 + \left(1 - \frac{4 \det(\mathbf{Q})}{(\text{tr}(\mathbf{Q}) - 1)^2} \right)^{1/2} \right),$$

provided that $\det(\mathbf{Q}) \neq 0$. The roots α_1 and α_2 are real if $(\text{tr}(\mathbf{Q}) - 1)^2 \geq 4 \det(\mathbf{Q})$. If $(\text{tr}(\mathbf{Q}) - 1)^2 < 4 \det(\mathbf{Q})$, α_1 and α_2 would produce complex entries and these have no meaning within the context of Markov chains. Thus, the case where the quadratic form $(1 - (\text{tr}(\mathbf{Q}) - 1)z + \det(\mathbf{Q})z^2)$ does not have linear factors is not considered. Moreover, it is difficult to simplify the reciprocal of $(1 - (\text{tr}(\mathbf{Q}) - 1)z + \det(\mathbf{Q})z^2)$ as a series in the form $\sum_{r=0}^{\infty} \theta^r z^r$, where θ is independent of z . More specifically,

$$\frac{1}{(1 - (\text{tr}(\mathbf{Q}) - 1)z + \det(\mathbf{Q})z^2)} = \sum_{r=0}^{\infty} \left(\sum_{s=0}^r (-1)^s \binom{r}{s} (\det(\mathbf{Q}))^s (\text{tr}(\mathbf{Q}) - 1)^{r-s} z^s \right) z^r.$$

However, the reciprocal of each of the factors in equation (2.8) when α_1 and α_2 are real can be expressed in the following series

$$\frac{1}{1 - z} = \sum_{t=0}^{\infty} z^t. \quad (2.9)$$

$$\frac{1}{\alpha - z} = \sum_{t=0}^{\infty} \alpha^{-(1+t)} z^t. \quad (2.10)$$

$$\frac{1}{(\alpha - z)^2} = \sum_{t=0}^{\infty} (1 + t) \alpha^{-(2+t)} z^t. \quad (2.11)$$

To obtain the $\text{adj}(\mathbf{I} - \mathbf{Q}z)$, we first find the cofactors of each entry in $(\mathbf{I} - \mathbf{Q}z)$. The cofactor of $1 - q_{11}z$ is $\Lambda_{11}(z) = 1 - (q_{22} + q_{33})z + (q_{22}q_{33} - q_{23}q_{32})z^2$, the cofactor of $-q_{12}z$ is $\Lambda_{12}(z) = q_{21}z - (q_{21}q_{33} - q_{23}q_{31})z^2$ and so on. Proceeding in this way, the entries in the $\text{adj}(\mathbf{I} - \mathbf{Q}z)$ are found to be a polynomial in z of degree two. More precisely,

$$\text{adj}(\mathbf{I} - \mathbf{Q}z) = \begin{bmatrix} \Lambda_{11}(z) & \Lambda_{21}(z) & \Lambda_{31}(z) \\ \Lambda_{12}(z) & \Lambda_{22}(z) & \Lambda_{32}(z) \\ \Lambda_{13}(z) & \Lambda_{23}(z) & \Lambda_{33}(z) \end{bmatrix},$$

where $\Lambda_{13}(z) = q_{31}z + (q_{21}q_{32} - q_{22}q_{31})z^2$, $\Lambda_{21}(z) = q_{12}z + (q_{13}q_{32} - q_{12}q_{33})z^2$, $\Lambda_{22}(z) = 1 - (q_{11} + q_{33})z + (q_{11}q_{33} - q_{13}q_{31})z^2$, $\Lambda_{23}(z) = q_{32}z - (q_{11}q_{32} - q_{12}q_{31})z^2$, $\Lambda_{31}(z) = q_{13}z + (q_{12}q_{23} - q_{13}q_{22})z^2$, $\Lambda_{32}(z) = q_{23}z - (q_{11}q_{23} - q_{13}q_{21})z^2$ and $\Lambda_{33}(z) = 1 - (q_{11} + q_{22})z + (q_{11}q_{22} - q_{12}q_{21})z^2$.

Resolving the quotient (2.4) into the sum of partial fractions and using the expressions (2.9) to (2.11), we obtain the following results for each case m according to whether $(\text{tr}(\mathbf{Q}) - 1)^2 > 4 \det(\mathbf{Q})$ or $(\text{tr}(\mathbf{Q}) - 1)^2 = 4 \det(\mathbf{Q})$.

Case 1

If $(tr(\mathbf{Q}) - 1)^2 > 4 \det(\mathbf{Q})$, then

$$[\mathbf{I} - \mathbf{Q}z]^{-1} = \sum_{t=0}^{\infty} \frac{1}{\det(\mathbf{Q})} \left(\frac{1}{(\alpha_1 - 1)(\alpha_2 - 1)} \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} + \frac{\alpha_1^{-(1+t)}}{(\alpha_1 - 1)(\alpha_1 - \alpha_2)} \times \right. \\ \left. \begin{bmatrix} b_{11} & b_{12} & b_{13} \\ b_{21} & b_{22} & b_{23} \\ b_{31} & b_{32} & b_{33} \end{bmatrix} + \frac{\alpha_2^{-(1+t)}}{(\alpha_2 - 1)(\alpha_2 - \alpha_1)} \begin{bmatrix} c_{11} & c_{12} & c_{13} \\ c_{21} & c_{22} & c_{23} \\ c_{31} & c_{32} & c_{33} \end{bmatrix} \right) z^t, \quad (2.12)$$

where $a_{11} = 1 - (q_{22} + q_{33}) + (q_{22}q_{33} - q_{23}q_{32})$, $a_{12} = q_{12} + (q_{13}q_{32} - q_{12}q_{33})$, $a_{13} = q_{13} + (q_{12}q_{23} - q_{13}q_{22})$, $a_{21} = q_{21} - (q_{21}q_{33} - q_{23}q_{31})$, $a_{22} = 1 - (q_{11} + q_{33}) + (q_{11}q_{33} - q_{13}q_{31})$, $a_{23} = q_{23} - (q_{11}q_{23} - q_{13}q_{21})$, $a_{31} = q_{31} + (q_{21}q_{32} - q_{22}q_{31})$, $a_{32} = q_{32} - (q_{11}q_{32} - q_{12}q_{31})$, $a_{33} = 1 - (q_{11} + q_{22}) + (q_{11}q_{22} - q_{12}q_{21})$, $b_{11} = 1 - (q_{22} + q_{33})\alpha_1 + (q_{22}q_{33} - q_{23}q_{32})\alpha_1^2$, $b_{21} = q_{12}\alpha_1 + (q_{13}q_{32} - q_{12}q_{33})\alpha_1^2$, $b_{31} = q_{13}\alpha_1 + (q_{12}q_{23} - q_{13}q_{22})\alpha_1^2$, $b_{12} = q_{21}\alpha_1 - (q_{21}q_{33} - q_{23}q_{31})\alpha_1^2$, $b_{22} = 1 - (q_{11} + q_{33})\alpha_1 + (q_{11}q_{33} - q_{13}q_{31})\alpha_1^2$, $b_{32} = q_{23}\alpha_1 - (q_{11}q_{23} - q_{13}q_{21})\alpha_1^2$, $b_{13} = q_{31}\alpha_1 + (q_{21}q_{32} - q_{22}q_{31})\alpha_1^2$, $b_{23} = q_{32}\alpha_1 - (q_{11}q_{32} - q_{12}q_{31})\alpha_1^2$, $b_{33} = 1 - (q_{11} + q_{22})\alpha_1 + (q_{11}q_{22} - q_{12}q_{21})\alpha_1^2$, $c_{11} = 1 - (q_{22} + q_{33})\alpha_2 + (q_{22}q_{33} - q_{23}q_{32})\alpha_2^2$, $c_{21} = q_{12}\alpha_2 + (q_{13}q_{32} - q_{12}q_{33})\alpha_2^2$, $c_{31} = q_{13}\alpha_2 + (q_{12}q_{23} - q_{13}q_{22})\alpha_2^2$, $c_{12} = q_{21}\alpha_2 - (q_{21}q_{33} - q_{23}q_{31})\alpha_2^2$, $c_{22} = 1 - (q_{11} + q_{33})\alpha_2 + (q_{11}q_{33} - q_{13}q_{31})\alpha_2^2$, $c_{32} = q_{23}\alpha_2 - (q_{11}q_{23} - q_{13}q_{21})\alpha_2^2$, $c_{13} = q_{31}\alpha_2 + (q_{21}q_{32} - q_{22}q_{31})\alpha_2^2$, $c_{23} = q_{32}\alpha_2 - (q_{11}q_{32} - q_{12}q_{31})\alpha_2^2$, $c_{33} = 1 - (q_{11} + q_{22})\alpha_2 + (q_{11}q_{22} - q_{12}q_{21})\alpha_2^2$.

Case 2

If $(tr(\mathbf{Q}) - 1)^2 = 4 \det(\mathbf{Q})$, then $\alpha_1 = \alpha_2 = \alpha$ and

$$[\mathbf{I} - \mathbf{Q}z]^{-1} = \sum_{t=0}^{\infty} \left(\frac{1}{(\alpha - 1)^2 \det(\mathbf{Q})} \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} + \frac{(1+t)\alpha^{-(2+t)}}{(\alpha - 1) \det(\mathbf{Q})} \times \right. \\ \left. \begin{bmatrix} b_{11} & b_{12} & b_{13} \\ b_{21} & b_{22} & b_{23} \\ b_{31} & b_{32} & b_{33} \end{bmatrix} + \frac{\alpha^{-(1+t)}}{\alpha} \begin{bmatrix} d_{11} & d_{12} & d_{13} \\ d_{21} & d_{22} & d_{23} \\ d_{31} & d_{32} & d_{33} \end{bmatrix} \right) z^t, \quad (2.13)$$

where $d_{11} = (1/\det(\mathbf{Q}) - \alpha^2 a_{11} - b_{11})$, $d_{12} = -(\alpha^2 a_{12} + b_{12})$, $d_{13} = -(\alpha^2 a_{13} + b_{13})$, $d_{21} = -(\alpha^2 a_{21} + b_{21})$, $d_{22} = (1/\det(\mathbf{Q}) - \alpha^2 a_{22} - b_{22})$, $d_{23} = -(\alpha^2 a_{23} + b_{23})$, $d_{31} = -(\alpha^2 a_{31} + b_{31})$, $d_{32} = -(\alpha^2 a_{32} + b_{32})$, $d_{33} = (1/\det(\mathbf{Q}) - \alpha^2 a_{33} - b_{33})$.

In the expression for Case 1, let

$$\mathbf{A}_1 = \frac{1}{(\alpha_1 - 1)(\alpha_2 - 1) \det(\mathbf{Q})} \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix},$$

$$\mathbf{B}_1 = \frac{1}{(\alpha_1 - 1)(\alpha_1 - \alpha_2) \det(\mathbf{Q})} \begin{bmatrix} b_{11} & b_{12} & b_{13} \\ b_{21} & b_{22} & b_{23} \\ b_{31} & b_{32} & b_{33} \end{bmatrix}$$

and

$$\mathbf{C} = \frac{1}{(\alpha_2 - 1)(\alpha_2 - \alpha_1) \det(\mathbf{Q})} \begin{bmatrix} c_{11} & c_{12} & c_{13} \\ c_{21} & c_{22} & c_{23} \\ c_{31} & c_{32} & c_{33} \end{bmatrix},$$

and for Case 2, let

$$\mathbf{A}_2 = \frac{1}{(\alpha - 1)^2 \det(\mathbf{Q})} \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix},$$

$$\mathbf{B}_2 = \frac{1}{(\alpha - 1) \det(\mathbf{Q})} \begin{bmatrix} b_{11} & b_{12} & b_{13} \\ b_{21} & b_{22} & b_{23} \\ b_{31} & b_{32} & b_{33} \end{bmatrix}$$

and

$$\mathbf{D} = \frac{1}{\alpha} \begin{bmatrix} d_{11} & d_{12} & d_{13} \\ d_{21} & d_{22} & d_{23} \\ d_{31} & d_{32} & d_{33} \end{bmatrix}.$$

Making the appropriate substitution for $[\mathbf{I} - \mathbf{Q}z]^{-1}$, it follows from equation (2.1) for any given $t = 1 + 1/n$, $n > 0$, that

$$\mathbf{Q}^{(1+1/n)} = \mathbf{A}_m + \mathbf{T}_m(1 + 1/n), \quad m = 1, 2,$$

where

$$\mathbf{T}_m(1+1/n) = \begin{cases} \alpha_1^{-(2+1/n)} \mathbf{B}_1 + \alpha_2^{-(2+1/n)} \mathbf{C}, & m = 1 \text{ if } (tr(\mathbf{Q}) - 1)^2 > 4 \det(\mathbf{Q}) \\ (2 + 1/n) \alpha^{-(3+1/n)} \mathbf{B}_2 + \alpha^{-(2+1/n)} \mathbf{D}, & m = 2 \text{ if } (tr(\mathbf{Q}) - 1)^2 = 4 \det(\mathbf{Q}). \end{cases}$$

As \mathbf{Q} is irreducible, it follows for large t that

$$\lim_{t \rightarrow \infty} \mathbf{Q}^t = \mathbf{A}_m + \lim_{t \rightarrow \infty} \mathbf{T}_m(t)$$

exists. This would hold only if $\alpha_1, \alpha_2 > 1$. With $\alpha_1, \alpha_2 > 1$, $\lim_{t \rightarrow \infty} \mathbf{T}_m(t) = \mathbf{0}$. In either case m , \mathbf{A}_m is a matrix of limiting-state probabilities.

To show that the matrix \mathbf{Q}^t is meaningful for any given $t = 1 + 1/n$, $n > 0$, if $\alpha_1, \alpha_2 > 1$, consider the doubly stochastic matrix in [5]:

$$\mathbf{P} + \mathbf{w}'\mathbf{r} = \begin{bmatrix} 0.5 & 0.5 & 0 \\ 0.5 & 0.25 & 0.25 \\ 0 & 0.25 & 0.75 \end{bmatrix},$$

which has the real roots $\alpha_1 = 1.4641$ and $\alpha_2 = -5.4641$. The additive representation is

$$\mathbf{Q}^{1+1/n} = \begin{bmatrix} 0.3333 & 0.3333 & 0.3333 \\ 0.3333 & 0.3333 & 0.3333 \\ 0.3333 & 0.3333 & 0.3333 \end{bmatrix} + (1.4641)^{-(2+1/n)} \begin{bmatrix} 0.4880 & 0.1786 & -0.6667 \\ 0.1786 & 0.0654 & -0.2440 \\ -0.6667 & -0.2440 & 0.9107 \end{bmatrix} \\ + (-5.4641)^{-(2+1/n)} \begin{bmatrix} -1.8214 & 2.4880 & -0.6667 \\ 2.4880 & -3.3987 & 0.9107 \\ -0.6667 & 0.9107 & -0.2440 \end{bmatrix}.$$

For any $n > 0$, the third term is a matrix of complex entries because the n th root, $(-5.4641)^{1/n}$, arising from the scalar $(-5.4641)^{-(2+1/n)}$, does not exist. Thus the fractional indicial matrix $(\mathbf{P} + \mathbf{w}'\mathbf{r})^{(1+1/n)}$ cannot be represented as a sum of constant matrices that is meaningful within the Markov chain framework.

3. Illustration

The applicability of the new representation for the irreducible stochastic matrix \mathbf{Q} is demonstrated in this section. We consider two test problems. The first problem is contained in [11] and the second one is in [12].

Example 1. Singer and Spilerman [11] expressed the following transition matrix

$$\tilde{\mathbf{P}} = \begin{bmatrix} 0.16 & 0.53 & 0.31 \\ 0.0525 & 0.49 & 0.4575 \\ 0.11 & 0.14 & 0.75 \end{bmatrix},$$

in terms of the intensity matrix as

$$\hat{\mathbf{P}} = \exp \left(\begin{bmatrix} -2.046 & 1.993 & 0.053 \\ 0.024 & -0.818 & 0.794 \\ 0.315 & 0.043 & -0.358 \end{bmatrix} \right),$$

where $\hat{\mathbf{P}}$ is an embeddable matrix of $\tilde{\mathbf{P}}$. Clearly, $\hat{\mathbf{P}}$ is an approximation of $\tilde{\mathbf{P}}$ as

$$\hat{\mathbf{P}} = \exp \left(\begin{bmatrix} -2.046 & 1.993 & 0.053 \\ 0.024 & -0.818 & 0.794 \\ 0.315 & 0.043 & -0.358 \end{bmatrix} \right) = \begin{bmatrix} 0.1601 & 0.5296 & 0.3103 \\ 0.0525 & 0.4894 & 0.4581 \\ 0.1105 & 0.1405 & 0.7489 \end{bmatrix}.$$

The additive representation is possible as $\det(\tilde{\mathbf{P}}) = 0.0399$ is non-singular, the difference $(tr(\tilde{\mathbf{P}}) - 1)^2 - 4\det(\tilde{\mathbf{P}}) = 0.16 - 0.1597 > 0$, and the roots of the determinant $\det(\mathbf{I} - \tilde{\mathbf{P}}z)$ are real and greater than one, viz.

$$\alpha_1 = \frac{0.4}{2(0.0399)} \left(1 - \left(1 - \frac{4(0.0399)}{(0.4)^2} \right)^{1/2} \right) = 4.7925$$

and

$$\alpha_2 = \frac{0.4}{2(0.0399)} \left(1 + \left(1 - \frac{4(0.0399)}{(0.4)^2} \right)^{1/2} \right) = 5.2263.$$

Using the additive representation, the $(1 + 1/n)$ -step transition matrix, $\mathbf{Q}^{(1+1/n)}$, for $n > 0$, is represented as:

$$\begin{aligned} \mathbf{Q}^{1+1/n} &= \begin{bmatrix} 0.0992 & 0.2749 & 0.6260 \\ 0.0992 & 0.2749 & 0.6260 \\ 0.0992 & 0.2749 & 0.6260 \end{bmatrix} \\ &+ (4.7925)^{-(2+1/n)} \begin{bmatrix} -30.8570 & 85.1439 & -54.2869 \\ -7.6590 & 21.1336 & -13.4746 \\ 8.2509 & -22.7667 & 14.5158 \end{bmatrix} \\ &+ (5.2263)^{-(2+1/n)} \begin{bmatrix} 38.3583 & -94.2879 & 55.9296 \\ 7.8341 & -19.2569 & 11.4228 \\ -9.5160 & 23.3910 & -13.8751 \end{bmatrix}. \end{aligned}$$

This representation does not require any form of perturbation as \mathbf{Q} is equal to $\tilde{\mathbf{P}}$.

Example 2. Tsaklidis [12] considered a continuous time homogeneous Markov system with fixed size, where the matrix of the transition intensities of the memberships is given as

$$\Phi = \begin{bmatrix} -1/2 & 0 & 1/2 \\ 1/8 & -1/2 & 3/8 \\ 0 & 1/2 & -1/2 \end{bmatrix}$$

In this example, the determinant $\det(\mathbf{I} - z \exp(\Phi))$ has equal roots, that is, $\alpha_1 = \alpha_2 = 2.1170$. We obtain a meaningful $(1 + 1/n)$ -step transition matrix for any given $n > 0$, using the additive representation as:

$$\begin{aligned} \mathbf{Q}^{1+1/n} &= \begin{bmatrix} 0.1111 & 0.4444 & 0.4444 \\ 0.1111 & 0.4444 & 0.4444 \\ 0.1111 & 0.4444 & 0.4444 \end{bmatrix} \\ &+ (2 + 1/n) (2.117)^{-(3+1/n)} \begin{bmatrix} 0.7469 & -1.4939 & 0.7469 \\ 0.1867 & -0.3735 & 0.1867 \\ -0.3735 & 0.7469 & -0.3735 \end{bmatrix} \\ &+ (2.117)^{-(2+1/n)} \begin{bmatrix} 1.5289 & -0.2352 & -1.2937 \\ -0.3234 & 1.3525 & -1.0291 \\ -0.0588 & -1.2937 & 1.3525 \end{bmatrix}. \end{aligned}$$

The matrix $\mathbf{Q}^{1+1/n}$ is a stochastic matrix and is compatible with the continuous-time representation, $\exp((1 + 1/n)\Phi)$, for any given $n > 0$.

Suppose that there exist an initial structure $\mathbf{n}(0) = [55, 40, 5]$. Then the results of using the additive representation for a shift in the unit interval of the Markov chain by 3 months, 6 months and 9 months are $\mathbf{n}(1+1/4) = [33, 33, 34]$, $\mathbf{n}(1+1/2) = [30, 33, 37]$

and $\mathbf{n}(1 + 3/4) = [28, 33, 39]$, respectively¹. These results are consistent with the continuous time process for $t = 5/4, 3/2, 7/4$.

4. Conclusion

This paper has provided the additive representation of stochastic matrices as a means for obtaining fractional indicial matrices for the manpower system where the personnel structure is to be projected for a few months beyond one year (for instance, one year and six months, one year and three months, etc.). As an alternative to the assertion that supports the continuous-time formulation in place of the discrete-time Markov framework [11], this study gives instances where certain discrete-time Markov framework for forecasting manpower structure could have a meaningful fractional indicial stochastic matrix without recourse to the continuous-time representation via the transition intensities. The approach in this paper circumvents the problem of non-uniqueness that exists in the earlier formulations [6, 11]. Even so three conditions should be satisfied: (i) the transition matrix \mathbf{Q} is irreducible, (ii) the determinant of \mathbf{Q} is non-singular, and (iii) the characteristic polynomial arising from the determinant $\det(\mathbf{I} - \mathbf{Q}z)$ has linear factors with real roots, which exceeds one. For instances where these conditions are violated, no substantive meaning can be attached in the additive context. In that case, the appropriate mathematical structure is a continuous-time formulation.

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¹The results are approximated to the nearest integer without violating the assumption of a fixed total size.

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Relative Order and Relative Type Oriented Growth Properties of Generalized Iterated Entire Functions

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ABSTRACT: The main aim of this paper is to study some growth properties of generalized iterated entire functions in the light of their relative orders, relative types and relative weak types.

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Keywords and Phrases: Entire function; Growth; Relative order; Relative type; Relative weak type; Composition; Property (A).

Let \mathbb{C} be the set of all finite complex numbers. For any entire function $f(z) = \sum_{n=0}^{\infty} a_n z^n$ defined in \mathbb{C} , the maximum modulus function $M_f(r)$ on $|z| = r$ is defined by $M_f(r) = \max_{|z|=r} |f(z)|$. If $f(z)$ is non-constant then $M_f(r)$ is strictly increasing and continuous. Also its inverse $M_f^{-1} : (|f(0)|, \infty) \rightarrow (0, \infty)$ exists and is such that $\lim_{s \rightarrow \infty} M_f^{-1}(s) = \infty$. Naturally, $M_f^{-1}(r)$ is also an increasing function of r . Also a non-constant entire function $f(z)$ is said to have the Property (A) if for any $\delta > 1$ and for all sufficiently large r , $[M_f(r)]^2 \leq M_f(r^\delta)$ holds (see [3]). For examples of functions with or without the Property (A), one may see [3]. In this connection Lahiri et al. (see [6]) prove that every entire function $f(z)$ satisfying the property (A) is transcendental. Moreover for any transcendental entire function $f(z)$, it is well known that $\lim_{r \rightarrow \infty} \frac{\log M_f(r)}{\log r} = \infty$ and for its application in growth measurement, one may see [8]. For another entire function $g(z)$, the ratio $\frac{M_f(r)}{M_g(r)}$ as $r \rightarrow \infty$ is called the growth of $f(z)$ with respect to $g(z)$ in terms of their maximum moduli. The notion of order and lower order which are the main tools to study the comparative growth properties of entire functions are very classical in complex analysis and their definitions are as follows:

Definition 1. The order and the lower order of an entire function $f(z)$ denoted by $\rho(f)$ and $\lambda(f)$ respectively are defined as

$$\rho(f) = \limsup_{r \rightarrow \infty} \frac{\log \log M_f(r)}{\log \log M_{\exp z}(r)} = \limsup_{r \rightarrow \infty} \frac{\log \log M_f(r)}{\log r}.$$

The rate of growth of an entire function generally depends upon order (respectively, lower order) of it. The entire function with higher order is of faster growth than that of lesser order. But if orders of two entire functions are same, then it is impossible to detect the function with faster growth. In that case, it is necessary to compute another class of growth indicators of entire functions called their types. Thus the type $\sigma(f)$ and lower type $\bar{\sigma}(f)$ of an entire function $f(z)$ are defined as:

Definition 2. Let $f(z)$ be an entire function with non zero finite order. Then the type $\sigma(f)$ and lower type $\bar{\sigma}(f)$ of an entire function $f(z)$ are defined as

$$\sigma(f) = \limsup_{r \rightarrow \infty} \frac{\log M_f(r)}{(\log M_{\exp z}(r))^{\rho(f)}} = \limsup_{r \rightarrow \infty} \frac{\log M_f(r)}{r^{\rho(f)}}.$$

In order to calculate the order, it is seen that we have compared the maximum modulus of entire function $f(z)$ with that of $\exp z$ but here a question may arise why should we compare the maximum modulus of any entire function with that of only $\exp z$ whose growth rate is too high. From this view point, the relative order of entire functions may be thought of by Bernal (see [2, 3]) who introduced the concept of relative order between two entire functions to avoid comparing growth just with $\exp z$. Thus the relative order of an entire function $f(z)$ with respect to an entire function $g(z)$, denoted by $\rho_g(f)$ is define as:

$$\begin{aligned} \rho_g(f) &= \inf \{ \mu > 0 : M_f(r) < M_g(r^\mu) \text{ for all } r > r_0(\mu) > 0 \} \\ &= \limsup_{r \rightarrow \infty} \frac{\log M_g^{-1}(M_f(r))}{\log r}. \end{aligned}$$

Similarly, one can define the relative lower order of $f(z)$ with respect to $g(z)$ denoted by $\lambda_g(f)$ as follows :

$$\lambda_g(f) = \liminf_{r \rightarrow \infty} \frac{\log M_g^{-1}(M_f(r))}{\log r}.$$

In the definition of relative order and relative lower order we generally compare the maximum modulus of any entire function $f(z)$ with that of any entire function $g(z)$ and it is quite natural that when $g(z) = \exp z$, both the definitions of relative order and relative lower order coincide with Definition 1.

In order to compare the relative growth of two entire functions having same non zero finite relative order with respect to another entire function, Roy [7] introduced the notion of relative type of two entire functions in the following way:

Definition 3. [7] Let $f(z)$ and $g(z)$ be any two entire functions such that $0 < \rho_g(f) < \infty$. Then the relative type $\sigma_g(f)$ of $f(z)$ with respect to $g(z)$ is defined as:

$$\begin{aligned} \sigma_g(f) &= \inf \left\{ k > 0 : M_f(r) < M_g \left(kr^{\rho_g(f)} \right) \text{ for all sufficiently large values of } r \right\} \\ &= \limsup_{r \rightarrow \infty} \frac{M_g^{-1}(M_f(r))}{r^{\rho_g(f)}}. \end{aligned}$$

Similarly, one can define the relative lower type of an entire function $f(z)$ with respect to another entire function $g(z)$ denoted by $\bar{\sigma}_g(f)$ when $0 < \rho_g(f) < \infty$ which is as follows:

$$\bar{\sigma}_g(f) = \liminf_{r \rightarrow \infty} \frac{M_g^{-1}(M_f(r))}{r^{\rho_g(f)}}.$$

It is obvious that $0 \leq \bar{\sigma}_g(f) \leq \sigma_g(f) \leq \infty$.

If we consider $g(z) = \exp z$, then one can easily verify that Definition 3 coincides with the classical definitions of type and lower type respectively.

Like wise, to determine the relative growth of two entire functions having same non zero finite relative lower order with respect to another entire function, one may introduce the definition of relative weak type of an entire function $f(z)$ with respect to another entire function $g(z)$ of finite positive relative lower order $\lambda_g(f)$ in the following way:

Definition 4. Let $f(z)$ and $g(z)$ be any two entire functions such that $0 < \lambda_g(f) < \infty$. The relative -weak type $\tau_g(f)$ and the growth indicator $\bar{\tau}_g(f)$ of an entire function $f(z)$ with respect to another entire function $g(z)$ are defined as:

$$\begin{aligned} \tau_g(f) &= \lim_{r \rightarrow \infty} \inf \frac{M_g^{-1}(M_f(r))}{r^{\lambda_g(f)}} \\ \bar{\tau}_g(f) &= \lim_{r \rightarrow \infty} \sup \frac{M_g^{-1}(M_f(r))}{r^{\lambda_g(f)}}. \end{aligned}$$

For any two entire functions $f(z), g(z)$ defined in \mathbb{C} and for any real number $\alpha \in (0, 1]$, Banerjee et al. [1] introduced the concept of generalized iteration of $f(z)$ with respect to $g(z)$ in the following manner:

$$\begin{aligned} f_{1,g}(z) &= (1 - \alpha)z + \alpha f(z) \\ f_{2,g}(z) &= (1 - \alpha)g_{1,f}(z) + \alpha f(g_{1,f}(z)) \\ f_{3,g}(z) &= (1 - \alpha)g_{2,f}(z) + \alpha f(g_{2,f}(z)) \\ \dots & \dots \dots \dots \dots \\ \dots & \dots \dots \dots \dots \\ f_{n,g}(z) &= (1 - \alpha)g_{n-1,f}(z) + \alpha f(g_{n-1,f}(z)) \end{aligned}$$

and so

$$\begin{aligned} g_{1,f}(z) &= (1 - \alpha)z + \alpha g(z) \\ g_{2,f}(z) &= (1 - \alpha)f_{1,g}(z) + \alpha g(f_{1,g}(z)) \\ g_{3,f}(z) &= (1 - \alpha)f_{2,g}(z) + \alpha g(f_{2,g}(z)) \\ \dots & \dots \dots \dots \dots \\ \dots & \dots \dots \dots \dots \\ g_{n,f}(z) &= (1 - \alpha)f_{n-1,g}(z) + \alpha g(f_{n-1,g}(z)). \end{aligned}$$

Clearly all $f_{n,g}(z)$ and $g_{n,f}(z)$ are entire functions.

Further for another two non constant entire functions $h(z)$ and $k(z)$, one may define the iteration of $M_h^{-1}(r)$ with respect to $M_k^{-1}(r)$ in the following manner:

$$\begin{aligned}
 &M_h^{-1}(r) = M_{h_1}^{-1}(r); \\
 &M_k^{-1}(M_h^{-1}(r)) = M_{h_1}^{-1}(M_k^{-1}(r)) = M_{h_2}^{-1}(r); \\
 &M_h^{-1}(M_k^{-1}(M_h^{-1}(r))) = M_{h_2}^{-1}(M_h^{-1}(r)) = M_{h_3}^{-1}(r); \\
 &\dots\dots\dots \\
 &\dots\dots\dots \\
 &M_h^{-1}(\dots\dots\dots(M_h^{-1}(M_k^{-1}(M_h^{-1}(r)))))) = M_{h_n}^{-1}(r) \text{ when } n \text{ is odd and} \\
 &M_k^{-1}(\dots\dots\dots(M_h^{-1}(M_k^{-1}(M_h^{-1}(r)))))) = M_{h_n}^{-1}(r) \text{ when } n \text{ is even.}
 \end{aligned}$$

Obviously $M_{h_n}^{-1}(r)$ is an increasing functions of r .

During the past decades, several researchers made close investigations on the growth properties of composite entire functions in different directions using their classical growth indicators such as order and type but the study of growth properties of composite entire functions using the concepts of relative order and relative type was mostly unknown to complex analysis which is and is the prime concern of the paper. The main aim of this paper is to study the growth properties of generalized iterated entire functions in almost a new direction in the light of their relative orders, relative types and relative weak types. Also our notation is standard within the theory of Nevanlinna’s value distribution of entire functions which are available in [5] and [10]. Hence we do not explain those in details.

1. Lemmas

In this section we present some lemmas which will be needed in the sequel.

Lemma 1. [4] *If $f(z)$ and $g(z)$ are any two entire functions with $g(0) = 0$. Let β satisfy $0 < \beta < 1$ and $c(\beta) = \frac{(1-\beta)^2}{4\beta}$. Then for all sufficiently large values of r ,*

$$M_f(c(\beta) M_g(\beta r)) \leq M_{f \circ g}(r) \leq M_f(M_g(r)) .$$

In addition if $\beta = \frac{1}{2}$, then for all sufficiently large values of r ,

$$M_{f \circ g}(r) \geq M_f\left(\frac{1}{8}M_g\left(\frac{r}{2}\right)\right) .$$

Lemma 2. [3] *Let $f(z)$ be an entire function which satisfies the Property (A). Then for any positive integer n and for all large r ,*

$$[M_f(r)]^n \leq M_f(r^\delta)$$

holds where $\delta > 1$.

Lemma 3. [3] Let $f(z)$ be an entire function, $\alpha > 1$ and $0 < \beta < \alpha$. Then

$$M_f(\alpha r) > \beta M_f(r) .$$

Lemma 4. Let $f(z)$, $g(z)$ are any two transcendental entire functions and $h(z)$, $k(z)$ are any two entire functions such that $0 < \rho_h(f) < \infty$, $0 < \rho_k(g) < \infty$ and $h(z)$, $k(z)$ satisfy the Property (A). Then for all sufficiently large values of r ,

$$(i) \quad (M_{h_n}^{-1}(M_{f_{n,g}}(r)))^{\frac{1}{\delta}} < M_k^{-1}(M_g(r)) \text{ when } n \text{ is even}$$

and

$$(ii) \quad (M_{h_n}^{-1}(M_{f_{n,g}}(r)))^{\frac{1}{\delta}} < M_h^{-1}(M_f(r)) \text{ when } n \text{ is odd}$$

where $\delta > 1$.

Proof. Let β be any positive integer such that $\max\{\rho_h(f), \rho_k(g)\} < \beta$ hold. Since for any transcendental entire function $f(z)$, $\frac{\log M_f(r)}{\log r} \rightarrow \infty$ as $r \rightarrow \infty$, in view of Lemma 1, Lemma 2 and for any even integer n , we get for all sufficiently large values of r that

$$\begin{aligned} & M_{f_{n,g}}(r) \leq (1 - \alpha) M_{g_{n-1,f}}(r) + \alpha M_{f(g_{n-1,f})}(r) \\ \Rightarrow & M_{f_{n,g}}(r) < (1 - \alpha) M_f(M_{g_{n-1,f}}(r)) + \alpha M_f(M_{g_{n-1,f}}(r)) \\ \Rightarrow & M_h^{-1}(M_{f_{n,g}}(r)) < M_h^{-1}(M_f(M_{g_{n-1,f}}(r))) \\ \Rightarrow & M_h^{-1}(M_{f_{n,g}}(r)) < (M_{g_{n-1,f}}(r))^\beta \\ \Rightarrow & (M_h^{-1}(M_{f_{n,g}}(r)))^{\frac{1}{\beta}} < M_{g_{n-1,f}}(r) \\ \Rightarrow & (M_h^{-1}(M_{f_{n,g}}(r)))^{\frac{1}{\beta}} < (1 - \alpha) M_{f_{n-2,g}}(r) + \alpha M_{g(f_{n-2,g})}(r) \\ \Rightarrow & (M_h^{-1}(M_{f_{n,g}}(r)))^{\frac{1}{\beta}} < (1 - \alpha) M_g(M_{f_{n-2,g}}(r)) + \alpha M_g(M_{f_{n-2,g}}(r)) \\ \Rightarrow & M_k^{-1} \left((M_h^{-1}(M_{f_{n,g}}(r)))^{\frac{1}{\beta}} \right) < M_k^{-1}(M_g(M_{f_{n-2,g}}(r))) \\ \Rightarrow & (M_k^{-1}(M_h^{-1}(M_{f_{n,g}}(r))))^{\frac{1}{\delta}} < M_k^{-1}(M_g(M_{f_{n-2,g}}(r))) \\ \Rightarrow & (M_k^{-1}(M_h^{-1}(M_{f_{n,g}}(r))))^{\frac{1}{\delta}} < (M_{f_{n-2,g}}(r))^\beta \\ \Rightarrow & (M_k^{-1}(M_h^{-1}(M_{f_{n,g}}(r))))^{\frac{1}{\delta \cdot \beta}} < M_{f_{n-2,g}}(r) \\ \Rightarrow & (M_{h_2}^{-1}(M_{f_{n,g}}(r)))^{\frac{1}{\delta \cdot \beta}} < M_{f_{n-2,g}}(r) \\ \Rightarrow & M_h^{-1} \left((M_{h_2}^{-1}(M_{f_{n,g}}(r)))^{\frac{1}{\delta \cdot \beta}} \right) < (M_{g_{n-3,f}}(r))^\beta \\ \Rightarrow & (M_h^{-1}(M_{h_2}^{-1}(M_{f_{n,g}}(r))))^{\frac{1}{\delta}} < (M_{g_{n-3,f}}(r))^\beta \\ \Rightarrow & (M_{h_3}^{-1}(M_{f_{n,g}}(r)))^{\frac{1}{\delta \cdot \beta}} < M_{g_{n-3,f}}(r) \end{aligned}$$

$$\begin{aligned} \Rightarrow M_k^{-1} \left((M_{h_3}^{-1} (M_{f_{n,g}}(r)))^{\frac{1}{\delta \cdot \beta}} \right) &< (M_{f_{n-4,g}}(r))^\beta \\ \Rightarrow (M_{h_4}^{-1} (M_{f_{n,g}}(r)))^{\frac{1}{\delta \cdot \beta}} &< M_{f_{n-4,g}}(r) \\ \dots & \dots \dots \dots \\ \dots & \dots \dots \dots \end{aligned}$$

Therefore

$$(M_{h_n}^{-1} (M_{f_{n,g}}(r)))^{\frac{1}{\delta}} < M_k^{-1} (M_g(r)) \text{ when } n \text{ is even.}$$

Similarly,

$$(M_{h_n}^{-1} (M_{f_{n,g}}(r)))^{\frac{1}{\delta}} < M_h^{-1} (M_f(r)) \text{ when } n \text{ is odd .}$$

Hence the lemma follows. □

Remark 1. If we consider $0 < \rho_h(f) \leq 1$ and $0 < \rho_k(g) \leq 1$ in Lemma 4, then it is not necessary for both $h(z)$ and $k(z)$ to satisfy Property (A) and in this case Lemma 4 holds with $\delta = 1$.

Lemma 5. Let $f(z), g(z)$ are any two transcendental entire functions and $h(z), k(z)$ are any two entire functions such that $0 < \lambda_h(f) < \infty, 0 < \lambda_k(g) < \infty$ and $h(z), k(z)$ satisfy the Property (A). Also let $\delta > 1, 0 < \beta < \alpha < 1, \omega$ is a positive integer such that $\min \{ \lambda_h(f), \lambda_k(g) \} > \frac{1}{\omega}$ and $\gamma_n > \frac{\gamma_{n-1}^\omega}{(\alpha - \beta)}$ where $\gamma_0 = 1$. Then for all sufficiently large values of r ,

$$(i) \gamma_n (M_{h_n}^{-1} (M_{f_{n,g}}(r)))^\delta > M_k^{-1} \left(M_g \left(\frac{r}{18^n} \right) \right) \text{ when } n \text{ is even}$$

and

$$(ii) \gamma_n (M_{h_n}^{-1} (M_{f_{n,g}}(r)))^\delta > M_h^{-1} \left(M_f \left(\frac{r}{18^n} \right) \right) \text{ when } n \text{ is odd .}$$

Proof. Since for any transcendental entire function $f, \frac{\log M_f(r)}{\log r} \rightarrow \infty$ as $r \rightarrow \infty$, therefore $\frac{\log \frac{\beta}{(1-\alpha)} M_f(r)}{\log r} \rightarrow \infty$ as $r \rightarrow \infty$ where $0 < \beta < \alpha$. Hence in view of Lemma 1, Lemma 2, Lemma 3 and for any even integer n , we get for all sufficiently large values of r that

$$M_{f_{n,g}}(r) \geq \alpha M_{f(g_{n-1,f})}(r) - (1 - \alpha) M_{g_{n-1,f}}(r)$$

$$\begin{aligned}
 &\Rightarrow M_{f_{n,g}}(r) > \alpha M_f \left(M_{g_{n-1,f}} \left(\frac{r}{18} \right) \right) - \beta M_f \left(M_{g_{n-1,f}} \left(\frac{r}{18} \right) \right) \\
 &\Rightarrow M_{f_{n,g}}(r) > (\alpha - \beta) M_f \left(M_{g_{n-1,f}} \left(\frac{r}{18} \right) \right) \\
 &\Rightarrow M_h^{-1} \left(\frac{1}{(\alpha - \beta)} M_{f_{n,g}}(r) \right) > M_h^{-1} \left(M_f \left(M_{g_{n-1,f}} \left(\frac{r}{18} \right) \right) \right) \\
 &\Rightarrow M_h^{-1} \left(\frac{1}{(\alpha - \beta)} M_{f_{n,g}}(r) \right) > \left(M_{g_{n-1,f}} \left(\frac{r}{18} \right) \right)^{\frac{1}{\omega}} \\
 &\Rightarrow (\gamma_1 M_h^{-1} (M_{f_{n,g}}(r)))^\omega > M_{g_{n-1,f}} \left(\frac{r}{18} \right) \\
 &\Rightarrow \gamma_1^\omega (M_h^{-1} (M_{f_{n,g}}(r)))^\omega > \alpha M_g \left(M_{f_{n-2,g}} \left(\frac{r}{18^2} \right) \right) - \beta M_g \left(M_{f_{n-2,g}} \left(\frac{r}{18^2} \right) \right) \\
 &\Rightarrow \gamma_1^\omega (M_h^{-1} (M_{f_{n,g}}(r)))^\omega > (\alpha - \beta) M_g \left(M_{f_{n-2,g}} \left(\frac{r}{18^2} \right) \right) \\
 &\Rightarrow \frac{\gamma_1^\omega}{(\alpha - \beta)} (M_h^{-1} (M_{f_{n,g}}(r)))^\omega > M_g \left(M_{f_{n-2,g}} \left(\frac{r}{18^2} \right) \right) \\
 &\Rightarrow M_k^{-1} \left(\frac{\gamma_1^\omega}{(\alpha - \beta)} (M_h^{-1} (M_{f_{n,g}}(r)))^\omega \right) > M_k^{-1} \left(M_g \left(M_{f_{n-2,g}} \left(\frac{r}{18^2} \right) \right) \right) \\
 &\Rightarrow \gamma_2 (M_k^{-1} (M_h^{-1} (M_{f_{n,g}}(r))))^\delta > \left(M_{f_{n-2,g}} \left(\frac{r}{18^2} \right) \right)^{\frac{1}{\omega}} \\
 &\Rightarrow \gamma_2^\omega (M_k^{-1} (M_h^{-1} (M_{f_{n,g}}(r))))^{\delta\omega} > M_{f_{n-2,g}} \left(\frac{r}{18^2} \right) \\
 &\Rightarrow M_h^{-1} \left(\frac{\gamma_2^\omega}{(\alpha - \beta)} (M_k^{-1} (M_h^{-1} (M_{f_{n,g}}(r))))^{\delta\omega} \right) > \left(M_{g_{n-3,f}} \left(\frac{r}{18^3} \right) \right)^{\frac{1}{\omega}} \\
 &\Rightarrow \gamma_3^\omega (M_h^{-1} (M_k^{-1} (M_h^{-1} (M_{f_{n,g}}(r)))))^{\delta\omega} > M_{g_{n-3,f}} \left(\frac{r}{18^3} \right) \\
 &\Rightarrow \gamma_3^\omega (M_{h_3}^{-1} (M_{f_{n,g}}(r)))^{\delta\omega} > M_{g_{n-3,f}} \left(\frac{r}{18^3} \right) \\
 &\Rightarrow \gamma_4^\omega (M_k^{-1} (M_{h_3}^{-1} (M_{f_{n,g}}(r)))))^{\delta\omega} > M_{f_{n-4,g}} \left(\frac{r}{18^4} \right) \\
 &\Rightarrow \gamma_4^\omega (M_{h_4}^{-1} (M_{f_{n,g}}(r)))^{\delta\omega} > M_{f_{n-4,g}} \left(\frac{r}{18^4} \right) \\
 &\dots\dots\dots \\
 &\dots\dots\dots
 \end{aligned}$$

Therefore

$$\gamma_n (M_{h_n}^{-1} (M_{f_{n,g}}(r)))^\delta > M_k^{-1} \left(M_g \left(\frac{r}{18^n} \right) \right) \text{ when } n \text{ is even.}$$

Similarly,

$$\gamma_n (M_{h_n}^{-1} (M_{f_{n,g}}(r)))^\delta > M_h^{-1} \left(M_f \left(\frac{r}{18^n} \right) \right) \text{ when } n \text{ is odd.}$$

Hence the lemma follows. □

Remark 2. If we consider $1 \leq \lambda_h(f) < \infty$ and $1 \leq \lambda_k(g) < \infty$ in Lemma 5, then it

is not necessary for both h and k to satisfy Property (A) and in this case Lemma 5 holds with $\delta = 1$.

2. Main Results

In this section we present the main results of the paper. Throughout the paper, we consider the entire functions $H(z)$, $K(z)$, $h(z)$, $k(z)$ satisfy the Property (A) as and when necessary. Also consider that $F(z)$, $G(z)$, $f(z)$, $g(z)$ are non constant entire functions.

Theorem 1. *Let $f(z)$, $g(z)$, $k(z)$ and $h(z)$ be any four entire functions such that $0 < \lambda_h(f) \leq \rho_h(f) < \infty$, $0 < \lambda_k(g) < \infty$ and $0 < \mu < \rho_k(g) < \infty$. Then for any even number n ,*

$$\limsup_{r \rightarrow \infty} \frac{M_{h_n}^{-1}(M_{f_{n,g}}(r))}{\log M_h^{-1} M_f(\exp r^{\delta\mu})} = \infty,$$

where $\delta < 1$.

Proof. From the first part of Lemma 5, we get for a sequence of values of r tending to infinity that

$$M_{h_n}^{-1}(M_{f_{n,g}}(r)) > \left(\frac{1}{\gamma_n}\right)^\delta \left(\frac{r}{18^n}\right)^{\delta(\rho_k(g)-\varepsilon)}, \quad (2.1)$$

where γ_n is defined in Lemma 5.

Again from the definition of $\rho_h(f)$, we obtain for all sufficiently large values of r that

$$\log M_h^{-1}(M_f(\exp r^{\delta\mu})) \leq (\rho_h(f) + \varepsilon) r^{\delta\mu}. \quad (2.2)$$

Now from (2.1) and (2.2), it follows for a sequence of values of r tending to infinity that

$$\frac{M_{h_n}^{-1}(M_{f_{n,g}}(r))}{\log M_h^{-1}(M_f(\exp r^{\delta\mu}))} > \frac{\left(\frac{1}{\gamma_n}\right)^\delta \left(\frac{r}{18^n}\right)^{\delta(\rho_k(g)-\varepsilon)}}{(\rho_h(f) + \varepsilon) r^{\delta\mu}}. \quad (2.3)$$

As $\mu < \rho_k(g)$, we can choose $\varepsilon (> 0)$ in such a way that

$$\mu < \rho_k(g) - \varepsilon. \quad (2.4)$$

Thus from (2.3) and (2.4) we get that

$$\limsup_{r \rightarrow \infty} \frac{M_{h_n}^{-1}(M_{f_{n,g}}(r))}{\log M_h^{-1}(M_f(\exp r^{\delta\mu}))} = \infty. \quad (2.5)$$

Hence the theorem follows from (2.5). \square

Theorem 2. Let $f(z)$, $g(z)$, $k(z)$ and $h(z)$ be any four entire functions such that $0 < \lambda_h(f) \leq \rho_h(f) < \infty$, $0 < \lambda_k(g) < \infty$ and $0 < \mu < \rho_k(g) < \infty$. Then for any even number n ,

$$\limsup_{r \rightarrow \infty} \frac{M_{h_n}^{-1}(M_{f_{n,g}}(r))}{\log M_k^{-1}(M_g(\exp r^{\delta\mu}))} = \infty,$$

where $\delta < 1$.

Proof. Let $0 < \mu < \mu_0 < \rho_k(g)$. Then from (2.5), we obtain for a sequence of values of r tending to infinity and $A > 1$ that

$$\begin{aligned} M_{h_n}^{-1}(M_{f_{n,g}}(r)) &> A \log M_h^{-1}(M_f(\exp r^{\delta\mu_0})) \\ \text{i.e., } M_{h_n}^{-1}(M_{f_{n,g}}(r)) &> A(\lambda_h(f) - \varepsilon)r^{\delta\mu_0}. \end{aligned} \quad (2.6)$$

Again from the definition of $\rho_k(g)$, we obtain for all sufficiently large values of r that

$$\log M_k^{-1}(M_g(\exp r^{\delta\mu})) \leq (\rho_k(g) + \varepsilon)r^{\delta\mu}. \quad (2.7)$$

So combining (2.6) and (2.7), we obtain for a sequence of values of r tending to infinity that

$$\frac{M_{h_n}^{-1}(M_{f_{n,g}}(r))}{\log M_k^{-1}(M_g(\exp r^{\delta\mu}))} > \frac{A(\lambda_h(f) - \varepsilon)r^{\delta\mu_0}}{(\rho_k(g) + \varepsilon)r^{\delta\mu}}. \quad (2.8)$$

Since $\mu_0 > \mu$, from (2.8) it follows that

$$\limsup_{r \rightarrow \infty} \frac{M_{h_n}^{-1}(M_{f_{n,g}}(r))}{\log M_k^{-1}(M_g(\exp r^{\delta\mu}))} = \infty.$$

Thus the theorem follows. \square

Now we state the following two theorems without their proofs as those can easily be carried out in the line of Theorem 1 and Theorem 2 respectively and with the help of the second part of Lemma 5.

Theorem 3. Let $f(z)$, $g(z)$, $k(z)$ and $h(z)$ be any four entire functions such that $0 < \lambda_k(g) \leq \rho_k(g) < \infty$, $0 < \lambda_h(f) < \infty$ and $0 < \mu < \rho_h(f) < \infty$. Then for any odd number n ,

$$\limsup_{r \rightarrow \infty} \frac{M_{h_n}^{-1}(M_{f_{n,g}}(r))}{\log M_h^{-1}(M_f(\exp r^{\delta\mu}))} = \infty,$$

where $\delta < 1$.

Theorem 4. Let $f(z)$, $g(z)$, $k(z)$ and $h(z)$ be any four entire functions such that $0 < \lambda_k(g) \leq \rho_k(g) < \infty$, $0 < \lambda_h(f) < \infty$ and $0 < \mu < \rho_h(f) < \infty$. Then for any odd number n ,

$$\limsup_{r \rightarrow \infty} \frac{M_{h_n}^{-1}(M_{f_{n,g}}(r))}{\log M_k^{-1}(M_g(\exp r^{\delta\mu}))} = \infty,$$

where $\delta < 1$.

Theorem 5. *Let $f(z)$, $g(z)$, $k(z)$ and $h(z)$ be any four entire functions such that $0 < \lambda_h(f) \leq \rho_h(f) < \infty$, $0 < \rho_k(g) < \infty$ and $\lambda_k(g) < \mu < \infty$. Then for any even number n ,*

$$\liminf_{r \rightarrow \infty} \frac{M_{h_n}^{-1}(M_{f_{n,g}}(r))}{\log M_h^{-1}(M_f(\exp r^{\delta\mu}))} = 0,$$

where $\delta > 1$.

Proof. From the first part of Lemma 4, it follows for a sequence of values of r tending to infinity that

$$M_{h_n}^{-1}(M_{f_{n,g}}(r)) < r^{\delta(\lambda_k(g)+\varepsilon)}. \quad (2.9)$$

Again for all sufficiently large values of r we get that

$$\log M_h^{-1}(M_f(\exp r^{\delta\mu})) \geq (\lambda_h(f) - \varepsilon) r^{\delta\mu}. \quad (2.10)$$

Now from (2.9) and (2.10), it follows for a sequence of values of r tending to infinity that

$$\frac{M_{h_n}^{-1}(M_{f_{n,g}}(r))}{\log M_h^{-1}(M_f(\exp r^{\delta\mu}))} < \frac{r^{\delta(\lambda_k(g)+\varepsilon)}}{(\lambda_h(f) - \varepsilon) r^{\delta\mu}}. \quad (2.11)$$

As $\lambda_k(g) < \mu$, we can choose $\varepsilon (> 0)$ in such a way that

$$\lambda_k(g) + \varepsilon < \mu. \quad (2.12)$$

Thus the theorem follows from (2.11) and (2.12). \square

In the line of Theorem 5, we may state the following theorem without its proof:

Theorem 6. *Let $f(z)$, $g(z)$, $k(z)$ and $h(z)$ be any four entire functions such that $0 < \rho_h(f) < \infty$, $0 < \rho_k(g) < \infty$ and $\lambda_k(g) < \mu < \infty$. Then for any even number n ,*

$$\liminf_{r \rightarrow \infty} \frac{M_{h_n}^{-1}(M_{f_{n,g}}(r))}{\log M_k^{-1}(M_g(\exp r^{\delta\mu}))} = 0,$$

where $\delta > 1$.

Theorem 7. *Let $f(z)$, $g(z)$, $k(z)$ and $h(z)$ be any four entire functions such that $0 < \lambda_k(g) \leq \rho_k(g) < \infty$, $0 < \rho_h(f) < \infty$ and $\lambda_h(f) < \mu < \infty$. Then for any odd number n ,*

$$\liminf_{r \rightarrow \infty} \frac{M_{h_n}^{-1}(M_{f_{n,g}}(r))}{\log M_k^{-1}(M_g(\exp r^{\delta\mu}))} = 0,$$

where $\delta > 1$.

Theorem 8. *Let $f(z)$, $g(z)$, $k(z)$ and $h(z)$ be any four entire functions such that $0 < \rho_k(g) < \infty$, $0 < \rho_h(f) < \infty$ and $\lambda_h(f) < \mu < \infty$. Then for any odd number n ,*

$$\liminf_{r \rightarrow \infty} \frac{M_{h_n}^{-1}(M_{f_{n,g}}(r))}{\log M_h^{-1}(M_f(\exp r^{\delta\mu}))} = 0,$$

where $\delta > 1$.

We omit the proofs of Theorem 7 and Theorem 8 as those can be carried out in the line of Theorem 5 and Theorem 6 respectively and with the help of the second part of Lemma 4.

Theorem 9. *Let $f(z)$, $g(z)$, $k(z)$ and $h(z)$ be any four entire functions such that $0 < \lambda_h(f) \leq \rho_h(f) < \infty$ and $0 < \lambda_k(g) < \infty$. Also let γ be a positive continuous on $[0, +\infty)$ function increasing to $+\infty$. Then for every real number κ and positive integer n*

$$\lim_{r \rightarrow \infty} \frac{M_{h_n}^{-1}(M_{f_{n,g}}(r))}{\{\log M_h^{-1}(M_f(\exp \gamma(r)))\}^{1+\kappa}} = \infty,$$

where

$$\lim_{r \rightarrow \infty} \frac{\log \gamma(r)}{\log r} = 0.$$

Proof. First let us consider n to be an even integer. If $\kappa \leq 0$ then the theorem is trivial. So we suppose that $1 + \kappa > 0$. Now it follows from the first part of Lemma 5, for all sufficiently large values of r that

$$M_{h_n}^{-1}(M_{f_{n,g}}(r)) > \left(\frac{1}{\gamma_n}\right)^{\frac{1}{\delta}} \left(\frac{r}{18^n}\right)^{\frac{\lambda_k(g)-\varepsilon}{\delta}}, \quad (2.13)$$

where δ and γ_n are defined in Lemma 5.

Again from the definition of $\rho_h(f)$, it follows for all sufficiently large values of r that

$$\{\log M_h^{-1}(M_f(\exp \gamma(r)))\}^{1+\kappa} \leq (\rho_h(f) + \varepsilon)^{1+\kappa} (\gamma(r))^{1+\kappa}. \quad (2.14)$$

Now from (2.13) and (2.14), it follows for all sufficiently large values of r that

$$\frac{M_{h_n}^{-1}(M_{f_{n,g}}(r))}{\{\log M_h^{-1}(M_f(\exp \gamma(r)))\}^{1+\kappa}} > \frac{\left(\frac{1}{\gamma_n}\right)^{\frac{1}{\delta}} \cdot \left(\frac{1}{18^n}\right)^{\frac{\lambda_k(g)-\varepsilon}{\delta}} \cdot r^{\frac{\lambda_k(g)-\varepsilon}{\delta}}}{(\rho_h(f) + \varepsilon)^{1+\kappa} (\gamma(r))^{1+\kappa}}.$$

Since $\lim_{r \rightarrow \infty} \frac{\log \gamma(r)}{\log r} = 0$, therefore $\frac{r^{\frac{\lambda_k(g)-\varepsilon}{\delta}}}{(\gamma(r))^{1+\kappa}} \rightarrow \infty$ as $r \rightarrow \infty$, then from above it follows that

$$\liminf_{r \rightarrow \infty} \frac{M_{h_n}^{-1}(M_{f_{n,g}}(r))}{\{\log M_h^{-1}(M_f(\exp \gamma(r)))\}^{1+\kappa}} = \infty \text{ for any even number } n.$$

Similarly, with the help of the second part of Lemma 5 one can easily derive the same conclusion for any odd integer n .

Hence the theorem follows. \square

Remark 3. Theorem 9 is still valid with “limit superior” instead of “limit” if we replace the condition “ $0 < \lambda_h(f) \leq \rho_h(f) < \infty$ ” by “ $0 < \lambda_h(f) < \infty$ ”.

In the line of Theorem 9, one may state the following theorem without its proof:

Theorem 10. Let $f(z)$, $g(z)$, $k(z)$ and $h(z)$ be any four entire functions such that $0 < \lambda_h(f) < \infty$ and $0 < \lambda_k(g) \leq \rho_k(g) < \infty$. Also let γ be a positive continuous on $[0, +\infty)$ function increasing to $+\infty$. Then for every real number κ and positive integer n

$$\lim_{r \rightarrow \infty} \frac{M_{h_n}^{-1}(M_{f_{n,g}}(r))}{\{\log M_k^{-1}(M_g(\exp \gamma(r)))\}^{1+\kappa}} = \infty,$$

where

$$\lim_{r \rightarrow \infty} \frac{\log \gamma(r)}{\log r} = 0.$$

Remark 4. In Theorem 10 if we take the condition $0 < \lambda_k(g) < \infty$ instead of $0 < \lambda_k(g) \leq \rho_k(g) < \infty$, then also Theorem 10 remains true with “limit superior” in place of “limit”.

Theorem 11. Let $f(z)$, $g(z)$, $k(z)$ and $h(z)$ be any four entire functions such that $0 < \lambda_h(f) \leq \rho_h(f) < \infty$ and $0 < \rho_k(g) < \infty$. Also let γ be a positive continuous on $[0, +\infty)$ function increasing to $+\infty$. Then for each $\kappa \in (-\infty, \infty)$ and positive integer n

$$\lim_{r \rightarrow \infty} \frac{(M_{h_n}^{-1}(M_{f_{n,g}}(r)))^{1+\kappa}}{\log M_h^{-1}(M_f(\exp \gamma(r)))} = 0,$$

where

$$\lim_{r \rightarrow \infty} \frac{\log \gamma(r)}{\log r} = \infty.$$

Proof. If $1 + \kappa \leq 0$, then the theorem is obvious. We consider that $1 + \kappa > 0$. Also let us consider n to be an even integer. Now it follows from the first part of Lemma 4 for all sufficiently large values of r that

$$M_{h_n}^{-1}(M_{f_{n,g}}(r)) < r^{\delta(\rho_k(g)+\varepsilon)}, \quad (2.15)$$

where $\delta > 1$.

Again for all sufficiently large values of r we get that

$$\log M_h^{-1}(M_f(\exp \gamma(r))) \geq (\lambda_h(f) - \varepsilon) \gamma(r). \quad (2.16)$$

Hence for all sufficiently large values of r , we obtain from (2.15) and (2.16) that

$$\frac{(M_{h_n}^{-1}(M_{f_{n,g}}(r)))^{1+\kappa}}{\log M_h^{-1}(M_f(\exp \gamma(r)))} < \frac{r^{\delta(\rho_k(g)+\varepsilon)(1+\kappa)}}{(\lambda_h(f) - \varepsilon) \gamma(r)}, \quad (2.17)$$

where we choose $0 < \varepsilon < \min\{\lambda_h(f), \rho_k(g)\}$.

Since $\lim_{r \rightarrow \infty} \frac{\log \gamma(r)}{\log r} = \infty$, therefore $\frac{r^{\delta(\rho_k(g)+\varepsilon)(1+\kappa)}}{\gamma(r)} \rightarrow \infty$ as $r \rightarrow \infty$, then from (2.17) we obtain that

$$\liminf_{r \rightarrow \infty} \frac{(M_{h_n}^{-1}(M_{f_{n,g}}(r)))^{1+\kappa}}{\log M_h^{-1}(M_f(\exp \gamma(r)))} = 0 \text{ for any even number } n.$$

Similarly, with the help of the second part of Lemma 4 one can easily derive the same conclusion for any odd integer n .

This proves the theorem. \square

Remark 5. In Theorem 11 if we take the condition $0 < \rho_h(f) < \infty$ instead of $0 < \lambda_h(f) \leq \rho_h(f) < \infty$, the theorem remains true with “limit inferior” in place of “limit”.

In view of Theorem 11, the following theorem can be carried out :

Theorem 12. Let $f(z)$, $g(z)$, $k(z)$ and $h(z)$ be any four entire functions such that $0 < \rho_h(f) < \infty$ and $0 < \lambda_k(g) \leq \rho_k(g) < \infty$. Also let γ be a positive continuous on $[0, +\infty)$ function increasing to $+\infty$. Then for each $\kappa \in (-\infty, \infty)$ and positive integer n

$$\lim_{r \rightarrow \infty} \frac{(M_{h_n}^{-1}(M_{f_{n,g}}(r)))^{1+\kappa}}{\log M_k^{-1}(M_g(\exp \gamma(r)))} = 0,$$

where

$$\lim_{r \rightarrow \infty} \frac{\log \gamma(r)}{\log r} = \infty.$$

The proof is omitted.

Remark 6. In Theorem 12 if we take the condition $0 < \rho_k(g) < \infty$ instead of $0 < \lambda_k(g) \leq \rho_k(g) < \infty$ then the theorem remains true with “limit inferior” in place of “limit”.

Theorem 13. Let $f(z)$, $g(z)$, $k(z)$ and $h(z)$ be any four entire functions such that $\lambda_k(g) < \lambda_h(f) \leq \rho_h(f) < \infty$ and $0 < \rho_k(g) < \infty$. Then for any even number n ,

$$\liminf_{r \rightarrow \infty} \frac{M_{h_n}^{-1}(M_{f_{n,g}}(r))}{M_h^{-1}(M_f(r^\delta))} = 0,$$

where $\delta > 1$.

Proof. From the first part of Lemma 4, we obtain for a sequence of values of r tending to infinity that

$$M_{h_n}^{-1}(M_{f_{n,g}}(r)) < r^{\delta(\lambda_k(g)+\varepsilon)}. \quad (2.18)$$

Again from the definition of relative order, we obtain for all sufficiently large values of r that

$$M_h^{-1}(M_f(r^\delta)) \geq r^{\delta(\lambda_h(f)-\varepsilon)}. \quad (2.19)$$

Now in view of (2.18) and (2.19), we get for a sequence of values of r tending to infinity that

$$\frac{M_{h_n}^{-1}(M_{f_{n,g}}(r))}{M_h^{-1}(M_f(r^\delta))} < \frac{r^{\delta(\lambda_k(g)+\varepsilon)}}{r^{\delta(\lambda_h(f)-\varepsilon)}}. \quad (2.20)$$

Since $\lambda_k(g) < \lambda_h(f)$, we can choose $\varepsilon (> 0)$ in such a way that $\lambda_k(g) + \varepsilon < \lambda_h(f) - \varepsilon$ and then the theorem follows from (2.20). \square

Remark 7. If we take $0 < \rho_k(g) < \lambda_h(f) \leq \rho_h(f) < \infty$ instead of “ $\lambda_k(g) < \lambda_h(f) \leq \rho_h(f) < \infty$ and $\rho_k(g) < \infty$ ” and the other conditions remain the same, the conclusion of Theorem 13 remains valid with “limit inferior” replaced by “limit”.

Theorem 14. Let $f(z)$, $g(z)$, $k(z)$ and $h(z)$ be any four entire functions such that $\lambda_h(f) < \lambda_k(g) \leq \rho_k(g) < \infty$ and $0 < \rho_h(f) < \infty$. Then for any odd number n ,

$$\liminf_{r \rightarrow \infty} \frac{M_{h_n}^{-1}(M_{f_{n,g}}(r))}{M_k^{-1}(M_g(r^\delta))} = 0,$$

where $\delta > 1$.

The proof of Theorem 14 is omitted as it can be carried out in the line of Theorem 13 and with the help of the second part of Lemma 4.

Remark 8. If we consider $0 < \rho_h(f) < \lambda_k(g) \leq \rho_k(g) < \infty$ instead of “ $\lambda_h(f) < \lambda_k(g) \leq \rho_k(g) < \infty$ and $\rho_h(f) < \infty$ ” and the other conditions remain the same, the conclusion of Theorem 13 remains valid with “limit inferior” replaced by “limit”.

Theorem 15. Let $f(z)$, $g(z)$, $k(z)$ and $h(z)$ be any four entire functions such that $0 < \lambda_h(f) \leq \rho_h(f) < \infty$ and $0 < \rho_k(g) < \infty$. Then

$$\limsup_{r \rightarrow \infty} \frac{\log M_{h_n}^{-1}(M_{f_{n,g}}(r))}{\log M_h^{-1}(M_f(r^\delta))} \leq \frac{\rho_k(g)}{\lambda_h(f)} \text{ when } n \text{ is even,}$$

and

$$\limsup_{r \rightarrow \infty} \frac{\log M_{h_n}^{-1}(M_{f_{n,g}}(r))}{\log M_h^{-1}(M_f(r^\delta))} \leq \frac{\rho_h(f)}{\lambda_h(f)} \text{ when } n \text{ is any odd integer}$$

where $\delta > 1$.

Proof. From the first part of Lemma 4, it follows for all sufficiently large values of r that

$$\begin{aligned} \frac{\log M_{h_n}^{-1}(M_{f_{n,g}}(r))}{\log M_h^{-1}(M_f(r^\delta))} &< \frac{\delta \log M_k^{-1}(M_g(r))}{\log M_h^{-1}(M_f(r^\delta))} \\ \text{i.e., } \frac{\log M_{h_n}^{-1}(M_{f_{n,g}}(r))}{\log M_h^{-1}(M_f(r^\delta))} &< \frac{\delta \log M_k^{-1}(M_g(r))}{\delta \log r} \cdot \frac{\log r^\delta}{\log M_h^{-1}(M_f(r^\delta))} \end{aligned}$$

$$\begin{aligned} \text{i.e., } \limsup_{r \rightarrow \infty} \frac{\log M_{h_n}^{-1}(M_{f_{n,g}}(r))}{\log M_h^{-1}(M_f(r^\delta))} &\leq \limsup_{r \rightarrow \infty} \frac{\log M_k^{-1}(M_g(r))}{\log r} \cdot \limsup_{r \rightarrow \infty} \frac{\log r^\delta}{\log M_h^{-1}(M_f(r^\delta))} \\ \text{i.e., } \limsup_{r \rightarrow \infty} \frac{\log M_{h_n}^{-1}(M_{f_{n,g}}(r))}{\log M_h^{-1}(M_f(r^\delta))} &\leq \rho_k(g) \cdot \frac{1}{\lambda_h(f)} = \frac{\rho_k(g)}{\lambda_h(f)}. \end{aligned}$$

Thus the first part of theorem follows from above.

Similarly, with the help of the second part of Lemma 4 one can easily derive conclusion of the second part of theorem.

Hence the theorem follows. \square

Theorem 16. *Let $f(z)$, $g(z)$, $k(z)$ and $h(z)$ be any four entire functions such that $0 < \lambda_k(g) \leq \rho_k(g) < \infty$ and $0 < \rho_h(f) < \infty$. Then*

$$\limsup_{r \rightarrow \infty} \frac{\log M_{h_n}^{-1}(M_{f_{n,g}}(r))}{\log M_k^{-1}(M_g(r^\delta))} \leq \frac{\rho_k(g)}{\lambda_k(g)} \text{ when } n \text{ is even,}$$

and

$$\limsup_{r \rightarrow \infty} \frac{\log M_{h_n}^{-1}(M_{f_{n,g}}(r))}{\log M_k^{-1}(M_g(r^\delta))} \leq \frac{\rho_h(f)}{\lambda_k(g)} \text{ when } n \text{ is any odd integer}$$

where $\delta > 1$.

The proof of Theorem 16 is omitted as it can be carried out in the line of Theorem 15.

Now we state the following two theorems without their proofs as those can easily be carried out in the line of Theorem 15 and Theorem 16 respectively and with the help of Lemma 4.

Theorem 17. *Let $f(z)$, $g(z)$, $k(z)$ and $h(z)$ be any four entire functions such that $0 < \lambda_h(f) \leq \rho_h(f) < \infty$ and $0 < \lambda_k(g) \leq \rho_k(g) < \infty$. Then*

$$\liminf_{r \rightarrow \infty} \frac{\log M_{h_n}^{-1}(M_{f_{n,g}}(r))}{\log M_h^{-1}(M_f(r^\delta))} \leq \frac{\lambda_k(g)}{\lambda_h(f)} \text{ when } n \text{ is even,}$$

and

$$\liminf_{r \rightarrow \infty} \frac{\log M_{h_n}^{-1}(M_{f_{n,g}}(r))}{\log M_h^{-1}(M_f(r^\delta))} \leq 1 \text{ when } n \text{ is any odd integer}$$

where $\delta > 1$.

Theorem 18. *Let $f(z)$, $g(z)$, $k(z)$ and $h(z)$ be any four entire functions such that $0 < \lambda_h(f) \leq \rho_h(f) < \infty$ and $0 < \lambda_k(g) \leq \rho_k(g) < \infty$. Then*

$$\liminf_{r \rightarrow \infty} \frac{\log M_{h_n}^{-1}(M_{f_{n,g}}(r))}{\log M_k^{-1}(M_g(r^\delta))} \leq 1 \text{ when } n \text{ is even,}$$

and

$$\liminf_{r \rightarrow \infty} \frac{\log M_{h_n}^{-1}(M_{f_{n,g}}(r))}{\log M_k^{-1}(M_g(r^\delta))} \leq \frac{\lambda_h(f)}{\lambda_k(g)} \text{ when } n \text{ is any odd integer}$$

where $\delta > 1$.

Theorem 19. *Let $f(z)$, $g(z)$, $k(z)$ and $h(z)$ be any four entire functions such that $0 < \lambda_h(f) < \infty$ and $0 < \lambda_k(g) < \infty$. Then for any even number n ,*

$$\liminf_{r \rightarrow \infty} \frac{\log M_{h_n}^{-1}(M_{f_{n,g}}(r))}{\log M_h^{-1}(M_f(r^\delta))} \geq \frac{\lambda_k(g)}{\rho_h(f)} \text{ when } 0 < \rho_h(f) < \infty$$

and

$$\liminf_{r \rightarrow \infty} \frac{\log M_{h_n}^{-1}(M_{f_{n,g}}(r))}{\log M_k^{-1}(M_g(r^\delta))} \geq \frac{\lambda_k(g)}{\rho_k(g)} \text{ when } 0 < \rho_k(g) < \infty,$$

where $\delta < 1$.

Proof. From the first part of Lemma 5, we obtain for all sufficiently large values of r that

$$\log M_{h_n}^{-1}(M_{f_{n,g}}(r)) > \delta(\lambda_k(g) - \varepsilon) \log\left(\frac{r}{18^n}\right) + \log\left(\frac{1}{\gamma_n}\right), \quad (2.21)$$

where γ_n is defined in Lemma 5.

Also from the definition of $\rho_h(f)$, we obtain for all sufficiently large values of r that

$$\log M_h^{-1}(M_f(r^\delta)) \leq \delta(\rho_h(f) + \varepsilon) \log r. \quad (2.22)$$

Analogously, from the definition of $\rho_k(g)$, it follows for all sufficiently large values of r that

$$\log M_k^{-1}(M_g(r^\delta)) \leq \delta(\rho_k(g) + \varepsilon) \log r. \quad (2.23)$$

Now from (2.21) and (2.22), it follows for all sufficiently large values of r that

$$\begin{aligned} \frac{\log M_{h_n}^{-1}(M_{f_{n,g}}(r))}{\log M_h^{-1}(M_f(r^\delta))} &> \frac{\delta(\lambda_k(g) - \varepsilon) \log\left(\frac{r}{18^n}\right) + \log\left(\frac{1}{\gamma_n}\right)}{\delta(\rho_h(f) + \varepsilon) \log r} \\ \text{i.e., } \liminf_{r \rightarrow \infty} \frac{\log M_{h_n}^{-1}(M_{f_{n,g}}(r))}{\log M_h^{-1}(M_f(r^\delta))} &\geq \frac{\lambda_k(g)}{\rho_h(f)}. \end{aligned} \quad (2.24)$$

Thus the first part of theorem follows from (2.24).

Similarly, the conclusion of the second part of theorem can easily be derived from (2.21) and (2.23).

Hence the theorem follows. \square

Theorem 20. Let $f(z)$, $g(z)$, $k(z)$ and $h(z)$ be any four entire functions such that $0 < \lambda_h(f) < \infty$ and $0 < \lambda_k(g) < \infty$. Then for any odd number n ,

$$\liminf_{r \rightarrow \infty} \frac{\log M_{h_n}^{-1}(M_{f_{n,g}}(r))}{\log M_h^{-1}(M_f(r^\delta))} \geq \frac{\lambda_h(f)}{\rho_h(f)} \text{ when } 0 < \rho_h(f) < \infty$$

and

$$\liminf_{r \rightarrow \infty} \frac{\log M_{h_n}^{-1}(M_{f_{n,g}}(r))}{\log M_k^{-1}(M_g(r^\delta))} \geq \frac{\lambda_h(f)}{\rho_k(g)} \text{ when } 0 < \rho_k(g) < \infty,$$

where $\delta < 1$.

The proof of Theorem 20 is omitted as it can be carried out in the line of Theorem 19 and with the help of the second part of Lemma 5.

Now we state the following two theorems without their proofs as those can easily be carried out in the line of Theorem 19 and Theorem 20 respectively and with the help of Lemma 5.

Theorem 21. Let $f(z)$, $g(z)$, $k(z)$ and $h(z)$ be any four entire functions such that $0 < \lambda_h(f) < \infty$ and $0 < \lambda_k(g) \leq \rho_k(g) < \infty$. Then for any even number n ,

$$\limsup_{r \rightarrow \infty} \frac{\log M_{h_n}^{-1}(M_{f_{n,g}}(r))}{\log M_h^{-1}(M_f(r^\delta))} \geq \frac{\rho_k(g)}{\rho_h(f)} \text{ when } 0 < \rho_h(f) < \infty$$

and

$$\limsup_{r \rightarrow \infty} \frac{\log M_{h_n}^{-1}(M_{f_{n,g}}(r))}{\log M_k^{-1}(M_g(r^\delta))} \geq 1, \text{ when } 0 < \rho_k(g) < \infty,$$

where $\delta < 1$.

Theorem 22. Let $f(z)$, $g(z)$, $k(z)$ and $h(z)$ be any four entire functions such that $0 < \lambda_h(f) \leq \rho_h(f) < \infty$ and $0 < \lambda_k(g) < \infty$. Then for any odd number n ,

$$\limsup_{r \rightarrow \infty} \frac{\log M_{h_n}^{-1}(M_{f_{n,g}}(r))}{\log M_k^{-1}(M_g(r^\delta))} \geq \frac{\rho_h(f)}{\rho_k(g)} \text{ when } 0 < \rho_k(g) < \infty$$

and

$$\limsup_{r \rightarrow \infty} \frac{\log M_{h_n}^{-1}(M_{f_{n,g}}(r))}{\log M_h^{-1}(M_f(r^\delta))} \geq 1 \text{ when } 0 < \rho_h(f) < \infty,$$

where $\delta < 1$.

Theorem 23. Let $F(z)$, $G(z)$, $H(z)$, $K(z)$, $f(z)$, $g(z)$, $h(z)$ and $k(z)$ are all entire functions such that $0 < \lambda_H(F) < \infty$, $0 < \lambda_K(G) < \infty$, $0 < \rho_h(f) < \infty$ and $0 < \rho_k(g) < \infty$. Then for any two integers m and n

$$(i) \lim_{r \rightarrow \infty} \frac{M_{H_m}^{-1}(M_{F_{m,G}}(r))}{M_{h_n}^{-1}(M_{f_{n,g}}(r)) \cdot \log M_h^{-1}(M_f(r))} = \infty$$

and

$$(ii) \lim_{r \rightarrow \infty} \frac{M_{H_m}^{-1}(M_{F_{m,G}}(r))}{M_{h_n}^{-1}(M_{f_{n,g}}(r)) \cdot \log M_k^{-1}(M_g(r))} = \infty,$$

when for any $\delta > 1$ be such that

$$\left\{ \begin{array}{l} \delta^2 \rho_k(g) < \lambda_K(G) \text{ for } m \text{ and } n \text{ both even} \\ \delta^2 \rho_h(f) < \lambda_H(F) \text{ for } m \text{ and } n \text{ both odd} \\ \delta^2 \rho_h(f) < \lambda_K(G) \text{ for } m \text{ even and } n \text{ odd} \\ \delta^2 \rho_k(g) < \lambda_H(F) \text{ for } m \text{ odd and } n \text{ even} . \end{array} \right. \quad (2.25)$$

Proof. We have from the definition of relative order and for all sufficiently large values of r that

$$\log M_h^{-1}(M_f(r)) \leq (\rho_h(f) + \varepsilon) \log r. \quad (2.26)$$

Case I. Let m and n are any two even numbers.

Therefore in view of first part of Lemma 4, we get for all sufficiently large values of r that

$$M_{h_n}^{-1}(M_{f_{n,g}}(r)) < (r)^{\delta(\rho_k(g)+\varepsilon)}, \quad (2.27)$$

where $\delta > 1$.

So from (2.26) and (2.27) it follows for all sufficiently large values of r that

$$M_{h_n}^{-1}(M_{f_{n,g}}(r)) \cdot \log M_h^{-1}(M_f(r)) < (r)^{\delta(\rho_k(g)+\varepsilon)} \cdot (\rho_h(f) + \varepsilon) \log r. \quad (2.28)$$

Also from first part of Lemma 5, we obtain for all sufficiently large values of r that

$$M_{H_m}^{-1}(M_{F_{m,G}}(r)) > \left(\frac{1}{\gamma_m}\right)^{\frac{1}{\delta}} \left(\frac{r}{18^m}\right)^{\frac{(\lambda_K(G)-\varepsilon)}{\delta}}, \quad (2.29)$$

where $\delta > 1$ and γ_m is defined in Lemma 5.

Hence combining (2.28) and (2.29) we get for all sufficiently large values of r that,

$$\frac{M_{H_m}^{-1}(M_{F_{m,G}}(r))}{M_{h_n}^{-1}(M_{f_{n,g}}(r)) \cdot \log M_h^{-1}(M_f(r))} > \frac{\left(\frac{1}{\gamma_m}\right)^{\frac{1}{\delta}} \left(\frac{r}{18^m}\right)^{\frac{(\lambda_K(G)-\varepsilon)}{\delta}}}{(r)^{\delta(\rho_k(g)+\varepsilon)} \cdot (\rho_h(f) + \varepsilon) \log r}. \quad (2.30)$$

Since $\delta^2 \rho_k(g) < \lambda_K(G)$, we can choose $\varepsilon(> 0)$ in such a manner that

$$\delta^2 (\rho_k(g) + \varepsilon) \leq (\lambda_K(G) - \varepsilon). \quad (2.31)$$

Thus from (2.30) and (2.31) we obtain that

$$\lim_{r \rightarrow \infty} \frac{M_{H_m}^{-1}(M_{F_{m,G}}(r))}{M_{h_n}^{-1}(M_{f_{n,g}}(r)) \cdot \log M_h^{-1}(M_f(r))} = \infty. \quad (2.32)$$

Case II. Let m and n are any two odd numbers .

Now in view of second part of Lemma 4, we get for all sufficiently large values of r that

$$M_{h_n}^{-1}(M_{f_{n,g}}(r)) < (r)^{\delta(\rho_h(f)+\varepsilon)}, \quad (2.33)$$

where $\delta > 1$.

So from (2.26) and (2.33) it follows for all sufficiently large values of r that

$$M_{h_n}^{-1}(M_{f_{n,g}}(r)) \cdot \log M_h^{-1}(M_f(r)) < (r)^{\delta(\rho_h(f)+\varepsilon)} \cdot (\rho_h(f) + \varepsilon) \log r. \quad (2.34)$$

Also from second part of Lemma 5, we obtain for all sufficiently large values of r that

$$M_{H_m}^{-1}(M_{F_{m,G}}(r)) > \left(\frac{1}{\gamma_m}\right)^{\frac{1}{\delta}} \left(\frac{r}{18^m}\right)^{\frac{(\lambda_H(F)-\varepsilon)}{\delta}}. \quad (2.35)$$

Hence combining (2.34) and (2.35) we get for all sufficiently large values of r that,

$$\frac{M_{H_m}^{-1}(M_{F_{m,G}}(r))}{M_{h_n}^{-1}(M_{f_{n,g}}(r)) \cdot \log M_h^{-1}(M_f(r))} > \frac{\left(\frac{1}{\gamma_m}\right)^{\frac{1}{\delta}} \left(\frac{r}{18^m}\right)^{\frac{(\lambda_H(F)-\varepsilon)}{\delta}}}{(r)^{\delta(\rho_h(f)+\varepsilon)} \cdot (\rho_h(f) + \varepsilon) \log r}. \quad (2.36)$$

As $\delta^2 \rho_h(f) < \lambda_H(F)$, we can choose $\varepsilon(> 0)$ in such a manner that

$$\delta^2 (\rho_h(f) + \varepsilon) \leq (\lambda_H(F) - \varepsilon). \quad (2.37)$$

Therefore from (2.36) and (2.37) it follows that

$$\lim_{r \rightarrow \infty} \frac{M_{H_m}^{-1}(M_{F_{m,G}}(r))}{M_{h_n}^{-1}(M_{f_{n,g}}(r)) \cdot \log M_h^{-1}(M_f(r))} = \infty. \quad (2.38)$$

Case III. Let m be any even number and n be any odd number.

Then combining (2.29) and (2.34) we get for all sufficiently large values of r that

$$\frac{M_{H_m}^{-1}(M_{F_{m,G}}(r))}{M_{h_n}^{-1}(M_{f_{n,g}}(r)) \cdot \log M_h^{-1}(M_f(r))} > \frac{\left(\frac{1}{\gamma_m}\right)^{\frac{1}{\delta}} \left(\frac{r}{18^m}\right)^{\frac{(\lambda_K(G)-\varepsilon)}{\delta}}}{(r)^{\delta(\rho_h(f)+\varepsilon)} \cdot (\rho_h(f) + \varepsilon) \log r}. \quad (2.39)$$

Since $\delta^2 \rho_h(f) < \lambda_K(G)$, we can choose $\varepsilon(> 0)$ in such a manner that

$$\delta^2(\rho_h(f) + \varepsilon) \leq (\lambda_K(G) - \varepsilon). \quad (2.40)$$

So from (2.39) and (2.40) we get that

$$\lim_{r \rightarrow \infty} \frac{M_{H_m}^{-1}(M_{F_{m,G}}(r))}{M_{h_n}^{-1}(M_{f_{n,g}}(r)) \cdot \log M_h^{-1}(M_f(r))} = \infty. \quad (2.41)$$

Case IV. Let m be any odd number and n be any even number .

Therefore combining (2.28) and (2.35) we obtain for all sufficiently large values of r that

$$\frac{M_{H_m}^{-1}(M_{F_{m,G}}(r))}{M_{h_n}^{-1}(M_{f_{n,g}}(r)) \cdot \log M_h^{-1}(M_f(r))} > \frac{\left(\frac{1}{\gamma_m}\right)^{\frac{1}{\delta}} \left(\frac{r}{18^m}\right)^{\frac{(\lambda_H(F)-\varepsilon)}{\delta}}}{(r)^{\delta(\rho_k(g)+\varepsilon)} \cdot (\rho_h(f) + \varepsilon) \log r}. \quad (2.42)$$

As $\delta^2 \rho_k(g) < \lambda_H(F)$, we can choose $\varepsilon(> 0)$ in such a manner that

$$\delta^2(\rho_k(g) + \varepsilon) \leq (\lambda_H(F) - \varepsilon). \quad (2.43)$$

Hence from (2.42) and (2.43) we have

$$\lim_{r \rightarrow \infty} \frac{M_{H_m}^{-1}(M_{F_{m,G}}(r))}{M_{h_n}^{-1}(M_{f_{n,g}}(r)) \cdot \log M_h^{-1}(M_f(r))} = \infty. \quad (2.44)$$

Thus the first part of the theorem follows from (2.32), (2.38), (2.41) and (2.44).

Similarly, from the definition of $\rho_k(g)$ one can easily derive the conclusion of the second part of the theorem.

Hence the theorem follows. \square

Remark 9. If we consider $\rho_K(G)$, $\rho_H(F)$, $\rho_K(G)$ and $\rho_H(F)$ instead of $\lambda_K(G)$, $\lambda_H(F)$, $\lambda_K(G)$ and $\lambda_H(F)$ respectively in (2.25) and the other conditions remain the same, the conclusion of Theorem 23 is remain valid with “limit superior” replaced by “limit”.

Theorem 24. Let $f(z)$, $g(z)$, $k(z)$ and $h(z)$ be any four entire functions such that $0 < \rho_h(f) < \infty$, $0 < \rho_k(g) < \infty$ and $\sigma_k(g) < \infty$. Then for any even number n ,

$$\limsup_{r \rightarrow \infty} \frac{M_{h_n}^{-1}(M_{f_{n,g}}(r))}{\log M_h^{-1}\left(M_f\left(\exp(r)^{\delta \rho_k(g)}\right)\right)} \leq \frac{(\sigma_k(g))^\delta}{\lambda_h(f)} \text{ if } \lambda_h(f) \neq 0$$

and

$$\limsup_{r \rightarrow \infty} \frac{M_{h_n}^{-1}(M_{f_{n,g}}(r))}{\log M_k^{-1}\left(M_g\left(\exp(r)^{\delta \rho_k(g)}\right)\right)} \leq \frac{(\sigma_k(g))^\delta}{\lambda_k(g)} \text{ if } \lambda_k(g) \neq 0,$$

where $\delta > 1$.

Proof. In view of the first part of Lemma 4 we have for all sufficiently large values of r that

$$\frac{M_{h_n}^{-1}(M_{f_{n,g}}(r))}{\log M_h^{-1}\left(M_f\left(\exp(r)^{\delta \rho_k(g)}\right)\right)} < \frac{(M_k^{-1}(M_g(r)))^\delta}{\log M_h^{-1}\left(M_f\left(\exp(r)^{\delta \rho_k(g)}\right)\right)}$$

i.e.,

$$\frac{M_{h_n}^{-1}(M_{f_{n,g}}(r))}{\log M_h^{-1}\left(M_f\left(\exp(r)^{\delta \rho_k(g)}\right)\right)} < \left(\frac{M_k^{-1}(M_g(r))}{r^{\rho_k(g)}}\right)^\delta \cdot \frac{\log \exp(r)^{\delta \rho_k(g)}}{\log M_h^{-1}\left(M_f\left(\exp(r)^{\delta \rho_k(g)}\right)\right)}$$

$$\begin{aligned} \text{i.e.,} \quad & \limsup_{r \rightarrow \infty} \frac{M_{h_n}^{-1}(M_{f_{n,g}}(r))}{\log M_h^{-1}\left(M_f\left(\exp(r)^{\delta \rho_k(g)}\right)\right)} \\ & \leq \left(\limsup_{r \rightarrow \infty} \frac{M_k^{-1}(M_g(r))}{r^{\rho_k(g)}}\right)^\delta \cdot \limsup_{r \rightarrow \infty} \frac{\log \exp(r)^{\delta \rho_k(g)}}{\log M_h^{-1}\left(M_f\left(\exp(r)^{\delta \rho_k(g)}\right)\right)} \end{aligned}$$

$$\text{i.e.,} \quad \limsup_{r \rightarrow \infty} \frac{M_{h_n}^{-1}(M_{f_{n,g}}(r))}{\log M_h^{-1}\left(M_f\left(\exp(r)^{\delta \rho_k(g)}\right)\right)} \leq (\sigma_k(g))^\delta \cdot \frac{1}{\lambda_h(f)} = \frac{(\sigma_k(g))^\delta}{\lambda_h(f)}.$$

Thus the first part of theorem is established.

Similarly, with the help of the first part of Lemma 4 one can easily derive conclusion of the second part of theorem.

Hence the theorem follows. \square

Theorem 25. Let $f(z)$, $g(z)$, $k(z)$ and $h(z)$ be any four entire functions such that $0 < \lambda_h(f) \leq \rho_h(f) < \infty$, $0 < \lambda_k(g) \leq \rho_k(g) < \infty$ and $\sigma_k(g) < \infty$. Then for any even number n ,

$$\liminf_{r \rightarrow \infty} \frac{M_{h_n}^{-1}(M_{f_{n,g}}(r))}{\log M_h^{-1}\left(M_f\left(\exp(r)^{\delta \rho_k(g)}\right)\right)} \leq \min \left\{ \frac{(\bar{\sigma}_k(g))^\delta}{\lambda_h(f)}, \frac{(\sigma_k(g))^\delta}{\rho_h(f)} \right\}$$

and

$$\liminf_{r \rightarrow \infty} \frac{M_{h_n}^{-1}(M_{f_{n,g}}(r))}{\log M_k^{-1}\left(M_g\left(\exp(r)^{\delta \rho_k(g)}\right)\right)} \leq \min \left\{ \frac{(\bar{\sigma}_k(g))^\delta}{\lambda_k(g)}, \frac{(\sigma_k(g))^\delta}{\rho_k(g)} \right\},$$

where $\delta > 1$.

Proof of Theorem 25 is omitted as it can be carried out in the line of Theorem 24 and with help of the first part of Lemma 4.

Now we state the following two theorems without their proofs as those can easily be carried out with the help of second part of Lemma 4 and in the line of Theorem 24 and Theorem 25 respectively.

Theorem 26. *Let $f(z)$, $g(z)$, $k(z)$ and $h(z)$ be any four entire functions such that $0 < \rho_h(f) < \infty$, $0 < \rho_k(g) < \infty$ and $\sigma_h(f) < \infty$. Then for any odd number n ,*

$$\limsup_{r \rightarrow \infty} \frac{M_{h_n}^{-1}(M_{f_{n,g}}(r))}{\log M_h^{-1}\left(M_f\left(\exp(r)^{\delta \rho_h(f)}\right)\right)} \leq \frac{(\sigma_h(f))^\delta}{\lambda_h(f)} \text{ if } \lambda_h(f) \neq 0$$

and

$$\limsup_{r \rightarrow \infty} \frac{M_{h_n}^{-1}(M_{f_{n,g}}(r))}{\log M_k^{-1}\left(M_g\left(\exp(r)^{\delta \rho_h(f)}\right)\right)} \leq \frac{(\sigma_h(f))^\delta}{\lambda_k(g)} \text{ if } \lambda_k(g) \neq 0,$$

where $\delta > 1$.

Theorem 27. *Let $f(z)$, $g(z)$, $k(z)$ and $h(z)$ be any four entire functions such that $0 < \lambda_h(f) \leq \rho_h(f) < \infty$, $0 < \lambda_k(g) \leq \rho_k(g) < \infty$ and $\sigma_h(f) < \infty$. Then for any odd number n ,*

$$\liminf_{r \rightarrow \infty} \frac{M_{h_n}^{-1}(M_{f_{n,g}}(r))}{\log M_h^{-1}\left(M_f\left(\exp(r)^{\delta \rho_h(f)}\right)\right)} \leq \min \left\{ \frac{(\bar{\sigma}_h(f))^\delta}{\lambda_h(f)}, \frac{(\sigma_h(f))^\delta}{\rho_h(f)} \right\}$$

and

$$\liminf_{r \rightarrow \infty} \frac{M_{h_n}^{-1}(M_{f_{n,g}}(r))}{\log M_k^{-1}\left(M_g\left(\exp(r)^{\delta \rho_h(f)}\right)\right)} \leq \min \left\{ \frac{(\bar{\sigma}_h(f))^\delta}{\lambda_k(g)}, \frac{(\sigma_h(f))^\delta}{\rho_k(g)} \right\},$$

where $\delta > 1$.

Analogously, one may state the following four theorems without their proofs on the basis of relative weak type of entire function with respect to another entire function :

Theorem 28. *Let $f(z)$, $g(z)$, $k(z)$ and $h(z)$ be any four entire functions such that $0 < \rho_h(f) < \infty$, $0 < \rho_k(g) < \infty$ and $\bar{\tau}_k(g) < \infty$. Then for any even number n ,*

$$\limsup_{r \rightarrow \infty} \frac{M_{h_n}^{-1}(M_{f_{n,g}}(r))}{\log M_h^{-1}\left(M_f\left(\exp(r)^{\delta \lambda_k(g)}\right)\right)} \leq \frac{(\bar{\tau}_k(g))^\delta}{\lambda_h(f)} \text{ if } \lambda_h(f) \neq 0$$

and

$$\limsup_{r \rightarrow \infty} \frac{M_{h_n}^{-1}(M_{f_{n,g}}(r))}{\log M_k^{-1}\left(M_g\left(\exp(r)^{\delta \lambda_k(g)}\right)\right)} \leq \frac{(\bar{\tau}_k(g))^\delta}{\lambda_k(g)} \quad \text{if } \lambda_k(g) \neq 0,$$

where $\delta > 1$.

Theorem 29. Let $f(z)$, $g(z)$, $k(z)$ and $h(z)$ be any four entire functions such that $0 < \lambda_h(f) \leq \rho_h(f) < \infty$, $0 < \lambda_k(g) \leq \rho_k(g) < \infty$ and $\bar{\tau}_k(g) < \infty$. Then for any even number n ,

$$\liminf_{r \rightarrow \infty} \frac{M_{h_n}^{-1}(M_{f_{n,g}}(r))}{\log M_h^{-1}\left(M_f\left(\exp(r)^{\delta \lambda_k(g)}\right)\right)} \leq \min \left\{ \frac{(\tau_k(g))^\delta}{\lambda_h(f)}, \frac{(\bar{\tau}_k(g))^\delta}{\rho_h(f)} \right\}$$

and

$$\liminf_{r \rightarrow \infty} \frac{M_{h_n}^{-1}(M_{f_{n,g}}(r))}{\log M_k^{-1}\left(M_g\left(\exp(r)^{\delta \lambda_k(g)}\right)\right)} \leq \min \left\{ \frac{(\tau_k(g))^\delta}{\lambda_k(g)}, \frac{(\bar{\tau}_k(g))^\delta}{\rho_k(g)} \right\},$$

where $\delta > 1$.

Theorem 30. Let $f(z)$, $g(z)$, $k(z)$ and $h(z)$ be any four entire functions such that $0 < \rho_h(f) < \infty$, $0 < \rho_k(g) < \infty$ and $\bar{\tau}_h(f) < \infty$. Then for any odd number n ,

$$\limsup_{r \rightarrow \infty} \frac{M_{h_n}^{-1}(M_{f_{n,g}}(r))}{\log M_h^{-1}\left(M_f\left(\exp(r)^{\delta \lambda_h(f)}\right)\right)} \leq \frac{(\bar{\tau}_h(f))^\delta}{\lambda_h(f)} \quad \text{if } \lambda_h(f) \neq 0$$

and

$$\limsup_{r \rightarrow \infty} \frac{M_{h_n}^{-1}(M_{f_{n,g}}(r))}{\log M_k^{-1}\left(M_g\left(\exp(r)^{\delta \lambda_h(f)}\right)\right)} \leq \frac{(\bar{\tau}_h(f))^\delta}{\lambda_k(g)} \quad \text{if } \lambda_k(g) \neq 0,$$

where $\delta > 1$.

Theorem 31. Let $f(z)$, $g(z)$, $k(z)$ and $h(z)$ be any four entire functions such that $0 < \lambda_h(f) \leq \rho_h(f) < \infty$, $0 < \lambda_k(g) \leq \rho_k(g) < \infty$ and $\bar{\tau}_h(f) < \infty$. Then for any odd number n ,

$$\liminf_{r \rightarrow \infty} \frac{M_{h_n}^{-1}(M_{f_{n,g}}(r))}{\log M_h^{-1}\left(M_f\left(\exp(r)^{\delta \lambda_h(f)}\right)\right)} \leq \min \left\{ \frac{(\tau_h(f))^\delta}{\lambda_h(f)}, \frac{(\bar{\tau}_h(f))^\delta}{\rho_h(f)} \right\}$$

and

$$\liminf_{r \rightarrow \infty} \frac{M_{h_n}^{-1}(M_{f_{n,g}}(r))}{\log M_k^{-1}\left(M_g\left(\exp(r)^{\delta \lambda_h(f)}\right)\right)} \leq \min \left\{ \frac{(\tau_h(f))^\delta}{\lambda_k(g)}, \frac{(\bar{\tau}_h(f))^\delta}{\rho_k(g)} \right\},$$

where $\delta > 1$.

Theorem 32. Let $f(z)$, $g(z)$, $k(z)$ and $h(z)$ be any four entire functions such that $0 < \lambda_h(f) < \infty$, $0 < \lambda_k(g) < \infty$ and $\bar{\sigma}_k(g) > 0$. Then for any even number n and $\delta < 1$

$$\liminf_{r \rightarrow \infty} \frac{M_{h_n}^{-1}(M_{f_{n,g}}(r))}{\log M_h^{-1}\left(M_f\left(\exp(r)^{\delta \rho_k(g)}\right)\right)} \geq A \frac{\bar{\sigma}_k(g)}{\rho_h(f)} \text{ if } \rho_h(f) < \infty$$

and

$$\liminf_{r \rightarrow \infty} \frac{M_{h_n}^{-1}(M_{f_{n,g}}(r))}{\log M_k^{-1}\left(M_g\left(\exp(r)^{\delta \rho_k(g)}\right)\right)} \geq A \frac{\bar{\sigma}_k(g)}{\rho_k(g)} \text{ if } \rho_k(g) < \infty,$$

where $A = \frac{1}{[18^{n\rho_k(g)} \cdot \gamma_n]^\delta}$ and γ_n is defined in Lemma 5.

Proof. From the first part of Lemma 5, we obtain for all sufficiently large values of r that

$$M_{h_n}^{-1}(M_{f_{n,g}}(r)) > \frac{1}{[18^{n\rho_k(g)} \cdot \gamma_n]^\delta} (\bar{\sigma}_k(g) - \varepsilon) r^{\delta \rho_k(g)}$$

i.e., $M_{h_n}^{-1}(M_{f_{n,g}}(r)) > A (\bar{\sigma}_k(g) - \varepsilon) r^{\delta \rho_k(g)}$. (2.45)

Also from the definition of $\rho_h(f)$, we obtain for all sufficiently large values of r that

$$\log M_h^{-1}\left(M_f\left(\exp(r)^{\delta \rho_k(g)}\right)\right) \leq (\rho_h(f) + \varepsilon) r^{\delta \rho_k(g)}. \quad (2.46)$$

Analogously, from the definition of $\rho_k(g)$, it follows for all sufficiently large values of r that

$$\log M_k^{-1}\left(M_g\left(\exp(r)^{\delta \rho_k(g)}\right)\right) \leq (\rho_k(g) + \varepsilon) r^{\delta \rho_k(g)}. \quad (2.47)$$

Now from (2.45) and (2.46), it follows for all sufficiently large values of r that

$$\frac{M_{h_n}^{-1}(M_{f_{n,g}}(r))}{\log M_h^{-1}\left(M_f\left(\exp(r)^{\delta \rho_k(g)}\right)\right)} > A \frac{(\bar{\sigma}_k(g) - \varepsilon) r^{\delta \rho_k(g)}}{(\rho_h(f) + \varepsilon) r^{\delta \rho_k(g)}}$$

i.e., $\liminf_{r \rightarrow \infty} \frac{M_{h_n}^{-1}(M_{f_{n,g}}(r))}{\log M_h^{-1}\left(M_f\left(\exp(r)^{\delta \rho_k(g)}\right)\right)} \geq A \frac{\bar{\sigma}_k(g)}{\rho_h(f)}$. (2.48)

Thus the first part of theorem follows from (2.48).

Like wise, the conclusion of the second part of theorem can easily be derived from (2.45) and (2.47).

Hence the theorem follows. □

Theorem 33. Let $f(z)$, $g(z)$, $k(z)$ and $h(z)$ be any four entire functions such that $0 < \lambda_h(f) \leq \rho_h(f) < \infty$, $0 < \lambda_k(g) \leq \rho_k(g) < \infty$ and $\bar{\sigma}_k(g) > 0$. Then for any even number n and $\delta < 1$

$$\limsup_{r \rightarrow \infty} \frac{M_{h_n}^{-1}(M_{f_{n,g}}(r))}{\log M_h^{-1}\left(M_f\left(\exp(r)^{\delta \rho_k(g)}\right)\right)} \geq A \cdot \max \left\{ \frac{\sigma_k(g)}{\rho_h(f)}, \frac{\bar{\sigma}_k(g)}{\lambda_h(f)} \right\}$$

and

$$\limsup_{r \rightarrow \infty} \frac{M_{h_n}^{-1}(M_{f_{n,g}}(r))}{\log M_k^{-1}\left(M_g\left(\exp(r)^{\delta \rho_k(g)}\right)\right)} \geq A \cdot \max \left\{ \frac{\sigma_k(g)}{\rho_k(g)}, \frac{\bar{\sigma}_k(g)}{\lambda_k(g)} \right\},$$

where $A = \frac{1}{[18^{n\rho_k(g) \cdot \gamma_n}]^\delta}$ and γ_n is defined in Lemma 5.

Proof of Theorem 33 is omitted as it can be carried out in the line of Theorem 32 and with help of the first part of Lemma 5.

Similarly, we state the following two theorems without their proofs as those can easily be carried out with the help of second part of Lemma 5 and in the line of Theorem 32 and Theorem 33 respectively.

Theorem 34. Let $f(z)$, $g(z)$, $k(z)$ and $h(z)$ be any four entire functions such that $0 < \lambda_h(f) < \infty$, $0 < \lambda_k(g) < \infty$ and $\bar{\sigma}_h(f) > 0$. Then for any odd number n and $\delta < 1$

$$\liminf_{r \rightarrow \infty} \frac{M_{h_n}^{-1}(M_{f_{n,g}}(r))}{\log M_h^{-1}\left(M_f\left(\exp(r)^{\delta \rho_h(f)}\right)\right)} \geq A \frac{\bar{\sigma}_h(f)}{\rho_h(f)} \text{ if } \rho_h(f) < \infty$$

and

$$\liminf_{r \rightarrow \infty} \frac{M_{h_n}^{-1}(M_{f_{n,g}}(r))}{\log M_k^{-1}\left(M_g\left(\exp(r)^{\delta \rho_h(f)}\right)\right)} \geq A \frac{\bar{\sigma}_h(f)}{\rho_k(g)} \text{ if } \rho_k(g) < \infty,$$

where $A = \frac{1}{[18^{n\rho_h(f) \cdot \gamma_n}]^\delta}$ and γ_n is defined in Lemma 5.

Theorem 35. Let $f(z)$, $g(z)$, $k(z)$ and $h(z)$ be any four entire functions such that $0 < \lambda_h(f) \leq \rho_h(f) < \infty$, $0 < \lambda_k(g) \leq \rho_k(g) < \infty$ and $\bar{\sigma}_h(f) > 0$. Then for any odd number n and $\delta < 1$

$$\limsup_{r \rightarrow \infty} \frac{M_{h_n}^{-1}(M_{f_{n,g}}(r))}{\log M_h^{-1}\left(M_f\left(\exp(r)^{\delta \rho_h(f)}\right)\right)} \geq A \cdot \max \left\{ \frac{\sigma_h(f)}{\rho_h(f)}, \frac{\bar{\sigma}_h(f)}{\lambda_h(f)} \right\}$$

and

$$\limsup_{r \rightarrow \infty} \frac{M_{h_n}^{-1}(M_{f_{n,g}}(r))}{\log M_k^{-1}\left(M_g\left(\exp(r)^{\delta \rho_h(f)}\right)\right)} \geq A \cdot \max \left\{ \frac{\sigma_h(f)}{\rho_k(g)}, \frac{\bar{\sigma}_h(f)}{\lambda_k(g)} \right\},$$

where $A = \frac{1}{[18^{n\rho_h(f) \cdot \gamma_n}]^\delta}$ and γ_n is defined in Lemma 5.

Similarly, one may state the following four theorems without their proofs on the basis of relative weak type of entire function with respect to another entire function:

Theorem 36. Let $f(z)$, $g(z)$, $k(z)$ and $h(z)$ be any four entire functions such that $0 < \lambda_h(f) < \infty$, $0 < \lambda_k(g) < \infty$ and $\tau_k(g) > 0$. Then for any even number n and $\delta < 1$

$$\liminf_{r \rightarrow \infty} \frac{M_{h_n}^{-1}(M_{f_{n,g}}(r))}{\log M_h^{-1}\left(M_f\left(\exp(r)^{\delta \lambda_k(g)}\right)\right)} \geq A \frac{\tau_k(g)}{\rho_h(f)} \text{ if } \rho_h(f) < \infty$$

and

$$\liminf_{r \rightarrow \infty} \frac{M_{h_n}^{-1}(M_{f_{n,g}}(r))}{\log M_k^{-1}\left(M_g\left(\exp(r)^{\delta \lambda_k(g)}\right)\right)} \geq A \frac{\tau_k(g)}{\rho_k(g)} \text{ if } \rho_k(g) < \infty,$$

where $A = \frac{1}{[18^{n \lambda_k(g) \cdot \gamma_n}]^\delta}$ and γ_n is defined in Lemma 5.

Theorem 37. Let $f(z)$, $g(z)$, $k(z)$ and $h(z)$ be any four entire functions such that $0 < \lambda_h(f) \leq \rho_h(f) < \infty$, $0 < \lambda_k(g) \leq \rho_k(g) < \infty$ and $\tau_k(g) > 0$. Then for any even number n and $\delta < 1$

$$\limsup_{r \rightarrow \infty} \frac{M_{h_n}^{-1}(M_{f_{n,g}}(r))}{\log M_h^{-1}\left(M_f\left(\exp(r)^{\delta \lambda_k(g)}\right)\right)} \geq A \cdot \max \left\{ \frac{\bar{\tau}_k(g)}{\rho_h(f)}, \frac{\tau_k(g)}{\lambda_h(f)} \right\}$$

and

$$\limsup_{r \rightarrow \infty} \frac{M_{h_n}^{-1}(M_{f_{n,g}}(r))}{\log M_k^{-1}\left(M_g\left(\exp(r)^{\delta \lambda_k(g)}\right)\right)} \geq A \cdot \max \left\{ \frac{\bar{\tau}_k(g)}{\rho_k(g)}, \frac{\tau_k(g)}{\lambda_k(g)} \right\},$$

where $A = \frac{1}{[18^{n \lambda_k(g) \cdot \gamma_n}]^\delta}$ and γ_n is defined in Lemma 5.

Theorem 38. Let $f(z)$, $g(z)$, $k(z)$ and $h(z)$ be any four entire functions such that $0 < \lambda_h(f) < \infty$, $0 < \lambda_k(g) < \infty$ and $\tau_h(f) > 0$. Then for any odd number n and $\delta < 1$

$$\liminf_{r \rightarrow \infty} \frac{M_{h_n}^{-1}(M_{f_{n,g}}(r))}{\log M_h^{-1}\left(M_f\left(\exp(r)^{\delta \lambda_h(f)}\right)\right)} \geq A \frac{\tau_h(f)}{\rho_h(f)} \text{ if } \rho_h(f) < \infty$$

and

$$\liminf_{r \rightarrow \infty} \frac{M_{h_n}^{-1}(M_{f_{n,g}}(r))}{\log M_k^{-1}\left(M_g\left(\exp(r)^{\delta \lambda_h(f)}\right)\right)} \geq A \frac{\tau_h(f)}{\rho_k(g)} \text{ if } \rho_k(g) < \infty,$$

where $A = \frac{1}{[18^{n \lambda_h(f) \cdot \gamma_n}]^\delta}$ and γ_n is defined in Lemma 5.

Theorem 39. Let $f(z)$, $g(z)$, $k(z)$ and $h(z)$ be any four entire functions such that $0 < \lambda_h(f) \leq \rho_h(f) < \infty$, $0 < \lambda_k(g) \leq \rho_k(g) < \infty$ and $\tau_h(f) > 0$. Then for any odd number n and $\delta < 1$

$$\limsup_{r \rightarrow \infty} \frac{M_{h_n}^{-1}(M_{f_{n,g}}(r))}{\log M_h^{-1}\left(M_f\left(\exp(r)^{\delta \lambda_h(f)}\right)\right)} \geq A \cdot \max \left\{ \frac{\bar{\tau}_h(f)}{\rho_h(f)}, \frac{\tau_h(f)}{\lambda_h(f)} \right\}$$

and

$$\limsup_{r \rightarrow \infty} \frac{M_{h_n}^{-1}(M_{f_{n,g}}(r))}{\log M_k^{-1}\left(M_g\left(\exp(r)^{\delta \lambda_h(f)}\right)\right)} \geq A \cdot \max \left\{ \frac{\bar{\tau}_h(f)}{\rho_k(g)}, \frac{\tau_h(f)}{\lambda_k(g)} \right\},$$

where $A = \frac{1}{[18^{n \lambda_h(f) \cdot \gamma_n}]^\delta}$ and γ_n is defined in Lemma 5.

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Solvability of a Quadratic Integral Equation of Fredholm Type Via a Modified Argument

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ABSTRACT: This article concerns with the existence of solutions of the a quadratic integral equation of Fredholm type with a modified argument,

$$x(t) = p(t) + (Fx)(t) \int_0^1 k(t, \tau)x(q(\tau))d\tau,$$

where p, k are functions and F is an operator satisfying the given conditions. Using the properties of the Hölder spaces and the classical Schauder fixed point theorem, we obtain the existence of solutions of the equation under certain assumptions. Also, we present two concrete examples in which our result can be applied.

AMS Subject Classification: 45G10, 45M99, 47H10.

Keywords and Phrases: Fredholm equation; Hölder condition; Schauder fixed point theorem.

1. Introduction

Integral equations arise from naturally in many applications in describing numerous real world problems (see, for instance, the books [2, 3] and references therein). Quadratic integral equations arise naturally in applications of real world problems. For example, problems in the theory of radiative transfer in the theory of neutron transport and in the kinetic theory of gases lead to the quadratic equation [12, 20]. There are many interesting existence results for all kinds of quadratic integral equations, one can refer to [6, 1].

The study of differential equations with a modified arguments arise in a wide variety of scientific and technical application, including the modelling of problems from the natural and social sciences such as physics, biological and economics sciences. A special class of these differential equations have linear modifications of their arguments, and have been studied by several authors, [7] - [23].

Recently, Banaś and Nalepa [7] have studied the space of real functions defined on a given bounded metric space and having the growths tempered by a given modulus of continuity, and derive the existence theorem in the space of functions satisfying the Hölder condition for some quadratic integral equations of Fredholm type

$$x(t) = p(t) + x(t) \int_a^b k(t, \tau) x(\tau) d\tau. \quad (1.1)$$

Further, Caballero et al. [9] have studied the solvability of the following quadratic integral equation of Fredholm type

$$x(t) = p(t) + x(t) \int_0^1 k(t, \tau) x(q(\tau)) d\tau \quad (1.2)$$

in Hölder spaces. The purpose of this paper is to investigate the existence of solutions of the following integral equation of Fredholm type with a modified argument in Hölder spaces

$$x(t) = p(t) + (Fx)(t) \int_0^1 k(t, \tau) x(q(\tau)) d\tau, \quad t \in I = [0, 1] \quad (1.3)$$

where p , k , q and F are functions satisfying the given conditions. To do this, we will use a recent result about the relative compactness in Hölder spaces and the classical Schauder fixed point theorem.

Notice that equation (1.1) in [9] is a particular case of (1.3), for $(Fx)(\tau) = x(\tau)$. The obtained result in this paper is more general than the result in [9].

2. Preliminaries

Let we introduce notations, definitions and theorems which are used throughout this paper.

By $C[a, b]$, we denote the space of continuous functions on $[a, b]$ equipped with usually the supremum norm

$$\|x\|_\infty = \sup\{|x(t)| : t \in [a, b]\}$$

for $x \in C[a, b]$. For a fixed α with $0 < \alpha \leq 1$, we write $H_\alpha[a, b]$ to denote the set of all the real valued functions x defined on $[a, b]$ and satisfying the Hölder condition with α , that is, there exists a constant H such that the inequality

$$|x(t) - x(s)| \leq H|t - s|^\alpha \quad (2.1)$$

holds for all $t, s \in [a, b]$. One can easily see that $H_\alpha[a, b]$ is a linear subspace of $C[a, b]$. In the sequel, for $x \in H_\alpha[a, b]$, by H_x^α we will denote the least possible constant for which inequality (2.1) is satisfied. Rather, we put

$$H_x^\alpha = \sup \left\{ \frac{|x(t) - x(s)|}{|t - s|^\alpha} : t, s \in [a, b], t \neq s \right\}. \quad (2.2)$$

The space $H_\alpha[a, b]$ with $0 < \alpha \leq 1$ can be equipped with the norm:

$$\|x\|_\alpha = |x(a)| + \sup \left\{ \frac{|x(t) - x(s)|}{|t - s|^\alpha} : t, s \in [a, b], t \neq s \right\} \quad (2.3)$$

for $x \in H_\alpha[a, b]$. In [7], the authors proved that $(H_\alpha[a, b], \|\cdot\|_\alpha)$ with $0 < \alpha \leq 1$ is a Banach space. The following lemmas in [7] present some results related to the Hölder spaces and norm.

Lemma 2.1. For $0 < \alpha \leq 1$ and $x \in H_\alpha[a, b]$, the following inequality is satisfied

$$\|x\|_\infty \leq \max \{1, (b - a)^\alpha\} \|x\|_\alpha.$$

In particular, the inequality $\|x\|_\infty \leq \|x\|_\alpha$ holds, for $a = 0$ and $b = 1$.

Lemma 2.2. For $0 < \alpha < \gamma \leq 1$, we have

$$H_\gamma[a, b] \subset H_\alpha[a, b] \subset C[a, b].$$

Moreover, for $x \in H_\gamma[a, b]$ the following inequality holds

$$\|x\|_\alpha \leq \max \{1, (b - a)^{\gamma - \alpha}\} \|x\|_\gamma.$$

In particular, the inequality $\|x\|_\infty \leq \|x\|_\alpha \leq \|x\|_\gamma$ is satisfied for $a = 0$ and $b = 1$.

Now we present the important theorem which is the sufficient condition for relative compactness in the spaces $H_\alpha[a, b]$ with $0 < \alpha \leq 1$.

Theorem 2.3. [9] Suppose that $0 < \alpha < \beta \leq 1$ and that A is a bounded subset of $H_\beta[a, b]$ (this means that $\|x\|_\beta \leq M$ for certain constant $M > 0$, for all $x \in A$) then A is a relatively compact subset of $H_\alpha[a, b]$.

Lemma 2.4. [9] Suppose that $0 < \alpha < \beta \leq 1$ and by B_r^β we denote the closed ball centered at θ with radius r in the space $H_\beta[a, b]$, i.e., $B_r^\beta = \{x \in H_\beta[a, b] : \|x\|_\beta \leq r\}$. Then B_r^β is a closed subset of $H_\alpha[a, b]$.

Corollary 2.5. Suppose that $0 < \alpha < \beta \leq 1$ then B_r^β is a compact subset of the space $H_\alpha[a, b]$, [9].

Theorem 2.6 (Schauder's fixed point theorem). Let L be a nonempty, convex, and compact subset of a Banach space $(X, \|\cdot\|)$ and let $T : L \rightarrow L$ be a continuity mapping. Then T has at least one fixed point in L , [24].

3. Main Result

In this section, we will study the solvability of the equation (1.3) in the space $H_\alpha[0, 1]$ ($0 < \alpha \leq 1$). We will use the following assumptions:

- (i) $p \in H_\beta[0, 1]$, $0 < \beta \leq 1$.
- (ii) $k : [0, 1] \times [0, 1] \rightarrow \mathbb{R}$ is a continuous function such that it satisfies the Hölder condition with exponent β with respect to the first variable, that is, there exists a constant $k_\beta > 0$ such that:

$$|k(t, \tau) - k(s, \tau)| \leq k_\beta |t - s|^\beta,$$

for any $t, s, \tau \in [0, 1]$.

- (iii) $q : [0, 1] \rightarrow [0, 1]$ is a measurable function.
- (iv) The operator $F : H_\beta[0, 1] \rightarrow H_\beta[0, 1]$ is continuous with respect to the norm $\|\cdot\|_\alpha$ for $0 < \alpha < \beta \leq 1$ and there exists a function $f : \mathbb{R}^+ \rightarrow \mathbb{R}^+ = [0, \infty)$ which is non-decreasing such that it holds the inequality

$$\|Fx\|_\beta \leq f(\|x\|_\beta),$$

for any $x \in H_\beta[0, 1]$.

- (v) There exists a positive solution r_0 of the inequality

$$\|p\|_\beta + (2K + k_\beta)rf(r) \leq r,$$

where K is a constant is satisfying the following inequality,

$$K = \sup \left\{ \int_0^1 |k(t, \tau)| d\tau : t \in [0, 1] \right\}.$$

Theorem 3.1. *Under the assumptions (i)-(v), Equation (1.3) has at least one solution belonging to the space $H_\alpha[0, 1]$.*

Proof. Consider the operator T below that defined on the space $H_\beta[0, 1]$ by

$$(Tx)(t) = p(t) + (Fx)(t) \int_0^1 k(t, \tau)x(q(\tau))d\tau, \quad t \in [0, 1].$$

We will firstly prove that T transforms the space $H_\beta[0, 1]$ into itself. For arbitrarily fixed $x \in H_\beta[0, 1]$ and $t, s \in [0, 1]$ with ($t \neq s$), taking into account assumptions

(i), (ii) and (iii), we obtain

$$\begin{aligned}
& \frac{|(Tx)(t) - (Tx)(s)|}{|t - s|^\beta} \\
&= \frac{\left| p(t) + (Fx)(t) \int_0^1 k(t, \tau)x(q(\tau)) d\tau - p(s) - (Fx)(s) \int_0^1 k(s, \tau)x(q(\tau)) d\tau \right|}{|t - s|^\beta} \\
&\leq \frac{1}{|t - s|^\beta} \left[|p(t) - p(s)| + \left| (Fx)(t) \int_0^1 k(t, \tau)x(q(\tau)) d\tau \right. \right. \\
&\quad \left. \left. - (Fx)(s) \int_0^1 k(s, \tau)x(q(\tau)) d\tau \right| \right] \\
&\leq \frac{|p(t) - p(s)|}{|t - s|^\beta} \\
&\quad + \frac{1}{|t - s|^\beta} \left| (Fx)(t) \int_0^1 k(t, \tau)x(q(\tau)) d\tau - (Fx)(s) \int_0^1 k(t, \tau)x(q(\tau)) d\tau \right| \\
&\quad + \frac{1}{|t - s|^\beta} \left| (Fx)(s) \int_0^1 k(t, \tau)x(q(\tau)) d\tau - (Fx)(s) \int_0^1 k(s, \tau)x(q(\tau)) d\tau \right| \\
&\leq \frac{|p(t) - p(s)|}{|t - s|^\beta} + \frac{|(Fx)(t) - (Fx)(s)|}{|t - s|^\beta} \int_0^1 |k(t, \tau)| |x(q(\tau))| d\tau \\
&\quad + \frac{|(Fx)(s)| \int_0^1 |k(t, \tau) - k(s, \tau)| |x(q(\tau))| d\tau}{|t - s|^\beta} \\
&\leq \frac{|p(t) - p(s)|}{|t - s|^\beta} + \frac{|(Fx)(t) - (Fx)(s)|}{|t - s|^\beta} \|x\|_\infty \int_0^1 |k(t, \tau)| d\tau \\
&\quad + \frac{\|Fx\|_\infty \|x\|_\infty \int_0^1 |k(t, \tau) - k(s, \tau)| d\tau}{|t - s|^\beta} \\
&\leq \frac{|p(t) - p(s)|}{|t - s|^\beta} + \frac{|(Fx)(t) - (Fx)(s)|}{|t - s|^\beta} \|x\|_\infty K + \frac{\|Fx\|_\infty \|x\|_\infty \int_0^1 k_\beta |t - s|^\beta d\tau}{|t - s|^\beta} \\
&\leq H_p^\beta + H_{Fx}^\beta \|x\|_\infty K + \|Fx\|_\infty \|x\|_\infty k_\beta.
\end{aligned}$$

By using the facts that $\|x\|_\infty \leq \|x\|_\beta$ and $H_x^\beta \leq \|x\|_\beta$ concluded Lemma 2.1 and the definition $\|x\|_\beta$, respectively we infer that

$$\frac{|(Tx)(t) - (Tx)(s)|}{|t - s|^\beta} \leq H_p^\beta + (K + k_\beta) \|x\|_\beta \|Fx\|_\beta.$$

From this inequality, we have $Tx \in H_\beta[0, 1]$. This proves that the operator T

maps the space $H_\beta[0, 1]$ into itself. On the other hand we can write

$$\begin{aligned}
\|Tx\|_\beta &= |(Tx)(0)| + \sup \left\{ \frac{|(Tx)(t) - (Tx)(s)|}{|t - s|^\beta} : t, s \in [0, 1], t \neq s \right\} \\
&\leq |(Tx)(0)| + H_p^\beta + (K + k_\beta)\|x\|_\beta \|Fx\|_\beta \\
&\leq |p(0)| + |(Fx)(0)| \int_0^1 |k(0, \tau)| |x(q(\tau))| d\tau + H_p^\beta + (K + k_\beta)\|x\|_\beta \|Fx\|_\beta \\
&\leq \|p\|_\beta + \|Fx\|_\infty \|x\|_\infty \int_0^1 |k(0, \tau)| d\tau + (K + k_\beta)\|x\|_\beta \|Fx\|_\beta \\
&\leq \|p\|_\beta + K \|Fx\|_\beta \|x\|_\beta + (K + k_\beta)\|x\|_\beta \|Fx\|_\beta \\
&= \|p\|_\beta + (2K + k_\beta)\|x\|_\beta \|Fx\|_\beta \\
&\leq \|p\|_\beta + (2K + k_\beta)\|x\|_\beta f(\|x\|_\beta), \tag{3.1}
\end{aligned}$$

for any $x \in H_\beta[0, 1]$. So, if we take x in $B_{r_0}^\beta$ then by assumption (v) we get $Tx \in B_{r_0}^\beta$. As a result, it follows that T transforms the ball

$$B_{r_0}^\beta = \{x \in H_\beta[0, 1] : \|x\|_\beta \leq r_0\}$$

into itself. That is,

$$T : B_{r_0}^\beta \rightarrow B_{r_0}^\beta.$$

Next, we will prove that the operator T is continuous on $B_{r_0}^\beta$, according to the induced norm by $\|\cdot\|_\alpha$, where $0 < \alpha < \beta \leq 1$. To do this, let us take any fixed $y \in B_{r_0}^\beta$ and arbitrary $\varepsilon > 0$. Since the operator $F : H_\beta[0, 1] \rightarrow H_\beta[0, 1]$ is continuous on $H_\beta[0, 1]$ with respect to the norm $\|\cdot\|_\alpha$, there exists $\delta > 0$ such that the inequality

$$\|Fx - Fy\|_\alpha < \frac{\varepsilon}{4(K + k_\beta)r_0}$$

is satisfied for all $x \in B_{r_0}^\beta$, such that $\|x - y\|_\alpha \leq \delta$ and

$$0 < \delta < \frac{\varepsilon}{2(2K + k_\beta)f(r_0)}.$$

Then, for any $x \in B_{r_0}^\beta$ and $t, s \in [0, 1]$ with $t \neq s$ and $0 < \alpha \leq 1$ we have

$$\begin{aligned}
& \left| \frac{[(Tx)(t) - (Ty)(t)] - [(Tx)(s) - (Ty)(s)]}{|t - s|^\alpha} \right. \\
&= \left| \frac{\left[(Fx)(t) \int_0^1 k(t, \tau) x(q(\tau)) d\tau - (Fy)(t) \int_0^1 k(t, \tau) y(q(\tau)) d\tau \right]}{|t - s|^\alpha} \right. \\
&\quad \left. - \frac{\left[(Fx)(s) \int_0^1 k(s, \tau) x(q(\tau)) d\tau - (Fy)(s) \int_0^1 k(s, \tau) y(q(\tau)) d\tau \right]}{|t - s|^\alpha} \right| \\
&= \frac{1}{|t - s|^\alpha} \left| \left[(Fx)(t) \int_0^1 k(t, \tau) x(q(\tau)) d\tau - (Fy)(t) \int_0^1 k(t, \tau) x(q(\tau)) d\tau \right] \right. \\
&\quad + \left[(Fy)(t) \int_0^1 k(t, \tau) x(q(\tau)) d\tau - (Fy)(t) \int_0^1 k(t, \tau) y(q(\tau)) d\tau \right] \\
&\quad - \left[(Fx)(s) \int_0^1 k(s, \tau) x(q(\tau)) d\tau - (Fy)(s) \int_0^1 k(s, \tau) x(q(\tau)) d\tau \right] \\
&\quad \left. - \left[(Fy)(s) \int_0^1 k(s, \tau) x(q(\tau)) d\tau - (Fy)(s) \int_0^1 k(s, \tau) y(q(\tau)) d\tau \right] \right| \\
&= \frac{1}{|t - s|^\alpha} \left| \left[[(Fx)(t) - (Fy)(t)] \int_0^1 k(t, \tau) x(q(\tau)) d\tau \right] \right. \\
&\quad + \left[(Fy)(t) \int_0^1 k(t, \tau) [x(q(\tau)) - y(q(\tau))] d\tau \right] \\
&\quad - \left[[(Fx)(s) - (Fy)(s)] \int_0^1 k(s, \tau) x(q(\tau)) d\tau \right] \\
&\quad \left. - \left[(Fy)(s) \int_0^1 k(s, \tau) [x(q(\tau)) - y(q(\tau))] d\tau \right] \right| \\
&= \frac{1}{|t - s|^\alpha} \left| \{ [(Fx)(t) - (Fy)(t)] - [(Fx)(s) - (Fy)(s)] \} \int_0^1 k(t, \tau) x(q(\tau)) d\tau \right. \\
&\quad + \left[[(Fx)(s) - (Fy)(s)] \int_0^1 k(t, \tau) x(q(\tau)) d\tau \right] \\
&\quad - \left[[(Fx)(s) - (Fy)(s)] \int_0^1 k(s, \tau) x(q(\tau)) d\tau \right] \\
&\quad + \left[(Fy)(t) \int_0^1 k(t, \tau) [x(q(\tau)) - y(q(\tau))] d\tau \right] \\
&\quad \left. - \left[(Fy)(s) \int_0^1 k(s, \tau) [x(q(\tau)) - y(q(\tau))] d\tau \right] \right|
\end{aligned}$$

$$\begin{aligned}
&= \frac{1}{|t-s|^\alpha} \left| \{[(Fx)(t) - (Fy)(t)] - [(Fx)(s) - (Fy)(s)]\} \int_0^1 k(t, \tau) x(q(\tau)) d\tau \right. \\
&+ \left. \left[[(Fx)(s) - (Fy)(s)] \int_0^1 (k(t, \tau) - k(s, \tau)) x(q(\tau)) d\tau \right] \right. \\
&+ \left. \left[(Fy)(t) \int_0^1 k(t, \tau) [x(q(\tau)) - y(q(\tau))] d\tau \right] \right. \\
&\left. - \left[(Fy)(s) \int_0^1 k(s, \tau) [x(q(\tau)) - y(q(\tau))] d\tau \right] \right|.
\end{aligned}$$

From the last inequality it follows that

$$\begin{aligned}
&\frac{|[(Tx)(t) - (Ty)(t)] - [(Tx)(s) - (Ty)(s)]|}{|t-s|^\alpha} \\
&\leq \frac{1}{|t-s|^\alpha} |[(Fx)(t) - (Fy)(t)] - [(Fx)(s) - (Fy)(s)]| \left| \int_0^1 k(t, \tau) x(q(\tau)) d\tau \right| \\
&+ \frac{1}{|t-s|^\alpha} |(Fx)(s) - (Fy)(s)| \left| \int_0^1 (k(t, \tau) - k(s, \tau)) x(q(\tau)) d\tau \right| \\
&+ \frac{1}{|t-s|^\alpha} \left| (Fy)(t) \int_0^1 k(t, \tau) [x(q(\tau)) - y(q(\tau))] d\tau \right. \\
&\left. - (Fy)(s) \int_0^1 k(s, \tau) [x(q(\tau)) - y(q(\tau))] d\tau \right| \\
&\leq \frac{|[(Fx)(t) - (Fy)(t)] - [(Fx)(s) - (Fy)(s)]|}{|t-s|^\alpha} \|x\|_\infty \int_0^1 |k(t, \tau)| d\tau \\
&+ |[(Fx)(s) - (Fy)(s)] - [(Fx)(0) - (Fy)(0)]| \|x\|_\infty \int_0^1 \frac{|k(t, \tau) - k(s, \tau)|}{|t-s|^\alpha} d\tau \\
&+ |(Fx)(0) - (Fy)(0)| \|x\|_\infty \int_0^1 \frac{|k(t, \tau) - k(s, \tau)|}{|t-s|^\alpha} d\tau \\
&+ \frac{1}{|t-s|^\alpha} \left| (Fy)(t) \int_0^1 k(t, \tau) [x(q(\tau)) - y(q(\tau))] d\tau \right. \\
&\left. - (Fy)(s) \int_0^1 k(s, \tau) [x(q(\tau)) - y(q(\tau))] d\tau \right| \\
&+ \frac{1}{|t-s|^\alpha} \left| (Fy)(s) \int_0^1 k(t, \tau) [x(q(\tau)) - y(q(\tau))] d\tau \right. \\
&\left. - (Fy)(s) \int_0^1 k(s, \tau) [x(q(\tau)) - y(q(\tau))] d\tau \right|
\end{aligned}$$

$$\begin{aligned}
&\leq H_{Fx-Fy}^\alpha \|x\|_\infty K \\
&+ \sup_{u,v \in [0,1]} |[(Fx)(u) - (Fy)(u)] - [(Fx)(v) - (Fy)(v)]| \|x\|_\infty \int_0^1 \frac{|k(t,\tau) - k(s,\tau)|}{|t-s|^\alpha} d\tau \\
&+ |(Fx)(0) - (Fy)(0)| \|x\|_\infty \int_0^1 \frac{|k(t,\tau) - k(s,\tau)|}{|t-s|^\alpha} d\tau \\
&+ \frac{|(Fy)(t) - (Fy)(s)|}{|t-s|^\alpha} \int_0^1 |k(t,\tau)| |x(q(\tau)) - y(q(\tau))| d\tau \\
&+ |(Fy)(s)| \int_0^1 \frac{|k(t,\tau) - k(s,\tau)|}{|t-s|^\alpha} |x(q(\tau)) - y(q(\tau))| d\tau \\
&\leq K \|x\|_\infty \|Fx - Fy\|_\alpha \\
&+ \sup_{u,v \in [0,1]} |[(Fx)(u) - (Fy)(u)] - [(Fx)(v) - (Fy)(v)]| \|x\|_\infty \int_0^1 \frac{k_\beta |t-s|^\beta}{|t-s|^\alpha} d\tau \\
&+ |(Fx)(0) - (Fy)(0)| \|x\|_\infty \int_0^1 \frac{k_\beta |t-s|^\beta}{|t-s|^\alpha} d\tau \\
&+ \frac{|(Fy)(t) - (Fy)(s)|}{|t-s|^\alpha} \int_0^1 |k(t,\tau)| |x(q(\tau)) - y(q(\tau))| d\tau \\
&+ |(Fy)(s)| \int_0^1 \frac{k_\beta |t-s|^\beta}{|t-s|^\alpha} |x(q(\tau)) - y(q(\tau))| d\tau.
\end{aligned}$$

In view of the inequalities $\|x\|_\infty \leq \|x\|_\alpha$, $H_x^\beta \leq \|x\|_\alpha$, we derive the following inequalities

$$\begin{aligned}
&\frac{|[(Tx)(t) - (Ty)(t)] - [(Tx)(s) - (Ty)(s)]|}{|t-s|^\alpha} \\
&\leq K \|x\|_\infty \|Fx - Fy\|_\alpha + k_\beta \|x\|_\infty |t-s|^{\beta-\alpha} \\
&\sup_{u,v \in [0,1], u \neq v} \left\{ \frac{|[(Fx)(u) - (Fy)(u)] - [(Fx)(v) - (Fy)(v)]|}{|u-v|^\alpha} |u-v|^\alpha \right\} \\
&+ k_\beta \|x\|_\infty |t-s|^{\beta-\alpha} |(Fx)(0) - (Fy)(0)| + KH_{Fy}^\alpha \|x - y\|_\infty \\
&+ k_\beta \|Fy\|_\infty \|x - y\|_\infty |t-s|^{\beta-\alpha} \\
&\leq K \|x\|_\beta \|Fx - Fy\|_\alpha + 2k_\beta \|x\|_\beta \|Fx - Fy\|_\alpha \\
&+ K \|Fy\|_\alpha \|x - y\|_\alpha + k_\beta \|Fy\|_\alpha \|x - y\|_\alpha
\end{aligned}$$

$$\begin{aligned}
&= (K + 2k_\beta) \|x\|_\beta \|Fx - Fy\|_\alpha \\
&+ (K + k_\beta) \|Fy\|_\alpha \|x - y\|_\alpha. \tag{3.2}
\end{aligned}$$

Since $\|y\|_\alpha \leq \|y\|_\beta \leq r_0$ (see Lemma 2.2) and from the assumption (iv) and (3.2) we deduce the following inequality

$$\begin{aligned}
&\frac{|[(Tx)(t) - (Ty)(t)] - [(Tx)(s) - (Ty)(s)]|}{|t - s|^\alpha} \\
&\leq (K + 2k_\beta) \|x\|_\beta \|Fx - Fy\|_\alpha + (K + k_\beta) \|Fy\|_\beta \|x - y\|_\alpha \\
&\leq (K + 2k_\beta) \|x\|_\beta \|Fx - Fy\|_\alpha + (K + k_\beta) f\left(\|y\|_\beta\right) \|x - y\|_\alpha \\
&\leq (K + 2k_\beta) r_0 \|Fx - Fy\|_\alpha + (K + k_\beta) f(r_0) \delta. \tag{3.3}
\end{aligned}$$

On the other hand,

$$\begin{aligned}
|(Tx)(0) - (Ty)(0)| &= \left| (Fx)(0) \int_0^1 k(0, \tau) x(q(\tau)) d\tau - (Fy)(0) \int_0^1 k(0, \tau) y(q(\tau)) d\tau \right| \\
&\leq \left| (Fx)(0) \int_0^1 k(0, \tau) x(q(\tau)) d\tau - (Fx)(0) \int_0^1 k(0, \tau) y(q(\tau)) d\tau \right| \\
&+ \left| (Fx)(0) \int_0^1 k(0, \tau) y(q(\tau)) d\tau - (Fy)(0) \int_0^1 k(0, \tau) y(q(\tau)) d\tau \right| \\
&\leq |(Fx)(0)| \int_0^1 |k(0, \tau)| |x(q(\tau)) - y(q(\tau))| d\tau \\
&+ |(Fx)(0) - (Fy)(0)| \int_0^1 |k(0, \tau)| |y(q(\tau))| d\tau
\end{aligned}$$

From the last inequality it follows that

$$\begin{aligned}
|(Tx)(0) - (Ty)(0)| &\leq K \|Fx\|_\infty \|x - y\|_\infty + K \|y\|_\infty \|Fx - Fy\|_\infty \\
&\leq K \|Fx\|_\beta \|x - y\|_\alpha + K \|y\|_\beta \|Fx - Fy\|_\alpha \\
&\leq K f\left(\|x\|_\beta\right) \|x - y\|_\alpha + K \|y\|_\beta \|Fx - Fy\|_\alpha \\
&\leq K f(r_0) \delta + K r_0 \|Fx - Fy\|_\alpha. \tag{3.4}
\end{aligned}$$

From (3.3) and (3.4), it follows that

$$\begin{aligned}
&\|Tx - Ty\|_\alpha \\
&= |(Tx)(0) - (Ty)(0)| + H_{Tx - Ty}^\alpha \\
&= |(Tx)(0) - (Ty)(0)| + \sup_{t, s \in [0, 1], t \neq s} \left\{ \frac{|[(Tx)(t) - (Ty)(t)] - [(Tx)(s) - (Ty)(s)]|}{|t - s|^\alpha} \right\}
\end{aligned}$$

$$\begin{aligned}
&\leq Kf(r_0)\delta + Kr_0\|Fx - Fy\|_\alpha + (K + 2k_\beta)r_0\|Fx - Fy\|_\alpha + (K + k_\beta)f(r_0)\delta \\
&= 2(K + k_\beta)r_0\|Fx - Fy\|_\alpha + (2K + k_\beta)f(r_0)\delta \\
&< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.
\end{aligned}$$

This show that the operator T is continuous at the point $y \in B_{r_0}^\beta$. We conclude that T is continuous on $B_{r_0}^\beta$ with respect to the norm $\|\cdot\|_\alpha$. In addition the set $B_{r_0}^\beta$ is compact subset of the space $H_\alpha[0, 1]$ from [9] (see [9; the appendix at the p. 9]). Therefore, applying the classical Schauder fixed point theorem, we complete the proof. \square

4. Examples

In this section, we provide an example illustrating the main results in the above.

Example 1. Let us consider the quadratic integral equation:

$$x(t) = \ln\left(\sqrt[4]{n \sin t + \hat{n}} + 1\right) + x^2(t) \int_0^1 \sqrt[3]{mt^3 + \tau x} \left(\frac{1}{\tau + 1}\right) d\tau \quad (4.1)$$

where $t \in [0, 1]$ and n, \hat{n}, m are the suitable non-negative constants.

Observe that (4.1) is a particular case of (1.3) if we put $p(t) = \ln\left(\sqrt[4]{n \sin t + \hat{n}} + 1\right)$, $k(t, \tau) = \sqrt[3]{mt^3 + \tau}$ and $q(\tau) = \frac{1}{\tau + 1}$. The operator F defined by $(Fx)(t) = x^2(t)$ for all $t \in [0, 1]$.

Since functions $s, h : \mathbb{R}^+ \rightarrow \mathbb{R}^+$ defined by $s(t) = \ln(t + 1)$, $h(t) = \sqrt[4]{t}$ are concav and $s(0) = 0$, $h(0) = 0$, then from Lemma 4.4 in [9] these functions are subadditive. If we consider the result of subadditivity and the inequalities $\ln x < x$ for $x > 0$ and $|\sin x - \sin y| \leq |x - y|$ for $x, y \in \mathbb{R}$, we can write

$$\begin{aligned}
|p(t) - p(s)| &= \left| \ln\left(\sqrt[4]{n \sin t + \hat{n}} + 1\right) - \ln\left(\sqrt[4]{n \sin s + \hat{n}} + 1\right) \right| \\
&\leq \ln \left| \sqrt[4]{n \sin t + \hat{n}} - \sqrt[4]{n \sin s + \hat{n}} \right| \\
&< \left| \sqrt[4]{n \sin t + \hat{n}} - \sqrt[4]{n \sin s + \hat{n}} \right| \\
&\leq \left| \sqrt[4]{n |\sin t - \sin s|} \right| \\
&\leq \sqrt[4]{n} |t - s|^{\frac{1}{4}}.
\end{aligned}$$

It means that $p \in H_{\frac{1}{4}}[0, 1]$ and, moreover, $H_p^{\frac{1}{4}} = \sqrt[4]{n}$. We can take the constants α and β as $0 < \alpha < \frac{1}{4}$ and $\beta = \frac{1}{4}$. Therefore, assumption (i) of Theorem (3.1) is

satisfied. Note that

$$\begin{aligned} \|p\|_{\frac{1}{4}} &= |p(0)| + \sup \left\{ \frac{|p(t) - p(s)|}{|t - s|^{\frac{1}{4}}} : t, s \in [0, 1], t \neq s \right\} \\ &= |p(0)| + H_p^{\frac{1}{4}} = \ln \left(\sqrt[4]{\hat{n}} + 1 \right) + \sqrt[4]{\hat{n}}. \end{aligned}$$

Further, we have

$$\begin{aligned} |k(t, \tau) - k(s, \tau)| &= \left| \sqrt[3]{mt^3 + \tau} - \sqrt[3]{ms^3 + \tau} \right| \\ &\leq \sqrt[3]{|mt^3 - ms^3|} \\ &= \sqrt[3]{m} \sqrt[3]{|t^3 - s^3|} \\ &= \sqrt[3]{m} \sqrt[3]{|t - s|} \sqrt[3]{|t^2 + ts + s^2|} \\ &\leq \sqrt[3]{3m} |t - s|^{\frac{1}{3}} \\ &= \sqrt[3]{3m} |t - s|^{\frac{1}{4}} |t - s|^{\frac{1}{12}} \\ &\leq \sqrt[3]{3m} |t - s|^{\frac{1}{4}} \end{aligned}$$

for all $t, s \in [0, 1]$. Assumption (ii) of Theorem (3.1) is satisfied with $k_\beta = k_{\frac{1}{4}} = \sqrt[3]{3m}$. It is clear that $q(\tau) = \frac{1}{\tau+1}$ satisfies assumption (iii). The constant K is given by

$$\begin{aligned} K &= \sup \left\{ \int_0^1 |k(t, \tau)| d\tau : t \in [0, 1] \right\} \\ &= \sup \left\{ \int_0^1 \left| \sqrt[3]{mt^3 + \tau} \right| d\tau : t \in [0, 1] \right\} \\ &= \int_0^1 \sqrt[3]{m + \tau} d\tau \\ &= \frac{3}{4} \left(\sqrt[3]{(m+1)^4} - \sqrt[3]{m^4} \right). \end{aligned}$$

For all $x \in H_\beta [0, 1]$,

$$\begin{aligned} \|Fx\|_\beta &= |(Fx)(0)| + \sup \left\{ \frac{|(Fx)(t) - (Fx)(s)|}{|t - s|^\beta} : t, s \in [0, 1], t \neq s \right\} \\ &= |x^2(0)| + \sup \left\{ \frac{|x^2(t) - x^2(s)|}{|t - s|^\beta} : t, s \in [0, 1], t \neq s \right\} \end{aligned}$$

$$\begin{aligned}
&= |x^2(0)| + \sup \left\{ \frac{|x(t) - x(s)| |x(t) + x(s)|}{|t - s|^\beta} : t, s \in [0, 1], t \neq s \right\} \\
&\leq |x^2(0)| + 2 \|x\|_\infty \sup \left\{ \frac{|x(t) - x(s)|}{|t - s|^\beta} : t, s \in [0, 1], t \neq s \right\} \\
&\leq |x^2(0)| + 2 \|x\|_\beta \sup \left\{ \frac{|x(t) - x(s)|}{|t - s|^\beta} : t, s \in [0, 1], t \neq s \right\} \\
&\leq \|x\|_\beta^2 + 2 \|x\|_\beta^2 = 3 \|x\|_\beta^2.
\end{aligned}$$

Therefore, F is an operator from $H_\beta [0, 1]$ into $H_\beta [0, 1]$ and we can choose the function $f : \mathbb{R}^+ \rightarrow \mathbb{R}^+$ as $f(x) = 3x^2$. This function is non-decreasing and satisfies the inequality in assumption (iv).

Now, we will show that the operator F is continuous on the $H_\beta [0, 1]$ with respect to the norm $\|\cdot\|_\alpha$. To this end, fix arbitrarily $y \in H_\beta [0, 1]$ and $\varepsilon > 0$. Assume that $x \in H_\beta [0, 1]$ is an arbitrary function and $\|x - y\|_\alpha < \delta$, where δ is a positive number such that $0 < \delta < \sqrt{\|y\|_\alpha^2 + \frac{\varepsilon}{3}} - \|y\|_\alpha$.

Then, for arbitrary $t, s \in [0, 1]$ we obtain

$$\begin{aligned}
&(Fx - Fy)(t) - (Fx - Fy)(s) \\
&= x^2(t) - y^2(t) - (x^2(s) - y^2(s)) \\
&= (x(t) - y(t))(x(t) + y(t)) - (x(s) - y(s))(x(s) + y(s)) \\
&= [x(t) - y(t) - (x(s) - y(s))](x(t) + y(t)) + (x(s) - y(s))(x(t) + y(t)) \\
&\quad - (x(s) - y(s))(x(s) + y(s)) \\
&= [x(t) - y(t) - (x(s) - y(s))](x(t) + y(t)) \\
&\quad + (x(s) - y(s))[x(t) + y(t) - (x(s) + y(s))].
\end{aligned} \tag{4.2}$$

By (4.2), we have

$$\begin{aligned}
&|(Fx - Fy)(t) - (Fx - Fy)(s)| \\
&\leq |x(t) - y(t) - (x(s) - y(s))| |x(t) + y(t)| + |x(s) - y(s)| |x(t) + y(t) - (x(s) + y(s))| \\
&\leq \|x + y\|_\infty |x(t) - y(t) - (x(s) - y(s))| + \|x - y\|_\infty |x(t) + y(t) - (x(s) + y(s))| \\
&\leq \|x + y\|_\alpha |x(t) - y(t) - (x(s) - y(s))| + \|x - y\|_\alpha |x(t) + y(t) - (x(s) + y(s))|.
\end{aligned} \tag{4.3}$$

By (4.3), we observe

$$\begin{aligned}
& \sup \left\{ \frac{|(Fx - Fy)(t) - (Fx - Fy)(s)|}{|t - s|^\alpha} : t, s \in [0, 1], t \neq s \right\} \\
& \leq \|x + y\|_\alpha \sup_{t, s \in [0, 1], t \neq s} \frac{|x(t) - y(t) - (x(s) - y(s))|}{|t - s|^\alpha} \\
& + \|x - y\|_\alpha \sup_{t, s \in [0, 1], t \neq s} \frac{|x(t) + y(t) - (x(s) + y(s))|}{|t - s|^\alpha} \\
& \leq \|x + y\|_\alpha \|x - y\|_\alpha + \|x - y\|_\alpha \|x + y\|_\alpha \\
& = 2 \|x + y\|_\alpha \|x - y\|_\alpha. \tag{4.4}
\end{aligned}$$

From (4.4), it follows

$$\begin{aligned}
\|Fx - Fy\|_\alpha &= |(Fx - Fy)(0)| + \sup_{t \neq s} \left\{ \frac{|(Fx - Fy)(t) - (Fx - Fy)(s)|}{|t - s|^\alpha} : t, s \in [0, 1] \right\} \\
&\leq |x^2(0) - y^2(0)| + 2 \|x + y\|_\alpha \|x - y\|_\alpha \\
&= |x(0) - y(0)| |x(0) + y(0)| + 2 \|x + y\|_\alpha \|x - y\|_\alpha \\
&\leq \|x - y\|_\infty \|x + y\|_\infty + 2 \|x + y\|_\alpha \|x - y\|_\alpha \\
&\leq 3 \|x + y\|_\alpha \|x - y\|_\alpha \\
&\leq 3 \|x - y\|_\alpha (\|x - y\|_\alpha + 2 \|y\|_\alpha) \\
&\leq 3\delta (\delta + 2 \|y\|_\alpha) \\
&< \varepsilon. \tag{4.5}
\end{aligned}$$

So that, the inequality

$$\|Fx - Fy\|_\alpha \leq 3\delta (\delta + 2 \|y\|_\alpha) < \varepsilon$$

is satisfied for all $x \in H_\beta [0, 1]$, where $0 < \delta < \sqrt{\|y\|_\alpha^2 + \varepsilon} - \|y\|_\alpha$. Therefore, we can choose the positive number δ as $\delta = \frac{1}{2} \sqrt{\|y\|_\alpha^2 + \varepsilon} - \|y\|_\alpha$. This shows that the operator F is continuous at the point $y \in H_\beta [0, 1]$. Since y is chosen arbitrarily, we deduce that F is continuous on $H_\beta [0, 1]$ with respect to the norm $\|\cdot\|_\alpha$.

In this case, the inequality appearing in assumption (v) of Theorem (3.1) takes the following form

$$\|p\|_{\frac{1}{4}} + (2K + k_{\frac{1}{4}})rf(r) \leq r$$

which is equivalent to

$$\ln \left(\sqrt[4]{\hat{n}} + 1 \right) + \sqrt[4]{\hat{n}} + \left[\frac{3}{2} \left(\sqrt[3]{(m+1)^4} - \sqrt[3]{m^4} \right) + \sqrt[3]{3m} \right] 3r^3 \leq r. \tag{4.6}$$

Obviously, there exists a positive number r_0 satisfying (4.6) provided that the constants n, \hat{n} and m can be chosen as suitable. For example, if one chose $n = \frac{1}{2^{16}}, \hat{n} = 0$ and $m = 1, r_0 = \frac{1}{4}$, then the inequality

$$\begin{aligned} & \|p\|_{\frac{1}{4}} + (2K + k_{\frac{1}{4}})rf(r) \\ &= \ln\left(\sqrt[4]{\hat{n}} + 1\right) + \sqrt[4]{n} + \left[\frac{3}{2}\left(\sqrt[3]{(m+1)^4} - \sqrt[3]{m^4}\right) + \sqrt[3]{3m}\right]3r^3 \\ &\approx 0,23696 < \frac{1}{4}. \end{aligned}$$

Finally, applying Theorem (3.1) we conclude that equation (4.1) has at least one solution in the space $H_\alpha[0, 1]$ with $0 < \alpha < \frac{1}{4}$.

Example 2. Let us consider the quadratic integral equation

$$x(t) = \ln\left(\frac{t}{7} + 1\right) + (ax(t) + b) \int_0^1 \sqrt{mt^2 + \tau x(e^\tau)} d\tau, \quad t \in [0, 1]. \quad (4.7)$$

Set $p(t) = \ln\left(\frac{t}{7} + 1\right)$, $k(t, \tau) = \sqrt{mt^2 + \tau}$, $q(\tau) = e^\tau$ for $t, \tau \in [0, 1]$ and m are non-negative constant. The operator F defined by $(Fx)(t) = ax(t) + b$, where a and b are any real number.

In what follows, we will prove that assumption (i)-(v) of Theorem (3.1) are satisfied. Since function $p : \mathbb{R}^+ \rightarrow \mathbb{R}^+$ defined by $p(t) = \ln\left(\frac{t}{7} + 1\right)$, is concave and $p(0) = 0$, then from Lemma 4.4 in [9] these functions are subadditive. By the result of subadditive

$$\begin{aligned} |p(t) - p(s)| &= \left| \ln\left(\frac{t}{7} + 1\right) - \ln\left(\frac{s}{7} + 1\right) \right| \\ &\leq \ln\left|\frac{t-s}{7}\right| \\ &< \frac{|t-s|}{7} \\ &\leq \frac{1}{7}|t-s|^{\frac{1}{2}} \end{aligned}$$

where we have used that $\ln x < x$ for $x > 0$. This says that $p \in H_{\frac{1}{2}}[0, 1]$ (i.e. $\beta = \frac{1}{2}$) and, moreover, $H_p^{\frac{1}{2}} = \frac{1}{7}$. Therefore, assumption (i) of Theorem (3.1) is satisfied. Note that

$$\begin{aligned} \|p\|_{\frac{1}{2}} &= |p(0)| + \sup \left\{ \frac{|p(t) - p(s)|}{|t-s|^{\frac{1}{2}}} : t, s \in [0, 1], t \neq s \right\} \\ &= |p(0)| + H_p^{\frac{1}{2}} = H_p^{\frac{1}{2}} = \frac{1}{7}. \end{aligned}$$

Further, we have

$$\begin{aligned}
|k(t, \tau) - k(s, \tau)| &= \left| \sqrt{mt^2 + \tau} - \sqrt{ms^2 + \tau} \right| \\
&\leq \sqrt{|mt^2 - ms^2|} \\
&= \sqrt{m} \sqrt{|t^2 - s^2|} \\
&= \sqrt{m} \sqrt{|t - s|} \sqrt{|t + s|} \\
&\leq \sqrt{m} \sqrt{2} |t - s|^{\frac{1}{2}} \\
&\leq \sqrt{2m} |t - s|^{\frac{1}{2}}
\end{aligned}$$

for all $t, s \in [0, 1]$. Assumption (ii) of Theorem (3.1) is satisfied with $k_\beta = k_{\frac{1}{2}} = \sqrt{2m}$. It is clear that $q(\tau) = e^\tau$ satisfies assumption (iii). In our case, the constant K is given by

$$\begin{aligned}
K &= \sup \left\{ \int_0^1 |k(t, \tau)| d\tau : t \in [0, 1] \right\} \\
&= \sup \left\{ \int_0^1 \left| \sqrt{mt^2 + \tau} \right| d\tau : t \in [0, 1] \right\} \\
&= \int_0^1 \sqrt{m + \tau} d\tau \\
&= \frac{2}{3} \left(\sqrt{(m+1)^3} - \sqrt{m^3} \right).
\end{aligned}$$

For all $x \in H_\beta [0, 1]$

$$\begin{aligned}
\|Fx\|_\beta &= |(Fx)(0)| + \sup \left\{ \frac{|(Fx)(t) - (Fx)(s)|}{|t - s|^\beta} : t, s \in [0, 1], t \neq s \right\} \\
&= |ax(0) + b| + \sup \left\{ \frac{|ax(t) + b - ax(s) - b|}{|t - s|^\beta} : t, s \in [0, 1], t \neq s \right\} \\
&= |a| |x(0)| + |b| + \sup \left\{ \frac{|x(t) - x(s)| |a|}{|t - s|^\beta} : t, s \in [0, 1], t \neq s \right\} \\
&\leq |a| |x(0)| + |b| + |a| \sup \left\{ \frac{|x(t) - x(s)|}{|t - s|^\beta} : t, s \in [0, 1], t \neq s \right\} \\
&\leq |a| \left(|x(0)| + \sup \left\{ \frac{|x(t) - x(s)|}{|t - s|^\beta} : t, s \in [0, 1], t \neq s \right\} \right) + |b| \\
&\leq |a| \|x\|_\beta + |b|.
\end{aligned}$$

Therefore, F is an operator from $H_\beta [0, 1]$ into $H_\beta [0, 1]$ and we can chose the function $f : \mathbb{R}^+ \rightarrow \mathbb{R}^+$ as $f(x) = |a|x + |b|$. This function is non-decreasing and satisfies the inequality in Assumption (iv).

Now, we will show that the operator F is continuous on the $H_\beta [0, 1]$ with respect to the norm $\|\cdot\|_\alpha$. To this end, fix arbitrarily $y \in H_\beta [0, 1]$ and $\varepsilon > 0$. Assume that $x \in H_\beta [0, 1]$ is an arbitrary function and $\|x - y\|_\alpha < \delta$, where δ is a positive number such that $0 < \delta < \frac{\varepsilon}{|a|}$ (in this place $a \neq 0$. It is obvious that if a is zero, the operator F is continuous).

Then, for arbitrary $t, s \in [0, 1]$ we obtain

$$\begin{aligned} \|Fx - Fy\|_\alpha &= |(Fx - Fy)(0)| + \sup_{t \neq s} \left\{ \frac{|(Fx - Fy)(t) - (Fx - Fy)(s)|}{|t - s|^\alpha} \right\} \\ &= |ax(0) - ay(0)| + \sup_{t \neq s} \left\{ \frac{|(ax(t) - ay(t)) - (ax(s) - ay(s))|}{|t - s|^\alpha} \right\} \\ &= |a| |x(0) - y(0)| + |a| \sup_{t \neq s} \left\{ \frac{|(x(t) - y(t)) - (x(s) - y(s))|}{|t - s|^\alpha} \right\} \\ &= |a| \left(|x(0) - y(0)| + \sup_{t \neq s} \left\{ \frac{|(x(t) - y(t)) - (x(s) - y(s))|}{|t - s|^\alpha} \right\} \right) \\ &= |a| \|x - y\|_\alpha \\ &\leq |a| \delta \\ &< \varepsilon. \end{aligned}$$

This shows that the operator F is continuous at the point $y \in H_\beta [0, 1]$. Since y was chosen arbitrarily, we deduce that F is continuous on $H_\beta [0, 1]$ with respect to the norm $\|\cdot\|_\alpha$.

In this case, the inequality appearing in assumption (v) of Theorem (3.1) takes the following form

$$\|p\|_{\frac{1}{2}} + (2K + k_{\frac{1}{2}})rf(r) \leq r$$

which is equivalent to

$$\frac{1}{7} + \left[\frac{4}{3} \left(\sqrt{(m+1)^3} - \sqrt{m^3} \right) + \sqrt{2m} \right] r (|a|r + |b|) \leq r. \quad (4.8)$$

Obviously, there exists a number positive r_0 satisfying (4.8) provided that the constants a, b and m can be chosen as suitable. For example, if one chose $a = \frac{1}{10}, b = \frac{1}{60}$ and $m = \frac{1}{2}, r_0 = \frac{1}{6}$, then the inequality

$$\begin{aligned} &\|p\|_{\frac{1}{2}} + (2K + k_{\frac{1}{2}})r_0f(r_0) \\ &= \frac{1}{7} + \left[\frac{4}{3} \left(\sqrt{(m+1)^3} - \sqrt{m^3} \right) + \sqrt{2m} \right] r_0 (|a|r_0 + |b|) \\ &\approx 0,15939 < \frac{1}{6}. \end{aligned}$$

Therefore, using Theorem (3.1), we conclude that equation (4.7) has at least one solution in the space $H_\alpha [0, 1]$ with $0 < \alpha < \frac{1}{2} = \beta$.

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On the Existence of Continuous Positive Monotonic Solutions of a Self-Reference Quadratic Integral Equation

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ABSTRACT: In this work we study the existence of positive monotonic solutions of a self-reference quadratic integral equation in the class of continuous real valued functions. The continuous dependence of the unique solution will be proved. Some examples will be given.

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Keywords and Phrases: Self-reference; Quadratic integral equation; Existence of solutions; Uniqueness of solution; Continuous dependence; Schauder fixed point theorem.

1. Introduction

Most papers of differential and integral equations with deviating arguments introduce the deviation of the arguments only on the time itself, however, the case of the deviating arguments depend on both the state variable x and the time t is important in theory and practice. These kinds of equations play an important role in nonlinear analysis and have many applications (see [1], [7]-[11] and [13]- [16]).

Buică [8] studied the existence, uniqueness and continuous dependence of the solution of the integral equation

$$x(t) = x_0 + \int_a^t f(s, x(x(s)))ds$$

corresponding to the initial value problem

$$\frac{d}{dt}x(t) = f(t, x(x(t))), \quad t \in (a, b], \quad x(a) = x_0$$

where $f \in C([a, b] \times [a, b])$ and Lipschitz continuous in the second argument. Here we relax the assumptions and generalize the results of [8] for the self-reference quadratic integral equation

$$x(t) = a(t) + \int_0^{\phi_1(t)} f_1(s, x(x(s))) ds + \int_0^{\phi_2(t)} f_2(s, x(x(s))) ds, \quad t \in [0, T]. \quad (1)$$

Quadratic integral equations have been studied by some authors, see for examples [2]-[6] and [9] and references therein.

Let $C[0, T]$ be the Banach space consisting of all functions which are defined and continuous on the interval $[0, T]$. Our aim in this paper is to study the existence of continuous positive monotonic solutions $x \in C[0, T]$ of the self-reference quadratic integral equation (1). The uniqueness of the solution will be studied also. Moreover we prove that the unique solution of (1) depends continuously on the the functions a , f_1 and f_2 .

2. Existence of solution

Consider the quadratic integral equation (1) under the following assumptions:

(i) $a: [0, T] \rightarrow R^+$ and there exists a positive constant a such that

$$|a(t_2) - a(t_1)| \leq a|t_2 - t_1|, \quad t_1, t_2 \in [0, T].$$

(ii) $f_i : [0, T] \times [0, T] \rightarrow R^+$ satisfies Carathéodory condition, i.e. f_i are measurable in t for all $x \in C[0, T]$ and continuous in x for almost all $t \in [0, T]$, $i = 1, 2$.

(iii) There exist two constants $b_1, b_2 \geq 0$ and two bounded measurable functions $m_i : [0, T] \rightarrow R$, $|m_i(t)| \leq c_i$ such that

$$|f_i(t, x)| \leq |m_i(t)| + b_i|x|, \quad i = 1, 2.$$

(iv) $\phi_i : [0, T] \rightarrow [0, T]$ such that $\phi_i(0) = 0$ and

$$|\phi_i(t) - \phi_i(s)| \leq |t - s|, \quad i = 1, 2.$$

This assumption implies that $\phi_i(t) \leq t$, $i = 1, 2$ and $x(0) = a(0)$.

(v) $LT + |a(0)| \leq T$ and $L = a + 2M_1M_2T < 1$ where

$$M_1 = c_1 + b_1T, \quad M_2 = c_2 + b_2T.$$

Define the set S_L by

$$S_L = \{x \in C[0, T] : |x(t) - x(s)| \leq L|t - s|\} \subset C[0, T].$$

It clear that S_L is nonempty, closed, bounded and convex subset of $C[0, T]$.

Now we can prove the following existence theorem

Theorem 1. *Let the assumptions (i) – (v) be satisfied, then the self-reference quadratic integral equation (1) has at least one positive solution $x \in S_L \subset C[0, T]$.*

Proof. Define the operator F associated with equation (1) by

$$Fx(t) = a(t) + \int_0^{\phi_1(t)} f_1(s, x(x(s)))ds \int_0^{\phi_2(t)} f_2(s, x(x(s)))ds, \quad t \in [0, T].$$

Let $x \in S_L \subset C[0, T]$, $t \in [0, T]$. Then, from our assumptions we have

$$\begin{aligned} |Fx(t)| &= \left| a(t) + \int_0^{\phi_1(t)} f_1(s, x(x(s)))ds \int_0^{\phi_2(t)} f_2(s, x(x(s)))ds \right| \\ &\leq |a(t)| + \int_0^{\phi_1(t)} |f_1(s, x(x(s)))|ds \int_0^{\phi_2(t)} |f_2(s, x(x(s)))|ds \\ &\leq |a(t)| + \int_0^{\phi_1(t)} \{ |m_1(s)| + b_1|x(x(s))| \} ds \int_0^{\phi_2(t)} \{ |m_2(s)| + b_2|x(x(s))| \} ds \\ &\leq |a(t)| + [c_1\phi_1(t) + b_1 \int_0^{\phi_1(t)} \{ L|x(s)| + |x(0)| \} ds] \\ &\quad [c_2\phi_2(t) + b_2 \int_0^{\phi_2(t)} \{ L|x(s)| + |x(0)| \} ds] \\ &\leq |a(t)| + [c_1T + b_1(LT + |a(0)|)\phi_1(t)] [c_2T + b_2(LT + |a(0)|)\phi_2(t)] \\ &\leq |a(t)| + [c_1 + b_1T] [c_2 + b_2T] T^2 \\ &\leq |a(t)| + M_1M_2T^2 \leq a T + |a(0)| + M_1M_2T^2 \\ &< L T + |a(0)| \leq T. \end{aligned}$$

This proves that the class $\{Fx\}$ is uniformly bounded.

Now let $x \in S_L$ and $t_1, t_2 \in [0, T]$ such that $t_1 < t_2$ and $|t_2 - t_1| < \delta$, then

$$\begin{aligned} |Fx(t_2) - Fx(t_1)| &= \left| a(t_2) + \int_0^{\phi_1(t_2)} f_1(s, x(x(s)))ds \int_0^{\phi_2(t_2)} f_2(s, x(x(s)))ds \right. \\ &\quad \left. - a(t_1) - \int_0^{\phi_1(t_1)} f_1(s, x(x(s)))ds \int_0^{\phi_2(t_1)} f_2(s, x(x(s)))ds \right| \\ &= |a(t_2) - a(t_1)| \\ &\quad + \left| \int_0^{\phi_1(t_2)} f_1(s, x(x(s)))ds \int_0^{\phi_2(t_2)} f_2(s, x(x(s)))ds \right. \\ &\quad \left. - \int_0^{\phi_1(t_2)} f_1(s, x(x(s)))ds \int_0^{\phi_2(t_1)} f_2(s, x(x(s)))ds \right. \\ &\quad \left. + \int_0^{\phi_1(t_2)} f_1(s, x(x(s)))ds \int_0^{\phi_2(t_1)} f_2(s, x(x(s)))ds \right. \\ &\quad \left. - \int_0^{\phi_1(t_1)} f_1(s, x(x(s)))ds \int_0^{\phi_2(t_1)} f_2(s, x(x(s)))ds \right| \end{aligned}$$

$$\begin{aligned}
&\leq |a(t_2) - a(t_1)| \\
&+ \left| \int_0^{\phi_1(t_2)} f_1(s, x(x(s))) ds \left[\int_0^{\phi_2(t_2)} f_2(s, x(x(s))) ds - \int_0^{\phi_2(t_1)} f_2(s, x(x(s))) ds \right] \right| \\
&+ \left| \int_0^{\phi_2(t_1)} f_2(s, x(x(s))) ds \left[\int_0^{\phi_1(t_2)} f_1(s, x(x(s))) ds - \int_0^{\phi_1(t_1)} f_1(s, x(x(s))) ds \right] \right| \\
&\leq |a(t_2) - a(t_1)| \\
&+ \int_0^{\phi_1(t_2)} |f_1(s, x(x(s)))| ds \left| \int_{\phi_2(t_1)}^{\phi_2(t_2)} f_2(s, x(x(s))) ds \right| \\
&+ \int_0^{\phi_2(t_1)} |f_2(s, x(x(s)))| ds \left| \int_{\phi_1(t_1)}^{\phi_1(t_2)} f_1(s, x(x(s))) ds \right| \\
&\leq a |t_2 - t_1| \\
&+ \int_0^{\phi_1(t_2)} \{|m_1(s)| + b_1|x(x(s))|\} ds \left(\left| \int_{\phi_2(t_1)}^{\phi_2(t_2)} \{|m_2(s)| + b_2|x(x(s))|\} ds \right| \right) \\
&+ \left(\int_0^{\phi_2(t_1)} \{|m_2(s)| + b_2|x(x(s))|\} ds \right) \left(\left| \int_{\phi_1(t_1)}^{\phi_1(t_2)} \{|m_1(s)| + b_1|x(x(s))|\} ds \right| \right) \\
&\leq a |t_2 - t_1| \\
&+ \left[c_1 \phi_1(t_2) + b_1 \int_0^{\phi_1(t_2)} \{L|x(s)| + |x(0)|\} ds \right] \\
&\quad \left[c_2 |\phi_2(t_2) - \phi_2(t_1)| + b_2 \left| \int_{\phi_2(t_1)}^{\phi_2(t_2)} \{L|x(s)| + |x(0)|\} ds \right| \right] \\
&+ \left[c_2 \phi_2(t_1) + b_2 \int_0^{\phi_2(t_1)} \{L|x(s)| + |x(0)|\} ds \right] \\
&\quad \left[c_1 |\phi_1(t_2) - \phi_1(t_1)| + b_1 \left| \int_{\phi_1(t_1)}^{\phi_1(t_2)} \{L|x(s)| + |x(0)|\} ds \right| \right] \\
&\leq a |t_2 - t_1| \\
&+ \left[c_1 + b_1 \{L T + |a(0)|\} \right] \left[c_2 + b_2 \{L T + |a(0)|\} \right] \phi_1(t_2) |\phi_2(t_2) - \phi_2(t_1)| \\
&+ \left[c_2 + b_2 \{L T + |a(0)|\} \right] \left[c_1 + b_1 \{L T + |a(0)|\} \right] \phi_2(t_1) |\phi_1(t_2) - \phi_1(t_1)| \\
&\leq a |t_2 - t_1| \\
&+ \left[c_1 + b_1 \{L T + |a(0)|\} \right] \left[c_2 + b_2 \{L T + |a(0)|\} \right] T |t_2 - t_1| \\
&+ \left[c_2 + b_2 \{L T + |a(0)|\} \right] \left[c_1 + b_1 \{L T + |a(0)|\} \right] T |t_2 - t_1| \\
&\leq a |t_2 - t_1| + 2T(c_1 + b_1 T)(c_2 + b_2 T)|t_2 - t_1| \\
&= a |t_2 - t_1| + 2TM_1M_2|t_2 - t_1| = L|t_2 - t_1|.
\end{aligned}$$

This proves that $F : S_L \rightarrow S_L$ and the class $\{Fx\}$ is equicontinuous.

Now the class of continuous functions $\{Fx\} \subset S_L \subset C[0, T]$ is uniformly bounded and equicontinuous on S_L . Hence, applying Arzela-Ascoli Theorem [12] we deduce that the operator F is compact.

Finally we show that F is continuous. Let $\{x_n\} \subset S_L$ such that $x_n \rightarrow x_0$ on $[0, T]$, then

$$\begin{aligned} |f_i(t, x_n(x_n(t)))| &\leq |m_i(t)| + b_i|x_n(x_n(t))| \\ &\leq |m_i(t)| + b_iT, \quad i = 1, 2 \end{aligned}$$

and

$$\begin{aligned} |x_n(x_n(t)) - x_0(x_0(t))| &= |x_n(x_n(t)) - x_n(x_0(t)) + x_n(x_0(t)) - x_0(x_0(t))| \\ &\leq |x_n(x_n(t)) - x_n(x_0(t))| + |x_n(x_0(t)) - x_0(x_0(t))| \\ &\leq L|x_n(t) - x_0(t)| + |x_n(x_0(t)) - x_0(x_0(t))|. \end{aligned}$$

This implies that

$$x_n(x_n(t)) \rightarrow (x_0(x_0(t))).$$

From the continuity of f_i , $i = 1, 2$ in the second argument we have

$$f(t, x_n(x_n(t))) \rightarrow f(t, x_0(x_0(t))).$$

Now by Lebesgue's dominated convergence Theorem [12] we obtain

$$\begin{aligned} \lim_{n \rightarrow \infty} (Fx_n)(t) &= \lim_{n \rightarrow \infty} a(t) + \lim_{n \rightarrow \infty} \int_0^{\phi_1(t)} f_1(s, x_n(x_n(s))) ds + \int_0^{\phi_2(t)} f_2(s, x_n(x_n(s))) ds \\ &= a(t) + \int_0^{\phi_1(t)} f_1(s, x_0(x_0(s))) ds + \int_0^{\phi_2(t)} f_2(s, x_0(x_0(s))) ds \\ &= (Fx_0)(t). \end{aligned}$$

Then F is continuous. Using Schauder fixed point Theorem ([12]), then the operator F has at least one fixed point $x \in S_L$. Consequently there exist at least one solution $x \in C[0, T]$ of equation (1).

Finally, from our assumptions we have

$$x(t) = a(t) + \int_0^{\phi_1(t)} f_1(s, x(x(s))) ds + \int_0^{\phi_2(t)} f_2(s, x(x(s))) ds > 0, \quad t \in [0, T].$$

and the solution of the quadratic integral equation (1) is positive. □

Now the following two corollaries can be easily proved.

Corollary 1. *Let the assumptions of Theorem 1 be satisfied. If the functions a , ϕ_1 and ϕ_2 are nondecreasing, then the solution of the quadratic integral equation (1) is positive and nondecreasing.*

Corollary 2. *Let the assumptions of Corollary 1 be satisfied. If, in addition $\phi_i(t) = t$, $i = 1, 2$, then the quadratic integral equation*

$$x(t) = a(t) + \int_0^t f_1(s, x(x(s))) ds + \int_0^t f_2(s, x(x(s))) ds, \quad t \in [0, T] \quad (2)$$

has at least one positive and nondecreasing solution $x \in C[0, T]$.

Example 1. Consider the following quadratic integral equation

$$\begin{aligned} x(t) = & \left(\frac{1}{4} + \frac{1}{8}t \right) + \int_0^{\beta_1 t} \left(\frac{1}{3} s^3 e^{-s^2} + \frac{\ln(1 + |x(x(s))|)}{4 + s^2} \right) ds \\ & \int_0^{\beta_2 t^\zeta} \left(\frac{1}{12} |\cos(3(s+1))| + \frac{3}{24} |x(x(s))| \right) ds, \end{aligned} \quad (3)$$

where $t \in [0, 1]$, $\beta_1 \in (0, 1]$, $\zeta > 1$ and $\beta_2 \zeta < 1$.

Here we have

$$f_1(t, x(x(t))) = \frac{1}{3} t^3 e^{-t^2} + \frac{\ln(1 + |x(x(t))|)}{4 + t^2},$$

$$|f_1(t, x(x(t)))| \leq \frac{1}{3} t^3 e^{-t^2} + \frac{1}{4} |x(x(t))| \quad \text{and} \quad m_1(t) = \frac{1}{3} t^3 e^{-t^2},$$

$$f_2(t, x(x(t))) = \frac{1}{12} \cos(3(t+1)) + \frac{3}{24} |x(x(t))|,$$

$$|f_2(t, x(x(t)))| = \frac{1}{12} |\cos(3(t+1))| + \frac{3}{24} |x(x(t))| \quad \text{and} \quad m_2(t) = \frac{1}{12} |\cos(3(t+1))|.$$

Also we have $\phi_1(t) = \beta_1 t$, $\phi_2(t) = \beta_2 t^\zeta$, $a(t) = \frac{1}{4} + \frac{1}{8}t$, $a = \frac{1}{8}$, $b_1 = \frac{1}{4}$, $b_2 = \frac{3}{24}$, $c_1 = \frac{1}{3}$, $c_2 = \frac{1}{12}$, and $M_1 = \frac{7}{12}$, $M_2 = \frac{5}{24}$.

Hence $L \simeq 0.368 < 1$ and $L T + |a(0)| = 0.618 \leq T = 1$.

Now it is clear that all assumptions of Theorem 1 are satisfied, then equation (3) has at least one solution.

3. Uniqueness of the solution

In this section we study the uniqueness of the solution $x \in C[0, T]$ of the quadratic integral equation (1).

Consider the following assumption

(ii^*) $f_i : [0, T] \times [0, T] \rightarrow R^+$ are measurable in t for all $x \in C[0, T]$, satisfy the Lipschitz condition

$$|f_i(t, x) - f_i(t, y)| \leq b_i |x - y| \quad i = 1, 2$$

$$|f_i(t, 0)| \leq c_i, \quad \forall t \in [0, T].$$

Theorem 2. Let the assumptions (i), (iv), (v) and (ii*) be satisfied, if

$$(\gamma_1 b_2 + \gamma_2 b_1) T (L + 1) < 1,$$

where $\gamma_i = (c_i + b_i T)T$, $i = 1, 2$, then equation (1) has a unique solution $x \in C[0, T]$.

Proof. From assumption (ii*) we can deduced that

$$|f_i(t, x)| \leq b_i |x| + |f_i(t, 0)| \leq b_i |x| + c_i, \quad i = 1, 2,$$

then all assumptions of Theorem 1 are satisfied and the integral equation (1) has at least one solution. Let x, y be two solutions of (1), then obtain

$$\begin{aligned} |x(t) - y(t)| &= \left| a(t) + \int_0^{\phi_1(t)} f_1(s, x(x(s))) ds \int_0^{\phi_2(t)} f_2(s, x(x(s))) ds \right. \\ &\quad \left. - a(t) - \int_0^{\phi_1(t)} f_1(s, y(y(s))) ds \int_0^{\phi_2(t)} f_2(s, y(y(s))) ds \right| \\ &= \left| \int_0^{\phi_1(t)} f_1(s, x(x(s))) ds \left[\int_0^{\phi_2(t)} \{f_2(s, x(x(s))) - f_2(s, y(y(s)))\} ds \right] \right. \\ &\quad \left. + \int_0^{\phi_2(t)} f_2(s, y(y(s))) ds \left[\int_0^{\phi_1(t)} \{f_1(s, x(x(s))) - f_1(s, y(y(s)))\} ds \right] \right| \\ &\leq \int_0^{\phi_1(t)} |f_1(s, x(x(s)))| ds \int_0^{\phi_2(t)} |f_2(s, x(x(s))) - f_2(s, y(y(s)))| ds \\ &\quad + \int_0^{\phi_2(t)} |f_2(s, y(y(s)))| ds \int_0^{\phi_1(t)} |f_1(s, x(x(s))) - f_1(s, y(y(s)))| ds \\ &\leq \int_0^{\phi_1(t)} |f_1(s, x(x(s)))| ds b_2 \int_0^{\phi_2(t)} |x(x(s)) - y(y(s))| ds \\ &\quad + \int_0^{\phi_2(t)} |f_2(s, y(y(s)))| ds b_1 \int_0^{\phi_1(t)} |x(x(s)) - y(y(s))| ds, \end{aligned} \quad (4)$$

$$\begin{aligned} \int_0^{\phi_i(t)} |f_i(s, x(x(s)))| ds &\leq b_i \int_0^{\phi_i(t)} |x(x(s))| ds + \int_0^{\phi_i(t)} |f_i(t, 0)| ds \\ &\leq b_i \int_0^{\phi_i(t)} \{L T + |x(0)|\} ds + c_i \phi_i(t) \\ &\leq b_i \phi_i(t) T + c_i \phi_i(t) \\ &\leq (b_i T + c_i) T = \gamma_i, \quad i = 1, 2 \end{aligned} \quad (5)$$

and

$$\begin{aligned} |x(x(s)) - y(y(s))| &= |x(x(s)) - y(y(s)) + x(y(s)) - x(y(s))| \\ &\leq |x(x(s)) - x(y(s))| + |x(y(s)) - y(y(s))| \\ &\leq L|x(s) - y(s)| + |x(y(s)) - y(y(s))|. \end{aligned} \quad (6)$$

Substituting (5) and (6) in (4) we can get

$$\begin{aligned} |x(t) - y(t)| &\leq \gamma_1 b_2 \int_0^{\phi_2(t)} \{L|x(s) - y(s)| + |x(y(s)) - y(y(s))|\} ds \\ &\quad + \gamma_2 b_1 \int_0^{\phi_1(t)} \{L|x(s) - y(s)| + |x(y(s)) - y(y(s))|\} ds \\ &\leq \gamma_1 b_2 \|x - y\| (L + 1) \phi_2(t) + \gamma_2 b_1 \|x - y\| (L + 1) \phi_1(t) \\ &\leq (\gamma_1 b_2 + \gamma_2 b_1) T (L + 1) \|x - y\| \end{aligned}$$

and

$$[1 - (\gamma_1 b_2 + \gamma_2 b_1) T (L + 1)] \|x - y\| \leq 0,$$

then $x(t) = y(t)$, $t \in [0, T]$ and equation (1) has a unique solution $x \in C[0, T]$. \square

Example 2. Let $T = 1$, $t \in [0, 1]$ and $\alpha, \beta, \mu, \rho \in (0, 1]$ are parameters. Consider the following quadratic integral equation

$$x(t) = \left(\frac{2}{7} + \frac{1}{7}t\right) + \int_0^{\alpha t} \left(\frac{\mu}{8-s} + \frac{1}{14}|x(x(s))|\right) ds \int_0^{\beta t} \left(\frac{\rho}{6} \ln(1+|s|) + \frac{1}{2}|x(x(s))|\right) ds. \quad (7)$$

Here we have

$$\begin{aligned} f_1(t, x(x(t))) &= \frac{\mu}{8-t} + \frac{1}{14}|x(x(t))|, \\ |f_1(t, x) - f_1(t, y)| &\leq \frac{1}{14}|x - y|, \\ f_2(t, x(x(t))) &= \frac{\rho}{6} \ln(1+|t|) + \frac{1}{2}|x(x(t))|, \end{aligned}$$

and

$$|f_2(t, x) - f_2(t, y)| \leq \frac{1}{2}|x - y|.$$

Also, $m_1(t) = \frac{\mu}{8-t}$, $c_1 = \frac{1}{7}$, $m_2(t) = \frac{\rho}{6} \ln(1+|t|)$, $c_2 = \frac{1}{6}$, $\phi_1(t) = \alpha t$, $\phi_2(t) = \beta t$ and $a(t) = \frac{2}{7} + \frac{1}{7}t$, then we obtain $a = \frac{1}{7}$, $b_1 = \frac{1}{14}$, $b_2 = \frac{1}{2}$, $M_1 = \frac{3}{14}$ and $M_2 = \frac{2}{3}$.

Hence $L = \frac{3}{7} < 1$ and $L T + |a(0)| = \frac{5}{7} \leq T = 1$.

Moreover we have $\gamma_1 = \frac{3}{14}$, $\gamma_2 = \frac{2}{3}$ and

$$(\gamma_1 b_2 + \gamma_2 b_1) T (L + 1) \simeq 0.2210 < 1.$$

Now all assumptions of Theorem 2 are satisfied, then equation (7) has a unique solution.

4. Continuous dependence

In this section we prove that the solution of equation (1) depends continuously on the functions a , f_1, f_2 .

4.1. Continuous dependence on the function a

Definition 1. The solution of the integral equation (1) depends continuously on the function a if $\forall \epsilon > 0 \exists \delta(\epsilon) > 0$ such that

$$|a(t) - a^*(t)| \leq \delta \quad \Rightarrow \quad \|x - x^*\| \leq \epsilon \quad (8)$$

where x^* is the unique solution of equation

$$x^*(t) = a^*(t) + \int_0^{\phi_1(t)} f_1(s, x^*(x^*(s))) ds + \int_0^{\phi_2(t)} f_2(s, x^*(x^*(s))) ds, \quad t \in [0, T]. \quad (9)$$

Theorem 3. Let the assumptions of Theorem 2 be satisfied, assume that $|a(t) - a^*(t)| \leq \delta$, then the solution of (1) depends continuously on the function a .

Proof. Let $|a(t) - a^*(t)| \leq \delta$, then we can get

$$\begin{aligned} |x(t) - x^*(t)| &= \left| a(t) + \int_0^{\phi_1(t)} f_1(s, x(x(s))) ds + \int_0^{\phi_2(t)} f_2(s, x(x(s))) ds \right. \\ &\quad \left. - a^*(t) - \int_0^{\phi_1(t)} f_1(s, x^*(x^*(s))) ds - \int_0^{\phi_2(t)} f_2(s, x^*(x^*(s))) ds \right| \\ &= |a(t) - a^*(t) + \int_0^{\phi_1(t)} f_1(s, x(x(s))) ds \\ &\quad \times \left[\int_0^{\phi_2(t)} f_2(s, x(x(s))) ds - \int_0^{\phi_2(t)} f_2(s, x^*(x^*(s))) ds \right] \\ &\quad + \int_0^{\phi_2(t)} f_2(s, x^*(x^*(s))) ds \\ &\quad \times \left[\int_0^{\phi_1(t)} f_1(s, x(x(s))) ds - \int_0^{\phi_1(t)} f_1(s, x^*(x^*(s))) ds \right]| \\ &\leq |a(t) - a^*(t)| \\ &\quad + \int_0^{\phi_1(t)} |f_1(s, x(x(s)))| ds \int_0^{\phi_2(t)} |f_2(s, x(x(s))) - f_2(s, x^*(x^*(s)))| ds \\ &\quad + \int_0^{\phi_2(t)} |f_2(s, x^*(x^*(s)))| ds \int_0^{\phi_1(t)} |f_1(s, x(x(s))) - f_1(s, x^*(x^*(s)))| ds \\ &\leq \delta + \int_0^{\phi_1(t)} (c_1 + b_1|x(x(s))|) ds + b_2 \int_0^{\phi_2(t)} |x(x(s)) - x^*(x^*(s))| ds \\ &\quad + \int_0^{\phi_2(t)} (c_2 + b_2|x^*(x^*(s))|) ds + b_1 \int_0^{\phi_1(t)} |x(x(s)) - x^*(x^*(s))| ds \end{aligned}$$

$$\begin{aligned}
&\leq \delta + M_1 \phi_1(t) b_2 \int_0^{\phi_2(t)} |x(x(s)) - x^*(x^*(s))| ds \\
&+ M_2 \phi_2(t) b_1 \int_0^{\phi_1(t)} |x(x(s)) - x^*(x^*(s))| ds \\
&\leq \delta + M_1 T b_2 (L+1) \|x - x^*\| \phi_2(t) \\
&+ M_2 T b_1 (L+1) \|x - x^*\| \phi_1(t) \\
&\leq \delta + (\gamma_1 b_2 + \gamma_2 b_1)(L+1) T \|x - x^*\|,
\end{aligned}$$

$$\|x - x^*\| (1 - (\gamma_1 b_2 + \gamma_2 b_1)(L+1) T) \leq \delta$$

and

$$\|x - x^*\| \leq \frac{\delta}{1 - (\gamma_1 b_2 + \gamma_2 b_1)(L+1)T} = \epsilon.$$

4.2. Continuous dependence on the functions f_1

Here we prove that the solution of the equation (1) depends continuously on the function f_1 .

Definition 2. The solution of the integral equation (1) depends continuously on the function f_1 if $\forall \epsilon > 0 \exists \delta(\epsilon) > 0$ such that

$$|f_1(t, x(x(t))) - f_1^*(t, x(x(t)))| \leq \delta \Rightarrow \|x - x^*\| \leq \epsilon \quad (10)$$

where x^* is the unique solution of equation

$$x^*(t) = a(t) + \int_0^{\phi_1(t)} f_1^*(s, x^*(x^*(s))) ds \int_0^{\phi_2(t)} f_2(s, x^*(x^*(s))) ds, \quad t \in [0, T].$$

Theorem 4. Let the assumptions of Theorem 2 be satisfied, assume that

$$|f_1(t, x(x(t))) - f_1^*(t, x(x(t)))| \leq \delta,$$

then the solution of (1) depends continuously on the functions f_1 .

Proof. Let $|f_1(t, x(x(t))) - f_1^*(t, x(x(t)))| \leq \delta$, then we obtain

$$\begin{aligned}
|x(t) - x^*(t)| &= \left| a(t) + \int_0^{\phi_1(t)} f_1(s, x(x(s))) ds \int_0^{\phi_2(t)} f_2(s, x(x(s))) ds \right. \\
&\quad \left. - a(t) - \int_0^{\phi_1(t)} f_1^*(s, x^*(x^*(s))) ds \int_0^{\phi_2(t)} f_2(s, x^*(x^*(s))) ds \right|
\end{aligned}$$

$$\begin{aligned}
 &= \left| \int_0^{\phi_1(t)} f_1(s, x(x(s))) ds \int_0^{\phi_2(t)} f_2(s, x(x(s))) ds \right. \\
 &- \int_0^{\phi_1(t)} f_1(s, x^*(x^*(s))) ds \int_0^{\phi_2(t)} f_2(s, x(x(s))) ds \\
 &+ \int_0^{\phi_1(t)} f_1(s, x^*(x^*(s))) ds \int_0^{\phi_2(t)} f_2(s, x(x(s))) ds \\
 &- \left. \int_0^{\phi_1(t)} f_1^*(s, x^*(x^*(s))) ds \int_0^{\phi_2(t)} f_2(s, x^*(x^*(s))) ds \right| \\
 &= \left| \int_0^{\phi_2(t)} f_2(s, x(x(s))) ds \right. \\
 &\times \left[\int_0^{\phi_1(t)} f_1(s, x(x(s))) ds - \int_0^{\phi_1(t)} f_1(s, x^*(x^*(s))) ds \right] \\
 &+ \int_0^{\phi_1(t)} f_1(s, x^*(x^*(s))) ds \int_0^{\phi_2(t)} f_2(s, x(x(s))) ds \\
 &- \int_0^{\phi_1(t)} f_1^*(s, x^*(x^*(s))) ds \int_0^{\phi_2(t)} f_2(s, x^*(x^*(s))) ds \\
 &+ \int_0^{\phi_1(t)} f_1(s, x^*(x^*(s))) ds \int_0^{\phi_2(t)} f_2(s, x^*(x^*(s))) ds \\
 &- \left. \int_0^{\phi_1(t)} f_1(s, x^*(x^*(s))) ds \int_0^{\phi_2(t)} f_2(s, x^*(x^*(s))) ds \right| \\
 &= \left| \int_0^{\phi_2(t)} f_2(s, x(x(s))) ds \left[\int_0^{\phi_1(t)} f_1(s, x(x(s))) ds - \int_0^{\phi_1(t)} f_1(s, x^*(x^*(s))) ds \right] \right. \\
 &+ \int_0^{\phi_1(t)} f_1(s, x^*(x^*(s))) ds \left[\int_0^{\phi_2(t)} f_2(s, x(x(s))) ds - \int_0^{\phi_2(t)} f_2(s, x^*(x^*(s))) ds \right] \\
 &+ \int_0^{\phi_2(t)} f_2(s, x^*(x^*(s))) ds \left[\int_0^{\phi_1(t)} f_1(s, x^*(x^*(s))) ds - \int_0^{\phi_1(t)} f_1^*(s, x^*(x^*(s))) ds \right] \left. \right| \\
 &\leq \int_0^{\phi_2(t)} |f_2(s, x(x(s)))| ds \int_0^{\phi_1(t)} |f_1(s, x(x(s))) - f_1(s, x^*(x^*(s)))| ds \\
 &+ \int_0^{\phi_1(t)} |f_1(s, x^*(x^*(s)))| ds \int_0^{\phi_2(t)} |f_2(s, x(x(s))) - f_2(s, x^*(x^*(s)))| ds \\
 &+ \int_0^{\phi_2(t)} |f_2(s, x^*(x^*(s)))| ds \int_0^{\phi_1(t)} |f_1(s, x^*(x^*(s))) - f_1^*(s, x^*(x^*(s)))| ds \\
 &\leq \int_0^{\phi_2(t)} |f_2(s, x(x(s)))| ds \int_0^{\phi_1(t)} b_1 |x(x(s)) - x^*(x^*(s))| ds \\
 &+ \int_0^{\phi_1(t)} |f_1(s, x^*(x^*(s)))| ds \int_0^{\phi_2(t)} b_2 |x(x(s)) - x^*(x^*(s))| ds \\
 &+ \int_0^{\phi_2(t)} |f_2(s, x^*(x^*(s)))| ds \int_0^{\phi_1(t)} |f_1(s, x^*(x^*(s))) - f_1^*(s, x^*(x^*(s)))| ds.
 \end{aligned}$$

Using (5) and (6) we obtain

$$|x(t) - x^*(t)| \leq \gamma_2 b_1 (L+1)T \|x - x^*\| + \gamma_1 b_2 (L+1)T \|x - x^*\| + \gamma_2 T \delta,$$

$$\|x - x^*\| [1 - (\gamma_2 b_1 + \gamma_1 b_2)(L+1)T] \leq \gamma_2 T \delta$$

and

$$\|x - x^*\| \leq \frac{\gamma_2 T \delta}{1 - (\gamma_2 b_1 + \gamma_1 b_2)(L+1)T} = \epsilon.$$

□

Corollary 3. *Let the assumptions of Theorem 4 be satisfied. In Example 2 if μ changed to μ^* , then the solution of equation (7) depends continuously on μ (the function f_1).*

4.3. Continuous dependence on the functions f_2

By the same way, as in Theorem 4 we can prove that the solution of equation (1) dependence continuously on the function f_2 .

Theorem 5. *Let the assumptions of Theorem 2 be satisfied, assume that*

$$|f_2(t, x(x(t))) - f_2^*(t, x(x(t)))| \leq \delta,$$

then the solution of (1) depends continuously on the functions f_2 .

Corollary 4. *Let the assumptions of Theorem 5 be satisfied. In Example 2 if ρ changed to ρ^* , then the solution of equation (7) depends continuously on ρ (the function f_2).*

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Inequality for Polynomials with Prescribed Zeros

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ABSTRACT: For a polynomial $p(z)$ of degree n with a zero at β , of order at least $k(\geq 1)$, it is known that

$$\int_0^{2\pi} \left| \frac{p(e^{i\theta})}{(e^{i\theta} - \beta)^k} \right|^2 d\theta \leq \left\{ \prod_{j=1}^k \left(1 + |\beta|^2 - 2|\beta| \cos \frac{\pi}{n+2-j} \right) \right\}^{-1} \int_0^{2\pi} |p(e^{i\theta})|^2 d\theta.$$

By considering polynomial $p(z)$ of degree n in the form

$p(z) = (z - \beta_1)(z - \beta_2) \dots (z - \beta_k)q(z)$, $k \geq 1$ and $q(z)$, a polynomial of degree $n - k$, with

$$S = \{ \gamma_{l_1} \gamma_{l_2} \dots \gamma_{l_k} : \gamma_{l_1} \gamma_{l_2} \dots \gamma_{l_k} \text{ is a permutation of } k \text{ objects} \\ \beta_1, \beta_2, \dots, \beta_k \text{ taken all at a time} \},$$

we have obtained

$$\int_0^{2\pi} \left| \frac{p(e^{i\theta})}{(e^{i\theta} - \beta_1)(e^{i\theta} - \beta_2) \dots (e^{i\theta} - \beta_k)} \right|^2 d\theta \\ \leq \left[\min_{\gamma_{l_1} \gamma_{l_2} \dots \gamma_{l_k} \in S} \left\{ \prod_{j=1}^k \left(1 + |\gamma_{l_j}|^2 - 2|\gamma_{l_j}| \cos \frac{\pi}{n+2-j} \right) \right\}^{-1} \right] \int_0^{2\pi} |p(e^{i\theta})|^2 d\theta,$$

a generalization of the known result.

AMS Subject Classification: 30C10, 30A10.

Keywords and Phrases: Inequality; Polynomial with prescribed zeros; Generalization.

1. Introduction and statement of result

While thinking of polynomials vanishing at β , Donaldson and Rahman [1] had considered the problem:

How large can $\left(\frac{1}{2\pi} \int_0^{2\pi} \left| \frac{p(e^{i\theta})}{e^{i\theta} - \beta} \right|^2 d\theta\right)^{1/2}$ be, for a polynomial $p(z)$ of degree n with

$$\left(\frac{1}{2\pi} \int_0^{2\pi} |p(e^{i\theta})|^2 d\theta\right)^{1/2} = 1?$$

and they had obtained

Theorem A. *If $p(z)$ is a polynomial of degree n such that $p(\beta) = 0$, where β is an arbitrary non-negative number then*

$$\int_0^{2\pi} \left| \frac{p(e^{i\theta})}{e^{i\theta} - \beta} \right|^2 d\theta \leq \left(1 + \beta^2 - 2\beta \cos\left(\frac{\pi}{n+1}\right)\right)^{-1} \int_0^{2\pi} |p(e^{i\theta})|^2 d\theta.$$

In [2] Jain had considered the zero of polynomial $p(z)$ at β to be of order at least $k(\geq 1)$, with β being an arbitrary complex number and had obtained the following generalization of Theorem A.

Theorem B. *If $p(z)$ is a polynomial of degree n such that $p(z)$ has a zero at β , of order at least $k(\geq 1)$, with β being an arbitrary complex number then*

$$\int_0^{2\pi} \left| \frac{p(e^{i\theta})}{(e^{i\theta} - \beta)^k} \right|^2 d\theta \leq \left\{ \prod_{j=1}^k \left(1 + |\beta|^2 - 2|\beta| \cos \frac{\pi}{n+2-j}\right) \right\}^{-1} \int_0^{2\pi} |p(e^{i\theta})|^2 d\theta.$$

In this paper we have obtained a generalization of Theorem B by considering polynomial $p(z)$ of degree n in the form

$$p(z) = (z - \beta_1)(z - \beta_2) \dots (z - \beta_k)q(z), k \geq 1.$$

More precisely we have proved

Theorem. *Let $p(z)$ be a polynomial of degree n such that*

$$p(z) = (z - \beta_1)(z - \beta_2) \dots (z - \beta_k)q(z), k \geq 1. \quad (1.1)$$

Further let

$$S = \{\gamma_{l_1} \gamma_{l_2} \dots \gamma_{l_k} : \gamma_{l_1} \gamma_{l_2} \dots \gamma_{l_k} \text{ is a permutation of } k \text{ objects } \beta_1, \beta_2, \dots, \beta_k \text{ taken all at a time}\}.$$

Then

$$\begin{aligned} & \int_0^{2\pi} \left| \frac{p(e^{i\theta})}{(e^{i\theta} - \beta_1)(e^{i\theta} - \beta_2) \dots (e^{i\theta} - \beta_k)} \right|^2 d\theta \\ & \leq \left[\min_{\gamma_{l_1} \gamma_{l_2} \dots \gamma_{l_k} \in S} \left\{ \prod_{j=1}^k \left(1 + |\gamma_{l_j}|^2 - 2|\gamma_{l_j}| \cos \frac{\pi}{n+2-j}\right) \right\}^{-1} \right] \int_0^{2\pi} |p(e^{i\theta})|^2 d\theta. \end{aligned}$$

2. Lemma

For the proof of Theorem we require the following lemma.

Lemma 1. *If $p(z)$ is a polynomial of degree n such that*

$$p(\beta) = 0,$$

where β is an arbitrary complex number then

$$\int_0^{2\pi} \left| \frac{p(e^{i\theta})}{e^{i\theta} - \beta} \right|^2 d\theta \leq \left(1 + |\beta|^2 - 2|\beta| \cos \frac{\pi}{n+1} \right)^{-1} \int_0^{2\pi} |p(e^{i\theta})|^2 d\theta.$$

This lemma is due to Jain [2].

3. Proof of Theorem

Theorem is trivially true for $k = 1$, by Lemma 1. Accordingly we assume that $k > 1$. The polynomial

$$T_1(z) = (z - \beta_1)q(z) \tag{3.1}$$

is of degree $n - k + 1$ and therefore by Lemma 1 we have

$$\int_0^{2\pi} |q(e^{i\theta})|^2 d\theta = \int_0^{2\pi} \left| \frac{T_1(e^{i\theta})}{e^{i\theta} - \beta_1} \right|^2 d\theta \leq \left(1 + |\beta_1|^2 - 2|\beta_1| \cos \frac{\pi}{n-k+2} \right)^{-1} \int_0^{2\pi} |T_1(e^{i\theta})|^2 d\theta. \tag{3.2}$$

Further the polynomial

$$T_2(z) = (z - \beta_2)T_1(z) = (z - \beta_1)(z - \beta_2)q(z), \quad (\text{by (3.1)}), \tag{3.3}$$

is of degree $n - k + 2$ and by Lemma 1 we have

$$\int_0^{2\pi} |T_1(e^{i\theta})|^2 d\theta = \int_0^{2\pi} \left| \frac{T_2(e^{i\theta})}{e^{i\theta} - \beta_2} \right|^2 d\theta \leq \left(1 + |\beta_2|^2 - 2|\beta_2| \cos \frac{\pi}{n-k+3} \right)^{-1} \int_0^{2\pi} |T_2(e^{i\theta})|^2 d\theta. \tag{3.4}$$

On combining (3.2) and (3.4) we get

$$\begin{aligned} & \int_0^{2\pi} |q(e^{i\theta})|^2 d\theta \\ & \leq \left\{ \left(1 + |\beta_1|^2 - 2|\beta_1| \cos \frac{\pi}{n-k+2} \right) \left(1 + |\beta_2|^2 - 2|\beta_2| \cos \frac{\pi}{n-k+3} \right) \right\}^{-1} \int_0^{2\pi} |T_2(e^{i\theta})|^2 d\theta. \end{aligned}$$

We can now continue and obtain similarly

$$\int_0^{2\pi} |q(e^{i\theta})|^2 \leq \left\{ \left(1 + |\beta_1|^2 - 2|\beta_1| \cos \frac{\pi}{n-k+2}\right) \left(1 + |\beta_2|^2 - 2|\beta_2| \cos \frac{\pi}{n-k+3}\right) \right. \\ \left. \times \left(1 + |\beta_3|^2 - 2|\beta_3| \cos \frac{\pi}{n-k+4}\right) \right\}^{-1} \int_0^{2\pi} |T_3(e^{i\theta})|^2 d\theta,$$

(with

$$T_3(z) = (z - \beta_3)T_2(z), = (z - \beta_1)(z - \beta_2)(z - \beta_3)q(z), \quad (\text{by (3.3)}), \quad (3.5)$$

.....

$$\int_0^{2\pi} |q(e^{i\theta})|^2 d\theta \leq \left\{ \left(1 + |\beta_1|^2 - 2|\beta_1| \cos \frac{\pi}{n-k+2}\right) \left(1 + |\beta_2|^2 - 2|\beta_2| \cos \frac{\pi}{n-k+3}\right) \dots \right. \\ \left. \dots \left(1 + |\beta_k|^2 - 2|\beta_k| \cos \frac{\pi}{n-k+k+1}\right) \right\}^{-1} \int_0^{2\pi} |T_k(e^{i\theta})|^2 d\theta, \quad (3.6)$$

(with

$$T_k(z) = (z - \beta_k)T_{k-1}(z), \\ = (z - \beta_1)(z - \beta_2) \dots (z - \beta_k)q(z), \quad (\text{similar to (3.3) and (3.5)}). \quad (3.7)$$

On using (1.1) and (3.7) in (3.6) we get

$$\int_0^{2\pi} \left| \frac{p(e^{i\theta})}{(e^{i\theta} - \beta_1)(e^{i\theta} - \beta_2) \dots (e^{i\theta} - \beta_k)} \right|^2 d\theta \leq \left\{ \left(1 + |\beta_1|^2 - 2|\beta_1| \cos \frac{\pi}{n-k+2}\right) \right. \\ \left(1 + |\beta_2|^2 - 2|\beta_2| \cos \frac{\pi}{n-k+3}\right) \dots \\ \dots \dots \left(1 + |\beta_k|^2 - 2|\beta_k| \cos \frac{\pi}{n+1}\right) \left. \right\}^{-1} \\ \times \int_0^{2\pi} |p(e^{i\theta})|^2 d\theta$$

and as the order of $\beta_1, \beta_2, \dots, \beta_k$ is immaterial, Theorem follows.

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Towards a Non-conformable Fractional Calculus of n-Variables

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ABSTRACT: In this paper we present an extension of the non-conformable local fractional derivative, to the case of functions of several variables. Results analogous to those known from the classic multivariate calculus are presented. To show the strength of this approach, we show an extension of the Second Lyapunov Method to the non-conformable local fractional case.

AMS Subject Classification: 26B12, 26A24, 35S05.

Keywords and Phrases: Functions of several variables; Fractional partial differential operators.

1. Preliminaries

The multivariate calculus presents a natural extension of the concepts of the one-dimensional calculus to real spaces of n dimensions. In itself the multivariate calculus is a particular expression of the most beautiful results of the analysis of several variables that have their climax in surface integration and that flaunt elegant coherence of the treatment of the theory of differential forms that summarize the simplicity and power of its physical applications. That's why from the point of view purely theoretical the multivariate calculus is the introduction to the analysis of several variables from a context particular; from the application point of view, his appearances are innumerable as a powerful tool resolute in problems in applied sciences. Thus, the calculus in several variables provides pure and applied researchers with the necessary knowledge to operate and apply mathematical functions with real variables in the approach and solution of practical situations. The partial derivative, is considered a fundamental axis for the approach and development of concepts that allow us to

understand and assimilate knowledge from almost all areas of applied science. Regarding the concept of multiple integration, reaches an interrelation with other areas of knowledge, especially physics, to finally to address general research topics, whether pure or applied. If we add to all the above the fact that the local fractional calculus has a very short development (conformable since 2014, [6], and non-conformable since last year, see [5] and [8]) we realize that a work where the fundamental foundations of the local fractional calculus can be established of several variables is necessary. Some results to the conformable case can be consulted in [3]. In this work we establish the first results to formalize the theoretical “corpus” necessary to develop this new mathematical branch and we extend the Second Method of Lyapunov to the non-conformable local fractional case of several variables.

2. Non-conformable partial derivative

Definition 1. Given a real valued function $f : \mathbb{R}^n \rightarrow \mathbb{R}$ and $\vec{a} = (a_1, \dots, a_n) \in \mathbb{R}^n$ a point whose i th component is positive. Then the non conformable partial N -derivative of f of order α in the point $\vec{a} = (a_1, \dots, a_n)$ is defined by

$$N_{x_i}^\alpha f(\vec{a}) = \lim_{\varepsilon \rightarrow 0} \frac{f(a_1, \dots, a_i + \varepsilon e^{\alpha_i^{-\alpha}}, \dots, a_n) - f(a_1, \dots, a_n)}{\varepsilon} \quad (1)$$

if it exists, is denoted $N_{x_i}^\alpha f(\vec{a})$, and called the i th non-conformable partial derivative of f of the order $\alpha \in (0, 1]$ at \vec{a} .

Remark 2. If a real valued function f with n variables has all non-conformable partial derivatives of the order $\alpha \in (0, 1]$ at \vec{a} , each $a_i > 0$, then the non-conformable α -gradient of f of the order $\alpha \in (0, 1]$ at \vec{a} is

$$\nabla_N^\alpha f(\vec{a}) = (N_{x_1}^\alpha f(\vec{a}), \dots, N_{x_n}^\alpha f(\vec{a})). \quad (2)$$

3. Applications of the Non-conformable Mean Value Theorem to the Multivariable Fractional Calculus

In this section, we will introduce the conformable version of two important properties of the classical partial derivative of the functions of several variables, [2]. Using the Non-conformable Mean Value Theorem, these results will be proven.

Theorem 3. (Function with a nonconformable partial zero derivative). Let $\alpha \in (0, 1]$, $f : X \rightarrow \mathbb{R}$ be a real valued function defined in an open and convex set $X \subset \mathbb{R}^n$, such that for all $\vec{x} = (x_1, \dots, x_n) \in X$, each $x_i > 0$. If the non-conformable partial derivative of f with respect to x_i , exist and is null on X , then $f(\vec{x}) = f(\vec{x}')$ for any points $\vec{x} = (x_1, \dots, x_i, \dots, x_n)$, $\vec{x}' = (x'_1, \dots, x'_i, \dots, x'_n) \in X$, i.e., the function f does not depend on the variable x_i .

Proof. Since X is a convex set and

$$\vec{x} = (x_1, \dots, x_i, \dots, x_n), \vec{x}' = (x'_1, \dots, x'_i, \dots, x'_n) \in X,$$

all points of the line segment $[\vec{x}, \vec{x}']$ are also in X , so the function g is defined in the interval of endpoints x_i and x'_i by $g(t) = f(x_1, \dots, x_{i-1}, t, x_{i+1}, \dots, x_n)$. This function is N -differentiable on above interval and its derivative at a point t , is given by $N_3^\alpha g(t) = N_{x_i}^\alpha f(x_1, \dots, t, \dots, x_n)$. Therefore, applying Theorem 2.7, [6], there is a point c_i between x_i and x'_i , such that $g(x'_i) - g(x_i) = \frac{(x'_i - x_i)}{e^{c_i - \alpha}} N_3^\alpha g(c_i)$, since point $c = (x_1, \dots, c_i, \dots, x_n) \in X$ and therefore $N_{x_i}^\alpha f(\vec{c}) = 0$, the above equality leads to $f(\vec{x}') - f(\vec{x}) = \frac{(x'_i - x_i)}{e^{c_i - \alpha}} N_{x_i}^\alpha f(\vec{c}) = 0$ then $f(\vec{x}) = f(\vec{x}')$, as we wanted to prove. \square

Now, we establish a first formula of finite increments for real valued functions of several variables, involving non-conformable partial derivatives.

Theorem 4. Let $\vec{a} = (a_1, a_2, \dots, a_n), \vec{b} = (b_1, b_2, \dots, b_n) \in \mathbb{R}^n$, x_0, x_1, \dots, x_n be points $\vec{x}_i = (b_1, \dots, b_i, a_{i+1}, \dots, a_n)$ (note that $\vec{x}_0 = \vec{a}$ and $\vec{x}_n = \vec{b}$) and line segment $S_i = [\vec{x}_{i-1}, \vec{x}_i]$, for $i = 1, 2, \dots, n$. Let $\alpha \in (0, 1]$ and $f : X \rightarrow \mathbb{R}$ be a real valued function defined in an open set $X \subset \mathbb{R}^n$ containing line segments S_1, S_2, \dots, S_n , such that for all $\vec{x} = (x_1, \dots, x_n) \in X$, each $x_i > 0$. If the non-conformable partial derivative of f with respect to x_i , exist on X , then there is a point c_i between a_i and b_i , for $i = 1, 2, \dots, n$, such that

$$\left. \begin{aligned} f(b_1, b_2, \dots, b_n) - f(a_1, a_2, \dots, a_n) = \\ = \sum_{i=1}^n \left((b_i - a_i) \frac{1}{e^{c_i - \alpha}} N_{x_i}^\alpha f(b_1, \dots, b_{i-1}, c_i, a_{i+1}, \dots, a_n) \right). \end{aligned} \right\} \quad (3)$$

Proof. First, we will express the difference $f(\vec{b}) - f(\vec{a})$ as follows

$$f(\vec{b}) - f(\vec{a}) = f(\vec{x}_n) - f(\vec{x}_{n-1}) = \sum_{i=1}^n [f(\vec{x}_i) - f(\vec{x}_{i-1})] \quad (4)$$

Consider now, for $i = 1, 2, \dots, n$, the real function g_i of the real variable t , defined on the closed interval of endpoints a_i and b_i , by

$$g(t) = f(x_1, \dots, x_{i-1}, t, x_{i+1}, \dots, x_n).$$

Since the non-conformable partial derivative of f with respect to x_i , exist on X and $S_i \subset X$, then g_i is N -differentiable on above interval and its derivative at a point t , is given by $N_3^\alpha g(t) = N_{x_i}^\alpha f(x_1, \dots, t, \dots, x_n)$. Therefore, applying Theorem 2.7, [6], there is a point c_i between a_i and b_i , such that $g_i(b_i) - g_i(a_i) = \frac{(b_i - a_i)}{e^{c_i - \alpha}} N_3^\alpha g_i(c_i)$. Then it is verified

$$f(\vec{x}_i) - f(\vec{x}_{i-1}) = \frac{(b_i - a_i)}{e^{c_i - \alpha}} N_{x_i}^\alpha f(b_1, \dots, b_{i-1}, c_i, a_{i+1}, \dots, a_n).$$

Taking the above expression to equation (4), our result is followed. \square

4. The Chain Rule

In [5] a version non-conformable of the classical chain rules is introduced as follows.

Theorem 5. *Let $\alpha \in (0, 1]$, g N -differentiable at $t > 0$ and f differentiable at $g(t)$ then*

$$N_3^\alpha(f \circ g)(t) = f'(g(t))N_3^\alpha g(t). \quad (5)$$

Remark 6. Using the fact that differentiability implies N -differentiability and assuming $g(t) > 0$, equation (5) can be written $N_3^\alpha(f \circ g)(t) = \frac{N_3^\alpha f(g(t))}{e^{g(t)-\alpha}} N_3^\alpha g(t)$.

Remark 7. Let f be a real valued function with n variables defined on an open set D , such that for all $(x_1, \dots, x_n) \in D$, each $x_i > 0$. The function f is said to be $C_\alpha(D, \mathbb{R})$ if all its non-conformable partial derivatives exist and are continuous on D .

We now show the chain rule for the functions of several variables, in two particular cases that are important in themselves. In the proof we will use the additional hypothesis of the continuity of non-conformable partial derivatives.

Theorem 8. (Chain Rule). *Let $\alpha \in (0, 1]$, $t \in \mathbb{R}$ and $\vec{x} = (x_1, \dots, x_n) \in \mathbb{R}^n$. If $\vec{f}(t) = (f_1(t), \dots, f_n(t))$ is N -differentiable at $a > 0$ and a real valued function g with n variables x_1, \dots, x_n , has all non-conformable partial derivatives of the order α at $\vec{f}(a) \in \mathbb{R}^n$, each $f_i(a) > 0$. Then the composition $(g \circ f)$ is N -differentiable at a and*

$$N_3^\alpha(g \circ f)(t) = \sum_{i=1}^n \frac{N_{x_i}^\alpha g(\vec{f}(a))}{e^{f_i(a)-\alpha}} N_3^\alpha f_i(a). \quad (6)$$

Proof. Assume $g \in C_\alpha(U(\vec{f}(a)), \mathbb{R})$, where $U(\vec{f}(a))$ is a neighborhood of the point $\vec{f}(a)$. Let $h(t) = (g \circ \vec{f})(t) = g(\vec{f}(t))$. From Definition 2.1, [5], we have that

$$N_3^\alpha h(a) = \lim_{\varepsilon \rightarrow 0} \frac{(h(a + \varepsilon e^{a-\alpha}) - h(a))}{\varepsilon} = \lim_{\varepsilon \rightarrow 0} \frac{(g(f(a + \varepsilon e^{a-\alpha})) - g(f(a)))}{\varepsilon}. \quad (7)$$

Without loss of generality we shall assume that $U(\vec{f}(a))$ is an open ball, $B(\vec{f}(a), r)$. Since \vec{f} is a continuous function, then together with the points $(f_1(a), \dots, f_n(a))$ and $(f_1(a + \varepsilon e^{a-\alpha}), \dots, f_n(a + \varepsilon e^{a-\alpha}))$, the points $(f_1(a), f_2(a + \varepsilon e^{a-\alpha}), \dots, f_n(a + \varepsilon e^{a-\alpha}))$, $(f_1(a), f_2(a), \dots, f_n(a + \varepsilon e^{a-\alpha}))$ and the lines connecting them must also to the ball $B(\vec{f}(a), r)$. We shall use this fact, applying Theorem 2.7, [6]:

$$\begin{aligned}
& \frac{(h(a + \varepsilon e^{a-\alpha}) - h(a))}{\varepsilon} = \frac{g(\vec{f}(a + \varepsilon e^{a-\alpha})) - g(\vec{f}(a))}{\varepsilon} = \\
& \frac{g(f_1(a + \varepsilon e^{a-\alpha}), \dots, f_n(a + \varepsilon e^{a-\alpha})) - g(f_1(a), f_2(a + \varepsilon e^{a-\alpha}), \dots, f_n(a + \varepsilon e^{a-\alpha}))}{\varepsilon} + \\
& + \dots + \frac{(g(f_1(a), f_2(a), \dots, f_n(a + \varepsilon e^{a-\alpha})) - g(f_1(a), f_2(a), \dots, f_n(a + \varepsilon e^{a-\alpha})))}{\varepsilon} = \\
& = N_{x_1}^\alpha g(c_1, f_2(a + \varepsilon e^{a-\alpha}), \dots, f_n(a + \varepsilon e^{a-\alpha})) \frac{1}{e^{c_1-\alpha}} \frac{f_1(a + \varepsilon e^{a-\alpha}) - f_1(a)}{\varepsilon} + \dots + \\
& + N_{x_n}^\alpha g(c_1, c_2, \dots, f_n(a + \varepsilon e^{a-\alpha})) \frac{1}{e^{c_n-\alpha}} \frac{f_n(a + \varepsilon e^{a-\alpha}) - f_n(a)}{\varepsilon}
\end{aligned}$$

where c_i is between $f_i(a)$ and $f_i(a + \varepsilon e^{a-\alpha})$ for all $i = 1, 2, \dots, n$. By taking limits as $\varepsilon \rightarrow 0$, using the continuity of non-conformable partial derivatives of g , and the fact that $c_i \rightarrow f_i(a)$ for all $i = 1, 2, \dots, n$, formula (7) can be written

$$\begin{aligned}
N_3^\alpha h(a) &= \lim_{\varepsilon \rightarrow 0} \frac{(h(a + \varepsilon e^{a-\alpha}) - h(a))}{\varepsilon} = \lim_{\varepsilon \rightarrow 0} \frac{(g(\vec{f}(a + \varepsilon e^{a-\alpha})) - g(\vec{f}(a)))}{\varepsilon} = \\
&= \lim_{\varepsilon \rightarrow 0} (N_{x_1}^\alpha g(c_1, f_2(a + \varepsilon e^{a-\alpha}), \dots, f_n(a + \varepsilon e^{a-\alpha})) \frac{f_1(a + \varepsilon e^{a-\alpha}) - f_1(a)}{\varepsilon e^{c_1-\alpha}} + \\
&+ N_{x_2}^\alpha g(f_1(a), c_2, \dots, f_n(a + \varepsilon e^{a-\alpha})) \frac{f_2(a + \varepsilon e^{a-\alpha}) - f_2(a)}{\varepsilon e^{c_2-\alpha}} + \dots + \\
&+ N_{x_n}^\alpha g(f_1(a), f_2(a), \dots, c_n)) \frac{f_n(a + \varepsilon e^{a-\alpha}) - f_n(a)}{\varepsilon e^{c_n-\alpha}} = \\
&= N_{x_1}^\alpha g(\vec{f}(a)) \frac{1}{e^{f_1(a)-\alpha}} N_3^\alpha f_1(a) + N_{x_2}^\alpha g(\vec{f}(a)) \frac{1}{e^{f_2(a)-\alpha}} N_3^\alpha f_2(a) + \dots + \\
&+ N_{x_n}^\alpha g(\vec{f}(a)) \frac{1}{e^{f_n(a)-\alpha}} N_3^\alpha f_n(a)
\end{aligned}$$

which completes the proof. \square

Remark 9. Also matrix form of equation (7) is given by the following

$$N_3^\alpha (g \circ \vec{f})(a) = (N_{x_1}^\alpha g(\vec{f}(a)), \dots, N_{x_n}^\alpha g(\vec{f}(a))) M(f, \alpha) \begin{pmatrix} N_3^\alpha f_1(a) \\ \dots \\ N_3^\alpha f_n(a) \end{pmatrix} \quad (8)$$

where $M(f, \alpha) = \begin{pmatrix} \frac{1}{e^{(f_1(a))-\alpha}} & \dots & 0 \\ \dots & \dots & \dots \\ 0 & \dots & \frac{1}{e^{(f_n(a))-\alpha}} \end{pmatrix}$ is the matrix corresponding to the linear transformation from R^n to R^n defined by

$$L_a^\alpha(x_1, \dots, x_n) = \begin{pmatrix} \frac{1}{e^{(f_1(a))-\alpha}} & \dots & 0 \\ \dots & \dots & \dots \\ 0 & \dots & \frac{1}{e^{(f_n(a))-\alpha}} \end{pmatrix} \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix}.$$

Theorem 10. (Chain Rule). Let $\alpha \in (0, 1]$, $\vec{x} = (x_1, \dots, x_n) \in \mathbb{R}^n$ and $\vec{y} = (y_1, \dots, y_m) \in \mathbb{R}^m$. If $\vec{f}(x_1, \dots, x_n) = (f_1(x_1, \dots, x_n), \dots, f_m(x_1, \dots, x_n))$ is a vector valued function such that each f_i has all non-conformable partial derivatives of the order α at $\vec{a} = (a_1, \dots, a_n) \in \mathbb{R}^n$, each $a_i > 0$, and a real valued function g with variables y_1, \dots, y_m has all non-conformable partial derivatives of the order α at $\vec{f}(a) \in \mathbb{R}^m$, all $f_i(a) > 0$. Then the composition $g \circ \vec{f}$ has all non-conformable partial derivatives of the order α at \vec{a} , which are given by

$$N_{x_i}^\alpha(g \circ \vec{f})(\vec{a}) = \sum_{j=1}^m N_{y_j}^\alpha g(\vec{f}(\vec{a})) \frac{1}{e^{(f_j(\vec{f}))^{-\alpha}}} N_{x_i}^\alpha f_j(\vec{a}) \quad (9)$$

for all $i = 1, 2, \dots, n$.

Proof. From definition of non-conformable partial derivative and the Theorem above, the result follows. \square

Remark 11. Also matrix form of equation (9) is given by the following

$$\left. \begin{aligned} & N_3^\alpha(g \circ \vec{f})(a) = \\ & = (N_{y_1}^\alpha g(\vec{f}(a)), \dots, N_{y_m}^\alpha g(\vec{f}(a))) N(f, \alpha) \begin{pmatrix} N_3^\alpha f_1(a) & \dots & 0 \\ \dots & \dots & \dots \\ 0 & \dots & N_3^\alpha f_m(a) \end{pmatrix} \end{aligned} \right\} \quad (10)$$

$$\left. \begin{aligned} & N_3^\alpha(g \circ \vec{f})(a) = \\ & = (N_{y_1}^\alpha g(\vec{f}(a)), \dots, N_{y_m}^\alpha g(\vec{f}(a))) N(f, \alpha) \begin{pmatrix} N_3^\alpha f_1(a) & \dots & 0 \\ \dots & \dots & \dots \\ 0 & \dots & N_3^\alpha f_m(a) \end{pmatrix} \end{aligned} \right\} \quad (11)$$

where $N(f, \alpha) = \begin{pmatrix} \frac{1}{e^{(f_1(a))^{-\alpha}}} & \dots & 0 \\ \dots & \dots & \dots \\ 0 & \dots & \frac{1}{e^{(f_m(a))^{-\alpha}}} \end{pmatrix}$ is the matrix corresponding to the linear transformation from R^m to R^m defined by

$$L_a^\alpha(y_1, \dots, y_m) = \begin{pmatrix} \frac{1}{e^{(f_1(a))^{-\alpha}}} & \dots & 0 \\ \dots & \dots & \dots \\ 0 & \dots & \frac{1}{e^{(f_m(a))^{-\alpha}}} \end{pmatrix} \begin{pmatrix} y_1 \\ \vdots \\ y_m \end{pmatrix}$$

and $\begin{pmatrix} N_3^\alpha f_1(a) & \dots & 0 \\ \dots & \dots & \dots \\ 0 & \dots & N_3^\alpha f_m(a) \end{pmatrix}$ is the non-conformable Jacobian of \vec{f} of order α at \vec{a} .

5. Non-Conformable Implicit Function Theorem

In this section, a non-conformable version of classical Implicit Function Theorem is obtained. The non-conformable implicit function result we prove concerns one equation and several variables.

Theorem 12. *Let $\alpha \in (0, 1]$, $F : X \rightarrow \mathbb{R}$ be a real valued function defined in an open set $X \subset \mathbb{R}^{n+1}$, such that for all $(x_1, \dots, x_n, y) \in X$, each $x_i, y > 0$, and the point $(a_1, \dots, a_n, b) \in X$. Suppose that*

$$i) F(a_1, \dots, a_n, b) = 0.$$

$$ii) F \in C_\alpha(X, \mathbb{R}).$$

$$iii) N_y^\alpha F(a_1, \dots, a_n, b) \neq 0.$$

Then there is a neighborhood, $U \subset \mathbb{R}^n$, of (a_1, \dots, a_n) such that there is a unique function $y = g(x_1, \dots, x_n)$ that satisfies

$$g(a_1, \dots, a_n) = b, F(x_1, \dots, x_n, g(x_1, \dots, x_n)) = 0, \forall (x_1, \dots, x_n) \in U.$$

Finally, $y = g(x_1, \dots, x_n)$ is C_α in U , and for every $i = 1, 2, \dots, n$, we have

$$N_{x_i}^\alpha g(x_1, \dots, x_n) = -\frac{N_{x_i}^\alpha F(x_1, \dots, x_n, g(x_1, \dots, x_n))e^{(g(x_1, \dots, x_n))^{-\alpha}}}{N_y^\alpha F(x_1, \dots, x_n, g(x_1, \dots, x_n))}. \quad (12)$$

Proof. Without loss of generality we shall assume that X is an open ball, $B((a_1, \dots, a_n, b), \varepsilon_0)$. Let $\rho \in (0, \varepsilon_0)$. If we call $\delta = \sqrt{(\varepsilon_0^2 - \rho^2)}$ it is verified that $\| (x_1, \dots, x_n) - (a_1, \dots, a_n) \| < \delta$ and $|y - b| < \rho$ implies

$$(x_1, \dots, x_n, y) \in B((a_1, \dots, a_n, b), \varepsilon_0).$$

Note that in particular if $|y - b| < \rho$ then $(a_1, \dots, a_n, y) \in B((a_1, \dots, a_n, b), \varepsilon_0)$. Since the function $y = F(a_1, \dots, a_n, y)$ is strictly monotone on $(b - \varepsilon_0, b + \varepsilon_0)$ and $F(a_1, \dots, a_n, b) = 0$, it follows that $F(a_1, \dots, a_n, b - \rho)$ and $F(a_1, \dots, a_n, b + \rho)$ have a different sign, [6]. Suppose that $F(a_1, \dots, a_n, b - \rho) < 0$ and $F(a_1, \dots, a_n, b + \rho) > 0$ (the same would be reasoned in the opposite case). By the continuity of F at $(a_1, \dots, a_n, b - \rho)$ and $(a_1, \dots, a_n, b + \rho)$, there exists $\delta' \in (0, \delta)$ (that depends of ρ), such that $\| (x_1, \dots, x_n) - (a_1, \dots, a_n) \| < \delta'$ implies $[F(x_1, \dots, x_n, b - \rho) < 0$ and $F(x_1, \dots, x_n, b + \rho) > 0]$. Since, the function $F(x_1, \dots, x_n, y)$ is continuous on the interval $[b - \rho, b + \rho]$, for all $(x_1, \dots, x_n) \in B((a_1, \dots, a_n), \delta')$, and using the classical Bolzano's Theorem it follows that there exist some $y_x \in (b - \rho, b + \rho)$ such that $F(x_1, \dots, x_n, y_x) = 0$, for each $x = (x_1, \dots, x_n)$. Furthermore, this value of y_x is unique, due to strict monotony of function $F(x_1, \dots, x_n, y)$. In other words, if we take $U = B((a_1, \dots, a_n), \delta')$, for each $(x_1, \dots, x_n) \in U$, there exists a unique $y = g(x_1, \dots, x_n)$ such that $F(x_1, \dots, x_n, y) = 0$. Now let's prove that g we can write $y = g(x_1, \dots, x_n)$ is a continuous function on $B((a_1, \dots, a_n), \delta')$. The continuity of

the function g at the point (a_1, \dots, a_n) is obvious, since for each $\rho > 0$ there exists a value $\delta' > 0$ such that $\|(x_1, \dots, x_n) - (a_1, \dots, a_n)\| < \delta'$ implies $|b - y_x| < \rho$ iff $|b - g(x_1, \dots, x_n)| < \rho$. To prove the continuity of the function g at any point $(x_1, \dots, x_n) \in B((a_1, \dots, a_n), \delta')$, simply substitute $B((a_1, \dots, a_n), \delta')$ for an open ball $B((x_1, \dots, x_n))$ contained in $B((a_1, \dots, a_n), \delta')$. Finally, let's show formula (11). Applying Non-conformable Chain Rule, to the equation $F(x_1, \dots, x_n, y) = 0$, we have

$$N_{x_i}^\alpha F(\vec{x}, g(\vec{x})) + N_y^\alpha F(\vec{x}, g(\vec{x})) \frac{1}{e^{(g(\vec{x}))^{-\alpha}}} N_{x_i}^\alpha g(\vec{x}) = 0 \quad (13)$$

for all $i = 1, 2, \dots, n$, where $\vec{x} = (x_1, \dots, x_n)$. Solving from this equation $N_{x_i}^\alpha g(\vec{x})$, we obtain (11). Also the right side of formula (11) is continuous, the continuity of the non-conformable partial derivatives $N_{x_i}^\alpha g(\vec{x})$ for all $i = 1, 2, \dots, n$, follows. \square

We will now see how Theorem 5.1 can be used to compute the non-conformable partial derivatives of implicit function of several variables.

Example 13. Consider the equation $F(x, y, z) = x^3 + 3y^2 + 4xz^2 - 3yz^2 - 5 = 0$ one solution of this equation is $(1, 1, 1)$. Clearly, F is C_α in an open ball, $B((1, 1, 1), \varepsilon_0)$, with $x, y, z > 0$, for some $\alpha \in (0, 1]$. Since $N_z^\alpha F(1, 1, 1) = \left[8xz e^{z^{-\alpha}} - 6yz e^{z^{-\alpha}} \right]_{(1,1,1)} = 2e \neq 0$.

Tells us that there is a neighbourhood, $U \subset \mathbb{R}^2$, of $(1, 1)$ such that there is a unique function $z = g(x, y)$ that satisfies $g(1, 1) = 1$ and $F(x, y, g(x, y)) = 0, \forall (x, y) \in U$. Moreover, $z = g(x, y)$ is C_α in U and

$$N_x^\alpha g(x, y) = -\frac{((3x^2 + 4z^2)e^{x^{-\alpha}})}{2(4x - 3y)z}, \quad N_y^\alpha g(x, y) = -\frac{(3(2y - z^2)e^{y^{-\alpha}})}{2(4x - 3y)z}.$$

Finally, we have $N_x^\alpha g(1, 1) = -7e/2$ and $N_y^\alpha g(1, 1) = -3e/2$.

6. An extension of the Second Method of Lyapunov

In the analysis of the stability of non-linear systems, the Second Method of Lyapunov has demonstrated its strength for more than 125 years. The technique is also called direct method because this method allows us to determine the stability and asymptotic stability of a system without explicitly integrating the nonlinear differential equation or system. Asymptotic stability is one of the stone areas of the qualitative theory of dynamical systems and is of fundamental importance in many applications of the theory in almost all fields where dynamical effects play a great role.

This method relies on the observation that asymptotic stability is very well linked to the existence of some functions, called Lyapunov's function, that is, a positive definite function, vanishing only on an invariant region and decreasing along those trajectories of the system not evolving in the invariant region. Lyapunov proved that the existence of a Lyapunov's function guarantees asymptotic stability and, for linear time-invariant systems, also showed the converse statement that asymptotic stability implies the existence of a Lyapunov's function in the region of stability.

In the case of non-linear autonomous systems, there are innumerable results and refinements. If we consider non-autonomous systems, the results are more complex and we must add additional conditions. It is therefore natural to ask whether the Second Method of Lyapunov can be extended to the case of non-integer derivatives. In the case of the global fractional derivatives (the classical ones) these extensions are far from being obtained, additional conditions must be imposed since the non-existence of a Chain Rule, makes it impossible to obtain the derivative of the Lyapunov Function along the solutions of the system considered, reason why different variants must be handled (in particular inequalities) that make possible the obtaining of similar results (see [4] for example).

In [7] we studied the stability of the Fractional Liénard Equation with derivative Caputo and, as we said, since the Chain Rule was not valid, the difficulties that we had to overcome were several.

In [1] the results obtained with Caputo fractional derivatives and Caputo fractional Dini derivatives of Lyapunov functions, are illustrated in examples. It is emphasized that in some cases these techniques cannot be used. In this regard, it can also be consulted [9].

We will show that if we consider local fractional derivatives, non-conformable in this case, similar results to those obtained in the Second Method of Lyapunov can be formulated in this framework. For this we consider the following equation:

$$N_3^\alpha(N_3^\alpha x) + a(t)g(x) = 0 \quad (14)$$

a natural generalization of the known equation:

$$x'' + a(t)g(x) = 0. \quad (15)$$

The prototype of the above equation is the so-called Emden-Fowler equation, which is used in mathematical physics, theoretical physics, and chemical physics. This equation has interesting mathematical and physical properties, and it has been investigated from various points of view, in particular, the solutions of this equation represent the Newton-Poisson gravitational potential of stars, such as the Sun, considered as spheres filled with polytropic gas.

The coefficient $a(t)$ is allowed to be negative for arbitrarily large values of t . Under this premise, in general not every solution to the second order nonlinear differential equation (14) is continuable throughout the entire half real axis. For this reason, and being the prolongability a property of paramount importance, we show that under natural conditions on the functions $a(t)$ and $g(x)$ of the equation (13), all the equations are continuable to the future.

Next to equation (13), we will consider the following equivalent system:

$$N_3^\alpha x(t) = y(t), \quad N_3^\alpha y(t) = -a(t)g(x), \quad (16)$$

with $a \in C([0, +\infty))$, $g \in C(\mathbb{R})$, $xg(x) > 0$ if $x \neq 0$ and $G(x) =_{N_3} J_0^\alpha g(s)$.

Later the following functions will be used

$$\left. \begin{aligned} b(t) &= \exp \left\{ -_{N_3} J_0^\alpha \left[\frac{N_3^\alpha a(s)_+}{a(s)} \right] (t) \right\}, \\ c(t) &= \exp \left\{ -_{N_3} J_0^\alpha \left[\frac{N_3^\alpha a(s)_-}{a(s)} \right] (t) \right\}. \end{aligned} \right\} \quad (17)$$

So

$$a(t) = b(t)c(t), \quad (18)$$

where $b(t)$ is non-increasing and $c(t)$ is non-decreasing function with $N_3^\alpha a(t)_+ = \max(N_3^\alpha a(t), 0)$ and $N_3^\alpha a(t)_- = \max(-N_3^\alpha a(t), 0)$, so that $N_3^\alpha a(t) = (N_3^\alpha a(t)_+) - (N_3^\alpha a(t)_-)$. Thus we can enunciate our result.

Theorem 14. *Under assumptions $a \in C([0, +\infty))$, $g \in C(\mathbb{R})$, $xg(x) > 0$ if $x \neq 0$, let a a continuous and positive function on $[0, +\infty)$ satisfying*

$$a(t) \rightarrow \infty, \quad t \rightarrow +\infty. \quad (19)$$

Then all solutions of (15) can be defined for all $t \geq t_0 > 0$.

Proof. We will develop an extension of Liapunov's Second Method in this proof. For this, we define the following functions.

$$W(t, x(t), y(t)) = b(t)V(t, x(t), y(t)) \quad (20)$$

where $b(t)$ is defined by (16) and V is given by

$$V(t, x(t), y(t)) = \frac{y^2}{2a(t)} + G(x) \quad (21)$$

where G is as before. Then along solutions of system (15), we have

$$N_3^\alpha W(t, x(t), y(t)) = V(t, x(t), y(t))N_3^\alpha b(t) + b(t)N_3^\alpha V(t, x(t), y(t))$$

and

$$N_3^\alpha V(t, x(t), y(t)) = -\frac{y^2}{2} \frac{N_3^\alpha a(t)}{a^2(t)}$$

Using (16), (17) and (18) we obtain

$$N_3^\alpha W(t, x(t), y(t)) \leq 0 \quad (22)$$

so W is non-increasing function. Suppose there is a non continuable solution of the system (15), i.e., suppose there is a time T for some solution of system (15), satisfying $\lim_{t \rightarrow T^-} |x(t)| = +\infty$. Now

$$b(T) \left[G(x) + \frac{y^2}{2M} \right] \leq W(t, x(t), y(t)) \leq W(t_0, x_0, y_0)$$

being $M = \max_{t \in [t_0, T]} a(t)$. From this we have $|y(t)|$ is uniformly bounded, say $|y(t)| \leq K$ for $t_0 \leq t \leq T$. But $N_3^\alpha x(t) = y(t)$ so $|x(t)| \leq x_0 + K(T - t_0)$. This completes the proof. \square

7. Epilogue

In this paper we have presented the first results related to the local non-conformable Fractional Calculus of several variables, as a necessary tool to expand the applications of this new mathematical area. We want to highlight the importance of the fundamentals presented here for the future development of this subject, both pure and applied. In particular, the Rule of the Chain and the Implicit Function Theorem, ensures that known results of the one-dimensional case can be extended in the immediate future (Taylor series, analysis of differentiability and its relation to the N -derivative, tangent plane, among others).

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On Nonlinear Fractional Neutral Differential Equation with the ψ -Caputo Fractional Derivative

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ABSTRACT: In this article, the solvability of fractional neutral differential equation involving ψ -Caputo fractional operator is considered using a Krasnoselskii's fixed point approach. Also, we establish the uniqueness of the solution under certain conditions. Ulam stabilities for the proposed problem are discussed. Finally, examples are displayed to aid the applicability of the theory results.

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Keywords and Phrases: Krasnoselskii's fixed point theory; ψ -Caputo operator; Neutral differential equation; Ulam stability; Existence of solution.

1. Introduction

Fractional calculus is strong tool of mathematical analysis that studies derivatives and integrals of fractional order. Fractional differential equations (FDE's, for short) are used in many fields of engineering and sciences such as dynamical of biological systems [12], economy [33], theory of control [7], automatic systems [36], signal processing [11], hydro-mechanics and non-linear elasticity [14, 32].

Various real life problems can be modeled as differential equation. The study of existence of solution of these differential equation is interest object of mathematical analysis. The fixed point theorems are powerful technique to obtain the existence of solution of these problem. There are many of fixed point theorems can be applied to obtain the solution of mathematical models [24, 25]. Krasnoselskii's and Banach fixed point theorems play an important role to obtain the existence of solution of a lot of mathematical problems [35].

In 1940, Ulam purposed new role of the stability analysis of the solutions for functional equations [34]. In the next year, Hyer [15] considered another type of stability in the Banach space which was more generalized than the kind of Ulam stability and applied this stability approach to obtain the stability certain conditions of some functional equations. After that, Rassias [27] considered another approach of stability, this approach is more improved than Hyers stability. Rassias used this approach to study stability of FDE's [16, 28].

Recently, many research articles study the Ulam stabilities, see [21, 20, 13, 10, 8, 2, 22, 3, 19, 17, 18, 30]. In 2011, Ardjouni and Djoudi [6] studied the stability for neutral ordinary differential equations via fixed points. In 2019, Akbulut and Tunc [1], established the stability of solutions of neutral ordinary differential equations with multiple time delay. In the same year, Niazi [26], discussed Ulam stabilities for nonlinear fractional neutral differential equations in Caputo sense via Picard operator.

There are many definitions are used to define the fractional derivative such as Riemann-Liouville, Caputo, Erdélyi-Kober and Hadamard [23]. More recently, Almeida [4] considers new investigation of the fractional operator and called it ψ -Caputo derivative. This new approach is more generalized than Riemann-Liouville, Caputo, Erdélyi-Kober and Hadamard derivative operator approaches. After one year, Almeida et al.[5] investigated the uniqueness of solution of initial value problem (I.V.P, for short) of FDE in ψ -Caputo sense.

In this paper, we discuss the existence and uniqueness of the following FDE with delay

$$\begin{cases} {}^*D_{0^+}^{\alpha, \psi}[x(t) - H(t, x(t - \vartheta(t)))] = F(x(t), x(t - \vartheta(t))); \\ \alpha \in (0, 1], t \in I = [0, 1]; \\ \text{subject to I.V.} \\ x(t) = \sigma(t), t \in [\rho, 0]; \end{cases} \quad (1)$$

where ${}^*D_{0^+}^{\alpha, \psi}$ is ψ -Caputo derivative operator, the delay $\rho = \inf\{t - \vartheta(t) : t \in [0, 1]\} \leq 0$, $\vartheta : \mathbb{R}^+ \rightarrow \mathbb{R}^+$ and $\sigma : [\rho, 0] \rightarrow \mathbb{R}$.

2. Preliminaries

In this section, we consider some facts and basic results. We recall the following definition [3].

Definition 2.1. Let $C([\rho, 1], \mathbb{R})$ be the vectorial space of all continuous functions $u : [\rho, 1] \rightarrow \mathbb{R}$. Clearly, $C([\rho, 1], \mathbb{R})$ is a complete normed space with the norm, $\|u\| = \max_{t \in [\rho, 1]} |u(t)|$. Therefore, $C^n([\rho, 1], \mathbb{R})$, $n \in \mathbb{N}$, be the vectorial space of all n -times continuous and differentiable functions from $[\rho, 1]$ to \mathbb{R} .

Next, we recall the definitions of ψ -fractional integral and derivative operators [4, 5].

Definition 2.2. Let $I = [0, 1]$ and $\psi \in C^n(I, \mathbb{R})$, be an increasing functions such that $\psi'(t) \neq 0$ for all $t \in I$. Consider an integrable function $u : I \rightarrow \mathbb{R}$. The

ψ -Riemann-Liouville fractional integral of order $\alpha > 0$, $\alpha \in \mathbb{R}$ of the function u is defined as

$$J_{0+}^{\alpha,\psi} u(t) = \frac{1}{\Gamma(\alpha)} \int_0^t \psi'(\zeta) (\psi(t) - \psi(\zeta))^{\alpha-1} u(\zeta) d\zeta,$$

and the ψ -Riemann-Liouville fractional derivative of order $\alpha > 0$, $\alpha \in \mathbb{R}$ of the function u is defined as

$$D_{0+}^{\alpha,\psi} u(t) = \frac{1}{\Gamma(n-\alpha)} \left(\frac{1}{\psi'(t)} \frac{d}{dt}\right)^n \int_0^t \psi'(\zeta) (\psi(t) - \psi(\zeta))^{n-\alpha-1} u(\zeta) d\zeta,$$

where $n = [\alpha] + 1$ and $[\alpha]$ denotes the integral part of α .

Definition 2.3. Let $\psi \in C^n(I, \mathbb{R})$, be an increasing function such that $\psi'(t) \neq 0$ for all $t \in I$. Consider an integrable function $u : I \rightarrow \mathbb{R}$. The ψ -Caputo fractional derivative of order $\alpha > 0$, $\alpha \in \mathbb{R}$ of the function u is defined as

$${}^*D_{0+}^{\alpha,\psi} u(t) = D_{0+}^{\alpha,\psi} [u(t) - \sum_{k=0}^{n-1} \frac{u_{\psi}^{[k]}(0)}{k!} (\psi(t) - \psi(0))^k],$$

where $n = [\alpha] + 1$, $[\alpha]$ denotes the integral part of α and $u_{\psi}^{[k]}(t) = \left(\frac{1}{\psi'(t)} \frac{d}{dt}\right)^k u(t)$.

We recall the following Lemma which was given in [5].

Lemma 2.4. Suppose that $u : I \rightarrow \mathbb{R}$, then

- (1) If $u \in C(I, \mathbb{R})$, then ${}^*D_{0+}^{\alpha,\psi} J_{0+}^{\alpha,\psi} u(t) = u(t)$.
- (2) If $u \in C^n(I, \mathbb{R})$, then

$$J_{0+}^{\alpha,\psi} {}^*D_{0+}^{\alpha,\psi} u(t) = u(t) - \sum_{k=0}^{n-1} \frac{u_{\psi}^{[k]}(0)}{k!} (\psi(t) - \psi(0))^k.$$

Now we recall Krasnoselskii's fixed point theorem which was given in [31].

Theorem 2.5. (Krasnoselskii's fixed point theorem) Let Υ be a Banach space. Suppose that Ω ($\Omega \neq \emptyset$) be a convex, bounded and closed subset of Υ . Consider $\mathcal{T}_1 : \Upsilon \rightarrow \Upsilon$ and $\mathcal{T}_2 : \Omega \rightarrow \Upsilon$ are such that

- (1) \mathcal{T}_1 be a contraction.
 - (2) \mathcal{T}_2 is completely continuous.
 - (3) $x = \mathcal{T}_1 x + \mathcal{T}_2 y \Rightarrow x \in \Omega$ for all $y \in \Omega$.
- Then, there exists $x^* \in \Omega$ such that $x^* = \mathcal{T}_1 x^* + \mathcal{T}_2 x^*$.

Now, we recall the definitions of these types of Ulam stability. For more details, see [29].

Definition 2.6. The Eq.(1) is said to be Ulam-Hyers stable (UHS for short) if, there exists $\lambda \in \mathbb{R}^+$ such that for every $\varepsilon > 0$ and each $u \in C([\rho, 1], \mathbb{R})$ solution of the inequality

$$|{}^*D_{0+}^{\alpha,\psi} [u(t) - H(t, u(t - \vartheta(t)))] - F(u(t), u(t - \vartheta(t)))| \leq \varepsilon, \quad t \in I,$$

there exists a unique solution $x \in C([\rho, 1], \mathbb{R})$ of Eq.(1) such that

$$|u(t) - x(t)| \leq \lambda \varepsilon, \quad \forall t \in [\rho, 1].$$

Definition 2.7. The Eq.(1) is said to be generalized Ulam-Hyers stable (GUHS for short) if, there exists $\varphi \in C([\rho, 1], \mathbb{R})$, $\varphi(0) = 0$, such that for every $\varepsilon > 0$ and each $u \in C([\rho, 1], \mathbb{R})$ solution of the inequality

$$|{}^*D_{0+}^{\alpha, \psi}[u(t) - H(t, u(t - \vartheta(t)))] - F(u(t), u(t - \vartheta(t)))| \leq \varepsilon, \quad t \in I,$$

there exists a unique solution $x \in C([\rho, 1], \mathbb{R})$ of Eq.(1) such that

$$|u(t) - x(t)| \leq \varphi(\varepsilon), \quad \forall t \in [\rho, 1].$$

Definition 2.8. The Eq.(1) is called Ulam-Hyers-Rassias stable (UHRS for short) w.r.t $\varphi \in C([\rho, 1], \mathbb{R})$, if there exists $\kappa_\varphi \in \mathbb{R}^+$ such that for every $\varepsilon > 0$ and each $u \in C([\rho, 1], \mathbb{R})$ solution of the inequality

$$|{}^*D_{0+}^{\alpha, \psi}[u(t) - H(t, u(t - \vartheta(t)))] - F(u(t), u(t - \vartheta(t)))| \leq \varepsilon \varphi(t), \quad t \in I, \quad (2)$$

there exists a unique solution $x \in C([\rho, 1], \mathbb{R})$ of Eq.(1) such that

$$|u(t) - x(t)| \leq \kappa_\varphi \varepsilon \varphi(t), \quad \forall t \in [\rho, 1].$$

Definition 2.9. The Eq.(1) is said to be generalized Ulam-Hyers-Rassias stable (GUHRS for short) w.r.t $\varphi \in C([\rho, 1], \mathbb{R})$, if there exists $\kappa_\varphi \in \mathbb{R}^+$ such that for each $u \in C([\rho, 1], \mathbb{R})$ solution of the inequalities

$$|{}^*D_{0+}^{\alpha, \psi}[u(t) - H(t, u(t - \vartheta(t)))] - F(u(t), u(t - \vartheta(t)))| \leq \varphi(t), \quad t \in I,$$

there exists a unique solution $u \in C([\rho, 1], \mathbb{R})$ of the Eq.(1) such that

$$|u(t) - x(t)| \leq \kappa_\varphi \varphi(t), \quad \forall t \in [\rho, t].$$

Let $H : I \times \mathbb{R} \rightarrow \mathbb{R}$ and $F : \mathbb{R} \times \mathbb{R} \rightarrow \mathbb{R}$. Then we study the Ulam stabilities of the following proposed problem

$$\begin{cases} {}^*D_{0+}^{\alpha, \psi}[x(t) - H(t, x(t - \vartheta(t)))] = F(x(t), x(t - \vartheta(t))); \\ \alpha \in (0, 1], \quad t \in I = [0, 1]; \\ \text{subject to initial value} \\ x(t) = \sigma(t), \quad t \in [\rho, 0]; \end{cases}$$

where ${}^*D^{\alpha, \psi}$ is ψ -Caputo derivative operator, $\rho = \inf\{t - \vartheta(t) : t \in [0, 1]\} \leq 0$, $\vartheta : \mathbb{R}^+ \rightarrow \mathbb{R}^+$, $\sigma : [\rho, 0] \rightarrow \mathbb{R}$ are continues and $\psi \in C^1(I, \mathbb{R})$ be an increasing function such that $\psi'(t) \neq 0$ for all $t \in I$. Then, we have the following lemma [9].

Lemma 2.10. *The solution of Eq.(1) is equivalent to the following nonlinear integral equation*

$$\begin{aligned} x(t) &= \sigma(0) - H(0, \sigma(-\vartheta(0))) + H(t, x(t - \vartheta(t))) \\ &\quad + \frac{1}{\Gamma(\alpha)} \int_0^t \psi'(s)(\psi(t) - \psi(s))^{\alpha-1} F(x(s), x(s - \vartheta(s))) ds. \end{aligned}$$

3. Existence and Uniqueness

In this section we will obtain the existence of solution and uniqueness of the proposed neutral FDE (1). suppose that $r_0 \in \mathbb{R}^+$ and $\Omega = \{x \in C([\rho, 1], \mathbb{R}) : \|x\| \leq r_0\}$. The Eq.(1) can be written as

$$(\mathcal{T}x)(t) = (\mathcal{T}_1x)(t) + (\mathcal{T}_2x)(t),$$

where

$$\mathcal{T}_1 : \Omega \rightarrow (CB([\rho, 1], \mathbb{R})) \quad , \quad \mathcal{T}_2 : \Omega \rightarrow (CB([\rho, 1], \mathbb{R})),$$

such that

$$\begin{aligned} (\mathcal{T}_1x)(t) &= \sigma(0) - H(0, \sigma(-\vartheta(0))) + H(t, x(t - \vartheta(t))), \\ (\mathcal{T}_2x)(t) &= \frac{1}{\Gamma(\alpha)} \int_0^t \psi'(s)(\psi(t) - \psi(s))^{\alpha-1} F(x(s), x(s - \vartheta(s))) ds, \end{aligned}$$

where $t \in [\rho, 1]$ and $x \in C([\rho, 1], \mathbb{R})$.

We will study Eq.(1) under the following conditions:

(C1) the functions $H : I \times \mathbb{R} \rightarrow \mathbb{R}$ and $F : \mathbb{R} \times \mathbb{R} \rightarrow \mathbb{R}$ are continuous and there exist $p \in (0, 1), q \in \mathbb{R}^+$ such that

$$|H(t, x_1) - H(t, x_2)| < L|x_1 - x_2|,$$

$$|F(x_1, y_1) - F(y_1, y_2)| < K \sum_{i=1}^2 |x_i - y_i|,$$

for all $x_1, x_2, y_1, y_2 \in \mathbb{R}$, and $t \in [0, 1]$;

(C2) let $A^* = |F(0, 0)|$ and $B^* = \max_{t \in I} |H(t, 0)|$ then

$$|\sigma(0) - H(0, \sigma(-\vartheta(0)))| + L r_0 + B^* + \frac{K r_0 + A^*}{\Gamma(\alpha + 1)} (\psi(1) - \psi(0))^\alpha \leq r_0.$$

Theorem 3.1. *Let the conditions (C1) and (C2) hold. Then Eq.(1) has at least one solution in Ω .*

Proof. The proof is done in the following 3 steps.

Step 1. \mathcal{T}_1 is contraction.

Let $x, y \in C([\rho, 1], \mathbb{R})$ are arbitrary and $t \in I$

$$|(\mathcal{T}_1x)(t) - (\mathcal{T}_1y)(t)| \leq L|x(t) - y(t)|,$$

which implies that

$$\|\mathcal{T}_1x - \mathcal{T}_1y\| \leq L\|x - y\|,$$

Thus, \mathcal{T}_1 is a contraction.

Step 2. \mathcal{T}_2 is completely continuous.

First, we will prove that \mathcal{T}_2 is continuous. Let $\{x_n\}$ be a sequence in $C([\rho, 1], \mathbb{R})$ such that $x_n \rightarrow x \in C([\rho, 1], \mathbb{R})$. Then, we get

$$\begin{aligned}
|(\mathcal{T}_2 x_n)(t) - (\mathcal{T}_2 x)(t)| &\leq \frac{1}{\Gamma(\alpha)} \int_0^t \psi'(s) (\psi(t) - \psi(s))^{\alpha-1} |F(x_n(s), x_n(s - \vartheta(s))) \\
&\quad - F(x(s), x(s - \vartheta(s)))| ds \\
&\leq \frac{K}{\Gamma(\alpha+1)} (\psi(t) - \psi(0))^\alpha \|x_n - x\| \\
&\leq \frac{K}{\Gamma(\alpha+1)} (\psi(1) - \psi(0))^\alpha \|x_n - x\|.
\end{aligned}$$

So, we have that

$$\|\mathcal{T}_2 x_n - \mathcal{T}_2 x\| \leq \frac{K}{\Gamma(\alpha+1)} (\psi(1) - \psi(0))^\alpha \|x_n - x\|.$$

Thus, $\|\mathcal{T}_2 x_n - \mathcal{T}_2 x\| \rightarrow 0$ as $n \rightarrow \infty$. Hence \mathcal{T}_2 is continuous operator. Therefore, for each $x, y \in C([\rho, 1], \mathbb{R})$ and $t \in I$, we have

$$\begin{aligned}
|F(x(t), y(t))| &\leq |F(x, y) - F(0, 0)| + |F(0, 0)| \\
&\leq K(\|x\| + \|y\|) + A^*.
\end{aligned}$$

Therefore,

$$\begin{aligned}
|(\mathcal{T}_2 x)(t)| &\leq \frac{1}{\Gamma(\alpha)} \int_0^t \psi'(s) (\psi(t) - \psi(s))^{\alpha-1} |F(x(s), x(s - \vartheta(s)))| ds \\
&\leq \frac{2K+A^*}{\Gamma(\alpha+1)} (\psi(t) - \psi(0))^\alpha,
\end{aligned}$$

for all $t \in I$. Hence we have

$$\|\mathcal{T}_2 x\| \leq \frac{2K+A^*}{\Gamma(\alpha+1)} (\psi(1) - \psi(0))^\alpha.$$

Thus \mathcal{T}_2 is bounded. Furthermore, if we choose $t_1, t_2 \in I$ such that $t_1 < t_2$, then we get

$$\begin{aligned}
&|(\mathcal{T}_2 x)(t_2) - (\mathcal{T}_2 x)(t_1)| \\
&= \left| \frac{1}{\Gamma(\alpha)} \int_0^{t_2} \psi'(s) (\psi(t_2) - \psi(s))^{\alpha-1} F(x(s), x(s - \vartheta(s))) ds \right. \\
&\quad \left. - \frac{1}{\Gamma(\alpha)} \int_0^{t_1} \psi'(s) (\psi(t_1) - \psi(s))^{\alpha-1} F(x(s), x(s - \vartheta(s))) ds \right| \\
&\leq \left| \frac{1}{\Gamma(\alpha)} \int_0^{t_2} \psi'(s) (\psi(t_2) - \psi(s))^{\alpha-1} F(x(s), x(s - \vartheta(s))) ds \right. \\
&\quad \left. - \frac{1}{\Gamma(\alpha)} \int_0^{t_2} \psi'(s) (\psi(t_1) - \psi(s))^{\alpha-1} F(x(s), x(s - \vartheta(s))) ds \right| \\
&\quad + \left| \frac{1}{\Gamma(\alpha)} \int_0^{t_2} \psi'(s) (\psi(t_1) - \psi(s))^{\alpha-1} F(x(s), x(s - \vartheta(s))) ds \right. \\
&\quad \left. - \frac{1}{\Gamma(\alpha)} \int_0^{t_1} \psi'(s) (\psi(t_1) - \psi(s))^{\alpha-1} F(x(s), x(s - \vartheta(s))) ds \right| \\
&\leq \frac{1}{\Gamma(\alpha)} \int_0^{t_2} \psi'(s) [(\psi(t_2) - \psi(s))^{\alpha-1} - (\psi(t_1) - \psi(s))^{\alpha-1}] |F(x(s), x(s - \vartheta(s)))| ds \\
&\quad + \frac{1}{\Gamma(\alpha)} \int_{t_1}^{t_2} \psi'(s) (\psi(t_1) - \psi(s))^{\alpha-1} |F(x(s), x(s - \vartheta(s)))| ds \\
&\leq \frac{2K}{\Gamma(\alpha+1)} [(\psi(t_2) - \psi(0))^\alpha - (\psi(t_1) - \psi(0))^\alpha].
\end{aligned}$$

Since ψ is continuous, then we have that $|(\mathcal{T}_2 x)(t_2) - (\mathcal{T}_2 x)(t_1)| \rightarrow 0$ as $t_1 \rightarrow t_2$. Thus $\mathcal{T}_2(\Omega)$ is relatively compact. From Arzela-Ascoli-theorem, we obtain \mathcal{T}_2 is compact. Hence \mathcal{T}_2 is completely continuous.

Step 3. Finding the fixed point of \mathcal{T} .

Let $x, y \in \Omega$. We get

$$\begin{aligned} & |(\mathcal{T}_1x)(t) + (\mathcal{T}_2y)(t)| \\ &= |\sigma(0) - H(0, \sigma(-\vartheta(0))) + H(t, x(t - \vartheta(t))) \\ &\quad + \frac{1}{\Gamma(\alpha)} \int_0^t \psi'(s)(\psi(t) - \psi(s))^{\alpha-1} F(x(s), x(s - \vartheta(s))) ds| \\ &\leq |\sigma(0) - H(0, \sigma(-\vartheta(0)))| + |H(t, x(t - \vartheta(t)))| + \\ &\quad + |\frac{1}{\Gamma(\alpha)} \int_0^t \psi'(s)(\psi(t) - \psi(s))^{\alpha-1} F(x(s), x(s - \vartheta(s))) ds| \\ &\leq |\sigma(0) - H(0, \sigma(-\vartheta(0)))| + L r_0 + B^* + \frac{K r_0 + A^*}{\Gamma(\alpha+1)} (\psi(t) - \psi(0))^\alpha \\ &\leq |\sigma(0) - H(0, \sigma(-\vartheta(0)))| + L r_0 + B^* + \frac{K r_0 + A^*}{\Gamma(\alpha+1)} (\psi(1) - \psi(0))^\alpha \\ &\leq r_0 \end{aligned}$$

Thus, the operators \mathcal{T}_1 and \mathcal{T}_2 satisfy all conditions of Theorem 2.5. Hence there exists $x^* \in \Omega$ such that x^* is solution of Eq.(1). □

Theorem 3.2. *Suppose that the conditions (C1) and (C2) hold. Let,*

$$(C3) \quad L + \frac{2K}{\Gamma(\alpha+1)} (\psi(1) - \psi(0))^\alpha < 1.$$

Then the Eq.(1) has unique solution.

Proof. We apply Banach contraction theorem to prove \mathcal{T} has a unique fixed point. Let $x, y \in C([\rho, 1], \mathbb{R})$. Then, we have

$$\begin{aligned} |(\mathcal{T}x)(t) - (\mathcal{T}y)(t)| &\leq L|x(t) - y(t)| + \frac{2K\|x-y\|}{\Gamma(\alpha)} \int_0^t \psi'(s)(\psi(t) - \psi(s))^{\alpha-1} ds \\ &\leq L + \frac{2K}{\Gamma(\alpha+1)} (\psi(t) - \psi(0))^\alpha \\ &\leq L + \frac{2K}{\Gamma(\alpha+1)} (\psi(1) - \psi(0))^\alpha \\ &\leq 1. \end{aligned}$$

Thus Eq.(1) has unique solution. □

4. Ulam Stabilities

In this part, various Ulam stability types will be considered.

Lemma 4.1. *Let $\alpha \in (0, 1)$, if $z \in C([\rho, 1], \mathbb{R})$ is the solution of the inequality of definition 2.6, then z is the solution of the following inequality*

$$|z(t) - N(t)| \leq \left(\frac{(\psi(1) - \psi(0))^\alpha}{\Gamma(\alpha+1)} \right) \varepsilon,$$

where

$$\begin{aligned} N(t) &= \sigma(0) - H(0, \sigma(-\vartheta(0))) + H(t, z(t - \vartheta(t))) \\ &\quad + \frac{1}{\Gamma(\alpha)} \int_0^t \psi'(s)(\psi(t) - \psi(s))^{\alpha-1} F(z(s), z(s - \vartheta(s))) ds. \end{aligned}$$

Proof. Let $z \in C([\rho, 1], \mathbb{R})$ be any solution of the inequality of definition 2.6, then there exists $\Theta \in C([\rho, 1], \mathbb{R})$ dependent on z such that

$$\begin{cases} *D_{0+}^{\alpha, \psi} [z(t) - H(t, z(t - \vartheta(t)))] = F(z(t), z(t - \vartheta(t))) + \Theta(t); \\ \alpha \in (0, 1], t \in I = [0, 1]; \\ \text{subject to initial value} \\ z(t) = \sigma(t), t \in [\rho, 0]; \end{cases} \quad (3)$$

and

$$|\Theta(t)| \leq \varepsilon, \forall t \in I.$$

Thus, Eq.(3) is equivalent to the following equation

$$\begin{aligned} z(t) = & \sigma(0) - H(0, \sigma(-\vartheta(0))) + H(t, z(t - \vartheta(t))) \\ & + \frac{1}{\Gamma(\alpha)} \int_0^t \psi'(s) (\psi(t) - \psi(s))^{\alpha-1} F(z(s), z(s - \vartheta(s))) ds \\ & + \frac{1}{\Gamma(\alpha)} \int_0^t \psi'(s) (\psi(t) - \psi(s))^{\alpha-1} \Theta(s) ds. \end{aligned}$$

Let

$$\begin{aligned} N(t) = & \sigma(0) - H(0, \sigma(-\vartheta(0))) + H(t, z(t - \vartheta(t))) \\ & + \frac{1}{\Gamma(\alpha)} \int_0^t \psi'(s) (\psi(t) - \psi(s))^{\alpha-1} F(z(s), z(s - \vartheta(s))) ds. \end{aligned}$$

Thus, we have

$$|z(t) - N(t)| \leq \frac{1}{\Gamma(\alpha)} \int_0^t \psi'(s) (\psi(t) - \psi(s))^{\alpha-1} |\Theta(s)| ds \leq \frac{1}{\Gamma(\alpha+1)} (\psi(1) - \psi(0))^\alpha \varepsilon.$$

□

Theorem 4.2. *Suppose that (C1)-(C3) hold. Then the Eq.(1) is UHS and consequently GUHS.*

Proof. Let $z \in C([\rho, 1], \mathbb{R})$ be a solution of the inequality of definition 2.6 and x be the unique solution of Eq.(1), then we get $|N(t)| \leq \varepsilon$ for all $t \in I$ and

$$|z(t) - x(t)| \leq |z(t) - N(t)| + |N(t) - x(t)|.$$

From Lemma 4.1, we get

$$\begin{aligned} |z(t) - x(t)| & \leq \left(\frac{(\psi(1) - \psi(0))^\alpha}{\Gamma(\alpha+1)} \right) \varepsilon_1 + L |z(t) - x(t)| \\ & \quad + \frac{1}{\Gamma(\alpha)} \int_0^t \psi'(s) (\psi(t) - \psi(s))^{\alpha-1} 2K |z(s) - x(s)| ds \\ & \leq \left(\frac{(\psi(1) - \psi(0))^\alpha}{\Gamma(\alpha+1)} \right) \varepsilon_1 + L |z(t) - x(t)| + \frac{2K}{\Gamma(\alpha+1)} (\psi(1) - \psi(0))^\alpha |z(t) - x(t)|, \end{aligned}$$

therefore, we get

$$\|z - x\| \leq \left(\frac{(\psi(1) - \psi(0))^\alpha}{\Gamma(\alpha+1)} \right) \varepsilon_1 + \mathbb{L} \|z - x\|,$$

where

$$\mathbb{L} = 1 - \left[L + \frac{2K}{\Gamma(\alpha+1)} (\psi(1) - \psi(0))^\alpha \right].$$

Then , we get

$$\|z - x\| \leq \lambda \varepsilon,$$

where

$$\lambda = \frac{\left(\frac{(\psi(1)-\psi(0))^\alpha}{\Gamma(\alpha+1)}\right)}{1 - \mathbb{L}}.$$

Thus the Eq.(1) is UHS. Therefore, if we put $\varphi(\varepsilon) = \lambda \varepsilon$, then we get that $\varphi(0) = 0$ and

$$\|z - x\| \leq \varphi(\varepsilon).$$

Then, the Eq.(1) is GUHS. □

Lemma 4.3. *Suppose that the following condition holds:*

(C4) *If $\phi \in C([\rho, 1], \mathbb{R})$ is increasing, then there exists $\mu_\phi \in \mathbb{R}^+$ such that for every $t \in I$, the following inequality hold*

$$*J_{0+}^{\alpha,\psi} \phi(t) \leq \mu_\phi \phi(t) .$$

If $z \in C([\rho, 1], \mathbb{R})$ is the solution of the inequality (2), then z is the solution of the following inequality

$$|z(t) - N(t)| \leq \mu_\phi \left(\frac{(\psi(1)-\psi(0))^\alpha}{\Gamma(\alpha+1)}\right) \phi(t) \varepsilon.$$

Proof. From Lemma 4.1, we get

$$|z(t) - N(t)| \leq \frac{1}{\Gamma(\alpha)} \int_0^t \psi'(s) (\psi(t) - \psi(s))^{\alpha-1} |\Theta(s)| ds.$$

From (C4), we have that

$$|z(t) - N(t)| \leq \mu_\phi \left(\frac{(\psi(T)-\psi(0))^\alpha}{\Gamma(\alpha+1)}\right) \phi(t) \varepsilon.$$

□

Theorem 4.4. *Consider the Conditions (C1)-(C4) hold. Then the Eq.(1) is UHRS and GUHRS .*

Proof. Let $z \in C([\rho, 1], \mathbb{R})$ be solution of the inequality (2) and x be the unique solution of Eq.(1). From Lemma 4.3, we get

$$\|z - x\| \leq \mu_\phi \left(\frac{(\psi(1)-\psi(0))^\alpha}{\Gamma(\alpha+1)}\right) \phi_1(t) \varepsilon + \mathbb{L} \|z - x\|.$$

So, we have that

$$\|z - x\| \leq \mu_\phi \lambda \phi(t) \varepsilon.$$

Thus the Eq.(1) is UHRS. Therefore, if we put $\varepsilon = 1$, then the Eq.(1) is GUHRS. □

5. Applications

The following examples are applications to the previous theoretical results.

Example 5.1. Consider the following ψ -Caputo FDE

$$\begin{cases} {}^*D_{0^+}^{\frac{1}{3}, \psi} [x(t) - \frac{te^{-t}}{10}x(t-0.1)] = \frac{1}{10} \tan^{-1}(x(t)) + \frac{|x(t-0.1)|}{14+|x(t-0.1)|}; \\ t \in I = [0, 1], \\ \text{subject to the nonlocal conditions} \\ x(t) = 0.2 \quad , t \in [-0.1, 0] \end{cases} \quad (4)$$

where $\psi(t) = \sqrt{1+t}$, for all $t \in [0, 1]$. Clearly, ψ is increasing on $[0, 1]$ and $\psi \in C^1([0, 1], \mathbb{R})$. Therefore,

$$H(t, x) = \frac{te^{-t}}{10}x,$$

also

$$F(t, x, y) = \frac{1}{10} \tan^{-1}(x) + \frac{|y|}{14+|y|}.$$

It is clear that, H, F are continuous. Since,

$$|H(t, x_1) - H(t, x_2)| \leq \frac{1}{10}|x_1 - x_2|,$$

$$|F(t, x_1, y_1) - F(t, x_2, y_2)| \leq \frac{1}{10}(|x_1 - x_2| + |y_1 - y_2|),$$

for all $x, y, x_1, y_1, x_2, y_2 \in \mathbb{R}$ and $t \in [0, 1]$. Thus, the condition (C1) holds with

$$L = K = \frac{1}{10},$$

therefore

$$A^* = 0 \quad , \quad B^* = 0 \quad , \quad \sigma(0) = 0.2.$$

The inequality of (C2)

$$|\sigma(0) - H(0, \sigma(-\vartheta(0)))| + L r_0 + B^* + \frac{K r_0 + A^*}{\Gamma(\alpha + 1)} (\psi(1) - \psi(0))^\alpha \leq r_0,$$

has the following form

$$0.2 + \frac{r_0}{10} + \frac{r_0(\sqrt{2}-1)^{\frac{1}{3}}}{10\Gamma(\frac{4}{3})} \leq r_0.$$

Hence (C2) is hold and $r_0 \geq 0.2447531$. Similarly, we get: $L + \frac{2K}{\Gamma(\alpha+1)} (\psi(1) - \psi(0))^\alpha = 0.11657002573 < 1$. Hence the condition (C3) holds. So, it is implies that, the Eq.(4) has a unique solution. Hence, the Eq.(4) is UHS, GUHS, UHRS and GUHRS.

Example 5.2. Consider the following ψ -Caputo FDE

$$\begin{cases} {}^*D_{0^+}^{\frac{1}{2}, \psi} [x(t) - \frac{t}{9} \sin(x(t - 0.1))] = \frac{1}{12}x(t) + \frac{1}{10}x(t - 0.1) \\ t \in I = [0, 1], \\ \text{subject to the nonlocal conditions} \\ x(t) = 0.2 \quad , t \in [-0.1, 0] \end{cases} \quad (5)$$

where $\psi(t) = \frac{t^2+t}{2}$, for all $t \in [0, 1]$. Clearly, ψ is increasing on $[0, 1]$ and $\psi \in C^1([0, 1], \mathbb{R})$. Therefore,

$$H(t, x) = \frac{t}{9} \sin(x),$$

also

$$F(t, x, y) = \frac{1}{12}x + \frac{1}{10}y.$$

It is clear that, H, F are continuous. Since,

$$|H(t, x_1) - H(t, x_2)| \leq \frac{1}{9}|x_1 - x_2|,$$

$$|F(t, x_1, y_1) - F(t, x_2, y_2)| \leq \frac{1}{10}(|x_1 - x_2| + |y_1 - y_2|),$$

for all $x, y, x_1, y_1, x_2, y_2 \in \mathbb{R}$ and $t \in [0, 1]$. Thus, the conditions (C1) holds with

$$L = \frac{1}{9} \quad , K = \frac{1}{10}$$

therefore

$$A^* = 0 \quad , \quad B^* = 0 \quad , \quad \sigma(0) = 0.2.$$

The inequality of (C2)

$$|\sigma(0) - H(0, \sigma(-\vartheta(0)))| + L r_0 + B^* + \frac{K r_0 + A^*}{\Gamma(\alpha + 1)} (\psi(1) - \psi(0))^\alpha \leq r_0,$$

has the following form

$$0.2 + \frac{r_0}{10} + \frac{r_0}{10\Gamma(\frac{3}{2})} \leq r_0.$$

Hence (C2) is hold and $r_0 \geq 0.25390377047$. Similarly, we get: $L + \frac{2K}{\Gamma(\alpha+1)} (\psi(1) - \psi(0))^\alpha = 0.32471910112 < 1$. Hence the condition (C3) holds . So, it is implies that, the Eq.(5) has a unique solution. Hence, the Eq.(5) is UHS, GUHS, UHRS and GUHRS.

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Boolean Algebra of One-Point Local Compactifications

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ABSTRACT: For a given locally compact Hausdorff space we introduce a Boolean algebra structure on the family of all its one-point local compactifications.

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Keywords and Phrases: Local; Compactification; Boolean; Algebra; Ends.

1. Introduction

Every locally compact, noncompact Hausdorff space X has a well known one-point compactification (Alexandroff compactification, [1]). In this paper we consider the set $\mathcal{B}(X)$ of all one-point local compactifications of X up to an equivalence. We prove that $\mathcal{B}(X)$ is a partially ordered set such that the order \leq induces a Boolean algebra. Moreover, the elements 0 and 1 of $\mathcal{B}(X)$ are respectively X and ωX . Then we focus on describing the algebra we get using topological notions and convergence and we provide examples by computing the algebra in some special cases. We also note the connection with the topic of ends of manifolds (see [2, pages 110-118]), as for a noncompact, connected, second countable manifold L with n ends, $n < \infty$, we have $|\mathcal{B}(L)| = 2^n$.

2. Notation and terminology

- Throughout the paper, ZFC is assumed.
- Given a locally compact Hausdorff space X we denote by ωX a one-point compactification of X if X is not compact and X otherwise,

- a clopen set is a set that is both closed and open,
- if Y is a one-point local compactification different from X , the unique point of $Y \setminus X$ will be denoted by ∞_Y ,
- a filter \mathcal{F} of open sets in a topological space X is a non-empty family of sets open in X such that $\emptyset \notin \mathcal{F}$ and, for all $V_1, V_2 \in \mathcal{F}$ and an open $V \subset X$ we have $V_1 \cap V_2 \in \mathcal{F} \Rightarrow V \in \mathcal{F}$.

3. Main results

Definition 1. If X is a locally compact Hausdorff space, we call (Y, f) an at most one-point local compactification of X iff Y is a locally compact Hausdorff and $f : X \rightarrow Y$ is a homeomorphic embedding such that $f(X)$ is dense in Y and $|Y \setminus f(X)| \leq 1$. If (Y, f) is an at most one point local compactification of X and $|Y \setminus f(X)| = 1$, we call (Y, f) a one-point local compactification of X .

For simplicity, we say that Y is a/an (at most) one-point local compactification of X iff (Y, id_X) is a/an (at most) one-point local compactification of X .

Definition 2. Let X be a locally compact Hausdorff space, (Y_1, f_1) and (Y_2, f_2) its at most one-point local compactifications. We will write $(Y_1, f_1) \leq (Y_2, f_2)$ (or, for simplicity, $Y_1 \leq Y_2$) iff one of the following conditions apply:

- $f_1(X) = Y_1$
- $Y_1 = f_1(X) \cup \{\infty_{Y_1}\}$, $Y_2 = f_2(X) \cup \{\infty_{Y_2}\}$ and the function

$$Y_1 \ni x \mapsto \begin{cases} f_2(f_1^{-1}(x)), & x \in f_1(X) \\ \infty_{Y_2}, & x = \infty_{Y_1} \end{cases} \in Y_2$$

is continuous.

Note that \leq is reflexive and transitive, with $0 = X$ and $1 = \omega X$. We can define an equivalence relation \equiv by

$$(Y_1, f_1) \equiv (Y_2, f_2) \text{ iff } (Y_1, f_1) \leq (Y_2, f_2) \text{ and } (Y_2, f_2) \leq (Y_1, f_1),$$

or, for simplicity,

$$Y_1 \equiv Y_2 \text{ iff } Y_1 \leq Y_2 \text{ and } Y_2 \leq Y_1.$$

We also define

$$\mathcal{B}(X) := \{Y\text{---one-point local compactification of } X\} / \equiv.$$

From now on instead of an equivalence class of Y in $\mathcal{B}(X)$ we will just write Y .

We are now ready to state the first result where we will prove that $\mathcal{B}(X)$ ordered by \leq is a Boolean algebra, by showing that it is in fact order isomorphic to a much simpler one.

Theorem 1. *Given a locally compact Hausdorff space X , $\mathcal{B}(X)$ is a partially ordered space with a lattice such that the order \leq induces a Boolean algebra, i.e., for Y_1, Y_2 one-point local compactifications of X :*

- $Y_1 \vee Y_2 = \sup_{\leq} \{Y_1, Y_2\}$,
- $Y_1 \wedge Y_2 = \inf_{\leq} \{Y_1, Y_2\}$,
- $0 = X$,
- $1 = \omega X$,
- for any space $Y \in \mathcal{B}(X)$ there exists a unique space $\setminus Y \in \mathcal{B}(X) : Y \wedge \setminus Y = 0, Y \vee \setminus Y = 1$.

In particular, $0 = 1$ iff X is compact.

Proof. First consider βX , a Čech–Stone compactification of X . We define $\mathcal{A}(X) := \{F \subset \beta X \setminus X : F \text{ clopen in } \beta X \setminus X\}$ (note that $\beta X \setminus X$ is compact). $\mathcal{A}(X)$ with standard set operations is a Boolean algebra. We will show an isomorphism between $\mathcal{B}(X)$ and $\mathcal{A}(X)$, proving that $\mathcal{B}(X)$ is also a Boolean algebra.

To this end, we will define $f : \mathcal{B}(X) \rightarrow \mathcal{A}(X)$. If X is compact, both $\mathcal{B}(X)$ and $\mathcal{A}(X)$ are trivial, therefore assume that X is not compact. Consider a clopen in $\beta X \setminus X$ set F such that $\emptyset \neq F \neq \beta X \setminus X$. We can now identify F and $(\beta X \setminus X) \setminus F$ with points, getting a compact space $X \cup \{\{F\}\} \cup \{\{(\beta X \setminus X) \setminus F\}\}$. Its subspace $X \cup \{\{F\}\}$ is then a one-point local compactification of X . Conversely, for any one-point local compactification Y of X there exists a unique clopen in $\beta X \setminus X$ set F_Y such that Y is equivalent with $X \cup \{\{F_Y\}\}$ (from the universal property of βX). We define $f(X) = \emptyset$ and for every one-point local compactification Y of X we put $f(Y) = F_Y$, where Y is the unique clopen in $\beta X \setminus X$ set such that Y is equivalent to $X \cup \{\{F_Y\}\}$. It can be easily seen that for one-point local compactifications Y_1, Y_2 of X we have $Y_1 \leq Y_2$ iff $F_{Y_1} \subset F_{Y_2}$, so f preserves the partial order and is indeed an isomorphism. Furthermore, for one-point local compactifications Y_1, Y_2 of X we have:

1. $Y_1 \vee Y_2 = X \cup \{\{F_{Y_1} \cup F_{Y_2}\}\}$.
2. $Y_1 \wedge Y_2 = X \cup \{\{F_{Y_1} \cap F_{Y_2}\}\}$ if $F_{Y_1} \cap F_{Y_2} \neq \emptyset$ and $Y_1 \wedge Y_2 = X$ otherwise.
3. $\setminus Y = X \cup \{\{(\beta X \setminus X) \setminus F_Y\}\}$ for $\emptyset \neq F_Y \neq \beta X \setminus X$.

□

Remark 1. The proof of Theorem 1 shows that $\mathcal{B}(X)$ is isomorphic (as a Boolean algebra) to the algebra of all clopen subsets of the remainder $\beta X \setminus X$ of X . One easily concludes that the Stone space of $\mathcal{B}(X)$ is homeomorphic to the space of all connected components of $\beta X \setminus X$ (that is, the space obtained from $\beta X \setminus X$ by identifying points that lie in a common connected component).

Now that we know that $\mathcal{B}(X)$ is a Boolean algebra, we will focus on describing it without using $\mathcal{A}(X)$. If we add a point $\{\infty_Y\}$ to a locally compact Hausdorff space X to get its one-point local compactification Y , we only need to know the neighborhood basis at $\{\infty_Y\}$ to know its topology. To this end, let us introduce the following characterization. For simplicity, we will also use one more definition.

Definition 3. Let X be a locally compact Hausdorff space, Y its one-point local compactification. Then

$$\tau(Y) := \{U \setminus \{\infty_Y\} : U \text{ open neighborhood of } \infty_Y \text{ in } Y\}.$$

$\tau(Y)$ uniquely determines $Y \neq X$, $Y \in \mathcal{B}(X)$.

Proposition 1. Let X be a locally compact Hausdorff space, $Y_1, Y_2 \in \mathcal{B}(X)$, $Y_1, Y_2 \neq 0$, $Y_1, Y_2 \neq 1$.

1. $\tau(Y_1 \wedge Y_2) = \{U_1 \cap U_2 : U_1 \in \tau(Y_1), U_2 \in \tau(Y_2)\}$, provided that the sets $U_1 \cap U_2$ are nonempty for all $U_1 \in \tau(Y_1), U_2 \in \tau(Y_2)$ and $Y_1 \wedge Y_2 = 0$ otherwise.
2. $\tau(Y_1 \vee Y_2) = \{U_1 \cup U_2 : U_1 \in \tau(Y_1), U_2 \in \tau(Y_2)\} = \tau(Y_1) \cap \tau(Y_2)$.
3. $\tau(\setminus Y_1) = \{X \setminus F : F \subset X, \text{ for any } U \in \tau(Y_1) \text{ } F \setminus U \text{ compact}\}$.

Or, in terms of convergence:

- (a) A net $(x_\gamma) \subset X$ in $Y_1 \wedge Y_2$ is convergent to $\infty_{Y_1 \wedge Y_2}$ iff (x_γ) is convergent to ∞_{Y_1} in Y_1 and to ∞_{Y_2} in Y_2 , and $Y_1 \wedge Y_2 = 0$ if there is no such net.
- (b) A net $(x_\gamma) \subset X$ in $Y_1 \vee Y_2$ is convergent to $\infty_{Y_1 \vee Y_2}$ iff every subnet of (x_γ) has a subnet convergent to ∞_{Y_1} in Y_1 or to ∞_{Y_2} in Y_2 .
- (c) A net $(x_\gamma) \subset X$ in $\setminus Y_1$ is convergent to $\infty_{\setminus Y_1}$ iff (x_γ) has no convergent subnets in Y_1 .

Proof. Again, let βX be a Čech–Stone compactification of X .

Note that if Y is a one-point local compactification of X and F_Y is a clopen set in $\beta X \setminus X$ such that Y is equivalent with $X \cup \{F_Y\}$, then

$$\tau(Y) = \{X \cap U : U \supset F_Y \text{ and } U \text{ open in } \beta X\}. \quad (*)$$

Following this notation consider F_{Y_1} and F_{Y_2} such that Y_1 and Y_2 are equivalent to $X \cup \{F_{Y_1}\}$ and $X \cup \{F_{Y_2}\}$ respectively.

Property (2) follows easily from (*).

To see that $\{U_1 \cup U_2 : U_1 \in \tau(Y_1), U_2 \in \tau(Y_2)\} = \tau(Y_1) \cap \tau(Y_2)$, take any $U_1 \in \tau(Y_1), U_2 \in \tau(Y_2)$. $U_2 = (U_2 \cup \{\infty_{Y_2}\}) \cap X$ is open in X , and thus open in Y_1 . $U_1 \cup \{\infty_{Y_1}\}$ is also open in Y_1 and thus so is $U_1 \cup \{\infty_{Y_1}\} \cup U_2$. Similarly, $U_1 \cup \{\infty_{Y_2}\} \cup U_2$ is open in Y_2 . The reverse inclusion is trivial.

We turn to (1). If $F_{Y_1} \cap F_{Y_2} = \emptyset$ we have $Y_1 \wedge Y_2 = 0$, assume the contrary. Consider U open in βX such that $F_{Y_1} \cap F_{Y_2} \subset U$ and take V_1, V_2 open in βX such

that $V_1 \cap V_2 = \emptyset$, and we have $F_{Y_1} \setminus U \subset V_1$ and $F_{Y_2} \setminus U \subset V_2$. Then $U_1 := V_1 \cup U$ and $U_2 := V_2 \cup U$ are open (in βX) supersets of respectively F_{Y_1} and F_{Y_2} such that $U_1 \cap U_2 = U$, which gives us (1).

We are left with (3). To see that

$$\tau(\setminus Y_1) \subset \{X \setminus F : F \subset X, \text{ for any } U \in \tau(Y_1) \ F \setminus U \text{ compact}\},$$

consider V open in βX such that $(\beta X \setminus X) \setminus F_{Y_1} \subset V$ and take any U open in βX such that $F_{Y_1} \subset U$. Then $(X \setminus V) \setminus U = X \setminus (U \cup V) = \beta X \setminus (U \cup V)$ is a closed subset of βX contained in X and therefore compact.

For the reverse inclusion, let V_0 and W_0 be open sets with disjoint closures in βX such that $(\beta X \setminus X) \setminus F_{Y_1} \subset V_0$ and $F_{Y_1} \subset W_0$. Consider $F \subset X$ such that for any $U \in \tau(Y_1)$ the set $F \setminus U$ is compact. Take any $x \in X$ and its closed (in X) neighborhood G such that G is compact. Then $X \setminus G \in \tau(Y_1)$, so $F \cap G$ is compact. Since x and its neighborhood G were arbitrary, this implies that F is closed in X (since if we take x from the boundary of F , we get that it must be in F). Similarly, since $F \cap \overline{V_0} \subset F \setminus W_0$ and $W_0 \cap X \in \tau(Y_1)$, we get that $F \cap \overline{V_0}$ is compact which implies that $F \cup F_{Y_1}$ is closed in βX . Therefore we have $X \setminus F = X \cap (\beta X \setminus (F \cup F_0)) \in \tau(\setminus Y_1)$ which ends the proof of (3).

Properties (a) – (c) follow easily from (1) – (3). □

On the other hand, one can wonder when a family \mathcal{F} of sets open in a locally compact Hausdorff space X induces its one-point local compactification. The following proposition answers that question.

Proposition 2. *Let \mathcal{F} be a filter of open sets in a locally compact Hausdorff space X . Then \mathcal{F} induces a one-point local compactification Y of X such that $\tau(Y) = \mathcal{F}$ iff:*

1. $\bigcap \mathcal{F} = \emptyset$,
2. there exists $U \in \mathcal{F}$ such that for every $V \in \mathcal{F}$, $\overline{U} \setminus V$ is compact,
3. for every $U \in \mathcal{F}$ there exists $V \in \mathcal{F}$ such that $\overline{V} \subset U$.

Proof. It follows from the definition of $\tau(Y)$ and the definition of a locally compact Hausdorff space that those conditions are necessary. We will prove that they are also sufficient. We take $Y := X \cup \{\infty_Y\}$. A set is open in Y iff it is open in X or it is of the form $U \cup \{\infty_Y\}$ for some $U \in \mathcal{F}$. It follows from (1) and (3) that the topology defined like that is Hausdorff. It remains to show that Y is locally compact. Take $U \in \mathcal{F}$ such that for every $V \in \mathcal{F}$ $\overline{U} \setminus V$ is compact and assume that \overline{U} (closure taken in Y) is not compact. It follows that there exists a net $(x_\gamma) \subset \overline{U}$ with no convergent subnets. In particular, (x_γ) is not convergent to ∞_Y , so there exists V_1 a neighborhood of ∞_Y and (y_γ) a subnet of (x_γ) such that $(y_\gamma) \subset \overline{U} \setminus V_1$ with no convergent subnets, a contradiction. □

We will now provide a characterization for $\mathcal{B}(\mathbb{R}^n)$. To this end, we will need facts about n -point Hausdorff compactifications (see [5] or [3, Theorem 6.8]).

Theorem 2 (Theorem 2.1 in [5]). *The following statements concerning a space X are equivalent:*

1. X has a N -point compactification.
2. X is locally compact and contains a compact subset K whose complement is the union of N mutually disjoint, open subsets $\{G_i\}_{i=1}^N$ such that $K \cup G_i$ is not compact for each i .
3. X is locally compact and contains a compact subset K whose complement is the union of N mutually disjoint, open subsets $\{G_i\}_{i=1}^N$ such that $K \cup G_i$ is contained in no compact subset for each i .

Using this, we can prove the following facts.

Lemma 1. *Let X be a locally compact, noncompact Hausdorff space such that for any $K \subset X$ compact there exists K_0 compact such that $K \subset K_0$ and $X \setminus K_0$ has exactly n connected components (for some fixed $n \in \mathbb{N}$ independent of the choice of K), all of them are open and have noncompact (in X) closures. Then X has an n -point Hausdorff compactification and does not have an $(n + 1)$ -point Hausdorff compactification.*

Lemma 2. *Let $n \in \mathbb{N}$ and X be a Hausdorff topological space that has an n -point Hausdorff compactification and does not have an $(n + 1)$ -point Hausdorff compactification. Then X is locally compact and $|\mathcal{B}(X)| = 2^n$.*

We will start with Lemma 1.

Proof. Applying the assumption of the lemma to the empty set we get that there exists $n \in \mathbb{N}$ and K_0 compact such that $X \setminus K_0$ has exactly n connected components, let us denote them by G_1, \dots, G_n . Therefore (by [5]) X has an n -point Hausdorff compactification. Suppose that X has an $(n + 1)$ -Hausdorff compactification. Again by [5], there exist H_1, \dots, H_{n+1} such that $K_1 := X \setminus \bigcup_{i=1}^{n+1} H_i$ is compact, but for each i the set $K_1 \cup H_i$ is not compact. Applying the assumption of the lemma again, this time to K_1 , we get that there exists a compact set K_2 such that $K_1 \subset K_2$ and $X \setminus K_2$ has n connected components, let us denote them by V_1, \dots, V_n . Then there exist $i_0 \in \{1, \dots, n\}$ and $j_1, j_2 \in \{1, \dots, n + 1\}$ such that $j_1 \neq j_2$ and H_{i_0} has nonempty intersection with both V_{j_1}, V_{j_2} , so it cannot be connected, a contradiction. \square

Now we turn to Lemma 2.

Proof. Since X has an n -point Hausdorff compactification, but does not have an $n + 1$ -point Hausdorff compactification, $\beta X \setminus X$ has exactly n connected components. From the proof of Theorem 1 we know that $|\mathcal{B}(X)| = |\mathcal{A}(X)|$. Each element of $\mathcal{A}(X)$ is a union of some connected components of $\beta X \setminus X$, so $|\mathcal{B}(X)| = |\mathcal{A}(X)| = 2^n$. \square

Remark 2. Note that if we assume that if X is a locally compact space such that $|\mathcal{B}(X)| = 2^n$, we also get that X has an n -point Hausdorff compactification and does not have an $(n + 1)$ -point Hausdorff compactification (see also [3, Theorem 6.32]).

From the above lemmas we immediately get the following.

Corollary 1.

- $\mathcal{B}(\mathbb{R}) = \{\mathbb{R}, [-\infty, \infty), (-\infty, \infty], \mathbb{S}^1\}$.
- $\mathcal{B}(\mathbb{R}^n) = \{\mathbb{R}^n, \mathbb{S}^n\}$ for $n \geq 2$.

We will now define the end of manifolds, as seen in [2].

Definition 4. Let L be a noncompact, connected manifold. Denote by $\{K_\alpha\}_{\alpha \in \mathcal{K}}$ the family of all compact subsets of L . We consider descending chains

$$U_{\alpha_1} \supseteq U_{\alpha_2} \supseteq \cdots \supseteq U_{\alpha_n} \supseteq \cdots$$

where each U_{α_k} is a connected component of $L \setminus K_{\alpha_k}$, has noncompact closure in L , satisfies $U_{\alpha_k} \supseteq \overline{U_{\alpha_{k+1}}}$ and

$$\bigcap_{k=1}^{\infty} U_{\alpha_k} = \emptyset.$$

We say that two such chains $\mathcal{U} = \{U_{\alpha_k}\}_{k=1}^{\infty}$ and $\mathcal{V} = \{U_{\beta_k}\}_{k=1}^{\infty}$ are equivalent ($\mathcal{U} \sim \mathcal{V}$) if for each $k \geq 1$ there is $n > k$ such that $U_{\alpha_k} \supseteq V_{\beta_n}$ and $V_{\beta_k} \supseteq U_{\alpha_n}$. It is easy to check that \sim is an equivalence relation. If

$$\mathcal{U} = \{U_{\alpha_1} \supseteq U_{\alpha_2} \supseteq \cdots \supseteq U_{\alpha_n} \supseteq \cdots\}$$

is as above, we call its equivalence class under \sim an *end* of L .

Corollary 2.

If L is a noncompact, connected, second countable manifold with n ends, $n < \infty$, then $|\mathcal{B}(L)| = 2^n$.

Proof. Let

$$\begin{aligned} \mathcal{U}_1 &= \{U_{\alpha_1}^1 \supseteq U_{\alpha_2}^1 \supseteq \cdots\} \\ &\vdots \\ \mathcal{U}_n &= \{U_{\alpha_1}^n \supseteq U_{\alpha_2}^n \supseteq \cdots\} \end{aligned}$$

be representatives of the ends of L .

For every $k \in \{1, 2, \dots\}, l \in \{1, 2, \dots, n\}$ let $K_{\alpha_k}^l$ be a compact set such that $U_{\alpha_k}^l$ is a connected component of $L \setminus K_{\alpha_k}^l$. We will show that by taking subsequences of $\mathcal{U}_2, \dots, \mathcal{U}_n$ we can assume that $U_{\alpha_k}^{l_2} \subset L \setminus K_{\alpha_k}^{l_1}$ for every $l_2 > l_1$ (note that a subsequence of a representative of an end is a representative of the same end).

Consider $K_{\alpha_1}^1$. Then $\{L \setminus \overline{U_{\alpha_1}^2}, L \setminus \overline{U_{\alpha_1}^3}, \dots\}$ is an open cover of $K_{\alpha_1}^1$ so there exists $N_1 > 0$ such that $K_{\alpha_1}^1 \subset L \setminus \overline{U_{\alpha_{N_1}}^2} \subset L \setminus U_{\alpha_{N_1}}^2$. Therefore $U_{\alpha_{N_1}}^2 \subset L \setminus K_{\alpha_1}^1$. Similarly, for each $m > 1$, we can define $N_m > N_{m-1}$ such that $U_{\alpha_{N_m}}^2 \subset L \setminus K_{\alpha_m}^1$. Replacing $U_{\alpha_m}^2$ by $U_{\alpha_{N_m}}^2$ for each $m > 0$ we get a subsequence we want for \mathcal{U}_2 . Now we proceed similarly for $\mathcal{U}_3, \dots, \mathcal{U}_n$.

We will now show that by again taking subsequences we can assume that for every $l_1 \neq l_2$ we have $U_{\alpha_1}^{l_1} \cap U_{\alpha_1}^{l_2} = \emptyset$. Assume the contrary. Then, without loss of generality, for each $k > 0$ we have $U_{\alpha_k}^1 \cap U_{\alpha_k}^2 \neq \emptyset$. Since $U_{\alpha_k}^2 \subset L \setminus K_{\alpha_k}^1$, the set $U_{\alpha_k}^2$ is connected, $U_{\alpha_k}^1$ is a connected component of $L \setminus K_{\alpha_k}^1$ and $U_{\alpha_k}^1 \cap U_{\alpha_k}^2 \neq \emptyset$, it follows that $U_{\alpha_k}^2 \subset U_{\alpha_k}^1$ for each $k > 0$. Now consider $K_{\alpha_k}^2$. As before, there exists $N_k > k$ such that $K_{\alpha_k}^2 \subset L \setminus U_{\alpha_{N_k}}^1$. It follows that $U_{\alpha_{N_k}}^1 \subset L \setminus K_{\alpha_k}^2$. If $U_{\alpha_{N_k}}^1 \not\subset U_{\alpha_k}^2$ then $U_{\alpha_{N_k}}^1 \cap U_{\alpha_k}^2 = \emptyset$, so $U_{\alpha_{N_k}}^1 \cap U_{\alpha_{N_k}}^2 = \emptyset$ (since $U_{\alpha_{N_k}}^2 \subset U_{\alpha_k}^2$). Therefore $U_{\alpha_{N_k}}^1 \subset U_{\alpha_k}^2$ and so \mathcal{U}_1 and \mathcal{U}_2 are representatives of the same end, a contradiction.

Now our aim is to use Lemmas 1 and 2, which will end the proof. To this end, we will construct a family of compact sets $\{K_j\}_{j=1}^\infty$. We will need some properties of manifolds, namely that a second countable manifold is metrizable and that the one-point compactification of a connected manifold is locally connected (see [4] or [6, page 104]). Let $\omega L = L \cup \{\infty\}$ be the one-point compactification of L . Since L is second countable we can choose a countable basis of its topology $\mathcal{B} = \{B_1, B_2, B_3, \dots\}$ consisting of open sets with compact closures. Take $A_1 := K_{\alpha_1}^1 \cup \dots \cup K_{\alpha_1}^n \cup \overline{B_1}$. Let K_1 be a compact superset of A_1 such that $\omega L \setminus K_1$ is connected (it exists because ωL is locally connected). Note that connected components of $L \setminus K_1$ are all open and have noncompact (in L) closures (because ∞ is in the closure taken in ωL of every one of them). Again, because L is locally compact we can take an open set A_2 with compact closure such that $K_1 \cup \overline{A_2} \subset A_2$. Let K_2 be a compact superset of A_2 such that $\omega L \setminus K_2$ is connected. As before, all connected components of $L \setminus K_2$ are open and have noncompact (in L) closures. Moreover, each of them is contained together with its closure in some connected component of $L \setminus K_1$. Note that since $\omega L \setminus K_2$ has non-empty intersection with every connected component of $L \setminus K_1$ (because ∞ is in the closure taken in ωL of every one of them), for every connected component of $L \setminus K_1$ there is at least one connected component of $L \setminus K_2$ contained in it. Continuing in this manner, we get $\{K_j\}_{j=1}^\infty$. Note that K_j is contained in the interior of K_{j+1} for each $j \geq 1$ and $\bigcup_{j=1}^\infty K_j = L$. Moreover, when j increases the number of connected components of $L \setminus K_j$ either increases or stays the same. Consider a connected component U_1 of $L \setminus K_1$. We want to show that $U_1 \cap U_{\alpha_1}^i \neq \emptyset$ for some i . Indeed, otherwise by choosing a connected component U_2 of $U_1 \setminus K_2$, then a connected U_3 of $U_2 \setminus K_3$ etc. we would get a representative of an end that is not among $\mathcal{U}_1, \dots, \mathcal{U}_n$, a contradiction. Suppose that $U_1 \cap U_{\alpha_1}^1 \neq \emptyset$. Since $K_{\alpha_1}^1 \subset K_1$ and $U_{\alpha_1}^1, U_1$ are connected components of their complements we get $U_1 \subset U_{\alpha_1}^1$. The sets $U_{\alpha_1}^i$ are pairwise disjoint, so $L \setminus K_1$ has at least n connected components. Moreover, the number of connected components of $L \setminus K_j$ cannot increase past n for any j . Indeed, if we had at least $n + 1$ connected components of $L \setminus K_j$ for some j , we could construct at least $n + 1$ different ends (similarly as before) which again contradicts the fact that $\mathcal{U}_1, \dots, \mathcal{U}_n$ are all of the ends in L . Lemma 1 ends the proof. \square

From this and Remark 2 we also get the following.

Corollary 3. *If L is a noncompact, connected, second countable manifold with n ends, $n < \infty$, then L has an n -point Hausdorff compactification and does not have an $(n + 1)$ -point Hausdorff compactification.*

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Finite Approximation of Continuous Noncooperative Two-person Games on a Product of Linear Strategy Functional Spaces

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ABSTRACT: A method of the finite approximation of continuous non-cooperative two-person games is presented. The method is based on sampling the functional spaces, which serve as the sets of pure strategies of the players. The pure strategy is a linear function of time, in which the trend-defining coefficient is variable. The spaces of the players' pure strategies are sampled uniformly so that the resulting finite game is a bimatrix game whose payoff matrices are square. The approximation procedure starts with not a great number of intervals. Then this number is gradually increased, and new, bigger, bimatrix games are solved until an acceptable solution of the bimatrix game becomes sufficiently close to the same-type solutions at the preceding iterations. The closeness is expressed as the absolute difference between the trend-defining coefficients of the strategies from the neighboring solutions. These distances should be decreasing once they are smoothed with respective polynomials of degree 2.

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1. Introduction

Continuous noncooperative two-person games model interactions of a pair of subjects (players or persons) possessing continuums of their pure strategies [5, 10]. A specificity

of such games consists in that finding and practicing a solution in mixed strategies is often intractable [11, 6, 9]. Even if a solution exists in pure strategies, it often is revealed not to be a single one. Thus, the problem of the single solution selection (or uniqueness) arises. However, even if the solution is unique, it is not guaranteed to be simultaneously profitable and symmetric [11, 9, 2, 1].

The solution search in continuous games is not a trivial task also. Analytic search generalization is possible only in special classes [10, 3]. Therefore, finite approximation of continuous noncooperative two-person games is not just preferable, but also is necessary.

2. Motivation

A special class of noncooperative two-person games is when the player's pure strategy is a time-varying function. Commonly, apart from the time, this function is determined by a few parameters (coefficients). These coefficients may vary through intervals. Therefore, the set of the player's pure strategies is a functional space. Such a game model is typical for economic interaction processes, where the player uses short-term time-varying strategies [11, 13, 12].

In the simplest case, the strategy is a linear function of time. The time interval is usually short, through which a short-term trend of economic activity is realized [11, 9]. Thus, a whole process is modeled as a series of those noncooperative games. Each game corresponds to its short term. Then, obviously, the games are required to be solved as fast as possible.

The problems of fast solution and solution uniqueness are addressed in studying finite approximations of continuous games. When the game is defined on finite-dimensional Euclidean subspaces, it can be approximated by appropriately sampling the sets of players' pure strategies [6, 7]. Then an approximating game is solved easily and faster. Besides, an approximated solution (with respect to the initial game) can be selected in order to meet demands and rules of the economic system [11, 9]. In the case when the game is defined on a product of functional spaces, a strict substantiation is required to sample the functional sets of players' pure strategies. As in the case of finite-dimensional Euclidean subspaces, this will allow sampling without significant losses.

3. Goals and tasks to be fulfilled

Due to above reasons, the goal is to develop a method of finite approximation of continuous noncooperative two-person games whose kernels are defined on a product of linear strategy functional spaces. For achieving the goal, the following tasks are to be fulfilled:

1. To formalize a continuous noncooperative two-person game whose kernel is defined on a product of linear strategy functional spaces. In such a game, the set of the player's pure strategies is a continuum of linear functions of time.

2. To formalize a method of finite approximation.
3. To discuss applicability and significance of the method.

4. A continuous noncooperative two-person game

Each of the players uses short-term time-varying strategies determined by two coefficients. The short-term trend is defined by a real-valued coefficient which is multiplied by time t . The other coefficient is presumed to be known (i. e., it is a constant). Herein, this real-valued constant is called an offset.

The pure strategy is valid on interval $[t_1; t_2]$ by $t_2 > t_1$, so pure strategies of the player belong to a functional space of linear functions of time:

$$L[t_1; t_2] \subset \mathbb{L}_2[t_1; t_2].$$

Denote the trend-defining coefficient of the first player by b_x , where

$$b_x \in [b_x^{(\min)}; b_x^{(\max)}] \text{ by } b_x^{(\max)} > b_x^{(\min)}. \quad (1)$$

If the first player's offset is a_x , then its set of pure strategies is

$$\begin{aligned} X = \left\{ x(t) = a_x + b_x t, t \in [t_1; t_2] : b_x \in [b_x^{(\min)}; b_x^{(\max)}] \subset \mathbb{R} \right\} \subset \\ \subset L[t_1; t_2] \subset \mathbb{L}_2[t_1; t_2]. \end{aligned} \quad (2)$$

For the second player, denote its offset by a_y and its trend-defining coefficient by b_y , where

$$b_y \in [b_y^{(\min)}; b_y^{(\max)}] \text{ by } b_y^{(\max)} > b_y^{(\min)}. \quad (3)$$

Then the set of pure strategies of the second player is

$$\begin{aligned} Y = \left\{ y(t) = a_y + b_y t, t \in [t_1; t_2] : b_y \in [b_y^{(\min)}; b_y^{(\max)}] \subset \mathbb{R} \right\} \subset \\ \subset L[t_1; t_2] \subset \mathbb{L}_2[t_1; t_2]. \end{aligned} \quad (4)$$

The players' payoffs in situation $\{x(t), y(t)\}$ are

$$K_x(x(t), y(t)) \text{ and } K_y(x(t), y(t)),$$

respectively. These payoffs are integral functionals:

$$K_x(x(t), y(t)) = \int_{t_1}^{t_2} f(x(t), y(t)) dt \quad (5)$$

and

$$K_y(x(t), y(t)) = \int_{t_1}^{t_2} g(x(t), y(t)) dt, \quad (6)$$

where $f(x(t), y(t))$ and $g(x(t), y(t))$ are algebraic functions of $x(t)$ and $y(t)$ defined everywhere on $[t_1; t_2]$. Therefore, the continuous noncooperative two-person game

$$\langle \{X, Y\}, \{K_x(x(t), y(t)), K_y(x(t), y(t))\} \rangle \quad (7)$$

is defined on product

$$X \times Y \subset L[t_1; t_2] \times L[t_1; t_2] \subset \mathbb{L}_2[t_1; t_2] \times \mathbb{L}_2[t_1; t_2] \quad (8)$$

of linear strategy functional spaces (2) and (4).

5. Acceptable solutions

Since a series of games (7) on product (8) is to be solved in practice, the only acceptable solutions are equilibrium or/and efficient situations in pure strategies. Such situations are defined similarly to those in games on finite-dimensional Euclidean subspaces [5, 10].

Definition 1. Situation $\{x^*(t), y^*(t)\}$ in game (7) on product (8) by conditions (1) — (6) is an equilibrium situation in pure strategies if inequalities

$$K_x(x(t), y^*(t)) \leq K_x(x^*(t), y^*(t)) \quad \forall x(t) \in X \quad (9)$$

and

$$K_y(x^*(t), y(t)) \leq K_y(x^*(t), y^*(t)) \quad \forall y(t) \in Y \quad (10)$$

are simultaneously true.

Definition 2. Situation $\{x^{**}(t), y^{**}(t)\}$ in game (7) on product (8) by conditions (1) — (6) is an efficient situation in pure strategies if both a pair of inequalities

$$\begin{aligned} K_x(x^{**}(t), y^{**}(t)) &\leq K_x(x(t), y(t)) \quad \text{and} \\ K_y(x^{**}(t), y^{**}(t)) &< K_y(x(t), y(t)) \end{aligned} \quad (11)$$

and a pair of inequalities

$$\begin{aligned} K_x(x^{**}(t), y^{**}(t)) &< K_x(x(t), y(t)) \quad \text{and} \\ K_y(x^{**}(t), y^{**}(t)) &\leq K_y(x(t), y(t)) \end{aligned} \quad (12)$$

are impossible for any $x(t) \in X$ and $y(t) \in Y$.

The continuous noncooperative two-person game can have the empty set of equilibria in pure strategies [10]. Moreover, every efficient situation can be too asymmetric, i. e. it is profitable for one player and unacceptably unprofitable for the other player. In such cases, the game does not have an acceptable solution. Then the concepts of ε -equilibrium and ε -efficiency are useful [10, 11].

Definition 3. Situation $\{x^{*(\varepsilon)}(t), y^{*(\varepsilon)}(t)\}$ in game (7) on product (8) by conditions (1) — (6) is an ε -equilibrium situation in pure strategies for some $\varepsilon > 0$ if inequalities

$$K_x(x(t), y^{*(\varepsilon)}(t)) \leq K_x(x^{*(\varepsilon)}(t), y^{*(\varepsilon)}(t)) + \varepsilon \quad \forall x(t) \in X \quad (13)$$

and

$$K_y(x^{*(\varepsilon)}(t), y(t)) \leq K_y(x^{*(\varepsilon)}(t), y^{*(\varepsilon)}(t)) + \varepsilon \quad \forall y(t) \in Y \quad (14)$$

are simultaneously true.

Definition 4. Situation $\{x^{**(\varepsilon)}(t), y^{**(\varepsilon)}(t)\}$ in game (7) on product (8) by conditions (1) — (6) is an ε -efficient situation in pure strategies for some $\varepsilon > 0$ if both a pair of inequalities

$$\begin{aligned} K_x(x^{**(\varepsilon)}(t), y^{**(\varepsilon)}(t)) + \varepsilon &\leq K_x(x(t), y(t)) \quad \text{and} \\ K_y(x^{**(\varepsilon)}(t), y^{**(\varepsilon)}(t)) + \varepsilon &< K_y(x(t), y(t)) \end{aligned} \quad (15)$$

and a pair of inequalities

$$\begin{aligned} K_x(x^{**(\varepsilon)}(t), y^{**(\varepsilon)}(t)) + \varepsilon &< K_x(x(t), y(t)) \quad \text{and} \\ K_y(x^{**(\varepsilon)}(t), y^{**(\varepsilon)}(t)) + \varepsilon &\leq K_y(x(t), y(t)) \end{aligned} \quad (16)$$

are impossible for any $x(t) \in X$ and $y(t) \in Y$.

The situations given by Definitions 1 — 4 are the acceptable solutions regardless of whether the game is finite or not. The best consequent is when a situation is simultaneously equilibrium (by Definition 1) and efficient (by Definition 2). If this is impossible, then the most preferable is an efficient situation at which the sum of players' payoffs is maximal. However, if the payoffs are unacceptably asymmetric, then the best consequent is to find such ε for which a situation is simultaneously equilibrium (by Definition 3) and efficient (by Definition 4). This approach relates to the method of solving games approximately by providing concessions [8]. Eventually, a payoff asymmetry may be smoothed by a compensation from the player whose payoff is unacceptably greater [11].

6. The finite approximation

It is obvious that, in game (7) on product (8) by conditions (1) — (6), the pure strategy of the player is determined by the trend-defining coefficient. Therefore, this game can be thought of as it is defined, instead of product (8) of linear strategy functional spaces (2) and (4), on rectangle

$$\left[b_x^{(\min)}; b_x^{(\max)} \right] \times \left[b_y^{(\min)}; b_y^{(\max)} \right] \subset \mathbb{R}^2. \quad (17)$$

This rectangle is easily sampled by using a number of equal intervals along each dimension. Denote this number by S , $S \in \mathbb{N} \setminus \{1\}$. Then

$$B_x = \left\{ b_x^{(\min)} + (s-1) \cdot \frac{b_x^{(\max)} - b_x^{(\min)}}{S} \right\}_{s=1}^{S+1} = \left\{ b_x^{(s)} \right\}_{s=1}^{S+1} \subset \left[b_x^{(\min)}; b_x^{(\max)} \right] \quad (18)$$

and

$$B_y = \left\{ b_y^{(\min)} + (s-1) \cdot \frac{b_y^{(\max)} - b_y^{(\min)}}{S} \right\}_{s=1}^{S+1} = \left\{ b_y^{(s)} \right\}_{s=1}^{S+1} \subset \left[b_y^{(\min)}; b_y^{(\max)} \right]. \quad (19)$$

So, rectangle (17) is substituted with grid $B_x \times B_y$. Set (18) leads to a finite set

$$\begin{aligned} X_B &= \left\{ x(t) = a_x + b_x t, t \in [t_1; t_2] : b_x \in B_x \subset \left[b_x^{(\min)}; b_x^{(\max)} \right] \subset \mathbb{R} \right\} = \\ &= \left\{ x_s(t) = a_x + b_x^{(s)} t \right\}_{s=1}^{S+1} \subset X \subset L[t_1; t_2] \subset \mathbb{L}_2[t_1; t_2] \end{aligned} \quad (20)$$

of pure strategies (linear functions of time) of the first player, where

$$x_1(t) = a_x + b_x^{(\min)} t, \quad x_{S+1}(t) = a_x + b_x^{(\max)} t,$$

and set (19) leads to a finite set

$$\begin{aligned} Y_B &= \left\{ y(t) = a_y + b_y t, t \in [t_1; t_2] : b_y \in B_y \subset \left[b_y^{(\min)}; b_y^{(\max)} \right] \subset \mathbb{R} \right\} = \\ &= \left\{ y_s(t) = a_y + b_y^{(s)} t \right\}_{s=1}^{S+1} \subset Y \subset L[t_1; t_2] \subset \mathbb{L}_2[t_1; t_2] \end{aligned} \quad (21)$$

of pure strategies (linear functions of time) of the second player, where

$$y_1(t) = a_y + b_y^{(\min)} t, \quad y_{S+1}(t) = a_y + b_y^{(\max)} t.$$

Subsequently, game (7) on product (8) by conditions (1) — (6) is substituted with a finite game

$$\begin{aligned} &\langle \{X_B, Y_B\}, \{K_x(x(t), y(t)), K_y(x(t), y(t))\} \rangle \\ &\text{by } x(t) \in X_B \text{ and } y(t) \in Y_B \end{aligned} \quad (22)$$

defined on product

$$X_B \times Y_B \subset X \times Y \subset L[t_1; t_2] \times L[t_1; t_2] \subset \mathbb{L}_2[t_1; t_2] \times \mathbb{L}_2[t_1; t_2] \quad (23)$$

of linear strategy functional subspaces (20) and (21). In fact, game (22) is a bimatrix $(S+1) \times (S+1)$ -game.

To perform an appropriate approximation via the sampling, number S is selected so that none of S^2 rectangles

$$\left[b_x^{(i)}; b_x^{(i+1)} \right] \times \left[b_y^{(j)}; b_y^{(j+1)} \right] \text{ by } i = \overline{1, S} \text{ and } j = \overline{1, S} \quad (24)$$

would contain significant specificities of payoff functionals (5) and (6). In fact, such specificities are extremals of these functionals.

Theorem 1. *In game (7) on product (8) by conditions (1) — (6), the player’s payoff functional achieves its maximal and minimal values on any closed subset of rectangle (17) of the trend-defining coefficients.*

Proof. Both $f(x(t), y(t))$ and $g(x(t), y(t))$ are algebraic functions of linear functions $x(t)$ and $y(t)$ defined everywhere on $[t_1; t_2]$. Therefore, both integrals in functionals (5) and (6) achieve some maximal and minimal values on any closed subset of rectangle (17) of the trend-defining coefficients. \square

Thus, Theorem 1 allows controlling extremals of payoff functionals (5) and (6) by the trend-defining coefficient. Moreover, Theorem 1 is easily expanded on closed rectangles (24) for any number S . Consequently, if inequalities

$$\begin{aligned} & \max_{\substack{b_x \in [b_x^{(i)}; b_x^{(i+1)}], \\ b_y \in [b_y^{(j)}; b_y^{(j+1)}]}} K_x(x(t), y(t)) - \min_{\substack{b_x \in [b_x^{(i)}; b_x^{(i+1)}], \\ b_y \in [b_y^{(j)}; b_y^{(j+1)}]}} K_x(x(t), y(t)) = \\ = & \max_{\substack{b_x \in [b_x^{(i)}; b_x^{(i+1)}], \\ b_y \in [b_y^{(j)}; b_y^{(j+1)}]}} \int_{t_1}^{t_2} f(x(t), y(t)) dt - \min_{\substack{b_x \in [b_x^{(i)}; b_x^{(i+1)}], \\ b_y \in [b_y^{(j)}; b_y^{(j+1)}]}} \int_{t_1}^{t_2} f(x(t), y(t)) dt \leq \mu \\ & \forall i = \overline{1, S} \text{ and } \forall j = \overline{1, S} \end{aligned} \tag{25}$$

and

$$\begin{aligned} & \max_{\substack{b_x \in [b_x^{(i)}; b_x^{(i+1)}], \\ b_y \in [b_y^{(j)}; b_y^{(j+1)}]}} K_y(x(t), y(t)) - \min_{\substack{b_x \in [b_x^{(i)}; b_x^{(i+1)}], \\ b_y \in [b_y^{(j)}; b_y^{(j+1)}]}} K_y(x(t), y(t)) = \\ = & \max_{\substack{b_x \in [b_x^{(i)}; b_x^{(i+1)}], \\ b_y \in [b_y^{(j)}; b_y^{(j+1)}]}} \int_{t_1}^{t_2} g(x(t), y(t)) dt - \min_{\substack{b_x \in [b_x^{(i)}; b_x^{(i+1)}], \\ b_y \in [b_y^{(j)}; b_y^{(j+1)}]}} \int_{t_1}^{t_2} g(x(t), y(t)) dt \leq \mu \\ & \forall i = \overline{1, S} \text{ and } \forall j = \overline{1, S} \end{aligned} \tag{26}$$

are simultaneously true for some sufficiently small $\mu > 0$, then those μ -variations can be ignored. Thus, for the properly selected S and μ , game (7) on product (8) by conditions (1) — (6) can be approximated by finite game (22). The quality of the approximation can be comprehended by the following assertions.

Theorem 2. *If $\{x^*(t), y^*(t)\}$ is an equilibrium in game (7) on product (8) by conditions (1) — (6), where functionals (5) and (6) are continuous, conditions (25) and (26) hold for some S and μ ,*

$$\begin{aligned} x^*(t) &= a_x + b_x^* t \quad \text{by} \quad b_x^* \in [b_x^{(h)}; b_x^{(h+1)}] \quad \text{and} \\ y^*(t) &= a_y + b_y^* t \quad \text{by} \quad b_y^* \in [b_y^{(k)}; b_y^{(k+1)}] \\ & \text{for } h \in \{\overline{1, S}\}, \quad k \in \{\overline{1, S}\}, \end{aligned} \tag{27}$$

then every situation $\{x^{*(\varepsilon)}(t), y^{*(\varepsilon)}(t)\}$ for which

$$\begin{aligned} x^{*(\varepsilon)}(t) &= a_x + b_x^{*(\varepsilon)}t \quad \text{by } b_x^{*(\varepsilon)} \in [b_x^{(h)}; b_x^{(h+1)}] \quad \text{and} \\ y^{*(\varepsilon)}(t) &= a_y + b_y^{*(\varepsilon)}t \quad \text{by } b_y^{*(\varepsilon)} \in [b_y^{(k)}; b_y^{(k+1)}] \\ &\text{for } h \in \{\overline{1, S}\}, \quad k \in \{\overline{1, S}\}, \end{aligned} \quad (28)$$

is an ε -equilibrium for some $\varepsilon > 0$. As number S is increased, the value of ε can be made smaller.

Proof. Whichever integer S and the corresponding μ are, the value of ε always can be chosen such that inequalities (13) and (14) be true for every situation composed of strategies (28) by (27). It is obvious that, owing to the continuity of the functionals, increasing number S allows decreasing the value of μ , which provides ε -equilibria to be closer to the equilibrium composed of strategies (27). \square

Theorem 3. If $\{x^{**}(t), y^{**}(t)\}$ is an efficient situation in game (7) on product (8) by conditions (1) – (6), where functionals (5) and (6) are continuous, conditions (25) and (26) hold for some S and μ ,

$$\begin{aligned} x^{**}(t) &= a_x + b_x^{**}t \quad \text{by } b_x^{**} \in [b_x^{(h)}; b_x^{(h+1)}] \quad \text{and} \\ y^{**}(t) &= a_y + b_y^{**}t \quad \text{by } b_y^{**} \in [b_y^{(k)}; b_y^{(k+1)}] \\ &\text{for } h \in \{\overline{1, S}\}, \quad k \in \{\overline{1, S}\}, \end{aligned} \quad (29)$$

then every situation $\{x^{**(\varepsilon)}(t), y^{**(\varepsilon)}(t)\}$ for which

$$\begin{aligned} x^{**(\varepsilon)}(t) &= a_x + b_x^{**(\varepsilon)}t \quad \text{by } b_x^{**(\varepsilon)} \in [b_x^{(h)}; b_x^{(h+1)}] \quad \text{and} \\ y^{**(\varepsilon)}(t) &= a_y + b_y^{**(\varepsilon)}t \quad \text{by } b_y^{**(\varepsilon)} \in [b_y^{(k)}; b_y^{(k+1)}] \\ &\text{for } h \in \{\overline{1, S}\}, \quad k \in \{\overline{1, S}\}, \end{aligned} \quad (30)$$

is an ε -efficient situation for some $\varepsilon > 0$. As number S is increased, the value of ε can be made smaller.

Proof. Whichever integer S and the corresponding μ are, value ε always can be chosen such that inequalities (15) and (16) be true for every situation composed of strategies (30) by (29). It is obvious that, owing to the continuity of the functionals, increasing number S allows decreasing the value of μ , which provides ε -efficient situations to be closer to the efficient situation composed of strategies (29). \square

Hence, the finite approximation should start from some integer S , for which a bimatrix $(S + 1) \times (S + 1)$ -game (22) is built and solved. Then this integer is gradually increased (although, the increment is not ascertained for general case), and new, bigger, bimatrix games are solved. The process can be stopped if the acceptable

solution (whether it is an equilibrium, efficient, ε -equilibrium, or ε -efficient situation) by the last iteration does not differ much from the acceptable solutions (of the same type) by a few preceding iterations. Thus, if

$$\{x^{<l>*}(t), y^{<l>*}(t)\} = \{a_x + b_x^{<l>*}t, a_y + b_y^{<l>*}t\} \in X_B \times Y_B \subset X \times Y \quad (31)$$

is an acceptable solution at the l -th iteration, then the conditions of the sufficient closeness to the solutions at the preceding and succeeding iterations are as follows:

$$\begin{aligned} \sqrt{\int_{t_1}^{t_2} (x^{<l-1>*}(t) - x^{<l>*}(t))^2 dt} &\geq \sqrt{\int_{t_1}^{t_2} (x^{<l>*}(t) - x^{<l+1>*}(t))^2 dt} \quad \text{and} \\ \sqrt{\int_{t_1}^{t_2} (y^{<l-1>*}(t) - y^{<l>*}(t))^2 dt} &\geq \sqrt{\int_{t_1}^{t_2} (y^{<l>*}(t) - y^{<l+1>*}(t))^2 dt} \quad (32) \end{aligned}$$

and

$$\begin{aligned} \max_{t \in [t_1; t_2]} |x^{<l-1>*}(t) - x^{<l>*}(t)| &\geq \max_{t \in [t_1; t_2]} |x^{<l>*}(t) - x^{<l+1>*}(t)| \quad \text{and} \\ \max_{t \in [t_1; t_2]} |y^{<l-1>*}(t) - y^{<l>*}(t)| &\geq \max_{t \in [t_1; t_2]} |y^{<l>*}(t) - y^{<l+1>*}(t)| \quad (33) \end{aligned}$$

by $l = 2, 3, 4, \dots$

Theorem 4. Conditions (32) and (33) of the sufficient closeness for game (7) on product (8) by conditions (1) – (6) are expressed as

$$|b_x^{<l-1>*} - b_x^{<l>*}| \geq |b_x^{<l>*} - b_x^{<l+1>*}| \quad \text{by } l = 2, 3, 4, \dots \quad (34)$$

and

$$|b_y^{<l-1>*} - b_y^{<l>*}| \geq |b_y^{<l>*} - b_y^{<l+1>*}| \quad \text{by } l = 2, 3, 4, \dots \quad (35)$$

Proof. Due to that

$$\begin{aligned} \sqrt{\int_{t_1}^{t_2} (x^{<l-1>*}(t) - x^{<l>*}(t))^2 dt} &= \sqrt{\int_{t_1}^{t_2} (a_x + b_x^{<l-1>*}t - a_x - b_x^{<l>*}t)^2 dt} = \\ &= \sqrt{\int_{t_1}^{t_2} (b_x^{<l-1>*} - b_x^{<l>*})^2 t^2 dt} = \sqrt{(b_x^{<l-1>*} - b_x^{<l>*})^2 \left(\frac{t_2^3}{3} - \frac{t_1^3}{3}\right)} = \\ &= |b_x^{<l-1>*} - b_x^{<l>*}| \sqrt{\frac{t_2^3 - t_1^3}{3}} \end{aligned}$$

and

$$\max_{t \in [t_1; t_2]} |x^{<l-1>*}(t) - x^{<l>*}(t)| = \max_{t \in [t_1; t_2]} |(b_x^{<l-1>*} - b_x^{<l>*})t| =$$

$$= |b_x^{<l-1>*} - b_x^{<l>*}| t_2$$

(where time is presumed to be nonnegative), inequalities (32) and (33) are simplified explicitly:

$$\begin{aligned} |b_x^{<l-1>*} - b_x^{<l>*}| \sqrt{\frac{t_2^3 - t_1^3}{3}} &\geq |b_x^{<l>*} - b_x^{<l+1>*}| \sqrt{\frac{t_2^3 - t_1^3}{3}} \text{ and} \\ |b_y^{<l-1>*} - b_y^{<l>*}| \sqrt{\frac{t_2^3 - t_1^3}{3}} &\geq |b_y^{<l>*} - b_y^{<l+1>*}| \sqrt{\frac{t_2^3 - t_1^3}{3}} \end{aligned}$$

and

$$\begin{aligned} |b_x^{<l-1>*} - b_x^{<l>*}| t_2 &\geq |b_x^{<l>*} - b_x^{<l+1>*}| t_2 \text{ and} \\ |b_y^{<l-1>*} - b_y^{<l>*}| t_2 &\geq |b_y^{<l>*} - b_y^{<l+1>*}| t_2, \end{aligned}$$

whence they are expressed as (34) and (35), respectively. \square

If inequalities (34) and (35) hold for at least three iterations, the approximation procedure can be stopped. Clearly, the closeness strengthens if inequalities (34) and (35) hold strictly. However, inequalities (34) and (35) may not hold for a wide range of iterations, so it is better to require that both polylines

$$\lambda_x(l) = |b_x^{<l>*} - b_x^{<l+1>*}| \text{ by } l = 1, 2, 3, \dots \quad (36)$$

and

$$\lambda_y(l) = |b_y^{<l>*} - b_y^{<l+1>*}| \text{ by } l = 1, 2, 3, \dots \quad (37)$$

be decreasing on average. Herein, term ‘‘on average’’ implies that, in the case when inequalities (34) and (35) do not hold, polylines (36) and (37) are smoothed (approximated) with the respective polynomials of degree 2.

7. Exemplification

To exemplify the method of the game finite approximation, consider a case in which $t \in [1; 30]$, the set of pure strategies of the first player is

$$\begin{aligned} X = \{x(t) = 100 + b_x t, t \in [1; 30] : b_x \in [-0.4; 0.4] \subset \mathbb{R}\} &\subset \\ &\subset L[1; 30] \subset \mathbb{L}_2[1; 30], \end{aligned} \quad (38)$$

and the set of pure strategies of the second player is

$$\begin{aligned} Y = \{y(t) = 120 + b_y t, t \in [1; 30] : b_y \in [-0.6; 0.6] \subset \mathbb{R}\} &\subset \\ &\subset L[1; 30] \subset \mathbb{L}_2[1; 30]. \end{aligned} \quad (39)$$

The payoff functionals are

$$K_x(x(t), y(t)) = \int_1^{30} 10000 \cdot \frac{5x^2(t) + x(t) - x(t)y(t) - y^2(t)}{x^3(t) + x^2(t) + x(t) - x(t)y(t) - y^2(t)} dt \quad (40)$$

and

$$K_y(x(t), y(t)) = \int_1^{30} (y(t) - 1.2x(t))^2 dt. \tag{41}$$

Consequently, this game can be thought of as it is defined on rectangle (17):

$$[-0.4; 0.4] \times [-0.6; 0.6] \subset \mathbb{R}^2. \tag{42}$$

It is easy to show that functional (40) is continuous (Figure 1). The continuity of functional (41) is quite clear (Figure 2). Therefore, Theorem 2 and Theorem 3 will ensure fast approximation here. At $S = 5$ the respective bimatrix 6×6 -game has a single equilibrium and two efficient situations. By increasing the number of intervals

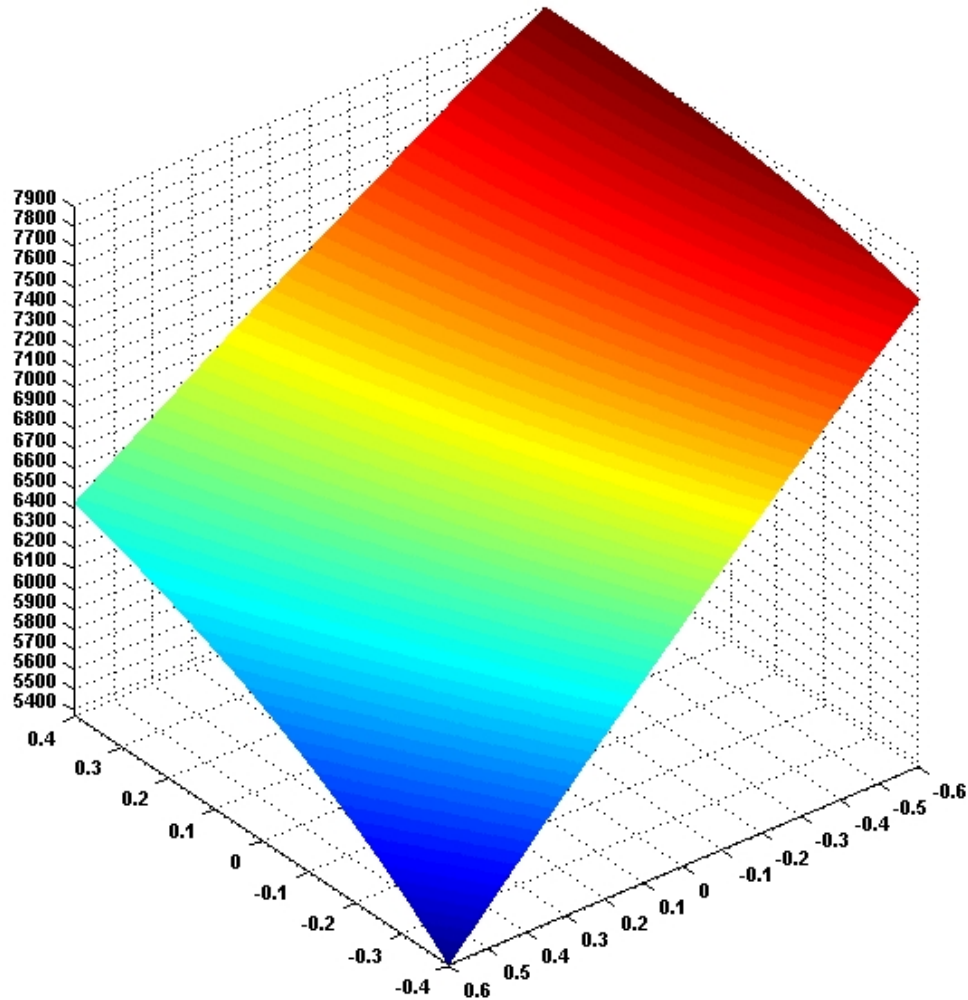


Figure 1: The first player's payoff functional (40) shown on rectangle (42)

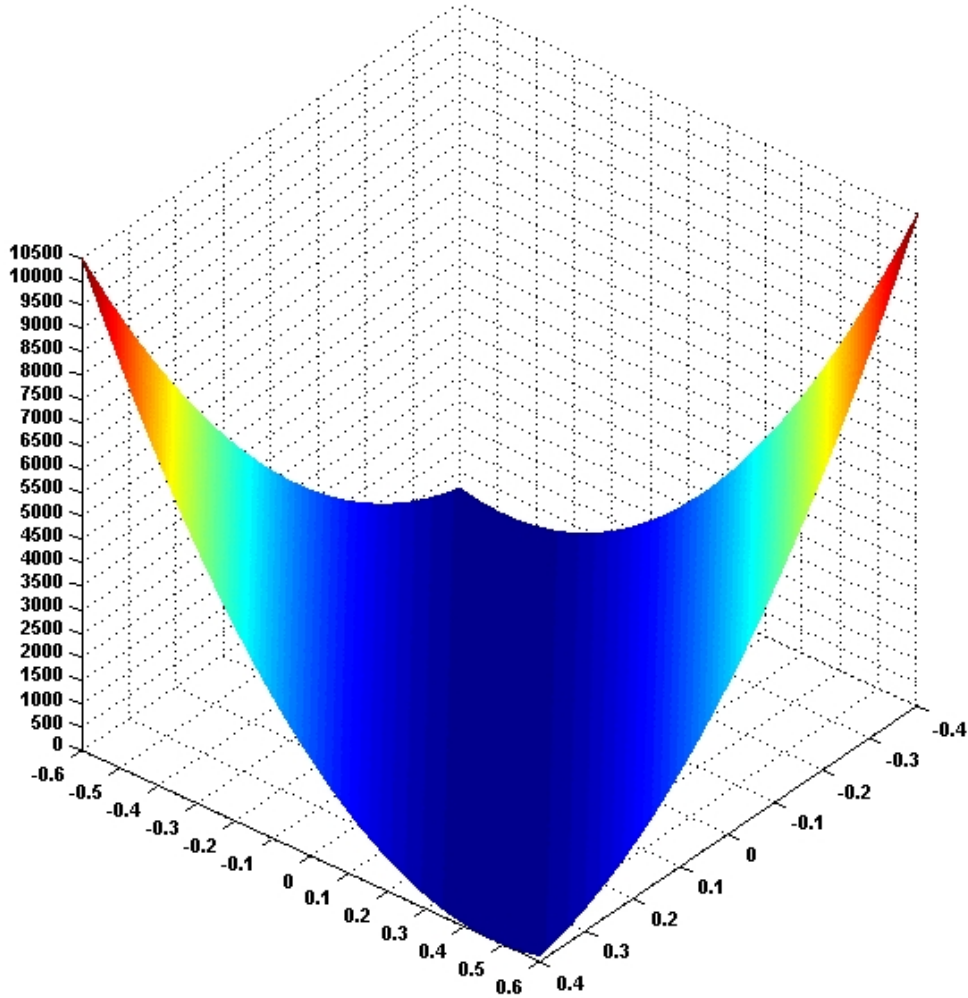


Figure 2: The second player's payoff functional (41) shown on rectangle (42)

with a step of 5 up to 100, a single equilibrium is still found, but the number of efficient situations grows. One of those efficient situations is equilibrium (by Definition 1). In such a situation, the equilibrium-and-efficient strategies of the first player become "stable" as S increases (Figure 3). Eventually,

$$x^{<20>*}(t) = 100 + 0.344t,$$

whereas the equilibrium-and-efficient strategy of the second player remains the same for all $S = 5, 10, 15, \dots, 100$ (Figure 4). So, condition (35) of the sufficient closeness of the second player's strategies holds trivially. After all, the first player's polyline by (36) decreases on average (Figure 5). This means that situation

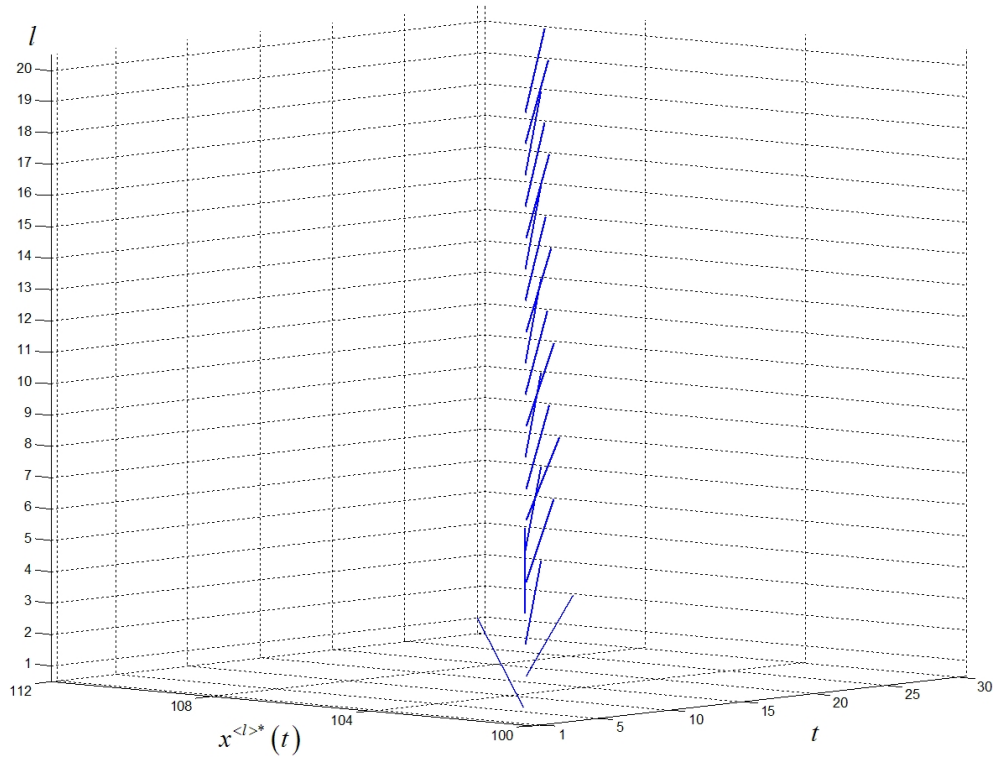


Figure 3: The series of 20 equilibrium-and-efficient strategies of the first player

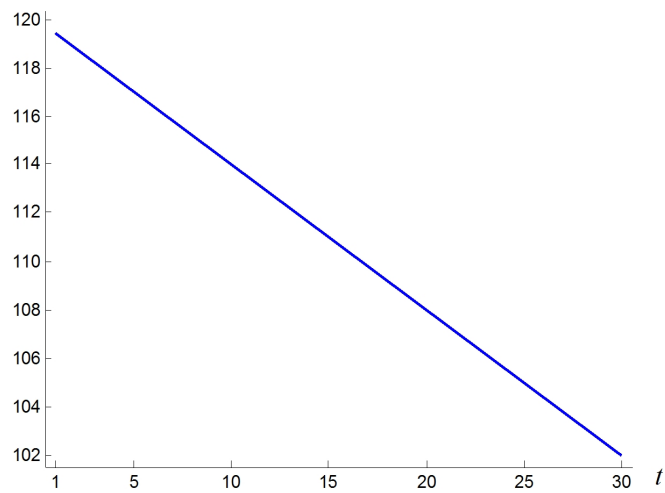


Figure 4: The second player's unvarying equilibrium-and-efficient strategy $y^{<l>*}(t) = 120 - 0.6t$ ($l = \overline{1, 20}$)

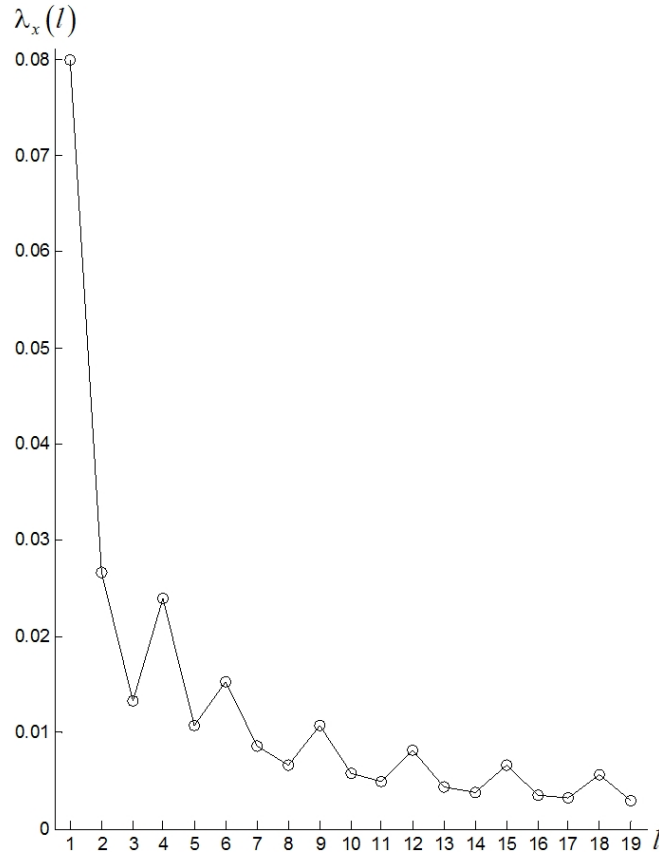


Figure 5: The first player's polyline from (36), which decreases on average

$$\{x^{<20>*}(t), y^{<20>*}(t)\} = \{100 + 0.344t, 120 - 0.6t\}$$

is the solution of the corresponding bimatrix 101×101 -game, which is the single acceptable approximate solution to the initial game with (38) — (41).

8. Discussion

Continuous games are approximated to finite games not just for the sake of simplicity itself. The matter is the finite approximation makes solutions tractable so that they can be easily implemented and practiced. So, the presented method of finite approximation specifies and, what is more important, establishes the applicability of continuous noncooperative two-person games on a product of linear strategy functional spaces. Mainly, it concerns modeling economic interaction processes, where the player can use a continuum of short-term time-varying strategies.

The presented method is quite significant for avoiding too complicated solutions resulting from game continuities and, moreover, functional spaces of pure strategies. This is similar to preventing Einstellung effect in modeling [4]. The transfer from a functional space product to a real-valued rectangle with subsequently sampling it into a grid herein “deinstellungizes” the continuous noncooperative two-person game.

9. Conclusion

For solving continuous noncooperative two-person games on a product of linear strategy functional spaces, a method of their finite approximation is presented, which is based on sampling the linear strategy functional spaces. The sets (i. e., the spaces) of the players’ pure strategies are sampled uniformly so that the resulting finite game is a bimatrix game whose payoff matrices are square. The approximation procedure starts with not a great number of intervals. Then this number is gradually increased, and new, bigger, bimatrix games are solved until an acceptable solution of the bimatrix game becomes sufficiently close to the same-type solutions at the preceding iterations. The closeness is expressed in terms of the respective functional spaces, which is simplified by Theorem 4, giving just the absolute difference between the trend-defining coefficients of the strategies from the neighboring solutions. These distances should be decreasing once they are smoothed with respective polynomials of degree 2.

A question of the game finite approximation for cases of nonlinear strategy spaces (when, say, the player’s strategy space is of parabolas or cubic polynomials) is believed to be answered in the similar manner. Nevertheless, some peculiarities concerning the continuity of the payoff functionals may weaken the impact of Theorem 2 and Theorem 3. Despite this, the game finite approximation will definitely have an expansion in order not to admit the above-mentioned Einstellung effect in modeling economic interaction processes, where players use short-term time-varying strategies of various types.

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Analogy of Classical and Dynamic Inequalities Merging on Time Scales

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ABSTRACT: In this paper, we present analogues of Radon's inequality and Nesbitt's inequality on time scales. Furthermore, we find refinements of some classical inequalities such as Bergström's inequality, the weighted power mean inequality, Cauchy–Schwarz's inequality and Hölder's inequality. Our investigations unify and extend some continuous inequalities and their corresponding discrete analogues.

AMS Subject Classification: 26D15, 26D20, 34N05.

Keywords and Phrases: Time scales; Radon's inequality; The weighted power mean inequality; Hölder's inequality; Nesbitt's inequality.

1. Introduction

We present here some well-known classical inequalities.

If $n \in \mathbb{N}$, $x_k \geq 0$ and $y_k > 0$ for $k \in \{1, 2, \dots, n\}$ and $\beta \geq 2$, then

$$n^{2-\beta} \frac{\left(\sum_{k=1}^n x_k\right)^\beta}{\sum_{k=1}^n y_k} \leq \sum_{k=1}^n \frac{x_k^\beta}{y_k}. \quad (1.1)$$

Inequality (1.1) is called Radon's inequality as given in [21, 22, 23, 24].

The weighted power mean inequality given in [9, pp. 111-112, Theorem 10.5], [11, pp. 12-15] and [15] is defined as follows:

Let x_1, x_2, \dots, x_n be nonnegative real numbers and p_1, p_2, \dots, p_n be positive real numbers. If $\eta_2 > \eta_1 > 0$, then

$$\left(\frac{p_1 x_1^{\eta_1} + p_2 x_2^{\eta_1} + \dots + p_n x_n^{\eta_1}}{p_1 + p_2 + \dots + p_n}\right)^{\frac{1}{\eta_1}} \leq \left(\frac{p_1 x_1^{\eta_2} + p_2 x_2^{\eta_2} + \dots + p_n x_n^{\eta_2}}{p_1 + p_2 + \dots + p_n}\right)^{\frac{1}{\eta_2}}. \quad (1.2)$$

If x_k and y_k for $k \in \{1, 2, \dots, n\}$ are sequences of real numbers, then Cauchy–Schwarz’s inequality is given by:

$$\sum_{k=1}^n x_k y_k \leq \left(\sum_{k=1}^n x_k^2 \right)^{\frac{1}{2}} \left(\sum_{k=1}^n y_k^2 \right)^{\frac{1}{2}}, \quad (1.3)$$

as given in [9].

We will prove these results on time scales. The calculus of time scales was initiated by Stefan Hilger as given in [12]. A *time scale* is an arbitrary nonempty closed subset of the real numbers. The theory of time scales is applied to combine results in one comprehensive form. The three most popular examples of calculus on time scales are differential calculus, difference calculus, and quantum calculus, i.e., when $\mathbb{T} = \mathbb{R}$, $\mathbb{T} = \mathbb{N}$ and $\mathbb{T} = q^{\mathbb{N}_0} = \{q^t : t \in \mathbb{N}_0\}$ where $q > 1$. The time scales calculus is studied as delta calculus, nabla calculus and diamond- α calculus. This hybrid theory is also widely applied on dynamic inequalities. The basic work on dynamic inequalities is done by Ravi Agarwal, George Anastassiou, Martin Bohner, Allan Peterson, Donal O’Regan, Samir Saker and many other authors.

In this paper, it is assumed that all considerable integrals exist and are finite and \mathbb{T} is a time scale, $a, b \in \mathbb{T}$ with $a < b$ and an interval $[a, b]_{\mathbb{T}}$ means the intersection of a real interval with the given time scale.

2. Preliminaries

We need here basic concepts of delta calculus. The results of delta calculus are adopted from monographs [6, 7].

For $t \in \mathbb{T}$, the *forward jump operator* $\sigma : \mathbb{T} \rightarrow \mathbb{T}$ is defined by

$$\sigma(t) := \inf\{s \in \mathbb{T} : s > t\}.$$

The mapping $\mu : \mathbb{T} \rightarrow \mathbb{R}_0^+ = [0, +\infty)$ such that $\mu(t) := \sigma(t) - t$ is called the *forward graininess function*. The *backward jump operator* $\rho : \mathbb{T} \rightarrow \mathbb{T}$ is defined by

$$\rho(t) := \sup\{s \in \mathbb{T} : s < t\}.$$

The mapping $\nu : \mathbb{T} \rightarrow \mathbb{R}_0^+ = [0, +\infty)$ such that $\nu(t) := t - \rho(t)$ is called the *backward graininess function*. If $\sigma(t) > t$, we say that t is *right-scattered*, while if $\rho(t) < t$, we say that t is *left-scattered*. Also, if $t < \sup \mathbb{T}$ and $\sigma(t) = t$, then t is called *right-dense*, and if $t > \inf \mathbb{T}$ and $\rho(t) = t$, then t is called *left-dense*. If \mathbb{T} has a left-scattered maximum M , then $\mathbb{T}^k = \mathbb{T} - \{M\}$, otherwise $\mathbb{T}^k = \mathbb{T}$.

For a function $f : \mathbb{T} \rightarrow \mathbb{R}$, the delta derivative f^Δ is defined as follows:

Let $t \in \mathbb{T}^k$. If there exists $f^\Delta(t) \in \mathbb{R}$ such that for all $\epsilon > 0$, there is a neighborhood U of t , such that

$$|f(\sigma(t)) - f(s) - f^\Delta(t)(\sigma(t) - s)| \leq \epsilon|\sigma(t) - s|,$$

for all $s \in U$, then f is said to be *delta differentiable* at t , and $f^\Delta(t)$ is called the *delta derivative* of f at t .

A function $f : \mathbb{T} \rightarrow \mathbb{R}$ is said to be *right-dense continuous* (*rd-continuous*), if it is continuous at each right-dense point and there exists a finite left-sided limit at every left-dense point. The set of all rd-continuous functions is denoted by $C_{rd}(\mathbb{T}, \mathbb{R})$.

The next definition is given in [6, 7].

Definition 2.1. A function $F : \mathbb{T} \rightarrow \mathbb{R}$ is called a delta antiderivative of $f : \mathbb{T} \rightarrow \mathbb{R}$, provided that $F^\Delta(t) = f(t)$ holds for all $t \in \mathbb{T}^k$. Then the delta integral of f is defined by

$$\int_a^b f(t)\Delta t = F(b) - F(a).$$

The following results of nabla calculus are taken from [2, 6, 7].

If \mathbb{T} has a right-scattered minimum m , then $\mathbb{T}_k = \mathbb{T} - \{m\}$, otherwise $\mathbb{T}_k = \mathbb{T}$. A function $f : \mathbb{T}_k \rightarrow \mathbb{R}$ is called *nabla differentiable* at $t \in \mathbb{T}_k$, with nabla derivative $f^\nabla(t)$, if there exists $f^\nabla(t) \in \mathbb{R}$ such that given any $\epsilon > 0$, there is a neighborhood V of t , such that

$$|f(\rho(t)) - f(s) - f^\nabla(t)(\rho(t) - s)| \leq \epsilon|\rho(t) - s|,$$

for all $s \in V$.

A function $f : \mathbb{T} \rightarrow \mathbb{R}$ is said to be *left-dense continuous* (*ld-continuous*), provided it is continuous at all left-dense points in \mathbb{T} and its right-sided limits exist (finite) at all right-dense points in \mathbb{T} . The set of all ld-continuous functions is denoted by $C_{ld}(\mathbb{T}, \mathbb{R})$.

The next definition is given in [2, 6, 7].

Definition 2.2. A function $G : \mathbb{T} \rightarrow \mathbb{R}$ is called a nabla antiderivative of $g : \mathbb{T} \rightarrow \mathbb{R}$, provided that $G^\nabla(t) = g(t)$ holds for all $t \in \mathbb{T}_k$. Then the nabla integral of g is defined by

$$\int_a^b g(t)\nabla t = G(b) - G(a).$$

Now we present short introduction of diamond- α derivative as given in [1, 19].

Let \mathbb{T} be a time scale and $f(t)$ be differentiable on \mathbb{T} in the Δ and ∇ senses. For $t \in \mathbb{T}_k^k$, where $\mathbb{T}_k^k = \mathbb{T}^k \cap \mathbb{T}_k$, the *diamond- α dynamic derivative* $f^{\diamond\alpha}(t)$ is defined by

$$f^{\diamond\alpha}(t) = \alpha f^\Delta(t) + (1 - \alpha)f^\nabla(t), \quad 0 \leq \alpha \leq 1.$$

Thus f is diamond- α differentiable if and only if f is Δ and ∇ differentiable.

The diamond- α derivative reduces to the standard Δ -derivative for $\alpha = 1$, or the standard ∇ -derivative for $\alpha = 0$. It represents a weighted dynamic derivative for $\alpha \in (0, 1)$.

Theorem 2.3 ([19]). *Let $f, g : \mathbb{T} \rightarrow \mathbb{R}$ be diamond- α differentiable at $t \in \mathbb{T}$ and we write $f^\sigma(t) = f(\sigma(t))$, $g^\sigma(t) = g(\sigma(t))$, $f^\rho(t) = f(\rho(t))$ and $g^\rho(t) = g(\rho(t))$. Then*

(i) $f \pm g : \mathbb{T} \rightarrow \mathbb{R}$ is diamond- α differentiable at $t \in \mathbb{T}$, with

$$(f \pm g)^{\diamond\alpha}(t) = f^{\diamond\alpha}(t) \pm g^{\diamond\alpha}(t).$$

(ii) $fg : \mathbb{T} \rightarrow \mathbb{R}$ is diamond- α differentiable at $t \in \mathbb{T}$, with

$$(fg)^{\diamond\alpha}(t) = f^{\diamond\alpha}(t)g(t) + \alpha f^{\sigma}(t)g^{\Delta}(t) + (1 - \alpha)f^{\rho}(t)g^{\nabla}(t).$$

(iii) For $g(t)g^{\sigma}(t)g^{\rho}(t) \neq 0$, $\frac{f}{g} : \mathbb{T} \rightarrow \mathbb{R}$ is diamond- α differentiable at $t \in \mathbb{T}$, with

$$\left(\frac{f}{g}\right)^{\diamond\alpha}(t) = \frac{f^{\diamond\alpha}(t)g^{\sigma}(t)g^{\rho}(t) - \alpha f^{\sigma}(t)g^{\rho}(t)g^{\Delta}(t) - (1 - \alpha)f^{\rho}(t)g^{\sigma}(t)g^{\nabla}(t)}{g(t)g^{\sigma}(t)g^{\rho}(t)}.$$

Definition 2.4 ([19]). Let $a, t \in \mathbb{T}$ and $h : \mathbb{T} \rightarrow \mathbb{R}$. Then the diamond- α integral from a to t of h is defined by

$$\int_a^t h(s) \diamond_{\alpha} s = \alpha \int_a^t h(s) \Delta s + (1 - \alpha) \int_a^t h(s) \nabla s, \quad 0 \leq \alpha \leq 1,$$

provided that there exist delta and nabla integrals of h on \mathbb{T} .

Theorem 2.5 ([19]). Let $a, b, t \in \mathbb{T}$, $c \in \mathbb{R}$. Assume that $f(s)$ and $g(s)$ are \diamond_{α} -integrable functions on $[a, b]_{\mathbb{T}}$. Then

$$(i) \int_a^t [f(s) \pm g(s)] \diamond_{\alpha} s = \int_a^t f(s) \diamond_{\alpha} s \pm \int_a^t g(s) \diamond_{\alpha} s.$$

$$(ii) \int_a^t cf(s) \diamond_{\alpha} s = c \int_a^t f(s) \diamond_{\alpha} s.$$

$$(iii) \int_a^t f(s) \diamond_{\alpha} s = - \int_t^a f(s) \diamond_{\alpha} s.$$

$$(iv) \int_a^t f(s) \diamond_{\alpha} s = \int_a^b f(s) \diamond_{\alpha} s + \int_b^t f(s) \diamond_{\alpha} s.$$

$$(v) \int_a^a f(s) \diamond_{\alpha} s = 0.$$

We need the following results.

Definition 2.6 ([10]). A function $f : \mathbb{T} \rightarrow \mathbb{R}$ is called convex on $I_{\mathbb{T}} = I \cap \mathbb{T}$, where I is an interval of \mathbb{R} (open or closed), if

$$f(\chi t + (1 - \chi)s) \leq \chi f(t) + (1 - \chi)f(s), \quad (2.1)$$

for all $t, s \in I_{\mathbb{T}}$ and all $\chi \in [0, 1]$ such that $\chi t + (1 - \chi)s \in I_{\mathbb{T}}$.

The function f is strictly convex on $I_{\mathbb{T}}$ if the inequality (2.1) is strict for distinct $t, s \in I_{\mathbb{T}}$ and $\chi \in (0, 1)$.

The function f is concave (respectively, strictly concave) on $I_{\mathbb{T}}$, if $-f$ is convex (respectively, strictly convex).

Theorem 2.7 ([1]). *Let $a, b \in \mathbb{T}$ and $c, d \in \mathbb{R}$. Suppose that $g \in C([a, b]_{\mathbb{T}}, (c, d))$ and $h \in C([a, b]_{\mathbb{T}}, \mathbb{R})$ with $\int_a^b |h(s)| \diamond_{\alpha} s > 0$. If $\Phi \in C((c, d), \mathbb{R})$ is convex, then generalized Jensen's inequality is*

$$\Phi \left(\frac{\int_a^b |h(s)|g(s) \diamond_{\alpha} s}{\int_a^b |h(s)| \diamond_{\alpha} s} \right) \leq \frac{\int_a^b |h(s)|\Phi(g(s)) \diamond_{\alpha} s}{\int_a^b |h(s)| \diamond_{\alpha} s}. \tag{2.2}$$

If Φ is strictly convex, then the inequality \leq can be replaced by $<$.

Theorem 2.8 ([16]). *Let $w, f, g \in C([a, b]_{\mathbb{T}}, \mathbb{R})$ be \diamond_{α} -integrable functions, where $w, g \neq 0$. If $\xi \geq 0$, then*

$$\frac{\left(\int_a^b |w(x)||f(x)| \diamond_{\alpha} x \right)^{\xi+1}}{\left(\int_a^b |w(x)||g(x)| \diamond_{\alpha} x \right)^{\xi}} \leq \int_a^b \frac{|w(x)||f(x)|^{\xi+1}}{|g(x)|^{\xi}} \diamond_{\alpha} x. \tag{2.3}$$

Inequality (2.3) is called Radon's inequality on time scales and is reversed for $-1 < \xi < 0$.

3. Main Results

In order to present our main results, first we present a simple proof for an extension of Radon's inequality on time scales.

Theorem 3.1. *Let $w, f, g \in C([a, b]_{\mathbb{T}}, \mathbb{R})$ be \diamond_{α} -integrable functions with $\int_a^b |w(x)| \diamond_{\alpha} x > 0$ and $g \neq 0$. If $\beta \geq 2$, then*

$$\left(\int_a^b |w(x)| \diamond_{\alpha} x \right)^{2-\beta} \frac{\left(\int_a^b |w(x)||f(x)| \diamond_{\alpha} x \right)^{\beta}}{\int_a^b |w(x)||g(x)| \diamond_{\alpha} x} \leq \int_a^b \frac{|w(x)||f(x)|^{\beta}}{|g(x)|} \diamond_{\alpha} x. \tag{3.1}$$

Proof. The right-hand side of (3.1) takes the form

$$\int_a^b \frac{|w(x)||f(x)|^{\beta}}{|g(x)|} \diamond_{\alpha} x = \int_a^b \frac{|w(x)||f(x)|^{\beta}}{\left(|g(x)|^{\frac{1}{\beta-1}} \right)^{\beta-1}} \diamond_{\alpha} x. \tag{3.2}$$

Applying Radon's inequality (2.3), the inequality (3.2) becomes

$$\int_a^b \frac{|w(x)||f(x)|^{\beta}}{|g(x)|} \diamond_{\alpha} x \geq \frac{\left(\int_a^b |w(x)||f(x)| \diamond_{\alpha} x \right)^{\beta}}{\left(\int_a^b |w(x)||g(x)|^{\frac{1}{\beta-1}} \diamond_{\alpha} x \right)^{\beta-1}}. \tag{3.3}$$

Note that

$$\int_a^b |w(x)||g(x)|^{\frac{1}{\beta-1}} \diamond_{\alpha} x = \int_a^b \frac{|w(x)||g(x)|^{\frac{1}{\beta-1}}}{1^{\frac{1}{\beta-1}-1}} \diamond_{\alpha} x. \tag{3.4}$$

Applying reverse Radon's inequality on right-hand side of (3.4), we get

$$\int_a^b \frac{|w(x)||g(x)|^{\frac{1}{\beta-1}}}{1^{\frac{1}{\beta-1}-1}} \diamond_\alpha x \leq \frac{\left(\int_a^b |w(x)||g(x)| \diamond_\alpha x\right)^{\frac{1}{\beta-1}}}{\left(\int_a^b |w(x)| \diamond_\alpha x\right)^{\frac{2-\beta}{\beta-1}}}. \quad (3.5)$$

From (3.3) and (3.5), we get the proof of the desired result. \square

Remark 3.2. Let $\alpha = 1$, $\mathbb{T} = \mathbb{Z}$, $a = 1$, $b = n + 1$, $w \equiv 1$, $f(k) = x_k \in [0, +\infty)$ and $g(k) = y_k \in (0, +\infty)$ for $k \in \{1, 2, \dots, n\}$. Then (3.1) reduces to (1.1).

Remark 3.3. Let $\alpha = 1$, $\mathbb{T} = \mathbb{Z}$, $a = 1$, $b = n + 1$, $w \equiv 1$, $f(k) = x_k \in \mathbb{R}$ and $g(k) = y_k \in (0, +\infty)$ for $k \in \{1, 2, \dots, n\}$. If $\beta = 2$, then (3.1) reduces to

$$\frac{\left(\sum_{k=1}^n x_k\right)^2}{\sum_{k=1}^n y_k} \leq \sum_{k=1}^n \frac{x_k^2}{y_k}, \quad (3.6)$$

which is called Bergström's inequality or Titu Andreescu's inequality, or Engel's inequality in literature as given in [4, 5, 8, 14] with equality if and only if $\frac{x_1}{y_1} = \frac{x_2}{y_2} = \dots = \frac{x_n}{y_n}$.

The following inequality is called the dynamic weighted power mean inequality on time scales.

Corollary 3.4. Let $w, f \in C([a, b]_{\mathbb{T}}, \mathbb{R})$ be \diamond_α -integrable functions with $\int_a^b |w(x)| \diamond_\alpha x > 0$. If $\eta \geq \eta_1 > 0$ and $\eta_2 = 2\eta$, then

$$\left(\frac{\int_a^b |w(x)||f(x)|^{\eta_1} \diamond_\alpha x}{\int_a^b |w(x)| \diamond_\alpha x}\right)^{\frac{1}{\eta_1}} \leq \left(\frac{\int_a^b |w(x)||f(x)|^{\eta_2} \diamond_\alpha x}{\int_a^b |w(x)| \diamond_\alpha x}\right)^{\frac{1}{\eta_2}}. \quad (3.7)$$

Proof. Set $\beta = 2\left(\frac{\eta}{\eta_1}\right) = \frac{\eta_2}{\eta_1} \geq 2$ and $g \equiv 1$. The inequality (3.1) reduces to

$$\left(\int_a^b |w(x)| \diamond_\alpha x\right)^{2-\frac{\eta_2}{\eta_1}} \frac{\left(\int_a^b |w(x)||f(x)| \diamond_\alpha x\right)^{\frac{\eta_2}{\eta_1}}}{\int_a^b |w(x)| \diamond_\alpha x} \leq \int_a^b |w(x)||f(x)|^{\frac{\eta_2}{\eta_1}} \diamond_\alpha x. \quad (3.8)$$

Replacing $|f(x)|$ by $|f(x)|^{\eta_1}$ and taking power $\frac{1}{\eta_2}$ on both sides of (3.8), we get

$$\begin{aligned} & \left(\int_a^b |w(x)| \diamond_\alpha x\right)^{\frac{1}{\eta_2} - \frac{1}{\eta_1}} \left(\int_a^b |w(x)||f(x)|^{\eta_1} \diamond_\alpha x\right)^{\frac{1}{\eta_1}} \\ & \leq \left(\int_a^b |w(x)||f(x)|^{\eta_2} \diamond_\alpha x\right)^{\frac{1}{\eta_2}}. \end{aligned} \quad (3.9)$$

This completes the desired result. \square

Remark 3.5. If we set $\alpha = 1$, $\mathbb{T} = \mathbb{Z}$, $a = 1$, $b = n + 1$, $w(k) = p_k \in (0, +\infty)$ and $f(k) = x_k \in [0, +\infty)$ for $k \in \{1, 2, \dots, n\}$, then (3.7) reduces to (1.2). Further, if $\sum_{k=1}^n p_k = 1$ and $\eta_1 = \eta$, then (1.2) reduces to

$$\left(\sum_{k=1}^n p_k x_k^{\eta_1} \right)^{\frac{1}{\eta_1}} \leq \left(\sum_{k=1}^n p_k x_k^{2\eta_1} \right)^{\frac{1}{2\eta_1}},$$

as given in [11].

Now we present Cauchy–Schwarz’s inequality on time scales.

Corollary 3.6. Let $w, f, g \in C([a, b]_{\mathbb{T}}, \mathbb{R})$ be \diamond_{α} -integrable functions. We have:

$$\begin{aligned} & \left(\int_a^b |w(x)||f(x)g(x)| \diamond_{\alpha} x \right)^2 \\ & \leq \left(\int_a^b |w(x)||f(x)|^2 \diamond_{\alpha} x \right) \left(\int_a^b |w(x)||g(x)|^2 \diamond_{\alpha} x \right). \end{aligned} \tag{3.10}$$

Proof. Setting $\beta = 2$ and replacing $|w(x)|$ by $|w(x)g(x)|$ in (3.1), the inequality (3.10) follows. \square

Remark 3.7. If we set $\alpha = 1$, $\mathbb{T} = \mathbb{Z}$, $a = 1$, $b = n + 1$, $w \equiv 1$, $f(k) = x_k \in \mathbb{R}$ and $g(k) = y_k \in \mathbb{R}$ for $k \in \{1, 2, \dots, n\}$, then (3.10) reduces to (1.3).

Corollary 3.8. Let $w, f \in C([a, b]_{\mathbb{T}}, \mathbb{R} - \{0\})$ be \diamond_{α} -integrable functions. If $\beta \geq 2$, then

$$\left(\int_a^b |w(x)||f(x)| \diamond_{\alpha} x \right)^{\beta} \leq \left(\int_a^b |w(x)|^{\beta} \diamond_{\alpha} x \right) \left(\int_a^b |f(x)|^{\frac{\beta}{\beta-1}} \diamond_{\alpha} x \right)^{\beta-1}. \tag{3.11}$$

Proof. Let $W, F, G \in C([a, b]_{\mathbb{T}}, \mathbb{R})$ be \diamond_{α} -integrable functions, neither $W \equiv 0$ nor $G \equiv 0$. If $\beta \geq 2$, then (3.1) takes the form

$$\left(\int_a^b |W(x)| \diamond_{\alpha} x \right)^{2-\beta} \frac{\left(\int_a^b |W(x)||F(x)| \diamond_{\alpha} x \right)^{\beta}}{\int_a^b |W(x)||G(x)| \diamond_{\alpha} x} \leq \int_a^b \frac{|W(x)||F(x)|^{\beta}}{|G(x)|} \diamond_{\alpha} x.$$

Putting $G \equiv 1$ and replacing $|W(x)|$ by $|f(x)|^{\frac{\beta}{\beta-1}}$ and $|F(x)|$ by $|w(x)||f(x)|^{\frac{-1}{\beta-1}}$, we get (3.11). \square

Remark 3.9. Let $\alpha = 1$, $\mathbb{T} = \mathbb{Z}$, $a = 1$, $b = n + 1$, $w(k) = p_k \in (0, +\infty)$ and $f(k) = x_k \in (0, +\infty)$ for $k \in \{1, 2, \dots, n\}$. If $\beta \geq 2$, then (3.11) reduces to

$$\left(\sum_{k=1}^n p_k x_k \right)^\beta \leq \left(\sum_{k=1}^n p_k^\beta \right) \left(\sum_{k=1}^n x_k^{\frac{\beta}{\beta-1}} \right)^{\beta-1}, \quad (3.12)$$

which is symmetric form of Hölder's inequality, as given in [13].

The following result is a generalization of Nesbitt's inequality on time scales.

Theorem 3.10. Let $w, f \in C([a, b]_{\mathbb{T}}, \mathbb{R} - \{0\})$ be \diamond_α -integrable functions. If $\gamma \geq 1$, $\eta \geq \eta_1 > 0$, $\eta_2 = 2\eta$, $\Omega = \int_a^b |w(x)| |f(x)|^{\eta_1} \diamond_\alpha x$ and $\Omega > \sup_{x \in [a, b]_{\mathbb{T}}} |f(x)|^{\eta_1}$, then

$$\begin{aligned} & \left(\frac{\int_a^b |w(x)| \diamond_\alpha x}{\left(\int_a^b |w(x)| \diamond_\alpha x - 1 \right)^\gamma} \right) \left(\frac{\Omega}{\int_a^b |w(x)| \diamond_\alpha x} \right)^{\gamma \left(\frac{\eta_2}{\eta_1} - 1 \right)} \\ & \leq \int_a^b |w(x)| \left(\frac{|f(x)|^{\eta_2}}{\Omega - |f(x)|^{\eta_1}} \right)^\gamma \diamond_\alpha x. \end{aligned} \quad (3.13)$$

Proof. Applying Jensen's inequality for $\gamma > 1$, we get

$$\begin{aligned} & \left(\int_a^b |w(x)| \left(\frac{|f(x)|^{\eta_2}}{\Omega - |f(x)|^{\eta_1}} \right) \diamond_\alpha x \right)^\gamma \\ & \leq \left(\int_a^b |w(x)| \diamond_\alpha x \right)^{\gamma-1} \int_a^b |w(x)| \left(\frac{|f(x)|^{\eta_2}}{\Omega - |f(x)|^{\eta_1}} \right)^\gamma \diamond_\alpha x. \end{aligned} \quad (3.14)$$

Now applying Radon's inequality (3.1), we get

$$\begin{aligned} & \int_a^b |w(x)| \left(\frac{|f(x)|^{\eta_2}}{\Omega - |f(x)|^{\eta_1}} \right) \diamond_\alpha x \\ & = \int_a^b |w(x)| \left(\frac{(|f(x)|^{\eta_1})^{\frac{\eta_2}{\eta_1}}}{\Omega - |f(x)|^{\eta_1}} \right) \diamond_\alpha x \\ & \geq \left(\int_a^b |w(x)| \diamond_\alpha x \right)^{2 - \frac{\eta_2}{\eta_1}} \frac{\left(\int_a^b |w(x)| |f(x)|^{\eta_1} \diamond_\alpha x \right)^{\frac{\eta_2}{\eta_1}}}{\int_a^b |w(x)| (\Omega - |f(x)|^{\eta_1}) \diamond_\alpha x} \\ & = \frac{\left(\int_a^b |w(x)| \diamond_\alpha x \right)}{\left(\int_a^b |w(x)| \diamond_\alpha x - 1 \right)} \left(\frac{\Omega}{\int_a^b |w(x)| \diamond_\alpha x} \right)^{\frac{\eta_2}{\eta_1} - 1}. \end{aligned}$$

Thus

$$\left(\int_a^b |w(x)| \left(\frac{|f(x)|^{\eta_2}}{\Omega - |f(x)|^{\eta_1}} \right) \diamond_\alpha x \right)^\gamma \geq \frac{\left(\int_a^b |w(x)| \diamond_\alpha x \right)^\gamma}{\left(\int_a^b |w(x)| \diamond_\alpha x - 1 \right)^\gamma} \left(\frac{\Omega}{\int_a^b |w(x)| \diamond_\alpha x} \right)^{\gamma \left(\frac{\eta_2}{\eta_1} - 1 \right)}. \quad (3.15)$$

Combining (3.14) and (3.15), we get the desired claim. □

Remark 3.11. If we set $\alpha = 1, \mathbb{T} = \mathbb{Z}, a = 1, b = n + 1, w \equiv 1, f(k) = x_k \in (0, +\infty)$ for $k \in \{1, 2, \dots, n\}$ and $\sum_{k=1}^n x_k^{\eta_1} > \max_{1 \leq k \leq n} x_k^{\eta_1}$, then (3.13) reduces to

$$\left(\frac{n}{(n-1)^\gamma}\right) \left(\frac{\sum_{k=1}^n x_k^{\eta_1}}{n}\right)^{\gamma\left(\frac{\eta_2}{\eta_1}-1\right)} \leq \sum_{k=1}^n \left(\frac{x_k^{\eta_2}}{\sum_{k=1}^n x_k^{\eta_1} - x_k^{\eta_1}}\right)^\gamma, \tag{3.16}$$

as given in [20].

Further, if we take $\eta_1 = 1, \gamma = 1, n = 3, x_1 = x, x_2 = y$ and $x_3 = z$, then (3.16) takes the form

$$\frac{3}{2} \left(\frac{x+y+z}{3}\right)^{\eta_2-1} \leq \frac{x^{\eta_2}}{y+z} + \frac{y^{\eta_2}}{z+x} + \frac{z^{\eta_2}}{x+y}. \tag{3.17}$$

Inequality (3.17) is called the generalized Nesbitt’s inequality as given in [20].

The following result is another consequence of Radon’s inequality on time scales.

Theorem 3.12. Let $w, f \in C([a, b]_{\mathbb{T}}, \mathbb{R} - \{0\})$ be \diamond_α -integrable functions. If $c_1 \in [0, +\infty), c_2, c_3, c_4 \in (0, +\infty), \gamma, \zeta, \kappa, \lambda \in [1, +\infty)$ and $c_3 \left(\int_a^b |w(x)||f(x)| \diamond_\alpha x\right)^\gamma > c_4 \sup_{x \in [a, b]_{\mathbb{T}}} |f(x)|^\gamma$, then

$$\begin{aligned} & \frac{\left(c_1 \left(\int_a^b |w(x)| \diamond_\alpha x\right)^\kappa + c_2\right)^\lambda}{\left(c_3 \left(\int_a^b |w(x)| \diamond_\alpha x\right)^\gamma - c_4\right)^\zeta} \left(\int_a^b |w(x)| \diamond_\alpha x\right)^{\gamma\zeta - \kappa\lambda + 1} \\ & \left(\int_a^b |w(x)||f(x)| \diamond_\alpha x\right)^{\kappa\lambda - \gamma\zeta} \leq \left(\frac{1}{\int_a^b |w(x)| \diamond_\alpha x}\right) \\ & \left\{ \int_a^b |w(x)| \left(c_1 \left(\int_a^b |w(x)||f(x)| \diamond_\alpha x\right)^\kappa + c_2 |f(x)|^\kappa\right)^\lambda \diamond_\alpha x \right\} \\ & \times \int_a^b |w(x)| \left\{ \frac{1}{\left(c_3 \left(\int_a^b |w(x)||f(x)| \diamond_\alpha x\right)^\gamma - c_4 |f(x)|^\gamma\right)^\zeta} \right\} \diamond_\alpha x. \tag{3.18} \end{aligned}$$

Proof. We obtain the following result by applying Radon’s inequality given in (2.3),

as

$$\begin{aligned} & \frac{\left(\int_a^b |w(x)| \diamond_\alpha x\right)^{\zeta+1}}{\left\{ \int_a^b |w(x)| \left(c_3 \left(\int_a^b |w(x)||f(x)| \diamond_\alpha x \right)^\gamma - c_4 |f(x)|^\gamma \right) \diamond_\alpha x \right\}^\zeta} \\ & \leq \int_a^b |w(x)| \left\{ \frac{1^{\zeta+1}}{\left(c_3 \left(\int_a^b |w(x)||f(x)| \diamond_\alpha x \right)^\gamma - c_4 |f(x)|^\gamma \right)^\zeta} \right\} \diamond_\alpha x. \end{aligned} \tag{3.19}$$

Applying (2.2) and (3.19), the right-hand side of (3.18) takes the form

$$\begin{aligned} & \left(\frac{1}{\int_a^b |w(x)| \diamond_\alpha x} \right) \left\{ \int_a^b |w(x)| \left(c_1 \left(\int_a^b |w(x)||f(x)| \diamond_\alpha x \right)^\kappa + c_2 |f(x)|^\kappa \right) \diamond_\alpha x \right\}^\lambda \\ & \quad \times \int_a^b |w(x)| \left\{ \frac{1}{\left(c_3 \left(\int_a^b |w(x)||f(x)| \diamond_\alpha x \right)^\gamma - c_4 |f(x)|^\gamma \right)^\zeta} \right\} \diamond_\alpha x \\ & \quad \geq \left(\int_a^b |w(x)| \diamond_\alpha x \right)^{\zeta+1-\lambda} \\ & \quad \times \frac{\left\{ c_1 \left(\int_a^b |w(x)| \diamond_\alpha x \right) \left(\int_a^b |w(x)||f(x)| \diamond_\alpha x \right)^\kappa + c_2 \int_a^b |w(x)||f(x)|^\kappa \diamond_\alpha x \right\}^\lambda}{\left\{ c_3 \left(\int_a^b |w(x)| \diamond_\alpha x \right) \left(\int_a^b |w(x)||f(x)| \diamond_\alpha x \right)^\gamma - c_4 \int_a^b |w(x)||f(x)|^\gamma \diamond_\alpha x \right\}^\zeta} \\ & \quad \geq \left(\int_a^b |w(x)| \diamond_\alpha x \right)^{\zeta+1-\lambda} \\ & \quad \times \frac{\left\{ c_1 \left(\int_a^b |w(x)| \diamond_\alpha x \right) \left(\int_a^b |w(x)||f(x)| \diamond_\alpha x \right)^\kappa + c_2 \frac{\left(\int_a^b |w(x)||f(x)| \diamond_\alpha x \right)^\kappa}{\left(\int_a^b |w(x)| \diamond_\alpha x \right)^{\kappa-1}} \right\}^\lambda}{\left\{ c_3 \left(\int_a^b |w(x)| \diamond_\alpha x \right) \left(\int_a^b |w(x)||f(x)| \diamond_\alpha x \right)^\gamma - c_4 \frac{\left(\int_a^b |w(x)||f(x)| \diamond_\alpha x \right)^\gamma}{\left(\int_a^b |w(x)| \diamond_\alpha x \right)^{\gamma-1}} \right\}^\zeta}. \end{aligned}$$

Therefore, the inequality (3.18) follows. □

Remark 3.13. If we set $\alpha = 1, \mathbb{T} = \mathbb{Z}, a = 1, b = n + 1, w \equiv 1, f(k) = x_k \in (0, +\infty)$ for $k \in \{1, 2, \dots, n\}, X_n = \sum_{k=1}^n x_k$ and $c_3 \left(\sum_{k=1}^n x_k \right)^\gamma > c_4 \max_{1 \leq k \leq n} x_k^\gamma$, then (3.18) reduces to

$$\begin{aligned} & \frac{(c_1 n^\kappa + c_2)^\lambda}{(c_3 n^\gamma - c_4)^\zeta} n^{\gamma\zeta - \kappa\lambda + 1} X_n^{\kappa\lambda - \gamma\zeta} \\ & \leq \frac{1}{n} \left(\sum_{k=1}^n (c_1 X_n^\kappa + c_2 x_k^\kappa)^\lambda \right) \sum_{k=1}^n \frac{1}{(c_3 X_n^\gamma - c_4 x_k^\gamma)^\zeta}, \end{aligned} \tag{3.20}$$

as given in [3].

Corollary 3.14. *Let $w, f \in C([a, b]_{\mathbb{T}}, \mathbb{R} - \{0\})$ be \diamond_{α} -integrable functions. If $c_1 \in [0, +\infty)$, $c_2, c_3, c_4 \in (0, +\infty)$, $\beta \in [2, +\infty)$ and $c_3 \int_a^b |w(x)||f(x)| \diamond_{\alpha} x > c_4 \sup_{x \in [a, b]_{\mathbb{T}}} |f(x)|$, then*

$$\begin{aligned} & \frac{\left(c_1 \int_a^b |w(x)| \diamond_{\alpha} x + c_2\right)^{\beta}}{c_3 \int_a^b |w(x)| \diamond_{\alpha} x - c_4} \left(\int_a^b |w(x)| \diamond_{\alpha} x\right)^{2-\beta} \left(\int_a^b |w(x)||f(x)| \diamond_{\alpha} x\right)^{\beta-1} \\ & \leq \int_a^b |w(x)| \left\{ \frac{\left(c_1 \int_a^b |w(x)||f(x)| \diamond_{\alpha} x + c_2|f(x)|\right)^{\beta}}{c_3 \int_a^b |w(x)||f(x)| \diamond_{\alpha} x - c_4|f(x)|} \right\} \diamond_{\alpha} x. \end{aligned} \tag{3.21}$$

Proof. By applying (3.1), the right-hand side of (3.21) becomes

$$\begin{aligned} & \int_a^b |w(x)| \left\{ \frac{\left(c_1 \int_a^b |w(x)||f(x)| \diamond_{\alpha} x + c_2|f(x)|\right)^{\beta}}{c_3 \int_a^b |w(x)||f(x)| \diamond_{\alpha} x - c_4|f(x)|} \right\} \diamond_{\alpha} x \\ & \geq \left(\int_a^b |w(x)| \diamond_{\alpha} x\right)^{2-\beta} \frac{\left\{\int_a^b |w(x)| \left(c_1 \int_a^b |w(x)||f(x)| \diamond_{\alpha} x + c_2|f(x)|\right) \diamond_{\alpha} x\right\}^{\beta}}{\int_a^b |w(x)| \left(c_3 \int_a^b |w(x)||f(x)| \diamond_{\alpha} x - c_4|f(x)|\right) \diamond_{\alpha} x}. \end{aligned} \tag{3.22}$$

Thus inequality (3.21) follows. □

Remark 3.15. If we set $\alpha = 1$, $\mathbb{T} = \mathbb{Z}$, $a = 1$, $b = n + 1$, $w \equiv 1$, $f(k) = x_k \in (0, +\infty)$ for $k \in \{1, 2, \dots, n\}$, $X_n = \sum_{k=1}^n x_k$ and $c_3 \left(\sum_{k=1}^n x_k\right) > c_4 \max_{1 \leq k \leq n} x_k$, then (3.21) reduces to

$$\frac{(c_1 n + c_2)^{\beta}}{c_3 n - c_4} n^{2-\beta} X_n^{\beta-1} \leq \sum_{k=1}^n \frac{(c_1 X_n + c_2 x_k)^{\beta}}{c_3 X_n - c_4 x_k}, \tag{3.23}$$

which is similar to an inequality given in [3].

Corollary 3.16. *Let $w, f \in C([a, b]_{\mathbb{T}}, \mathbb{R} - \{0\})$ be \diamond_{α} -integrable functions. If $c_3, c_4 \in (0, +\infty)$, $\beta \in [2, +\infty)$ and $c_3 \int_a^b |w(x)||f(x)| \diamond_{\alpha} x > c_4 \sup_{x \in [a, b]_{\mathbb{T}}} |f(x)|$, then*

$$\begin{aligned} & \frac{\left(\int_a^b |w(x)| \diamond_{\alpha} x\right)^{1-\beta}}{c_3 \int_a^b |w(x)| \diamond_{\alpha} x - c_4} \left(\int_a^b |w(x)||f(x)| \diamond_{\alpha} x\right)^{\beta} \\ & \leq \int_a^b |w(x)| \left\{ \frac{|f(x)|^{\beta+1}}{c_3 \int_a^b |w(x)||f(x)| \diamond_{\alpha} x - c_4|f(x)|} \right\} \diamond_{\alpha} x. \end{aligned} \tag{3.24}$$

Proof. Putting $c_1 = 0$, $c_2 = 1$ and replacing β by $\beta + 1$ in (3.21), the inequality (3.24) follows. \square

Remark 3.17. If we set $\alpha = 1$, then we get delta versions and if we set $\alpha = 0$, then we get nabla versions of diamond- α integral operator inequalities presented in this article.

Also, if we set $\mathbb{T} = \mathbb{Z}$, then we get discrete versions and if we set $\mathbb{T} = \mathbb{R}$, then we get continuous versions of diamond- α integral operator inequalities presented in this article.

4. Conclusion and Future Work

There have been recent developments of the theory and applications of dynamic inequalities on time scales. In this research article, we have presented some dynamic inequalities on diamond- α calculus, which is the linear combination of the delta and nabla integrals. Some generalizations and applications of Radon's inequality, Bergström's inequality, Nesbitt's inequality and other dynamic inequalities on time scales are also given in [17, 18].

In the future research, we can generalize the well-known inequalities using functional generalization, n -tuple diamond- α integral, fractional Riemann–Liouville integral, quantum calculus and α, β -symmetric quantum calculus.

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