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# Properties of differences in B-rings

*Aneta Dadej and Katarzyna Halik*

Submitted by: *Andrzej Kamiński*

ABSTRACT: Motivated by Pettis' extensions of Sierpiński theorems on generated families of sets, we consider B-rings, a generalization of the notion of Boolean algebras, and present their various properties. In particular, we discuss properties of differences which will be used in the proofs of results given in our forthcoming papers.

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## 1 Introduction

The notion of a ring of subsets of a given set used in measure theory is a generalization of an algebra of subsets of the set. It is natural to consider its counterpart in terms of a partially ordered set as a generalization of a *Boolean algebra* defined in terms of a partially ordered set as a distributive 0-1-lattice with complements. We consider such a generalization in this note under the name *B-ring* (see Definition 4) to avoid a possible misunderstanding connected with the common use of the term *Boolean ring* in the sense of an algebraic ring with a unit and commutative idempotent multiplication.

As well known (see e.g. [1]) there is a one-to-one correspondence between Boolean algebras (being a generalization of algebras of sets) and Boolean rings just mentioned. One may ask whether a similar correspondence takes place in case of B-rings and respective algebraic rings. Such a more general situation is not discussed in the classical monographs [1], [9], [10] and [11]. We study in this article various properties of B-rings that will be used in the proofs of results, presented in [2] and [3], which give an answer to the above question.

The consideration of B-rings was inspired by the results of B. J. Pettis who extended in [6] theorems of W. Sierpiński on generated families of subsets of a given set (see [8] and [7]) to Boolean  $\sigma$ -rings (for generalizations of Sierpiński's theorems in another direction see [5]). The main theorems from [6] were reformulated and extended

by the authors in their master theses, the results of which were partially published in [4]. It is clear that Pettis' term "Boolean  $\sigma$ -ring", though used in [6] without giving its strict definition, corresponds to a  $\sigma$ -ring of subsets of a given set. However we are forced to use the name "B-ring" instead of "Boolean ring" to avoid a collision with the traditional meaning of this term.

The notation used in the paper is mostly standard.

## 2 Boolean algebras and B-rings

Let us recall that by a partially ordered set we mean a non-empty set  $X$  with a binary relation which is reflexive ( $x \leq x$  for  $x \in X$ ), antisymmetric ( $x \leq y, y \leq x$  implies  $x = y$  for  $x, y \in X$ ) and transitive ( $x \leq y, y \leq z$  implies  $x \leq z$  for  $x, y, z \in X$ ).

We will say that  $\sup A$  exists in  $X$  if there is an  $x \in X$  ( $x =: \sup A$ ) such that 1°  $y \leq x$  for all  $y \in A$  and 2°  $y \leq x_1 \in X$  for all  $y \in A$  implies  $x \leq x_1$ . Clearly, if  $\sup A$  exists, it is unique. An analogous definition and comment concern  $\inf A$ .

**Definition 1** *A partially ordered set  $(X, \leq)$  is called a lattice if*

$$\forall_{x,y \in X} \quad \sup \{x, y\} \quad \text{and} \quad \inf \{x, y\} \quad \text{exist in } X. \quad (L)$$

Instead of the symbols  $\sup \{x, y\}$  and  $\inf \{x, y\}$  we will use in this note the standard notation:  $x \cup y := \sup \{x, y\}$ ,  $x \cap y := \inf \{x, y\}$  and call  $\cup$  and  $\cap$  the lattice operations.

The assertions in the below statement are simple consequences of properties of the relation of partial order and Definition 1 (see [1], section I).

**Statement 1** *In an arbitrary lattice  $(X, \leq)$  the following are true:*

$$(l_1) \quad \inf \{x_1, \dots, x_n\} \in X, \sup \{x_1, \dots, x_n\} \in X, \quad x_1, \dots, x_n \in X, \quad n \in \mathbb{N};$$

$$(l_2) \quad x \cap x = x, \quad x \cup x = x, \quad x \in X;$$

$$(l_3) \quad x_1 \cap x_2 = x_2 \cap x_1, \quad x_1 \cup x_2 = x_2 \cup x_1, \quad x_1, x_2 \in X;$$

$$(l_4) \quad (x_1 \cap x_2) \cap x_3 = x_1 \cap (x_2 \cap x_3), \quad x_1, x_2, x_3 \in X;$$

$$(l_5) \quad (x_1 \cup x_2) \cup x_3 = x_1 \cup (x_2 \cup x_3), \quad x_1, x_2, x_3 \in X;$$

$$(l_6) \quad x \cap y \leq x \leq x \cup y, \quad x \cap y \leq y \leq x \cup y, \quad x, y \in X;$$

$$(l_7) \quad x \leq y \Leftrightarrow x \cap y = x \Leftrightarrow x \cup y = y, \quad x, y \in X;$$

$$(l_8) \quad y_1 \leq y_2 \Rightarrow x \cap y_1 \leq x \cap y_2, \quad x \cup y_1 \leq x \cup y_2, \quad x, y_1, y_2 \in X;$$

$$(l_9) \quad (x_1 \cup x_2) \cap y \geq (x_1 \cap y) \cup (x_2 \cap y), \quad x_1, x_2, y \in X;$$

$$(l_{10}) \quad (x_1 \cap x_2) \cup y \leq (x_1 \cup y) \cap (x_2 \cup y), \quad x_1, x_2, y \in X.$$

**Definition 2** *A lattice  $(X, \leq)$  is called distributive if the following condition holds:*

$$\forall_{x_1, x_2, y \in X} \quad (x_1 \cup x_2) \cap y = (x_1 \cap y) \cup (x_2 \cap y). \quad (D)$$

**Remark 1** It is easy to show (see [1], section I.6) that in any lattice  $(X, \leq)$  condition (D) can be equivalently replaced by

$$\forall_{x_1, x_2, y \in X} \quad (x_1 \cap x_2) \cup y = (x_1 \cup y) \cap (x_2 \cup y). \quad (D')$$

**Definition 3** A lattice  $(X, \leq)$  will be called, respectively: (0) 0-lattice; (1) 1-lattice; (2) 0-1-lattice, whenever

$$\inf X \text{ exists in } X; \quad (L0)$$

$$\sup X \text{ exists in } X; \quad (L1)$$

$$\inf X \text{ and } \sup X \text{ exist in } X, \quad (L2)$$

respectively, and the elements  $0 := \inf X$ ,  $1 := \sup X$  are called the zero and the unit in  $X$ .

**Definition 4** A distributive 0-lattice  $(X, \leq)$  is called a B-ring if

$$\forall_{x, y \in X, x \leq y} \exists_{z \in X} \quad z \cap x = 0 \quad \text{and} \quad z \cup x = y, \quad (R)$$

i.e. if  $(X, \leq)$  satisfies conditions (L), (L0), (D) and (R).

A B-ring  $(X, \leq)$  is called a Boolean algebra if  $\sup X =: 1$  exists in  $X$ , i.e. if  $(X, \leq)$  satisfies conditions (L), (L2), (D) and (R).

Below we formulate various properties of the notions already introduced and those which will be defined later. Some of the assertions follow easily from the above and next definitions or properties formulated subsequently, so we omit their proofs.

**Statement 2** If  $(X, \leq)$  is a 0-lattice, then

$$0 \leq x, \quad x \cup 0 = 0 \cup x = x, \quad x \cap 0 = 0 \cap x = 0, \quad x \in X. \quad (l0)$$

If  $(X, \leq)$  is a 1-lattice, then

$$x \leq 1, \quad x \cup 1 = 1 \cup x = 1, \quad x \cap 1 = 1 \cap x = x, \quad x \in X. \quad (l1)$$

If  $(X, \leq)$  is a distributive 0-lattice, then

$$(x \cap c = y \cap c, \quad x \cup c = y \cup c) \Rightarrow x = y, \quad x, y, c \in X. \quad (l2)$$

**Statement 3** If  $(X, \leq)$  is a B-ring, then the element  $z$  in condition (R) is unique, i.e. for arbitrary  $x, y \in X$  and  $x \leq y$  there is a unique  $z$  such that:

$$z \cap x = 0 \quad \text{and} \quad z \cup x = y. \quad (r)$$

**Definition 5** If  $(X, \leq)$  is a B-ring and  $x, y$  are elements of  $X$  satisfying  $x \leq y$ , then by the proper difference  $y \ominus x$  of  $y$  and  $x$  we mean an element  $z$  satisfying (R). By Statement 3, we see that  $y \ominus x$  is defined uniquely.



**Statement 4** If  $(X, \leq)$  is a B-ring, then

$$\forall x, y \in X, x \leq y \quad y \ominus x \leq y, (y \ominus x) \cap x = 0, (y \ominus x) \cup x = y. \quad (r_1)$$

**Statement 5** If  $(X, \leq)$  is a B-ring then

$$\forall x, y \in X, x \leq y \quad y \ominus (y \ominus x) = x. \quad (r_2)$$

**Remark 2** If  $(X, \leq)$  is a Boolean algebra, then the following particular case of condition (R) holds:

$$\forall x \in X \quad \exists z \in X \quad z \cap x = 0 \quad \text{and} \quad z \cup x = 1. \quad (Ac)$$

It can be proved (cf. Statement 6) that in Boolean algebras condition (R) can be equivalently replaced by a seemingly weaker condition (Ac).

**Definition 6** If  $(X, \leq)$  is a Boolean algebra and  $x$  is an element of  $X$ , then by the complement  $x'$  of  $x$  we mean the unique (see Statement 3 and Remark 2) element  $z$  satisfying (Ac), i.e.  $x' := 1 \ominus x$ .

**Statement 6** If a distributive 0-1-lattice  $(X, \leq)$  satisfies (Ac), then condition (R) is satisfied, i.e.  $(X, \leq)$  is a Boolean algebra.

**Proof.** For arbitrary  $x, y \in X$  such that  $x \leq y$ , define  $z := y \cap x'$ . By  $(l_3)$ ,  $(l_4)$ ,  $(l_7)$ ,  $(l_0)$ ,  $(l_1)$  and  $(D')$ , we have

$$z \cap x = (y \cap x') \cap x = y \cap (x' \cap x) = y \cap 0 = 0$$

and

$$z \cup x = (y \cap x') \cup x = (y \cup x) \cap (x' \cup x) = (y \cup x) \cap 1 = y \cup x = y,$$

which means that condition (R) is satisfied.  $\square$

**Definition 7** If  $(X, \leq)$  is a B-ring, then by the difference  $y \setminus x$  we mean

$$y \setminus x := y \ominus (x \cap y)$$

for arbitrary  $x, y \in X$ . The definition makes sense since  $x \cap y \leq y$ , in view of Statement 3 and Definition 5.

Notice that Definitions 5 and 7 are consistent, due to the following obvious assertion:

**Statement 7** Let  $(X, \leq)$  be a B-ring. If  $x, y \in X$  and  $x \leq y$ , then  $x \setminus y = x \ominus y$ .

### 3 Differences in B-rings

In this section we will assume that  $(X, \leq)$  is a B-ring and we will collect several properties of differences in  $X$ . The first four properties are obvious.

**Property 1** For arbitrary  $x, y \in X$  we have

$$y \setminus x \leq y. \quad (d_1)$$

**Property 2** For arbitrary  $x, y \in X$ , we have

$$x = (x \cap y) \cup (x \setminus y), \quad (x \cap y) \cap (x \setminus y) = 0. \quad (d_2)$$

**Property 3** For arbitrary  $x, y \in X$ , we have

$$(x \setminus y) \cap (y \setminus x) = 0. \quad (d_3)$$

**Property 4** For arbitrary  $x, y \in X$ , we have

$$(x \setminus y) \cap y = 0. \quad (d_4)$$

Now we will prove the assertion which together with Property 2 yields Property 6. The next three assertions, Properties 7, 8 and 9, are given without proofs.

**Property 5** For arbitrary  $x, y \in X$ , we have

$$(x \setminus y) \cup y = x \cup y. \quad (d_5)$$

**Proof.** By  $(d_1)$  and  $(l_8)$ , we have

$$(x \setminus y) \cup y \leq x \cup y.$$

On the other hand, by  $(d_2)$ ,  $(l_3)$ ,  $(l_6)$  and  $(l_8)$ , we get

$$x = (x \setminus y) \cup (x \cap y) \leq (x \setminus y) \cup y,$$

so, by  $(l_8)$ ,  $(l_5)$ ,  $(l_2)$ , we have

$$x \cup y \leq [(x \setminus y) \cup y] \cup y = (x \setminus y) \cup y$$

and the assertion follows, due to symmetry of the partial order  $\leq$ .  $\square$

**Property 6** For arbitrary  $x, y \in X$ , we have

$$(x \setminus y) \cup (y \setminus x) \cup (x \cap y) = x \cup y. \quad (d_6)$$

**Property 7** For arbitrary  $x, y \in X$  there exists a unique  $z$ , namely  $z := x \setminus y$ , such that  $z \cap y = 0$  and  $z \cup y = x \cup y$ . Consequently,

$$(x \setminus y) \cap y = 0, \quad (x \setminus y) \cup y = x \cup y. \quad (d_7)$$

**Property 8** For arbitrary  $x, y_1, y_2 \in X$ , we have

$$y_1 \leq y_2 \Rightarrow (x \setminus y_2) \cap y_1 = 0. \quad (d_8)$$

**Property 9** For arbitrary  $x, y_1, y_2 \in X$ , we have

$$y_1 \leq y_2 \Rightarrow (x \setminus y_2) \setminus y_1 = x \setminus y_2. \quad (d_9)$$

We will prove the following assertion and the next two properties follow from Definition 7 and  $(r_3)$ .

**Property 10** For arbitrary  $x, y_1, y_2 \in X$ , we have

$$y_1 \leq y_2 \Rightarrow x \setminus y_2 \leq x \setminus y_1. \quad (d_{10})$$

**Proof.** We begin with a particular case, assuming that  $y_1 \leq y_2 \leq x$ . Then  $x \setminus y_2 \leq x$ , by Definition 5 and  $(r_1)$ , so

$$x \setminus y_2 = (x \setminus y_2) \cap x = (x \setminus y_2) \cap [(x \ominus y_1) \cup y_1], \quad (1)$$

in view of  $(l_7)$  and Statement 4. By (1),  $(D)$ ,  $(d_8)$  and Statement 7, we have

$$x \setminus y_2 = (x \setminus y_2) \cap (x \setminus y_1)$$

and the assertion in the considered case follows from  $(l_6)$ .

In the general case, we deduce from the particular case that

$$x \setminus y_2 = x \setminus (x \cap y_2) \leq x \setminus (x \cap y_1) = x \setminus y_1,$$

because, by  $(l_7)$  and  $(l_6)$ , we have

$$y_1 \leq y_2 \Rightarrow x \cap y_1 \leq x \cap y_2 \leq x. \quad \square$$

**Property 11** For arbitrary  $a, x, y \in X$ , we have

$$a \setminus (x \cup y) = (a \setminus x) \cap (a \setminus y). \quad (d_{11})$$

**Property 12** For arbitrary  $a, x, y \in X$ , we have

$$a \setminus (x \cap y) = (a \setminus x) \cup (a \setminus y). \quad (d_{12})$$

Our list of properties is concluded by the three assertions presented below with complete proofs.

**Property 13** For arbitrary  $a, x, y \in X$ , we have

$$a \setminus (x \cup y) = (a \setminus x) \setminus y. \quad (d_{13})$$

**Proof.** Denote  $z := a \setminus (x \cup y)$ . We have

$$z \cap y = 0, \quad (2)$$

because

$$\begin{aligned} z \cap y &= [a \setminus (x \cup y)] \cap y = [(a \setminus x) \cap (a \setminus y)] \cap y \\ &= (a \setminus x) \cap [(a \setminus y) \cap y] = (a \setminus x) \cap 0 = 0, \end{aligned}$$

by  $(d_{11})$ ,  $(d_4)$  and  $(l0)$ .

On the other hand, we will show that

$$z \cup y = (a \setminus x) \cup y. \quad (3)$$

To this aim notice first that

$$z \cup y = [(a \setminus x) \cap (a \setminus y)] \cup y = (a \setminus x) \cup y \quad (4)$$

in view of  $(d_{11})$ ,  $(D')$  and  $(d_4)$ . Due to  $(d_7)$ , we have  $(a \setminus y) \cup y = a \cup y$ , so  $(4)$  and  $(D)$  yield

$$z \cup y = [(a \setminus x) \cup y] \cap (a \cup y) = [(a \setminus x) \cap (a \cup y)] \cup y,$$

since  $y \cap (a \cup y) = y$ . Hence, by  $(D)$ ,  $(l_2)$ ,  $(l_7)$  and  $(d_1)$ , we have

$$z \cup y = [(a \setminus x) \cap a] \cup [(a \setminus x) \cap y] \cup y = (a \setminus x) \cup y$$

This proves identity  $(3)$ .

In view of  $(d_7)$ , equalities  $(2)$  and  $(3)$  imply  $z = (a \setminus x) \setminus y$ , i.e. the assertion is true.  $\square$

**Property 14** For arbitrary  $x, y, z \in X$ , we have

$$(x \setminus z) \cup (y \setminus z) = (x \cup y) \setminus z. \quad (d_{14})$$

**Proof.** To prove the assertion, in view of  $(d_7)$ , it suffices to show that

$$z \cap [(x \setminus z) \cup (y \setminus z)] = 0 \quad (5)$$

and

$$z \cup [(x \setminus z) \cup (y \setminus z)] = (x \cup y) \cup z. \quad (6)$$

The property  $(5)$  is obvious, due to  $(D)$  and  $(d_4)$ , we have

$$z \cap [(x \setminus z) \cup (y \setminus z)] = [z \cap (x \setminus z)] \cup [z \cap (y \setminus z)] = 0.$$

We apply  $(d_7)$ , we have

$$z \cup (x \setminus z) = x \cup z \quad \text{and} \quad z \cup (y \setminus z) = y \cup z$$

and thus

$$\begin{aligned} z \cup [(x \setminus z) \cup (y \setminus z)] &= [z \cup (x \setminus z)] \cup [z \cup (y \setminus z)] \\ &= (x \cup z) \cup (y \cup z) = (x \cup y) \cup z, \end{aligned}$$

which proves  $(6)$ .  $\square$

**Property 15** For arbitrary  $x, y, z \in X$ , we have

$$(x \setminus y) \cap z = (x \cap z) \setminus (y \cap z). \quad (d_{15})$$

**Proof.** In view of  $(d_7)$ , to prove  $(d_{15})$  it is enough to show that

$$(y \cap z) \cap [(x \setminus y) \cap z] = 0 \quad (7)$$

and

$$(y \cap z) \cup [(x \setminus y) \cap z] = (x \cap z) \cup (y \cap z). \quad (8)$$

To show (7) notice that

$$(x \setminus y) \cap z \leq x \setminus y \leq x \setminus (y \cap z),$$

by  $(l_5)$  and  $(d_{10})$ , so

$$(y \cap z) \cap [(x \setminus y) \cap z] \leq (y \cap z) \cap [x \setminus (y \cap z)] = 0 \quad (9)$$

in view of  $(l_6)$  and  $(d_4)$ . Since  $0 = \inf X$ , equality (7) follows from (9).

By  $(D')$ , the left side of (8) is equal to

$$[y \cup (x \setminus y)] \cap z = (x \cup y) \cap z = (x \cap z) \cup (y \cap z)$$

due to  $(d_7)$  and  $(D)$ .  $\square$

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## Local dynamics in a Leslie-Gower system model

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**ABSTRACT:** Mathematical ecology or/and biology requires the study of populations that interact. This is the reason for the intensive study of the predator-prey models. A Leslie-Gower model of such type is considered here and the stability properties of its equilibrium points are analytically and numerically investigated. Dynamics and bifurcations are deduced. Level curves for corresponding Lyapunov functions for various values of the physical parameters in the parameter space are graphically presented emphasizing the stability regions.

AMS Subject Classification: *65L10; 65L15; 65L60; 76E06*

Key Words and Phrases: *Lyapunov functions; bifurcation problems; predator-prey model*

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## Coefficient inequalities for certain classes of analytic functions associated with the Wright generalized hypergeometric function

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ABSTRACT: In this paper, we obtain sufficient condition involving coefficient inequalities for  $f(z)$  to in the class

$$\mathcal{W}_m^l(\alpha_1, \alpha_2, \dots, \alpha_l, \beta_1, \beta_2, \dots, \beta_m, \eta)$$

of analytic functions defined in the open unit disk and satisfying the analytic criterion

$$\operatorname{Re} \left\{ \frac{z(\mathcal{W}_m^l[\alpha_1]f(z))'}{\mathcal{W}_m^l[\alpha_1]f(z)} \right\} > \eta.$$

Our main result contain some interesting corollaries as special cases.

AMS Subject Classification: *30C45*

Key Words and Phrases: *Analytic functions, Coefficient inequalities, Hadamard product, Wright generalized hypergeometric functions*

### 1 Introduction and definitions

Let  $\mathcal{A}$  denote the class of functions of the form

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n \quad (1.1)$$

which are analytic and univalent in the open disk  $\mathcal{U} = \{z : |z| < 1\}$ . For functions  $\Phi \in \mathcal{A}$  given by  $\Phi(z) = z + \sum_{n=2}^{\infty} \phi_n z^n$  and  $\Psi \in \mathcal{A}$  given by  $\Psi(z) = z + \sum_{n=2}^{\infty} \psi_n z^n$ , we define the Hadamard product (or convolution) of  $\Phi$  and  $\Psi$  by

$$(\Phi * \Psi)(z) = z + \sum_{n=2}^{\infty} \phi_n \psi_n z^n, \quad z \in \mathcal{U}.$$

For positive real parameters  $\alpha_1, A_1, \dots, \alpha_l, A_l$  and  $\beta_1, B_1, \dots, \beta_m, B_m$  ( $l, m \in \mathbb{N} = 1, 2, 3, \dots$ ) such that

$$1 + \sum_{n=1}^m B_n - \sum_{n=1}^l A_n \geq 0, \quad z \in \mathcal{U},$$

the Wright generalized hypergeometric function [12]

$${}_l\Psi_m[(\alpha_1, A_1), \dots, (\alpha_l, A_l); (\beta_1, B_1), \dots, (\beta_m, B_m); z] = {}_l\Psi_m[(\alpha_n, A_n)_{1,l}; (\beta_n, B_n)_{1,m}; z]$$

is defined by

$${}_l\Psi_m[(\alpha_t, A_t)_{1,l}; (\beta_t, B_t)_{1,m}; z] = \sum_{n=0}^{\infty} \left\{ \prod_{t=1}^l \Gamma(\alpha_t + nA_t) \right\} \left\{ \prod_{t=1}^m \Gamma(\beta_t + nB_t) \right\}^{-1} \frac{z^n}{n!}, \quad z \in \mathcal{U}.$$

If  $A_t = 1$  ( $t = 1, 2, \dots, l$ ) and  $B_t = 1$  ( $t = 1, 2, \dots, m$ ) we have the relationship:

$$\Omega {}_l\Psi_m[(\alpha_t, 1)_{1,l}; (\beta_t, 1)_{1,m}; z] \equiv {}_lF_m(\alpha_1, \dots, \alpha_l; \beta_1, \dots, \beta_m; z) = \sum_{k=0}^{\infty} \frac{(\alpha_1)_k \dots (\alpha_l)_k}{(\beta_1)_k \dots (\beta_m)_k} \frac{z^k}{k!}$$

( $l \leq m + 1$ ;  $l, m \in \mathbb{N}_0 = \mathbb{N} \cup \{0\}$ ;  $z \in \mathcal{U}$ ) where  ${}_lF_m(\alpha_1, \dots, \alpha_l; \beta_1, \dots, \beta_m; z)$  is the generalized hypergeometric function (see for details [12]) where  $(\lambda)_n$  is the Pochhammer symbol and

$$\Omega = \left( \prod_{t=1}^l \Gamma(\alpha_t) \right)^{-1} \left( \prod_{t=1}^m \Gamma(\beta_t) \right). \quad (1.2)$$

By using the generalized hypergeometric function, Dziok and Srivastava [3] introduced a linear operator which was subsequently extended by Dziok and Raina [4] using the Wright's generalized hypergeometric function.

Let  $\mathcal{W}_m^l[(\alpha_t, A_t)_{1,l}; (\beta_t, B_t)_{1,m}] : \mathcal{A} \rightarrow \mathcal{A}$  be a linear operator defined by

$$\mathcal{W}_m^l[(\alpha_t, A_t)_{1,l}; (\beta_t, B_t)_{1,m}](f)(z) := \{\Omega z {}_l\phi_m[(\alpha_t, A_t)_{1,l}; (\beta_t, B_t)_{1,m}; z]\} * f(z).$$

We observe that, for  $f(z)$  of the form (1.1), we have

$$\mathcal{W}_m^l[(\alpha_t, A_t)_{1,l}; (\beta_t, B_t)_{1,m}](f)(z) = z + \sum_{n=2}^{\infty} \sigma_n(\alpha_1) a_n z^n$$

where  $\Omega$  is given by (1.2) and  $\sigma_n(\alpha_1)$  is defined by

$$\sigma_n(\alpha_1) = \frac{\Omega \Gamma(\alpha_1 + A_1(n-1)) \dots \Gamma(\alpha_l + A_l(n-1))}{(n-1)! \Gamma(\beta_1 + B_1(n-1)) \dots \Gamma(\beta_m + B_m(n-1))}. \quad (1.3)$$

For convenience sake, we use contracted notation  $\mathcal{W}_m^l[\alpha_1]$  to represent the following:

$$\mathcal{W}_m^l[\alpha_1]f(z) = \mathcal{W}[(\alpha_1, A_1), \dots, (\alpha_l, A_l); (\beta_1, B_1), \dots, (\beta_m, B_m)]f(z),$$

which is used in the sequel throughout.

The linear operator  $\mathcal{W}_m^l[\alpha_1]$  includes the Dziok-Srivastava operator (see [3]), so that it includes (as its special cases) various other linear operators introduced and studied by Bernardi [1], Carlson and Shaffer [2], Libera [8], Livingston [9], Ruscheweyh [10] and Srivastava-Owa [11].

For  $0 \leq \eta < 1$ , we let  $\mathcal{W}_m^l(\alpha_1, \alpha_2, \dots, \alpha_l, \beta_1, \beta_2, \dots, \beta_m, \eta)$  be the subclass of  $\mathcal{A}$  consisting of functions of the form (1.1) and satisfying the analytic criterion

$$\operatorname{Re} \left\{ \frac{z(\mathcal{W}_m^l[\alpha_1]f(z))'}{\mathcal{W}_m^l[\alpha_1]f(z)} \right\} > \eta, \quad z \in \mathcal{U}. \quad (1.4)$$

By suitably specializing the values of  $A_t, B_t, l, m, \alpha_1, \alpha_2, \dots, \alpha_l, \beta_1, \beta_2, \dots, \beta_m, \eta$  and  $\gamma$  the class  $\mathcal{W}_m^l(\alpha_1, \alpha_2, \dots, \alpha_l, \beta_1, \beta_2, \dots, \beta_m, \eta)$ , leads to various new subclasses of analytic functions.

As illustrations, we present some examples for the case when  $A_t = 1 (t = 1, 2, \dots, l)$  and  $B_t = 1 (t = 1, 2, \dots, m)$ .

**Example 1.1** If  $l = 2$  and  $m = 1$  with  $\alpha_1 = 1, \alpha_2 = 1, \beta_1 = 1$  then

$$\begin{aligned} \mathcal{W}_1^2(1, 1, 1, \eta) &\equiv \mathcal{S}^*(\eta) \\ &: = \left\{ f \in \mathcal{A} : \operatorname{Re} \left\{ \frac{zf'(z)}{f(z)} \right\} > \eta, \quad z \in \mathcal{U} \right\}. \end{aligned}$$

where  $\mathcal{S}^*(\eta)$  is the well-known starlike function of order  $\eta (0 \leq \eta < 1)$ .

**Example 1.2** If  $l = 2$  and  $m = 1$  with  $\alpha_1 = \zeta + 1 (\zeta > -1), \alpha_2 = 1, \beta_1 = 1$ , then

$$\begin{aligned} \mathcal{W}_1^2(\delta + 1, 1, 1, \eta) &\equiv \mathcal{R}_\zeta(\eta) \\ &: = \left\{ f \in \mathcal{A} : \operatorname{Re} \left\{ \frac{z(D^\zeta f(z))'}{D^\zeta f(z)} \right\} > \eta, \quad z \in \mathcal{U} \right\}, \end{aligned}$$

where  $D^\zeta$  is called Ruscheweyh derivative of order  $\zeta (\zeta > -1)$  defined by

$$D^\zeta f(z) := \frac{z}{(1-z)^{\zeta+1}} * f(z).$$

**Example 1.3** If  $l = 2$  and  $m = 1$  with  $\alpha_1 = \mu + 1 (\mu > -1), \alpha_2 = 1, \beta_1 = \mu + 2$ , then

$$\begin{aligned} \mathcal{W}_1^2(\mu + 1, 1, \mu + 2, \eta) &\equiv \mathcal{B}_\mu(\eta) \\ &: = \left\{ f \in \mathcal{A} : \operatorname{Re} \left( \frac{z(J_\mu f(z))'}{J_\mu f(z)} \right) > \eta, \quad z \in \mathcal{U} \right\}, \end{aligned}$$

where  $J_\mu$  is a Bernardi operator [1] defined by

$$J_\mu f(z) := \frac{\mu + 1}{z^\mu} \int_0^z t^{\mu-1} f(t) dt.$$

Note that the operator  $J_1$  was studied earlier by Libera [8] and Livingston [9].

**Example 1.4** If  $l = 2$  and  $m = 1$  with  $\alpha_1 = a$  ( $a > 0$ ),  $\alpha_2 = 1$ ,  $\beta_1 = c$  ( $c > 0$ ), then

$$\begin{aligned} \mathcal{W}_1^2(a, 1, c, \eta) &\equiv \mathcal{L}_c^a(\eta) \\ &:= \left\{ f \in \mathcal{A} : \operatorname{Re} \left( \frac{z(L(a, c)f(z))'}{L(a, c)f(z)} \right) > \eta, z \in \mathcal{U} \right\}, \end{aligned}$$

where  $L(a, c)$  is a well-known Carlson-Shaffer linear operator [2] defined by

$$L(a, c)f(z) := \left( \sum_{k=0}^{\infty} \frac{(a)_k}{(c)_k} z^{k+1} \right) * f(z).$$

Following the earlier works of [7] (see also [5], [6]), we obtain sufficient condition involving coefficient inequalities for  $f(z)$  to in the class  $\mathcal{W}_m^l(\alpha_1, \alpha_2, \dots, \alpha_l, \beta_1, \beta_2, \dots, \beta_m, \eta)$ . Several special cases and consequences of these coefficient inequalities are also pointed out.

In order to derive our main results, we have to recall here the following lemmas:

**Lemma 1.1** ([7]) A function  $p(z) \in \mathcal{P}$  satisfies  $\operatorname{Re} p(z) > 0$  ( $z \in \mathcal{U}$ ) if and only if

$$p(z) \neq \frac{x-1}{x+1} \quad (z \in \mathcal{U})$$

for all  $|x| = 1$ .

**Lemma 1.2** A function  $f(z) \in \mathcal{A}$  is in  $\mathcal{W}_m^l(\alpha_1, \alpha_2, \dots, \alpha_l, \beta_1, \beta_2, \dots, \beta_m, \eta)$  if and only if

$$1 + \sum_{n=2}^{\infty} A_n z^{n-1} \neq 0 \quad (1.5)$$

where

$$A_n = \frac{[n+1-2\eta + (n-1)x]\Omega\sigma_n(\alpha_1)}{2(1-\eta)} a_n \quad (1.6)$$

and  $\Omega, \sigma_n(\alpha_1)$  are given by (1.2) and (1.3).

**Proof.** Applying Lemma 1.1, we have

$$\frac{\frac{z(\mathcal{W}_m^l[\alpha_1]f(z))'}{\mathcal{W}_m^l[\alpha_1]f(z)} - \eta}{1-\eta} \neq \frac{x-1}{x+1} \quad (z \in \mathcal{U}; x \in \mathbb{C}; |x| = 1). \quad (1.7)$$

where  $f(z) \in \mathcal{W}_m^l(\alpha_1, \alpha_2, \dots, \alpha_l, \beta_1, \beta_2, \dots, \beta_m, \eta)$ . Then, we not need consider Lemma 1.1 for  $z = 0$ , because it follows that

$$p(0) = 1 \neq \frac{x-1}{x+1}$$

for all  $|x| = 1$ . From (1.7), it follows that

$$(x+1)z(\mathcal{W}_m^l[\alpha_1]f(z))' + (1-2\eta-x)\mathcal{W}_m^l[\alpha_1]f(z) \neq 0.$$

Thus, we have

$$2(1-\eta)z + \sum_{n=2}^{\infty} [n+1-2\eta+(n-1)x]\Omega\sigma_n(\alpha_1)a_n z^n \neq 0$$

$$(z \in \mathcal{U}; x \in \mathbb{C}; |x| = 1),$$

or, equivalently,

$$2(1-\eta)z \left( 1 + \sum_{n=2}^{\infty} \frac{[n+1-2\eta+(n-1)x]\Omega\sigma_n(\alpha_1)a_n z^{n-1}}{2(1-\eta)} \right) \neq 0 \quad (1.8)$$

$$(z \in \mathcal{U}; x \in \mathbb{C}; |x| = 1).$$

Now, dividing both sides of (1.8) by  $2(1-\eta)z$  ( $z \neq 0$ ), we obtain

$$1 + \sum_{n=2}^{\infty} \frac{[n+1-2\eta+(n-1)x]\Omega\sigma_n(\alpha_1)a_n z^{n-1}}{2(1-\eta)} \neq 0$$

$$(z \in \mathcal{U}; x \in \mathbb{C}; |x| = 1),$$

which completes the proof of Lemma 1.2. ■

## 2 Coefficient conditions

With the help of Lemma 1.2, we have

**Theorem 2.1** *If  $f(z) \in \mathcal{A}$  satisfies the following condition:*

$$\sum_{n=2}^{\infty} \left( \left| \sum_{l=1}^n \left\{ \sum_{k=1}^l (k+1-2\eta)\Omega\sigma_k(\alpha_1)(-1)^{l-k} \binom{\gamma}{l-k} a_k \right\} \binom{\delta}{n-l} \right| \right. \\ \left. + \left| \sum_{l=1}^n \left\{ \sum_{k=1}^l (k-1)(-1)^{l-k}\Omega\sigma_k(\alpha_1) \binom{\gamma}{l-k} a_k \right\} \binom{\delta}{n-l} \right| \right) \\ \leq 2(1-\eta), \quad (0 \leq \eta < 1; \gamma, \delta \in \mathbb{R}),$$

then  $f(z) \in \mathcal{W}_m^l(\alpha_1, \alpha_2, \dots, \alpha_l, \beta_1, \beta_2, \dots, \beta_m, \eta)$ .

**Proof.** To prove that  $1 + \sum_{n=2}^{\infty} A_n z^{n-1} \neq 0$ , it is sufficient that

$$\left( 1 + \sum_{n=2}^{\infty} A_n z^{n-1} \right) (1-z)^\gamma (1+z)^\delta \\ = 1 + \sum_{n=2}^{\infty} \left[ \sum_{l=1}^n \left\{ \sum_{k=1}^l A_k (-1)^{l-k} \binom{\gamma}{l-k} \right\} \binom{\delta}{n-l} \right] z^{n-1} \\ \neq 0,$$

where  $A_0 = 0, A_1 = 1$  and  $\gamma, \delta \in \mathbb{R}, z \in \mathcal{U}$ . Thus, if  $f(z)$  satisfies

$$\sum_{n=2}^{\infty} \left| \sum_{l=1}^n \left\{ \sum_{k=1}^l A_k (-1)^{l-k} \binom{\gamma}{l-k} \right\} \binom{\delta}{n-l} \right| \leq 1,$$

that is, if

$$\begin{aligned} & \frac{1}{2(1-\eta)} \sum_{n=2}^{\infty} \left| \sum_{l=1}^n \left\{ \sum_{k=1}^l [(k+1-2\eta) \right. \right. \\ & \quad \left. \left. + x(k-1)] \Omega \sigma_k(\alpha_1) (-1)^{l-k} \binom{\gamma}{l-k} a_k \right\} \binom{\delta}{n-l} \right| \\ & \leq \frac{1}{2(1-\eta)} \sum_{n=2}^{\infty} \left( \left| \sum_{l=1}^n \left\{ \sum_{k=1}^l (k+1-2\eta) \Omega \sigma_k(\alpha_1) (-1)^{l-k} \binom{\gamma}{l-k} a_k \right\} \binom{\delta}{n-l} \right| \right. \\ & \quad \left. + |x| \left| \sum_{l=1}^n \left\{ \sum_{k=1}^l (k-1) \Omega \sigma_k(\alpha_1) (-1)^{l-k} \binom{\gamma}{l-k} a_k \right\} \binom{\delta}{n-l} \right| \right) \\ & \leq 1 \quad (0 \leq \eta < 1; x \in \mathbb{C}; |x| = 1; \gamma, \delta \in \mathbb{R}), \end{aligned}$$

then  $f(z) \in \mathcal{W}_m^l(\alpha_1, \alpha_2, \dots, \alpha_l, \beta_1, \beta_2, \dots, \beta_m, \eta)$  and so the proof is complete. ■

Letting  $A_t = 1 (t = 1, 2, \dots, l)$ ,  $B_t = 1 (t = 1, 2, \dots, m)$ ,  $l = 2, m = 1$  with  $\alpha_1 = 1$ ,  $\alpha_2 = 1$  and  $\beta_1 = 1$  in Theorem 2.1, we have the following result obtained by Hayami *et al.* [7].

**Corollary 2.1** ([7]) *If  $f(z) \in \mathcal{A}$  satisfies the following condition:*

$$\begin{aligned} & \sum_{n=2}^{\infty} \left( \left| \sum_{l=1}^n \left\{ \sum_{k=1}^l (k+1-2\eta) (-1)^{l-k} \binom{\gamma}{l-k} a_k \right\} \binom{\delta}{n-l} \right| \right. \\ & \quad \left. + \left| \sum_{l=1}^n \left\{ \sum_{k=1}^l (k-1) (-1)^{l-k} \binom{\gamma}{l-k} a_k \right\} \binom{\delta}{n-l} \right| \right) \\ & \leq 2(1-\eta) \end{aligned}$$

for some  $\eta (0 \leq \eta < 1)$  and  $\gamma, \delta \in \mathbb{R}$ , then  $f(z) \in \mathcal{S}^*(\eta)$ . In particular, for  $\gamma = \delta = 0$ , if  $f(z) \in \mathcal{A}$  satisfies the following condition:

$$\sum_{n=2}^{\infty} (n-\eta) |a_n| \leq 1-\eta \quad (0 \leq \eta < 1)$$

then  $f(z) \in \mathcal{S}^*(\eta)$ .

Letting  $A_t = 1 (t = 1, 2, \dots, l)$ ,  $B_t = 1 (t = 1, 2, \dots, m)$ ,  $l = 2, m = 1$  with  $\alpha_1 = \zeta + 1 (\zeta > -1)$ ,  $\alpha_2 = 1$  and  $\beta_1 = 1$  in Theorem 2.1, we have

**Corollary 2.2** *If  $f(z) \in \mathcal{A}$  satisfies the following condition:*

$$\begin{aligned} & \sum_{n=2}^{\infty} \left( \left| \sum_{l=1}^n \left\{ \sum_{k=1}^l (k+1-2\eta) \frac{\Gamma(\zeta+k)}{(k-1)!\Gamma(\zeta+1)} (-1)^{l-k} \binom{\gamma}{l-k} a_k \right\} \binom{\delta}{n-l} \right| \right. \\ & \quad \left. + \left| \sum_{l=1}^n \left\{ \sum_{k=1}^l (k-1) (-1)^{l-k} \frac{\Gamma(\zeta+k)}{(k-1)!\Gamma(\zeta+1)} \binom{\gamma}{l-k} a_k \right\} \binom{\delta}{n-l} \right| \right) \\ & \leq 2(1-\eta), \quad (0 \leq \eta < 1; \gamma, \delta \in \mathbb{R}), \end{aligned}$$

then  $f(z) \in \mathcal{R}_{\zeta}(\eta)$ .

Letting  $A_t = 1(t = 1, 2, \dots, l)$ ,  $B_t = 1(t = 1, 2, \dots, m)$ ,  $l = 2$ ,  $m = 1$  with  $\alpha_1 = \mu + 1 (\mu > -1)$ ,  $\alpha_2 = 1$ ,  $\beta_1 = \mu + 2$ , in Theorem 2.1, we have

**Corollary 2.3** *If  $f(z) \in \mathcal{A}$  satisfies the following condition:*

$$\begin{aligned} & \sum_{n=2}^{\infty} \left( \left| \sum_{l=1}^n \left\{ \sum_{k=1}^l (k+1-2\eta) \frac{\mu+1}{\mu+k} (-1)^{l-k} \binom{\gamma}{l-k} a_k \right\} \binom{\delta}{n-l} \right| \right. \\ & \quad \left. + \left| \sum_{l=1}^n \left\{ \sum_{k=1}^l (k-1) \frac{\mu+1}{\mu+k} (-1)^{l-k} \binom{\gamma}{l-k} a_k \right\} \binom{\delta}{n-l} \right| \right) \\ & \leq 2(1-\eta), \quad (0 \leq \eta < 1; \gamma, \delta \in \mathbb{R}), \end{aligned}$$

then  $f(z) \in \mathcal{B}_{\mu}(\eta)$ .

Letting  $A_t = 1(t = 1, 2, \dots, l)$ ,  $B_t = 1(t = 1, 2, \dots, m)$ ,  $l = 2$ ,  $m = 1$  with  $\alpha_1 = a (a > 0)$ ,  $\alpha_2 = 1$  and  $\beta_1 = c (c > 0)$  in Theorem 2.1, we have

**Corollary 2.4** *If  $f(z) \in \mathcal{A}$  satisfies the following condition:*

$$\begin{aligned} & \sum_{n=2}^{\infty} \left( \left| \sum_{l=1}^n \left\{ \sum_{k=1}^l (k+1-2\eta) \frac{(a)_{k-1}}{(c)_{k-1}} (-1)^{l-k} \binom{\gamma}{l-k} a_k \right\} \binom{\delta}{n-l} \right| \right. \\ & \quad \left. + \left| \sum_{l=1}^n \left\{ \sum_{k=1}^l (k-1) (-1)^{l-k} \frac{(a)_{k-1}}{(c)_{k-1}} \binom{\gamma}{l-k} a_k \right\} \binom{\delta}{n-l} \right| \right) \\ & \leq 2(1-\eta), \quad (0 \leq \eta < 1; \gamma, \delta \in \mathbb{R}), \end{aligned}$$

then  $f(z) \in \mathcal{L}_c^a(\eta)$ .

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# On meromorphic multivalent functions defined with the use of linear operator

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ABSTRACT: In the present paper we introduce two classes of meromorphically multivalent functions and application of linear operators on these classes. We study various properties and coefficients bounds, the concept of neighbourhood also investigated.

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## 1 Introduction

Let  $T^*(p)$  be the class of functions  $f(z)$  of the form

$$f(z) = z^{-p} + \sum_{n=1}^{\infty} a_n z^{n-p}, \quad p \in \mathbb{N} = \{1, 2, \dots\} \quad (1)$$

which are analytic and multivalent in the punctured unit disk  $\mathcal{U}^* = \{z : z \in \mathbb{C} \text{ and } 0 < |z| < 1\}$ .

The Hadamard product of  $f$  and  $g$  where  $f$  defined by (1) and  $g(z) = z^{-p} + \sum_{n=1}^{\infty} b_n z^{n-p}$  denote by  $f * g$  define as

$$(f * g)(z) = z^{-p} + \sum_{n=1}^{\infty} a_n b_n z^{n-p}. \quad (2)$$

Now let

$$\varphi_p(a, c; z) = z^{-p} + \sum_{n=1}^{\infty} \frac{(a)_n}{(c)_n} z^{n-p} \quad (3)$$

( $z \in \mathcal{U}^*$ ,  $a \in \mathbb{R}$ ,  $c \in \mathbb{R}$ ,  $c \neq 0, -1, -2, \dots$ ) ( $a$ ) $_0 = 1$  and ( $a$ ) $_n = a(a+1)\dots(a+n-1)$ ,  $n \in \mathbb{N}$  which is called shifted factorial.

Consider the class  $\mathcal{K}_{a,c}(p; A, B, \delta)$ , a function  $f \in T^*(p)$  belongs to  $\mathcal{K}_{a,c}(p; A, B, \delta)$  if it satisfies the following condition

$$\left| \frac{z(\mathcal{L}_p(a, c)f(z))' + p\mathcal{L}_p(a, c)f(z)}{Bz(\mathcal{L}_p(a, c)f(z))' + p[B + (A - B)(1 - \delta)]\mathcal{L}_p(a, c)f(z)} \right| < 1 \quad (4)$$

where  $-1 \leq B < A \leq 1, 0 \leq \delta < 1, p \in \mathbb{N}, z \in \mathcal{U}, a \in \mathbb{R}, c \in \mathbb{R}, c \neq 0, -1, -2, \dots$  and

$$\mathcal{L}_p(a, c)f(z) = \varphi_p(a, c; z) * f(z), \quad f \in T^*(p). \quad (5)$$

The definition of  $\mathcal{L}_p(a, c)f(z)$  is motivated by Carlson - Shaffer [2] and the class  $\mathcal{K}_{a,c}(p; A, B, \delta)$  is generalized to the class studied by Liu and Srivastava [5].

The function  $f(z) \in \mathcal{K}_{a,c}(p; A, B, \delta)$  is in the class  $\mathcal{K}_{a,c}^+(p, A, B, \delta)$  such that

$$f(z) = z^{-p} + \sum_{n=p}^{\infty} |a_n|z^n, \quad (p \in \mathbb{N}). \quad (6)$$

Special cases of the classes  $\mathcal{K}_{a,c}^+(p; A, B, \delta)$  and  $\mathcal{K}_{a,c}(p; A, B, \delta)$

- (1) If  $a = c = 1, \delta = 0$  we get the class  $\mathcal{K}_{1,1}^+(p; A, B)$  was investigated by Mogra [6].
- (2) If  $\delta = 0$  we get the class  $\mathcal{K}_{a,c}(p; A, B)$  was studied by Liu and Srivastava [5].

## 2 Inclusion Properties of the Class $\mathcal{K}_{a,c}(p; A, B, \delta)$

In order to prove our results we need the following Lemma.

**Lemma (Jack [4])** Let  $w(z)$  be analytic non constant function in  $\mathcal{U}$  with  $w(0) = 0$ . If  $w(z)$  attains its maximum value on the circle  $|z| = r < 1$  at a point  $z_0 \in \mathcal{U}$ , then

$$z_0 w'(z_0) = \mu w(z_0), \quad \text{where } \mu \in \mathbb{R} \text{ and } \mu \geq 1. \quad (7)$$

**Theorem 2.1** Let  $a \geq \frac{p(1-\delta)(A-B)}{B+1}$ . Then  $\mathcal{K}_{a+1,c}(p; A, B, \delta) \subset \mathcal{K}_{a,c}(p; A, B, \delta)$  where  $-1 < B < A \leq 1, 0 \leq \delta < 1, p \in \mathbb{N}$ .

**Proof.** Assume that  $f \in \mathcal{K}_{a+1,c}(p; A, B, \delta)$  and suppose that

$$\frac{z(\mathcal{L}_p(a, c)f(z))'}{\mathcal{L}_p(a, c)f(z)} = -p \left( \frac{1 + [B + (A - B)(1 - \delta)]w(z)}{1 + Bw(z)} \right) \quad (8)$$

for  $w(z)$  is analytic or meromorphic in  $\mathcal{U}$ , with  $w(0) = 0$ . From (3) and (5) we have

$$z(\mathcal{L}_p(a, c)f(z))' = a\mathcal{L}_p(a+1, c)f(z) - (a+p)\mathcal{L}_p(a, c)f(z). \quad (9)$$

Now from (9) and (8), we get

$$\frac{a\mathcal{L}_p(a+1, c)f(z)}{\mathcal{L}_p(a, c)f(z)} = \frac{a + [aB - p(A - B)(1 - \delta)]w(z)}{1 + Bw(z)}, \quad (10)$$

then

$$\begin{aligned} \frac{(\mathcal{L}_p(a+1, c)f(z))'}{\mathcal{L}_p(a+1, c)f(z)} &= \frac{z(\mathcal{L}_p(a, c)f(z))'}{\mathcal{L}_p(a, c)f(z)} \\ &+ \frac{[aB - p(A - B)(1 - \delta)]zw'(z)}{a + [aB - p(A - B)(1 - \delta)]w(z)} - \frac{Bzw'(z)}{1 + Bw(z)}. \end{aligned} \quad (11)$$

The last expression obtained by differentiating logarithmically with respect to  $z$  of (10), so

$$\begin{aligned} \frac{z(\mathcal{L}_p(a+1, c)f(z))'}{\mathcal{L}_p(a+1, c)f(z)} &= -p \left[ \frac{1 + [B + (A - B)(1 - \delta)]w(z)}{1 + Bw(z)} \right] \\ &- \frac{p(1 - \delta)(A - B)zw'(z)}{(1 + Bw(z))[a + (aB - p(A - B)(1 - \delta))w(z)]}. \end{aligned} \quad (12)$$

Now suppose that there exists  $z_0 \in \mathcal{U}$  such that  $\max_{|z| \leq |z_0|} |w(z)| = |w(z_0)| = 1$ , then by Jack's lemma we have  $z_0w'(z_0) = \mu w(z_0)$ , ( $\mu \geq 1$ ).

Let  $w(z_0) = e^{i\theta}$  ( $0 \leq \theta < 2\pi$ ) in (12), we get after setting  $z = z_0$

$$\begin{aligned} &\left| \frac{z_0(\mathcal{L}_p(a, c)f(z_0))' + p\mathcal{L}_p(a, c)f(z_0)}{Bz_0(\mathcal{L}_p(a, c)f(z_0))' + [Bp + (A - B)(1 - \delta)p]\mathcal{L}_p(a, c)f(z_0)} \right|^2 - 1 \\ &= \left| \frac{-p(a + \mu) + [(aB - p(A - B)(1 - \delta))]e^{i\theta}}{a + [aB - \mu B - p(A - B)(1 - \delta)]e^{i\theta}} \right|^2 - 1 \\ &\geq \left| \frac{a + \mu + [aB - p(A - B)(1 - \delta)]e^{i\theta}}{a + [aB - \mu - p(A - B)(1 - \delta)]e^{i\theta}} \right|^2 - 1 \\ &= \frac{2\mu(1 + \cos \theta)[a(B + 1) - p(A - B)(1 - \delta)]}{|a + [aB - \mu - p(A - B)(1 - \delta)]e^{i\theta}|^2} \geq 0, \end{aligned}$$

since  $a \geq \frac{p(A-B)(1-\delta)}{1+B}$ .

This is a contradiction with our hypothesis that  $f \in \mathcal{K}_{a+1, c}(p; A, B, \delta)$ , then  $|w(z)| < 1$ , ( $z \in \mathcal{U}$ ) and we have  $f \in \mathcal{K}_{a, c}(p; A, B, \delta)$ .

**Theorem 2.2** Let  $f(z) \in \mathcal{K}_{a, c}(p; A, B, \delta)$ . Then  $g(z)$  defined by

$$\mathcal{L}_p(a, c)g(z) = \left( \frac{k - p\alpha}{z^k} \int_0^z t^{k-1} [\mathcal{L}_p(a, c)f(t)]^\alpha dt \right)^{1/\alpha} \quad (13)$$

where

$$\alpha > 0, \quad R(k) \geq p\alpha \left( \frac{1 + [B + (A - B)(1 - \delta)]}{1 + B} \right) > 0; \quad p \in \mathbb{N}$$

is also in the class  $\mathcal{K}_{a,c}(p; A, B, \delta)$ .

**Proof.** Consider  $f(z) \in \mathcal{K}_{a,c}(p; A, B, \delta)$  and by using (13), we have

$$[\mathcal{L}_p(a, c)g(z)]^\alpha = \frac{k - p\alpha}{z^k} \int_0^z t^{k-1} [\mathcal{L}_p(a, c)f(t)]^\alpha dt. \quad (14)$$

After differentiating logarithmically both sides of (14), we get

$$\frac{z(\mathcal{L}_p(a, c)g(z))'}{\mathcal{L}_p(a, c)g(z)} = -\frac{k}{\alpha} + \frac{k - p\alpha}{\alpha} \left[ \frac{\mathcal{L}_p(a, c)f(z)}{\mathcal{L}_p(a, c)g(z)} \right]^\alpha. \quad (15)$$

Let

$$\frac{z(\mathcal{L}_p(a, c)g(z))'}{\mathcal{L}_p(a, c)g(z)} = -p \left( \frac{1 + [B + (A - B)(1 - \delta)]w(z)}{1 + Bw(z)} \right), \quad (16)$$

then from (15) and (16), we get

$$\frac{k(\mathcal{L}_p(a, c)f(z))^\alpha + (\alpha p - k)(\mathcal{L}_p(a, c)f(z))^\alpha}{(\mathcal{L}_p(a, c)g(z))^\alpha} = \frac{\alpha p + \alpha p[B + (A - B)(1 - \delta)]w(z)}{1 + Bw(z)}. \quad (17)$$

Differentiating both sides of (17), we have

$$\begin{aligned} \frac{z\mathcal{L}_p(a, c)f(z)'}{\mathcal{L}_p(a, c)f(z)} &= \frac{p(1 + [B + (A - B)(1 - \delta)]w(z))}{\alpha p(1 + [B + (A - B)(1 - \delta)]w(z)) - k(1 + Bw(z))} \\ &\times \left[ k - \alpha p \left\{ \frac{1 + [B + (A - B)(1 - \delta)]w(z) + Bzw'(z)}{1 + Bw(z)} \right\} \right. \\ &\left. + \frac{[B + (A - B)(1 - \delta)]zw'(z)}{1 + [B + (A - B)(1 - \delta)]w(z)} \right]. \end{aligned} \quad (18)$$

By making necessary changes in previous theorem and suppose that

$$\max_{|z| \leq |z_0|} |w(z)| = |w(z_0)| = 1,$$

we find  $z_0 w'(z_0) = \mu w(z_0)$  by applying Jack's Lemma, where  $z_0 \in \mathcal{U}, \mu \geq 1$  and  $\mu \in \mathbb{R}$ . Let  $w(z_0) = e^{i\theta} (\theta \neq \pi)$ , in (18), we have

$$\begin{aligned} &\left| \frac{z_0(\mathcal{L}_p(a, c)f(z_0))' + \mathcal{L}_p(a, c)f(z_0)}{Bz_0(\mathcal{L}_p(a, c)f(z_0))' + p[B + (A - B)(1 - \delta)]\mathcal{L}_p(a, c)f(z_0)} \right|^2 - 1 \\ &= \left| \frac{k + \mu - \alpha p + [Bk - \alpha p(B + (A - B)(1 - \delta))]e^{i\theta}}{k - \alpha p + [Bk - B\mu - \alpha p(B + (A - B)(1 - \delta))]e^{i\theta}} \right|^2 - 1 \\ &= \frac{h(\theta)}{|(k - p\alpha) + [Bk - B\mu - \alpha p(B + (A - B)(1 - \delta))]e^{i\theta}|^2} \end{aligned}$$

where

$$\begin{aligned} h(\theta) &= \mu^2(1 - B^2) + 2\mu[(1 + B^2)k - \alpha p(1 + B(B + (A - B)(1 - \delta)))] \\ &\quad + 2\mu[2B\operatorname{Re}(k) - p\alpha(2B + (A - B)(1 - \delta))] \cos \theta \end{aligned}$$

where  $0 \leq \theta < 2\pi$ ,  $-1 \leq B < A \leq 1$ ,  $\mu \geq 1$ ,  $0 \leq \delta < 1$ .

By hypothesis we have  $Re(k) \geq p\alpha \left( \frac{1+[B+(A-B)(1-\delta)]}{1+B} \right)$  thus  $h(0) \geq 0$  and  $h(\pi) \geq 0$  which shows that  $h(\theta) \geq 0$  ( $0 \leq \theta < 2\pi$ ). So we get contradiction with our hypothesis. Therefore,  $|w(z)| < 1$ ,  $z \in \mathcal{U}$ , then  $g(z) \in \mathcal{K}_{a,c}(p; A, B, \delta)$ .

### 3 Coefficient Bounds

To investigate the coefficient bounds and some other results we assume that  $a > 0$ ,  $c > 0$  and  $A + B \leq 0$ , ( $-1 \leq B < A \leq 1$ ).

**Theorem 3.1** *If  $f(z) \in T^*(p)$  defined by (1), then  $f \in \mathcal{K}_{a,c}^+(p; A, B, \delta)$  if and only if*

$$\sum_{n=p}^{\infty} [(1-B)(n+p) - p(A-B)(1-\delta)] \frac{(a)_{n+p}}{(c)_{n+p}} |a_n| \leq p(1-\delta)(A-B). \quad (19)$$

The result is sharp for  $f(z)$  given by

$$f(z) = z^{-p} + \left( \frac{p(1-\delta)(A-B)}{n(1-B) + p(1-B - (A-B)(1-\delta))} \right) \frac{(c)_{n+p}}{(a)_{n+p}} z^n, \quad (20)$$

$n = p, p+1, \dots$ .

**Proof.** Let  $f \in \mathcal{K}_{a,c}^+(p; A, B, \delta)$  given by (6). Then

$$\begin{aligned} & \left| \frac{z(\mathcal{L}_p(a, c)f(z))' + p\mathcal{L}_p(a, c)f(z)}{Bz(\mathcal{L}_p(a, c)f(z))' + p(B + (A-B)(1-\delta))\mathcal{L}_p(a, c)f(z)} \right| \\ &= \left| \frac{\sum_{n=p}^{\infty} (n+p)|a_n| \frac{(a)_{n+p}}{(c)_{n+p}} z^{n+p}}{p(A-B)(1-\delta) + \sum_{n=p}^{\infty} (B(n+p) + p(A-B)(1-\delta))|a_n| \frac{(a)_{n+p}}{(c)_{n+p}} z^{n+p}} \right| < 1, \end{aligned}$$

choose  $z$  to be real and  $z \rightarrow 1^-$ , we obtain

$$\begin{aligned} & \sum_{n=p}^{\infty} \frac{(a)_{n+p}}{(c)_{n+p}} (n+p)|a_n| \leq p(A-B)(1-\delta) \\ & + \sum_{n=p}^{\infty} (B(n+p) + p(A-B)(1-\delta))|a_n| \frac{(a)_{n+p}}{(c)_{n+p}}, \end{aligned}$$

then

$$\sum_{n=p}^{\infty} [(1-B)(n+p) - p(A-B)(1-\delta)] \frac{(a)_{n+p}}{(c)_{n+p}} |a_n| \leq p(A-B)(1-\delta).$$

Conversely, assume that the inequality (19) holds true then

$$\begin{aligned} & \left| \frac{z(\mathcal{L}_p(a, c)f(z))' + p\mathcal{L}_p(a, c)f(z)}{Bz(\mathcal{L}_p(a, c)f(z))' + p(B + (A - B)(1 - \delta))\mathcal{L}_p(a, c)f(z)} \right| \\ & \leq \frac{\sum_{n=p}^{\infty} (n + p)|a_n| \frac{(a)_{n+p}}{(c)_{n+p}} |a_n|}{p(A - B)(1 - \delta) + \sum_{n=p}^{\infty} (B(n + p) + p(A - B)(1 - \delta)) \frac{(a)_{n+p}}{(c)_{n+p}} |a_n|} < 1 \end{aligned}$$

( $z \in \mathcal{U}, z \in \mathbb{C}, |z| = 1$ ).

Here, by Maximum Modulus Theorem we get  $f(z) \in \mathcal{K}_{a,c}^+(p; A, B, \delta)$ . Finally, we observe that the function given by (20) is an extremal function.

Next we investigate the extreme points of the class  $\mathcal{K}_{a,c}^+(p; A, B, \delta)$ .

**Theorem 3.2**  $f(z) \in \mathcal{K}_{a,c}^+(p; A, B, \delta)$  of the form (6) if and only if it can be expressed of the form

$$f(z) = \sum_{n=p-1}^{\infty} \lambda_n f_n(z), \lambda_n \geq 0, n = p - 1, p, \dots \quad (21)$$

where  $f_{p-1}(z) = z^{-p}$ ,  $f_n(z) = z^{-p} + \frac{p(1-\delta)(A-B)}{n(1-B)+p(1-B-(A-B)(1-\delta))} \frac{(c)_{n+p}}{(a)_{n+p}} z^n$ ,  $n = p, p + 1, \dots$  and  $\sum_{n=p-1}^{\infty} \lambda_n = 1$ .

**Proof.** Let  $f(z)$  of the form (21). Then

$$\begin{aligned} f(z) &= \lambda_{p-1} z^{-p} + \sum_{n=p}^{\infty} \lambda_n \left[ z^{-p} + \frac{p(1-\delta)(A-B)(c)_{n+p}}{n(1-B) + p(1-B - (A-B)(1-\delta))(a)_{n+p}} z^n \right] \\ &= \left[ z^{-p} + \sum_{n=p}^{\infty} \frac{p(1-\delta)(A-B)(c)_{n+p}}{[n(1-B) + p(1-B - (A-B)(1-\delta))](a)_{n+p}} \lambda_n z^n \right], \end{aligned}$$

then by Theorem 3.1 we have  $f \in \mathcal{K}_{a,c}^+(p; A, B, \delta)$ .

Conversely, let  $f(z) \in \mathcal{K}_{a,c}^+(p; A, B, \delta)$  where  $f(z)$  given by (6) then

$$\sum_{n=p}^{\infty} \frac{[n(1-B) + p(1-B - (A-B)(1-\delta))](a)_{n+p}}{p(1-\delta)(A-B)(c)_{n+p}} |a_n| \leq 1,$$

so we obtain  $\lambda_{p-1} = 1 - \sum_{n=p}^{\infty} \lambda_n$  where

$$\lambda_n = \frac{[n(1-\beta) + p(1-B - (A-B)(1-\delta))](a)_{n+p}}{p(1-\delta)(A-B)(c)_{n+p}} |a_n|, n = p, p + 1, \dots$$

then

$$f(z) = \lambda_{p-1} z^{-p} + \sum_{n=p}^{\infty} \lambda_n f_n(z) = \sum_{n=p-1}^{\infty} \lambda_n f_n(z).$$

**Theorem 3.3** Let  $f_i(z) = z^{-p} + \sum_{n=p}^{\infty} |a_{n,i}|z^n$  for  $i = 1, \dots, \ell$  belongs to  $\mathcal{K}_{a,c}^+(p; A, B, \delta)$ .

Then  $G(z) = \sum_{i=1}^{\ell} g_i f_i(z) \in \mathcal{K}_{a,c}^+(p; A, B, \delta)$  where  $\sum_{i=1}^{\ell} g_i = 1$ .

**Proof.** By Theorem 3.1 and for every  $i = 1, \dots, \ell$ , we have

$$\sum_{n=p}^{\infty} [(1-B)(n+p) - p(A-B)(1-\delta)] \frac{(a)_{n+p}}{(c)_{n+p}} |a_{n,i}| \leq p(A-B)(1-\delta)$$

then

$$G(z) = \sum_{i=1}^{\ell} g_i (z^{-p} + \sum_{n=p}^{\infty} |a_{n,i}|z^n) = z^{-p} + \sum_{n=p}^{\infty} \left( \sum_{i=1}^{\ell} g_i |a_{n,i}| \right) z^n.$$

Since

$$\begin{aligned} & \sum_{n=p}^{\infty} \left( \frac{(1-B)(n+p) - p(A-B)(1-\delta)}{P(A-B)(1-\delta)} \right) \left( \sum_{i=1}^{\ell} g_i |a_{n,i}| \right) \frac{(a)_{n+p}}{(c)_{n+p}} \\ &= \sum_{i=1}^{\ell} g_i \left( \sum_{n=p}^{\infty} \left( \frac{(1-B)(n+p) - p(A-B)(1-\delta)}{P(A-B)(1-\delta)} \right) \frac{(a)_{n+p}}{(c)_{n+p}} |a_{n,i}| \right) \leq 1 \end{aligned}$$

then  $G(z) \in \mathcal{K}_{a,c}^+(p; A, B, \delta)$ .

## 4 Neighbourhoods

**Definition 4.1** Let  $a > 0, c > 0, -1 \leq B < A \leq 1$  and  $\gamma \geq 0$ , we define  $\gamma$ -neighbourhood of a function  $f \in T^*(p)$  and denote by  $N_\gamma(f)$  contains all functions

$g(z) = z^{-p} + \sum_{n=1}^{\infty} b_n z^{n-p} \in T^*(p)$  satisfying

$$\sum_{n=1}^{\infty} \frac{[(1+|B|)n + p(A-B)(1-\delta)] (a)_n}{P(A-B)(1-\delta)} |a_n - b_n| \leq \gamma. \quad (22)$$

**Theorem 4.1** Let  $f \in \mathcal{K}_{a,c}(p; A, B, \delta)$ . Then  $N_\gamma(f) \subset \mathcal{K}_{a,c}(p; A, B, \delta)$  for every  $\mu \in \mathbb{C}$  with  $|\mu| < \gamma, \gamma > 0, \frac{f(z) + \mu z^{-p}}{1+\mu} \in \mathcal{K}_{a,c}(p; A, B, \delta)$ .

**Proof.** Let  $g \in \mathcal{K}_{a,c}(p; A, B, \delta)$ . Then by (4) we have

$$\left| \frac{z(\mathcal{L}_p(a, c)g(z))' + p\mathcal{L}_p(a, c)g(z)}{Bz(\mathcal{L}_p(a, c)g(z))' + p(B + (A-B)(1-\delta))\mathcal{L}_p(a, c)g(z)} \right| \neq \zeta \quad (23)$$



( $\zeta \in \mathbb{C}, |\zeta| = 1$ ), equivalently we must have  $\frac{(f*\varphi)(z)}{z^{-p}} \neq 0, z \in \mathcal{U}^*$ , where

$$\begin{aligned} \phi(z) &= z^{-p} + \sum_{n=1}^{\infty} d_n z^{n-p}, \\ &= z^{-p} + \sum_{n=1}^{\infty} \left( \frac{n(1 - \zeta B) - p\zeta(A - B)(1 - \delta)}{p\zeta(A - B)(1 - \delta)} \right) \frac{(a)_n}{(c)_n} z^{n-p}. \end{aligned}$$

So that

$$|d_n| \leq \frac{n(1 + |B|) + p(A - B)(1 - \delta)}{p(A - B)(1 - \delta)} \frac{(a)_n}{(c)_n}, n = p, p + 1, \dots.$$

Hence we have  $\left( z^p \left( \frac{f(z) + \mu z^{-p}}{1 + \mu} * \varphi(z) \right) \right) \neq 0$ , then

$$\frac{1}{1 + \mu} \frac{(f * \varphi)(z)}{z^{-p}} + \frac{\mu}{1 + \mu} \neq 0, \tag{24}$$

then

$$\begin{aligned} \left| \frac{1}{1 + \mu} \frac{(f * \varphi)(z)}{z^{-p}} + \frac{\mu}{1 + \mu} \right| &\geq \frac{1}{|1 + \mu|} \left| \frac{(f * \varphi)(z)}{z^{-p}} \right| \\ - \frac{|\mu|}{|1 + \mu|} &> \frac{1}{1 + \gamma} \left| \frac{(f * \varphi)(z)}{z^{-p}} \right| - \frac{\gamma}{1 + \gamma} \end{aligned}$$

to hold (24) we must have  $\frac{1}{1 + \gamma} \left| \frac{(f * \varphi)(z)}{z^{-p}} \right| - \frac{\gamma}{1 + \gamma} \geq 0$  then  $\left| \frac{(f * \varphi)(z)}{z^{-p}} \right| \geq \gamma$ . Now

$$\begin{aligned} \gamma - \left| \frac{(g * \varphi)(z)}{z^{-p}} \right| &\leq \left| \frac{((f - g) * \varphi)(z)}{z^{-p}} \right| \leq \sum_{n=1}^{\infty} |a_n - b_n| |d_n| |z|^n \\ &< \sum_{n=1}^{\infty} \frac{n(1 + |B|) + p(A - B)(1 - \delta)}{p(A - B)(1 - \delta)} \frac{(a)_n}{(c)_n} |a_n - b_n| \leq \gamma, \end{aligned}$$

thus  $\frac{(g*\varphi)(z)}{z^{-p}} \neq 0$  and  $g \in \mathcal{K}_{a,c}(p; A, B, \delta)$ .

**Theorem 4.2.** Let  $f \in T^*(p)$  and let  $s_1(z) = z^{-p}$  and  $s_\ell(z) = z^{-p} + \sum_{n=1}^{\ell-1} a_n z^{n-p}, \ell =$

$2, 3, \dots$ , suppose that  $\sum_{n=1}^{\infty} d_n |a_n| \leq 1$  where

$$d_n = \frac{n(1 + |B|) + p(A - B)(1 - \delta)}{p(A - B)(1 - \delta)} \frac{(a)_n}{(c)_n}$$

- (i) if  $a > 0, c > 0$ , then  $f \in \mathcal{K}_{a,c}(p; A, B, \delta)$ , and
- (ii) if  $a > c > 0$ , then

$$\operatorname{Re} \left( \frac{f(z)}{s_\ell(z)} \right) > 1 - \frac{1}{d_\ell}, \operatorname{Re} \frac{s_\ell(z)}{f(z)} > \frac{d_\ell}{1 - d_\ell}, z \in \mathcal{U}, \ell \in \mathbb{N}.$$

**Proof.** It is clear that  $N_1(z^{-p}) \subset \mathcal{K}_{a,c}(p; A, B, \delta)$ , since

$$\left( \frac{z^{-p} + z^{-p}\mu}{1 + \mu} \right) = z^{-p} \in \mathcal{K}_{a,c}(p; A, B, \delta),$$

then we have  $f \in \mathcal{K}_{a,c}(p; A, B, \delta)$ , also  $d_{n+1} > d_n > 1$  thus  $\sum_{n=1}^{\ell-1} |a_n| + d_\ell \sum_{n=\ell}^{\infty} |a_n| \leq 1$ .

Consider  $G(z) = d_\ell \left[ \frac{f(z)}{s_\ell(z)} - \left(1 - \frac{1}{d_\ell}\right) \right]$  and use the last expression we get

$$\left| \frac{G(z) - 1}{G(z) + 1} \right| \leq \frac{d_\ell \sum_{n=\ell}^{\infty} |a_n|}{2 - \sum_{n=1}^{\infty} |a_n| - d_\ell \sum_{n=\ell}^{\infty} |a_n|} \leq 1$$

then (i) is complete, to prove (ii).

Let  $F(z) = (1 + d_\ell) \left[ \frac{s_\ell(z)}{f(z)} - \frac{d_\ell}{1-d_\ell} \right]$  so we have

$$\left| \frac{F(z) - 1}{F(z) + 1} \right| \leq \frac{(1 - d_\ell) \sum_{n=\ell}^{\infty} |a_n|}{2 - 2 \sum_{n=1}^{\ell-1} |a_n| + (1 - d_\ell) \sum_{n=\ell}^{\infty} |a_n|} \leq 1$$

then the proof is complete.

**Definition 4.2.** Let  $f(z) \in T^*(p)$  given by (6). Then  $\gamma$ -neighbourhood of  $f$  and is denoted by  $N_\gamma^+(f)$  contains all functions  $g(z) = z^{-p} + \sum_{n=p}^{\infty} b_n z^n$  satisfying

$$\sum_{n=p}^{\infty} \frac{(1-B)(n+p) - p(A-B)(1-\delta)}{p(A-B)(1-\delta)} \frac{(a)_{n+p}}{(c)_{n+p}} |a_n - b_n| \leq \gamma,$$

where  $a > 0, c > 0, -1 \leq B < A \leq 1, 0 \leq \delta < 1, \gamma \geq 0$ .

**Theorem 4.3.** If  $f \in \mathcal{K}_{a+1,c}^+(p; A, B, \delta)$ , then  $N_\gamma^+(f) \subset \mathcal{K}_{a,c}^+(p; A, B, \delta)$  where  $A + B \leq 0$  and  $\gamma = \frac{2p}{a+2p}$ . The result is sharp.

**Proof.** By using the same procedure as in the proof of Theorem 4.1, with

$$h(z) = z^{-p} + \sum_{n=p}^{\infty} e_n z^n = z^{-p} + \sum_{n=p}^{\infty} \left[ \frac{(1-\zeta B)(n+p) - p\zeta(A-B)(1-\delta)}{\zeta p(A-B)(1-\delta)} \frac{(a)_{n+p}}{(c)_{n+p}} z^n \right]$$

where  $A + B \leq 0$  and  $f \in \mathcal{K}_{a+1,c}^+(p; A, B, \delta)$ , we have  $\left| \frac{(f*h)(z)}{z^{-p}} \right| \geq \frac{2p}{a+2p} = \gamma$ . For sharpness, let

$$f(z) = z^{-p} + \left( \frac{(A-B)(1-\delta)}{2-2B-(A-B)(1-\delta)} \right) \frac{(c)_{2p}}{(a+1)_{2p}} z^p \in \mathcal{K}_{a+1,c}^+(p; A, B, \delta)$$

and

$$g(z) = z^{-p} + \left[ \frac{(A-B)(1-\delta)}{2-2B-(A-B)(1-\delta)} \cdot \frac{(c)_{2p}}{(a+1)_{2p}} + \frac{\gamma'(A-B)(1-\delta)}{2-2B-(A-B)(1-\delta)} \frac{(c)_{2p}}{(a)_{2p}} \right] z^p$$

where  $\gamma' > \gamma = \frac{2p}{a+2p}$ , we get  $g(z) \in N_{\gamma'}^+(f)$  but not in  $\mathcal{K}_{a,c}^+(p; A, B, \delta)$ .

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## On some new quasi almost $\Delta^m$ -lacunary strongly P-convergent double sequences defined by Orlicz functions

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ABSTRACT: The idea of quasi almost P-convergent sequences defined as  $\|P - \lim_{p,q \rightarrow \infty} \sup_{m,n \geq 0} \frac{1}{pq} \sum_{k=m}^{m+p-1} \sum_{l=n}^{n+q-1} x_{k,l} - L\|_X = 0$  was introduced by V.A.Khan and Q.M.D.Lohani [ *Mathematicki Vesnik*, (60), 95-100 (2008)]. In this paper we introduce a new concept for quasi almost  $\Delta^m$ -lacunary strongly P-convergent double sequences defined by Orlicz function and give inclusion relations.

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Key Words and Phrases: *Lacunary Sequence, Differences Double sequences; Orlicz Function,, Quasi almost P-convergence*

### 1 Introduction

The difference sequence space  $X(\Delta)$  was introduced by Kizmaz[4] as follows:  $X(\Delta) = \{x = (x_k) : (\Delta x_k) \in X\}$  for  $X = l_\infty, c, c_0$ , where  $\Delta x = (\Delta x_k) = (x_k - x_{k+1})$ . After Et. and Colak [1] generalized the difference sequence spaces to the sequence spaces  $X(\Delta^m) = \{x = (x_k) : (\Delta^m x_k) \in X\}$  for  $X = l_\infty, c, c_0$ , where  $m \in \mathbb{N}$ ,  $\Delta^0 x = (x_k)$ ,  $\Delta x = (x_k - x_{k+1})$ ,  $\Delta^m x = (\Delta^m x_k) = (\Delta^{m-1} x_k - \Delta^{m-1} x_{k+1})$ , and so that

$$\Delta^m x_k = \sum_{v=0}^m (-1)^v \binom{m}{v} x_v.$$

A double sequence  $x = (x_{k,l})$  is a double infinite array of elements  $x_{kl}$ . for  $k, l \in \mathbb{N}$ . By the convergence of a double sequence we mean the convergence on the Pringsheim sense that is, a double sequence  $x = (x_{kl})$  is said to be Pringsheim convergent (or

P-convergent) if for  $\epsilon > 0$  there exists an integer  $N$  such that  $|x_{k,l} - L| < \epsilon$  whenever  $k, l > N$  [9]. We shall write this as

$$P - \lim_{k,l \rightarrow \infty} x_{k,l} = L, \quad \text{where } j, k \text{ tends to infinity independent of each other.}$$

By a lacunary sequence  $\theta = (k_r)$ ,  $r=0,1,2,\dots$  where  $k_0 = 0$ , we mean an increasing sequence of non negative integers  $h_r = (k_r - k_{r-1}) \rightarrow \infty (r \rightarrow \infty)$ . The intervals determined by  $\theta$  are denoted by  $I_r = (k_{r-1}, k_r]$  and ratio  $\frac{k_r}{k_{r-1}}$  will be denoted by  $q_r$ . The space of lacunary strongly convergent sequence  $N_\theta$  was defined by Freedman et al.[2] as follows

$$N_\theta = \left\{ x = (x_k) : \lim_{r \rightarrow \infty} \frac{1}{h_r} \sum_{k \in I_r} |x_k - L| = 0, \text{ for some } L \right\}.$$

The double lacunary sequence was defined by E.Savas and R.F.Patterson[13] as follows: The double sequence  $\theta_{r,s} = \{(k_r, l_s)\}$  is called double lacunary if there exist two increasing sequence of integers such that

$$k_0 = 0, h_r = k_r - k_{r-1} \rightarrow \infty \text{ as } r \rightarrow \infty$$

and

$$l_0 = 0, \bar{h}_s = l_s - l_{s-1} \rightarrow \infty \text{ as } s \rightarrow \infty.$$

**Notations :**  $k_{r,s} = k_r l_s$ ,  $h_{r,s} = h_r \bar{h}_s$ .

The following intervals are determined by  $\theta$ .

$$I_r = \{(k_r) : k_{r-1} < k < k_r\}, I_s = \{(l) : l_{s-1} < l < l_s\}$$

$$I_{r,s} = \{(k, l) : k_{r-1} < k < k_r \text{ and } l_{s-1} < l < l_s\},$$

$q_r = \frac{k_r}{k_{r-1}}$ ,  $\bar{q}_s = \frac{l_s}{l_{s-1}}$  and  $q_{r,s} = q_r \bar{q}_s$ . We will denote the set of all lacunary sequences by  $N_{\theta_{r,s}}$ . The space of double lacunary strongly convergent sequence is defined as follows:

$$N_{\theta_{r,s}} = \left\{ x = (x_{k,l}) : \lim_{r,s} \frac{1}{h_{r,s}} \sum_{(k,l) \in I_{r,s}} |x_{k,l} - L| = 0 \text{ for some } L \right\} \quad [\text{see (13)}].$$

An *Orlicz Function* is a function  $M : [0, \infty) \rightarrow [0, \infty)$  which is continuous, nondecreasing and convex with  $M(0) = 0$ ,  $M(x) > 0$  for  $x > 0$  and  $M(x) \rightarrow \infty$ , as  $x \rightarrow \infty$ . If convexity of  $M$  is replaced by  $M(x+y) \leq M(x) + M(y)$  then it is called Modulus function which is defined and characterized by Ruckle[12].

An Orlicz function  $M$  satisfies the  $\Delta_2$  - condition ( $M \in \Delta_2$  for short ) if there exist constant  $k \geq 2$  and  $u_0 > 0$  such that

$$M(2u) \leq KM(u)$$

whenever  $|u| \leq u_0$ .

An Orlicz function  $M$  can always be represented in the integral form  $M(x) = \int_0^x q(t)dt$ , where  $q$  known as the kernel of  $M$ , is right differentiable for  $t \geq 0, q(t) > 0$  for  $t > 0, q$  is non-decreasing and  $q(t) \rightarrow \infty$  as  $t \rightarrow \infty$ .

Note that an Orlicz function satisfies the inequality

$$M(\lambda x) \leq \lambda M(x) \text{ for all } \lambda \text{ with } 0 < \lambda < 1,$$

since  $M$  is convex and  $M(0) = 0$ .

W. Orlicz used the idea of Orlicz function to construct the space  $(L^M)$ . Lindenstrauss and Tzafriri [7] used the idea of Orlicz sequence space;

$$l_M := \left\{ x \in w : \sum_{k=1}^{\infty} M\left(\frac{|x_k|}{\rho}\right) < \infty, \text{ for some } \rho > 0 \right\}$$

which is Banach space with the norm

$$\|x\|_M = \inf \left\{ \rho > 0 : \sum_{k=1}^{\infty} M\left(\frac{|x_k|}{\rho}\right) \leq 1 \right\}.$$

The space  $l_M$  is closely related to the space  $l_p$ , which is an Orlicz sequence space with  $M(x) = x^p$  for  $1 \leq p < \infty$ .

Recently Moricz and Rhoads[9] defined almost P-convergent sequences as follows: A double sequence  $x = (x_{k,l})$  of real numbers is called almost P-convergent to a limit L if

$$P - \lim_{p,q \rightarrow \infty} \sup_{m,n \geq 0} \frac{1}{pq} \sum_{k=m}^{m+p-1} \sum_{l=n}^{n+q-1} |x_{k,l} - L| = 0.$$

Later on V.A.Khan and Q.M.D.Lohani[3], defined quasi almost P-convergent sequences as follows: A double sequence  $x = (x_{k,l})$  of elements of real normed space  $X$  is said to be quasi almost P-convergent to a limit  $L$  if

$$\|P - \lim_{p,q \rightarrow \infty} \sup_{m,n \geq 0} \frac{1}{pq} \sum_{k=m}^{m+p-1} \sum_{l=n}^{n+q-1} x_{k,l} - L\|_X = 0.$$

and denoted the above set of sequence as  $\bar{l}^2$ .

Let  $M$  be an Orlicz function and  $Q = (q_{k,l})$  be any factorable double sequence of strictly positive real numbers, we define the following sequence spaces:

$$\begin{aligned} & [L_{\theta_{r,s}}, \Delta^m, M, Q] \\ & = \left\{ x = (x_{k,l}) : P - \lim_{r,s \rightarrow \infty} \frac{1}{h_{r,s}} \sum_{(k,l) \in I_{r,s}} \left[ M\left(\frac{\|\Delta^m x_{k+m,l+n} - L\|}{\rho}\right) \right]^{q_{k,l}} = 0 \right. \\ & \quad \left. \text{uniformly in } m \text{ and } n \text{ for some } L \text{ and } \rho > 0 \right\}. \end{aligned}$$

$$\begin{aligned}
& [L_{\theta_{r,s}}, \Delta^m, M, Q]_0 \\
&= \left\{ x = (x_{k,l}) : P - \lim_{r,s \rightarrow \infty} \frac{1}{h_{r,s}} \sum_{(k,l) \in I_{r,s}} \left[ M \left( \frac{\|\Delta^m x_{k+m,l+n}\|}{\rho} \right) \right]^{q_{k,l}} = 0 \right. \\
&\quad \left. \text{uniformly in } m \text{ and } n \text{ and } \rho > 0 \right\}.
\end{aligned}$$

We shall denote  $[L_{\theta_{r,s}}, \Delta^m, M, Q]$  and  $[L_{\theta_{r,s}}, \Delta^m, M, Q]_0$  as  $[L_{\theta_{r,s}}, \Delta^m, M]$  and  $[L_{\theta_{r,s}}, \Delta^m, M]_0$  respectively when  $q_{k,l} = 1$  for all  $k, l$ . If  $x$  is in  $[L_{\theta_{r,s}}, \Delta^m, M]$ , we say that  $x$  is quasi almost lacunary strongly P-convergent with respect to the Orlicz function  $M$ . Also if  $M(x) = x$ ,  $q_{k,l} = 1$  for all  $k, l$ , then  $[L_{\theta_{r,s}}, \Delta^m, M, Q] = [L_{\theta_{r,s}}, \Delta^m]$  and  $[L_{\theta_{r,s}}, \Delta^m, M, Q]_0 = [L_{\theta_{r,s}}^0, \Delta^m]$  which are defined as follows:

$$\begin{aligned}
& [L_{\theta_{r,s}}, \Delta^m] = \left\{ x = (x_{k,l}) : P - \lim_{r,s \rightarrow \infty} \frac{1}{h_{r,s}} \sum_{(k,l) \in I_{r,s}} \|\Delta^m x_{k+m,l+n} - L\| = 0 \right. \\
&\quad \left. \text{uniformly in } m \text{ and } n \text{ for some } L \right\}.
\end{aligned}$$

and

$$\begin{aligned}
& [L_{\theta_{r,s}}^0, \Delta^m] = \left\{ x = (x_{k,l}) : P - \lim_{r,s \rightarrow \infty} \frac{1}{h_{r,s}} \sum_{(k,l) \in I_{r,s}} \|\Delta^m x_{k+m,l+n}\| = 0 \right. \\
&\quad \left. \text{uniformly in } m \text{ and } n \right\}.
\end{aligned}$$

Again note that if  $q_{k,l} = 1$  for all  $k$  and  $l$  then

$$[L_{\theta_{r,s}}, \Delta^m, M, Q] = [L_{\theta_{r,s}}, \Delta^m, M] \text{ and } [L_{\theta_{r,s}}, \Delta^m, M, Q]_0 = [L_{\theta_{r,s}}, \Delta^m, M]_0$$

We define

$$\begin{aligned}
& [L_{\theta_{r,s}}, \Delta^m, M] \\
&= \left\{ x = (x_{k,l}) : P - \lim_{r,s \rightarrow \infty} \frac{1}{h_{r,s}} \sum_{(k,l) \in I_{r,s}} \left[ M \left( \frac{\|\Delta^m x_{k+m,l+n} - L\|}{\rho} \right) \right] = 0 \right. \\
&\quad \left. \text{uniformly in } m \text{ and } n \text{ for some } L \text{ and } \rho > 0 \right\}.
\end{aligned}$$

and

$$\begin{aligned}
& [L_{\theta_{r,s}}, \Delta^m, M]_0 \\
&= \left\{ x = (x_{k,l}) : P - \lim_{r,s \rightarrow \infty} \frac{1}{h_{r,s}} \sum_{(k,l) \in I_{r,s}} \left[ M \left( \frac{\|\Delta^m x_{k+m,l+n} - L\|}{\rho} \right) \right] = 0 \right. \\
&\quad \left. \text{uniformly in } m \text{ and } n \text{ and } \rho > 0 \right\}.
\end{aligned}$$

Let us consider another extension of quasi almost P-convergence of double sequences to Orlicz function ; Let  $M$  be an Orlicz function and  $Q = (q_{k,l})$  be any factorable sequence of strictly positive real numbers, we define the following sequence space:

$$[T, \Delta^m, M, Q] = \left\{ x = (x_{k,l}) : P - \lim_{p,q \rightarrow \infty} \sum_{k,l=1,1}^{p,q} \left[ M \left( \frac{\|\Delta^m x_{k+m,l+n} - L\|}{\rho} \right) \right]^{q_{k,l}} = 0 \right. \\ \left. \text{uniformly in } m \text{ and } n \text{ for some } L \text{ and } \rho > 0 \right\}.$$

$$[T, \Delta^m, M, Q]_0 = \left\{ x = (x_{k,l}) : P - \lim_{p,q \rightarrow \infty} \sum_{k,l=1,1}^{p,q} \left[ M \left( \frac{\|\Delta^m x_{k+m,l+n}\|}{\rho} \right) \right]^{q_{k,l}} = 0 \right. \\ \left. \text{uniformly in } m \text{ and } n \text{ and for some } \rho > 0 \right\}.$$

If we take  $M(x) = x$ ,  $q_{k,l} = 1$  for all  $k$  and  $l$ , then  $[T, \Delta^m, M, Q] = [T, \Delta^m]$

With these new concepts we can consider the following theorem: The proof of first theorem is standard thus omitted

**Theorem 1.1.** For any Orlicz function  $M$  and a bounded factorable positive double number sequence  $(q_{k,l})$   $[L_{\theta_{r,s}}, \Delta^m, M, Q]$  and  $[L_{\theta_{r,s}}, \Delta^m, M, Q]_0$  are linear spaces.

**Theorem 1.2.** Let  $\theta_{r,s} = \{k_r, l_s\}$  be a double lacunary sequence with  $\liminf q_r > 1$  and  $\liminf \bar{q}_s > 1$ . Then for any Orlicz function  $M$ ,  $[T, \Delta^m, M, Q] \subset [L_{\theta_{r,s}}, \Delta^m, M, Q]$ .

**Proof.** It is sufficient to show that  $[T, \Delta^m, M, Q]_0 \subset [L_{\theta_{r,s}}, \Delta^m, M, Q]_0$ . The general inclusion follows by linearity. Suppose  $\liminf q_r > 1$  and  $\liminf \bar{q}_s > 1$ ; then there exists  $\delta > 0$  such that  $q_r > 1 + \delta$  and  $\bar{q}_s > 1 + \delta$ . This implies  $\frac{h_r}{k_r} \geq \frac{\delta}{1+\delta}$  and  $\frac{\bar{h}_r}{l_s} \geq \frac{\delta}{1+\delta}$ . Then for  $x \in [T, \Delta^m, M, Q]_0$ , we can write for each  $m$  and  $n$ ,

$$B_{r,s} = \frac{1}{h_{r,s}} \sum_{(k,l) \in I_{r,s}} \left[ M \left( \frac{\|\Delta^m x_{k+m,l+n}\|}{\rho} \right) \right]^{q_{k,l}} \\ = \frac{1}{h_{r,s}} \sum_{k=1}^{k_r} \sum_{l=1}^{l_s} \left[ M \left( \frac{\|\Delta^m x_{k+m,l+n}\|}{\rho} \right) \right]^{q_{k,l}} \\ - \frac{1}{h_{r,s}} \sum_{k=1}^{k_r-1} \sum_{l=1}^{l_s-1} \left[ M \left( \frac{\|\Delta^m x_{k+m,l+n}\|}{\rho} \right) \right]^{q_{k,l}} \\ - \frac{1}{h_{r,s}} \sum_{k=k_{r-1}+1}^{k_r} \sum_{l=1}^{l_s-1} \left[ M \left( \frac{\|\Delta^m x_{k+m,l+n}\|}{\rho} \right) \right]^{q_{k,l}}$$



$$\begin{aligned}
& -\frac{1}{h_{r,s}} \sum_{l=l_{s-1}+1}^{l_s} \sum_{k=1}^{k_{r-1}} \left[ M \left( \frac{\|\Delta^m x_{k+m,l+n}\|}{\rho} \right) \right]^{q_{k,l}} \\
= & \frac{k_r l_s}{h_{r,s}} \left( \frac{1}{k_r l_s} \sum_{k=1}^{k_r} \sum_{l=1}^{l_s} \left[ M \left( \frac{\|\Delta^m x_{k+m,l+n}\|}{\rho} \right) \right]^{q_{k,l}} \right) \\
& - \frac{k_{r-1} l_{s-1}}{h_{r,s}} \left( \frac{1}{k_{r-1} l_{s-1}} \sum_{k=1}^{k_{r-1}} \sum_{l=1}^{l_{s-1}} \left[ M \left( \frac{\|\Delta^m x_{k+m,l+n}\|}{\rho} \right) \right]^{q_{k,l}} \right) \\
& - \frac{1}{h_r} \sum_{k=k_{r-1}+1}^{k_r} \frac{l_{s-1}}{h_s} \frac{1}{l_{s-1}} \sum_{l=1}^{l_{s-1}} \left[ M \left( \frac{\|\Delta^m x_{k+m,l+n}\|}{\rho} \right) \right]^{q_{k,l}} \\
& - \frac{1}{h_s} \sum_{l=l_{s-1}+1}^{l_s} \frac{k_{r-1}}{h_r} \frac{1}{k_{r-1}} \sum_{k=1}^{k_{r-1}} \left[ M \left( \frac{\|\Delta^m x_{k+m,l+n}\|}{\rho} \right) \right]^{q_{k,l}}
\end{aligned}$$

Since  $x \in [T, \Delta^m, M, Q]_0$  the last two terms tends to zero uniformly in  $m, n$  in Pringsheim senses, thus for each  $m$  and  $n$

$$\begin{aligned}
B_{r,s} & = \frac{k_r l_s}{h_{r,s}} \left( \frac{1}{k_r l_s} \sum_{k=1}^{k_r} \sum_{l=1}^{l_s} \left[ M \left( \frac{\|\Delta^m x_{k+m,l+n}\|}{\rho} \right) \right]^{q_{k,l}} \right) \\
& - \frac{k_{r-1} l_{s-1}}{h_{r,s}} \left( \frac{1}{k_{r-1} l_{s-1}} \sum_{k=1}^{k_{r-1}} \sum_{l=1}^{l_{s-1}} \left[ M \left( \frac{\|\Delta^m x_{k+m,l+n}\|}{\rho} \right) \right]^{q_{k,l}} \right) + 0(1).
\end{aligned}$$

Since  $h_{r,s} = k_r l_s - k_{r-1} l_{s-1}$  we are granted for each  $m$  and  $n$  the following:

$$\frac{k_r l_s}{h_{r,s}} \leq \frac{1+\delta}{\delta} \quad \text{and} \quad \frac{k_{r-1} l_{s-1}}{h_{r,s}} \leq \frac{1}{\delta}.$$

The terms

$$\frac{1}{k_r l_s} \sum_{k=1}^{k_r} \sum_{l=1}^{l_s} \left[ M \left( \frac{\|\Delta^m x_{k+m,l+n}\|}{\rho} \right) \right]^{q_{k,l}}$$

and

$$\frac{1}{k_{r-1} l_{s-1}} \sum_{k=1}^{k_{r-1}} \sum_{l=1}^{l_{s-1}} \left[ M \left( \frac{\|\Delta^m x_{k+m,l+n}\|}{\rho} \right) \right]^{q_{k,l}}$$

are both Pringsheim null sequences for all  $m$  and  $n$ . Thus  $B_{r,s}$  is Pringsheim.

**Theorem 1.3** Let  $\theta_{r,s} = \{k, l\}$  be a double lacunary sequence with  $\limsup_r q_r < \infty$  and  $\limsup_s \bar{q}_s < \infty$ . Then for any Orlicz function  $M$

$$[L_{\theta_{r,s}}, \Delta^m, M, Q] \subset [T, \Delta^m, M, Q].$$

**Proof.** Since  $\limsup_r q_r < \infty$  and  $\limsup_s \bar{q}_s < \infty$ , there exists  $H > 0$  such that

$q_r < H$  and  $\bar{q}_s < H$  for all  $r$  and  $s$ . Let  $x \in [L_{\theta_{r,s}}, \Delta^m, M, Q]$  and  $\epsilon > 0$ . There exists  $r_0 > 0$  and  $s_0 > 0$  such that for every  $i \geq r_0$  and  $j \geq s_0$  and  $m$  and  $n$ ,

$$C_{i,j} = \frac{1}{h_{ij}} \sum_{(k,l) \in I_{i,j}} \left[ M \left( \frac{\|\Delta^m x_{k+m,l+n}\|}{\rho} \right) \right]^{q_{k,l}} < \epsilon.$$

Let  $D = \max\{C_{i,j} : 1 \leq i \leq r_0 \text{ and } 1 \leq j \leq s_0\}$  and  $p$  and  $q$  be such that  $k_{r-1} < p \leq k_r$  and  $l_{s-1} < q \leq l_s$ . Thus we obtain the following

$$\begin{aligned} \frac{1}{pq} \sum_{k,l=1,1}^{p,q} \left[ M \left( \frac{\|\Delta^m x_{k+m,l+n}\|}{\rho} \right) \right]^{q_{k,l}} &\leq \frac{1}{k_{r-1}l_{s-1}} \sum_{k,l=1,1}^{k_r,l_s} \left[ M \left( \frac{\|\Delta^m x_{k+m,l+n}\|}{\rho} \right) \right]^{q_{k,l}} \\ &\leq \frac{1}{k_{r-1}l_{s-1}} \sum_{t,u=1,1}^{r,s} \left( \sum_{(k,l) \in I_{t,u}} \left[ M \left( \frac{\|\Delta^m x_{k+m,l+n}\|}{\rho} \right) \right]^{q_{k,l}} \right) \\ &= \frac{1}{k_{r-1}l_{s-1}} \sum_{t,u=1,1}^{r_0,s_0} h_{t,u} + \frac{1}{k_{r-1}l_{s-1}} \sum_{t,u=1,1}^{r_0,s_0} h_{t,u} C_{t,u}. \\ &= \frac{D}{k_{r-1}l_{s-1}} \sum_{t,u=1,1}^{r_0,s_0} h_{t,u} + \frac{1}{k_{r-1}l_{s-1}} \sum_{(r_0 < t \leq r) \cup (s_0 < u \leq s)} h_{t,u} C_{t,u}. \\ &= \frac{Dk_{r_0}l_{s_0}r_0s_0}{k_{r-1}l_{s-1}} + \left( \sup_{t \geq r_0 \cup u \geq s_0} C_{t,u} \right) \frac{1}{k_{r-1}l_{s-1}} \sum_{(r_0 < t \leq r) \cup (s_0 < u \leq s)} h_{t,u}. \\ &= \frac{Dk_{r_0}l_{s_0}r_0s_0}{k_{r-1}l_{s-1}} + \frac{1}{k_{r-1}l_{s-1}} \epsilon \sum_{(r_0 < t \leq r) \cup (s_0 < u \leq s)} h_{t,u}. \\ &= \frac{Dk_{r_0}l_{s_0}r_0s_0}{k_{r-1}l_{s-1}} + \epsilon H^2 \end{aligned}$$

Since  $k_r$  and  $l_s$  both approach infinity as both  $p$  and  $q$  approach infinity, it follows that

$$\frac{1}{pq} \sum_{k,l=1,1}^{p,q} \left[ M \left( \frac{\|\Delta^m x_{k+m,l+n}\|}{\rho} \right) \right]^{q_{k,l}} \rightarrow 0, \text{ uniformly in } m \text{ and } n \text{ for some } \rho > 0.$$

Therefore  $x \in [T, \Delta^m, M, Q]$ .

**Theorem .1.4.** Let  $\theta_{r,s} = \{k_r, l_s\}$  be a double lacunary sequence with  $\liminf_{r,s} q_{r,s} \leq \limsup_{r,s} q_{r,s} < \infty$ . Then for any Orlicz function  $M$ ,  $[L_{\theta_{r,s}}, \Delta^m, M, Q] = [T, \Delta^m, M, Q]$ .

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# Inclusion relationship and Fekete-Szegő like inequalities for a subclass of meromorphic functions

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ABSTRACT: In this paper using a differential operator, we define a new subclass of meromorphic functions. Sharp upper bounds for the functional  $|a_1 - \mu a_0^2|$  in this class are obtained. An inclusion property is also given.

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## 1 Introduction

Denote by  $\Sigma$  the class of functions of the form

$$f(z) = \frac{1}{z} + \sum_{n=0}^{\infty} a_n z^n \quad (1)$$

which are analytic and univalent in the punctured disk  $\mathbb{U}^* = \{z \in \mathbb{C} : 0 < |z| < 1\}$ .

A function  $f \in \Sigma$  is said to be meromorphic starlike if

$$\Re \frac{z f'(z)}{f(z)} < 0, \quad z \in \mathbb{U}^*. \quad (2)$$

We denote by  $\Sigma^*$  the class of all meromorphic starlike functions.

A function  $f \in \Sigma$  is said to be meromorphic convex if

$$\Re \left( 1 + \frac{z f''(z)}{f'(z)} \right) < 0, \quad z \in \mathbb{U}^*. \quad (3)$$

The class of all meromorphic convex functions will be denoted by  $\Sigma^c$ .

Let  $f \in \Sigma$  be of the form (1) and let  $\alpha, \beta$  be real numbers with  $\alpha \geq \beta \geq 0$ . We define the analogue of the differential operator given in [13] as follows

$$D_{\alpha, \beta}^0 f(z) = f(z)$$

$$\begin{aligned} D_{\alpha, \beta}^1 f(z) &= D_{\alpha, \beta} f(z) = \\ &= \alpha \beta (z^2 f(z))'' + (\alpha - \beta) \frac{(z^2 f(z))'}{z} + (1 - \alpha + \beta) f(z) \end{aligned} \quad (4)$$

$$D_{\alpha, \beta}^m f(z) = D_{\alpha, \beta} \left( D_{\alpha, \beta}^{m-1} f(z) \right), \quad z \in \mathbb{U}^*, \quad m \in \mathbb{N} = \{1, 2, \dots\}. \quad (5)$$

If  $f \in \Sigma$  is given by (1), then from (4) and (5) we get

$$D_{\alpha, \beta}^m f(z) = \frac{1}{z} + \sum_{n=0}^{\infty} A(\alpha, \beta, n)^m a_n z^n, \quad z \in \mathbb{U}^* \quad (6)$$

where

$$A(\alpha, \beta, n) = [(n+2)\alpha\beta + \alpha - \beta](n+1) + 1. \quad (7)$$

Note that for  $\alpha = 1$  and  $\beta = 0$  we obtain the differential operator defined in [1].

Making use of the operator  $D_{\alpha, \beta}^m f(z)$  we introduce the following subclasses of meromorphic functions.

**Definition 1.1** Let  $\Sigma_m^*(\alpha, \beta)$  be the class of functions  $f \in \Sigma$  for which  $D_{\alpha, \beta}^m f(z) \in \Sigma^*$ , that is

$$\Re \frac{z(D_{\alpha, \beta}^m f(z))'}{D_{\alpha, \beta}^m f(z)} < 0, \quad z \in \mathbb{U}^*.$$

Note that  $\Sigma_0^*(\alpha, \beta) = \Sigma^*$ .

**Definition 1.2** Let  $\gamma$  be a complex number. We say that a function  $f \in \Sigma$  belongs to the class  $H\Sigma_m^*(\alpha, \beta, \gamma)$  if the function  $F$  defined by

$$\frac{1}{F(z)} = \frac{1 - \gamma}{D_{\alpha, \beta}^m f(z)} - \frac{\gamma}{z(D_{\alpha, \beta}^m f(z))'}, \quad z \in \mathbb{U}^* \quad (8)$$

is a meromorphic starlike function.

By specializing parameters  $\gamma$  and  $m$  we obtain the following subclasses:

1.  $H\Sigma_m^*(\alpha, \beta, 0) = \Sigma_m^*(\alpha, \beta)$ .
2.  $H\Sigma_0^*(\alpha, \beta, 0) = \Sigma^*$ .
3.  $H\Sigma_0^*(\alpha, \beta, 1) = \Sigma^c$ .

Also, if we consider  $m = 0$  in Definition 1.2, we obtain another subclass of  $\Sigma$  consisting of functions  $f$  for which the function  $F$  given by

$$\frac{1}{F(z)} = \frac{1-\gamma}{f(z)} - \frac{\gamma}{zf'(z)}$$

is in the class  $\Sigma^*$ . We denote this class of functions by  $H\Sigma^*(\gamma)$ .

In this paper we find the relationship between the classes  $H\Sigma_m^*(\alpha, \beta, \gamma)$  and  $\Sigma_m^*(\alpha, \beta)$ . Sharp upper bounds for the Fekete-Szegő like functional  $|a_1 - \mu a_0^2|$  are also obtained.

## 2 Relationship property

In order to prove the relationship between the classes  $H\Sigma_m^*(\alpha, \beta, \gamma)$  and  $\Sigma_m^*(\alpha, \beta)$  we need the following lemma.

**Lemma 2.1** ([7]) *Let  $p(z)$  be an analytic function in the open unit disk  $\mathbb{U} = \{z \in \mathbb{C} : |z| < 1\}$  with  $p(0) = 1$  and  $p(z) \neq 1$ . If  $0 < |z_0| < 1$  and*

$$\Re p(z_0) = \min_{|z| \leq |z_0|} \Re p(z)$$

then

$$z_0 p'(z_0) \leq -\frac{|1 - p(z_0)|^2}{2[1 - \Re p(z_0)]}.$$

**Theorem 2.1** *Let  $\gamma$  be a complex number such that  $\left| \gamma - \frac{1}{2} \right| \leq \frac{1}{2}$ . Then*

$$H\Sigma_m^*(\alpha, \beta, \gamma) \subset \Sigma_m^*(\alpha, \beta).$$

**Proof.** Assume that  $f$  belongs to the class  $H\Sigma_m^*(\alpha, \beta, \gamma)$ . Elementary calculations show that if  $f \in H\Sigma_m^*(\alpha, \beta, \gamma)$ , then

$$\Re \left[ 1 + \frac{z(D_{\alpha, \beta}^m f(z))'}{D_{\alpha, \beta}^m f(z)} + \frac{z(D_{\alpha, \beta}^m f(z))''}{(D_{\alpha, \beta}^m f(z))'} - \frac{(1-2\gamma)z(D_{\alpha, \beta}^m f(z))' + (1-\gamma)z^2(D_{\alpha, \beta}^m f(z))''}{(1-\gamma)(D_{\alpha, \beta}^m f(z))' - \gamma D_{\alpha, \beta}^m f(z)} \right] < 0, \quad z \in \mathbb{U}^*. \quad (9)$$

Consider the analytic function  $p(z) \in \mathbb{U}$ , given by

$$p(z) = -\frac{z(D_{\alpha, \beta}^m f(z))'}{D_{\alpha, \beta}^m f(z)}. \quad (10)$$

Then, the inequality (9) becomes

$$\Re \left[ p(z) - \frac{zp'(z)}{p(z)} + \frac{(1-\gamma)zp'(z)}{(1-\gamma)p(z) + \gamma} \right] > 0, \quad z \in \mathbb{U}. \quad (11)$$

Suppose that there exists a point  $z_0$  ( $0 < |z_0| < 1$ ) such that

$$\Re p(z) > 0 \quad (|z| < |z_0|) \text{ and } p(z_0) = i\rho, \quad (12)$$

where  $\rho$  is real and  $\rho \neq 0$ . Then, making use of Lemma 2.1, we get

$$z_0 p'(z_0) \leq -\frac{1 + \rho^2}{2}. \quad (13)$$

By virtue of (11), (12) and (13) it follows that

$$\begin{aligned} R_0 &:= \Re \left[ p(z_0) - \frac{z_0 p'(z_0)}{p(z_0)} + \frac{(1 - \gamma) z_0 p'(z_0)}{(1 - \gamma) p(z_0) + \gamma} \right] \\ &= \Re \left[ i\rho - \frac{z_0 p'(z_0)}{i\rho} + \frac{(1 - \gamma) z_0 p'(z_0)}{(1 - \gamma) i\rho + \gamma} \right]. \end{aligned}$$

Hence

$$R_0 = \frac{z_0 p'(z_0)}{|(1 - \gamma) i\rho + \gamma|^2} \Re[\bar{\gamma} - |\gamma|^2]. \quad (14)$$

Since  $\left| \gamma - \frac{1}{2} \right| \leq \frac{1}{2}$  it follows that  $\Re[\bar{\gamma} - |\gamma|^2] \geq 0$ . From (13) and (14) we get

$$R_0 \leq -\frac{1 + \rho^2}{2|(1 - \gamma) i\rho + \gamma|^2} \Re[\bar{\gamma} - |\gamma|^2] \leq 0,$$

which contradicts the assumption  $f \in H\Sigma_m^*(\alpha, \beta, \gamma)$ . Therefore, we must have

$$\Re p(z) = -\Re \frac{z(D_{\alpha, \beta}^m f(z))'}{D_{\alpha, \beta}^m f(z)} > 0$$

or

$$\Re \frac{z(D_{\alpha, \beta}^m f(z))'}{D_{\alpha, \beta}^m f(z)} < 0, \quad z \in \mathbb{U}^*$$

which shows that  $f \in \Sigma_m^*(\alpha, \beta)$ . Thus, the proof of our theorem is completed.

If we consider  $m = 0$  in Theorem 2.1, we obtain the following result.

**Corollary 2.1** *Let  $\gamma$  be a complex number such that  $\left| \gamma - \frac{1}{2} \right| \leq \frac{1}{2}$ . Then*

$$H\Sigma^*(\gamma) \subset \Sigma^*.$$

### 3 Fekete-Szegö like functional

Let  $S$  denotes the class of functions of the form

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n$$

which are analytic and univalent in  $\mathbb{U}$ .

In 1933, M. Fekete and G. Szegö [3] obtained sharp upper bounds for  $|a_3 - \mu a_2^2|$  for  $f \in S$  and  $\mu$  real. For different subclasses of  $S$ , Fekete-Szegö problem has been investigated by many authors including [2], [8], [9], [11], [15].

Recently, H.Silverman et al. [14] has obtained sharp upper bounds for Fekete-Szegö like functional  $|a_1 - \mu a_0^2|$  for certain subclasses of  $\Sigma$ . In this section we will find sharp upper bounds for  $|a_1 - \mu a_0^2|$  for the class  $H\Sigma_m^*(\alpha, \beta, \gamma)$ .

The following lemma will be used in order to obtain our result.

**Lemma 3.1** ([4]) *If  $p_1(z) = 1 + c_1z + c_2z^2 + \dots$  is a function with positive real part in  $\mathbb{U}$ , then*

$$|c_2 - vc_1^2| \leq 2 \max \{1; |2v - 1|\}. \tag{15}$$

The result is sharp for the functions  $p_1(z) = \frac{1+z}{1-z}$ ,  $p_1(z) = \frac{1+z^2}{1-z^2}$ .

**Theorem 3.1** *Let  $f(z)$  given by (1) be in the class  $H\Sigma_m^*(\alpha, \beta, \gamma)$ . Then, for any complex number  $\mu$*

$$|a_1 - \mu a_0^2| \leq \frac{1}{|1 - 2\gamma|(1 + 6\alpha\beta + 2\alpha - 2\beta)^m} \max \{1; \Lambda(\alpha, \beta, \gamma, \mu, m)\}, \tag{16}$$

if  $\gamma \notin \left\{ \frac{1}{2}, 1 \right\}$

$$|a_1 - \mu a_0^2| \leq \frac{1}{(1 + 6\alpha\beta + 2\alpha - 2\beta)^m}, \text{ if } \gamma = 1 \tag{17}$$

and

$$|a_1 - \mu a_0^2| \leq \frac{2\sqrt{6}|\mu|}{3(1 + 2\alpha\beta + \alpha - \beta)^m}, \text{ if } \gamma = \frac{1}{2} \tag{18}$$

where

$$\Lambda(\alpha, \beta, \gamma, \mu, m) = \frac{|(3\gamma^2 - 2\gamma - 1)(1 + 2\alpha\beta + \alpha - \beta)^{2m} + 4(1 - 2\gamma)(1 + 6\alpha\beta + 2\alpha - 2\beta)^m \mu|}{|1 - \gamma|^2(1 + 2\alpha\beta + \alpha - \beta)^{2m}}.$$

The bounds are sharp.

**Proof.** Suppose  $f(z)$  given by (1) belongs to the class  $H\Sigma_m^*(\alpha, \beta, \gamma)$ . Let  $p_1(z) = 1 + c_1z + c_2z^2 + \dots$  be an analytic function with positive real part in  $\mathbb{U}$ . From (9) we get

$$1 + \frac{z(D_{\alpha,\beta}^m f(z))'}{D_{\alpha,\beta}^m f(z)} + \frac{z(D_{\alpha,\beta}^m f(z))''}{(D_{\alpha,\beta}^m f(z))'} - \frac{(1 - 2\gamma)z(D_{\alpha,\beta}^m f(z))' + (1 - \gamma)z^2(D_{\alpha,\beta}^m f(z))''}{(1 - \gamma)(D_{\alpha,\beta}^m f(z))' - \gamma D_{\alpha,\beta}^m f(z)} = 1 + c_1z + c_2z^2 \dots \tag{19}$$



We have

$$\begin{aligned} \frac{z(D_{\alpha,\beta}^m f(z))'}{D_{\alpha,\beta}^m f(z)} &= -1 + (1 + 2\alpha\beta + \alpha - \beta)^m a_0 z \\ &+ [2(1 + 6\alpha\beta + 2\alpha - 2\beta)^m a_1 - (1 + 2\alpha\beta + \alpha - \beta)^{2m} a_0^2] z^2 \end{aligned} \quad (20)$$

$$+ \dots \frac{z(D_{\alpha,\beta}^m f(z))''}{(D_{\alpha,\beta}^m f(z))'} = -2 - 2(1 + 6\alpha\beta + 2\alpha - 2\beta)^m a_1 z^2 + \dots \quad (21)$$

and

$$\begin{aligned} \frac{(1 - 2\gamma)z(D_{\alpha,\beta}^m f(z))' + (1 - \gamma)z^2(D_{\alpha,\beta}^m f(z))''}{(1 - \gamma)(D_{\alpha,\beta}^m f(z))' - \gamma D_{\alpha,\beta}^m f(z)} &= -1 + \gamma(1 + 2\alpha\beta + \alpha - \beta)^m a_0 z \\ &- [2(1 - 2\gamma)(1 + 6\alpha\beta + 2\alpha - 2\beta)^m a_1 + \gamma^2(1 + 2\alpha\beta + \alpha - \beta)^{2m} a_0^2] z^2 + \dots \end{aligned} \quad (22)$$

Using (20), (21) and (22) in (19) we find

$$c_1 = -(1 - \gamma)(1 + 2\alpha\beta + \alpha - \beta)^m a_0$$

and

$$c_2 = -2(1 - 2\gamma)(1 + 6\alpha\beta + 2\alpha - 2\beta)^m a_1 - (1 - \gamma^2)(1 + 2\alpha\beta + \alpha - \beta)^{2m} a_0^2$$

which give

$$a_0 = -\frac{c_1}{(1 - \gamma)(1 + 2\alpha\beta + \alpha - \beta)^m}, \quad \text{if } \gamma \neq 1 \quad (23)$$

and

$$a_1 = \frac{-1}{2(1 - 2\gamma)(1 + 6\alpha\beta + 2\alpha - 2\beta)^m} \left[ c_2 - \frac{1 + \gamma}{1 - \gamma} c_1^2 \right] \quad (24)$$

if  $\gamma \notin \left\{ \frac{1}{2}, 1 \right\}$ .

Therefore, we have

$$a_1 - \mu a_0^2 = \frac{-1}{2(1 - 2\gamma)(1 + 6\alpha\beta + 2\alpha - 2\beta)^m} (c_2 - \nu c_1^2)$$

where

$$\nu = \frac{(1 - \gamma^2)(1 + 2\alpha\beta + \alpha - \beta)^{2m} - 2(1 - 2\gamma)(1 + 6\alpha\beta + 2\alpha - 2\beta)^m \mu}{(1 - \gamma)^2(1 + 2\alpha\beta + \alpha - \beta)^{2m}}.$$

Now, the result (16) follows by an application of Lemma 3.1.

If  $\gamma = 1$ , then  $a_0 = 0$  and  $a_1 = \frac{-c_2}{2(1 + 6\alpha\beta + 2\alpha - 2\beta)^m}$ . Since  $|c_2| \leq 2$  it follows that  $|a_1| \leq \frac{1}{(1 + 6\alpha\beta + 2\alpha - 2\beta)^m}$  which proves (17). Also, if  $\gamma = \frac{1}{2}$ , then  $a_1 = 0$  and

$$c_1 = -\frac{1}{2}(1 + 2\alpha\beta + \alpha - \beta)^m a_0$$

and

$$c_2 = \frac{3}{4}(1 + 2\alpha\beta + \alpha - \beta)^{2m} a_0^2.$$

Since  $|c_1| \leq 2$  and  $|c_2| \leq 2$  it follows that  $|a_0| \leq \frac{2\sqrt{6}}{3(1 + 2\alpha\beta + \alpha - \beta)^m}$  and thus, (18) is proved.

The bounds are sharp for the functions  $f_1(z)$  and  $f_2(z)$  defined by

$$\frac{1 - \gamma}{D_{\alpha, \beta}^m f_1(z)} - \frac{\gamma}{z(D_{\alpha, \beta}^m f_1(z))'} = \frac{1}{F_1(z)}, \text{ where } -\frac{zF_1'(z)}{F_1(z)} = \frac{1 + z}{1 - z}$$

respectively,

$$\frac{1 - \gamma}{D_{\alpha, \beta}^m f_2(z)} - \frac{\gamma}{z(D_{\alpha, \beta}^m f_2(z))'} = \frac{1}{F_2(z)}, \text{ where } -\frac{zF_2'(z)}{F_2(z)} = \frac{1 + z^2}{1 - z^2}.$$

Obviously, the functions  $F_1, F_2 \in \Sigma^*$  and  $f_1, f_2 \in H\Sigma_m^*(\alpha, \beta, \gamma)$ .

If we consider first  $m = 0$ , then  $\gamma = 0$  and then  $m = \gamma = 0$ , respectively in Theorem 3.1, we obtain the following consequences.

**Corollary 3.1** *Let  $f(z)$  given by (1) be in the class  $H\Sigma^*(\gamma)$ . Then, for any complex number  $\mu$*

$$|a_1 - \mu a_0^2| \leq \frac{1}{|1 - 2\gamma|} \max \left\{ 1; \frac{|3\gamma^2 - 2\gamma - 1 + 4(1 - 2\gamma)\mu|}{|1 - \gamma|^2} \right\},$$

if  $\gamma \notin \left\{ \frac{1}{2}, 1 \right\}$

$$|a_1 - \mu a_0^2| \leq 1, \text{ if } \gamma = 1$$

$$|a_1 - \mu a_0^2| \leq \frac{2\sqrt{6}}{3} |\mu|, \text{ if } \gamma = 0.$$

The bounds are sharp.

**Corollary 3.2** *If  $f(z)$  given by (1) belongs to the class  $\Sigma_m^*(\alpha, \beta)$ , then for any complex number  $\mu$*

$$|a_1 - \mu a_0^2| \leq \frac{1}{(1 + 6\alpha\beta + 2\alpha - 2\beta)^m} \max \{1; \Phi(\alpha, \beta, \mu, m)\}$$

where

$$\Phi(\alpha, \beta, \mu, m) = \frac{|4(1 + 6\alpha\beta + 2\alpha - 2\beta)^m \mu - (1 + 2\alpha\beta + \alpha - \beta)^{2m}|}{(1 + 2\alpha\beta + \alpha - \beta)^{2m}}.$$

The bound is sharp.

**Corollary 3.3** ([14]) *Let  $f(z)$  given by (1) be a meromorphic starlike function. Then, for any complex number  $\mu$*

$$|a_1 - \mu a_0^2| \leq \max\{1; |4\mu - 1|\}.$$

*The bound is sharp.*

Finally, if we consider  $\gamma = 1$  and  $m = 0$  in Theorem 3.1, we obtain the following result.

**Corollary 3.4** *Let  $f(z)$  given by (1) be a meromorphic convex function. Then, for any complex number  $\mu$*

$$|a_1 - \mu a_0^2| \leq 1.$$

*The bound is sharp.*

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# Application of generalized Ruscheweyh derivatives on p-valent functions

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ABSTRACT: In this paper, we define a new class of p-valent analytic functions with finitely many coefficients by making use of the generalized Ruscheweyh derivatives involving a general fractional derivative operator. Some properties of this class are also investigated. e.g. Coefficient estimates, convex combination, arithmetic mean, extreme points, radii of starlikeness and convexity. Many known results are as a special case of our results.

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Key Words and Phrases: *Starlike function, p-valent function, Convolution*

## 1 Introduction

Let  $A$  denote the class of functions that are analytic in the open unit disk  $U = \{z \in C : |z| < 1\}$  and let  $A_{n,p}$  be the subclass of  $A$  consisting of the functions  $f$  of the form

$$f(z) = z^p + \sum_{k=n+p}^{\infty} a_k z^k, \quad (n \in N) \quad (1.1)$$

where  $p$  is some positive integer and  $f$  is analytic and p-valent in  $U$ . Then function  $f \in A_{n,p}$  is said to be in class  $S_n(p, \delta)$  if and only if

$$Re \left( \frac{zf'(z)}{f(z)} \right) > \delta, \quad (z \in U) \quad (1.2)$$

for some  $\delta$  ( $0 \leq \delta < p$ ).

A function  $f \in S_n(p, \delta)$  is called  $p$ -valent starlike of order  $\delta$ . On the other hand a function  $f \in A_{n,p}$  is said to be in the class  $K_n(p, \delta)$  iff

$$\operatorname{Re} \left( 1 + \frac{zf''(z)}{f'(z)} \right) > \delta, \quad (z \in U) \quad (1.3)$$

for some  $\delta (0 \leq \delta < p)$ .

A function  $f \in K_n(p, \delta)$  is called  $p$ -valent convex function of order  $\delta$ . It is observed that

$$f \in K_n(p, \delta) \Leftrightarrow zf' \in S_n(p, \delta) \quad \forall n \in \{1, 2, \dots\} \quad (1.4)$$

The fractional derivative operator occurring in this paper is defined as (see, e.g. [10], [11]):

Let  $f$  is an analytic function in a simply connected region of the  $z$ -plane containing the origin, and the multiplicity of  $(z - \zeta)^{-\lambda}$  is removed by requiring  $\log(z - \zeta)$  to be real when  $z - \zeta > 0$ . Then the generalized fractional derivative of order  $\lambda$  is defined for a function  $f(z)$  by

$$J_{0,z}^{\lambda,\mu,\nu} f(z) = \begin{cases} \frac{1}{\Gamma(1-\lambda)} \frac{d}{dz} \left\{ z^{\lambda-\mu} \int_0^z (z-\zeta)^{-\lambda} \cdot {}_2F_1 \left( \mu - \lambda, 1 - \nu; 1 - \lambda; 1 - \frac{\zeta}{z} \right) f(\zeta) d\zeta \right\}, & (0 \leq \lambda < 1) \\ \frac{d^n}{dz^n} J_{0,z}^{\lambda-n,\mu,\nu} f(z), & (n \leq \lambda < n+1, n \in \mathbb{N}) \end{cases} \quad (1.5)$$

$(k > \max\{0, \mu - \nu - 1\} - 1)$

provided further that

$$f(z) = O(|z|^k), \quad (z \rightarrow 0) \quad (1.6)$$

It follows at once from the above definition that

$$J_{0,z}^{\lambda,\lambda,\nu} f(z) = D_z^\lambda f(z), \quad (0 \leq \lambda < 1) \quad (1.7)$$

Furthermore, in terms of gamma function, we have

$$J_{0,z}^{\lambda,\mu,\nu} z^\rho = \frac{\Gamma(\rho+1)\Gamma(\rho-\mu+\nu+2)}{\Gamma(\rho-\mu+1)\Gamma(\rho-\lambda+\nu+2)} z^{\rho-\mu}, \quad (1.8)$$

$(0 \leq \lambda < 1, \rho > \max\{0, \mu - \nu - 1\} - 1)$

In a recent paper, Goyal and Goyal [2] defined a generalized Ruscheweyh derivatives  $\mathbb{J}_p^{\lambda,\mu} f_{n,p}$ ,  $\mu > -1$  as

$$\begin{aligned} \mathbb{J}_p^{\lambda,\mu} f_{n,p}(z) &= \frac{\Gamma(\mu - \lambda + \nu + 2)}{\Gamma(\nu + 2)\Gamma(\mu + 1)} z^p J_{0,z}^{\lambda,\mu,\nu} (z^{\mu-p} f_{n,p}(z)) \\ &= z^p + \sum_{k=n+p}^{\infty} a_k B_p^{\lambda,\mu}(k) z^k \end{aligned} \quad (1.9)$$

where

$$B_p^{\lambda, \mu}(k) = \frac{\Gamma(k - p + 1 + \mu)\Gamma(\nu + 2 + \mu - \lambda)\Gamma(k + \nu - p + 2)}{\Gamma(k - p + 1)\Gamma(k + \nu - p + 2 + \mu - \lambda)\Gamma(\nu + 2)\Gamma(1 + \mu)} \quad (1.10)$$

For  $\lambda = \mu$ , this generalized Ruscheweyh derivatives get reduced to Ruscheweyh derivatives of  $f(z)$  of order  $\lambda$  (see, e.g. [7]):

$$\begin{aligned} D^\lambda f_{n,p}(z) &= \frac{z^p}{\Gamma(\lambda + 1)} \frac{d^\lambda}{dz^\lambda} (z^{\lambda-p} f_{n,p}(z)) \\ &= z^p + \sum_{k=n+p}^{\infty} a_k B_k(\lambda) z^k \end{aligned} \quad (1.11)$$

where

$$B_k(\lambda) = \frac{\Gamma(\lambda + k)}{\Gamma(\lambda + p)\Gamma(k - p + 1)} \quad (1.12)$$

For  $p=1$ , (1.12) reduces to ordinary Ruscheweyh derivatives for univalent functions [9].

Let  $T_{n,p}(\theta)$  be the subclass of  $A_{n,p}$  consisting of functions  $f$  of the form

$$f(z) = z^p - \sum_{k=n+p}^{\infty} e^{i(k-p)\theta} a_k z^k, \quad (a_k \geq 0, n \in N, -\pi \leq \theta \leq \pi) \quad (1.13)$$

For  $\alpha \geq 0, 0 \leq \beta < 1$  and  $\mu > -1$ , we define  $\mathcal{W}_{\lambda, \mu}^{n,p}(\alpha, \beta)$  subclass of  $\mathcal{T}_{n,p}(\theta)$  consisting of functions  $f$  of the form (1.13) satisfying

$$Re \left\{ \frac{p \mathbb{J}_p^{\lambda, \mu} f(z)}{z (\mathbb{J}_p^{\lambda, \mu} f(z))'} \right\} > \alpha \left| \frac{p \mathbb{J}_p^{\lambda, \mu} f(z)}{z (\mathbb{J}_p^{\lambda, \mu} f(z))'} - 1 \right| + \beta \quad (1.14)$$

Similarly, we define  $\mathcal{W}_{\lambda, \mu}^{1,p}(\alpha, \beta)$  subclass of  $\mathcal{T}_{1,p}(\theta)$  consisting of functions  $f$  of the form (1.13) satisfying (1.14).

Next we introduce the class  $\mathcal{W}_{\lambda, \mu}^{C_{n,p}}(\alpha, \beta) \subset \mathcal{W}_{\lambda, \mu}^{1,p}(\alpha, \beta)$  consisting of the functions of the form

$$f_{n,p}(z) = z^p - \sum_{m=p+1}^{n+p-1} \frac{p(1 - \beta)e^{i(m-p)\theta} C_m z^m}{[p(1 + \alpha) - m(\alpha + \beta)] B_p^{\lambda, \mu}(m)} - \sum_{k=n+p}^{\infty} e^{i(k-p)\theta} a_k z^k, \quad (1.15)$$

( $a_k \geq 0, n \in N, -\pi \leq \theta \leq \pi$ )

For  $\mu = \lambda$ , the class  $\mathcal{W}_{\lambda, \mu}^{n,p}(\alpha, \beta)$  get reduced to the class  $\mathcal{W}_\lambda^{n,p}(\alpha, \beta)$  consisting of the functions  $f$  of the form (1.13) so that

$$Re \left\{ \frac{p \mathbb{D}_p^\lambda f(z)}{z (\mathbb{D}_p^\lambda f(z))'} \right\} > \alpha \left| \frac{p \mathbb{D}_p^\lambda f(z)}{z (\mathbb{D}_p^\lambda f(z))'} - 1 \right| + \beta \quad (1.16)$$

For  $p = 1$ , the above class get reduced to the class  $\mathcal{W}_\lambda^n(\alpha, \beta)$  defined by Najafzadeh et al. [4].

## 2 Main Results

We shall need the following Lemmas in the sequel to prove our theorems:

**Lemma 2.1** Let  $f_{1,p}(z) = z^p - \sum_{k=n+1}^{\infty} e^{i(k-p)\theta} a_k z^k \in T_{1,p}(\theta)$ .  
Then  $f_{1,p} \in \mathcal{W}_{\lambda,\mu}^{1,p}(\alpha, \beta)$  iff

$$\sum_{k=p+1}^{\infty} \frac{p(1+\alpha) - k(\alpha+\beta)}{p(1-\beta)} B_p^{\lambda,\mu}(k) a_k < 1 \quad (2.1)$$

Proof. Firstly suppose that  $f_{1,p} \in \mathcal{W}_{\lambda,\mu}^{1,p}(\alpha, \beta)$ . Using the fact that  $Re(u) > \alpha|u-1|+\beta$  if and only if  $Re(u(1+\alpha e^{i\gamma}) - \alpha e^{i\gamma}) > \beta$  for real  $\gamma$ . Let

$$u = \frac{p\mathbb{J}_p^{\lambda,\mu} f_{1,p}(z)}{z(\mathbb{J}_p^{\lambda,\mu} f_{1,p}(z))'}$$

we have

$$R \left\{ \frac{(1-\beta) - \alpha e^{i\gamma} \sum_{k=p+1}^{\infty} \frac{p-k}{p} a_k B_p^{\lambda,\mu}(k) z^{k-p} - \sum_{k=p+1}^{\infty} \frac{p-\beta k}{p} a_k B_p^{\lambda,\mu}(k) z^{k-p}}{1 - \sum_{k=p+1}^{\infty} \frac{k}{p} a_k B_p^{\lambda,\mu}(k) z^{k-p}} \right\} > 0 \quad (2.2)$$

The above inequality must hold true for all  $z \in U$ . Let  $z \rightarrow 1^-$ , we easily get

$$\begin{aligned} R \left\{ \frac{(1-\beta) - \alpha e^{i\gamma} \sum_{k=p+1}^{\infty} \frac{p-k}{p} a_k B_p^{\lambda,\mu}(k) - \sum_{k=p+1}^{\infty} \frac{p-\beta k}{p} a_k B_p^{\lambda,\mu}(k)}{1 - \sum_{k=p+1}^{\infty} \frac{k}{p} a_k B_p^{\lambda,\mu}(k)} \right\} &> 0 \\ \Rightarrow Re\{p(1-\beta) - \alpha e^{i\gamma} \sum_{k=p+1}^{\infty} (p-k) a_k B_p^{\lambda,\mu}(k) - \sum_{k=p+1}^{\infty} (p-\beta k) a_k B_p^{\lambda,\mu}(k)\} &> 0 \\ \Rightarrow \sum_{k=p+1}^{\infty} \frac{p(1+\alpha) - k(\alpha+\beta)}{p(1-\beta)} B_p^{\lambda,\mu}(k) a_k &< 1 \end{aligned} \quad (2.3)$$

This proves the result.

**Theorem 2.1** Let  $f_{n,p}$  be defined by (1.15), then  $f \in \mathcal{W}_{\lambda,\mu}^{C_{n,p}}(\alpha, \beta)$  iff

$$\sum_{k=p+n}^{\infty} \frac{p(1+\alpha) - k(\alpha+\beta)}{p(1-\beta)} B_p^{\lambda,\mu}(k) a_k < 1 - \sum_{m=p+1}^{n+p-1} C_m \quad (2.4)$$

*Proof.* Let

$$a_m = \frac{p(1-\beta)C_m}{[p(1+\alpha) - m(\alpha+\beta)]B_p^{\lambda,\mu}(m)}, \quad p+1 \leq m \leq n+p-1 \quad (2.5)$$



Since  $\mathcal{W}_{\lambda,\mu}^{C_{n,p}}(\alpha, \beta) \subset \mathcal{W}_{\lambda,\mu}^{1,p}(\alpha, \beta)$ , therefore  $f \in \mathcal{W}_{\lambda,\mu}^{C_{n,p}}(\alpha, \beta)$  if and only if

$$\begin{aligned} & \sum_{m=p+1}^{n+p-1} \frac{p(1+\alpha) - m(\alpha+\beta)}{p(1-\beta)} B_p^{\lambda,\mu}(m) a_m \\ & + \sum_{k=p+n}^{\infty} \frac{p(1+\alpha) - k(\alpha+\beta)}{p(1-\beta)} B_p^{\lambda,\mu}(k) a_k < 1 \\ \Rightarrow & \sum_{k=p+n}^{\infty} \frac{p(1+\alpha) - k(\alpha+\beta)}{p(1-\beta)} B_p^{\lambda,\mu}(k) a_k < 1 - \sum_{m=p+1}^{n+p-1} C_m \end{aligned} \quad (2.6)$$

which proves the required result.

**Corollary 2.1** *If  $f_{n,p}(z) \in \mathcal{W}_{\lambda,\mu}^{C_{n,p}}(\alpha, \beta)$  then for  $k \geq n+p$ , we have*

$$a_k \leq \frac{p(1-\beta) \left(1 - \sum_{m=p+1}^{n+p-1} C_m\right)}{[p(1+\alpha) - k(\alpha+\beta)] B_p^{\lambda,\mu}(k)} \quad (2.7)$$

and this result is sharp for  $g_k(z)$ ,  $k \geq n+p$  defined by

$$\begin{aligned} g_k(z) = z^p - & \sum_{m=p+1}^{n+p-1} \frac{p(1-\beta) C_m e^{i(m-p)\theta}}{[p(1+\alpha) - m(\alpha+\beta)] B_p^{\lambda,\mu}(m)} z^m \\ & - \frac{p(1-\beta) \left(1 - \sum_{m=p+1}^{n+p-1} C_m\right)}{[p(1+\alpha) - k(\alpha+\beta)] B_p^{\lambda,\mu}(k)} z^k, \end{aligned} \quad (2.8)$$

**Theorem 2.2** *Let*

$$f_j(z) = z^p - \sum_{m=p+1}^{n+p-1} \frac{p(1-\beta) C_m e^{i(m-p)\theta}}{[p(1+\alpha) - m(\alpha+\beta)] B_p^{\lambda,\mu}(m)} z^m - \sum_{k=p+n}^{\infty} e^{i(k-p)\theta} a_{k,j} z^k \quad (2.9)$$

for  $j = 1, 2, \dots, l$  be in  $\mathcal{W}_{\lambda,\mu}^{C_{n,p}}(\alpha, \beta)$ . Then the function  $F(z) = \sum_{j=1}^l \eta_j f_j(z)$  is also in  $\mathcal{W}_{\lambda,\mu}^{C_{n,p}}(\alpha, \beta)$ , where  $\sum_{j=1}^l \eta_j = 1$ ,  $0 \leq \sum_{m=p+1}^{n+p-1} C_m \leq 1$  and  $0 \leq C_m \leq 1$ .

Proof. By Theorem 2.1, for every  $j = 1, 2, \dots, l$ , we have

$$\sum_{k=p+n}^{\infty} \frac{p(1+\alpha) - k(\alpha+\beta)}{p(1-\beta)} B_p^{\lambda,\mu}(k) a_{k,j} < 1 - \sum_{m=p+1}^{n+p-1} C_m \quad (2.10)$$

but

$$\begin{aligned} f(z) &= \sum_{j=1}^l \eta_j f_j z \\ &= z^p - \sum_{m=p+1}^{n+p-1} \frac{p(1-\beta) e^{i(m-p)\theta} C_m z^m}{[p(1+\alpha) - m(\alpha+\beta)] B_p^{\lambda,\mu}(m)} - \sum_{k=n+p}^{\infty} e^{i(k-p)\theta} \left( \sum_{j=1}^l \eta_j a_{k,j} \right) z^k, \end{aligned} \quad (2.11)$$

Therefore

$$\begin{aligned}
& \sum_{k=p+n}^{\infty} \frac{p(1+\alpha) - k(\alpha+\beta)}{p(1-\beta)} B_p^{\lambda,\mu}(k) \left( \sum_{j=1}^l \eta_j a_{k,j} \right) \\
&= \sum_{j=1}^l \sum_{k=p+n}^{\infty} \frac{p(1+\alpha) - k(\alpha+\beta)}{p(1-\beta)} B_p^{\lambda,\mu}(k) a_{k,j} \eta_j \\
&< \sum_{j=1}^l \left( 1 - \sum_{m=p+1}^{n+p-1} C_m \right) \eta_j \\
&= 1 - \sum_{m=p+1}^{n+p-1} C_m
\end{aligned} \tag{2.12}$$

**Remark 2.1** If  $f_1(z)$  and  $f_2(z)$  be in  $\mathcal{W}_{\lambda,\mu}^{C_n,p}(\alpha, \beta)$ , then the function  $F(z) = \frac{1}{2}(f_1(z) + f_2(z))$  is also in  $\mathcal{W}_{\lambda,\mu}^{C_n,p}(\alpha, \beta)$ .

**Remark 2.2** The class  $\mathcal{W}_{\lambda,\mu}^{C_n,p}(\alpha, \beta)$  is a convex set.

In the next two theorems, we shall prove the arithmetic mean property and find extreme points respectively for the class  $\mathcal{W}_{\lambda,\mu}^{C_n,p}(\alpha, \beta)$ .

**Theorem 2.3** Let  $f_j(z)$ ,  $j = (1, 2, \dots, l)$  defined by (2.9) be in  $\mathcal{W}_{\lambda,\mu}^{C_n,p}(\alpha, \beta)$  then the function

$$H(z) = z^p - \sum_{m=p+1}^{n+p-1} \frac{p(1-\beta)e^{i(m-p)\theta} C_m z^m}{[p(1+\alpha) - m(\alpha+\beta)] B_p^{\lambda,\mu}(m)} - \sum_{k=n+p}^{\infty} e^{i(k-p)\theta} b_k z^k, \tag{2.13}$$

is also in  $\mathcal{W}_{\lambda,\mu}^{C_n,p}(\alpha, \beta)$ ,

where  $b_k = \frac{1}{l} \sum_{j=1}^l a_{k,j}$  ( $b_k \geq 0$ )

Proof. We have

$$\begin{aligned}
& \sum_{k=p+n}^{\infty} \frac{p(1+\alpha) - k(\alpha+\beta)}{p(1-\beta)} B_p^{\lambda,\mu}(k) b_k \\
&= \sum_{k=p+n}^{\infty} \frac{p(1+\alpha) - k(\alpha+\beta)}{p(1-\beta)} B_p^{\lambda,\mu}(k) \frac{1}{l} \sum_{j=1}^l a_{k,j} \\
&= \frac{1}{l} \sum_{j=1}^l \left[ \sum_{k=p+n}^{\infty} \frac{p(1+\alpha) - k(\alpha+\beta)}{p(1-\beta)} B_p^{\lambda,\mu}(k) a_{k,j} \right] \\
&< \frac{1}{l} \sum_{j=1}^l \left( 1 - \sum_{m=p+1}^{n+p-1} C_m \right) \\
&= 1 - \sum_{m=p+1}^{n+p-1} C_m
\end{aligned} \tag{2.14}$$

Hence proved.

**Theorem 2.4** *Let*

$$f_{n,p}(z) = z^p - \sum_{m=p+1}^{n+p-1} \frac{p(1-\beta)e^{i(m-p)\theta}C_m z^m}{[p(1+\alpha) - m(\alpha+\beta)]B_p^{\lambda,\mu}(m)}. \quad (2.15)$$

and for  $k \geq n+p$

$$f_{k,p}(z) = z^p - \sum_{m=p+1}^{n+p-1} \frac{p(1-\beta)e^{i(m-p)\theta}C_m z^m}{[p(1+\alpha) - m(\alpha+\beta)]B_p^{\lambda,\mu}(m)} - \frac{p(1-\beta)(1 - \sum_{m=p+1}^{n+p-1} C_m)e^{i(k-p)\theta}}{[p(1+\alpha) - k(\alpha+\beta)]B_p^{\lambda,\mu}(k)} z^k. \quad (2.16)$$

Then the function  $F(z)$  is in  $\mathcal{W}_{\lambda,\mu}^{C_{n,p}}(\alpha, \beta)$  iff it can be expressed in the form

$$F(z) = \sum_{k=p+n-1}^{\infty} \sigma_k f_{k,p}(z) \quad (2.17)$$

where  $\sigma_k \geq 0$  ( $k \geq n+p-1$ ) and  $\sum_{k=p+n-1}^{\infty} \sigma_k = 1$ .

Proof. Consider

$$\begin{aligned} F(z) &= \sum_{k=p+n-1}^{\infty} \sigma_k f_{k,p}(z) \\ &= \sigma_n f_{n,p}(z) + \sum_{k=p+n}^{\infty} \sigma_k f_{k,p}(z) \\ &= z^p - \sum_{m=p+1}^{n+p-1} \frac{p(1-\beta)e^{i(m-p)\theta}C_m z^m}{[p(1+\alpha) - m(\alpha+\beta)]B_p^{\lambda,\mu}(m)} \\ &\quad - \sum_{k=n+p}^{\infty} \frac{p(1-\beta)\sigma_k(1 - \sum_{m=p+1}^{n+p-1} C_m)e^{i(k-p)\theta}}{[p(1+\alpha) - k(\alpha+\beta)]B_p^{\lambda,\mu}(k)} z^k \end{aligned} \quad (2.18)$$

Finally, we have

$$\begin{aligned} &\sum_{k=n+p}^{\infty} \frac{[p(1+\alpha) - k(\alpha+\beta)](1 - \sum_{m=p+1}^{n+p-1} C_m)p(1-\beta)\sigma_k B_p^{\lambda,\mu}(k)}{p(1-\beta)[p(1+\alpha) - k(\alpha+\beta)]B_p^{\lambda,\mu}(k)} \\ &= (1 - \sum_{m=p+1}^{n+p-1} C_m) \sum_{k=n+p}^{\infty} \sigma_k = 1 - \sum_{m=p+1}^{n+p-1} C_m(1 - \sigma_{n+p-1}) < 1 - \sum_{m=p+1}^{n+p-1} C_m \end{aligned} \quad (2.19)$$

Conversely suppose that  $f_j(z) \in \mathcal{W}_{\lambda, \mu}^{C_n, p}(\alpha, \beta)$  then

$$F(z) = z^p - \sum_{m=p+1}^{n+p-1} \frac{p(1-\beta)e^{i(m-p)\theta} C_m z^m}{[p(1+\alpha) - m(\alpha+\beta)] B_p^{\lambda, \mu}(m)} - \sum_{k=n+p}^{\infty} e^{i(k-p)\theta} a_k z^k$$

By putting

$$\sigma_k = \frac{[p(1+\alpha) - k(\alpha+\beta)] B_p^{\lambda, \mu}(k) a_k}{p(1-\beta)(1 - \sum_{m=p+1}^{n+p-1} C_m)}, \quad (k \geq n+p)$$

Where  $\sigma_k \geq 0$  and if we put  $\sigma_{n+p-1} = 1 - \sum_{k=n+p}^{\infty} \sigma_k$ , we obtain

$$\begin{aligned} F(z) &= z^p - \sum_{m=p+1}^{n+p-1} \frac{p(1-\beta)e^{i(m-p)\theta} C_m z^m}{[p(1+\alpha) - m(\alpha+\beta)] B_p^{\lambda, \mu}(m)} \\ &\quad - \sum_{k=n+p}^{\infty} \frac{p(1-\beta)e^{i(k-p)\theta} (1 - \sum_{m=p+1}^{n+p-1} C_m) \sigma_k z^k}{[p(1+\alpha) - k(\alpha+\beta)] B_p^{\lambda, \mu}(k)} \\ &= f_{n,p}(z) - \sum_{k=n+p}^{\infty} [f_{n,p}(z) - f_{k,p}(z)] \sigma_k \\ &= \sum_{k=n+p-1}^{\infty} \sigma_k f_{k,p}(z). \end{aligned} \tag{2.20}$$

### 3 Radii of starlikeness and convexity

Now we obtain the radii of starlikeness and convexity for the elements of the class  $\mathcal{W}_{\lambda, \mu}^{C_n, p}(\alpha, \beta)$ .

**Theorem 3.1** *Let the function  $f_{n,p}(z)$  defined by (1.15) be in the class  $\mathcal{W}_{\lambda, \mu}^{C_n, p}(\alpha, \beta)$ , then  $f_{n,p}(z)$  is starlike of order  $\gamma$  ( $0 \leq \gamma < p$ ) in  $|z| < r$ , where  $r$  is the largest value such that*

$$\begin{aligned} &\sum_{m=p+1}^{n+p-1} \frac{C_m r^{m-p}}{[p(1+\alpha) - m(\alpha+\beta)] B_p^{\lambda, \mu}(m)} + \frac{(1 - \sum_{m=p+1}^{n+p-1} C_m) r^{k-p}}{[p(1+\alpha) - m(\alpha+\beta)] B_p^{\lambda, \mu}(k)} \\ &\leq \frac{1}{p(1-\beta)}, \quad (k \geq n+p) \end{aligned} \tag{3.1}$$

Proof. We must show that

$$\left| \frac{z(f_{n,p}(z))'}{f_{n,p}(z)} - p \right| = \left| \frac{z(f_{n,p}(z))' - p f_{n,p}(z)}{f_{n,p}(z)} \right| < p - \gamma. \tag{3.2}$$

But substituting for  $f_{n,p}(z)$  from (1.15) and using triangular inequality in left-hand side of above inequality, we have

$$\begin{aligned} & \left| \frac{z(f_{n,p}(z))'}{f_{n,p}(z)} - p \right| \\ & < \frac{\sum_{m=p+1}^{n+p-1} \frac{p(1-\beta)(m-p)C_m}{[p(1+\alpha)-m(\alpha+\beta)]B_p^{\lambda,\mu}(m)} |z|^{m-p} + \sum_{k=n+p}^{\infty} (k-p)a_k |z|^{k-p}}{1 - \sum_{m=p+1}^{n+p-1} \frac{p(1-\beta)C_m}{[p(1+\alpha)-m(\alpha+\beta)]B_p^{\lambda,\mu}(m)} |z|^{m-p} - \sum_{k=n+p}^{\infty} a_k |z|^{k-p}} \quad (3.3) \\ & < \frac{\sum_{m=p+1}^{n+p-1} \frac{p(1-\beta)(m-p)C_m}{[p(1+\alpha)-m(\alpha+\beta)]B_p^{\lambda,\mu}(m)} r^{m-p} + \sum_{k=n+p}^{\infty} (k-p)a_k r^{k-p}}{1 - \sum_{m=p+1}^{n+p-1} \frac{p(1-\beta)C_m}{[p(1+\alpha)-m(\alpha+\beta)]B_p^{\lambda,\mu}(m)} r^{m-p} - \sum_{k=n+p}^{\infty} a_k r^{k-p}} \end{aligned}$$

Then (3.2) holds true if the above term is less than  $p - \gamma$  or equivalently

$$\begin{aligned} & \sum_{m=p+1}^{n+p-1} \frac{p(1-\beta)(m-\gamma)C_m}{[p(1+\alpha)-m(\alpha+\beta)]B_p^{\lambda,\mu}(m)} r^{m-p} \\ & + \sum_{k=n+p}^{\infty} (k-p) \frac{p(1-\beta) \left(1 - \sum_{m=p+1}^{n+p-1} C_m\right)}{[p(1+\alpha)-k(\alpha+\beta)]B_p^{\lambda,\mu}(k)} (k-\gamma)r^{k-p} < p - \gamma \quad (3.4) \\ \Rightarrow & \sum_{m=p+1}^{n+p-1} \frac{C_m r^{m-p}}{[p(1+\alpha)-m(\alpha+\beta)]B_p^{\lambda,\mu}(m)} + \frac{(1 - \sum_{m=p+1}^{n+p-1} C_m) r^{k-p}}{[p(1+\alpha)-m(\alpha+\beta)]B_p^{\lambda,\mu}(k)} \\ & \leq \frac{1}{p(1-\beta)} \end{aligned}$$

This proves the result.

**Theorem 3.2** Let  $f_{n,p}(z) \in \mathcal{W}_{\lambda,\mu}^{C_n,p}(\alpha, \beta)$  and  $\lambda > 0$  if

$$d_m = \frac{p(1-\beta)C_m^2}{[p(1+\alpha)-m(\alpha+\beta)]B_p^{\lambda,\mu}(m)}, \quad (p+1 \leq m \leq n+p-1) \quad (3.5)$$

then the function

$$G(z) = z^p - \sum_{m=p+1}^{n+p-1} \frac{p(1-\beta)d_m e^{\iota(m-p)\theta} z^m}{[p(1+\alpha)-m(\alpha+\beta)]B_p^{\lambda,\mu}(m)} - \sum_{k=n+p}^{\infty} a_k e^{\iota(k-p)\theta} z^k \quad (3.6)$$

is also in  $\mathcal{W}_{\lambda,\mu}^{C_n,p}(\alpha, \beta)$ .

Proof. Since  $\lambda > 0$  so  $[p(1 + \alpha) - m(\alpha + \beta)]B_p^{\lambda, \mu}(m) > 1$ , therefore

$$\begin{aligned} d_m &= \frac{p(1 - \beta)C_m^2}{[p(1 + \alpha) - m(\alpha + \beta)]B_p^{\lambda, \mu}(m)} < C_m \leq 1 \\ 0 \leq \sum_{m=p+1}^{n+p-1} d_m &< \sum_{m=p+1}^{n+p-1} C_m \leq 1, \end{aligned} \quad (3.7)$$

thus

$$\begin{aligned} &\sum_{k=n+p}^{\infty} \frac{[p(1 + \alpha) - m(\alpha + \beta)]a_k B_p^{\lambda, \mu}(k)}{p(1 - \beta)(1 - \sum_{m=p-1}^{n+p-1} d_m)} \\ &< \frac{[p(1 + \alpha) - m(\alpha + \beta)]a_k B_p^{\lambda, \mu}(k)}{p(1 - \beta)(1 - \sum_{m=p-1}^{n+p-1} C_m)} < 1 \end{aligned} \quad (3.8)$$

This complete the proof.

**Theorem 3.3** Let  $f_{n,p}, g_{n,p} \in \mathcal{W}_{\lambda, \mu}^{C_n, p}(\alpha, \beta)$  and  $\lambda > 0$  then

$$\begin{aligned} f_{n,p} * g_{n,p}(z) &= z^p - \sum_{m=p+1}^{n+p-1} \frac{p^2(1 - \beta)^2 C_m^2 e^{\iota(m-p)\theta} z^m}{[p(1 + \alpha) - m(\alpha + \beta)]^2 B_p^{\lambda, \mu}(m)^2} \\ &\quad - \sum_{k=n+p}^{\infty} a_k b_k e^{\iota(k-p)\theta} z^k \end{aligned} \quad (3.9)$$

is also in  $\mathcal{W}_{\lambda_1, \mu}^{C_n, p}(\alpha, \beta)$  if

$$\lambda_1 < \inf_k \left[ \frac{[B_p^{\lambda, \mu}(k)]^2}{1 - \sum_{m=p+1}^{n+p-1} d_m} - 1 \right]$$

where  $d_m (p + 1 \leq m \leq n + p - 1)$  are defined by (3.5).

Proof. By using (3.5) we obtain

$$f_{n,p} * g_{n,p}(z) = z^p - \sum_{m=p+1}^{n+p-1} \frac{p(1 - \beta)d_m e^{\iota(m-p)\theta} z^m}{[p(1 + \alpha) - m(\alpha + \beta)]B_p^{\lambda, \mu}(m)} - \sum_{k=n+p}^{\infty} a_k b_k e^{\iota(k-p)\theta} z^k$$

By Theorem 3.2 and Eq. (2.4), we have

$$\sum_{k=n+p}^{\infty} \frac{[p(1 + \alpha) - m(\alpha + \beta)]a_k B_p^{\lambda, \mu}(k)}{p(1 - \beta)(1 - \sum_{m=p-1}^{n+p-1} d_m)} < 1$$

$$\sum_{k=n+p}^{\infty} \frac{[p(1+\alpha) - m(\alpha+\beta)]b_k B_p^{\lambda, \mu}(k)}{p(1-\beta)(1 - \sum_{m=p-1}^{n+p-1} d_m)} < 1$$

by using Cauchy Schwarz inequality, we have

$$\sum_{k=n+p}^{\infty} \frac{[p(1+\alpha) - k(\alpha+\beta)]\sqrt{a_k b_k} B_p^{\lambda, \mu}(k)}{p(1-\beta)(1 - \sum_{m=p-1}^{n+p-1} d_m)} < 1 \tag{3.10}$$

We must prove

$$\sum_{k=n+p}^{\infty} \frac{[p(1+\alpha) - m(\alpha+\beta)]b_k B_p^{\lambda_1, \mu}(k)}{p(1-\beta)(1 - \sum_{m=p-1}^{n+p-1} d_m)} < 1. \tag{3.11}$$

According to (3.10) the inequality (3.11) holds true if

$$\sqrt{a_k b_k} \frac{B_p^{\lambda_1, \mu}(k)}{B_p^{\lambda, \mu}(k)} < 1. \tag{3.12}$$

But we have

$$\begin{aligned} \sqrt{a_k b_k} &< \frac{1 - \sum_{m=p+1}^{n+p-1} d_m}{B_p^{\lambda, \mu}(k)} \\ \Rightarrow \frac{1 - \sum_{m=p+1}^{n+p-1} d_m}{B_p^{\lambda, \mu}(k)} &< \frac{B_p^{\lambda, \mu}(k)}{B_p^{\lambda_1, \mu}(k)} \end{aligned}$$

Or equivalently

$$\frac{(\mu+1)(\nu+2)}{(\mu+\nu-\lambda_1+2)} < B_p^{\lambda_1, \mu}(k) < \frac{[B_p^{\lambda, \mu}(k)]^2}{1 - \sum_{m=p+1}^{n+p-1} d_m}.$$

Therefore

$$\lambda_1 < \inf_k \left[ \frac{[B_p^{\lambda, \mu}(k)]^2}{1 - \sum_{m=p+1}^{n+p-1} d_m} \left( \frac{\mu - \lambda_1}{\nu + 2} + 1 \right) - 1 \right]$$

and this gives the result.

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# Inclusion relationship and Fekete-Szegő like inequalities for a subclass of meromorphic functions

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ABSTRACT: In this paper using a differential operator, we define a new subclass of meromorphic functions. Sharp upper bounds for the functional  $|a_1 - \mu a_0^2|$  in this class are obtained. An inclusion property is also given.

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## 1 Introduction

Denote by  $\Sigma$  the class of functions of the form

$$f(z) = \frac{1}{z} + \sum_{n=0}^{\infty} a_n z^n \quad (1)$$

which are analytic and univalent in the punctured disk  $\mathbb{U}^* = \{z \in \mathbb{C} : 0 < |z| < 1\}$ .

A function  $f \in \Sigma$  is said to be meromorphic starlike if

$$\Re \frac{z f'(z)}{f(z)} < 0, \quad z \in \mathbb{U}^*. \quad (2)$$

We denote by  $\Sigma^*$  the class of all meromorphic starlike functions.

A function  $f \in \Sigma$  is said to be meromorphic convex if

$$\Re \left( 1 + \frac{z f''(z)}{f'(z)} \right) < 0, \quad z \in \mathbb{U}^*. \quad (3)$$

The class of all meromorphic convex functions will be denoted by  $\Sigma^c$ .

Let  $f \in \Sigma$  be of the form (1) and let  $\alpha, \beta$  be real numbers with  $\alpha \geq \beta \geq 0$ . We define the analogue of the differential operator given in [13] as follows

$$D_{\alpha, \beta}^0 f(z) = f(z)$$

$$\begin{aligned} D_{\alpha, \beta}^1 f(z) &= D_{\alpha, \beta} f(z) = \\ &= \alpha \beta (z^2 f(z))'' + (\alpha - \beta) \frac{(z^2 f(z))'}{z} + (1 - \alpha + \beta) f(z) \end{aligned} \quad (4)$$

$$D_{\alpha, \beta}^m f(z) = D_{\alpha, \beta} \left( D_{\alpha, \beta}^{m-1} f(z) \right), \quad z \in \mathbb{U}^*, \quad m \in \mathbb{N} = \{1, 2, \dots\}. \quad (5)$$

If  $f \in \Sigma$  is given by (1), then from (4) and (5) we get

$$D_{\alpha, \beta}^m f(z) = \frac{1}{z} + \sum_{n=0}^{\infty} A(\alpha, \beta, n)^m a_n z^n, \quad z \in \mathbb{U}^* \quad (6)$$

where

$$A(\alpha, \beta, n) = [(n+2)\alpha\beta + \alpha - \beta](n+1) + 1. \quad (7)$$

Note that for  $\alpha = 1$  and  $\beta = 0$  we obtain the differential operator defined in [1].

Making use of the operator  $D_{\alpha, \beta}^m f(z)$  we introduce the following subclasses of meromorphic functions.

**Definition 1.1** Let  $\Sigma_m^*(\alpha, \beta)$  be the class of functions  $f \in \Sigma$  for which  $D_{\alpha, \beta}^m f(z) \in \Sigma^*$ , that is

$$\Re \frac{z(D_{\alpha, \beta}^m f(z))'}{D_{\alpha, \beta}^m f(z)} < 0, \quad z \in \mathbb{U}^*.$$

Note that  $\Sigma_0^*(\alpha, \beta) = \Sigma^*$ .

**Definition 1.2** Let  $\gamma$  be a complex number. We say that a function  $f \in \Sigma$  belongs to the class  $H\Sigma_m^*(\alpha, \beta, \gamma)$  if the function  $F$  defined by

$$\frac{1}{F(z)} = \frac{1 - \gamma}{D_{\alpha, \beta}^m f(z)} - \frac{\gamma}{z(D_{\alpha, \beta}^m f(z))'}, \quad z \in \mathbb{U}^* \quad (8)$$

is a meromorphic starlike function.

By specializing parameters  $\gamma$  and  $m$  we obtain the following subclasses:

1.  $H\Sigma_m^*(\alpha, \beta, 0) = \Sigma_m^*(\alpha, \beta)$ .
2.  $H\Sigma_0^*(\alpha, \beta, 0) = \Sigma^*$ .
3.  $H\Sigma_0^*(\alpha, \beta, 1) = \Sigma^c$ .

Also, if we consider  $m = 0$  in Definition 1.2, we obtain another subclass of  $\Sigma$  consisting of functions  $f$  for which the function  $F$  given by

$$\frac{1}{F(z)} = \frac{1-\gamma}{f(z)} - \frac{\gamma}{zf'(z)}$$

is in the class  $\Sigma^*$ . We denote this class of functions by  $H\Sigma^*(\gamma)$ .

In this paper we find the relationship between the classes  $H\Sigma_m^*(\alpha, \beta, \gamma)$  and  $\Sigma_m^*(\alpha, \beta)$ . Sharp upper bounds for the Fekete-Szegő like functional  $|a_1 - \mu a_0^2|$  are also obtained.

## 2 Relationship property

In order to prove the relationship between the classes  $H\Sigma_m^*(\alpha, \beta, \gamma)$  and  $\Sigma_m^*(\alpha, \beta)$  we need the following lemma.

**Lemma 2.1** ([7]) *Let  $p(z)$  be an analytic function in the open unit disk  $\mathbb{U} = \{z \in \mathbb{C} : |z| < 1\}$  with  $p(0) = 1$  and  $p(z) \neq 1$ . If  $0 < |z_0| < 1$  and*

$$\Re p(z_0) = \min_{|z| \leq |z_0|} \Re p(z)$$

then

$$z_0 p'(z_0) \leq -\frac{|1 - p(z_0)|^2}{2[1 - \Re p(z_0)]}.$$

**Theorem 2.1** *Let  $\gamma$  be a complex number such that  $\left| \gamma - \frac{1}{2} \right| \leq \frac{1}{2}$ . Then*

$$H\Sigma_m^*(\alpha, \beta, \gamma) \subset \Sigma_m^*(\alpha, \beta).$$

**Proof.** Assume that  $f$  belongs to the class  $H\Sigma_m^*(\alpha, \beta, \gamma)$ . Elementary calculations show that if  $f \in H\Sigma_m^*(\alpha, \beta, \gamma)$ , then

$$\Re \left[ 1 + \frac{z(D_{\alpha, \beta}^m f(z))'}{D_{\alpha, \beta}^m f(z)} + \frac{z(D_{\alpha, \beta}^m f(z))''}{(D_{\alpha, \beta}^m f(z))'} - \frac{(1-2\gamma)z(D_{\alpha, \beta}^m f(z))' + (1-\gamma)z^2(D_{\alpha, \beta}^m f(z))''}{(1-\gamma)(D_{\alpha, \beta}^m f(z))' - \gamma D_{\alpha, \beta}^m f(z)} \right] < 0, \quad z \in \mathbb{U}^*. \quad (9)$$

Consider the analytic function  $p(z) \in \mathbb{U}$ , given by

$$p(z) = -\frac{z(D_{\alpha, \beta}^m f(z))'}{D_{\alpha, \beta}^m f(z)}. \quad (10)$$

Then, the inequality (9) becomes

$$\Re \left[ p(z) - \frac{zp'(z)}{p(z)} + \frac{(1-\gamma)zp'(z)}{(1-\gamma)p(z) + \gamma} \right] > 0, \quad z \in \mathbb{U}. \quad (11)$$

Suppose that there exists a point  $z_0$  ( $0 < |z_0| < 1$ ) such that

$$\Re p(z) > 0 \quad (|z| < |z_0|) \text{ and } p(z_0) = i\rho, \quad (12)$$

where  $\rho$  is real and  $\rho \neq 0$ . Then, making use of Lemma 2.1, we get

$$z_0 p'(z_0) \leq -\frac{1 + \rho^2}{2}. \quad (13)$$

By virtue of (11), (12) and (13) it follows that

$$\begin{aligned} R_0 &:= \Re \left[ p(z_0) - \frac{z_0 p'(z_0)}{p(z_0)} + \frac{(1 - \gamma) z_0 p'(z_0)}{(1 - \gamma) p(z_0) + \gamma} \right] \\ &= \Re \left[ i\rho - \frac{z_0 p'(z_0)}{i\rho} + \frac{(1 - \gamma) z_0 p'(z_0)}{(1 - \gamma) i\rho + \gamma} \right]. \end{aligned}$$

Hence

$$R_0 = \frac{z_0 p'(z_0)}{|(1 - \gamma) i\rho + \gamma|^2} \Re[\bar{\gamma} - |\gamma|^2]. \quad (14)$$

Since  $\left| \gamma - \frac{1}{2} \right| \leq \frac{1}{2}$  it follows that  $\Re[\bar{\gamma} - |\gamma|^2] \geq 0$ . From (13) and (14) we get

$$R_0 \leq -\frac{1 + \rho^2}{2|(1 - \gamma) i\rho + \gamma|^2} \Re[\bar{\gamma} - |\gamma|^2] \leq 0,$$

which contradicts the assumption  $f \in H\Sigma_m^*(\alpha, \beta, \gamma)$ . Therefore, we must have

$$\Re p(z) = -\Re \frac{z(D_{\alpha, \beta}^m f(z))'}{D_{\alpha, \beta}^m f(z)} > 0$$

or

$$\Re \frac{z(D_{\alpha, \beta}^m f(z))'}{D_{\alpha, \beta}^m f(z)} < 0, \quad z \in \mathbb{U}^*$$

which shows that  $f \in \Sigma_m^*(\alpha, \beta)$ . Thus, the proof of our theorem is completed.

If we consider  $m = 0$  in Theorem 2.1, we obtain the following result.

**Corollary 2.1** *Let  $\gamma$  be a complex number such that  $\left| \gamma - \frac{1}{2} \right| \leq \frac{1}{2}$ . Then*

$$H\Sigma^*(\gamma) \subset \Sigma^*.$$

### 3 Fekete-Szegö like functional

Let  $S$  denotes the class of functions of the form

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n$$

which are analytic and univalent in  $\mathbb{U}$ .

In 1933, M. Fekete and G. Szegö [3] obtained sharp upper bounds for  $|a_3 - \mu a_2^2|$  for  $f \in S$  and  $\mu$  real. For different subclasses of  $S$ , Fekete-Szegö problem has been investigated by many authors including [2], [8], [9], [11], [15].

Recently, H.Silverman et al. [14] has obtained sharp upper bounds for Fekete-Szegö like functional  $|a_1 - \mu a_0^2|$  for certain subclasses of  $\Sigma$ . In this section we will find sharp upper bounds for  $|a_1 - \mu a_0^2|$  for the class  $H\Sigma_m^*(\alpha, \beta, \gamma)$ .

The following lemma will be used in order to obtain our result.

**Lemma 3.1** ([4]) *If  $p_1(z) = 1 + c_1z + c_2z^2 + \dots$  is a function with positive real part in  $\mathbb{U}$ , then*

$$|c_2 - vc_1^2| \leq 2 \max \{1; |2v - 1|\}. \tag{15}$$

The result is sharp for the functions  $p_1(z) = \frac{1+z}{1-z}$ ,  $p_1(z) = \frac{1+z^2}{1-z^2}$ .

**Theorem 3.1** *Let  $f(z)$  given by (1) be in the class  $H\Sigma_m^*(\alpha, \beta, \gamma)$ . Then, for any complex number  $\mu$*

$$|a_1 - \mu a_0^2| \leq \frac{1}{|1 - 2\gamma|(1 + 6\alpha\beta + 2\alpha - 2\beta)^m} \max \{1; \Lambda(\alpha, \beta, \gamma, \mu, m)\}, \tag{16}$$

if  $\gamma \notin \left\{ \frac{1}{2}, 1 \right\}$

$$|a_1 - \mu a_0^2| \leq \frac{1}{(1 + 6\alpha\beta + 2\alpha - 2\beta)^m}, \text{ if } \gamma = 1 \tag{17}$$

and

$$|a_1 - \mu a_0^2| \leq \frac{2\sqrt{6}|\mu|}{3(1 + 2\alpha\beta + \alpha - \beta)^m}, \text{ if } \gamma = \frac{1}{2} \tag{18}$$

where

$$\Lambda(\alpha, \beta, \gamma, \mu, m) = \frac{|(3\gamma^2 - 2\gamma - 1)(1 + 2\alpha\beta + \alpha - \beta)^{2m} + 4(1 - 2\gamma)(1 + 6\alpha\beta + 2\alpha - 2\beta)^m \mu|}{|1 - \gamma|^2(1 + 2\alpha\beta + \alpha - \beta)^{2m}}.$$

The bounds are sharp.

**Proof.** Suppose  $f(z)$  given by (1) belongs to the class  $H\Sigma_m^*(\alpha, \beta, \gamma)$ . Let  $p_1(z) = 1 + c_1z + c_2z^2 + \dots$  be an analytic function with positive real part in  $\mathbb{U}$ . From (9) we get

$$1 + \frac{z(D_{\alpha,\beta}^m f(z))'}{D_{\alpha,\beta}^m f(z)} + \frac{z(D_{\alpha,\beta}^m f(z))''}{(D_{\alpha,\beta}^m f(z))'} - \frac{(1 - 2\gamma)z(D_{\alpha,\beta}^m f(z))' + (1 - \gamma)z^2(D_{\alpha,\beta}^m f(z))''}{(1 - \gamma)(D_{\alpha,\beta}^m f(z))' - \gamma D_{\alpha,\beta}^m f(z)} = 1 + c_1z + c_2z^2 \dots \tag{19}$$

We have

$$\begin{aligned} \frac{z(D_{\alpha,\beta}^m f(z))'}{D_{\alpha,\beta}^m f(z)} &= -1 + (1 + 2\alpha\beta + \alpha - \beta)^m a_0 z \\ &+ [2(1 + 6\alpha\beta + 2\alpha - 2\beta)^m a_1 - (1 + 2\alpha\beta + \alpha - \beta)^{2m} a_0^2] z^2 \end{aligned} \quad (20)$$

$$+ \dots \frac{z(D_{\alpha,\beta}^m f(z))''}{(D_{\alpha,\beta}^m f(z))'} = -2 - 2(1 + 6\alpha\beta + 2\alpha - 2\beta)^m a_1 z^2 + \dots \quad (21)$$

and

$$\begin{aligned} \frac{(1 - 2\gamma)z(D_{\alpha,\beta}^m f(z))' + (1 - \gamma)z^2(D_{\alpha,\beta}^m f(z))''}{(1 - \gamma)(D_{\alpha,\beta}^m f(z))' - \gamma D_{\alpha,\beta}^m f(z)} &= -1 + \gamma(1 + 2\alpha\beta + \alpha - \beta)^m a_0 z \\ &- [2(1 - 2\gamma)(1 + 6\alpha\beta + 2\alpha - 2\beta)^m a_1 + \gamma^2(1 + 2\alpha\beta + \alpha - \beta)^{2m} a_0^2] z^2 + \dots \end{aligned} \quad (22)$$

Using (20), (21) and (22) in (19) we find

$$c_1 = -(1 - \gamma)(1 + 2\alpha\beta + \alpha - \beta)^m a_0$$

and

$$c_2 = -2(1 - 2\gamma)(1 + 6\alpha\beta + 2\alpha - 2\beta)^m a_1 - (1 - \gamma^2)(1 + 2\alpha\beta + \alpha - \beta)^{2m} a_0^2$$

which give

$$a_0 = -\frac{c_1}{(1 - \gamma)(1 + 2\alpha\beta + \alpha - \beta)^m}, \quad \text{if } \gamma \neq 1 \quad (23)$$

and

$$a_1 = \frac{-1}{2(1 - 2\gamma)(1 + 6\alpha\beta + 2\alpha - 2\beta)^m} \left[ c_2 - \frac{1 + \gamma}{1 - \gamma} c_1^2 \right] \quad (24)$$

if  $\gamma \notin \left\{ \frac{1}{2}, 1 \right\}$ .

Therefore, we have

$$a_1 - \mu a_0^2 = \frac{-1}{2(1 - 2\gamma)(1 + 6\alpha\beta + 2\alpha - 2\beta)^m} (c_2 - \nu c_1^2)$$

where

$$\nu = \frac{(1 - \gamma^2)(1 + 2\alpha\beta + \alpha - \beta)^{2m} - 2(1 - 2\gamma)(1 + 6\alpha\beta + 2\alpha - 2\beta)^m \mu}{(1 - \gamma)^2(1 + 2\alpha\beta + \alpha - \beta)^{2m}}.$$

Now, the result (16) follows by an application of Lemma 3.1.

If  $\gamma = 1$ , then  $a_0 = 0$  and  $a_1 = \frac{-c_2}{2(1 + 6\alpha\beta + 2\alpha - 2\beta)^m}$ . Since  $|c_2| \leq 2$  it follows that  $|a_1| \leq \frac{1}{(1 + 6\alpha\beta + 2\alpha - 2\beta)^m}$  which proves (17). Also, if  $\gamma = \frac{1}{2}$ , then  $a_1 = 0$  and

$$c_1 = -\frac{1}{2}(1 + 2\alpha\beta + \alpha - \beta)^m a_0$$

and

$$c_2 = \frac{3}{4}(1 + 2\alpha\beta + \alpha - \beta)^{2m} a_0^2.$$

Since  $|c_1| \leq 2$  and  $|c_2| \leq 2$  it follows that  $|a_0| \leq \frac{2\sqrt{6}}{3(1 + 2\alpha\beta + \alpha - \beta)^m}$  and thus, (18) is proved.

The bounds are sharp for the functions  $f_1(z)$  and  $f_2(z)$  defined by

$$\frac{1 - \gamma}{D_{\alpha, \beta}^m f_1(z)} - \frac{\gamma}{z(D_{\alpha, \beta}^m f_1(z))'} = \frac{1}{F_1(z)}, \text{ where } -\frac{zF_1'(z)}{F_1(z)} = \frac{1 + z}{1 - z}$$

respectively,

$$\frac{1 - \gamma}{D_{\alpha, \beta}^m f_2(z)} - \frac{\gamma}{z(D_{\alpha, \beta}^m f_2(z))'} = \frac{1}{F_2(z)}, \text{ where } -\frac{zF_2'(z)}{F_2(z)} = \frac{1 + z^2}{1 - z^2}.$$

Obviously, the functions  $F_1, F_2 \in \Sigma^*$  and  $f_1, f_2 \in H\Sigma_m^*(\alpha, \beta, \gamma)$ .

If we consider first  $m = 0$ , then  $\gamma = 0$  and then  $m = \gamma = 0$ , respectively in Theorem 3.1, we obtain the following consequences.

**Corollary 3.1** *Let  $f(z)$  given by (1) be in the class  $H\Sigma^*(\gamma)$ . Then, for any complex number  $\mu$*

$$|a_1 - \mu a_0^2| \leq \frac{1}{|1 - 2\gamma|} \max \left\{ 1; \frac{|3\gamma^2 - 2\gamma - 1 + 4(1 - 2\gamma)\mu|}{|1 - \gamma|^2} \right\},$$

if  $\gamma \notin \left\{ \frac{1}{2}, 1 \right\}$

$$|a_1 - \mu a_0^2| \leq 1, \text{ if } \gamma = 1$$

$$|a_1 - \mu a_0^2| \leq \frac{2\sqrt{6}}{3} |\mu|, \text{ if } \gamma = 0.$$

The bounds are sharp.

**Corollary 3.2** *If  $f(z)$  given by (1) belongs to the class  $\Sigma_m^*(\alpha, \beta)$ , then for any complex number  $\mu$*

$$|a_1 - \mu a_0^2| \leq \frac{1}{(1 + 6\alpha\beta + 2\alpha - 2\beta)^m} \max \{1; \Phi(\alpha, \beta, \mu, m)\}$$

where

$$\Phi(\alpha, \beta, \mu, m) = \frac{|4(1 + 6\alpha\beta + 2\alpha - 2\beta)^m \mu - (1 + 2\alpha\beta + \alpha - \beta)^{2m}|}{(1 + 2\alpha\beta + \alpha - \beta)^{2m}}.$$

The bound is sharp.

**Corollary 3.3** ([14]) *Let  $f(z)$  given by (1) be a meromorphic starlike function. Then, for any complex number  $\mu$*

$$|a_1 - \mu a_0^2| \leq \max\{1; |4\mu - 1|\}.$$

*The bound is sharp.*

Finally, if we consider  $\gamma = 1$  and  $m = 0$  in Theorem 3.1, we obtain the following result.

**Corollary 3.4** *Let  $f(z)$  given by (1) be a meromorphic convex function. Then, for any complex number  $\mu$*

$$|a_1 - \mu a_0^2| \leq 1.$$

*The bound is sharp.*

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## Spaces of entire functions represented by vector valued Dirichlet series

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ABSTRACT: Spaces of entire functions  $f$  represented by vector valued Dirichlet series and having finite order and finite type are considered. These are endowed with a certain topology under which they become a Frechet space. On this space the form of linear continuous functionals is characterized. Proper bases are also characterized in terms of growth parameters.

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### Section 1

Let

$$f(s) = \sum_{n=1}^{\infty} a_n e^{s\lambda_n}, \quad s = \sigma + it, \quad (\sigma, t \text{ are real variables}), \quad (1)$$

where  $a'_n s$  belong to a Banach space  $(E, \|\cdot\|)$  and  $\lambda'_n s \in R$  satisfy the conditions  $0 < \lambda_1 < \lambda_2 < \lambda_3 \dots < \lambda_n \dots$ ,  $\lambda_n \rightarrow \infty$  as  $n \rightarrow \infty$  and

$$\limsup_{n \rightarrow \infty} \{n/\lambda_n\} = D' < \infty, \quad (2)$$

$$\limsup_{n \rightarrow \infty} (\lambda_{n+1} - \lambda_n) = h > 0, \quad (3)$$

and

$$\limsup_{n \rightarrow \infty} \{(\log \|a_n\|)/\lambda_n\} = -\infty. \quad (4)$$

Then, the vector valued Dirichlet series in (1) represents an entire function  $f(s)$ . By giving different topologies on the set of entire functions defined by Dirichlet series,

Kamthan and Hussain [2] have studied various topological properties. In this paper we obtain these properties for a space of entire functions defined by vector valued Dirichlet series.

## Section 2

Let for entire functions  $f$  defined as above by (1)

$$M(\sigma, f) = M(\sigma) = \sup_{-\infty < t < \infty} \|f(\sigma + it)\|.$$

Then  $M(\sigma)$  is called the maximum modulus of  $f(s)$ . The order  $\rho$  of  $f(s)$  is defined as [1]

$$\rho = \lim_{\sigma \rightarrow \infty} \sup \frac{\log \log M(\sigma)}{\sigma}, \quad (0 \leq \rho \leq \infty). \quad (5)$$

Also, for  $0 < \rho < \infty$  the type  $T$  of  $f(s)$  is defined by [1]

$$T = \lim_{\sigma \rightarrow \infty} \sup \frac{\log M(\sigma)}{e^{\sigma\rho}}, \quad (0 \leq T \leq \infty) \quad (6)$$

It was proved, (see [1]) that if  $f(s)$  is of order  $\rho$  ( $0 < \rho < \infty$ ) and

$$\lim_{n \rightarrow \infty} \sup \{(\log n)/\lambda_n\} = D = 0$$

then it is of type  $T$  if and only if

$$T = \lim_{n \rightarrow \infty} \sup \frac{\lambda_n}{\rho e} \|a_n\|^{\rho/\lambda_n}. \quad (7)$$

This implies

$$\lim_{n \rightarrow \infty} \sup \lambda_n^{1/\rho} \|a_n\|^{1/\lambda_n} = (T\rho e)^{1/\rho}. \quad (8)$$

We now denote by  $X$  the set of all entire functions  $f(s)$  given by (1) and satisfying (2) to (4), for which

$$\lim_{\sigma \rightarrow \infty} \sup \frac{\log M(\sigma)}{e^{\sigma\rho}} \leq T < \infty, \quad 0 < \rho < \infty. \quad (9)$$

Then from (8), we have

$$\lim_{n \rightarrow \infty} \sup \lambda_n^{1/\rho} \|a_n\|^{1/\lambda_n} \leq (T\rho e)^{1/\rho}. \quad (10)$$

From (10), for arbitrary  $\varepsilon > 0$  and all  $n > n_0(\varepsilon)$ ,

$$\|a_n\| \left[ \frac{\lambda_n}{(T + \varepsilon)e\rho} \right]^{\lambda_n/\rho} < 1. \quad (11)$$

For a fixed positive integer  $q \geq 1$ , there exists  $0 < \varepsilon < q^{-1}$ . Hence from (11),

$$\sum_{n=1}^{\infty} \|a_n\| \left[ \frac{\lambda_n}{(T+q^{-1})e\rho} \right]^{\lambda_n/\rho} < \sum_{n=1}^{n_0} \left[ \frac{\lambda_n}{(T+q^{-1})e\rho} \right]^{\lambda_n/\rho} + \sum_{n=n_0+1}^{\infty} \left[ \frac{T+\varepsilon}{T+q^{-1}} \right]^{\lambda_n/\rho} \leq O(1).$$

Hence, if we put

$$\|f\|_q = \sum_{n \geq 1} \|a_n\| \left[ \frac{\lambda_n}{(T+q^{-1})e\rho} \right]^{\lambda_n/\rho}; \quad q \geq 1, \quad (12)$$

then  $\|f\|_q$  is well defined and for  $q_1 \leq q_2$ ,  $\|f\|_{q_1} \leq \|f\|_{q_2}$ . This norm induces a metric topology on  $X$ . We define

$$\lambda(f, g) = \sum_{q \geq 1} \frac{1}{2^q} \cdot \frac{\|f - g\|_q}{1 + \|f - g\|_q}$$

We denote the space  $X$  with the above metric  $\lambda$  by  $X_\lambda$ .

Now we prove

**Theorem 2.1.** *The space  $X_\lambda$  is a Frechet space.*

*Proof.* Here,  $X_\lambda$  is a normed linear metric space. For showing that  $X_\lambda$  is a Frechet space, we need to show that  $X_\lambda$  is complete. Hence, let  $\{f_\alpha\}$  be a  $\lambda$ -Cauchy sequence in  $X$ . Therefore, for any given  $\varepsilon > 0$  there exists an integer  $n_0 = n_0(\varepsilon)$  such that

$$\lambda(f_\alpha, f_\beta) < \varepsilon/2 \text{ for all } \alpha, \beta > n_0.$$

Hence

$$\|f_\alpha - f_\beta\|_q < \varepsilon/2 \text{ for all } \alpha, \beta > n_0, q \geq 1.$$

Denoting by  $f_\alpha(s) = \sum_{n=1}^{\infty} a_n^{(\alpha)} e^{s\lambda_n}$ ,  $f_\beta(s) = \sum_{n=1}^{\infty} a_n^{(\beta)} e^{s\lambda_n}$ , we have therefore

$$\sum_{n=1}^{\infty} \|a_n^{(\alpha)} - a_n^{(\beta)}\| \left[ \frac{\lambda_n}{(T+q^{-1})e\rho} \right]^{\lambda_n/\rho} < \varepsilon/2 \quad (13)$$

for all  $\alpha, \beta > n_0$ ,  $q \geq 1$ . Since,  $\lambda_n \rightarrow \infty$  as  $n \rightarrow \infty$ , therefore we have

$$\|a_n^{(\alpha)} - a_n^{(\beta)}\| < \varepsilon/2 \quad \forall \alpha, \beta > n_0, \text{ and } n = 1, 2, \dots,$$

i.e. for each fixed  $n = 1, 2, \dots$ ,  $\{a_n^{(\alpha)}\}$  is a Cauchy sequence in the Banach space  $E$ . Hence there exists a sequence  $\{a_n\} \subseteq E$  such that

$$\lim_{\alpha \rightarrow \infty} a_n^{(\alpha)} = a_n, \quad n \geq 1.$$

Now letting  $\beta \rightarrow \infty$  in (13), we have for  $\alpha > n_0$ ,

$$\sum_{n=1}^{\infty} \|a_n^{(\alpha)} - a_n\| \left[ \frac{\lambda_n}{(T+q^{-1})e\rho} \right]^{\lambda_n/\rho} < \varepsilon/2. \quad (14)$$

Taking  $\alpha = n_0$ , we get for a fixed  $q$  in (14)

$$\|a_n\| \left[ \frac{\lambda_n}{(T+q^{-1})e\rho} \right]^{\lambda_n/\rho} < \|a_n^{(n_0)}\| \left[ \frac{\lambda_n}{(T+q^{-1})e\rho} \right]^{\lambda_n/\rho} + \varepsilon/2$$

Now  $f^{(n_0)} = \sum_{n=1}^{\infty} a_n^{(n_0)} e^{s\lambda_n} \in X_\lambda$ , hence the condition (11) is satisfied. For arbitrary  $p > q$ , we have,  $\|a_n^{(n_0)}\| < \left[ \frac{(T+p^{-1})e\rho}{\lambda_n} \right]^{\lambda_n/\rho}$  for arbitrarily large  $n$ . Hence we have,

$$\begin{aligned} \|a_n\| \left[ \frac{\lambda_n}{(T+q^{-1})e\rho} \right]^{\lambda_n/\rho} &< \left[ \frac{(T+p^{-1})e\rho}{\lambda_n} \right]^{\lambda_n/\rho} \left[ \frac{\lambda_n}{(T+q^{-1})e\rho} \right]^{\lambda_n/\rho} + \varepsilon/2 \\ &< \left[ \frac{T+p^{-1}}{T+q^{-1}} \right]^{\lambda_n/\rho} + \varepsilon/2 \\ &< \varepsilon \end{aligned}$$

for sufficiently large values of  $n$  since  $p > q$ . We find that the sequence  $\{a_n\}$  satisfies (11) and therefore  $f(s) = \sum_{n=1}^{\infty} a_n e^{s\lambda_n}$  belongs to  $X_\lambda$ . Using (14) again, we have for  $q = 1, 2, \dots$ ,

$$\|f_\alpha - f\|_q < \varepsilon/2.$$

Hence

$$\lambda(f_\alpha, f) = \sum_{q=1}^{\infty} \frac{1}{2^q} \frac{\|f_\alpha - f\|_q}{1 + \|f_\alpha - f\|_q} \leq \frac{\varepsilon}{2 + \varepsilon} \sum_{q=1}^{\infty} \frac{1}{2^q} < \varepsilon.$$

Since above inequality holds for all  $\alpha > n_0$ , we finally get  $f_\alpha \rightarrow f$  with respect to the metric  $\lambda$ , where  $f \in X_\lambda$ . Hence  $X_\lambda$  is complete. This proves Theorem 2.1  $\square$

Now, we characterize the linear continuous functional on  $X_\lambda$ . We prove

**Theorem 2.2.** *A continuous linear functional  $\psi$  on  $X_\lambda$  is of the form*

$$\psi(f) = \sum_{n=1}^{\infty} a_n C_n$$

if and only if

$$|C_n| \leq A \left[ \frac{\lambda_n}{(T+q^{-1})e\rho} \right]^{\lambda_n/\rho} \text{ for all } n \geq 1, q \geq 1, \quad (15)$$

where  $A$  is finite, positive number,  $f = f(s) = \sum_{n=1}^{\infty} a_n e^{s\lambda_n}$  and  $\lambda_1$  is sufficiently large.

*Proof.* Let  $\psi \in X'_\lambda$ , the dual space of  $X_\lambda$ . Then for any sequence  $\{f_m\} \subseteq X_\lambda$  such that  $f_m \rightarrow f$ , we have  $\psi(f_m) \rightarrow \psi(f)$  as  $m \rightarrow \infty$ . Now, let  $f(s) = \sum_{n=1}^{\infty} a_n e^{s \cdot \lambda_n}$  where  $a'_n s \in E$  satisfy (11). Then  $f \in X_\lambda$ . Also, let  $f_k(s) = \sum_{n=1}^k a_n e^{s \cdot \lambda_n}$ . Then  $f_k \in X_\lambda$  for  $k=1,2,\dots$ . Let  $q$  be any fixed positive integer and let  $0 < \varepsilon < q^{-1}$ . From (11) we can find an integer  $m$  such that

$$\|a_n\| < \left[ \frac{(T + \varepsilon) e \rho}{\lambda_n} \right]^{\lambda_n / \rho}, \quad \forall n > m.$$

Then, for sufficiently large value of  $m$ .

$$\begin{aligned} \left\| f - \sum_{n=1}^m a_n e^{s \cdot \lambda_n} \right\|_q &= \left\| \sum_{n=m+1}^{\infty} a_n e^{s \cdot \lambda_n} \right\|_q \\ &= \sum_{n=m+1}^{\infty} \|a_n\| \left[ \frac{\lambda_n}{(T + q^{-1}) e \rho} \right]^{\lambda_n / \rho} \\ &< \sum_{n=m+1}^{\infty} \left[ \frac{(T + \varepsilon) e \rho}{\lambda_n} \right]^{\lambda_n / \rho} \left[ \frac{\lambda_n}{(T + q^{-1}) e \rho} \right]^{\lambda_n / \rho} \end{aligned}$$

or

$$\left\| f - \sum_{n=1}^m a_n e^{s \cdot \lambda_n} \right\|_q < \sum_{n=m+1}^{\infty} \left[ \frac{(T + \varepsilon)}{(T + q^{-1})} \right]^{\lambda_n / \rho} < \varepsilon.$$

Hence

$$\lambda(f, f_m) = \sum_{q=1}^{\infty} \frac{1}{2^q} \frac{\|f - f_m\|_q}{1 + \|f - f_m\|_q} \leq \frac{\varepsilon}{1 + \varepsilon} < \varepsilon,$$

i.e.  $f_m \rightarrow f$  as  $m \rightarrow \infty$  in  $X_\lambda$ . Hence by assumption that  $\psi \in X'_\lambda$ , we have

$$\lim_{m \rightarrow \infty} \psi(f_m) = \psi(f).$$

Let us denote by  $C_n = \psi(e^{s \cdot \lambda_n})$ . Then

$$\psi(f_m) = \sum_{n=1}^m a_n \psi(e^{s \cdot \lambda_n}) = \sum_{n=1}^m a_n C_n.$$

Also  $|C_n| = |\psi(e^{s \cdot \lambda_n})|$ . Since  $\psi$  is continuous on  $X_\lambda$ , it is continuous on  $X_{|\cdot|_q}$  for each  $q = 1, 2, 3, \dots$ . Hence there exists a positive constant  $A$  independent of  $q$  such that

$$|\psi(e^{s \cdot \lambda_n})| = |C_n| \leq A \|\alpha\|_q \quad ; \quad q \geq 1,$$

where  $\alpha(s) = e^{s \cdot \lambda_n}$ . Now using the definition of the form for  $\alpha(s)$ , we get

$$|C_n| \leq A \left[ \frac{\lambda_n}{(T + q^{-1}) e \rho} \right]^{\lambda_n / \rho}, \quad n \geq 1, \quad q \geq 1.$$

Hence we get  $\psi(f) = \sum_{n=1}^{\infty} a_n C_n$ , where the sequence  $\{C_n\}$  satisfies (15). Conversely, suppose that  $\psi(f) = \sum_{n=1}^{\infty} a_n C_n$  and  $C_n$  satisfies (15). Then for  $q \geq 1$ ,

$$|\psi(f)| \leq A \sum_{n=1}^{\infty} \|a_n\| \left[ \frac{\lambda_n}{(T + q^{-1}) e \rho} \right]^{\lambda_n / \rho}$$

i.e.

$$|\psi(f)| \leq A \|f\|_q, \quad q \geq 1,$$

i.e.

$$\psi \in X'_{\|\cdot\|_q}, \quad q \geq 1.$$

Now, since

$$\lambda(f, g) = \sum_{q \geq 1} \frac{1}{2^q} \frac{\|f - g\|_q}{1 + \|f - g\|_q},$$

therefore

$$X'_\lambda = \bigcup_{q=1}^{\infty} X'_{\|\cdot\|_q}.$$

Hence  $\psi \in X'_\lambda$ . This completes the proof of Theorem 2.2 □

### Section 3

Following Kamthan and Gautam [3] we give some definitions. A sequence  $\{\alpha_n\} \subseteq X$  is said to be linearly independent if  $\sum_{n=1}^{\infty} c_n \alpha_n = 0$  implies that  $c_n = 0 \forall n$ , for all sequences of complex numbers  $\{c_n\}$  for which  $\sum_{n=1}^{\infty} c_n \alpha_n$  converges in  $X$ . A subspace  $X_0$  of  $X$  is said to be spanned by a sequence  $\{\alpha_n\} \subseteq X$  if  $X_0$  consists of all linear combinations  $\sum_{n=1}^{\infty} c_n \alpha_n$  such that  $\sum_{n=1}^{\infty} c_n \alpha_n$  converges in  $X$ . A sequence  $\{\alpha_n\} \subseteq X$  which is linearly independent and spans a subspace  $X_0$  of  $X$  is said to be a base in  $X_0$ . In particular, if  $e_n \in X$ ,  $e_n(s) = e^{s\lambda_n}$ ,  $n \geq 1$ , then  $\{e_n\}$  is a base in  $X$ . A sequence  $\{\alpha_n\} \subseteq X$  will be called a 'proper base' if it is a base and it satisfies the condition:

"for all sequences  $\{a_n\} \subseteq E$ , convergence of  $\sum_{n=1}^{\infty} \|a_n\| \alpha_n$  in  $X$  implies the convergence of  $\sum_{n=1}^{\infty} a_n e_n$  in  $X$ ".

We now prove

**Theorem 3.1.** *A necessary and sufficient condition that there exists a continuous linear transformation  $F : X \rightarrow X$  with  $F(e_n) = \alpha_n$ ,  $n = 1, 2, \dots$ , where  $\alpha_n \in X$ , is that for each  $\delta > 0$ .*

$$\limsup_{n \rightarrow \infty} \frac{\|\alpha_n; T + \delta\|^{1/\lambda_n}}{\lambda_n^{1/\rho}} < \frac{1}{(\epsilon\rho T)^{1/\rho}}. \quad (16)$$

*Proof.* Let  $F$  be a continuous linear transformation from  $X$  into  $X$  with  $F(e_n) = \alpha_n$ ,  $n = 1, 2, \dots$ . Then for any given  $\delta > 0$ , there exists a  $\delta_1 > 0$  and a constant  $K' = K'(\delta)$  depending on  $\delta$  only, such that

$$\begin{aligned} \|F(e_n); T + \delta\| &\leq K' \|T + \delta_1\| \\ \Rightarrow \|\alpha_n; T + \delta\| &\leq K' \left\{ \frac{\lambda_n}{(T + \delta_1)\epsilon\rho} \right\}^{\lambda_n/\rho} \\ \Rightarrow \frac{\|\alpha_n; T + \delta\|^{1/\lambda_n}}{\lambda_n^{1/\rho}} &\leq \frac{1 + o(1)}{\{(T + \delta_1)\epsilon\rho\}^{1/\rho}}, \text{ for } n \geq N \\ \Rightarrow \limsup_{n \rightarrow \infty} \frac{\|\alpha_n; T + \delta\|^{1/\lambda_n}}{\lambda_n^{1/\rho}} &< \frac{1}{(\epsilon\rho T)^{1/\rho}}. \end{aligned}$$

Conversely, let the sequence  $\{\alpha_n\}$  satisfy (16) and let

$$\alpha(s) = \sum_{n=1}^{\infty} a_n e_n.$$

Then this implies that

$$\limsup_{n \rightarrow \infty} \lambda_n^{1/\rho} \|a_n\|^{1/\lambda_n} \leq (\rho e T)^{1/\rho}.$$

Hence, given  $\eta > 0$ , there exists  $N_0 = N_0(\eta)$ , such that

$$\lambda_n^{1/\rho} \|a_n\|^{1/\lambda_n} \leq ((T + \eta)\rho e)^{1/\rho} \quad \forall n \geq N_0.$$

Further, for a given  $\eta_1 > \eta$ , we can find  $N_1 = N_1(\eta_1)$  from (16), such that for  $n \geq N_1$

$$\frac{\|\alpha_n; T + \delta\|^{1/\lambda_n}}{\lambda_n^{1/\rho}} \leq \frac{1}{((T + \eta_1)\epsilon\rho)^{1/\rho}}.$$

Choose  $n \geq \max(N_0, N_1)$ . Then

$$\begin{aligned} \|a_n\|^{1/\lambda_n} \|\alpha_n; T + \delta\|^{1/\lambda_n} &\leq \left\{ \frac{((T + \eta)\epsilon\rho)^{1/\rho}}{\lambda_n^{1/\rho}} \right\} \left\{ \frac{\lambda_n^{1/\rho}}{((T + \eta_1)\epsilon\rho)^{1/\rho}} \right\} \\ &\leq \left\{ \frac{(T + \eta)}{(T + \eta_1)} \right\}^{\lambda_n/\rho}. \end{aligned}$$



Since  $\eta_1 > \eta$ , the series  $\sum_{n=1}^{\infty} \|a_n\| \|\alpha_n; T + \delta\|$  converges for each  $\delta > 0$ . So  $\sum_{n=1}^{\infty} a_n \alpha_n$  converges to an element of  $X$ . Define  $F(\alpha) = \sum_{n=1}^{\infty} a_n \alpha_n$  for each  $\alpha \in X$ , then  $F(e_n) = \alpha_n$ . Now we have only to prove the continuity of  $F$ . Given  $\delta > 0$ , there exists  $\delta_1 > 0$  such that

$$\frac{\|\alpha_n; T + \delta\|^{1/\lambda_n}}{\lambda_n^{1/\rho}} \leq \frac{1}{((T + \delta_1) e\rho)^{1/\rho}}, \forall n \geq N = N(\delta, \delta_1).$$

Therefore,

$$\|\alpha_n; T + \delta\| \leq K' \left\{ \frac{\lambda_n}{(T + \delta_1) e\rho} \right\}^{\lambda_n/\rho}, \text{ where } K' = K'(\delta)$$

and the inequality is true for all  $n \geq 0$ . Now,

$$\begin{aligned} \|F(\alpha); T + \delta\| &\leq \sum_{n=1}^{\infty} \|a_n\| \|\alpha_n; T + \delta\| \\ &\leq K' \sum_{n=1}^{\infty} \|a_n\| \left\{ \frac{\lambda_n}{(T + \delta_1) e\rho} \right\}^{\lambda_n/\rho} \\ &= K' \|\alpha; T + \delta_1\|. \end{aligned}$$

Hence  $F$  is continuous. This proves Theorem 3.1.  $\square$

We now give the characterization of proper bases.

**Lemma 3.1.** *In the space  $X_\lambda$ , the following three conditions are equivalent:*

(i) *For any sequence  $\{a_n\} \subseteq E$ ,  $\sum_{n=1}^{\infty} a_n e_n$  converges in  $X$  implies  $\sum_{n=1}^{\infty} \|a_n\| \alpha_n$  converges in  $X$ ,*

(ii) *For any sequence  $\{a_n\} \subseteq E$ , the convergence of  $\sum_{n=1}^{\infty} a_n e_n$  in  $X$  implies that  $\lim_{n \rightarrow \infty} \|a_n\| \alpha_n = 0$  in  $X$ ,*

(iii)  $\limsup_{n \rightarrow \infty} \frac{\|\alpha_n; T + \delta\|^{1/\lambda_n}}{\lambda_n^{1/\rho}} < \frac{1}{(T e\rho)^{1/\rho}}$ , for each  $\delta > 0$ ,  $\alpha_n \in X, \alpha_n \in X$ .

*Proof.* First suppose that (i) holds. Then for any sequence  $\{a_n\}$ , where  $a_n^s$  belongs to Banach space  $E$ ,  $\sum_{n=1}^{\infty} a_n e_n$  converges in  $X$  implies that  $\sum_{n=1}^{\infty} \|a_n\| \alpha_n$  converges in  $X$  which in turn implies that  $\|a_n\| \alpha_n \rightarrow 0$  as  $n \rightarrow \infty$ . Hence (i)  $\Rightarrow$  (ii).

Now we assume that (ii) is true but (iii) is false. This implies that for some  $\delta > 0$ , there exists a sequence  $\{n_k\}$  of positive integers such that

$$\frac{\|\alpha_{n_k}; T + \delta\|^{1/\lambda_{n_k}}}{\lambda_{n_k}^{1/\rho}} \geq \frac{1}{((T + k^{-1}) e\rho)^{1/\rho}}, \forall n_k, k = 1, 2, \dots$$

Define a sequence  $\{a_n\}$ , as

$$\|a_n\| = \begin{cases} \|\alpha_n; T + \delta\|^{-1} & ; n = n_k \\ 0 & ; n \neq n_k \end{cases}. \quad (17)$$

Then, we have for large  $k$

$$\begin{aligned} \|a_{n_k}\|^{1/\lambda_{n_k}} \lambda_{n_k}^{1/\rho} &= \frac{\lambda_{n_k}^{1/\rho}}{\|\alpha_{n_k}; T+\delta\|^{1/\lambda_{n_k}}} \\ &\leq ((T+k^{-1})e\rho)^{1/\rho}, \quad \forall k \geq k_0 \end{aligned}$$

Hence,

$$\limsup_{k \rightarrow \infty} \lambda_{n_k}^{1/\rho} \|a_{n_k}\|^{1/\lambda_{n_k}} \leq (T\rho e)^{1/\rho}.$$

Thus  $\{a_n\}$  defined by (17) satisfies the condition

$$\limsup_{n \rightarrow \infty} \lambda_n^{1/\rho} \|a_n\|^{1/\lambda_n} \leq (T\rho e)^{1/\rho}$$

which is equivalent the condition that  $\sum a_n e_n$  converges in  $X$  (see Theorem 2.1 above). Hence by (ii),  $\lim_{n \rightarrow \infty} \|a_n\| \alpha_n = 0$ .

However,

$$\| \|a_{n_k}\| \alpha_{n_k}; T + \delta \| = \|a_{n_k}\| \| \alpha_{n_k}; T + \delta \| = 1.$$

Hence  $\lim_{n \rightarrow \infty} \|a_n\| \alpha_n = 0$  in  $X$ . This is a contradiction. Hence (ii)  $\Rightarrow$  (iii). In course of the proof of Theorem 3.1 above, we have already proved that (iii)  $\Rightarrow$  (i). Thus the proof of Lemma 3.1 is complete.  $\square$

**Lemma 3.2.** *Let  $\{a_n\} \subseteq E$  and  $\{\alpha_n\} \subseteq X_\lambda$ . The following three properties are equivalent:*

- (a)  $\lim_{n \rightarrow \infty} \|a_n\| \alpha_n = 0$  in  $X$  implies that  $\sum_{n=1}^{\infty} a_n e_n$  converges in  $X$ ,
- (b) Convergence of  $\sum_{n=1}^{\infty} \|a_n\| \alpha_n$  in  $X$  implies that  $\sum_{n=1}^{\infty} a_n e_n$  converges in  $X$ ,
- (c)  $\lim_{\delta \rightarrow 0} \left\{ \liminf_{n \rightarrow \infty} \frac{\|\alpha_n; T+\delta\|^{1/\lambda_n}}{\lambda_n^{1/\rho}} \right\} \geq \frac{1}{(T\rho e)^{1/\rho}}$ .

*Proof.* Obviously (a)  $\Rightarrow$  (b). We now prove that (b)  $\Rightarrow$  (c). To prove this, we suppose that (b) holds but (c) does not hold. Hence

$$\lim_{\delta \rightarrow 0} \left\{ \liminf_{n \rightarrow \infty} \frac{\|\alpha_n; T + \delta\|^{1/\lambda_n}}{\lambda_n^{1/\rho}} \right\} < \frac{1}{(T\rho e)^{1/\rho}}.$$

Since  $\|\alpha_n; T + \delta\|$  increases as  $\delta$  decreases, this implies that for each  $\delta > 0$

$$\left\{ \liminf_{n \rightarrow \infty} \frac{\|\alpha_n; T + \delta\|^{1/\lambda_n}}{\lambda_n^{1/\rho}} \right\} < \frac{1}{(T\rho e)^{1/\rho}}.$$

Hence, if  $\eta > 0$  be a fixed small positive number, then for each  $r > 0$ , we can find a positive number  $n_r$  such that  $\forall r$ , we have  $n_{r+1} > n_r$  and

$$\frac{\|\alpha_{n_r}; T + r^{-1}\|^{1/\lambda_{n_r}}}{\lambda_{n_r}^{1/\rho}} \leq \frac{1}{((T + \eta) e\rho)^{1/\rho}}. \quad (18)$$

Now we choose a positive number  $\eta_1 < \eta$ , and define a sequence  $\{a_n\} \subseteq E$  as

$$\|a_n\| = \begin{cases} \left[ \frac{\{(T+\eta_1)e\rho\}^{1/\rho}}{\lambda_{n_r}^{1/\rho}} \right]^{\lambda_n} & ; n = n_r, r > r_0 \\ 0 & ; n \neq n_r \end{cases} .$$

Then, for any  $\delta > 0$

$$\sum_{n=1}^{\infty} \|a_n\| \|\alpha_n; T + \delta\| = \sum_{r=1}^{\infty} \|a_{n_r}\| \|\alpha_{n_r}; T + \delta\| \quad (19)$$

For any given  $\delta > 0$ , omit from the above series those finite number of terms, which correspond to those number  $n_r$  for which  $1/r$  is greater than  $\delta$ . The remainder of the series in (19) is dominated by  $\sum_{r=1}^{\infty} \|a_{n_r}\| \|\alpha_{n_r}; T + r^{-1}\|$ . Now by (18) and (19), we find that

$$\begin{aligned} & \sum_{r=1}^{\infty} \|a_{n_r}\| \|\alpha_{n_r}; T + r^{-1}\| \\ & \leq \sum_{r=1}^{\infty} \left\{ \frac{\{(T+\eta_1)e\rho\}^{1/\rho}}{\lambda_{n_r}^{1/\rho}} \right\}^{\lambda_{n_r}} \left\{ \frac{\lambda_{n_r}^{1/\rho}}{\{(T+\eta)e\rho\}^{1/\rho}} \right\}^{\lambda_{n_r}} \\ & = \sum_{r=1}^{\infty} \left( \frac{T+\eta_1}{T+\eta} \right)^{\lambda_{n_r}/\rho} . \end{aligned}$$

Since  $\eta_1 < \eta$ , so above series on R.H.S. is convergent. For this sequence  $\{a_n\}$ ,  $\sum_{n=1}^{\infty} \|a_n\| \alpha_n$  converges in  $X(\rho, T, \delta)$  for each  $\delta > 0$  and hence converges in  $X$ . But we have,

$$\begin{aligned} \limsup_{n \rightarrow \infty} \lambda_n^{1/\rho} \|a_n\|^{1/\lambda_n} &= \limsup_{r \rightarrow \infty} \left\{ \frac{\{(T+\eta_1)e\rho\}^{1/\rho}}{\lambda_{n_r}^{1/\rho}} \right\}^{\lambda_{n_r}} \\ &= [(T + \eta_1) e \rho]^{1/\rho} > (T \rho e)^{1/\rho} , \end{aligned}$$

which is a contradiction. This proves (b) $\Rightarrow$ (c).

Now we prove that (c) $\Rightarrow$ (a). We assume (c) is true but (b) is not true. Then there exists a sequences  $\{a_n\}$ , where  $a_n^s$  belongs to Banach space  $E$ , for which  $\|a_n\| \alpha_n \rightarrow 0$  in  $X$ , but  $\sum_{n=1}^{\infty} a_n e_n$  does not converge in  $X$ . This implies that

$$\limsup_{n \rightarrow \infty} \lambda_n^{1/\rho} \|a_n\|^{1/\lambda_n} > (T \rho e)^{1/\rho} .$$

Hence there exists a positive number  $\varepsilon$  and a sequence  $\{n_k\}$  of positive integers such that

$$\lambda_n^{1/\rho} \|a_n\|^{1/\lambda_n} > [(T + \varepsilon) \rho e]^{1/\rho}, \quad \forall n = n_k .$$

We choose another positive number  $\eta < \varepsilon/2$ , by assumption we can find a positive number  $\delta$  i.e.  $\delta = \delta(\eta)$  such that

$$\left\{ \liminf_{n \rightarrow \infty} \frac{\|\alpha_n; T + \delta\|^{1/\lambda_n}}{\lambda_n^{1/\rho}} \right\} \geq \frac{1}{\{(T + \eta) e \rho\}^{1/\rho}} .$$

Hence there exists  $N = N(\eta)$ , such that

$$\frac{\|\alpha_n; T + \delta\|^{1/\lambda_n}}{\lambda_n^{1/\rho}} \geq \frac{1}{((T + 2\eta)e\rho)^{1/\rho}}, \quad \forall n \geq N.$$

Therefore

$$\begin{aligned} \max_n \|\alpha_n; T + \delta\| &= \max_n \|a_n\| \|\alpha_n; T + \delta\| \\ &\geq \max_k \|a_{n_k}\| \|\alpha_{n_k}; T + \delta\| \\ &\geq \left\{ \frac{((T + \varepsilon)\rho e)^{1/\rho}}{\lambda_{n_k}^{1/\rho}} \right\}^{\lambda_{n_k}} \left\{ \frac{\lambda_{n_k}^{1/\rho}}{((T + 2\eta)e\rho)^{1/\rho}} \right\}^{\lambda_{n_k}} \\ &> 1 \text{ for } n_k > N \text{ as } \varepsilon > 2\eta. \end{aligned}$$

Thus  $\{\|a_n\| \alpha_n\}$  does not tend to zero in  $X(\rho, T, \delta)$  for the  $\delta$  chosen above. Hence  $\{\|a_n\| \alpha_n\}$  does not tend to 0 in  $X$  and this is a contradiction. Thus (c) $\Rightarrow$ (a) is proved. This proves Lemma 3.2.  $\square$

**Remark:** In view of Lemma 2, it follows that a sequence  $\{\alpha_n\}$  is a proper base in  $X$  if and only if it satisfies the condition (17)

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## Subclasses of univalent functions related with circular domains

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ABSTRACT: We investigate the family of functions normalized by the condition  $f(0) = f'(0) - 1 = 0$ , that are analytic in the unit disk, with the property that the domain of values  $f'(z) + \frac{1+e^{i\alpha}}{2}zf''(z)$ , ( $\alpha \in (-\pi, \pi]$ ) is the disk  $|z - b| < b$ ,  $b \geq 1$ . Integral and convolution characterizations are found and coefficients bounds are given.

### Theorem 0.1 *d*

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Key Words and Phrases: *Convex function; Starlike function; Subordination; Convolution; Hadamard product*

## 1 Introduction

Let  $\mathcal{A}$  denote the class of functions of the form  $f(z) = z + \sum_{n=2}^{\infty} a_n z^n$  that are analytic in the unit disk  $\Delta = \{z \in \mathbb{C} : |z| < 1\}$  and let  $\mathcal{S}, \mathcal{K}$  be the subclasses of  $\mathcal{A}$  consisting of functions respectively starlike and convex in  $\Delta$ .

For the functions  $f$  and  $g$  with the series expansions  $f(z) = \sum_{k=0}^{\infty} a_k z^k$  and  $g(z) = \sum_{k=0}^{\infty} b_k z^k$  the Hadamard product (or convolution)  $f * g$  is defined by

$$(f * g)(z) = \sum_{k=0}^{\infty} a_k b_k z^k.$$

This product is associative, commutative and distributive over addition and the function  $\frac{1}{1-z}$  is an identity for it.

In [4] H. Silverman and H.M. Silvia introduced the classes

$$\mathcal{L}_\alpha = \left\{ f \in \mathcal{A} : \operatorname{Re} \left( f'(z) + \frac{1 + e^{i\alpha}}{2} z f''(z) \right) > 0, z \in \Delta \right\},$$

where  $\alpha \in (-\pi, \pi]$ .

For each  $\alpha \in (-\pi, \pi]$  and  $b, b \geq 1$ , let

$$\mathcal{L}_\alpha(b) = \left\{ f \in \mathcal{A} : \left| f'(z) + \frac{1 + e^{i\alpha}}{2} z f''(z) - b \right| < b, z \in \Delta \right\}.$$

Observe that if  $b \rightarrow \infty$ , then  $\mathcal{L}_\alpha(b) \rightarrow \mathcal{L}_\alpha$ .

## 2 Characterization results for $\mathcal{L}_\alpha(b)$ .

We give two characterization conditions for the considered classes  $\mathcal{L}_\alpha(b)$ . Let  $\mathcal{P}$  denote the class of holomorphic functions with the normalization  $p(0) = 1$ , having positive real part in  $\Delta$ . For  $b \geq 1$  let

$$\mathcal{P}(b) = \{p \in \mathcal{P} : |p(z) - b| < b, z \in \Delta\}.$$

**Theorem 2.1** For  $\alpha \neq \pi$ , let  $c = \frac{1 - e^{i\alpha}}{1 + e^{i\alpha}}$ . Then  $f \in \mathcal{L}_\alpha(b)$  if and only if there exists  $p \in \mathcal{P}(b)$  such that the following equality holds for all  $z \in \Delta$

$$(1) \quad f(z) = \int_0^z \frac{c+1}{\eta^{c+1}} \left[ \int_0^\eta \zeta^c p(\zeta) d\zeta \right] d\eta.$$

**Proof.** It is easily seen that  $f \in \mathcal{L}_\alpha(b)$  if and only if there exists  $p \in \mathcal{P}(b)$  such that

$$f'(z) + \frac{1 + e^{i\alpha}}{2} z f''(z) = p(z), z \in \Delta.$$

Since

$$f'(z) + \frac{1 + e^{i\alpha}}{2} z f''(z) = \frac{1 - e^{i\alpha}}{2} f'(z) + \frac{1 + e^{i\alpha}}{2} (z f'(z))'$$

we have that

$$\frac{1 - e^{i\alpha}}{1 + e^{i\alpha}} f'(z) + (z f'(z))' = \frac{2}{1 + e^{i\alpha}} p(z)$$

which is equivalent to

$$c z^c f'(z) + z^c (z f'(z))' = \frac{2}{1 + e^{i\alpha}} z^c p(z)$$

where  $c = \frac{1 - e^{i\alpha}}{1 + e^{i\alpha}}$ . This leads to

$$[z^c (z f'(z))] = \frac{2}{1 + e^{i\alpha}} z^c p(z).$$

Therefore

$$z^{c+1}f'(z) = \frac{2}{1+e^{i\alpha}} \int_0^z \zeta^c p(\zeta) d\zeta$$

which is equivalent to (1). The proof is completed.

Note that for a function

$$(2) \quad q_b(z) = \frac{b+bz}{b+(1-b)z}, \quad z \in \Delta, \quad b \geq 1$$

we have  $q_b(\Delta) = \{w \in \mathbf{C} : |w-b| < b\}$ . It is easy to observe that

$$\mathcal{P}(b) = \{p \in \mathcal{P} : p \prec q_b, \quad z \in \Delta\}.$$

Therefore we obtain another characterization of the class  $\mathcal{L}_\alpha(b)$  in terms of subordination.

**Corollary 2.1** *A necessary and sufficient condition for  $f$  to be in the class  $\mathcal{L}_\alpha(b)$  is*

$$f'(z) + \frac{1+e^{i\alpha}}{2} z f''(z) \prec q_b(z), \quad z \in \Delta,$$

where  $q_b$  is given by (2).

Note that

$$q_b(z) = 1 + \frac{2b-1}{b}z + \frac{b-1}{b} \frac{2b-1}{b} z^2 + \left(\frac{b-1}{b}\right)^2 \frac{2b-1}{b} z^3 + \dots, \quad z \in \Delta.$$

Thus from (1) we obtain a function  $f_{\alpha,b}$ , related to  $q_b$ , of the form

$$(3) \quad f_{\alpha,b}(z) = z + \frac{2b-1}{b} \sum_{n=2}^{\infty} \left(\frac{b-1}{b}\right)^{n-2} \frac{2z^n}{n[n+1+(n-1)e^{i\alpha}]}.$$

We will use the notion of convolution in our next characterization result.

**Theorem 2.2** *Let  $\alpha \in (-\pi, \pi]$ ,  $b \geq 1$  and let  $C(t) := b(1+e^{it})$ ,  $t \in [0, 2\pi)$ . Then the following conditions are equivalent:*

(i)  $f \in \mathcal{L}_\alpha(b)$

(ii)  $\frac{1}{z} \left[ f * \frac{z+e^{i\alpha}z^2}{(1-z)^3} \right] - C(t) \neq 0$  for all  $z \in \Delta$  and for all  $t \in [0, 2\pi)$ .

**Proof.** Let  $p(z) = f'(z) + \frac{1+e^{i\alpha}}{2} z f''(z)$ . Observe that

$$f \in \mathcal{L}_\alpha(b) \iff \{p(z) \in q_b(\Delta), \quad z \in \Delta\}.$$

Moreover,  $\partial(q_b(\Delta))$  is a curve with parametrization  $C(t) = b(1+e^{it})$ ,  $t \in [0, 2\pi)$ . Since

$$f'(z) + \frac{1+e^{i\alpha}}{2} z f''(z) = \frac{1-e^{i\alpha}}{2} f'(z) + \frac{1+e^{i\alpha}}{2} (z f'(z))',$$

we can write

$$\begin{aligned} f'(z) + \frac{1+e^{i\alpha}}{2}zf''(z) &= \left( f * \left( \frac{1-e^{i\alpha}}{2} \frac{z}{1-z} + \frac{1+e^{i\alpha}}{2} \frac{z}{(1-z)^2} \right) \right)' = \\ &= \left( f * \frac{z - \left( \frac{1-e^{i\alpha}}{2} \right) z^2}{(1-z)^2} \right)' = \frac{1}{z} \left\{ f * \frac{z + e^{i\alpha}z^2}{(1-z)^3} \right\}. \end{aligned}$$

Now, let  $f \in \mathcal{L}_\alpha(b)$ . We remark that it is equivalent to the condition

$$\frac{1}{z} \left\{ f * \frac{z + e^{i\alpha}z^2}{(1-z)^3} \right\} \in q_b(\Delta), \quad z \in \Delta.$$

Consequently, for every  $z \in \Delta$  the value  $\frac{1}{z} \left\{ f * \frac{z + e^{i\alpha}z^2}{(1-z)^3} \right\}$  is not a boundary point of  $q_b(\Delta)$  so the result (ii) follows immediately. Next, let  $\frac{1}{z} \left\{ f(z) * \frac{z + e^{i\alpha}z^2}{(1-z)^3} \right\} - C(t) \neq 0$  for all  $z \in \Delta$  and  $t \in [0, 2\pi)$ . The last inequality may be rewritten in the following equivalent form:  $f'(z) + \frac{1+e^{i\alpha}}{2}zf''(z) - C(t) \neq 0$ . Since  $p(0) = 1$  and  $p(z) \neq C(t)$  for all  $z \in \Delta$  and all  $t \in [0, 2\pi)$  thus  $p(z) \in q_b(\Delta)$ . The proof is thus completed.

**Remark 2.1** *It is easy to observe that for  $b_1 < b_2$  is  $\mathcal{L}_\alpha(b_1) \subset \mathcal{L}_\alpha(b_2)$  for each  $\alpha \in (-\pi, \pi]$ . Consequently we obtain*

$$\bigcup_{b \geq 1} \mathcal{L}_\alpha(b) = \mathcal{L}_\alpha$$

and

$$\bigcap_{b \geq 1} \mathcal{L}_\alpha(b) = \mathcal{L}_\alpha(1) = \left\{ f \in \mathcal{A} : \left| f'(z) + \frac{1+e^{i\alpha}}{2}zf''(z) - 1 \right| < 1, \quad z \in \Delta \right\}.$$

**Remark 2.2.** *H. Silverman and E.M. Silvia ([4]) proved that  $\mathcal{L}_\pi$  contains  $\mathcal{L}_\alpha$  for each  $\alpha$ . Thus we have  $\mathcal{L}_\alpha(b) \subset \mathcal{L}_\pi$  for every  $\alpha \in (-\pi, \pi]$  and  $b \geq 1$ , where  $\mathcal{L}_\pi$  is the well known class  $\mathcal{R}$ , consisting of univalent functions in  $\mathcal{A}$  whose derivatives have positive real part in  $\Delta$  ([1]).*

### 3 Special members of $\mathcal{L}_\alpha(b)$

In this section we give some examples of functions belonging to the considered classes.

**Theorem 3.1** *A function  $f(z) = z + a_n z^n \in \mathcal{L}_\alpha(1)$  if and only if*

$$|a_n| \leq \frac{\sqrt{2}}{n\sqrt{n^2 + 1 + (n^2 - 1)\cos\alpha}}.$$



**Proof.** It suffices to consider  $|z| = 1$ . Assume that  $f(z) = z + a_n z^n \in \mathcal{L}_\alpha(1)$ . Using the definition of  $\mathcal{L}_\alpha(b)$  we obtain the equivalent condition

$$\left| na_n z^{n-1} \left[ 1 + \frac{1}{2}(n-1)(1 + e^{i\alpha}) \right] \right| \leq 1.$$

Note that the above inequality is equivalent to

$$n|a_n| \sqrt{1 + \frac{1}{2}(n^2 - 1)(1 + \cos \alpha)} \leq 1$$

and this gives the required result.

**Theorem 3.2** Let  $b > 1$ . A function  $f(z) = z + a_n z^n \in \mathcal{L}_\alpha(b)$  if

$$|a_n| \leq \frac{\sqrt{2}}{n\sqrt{n^2 + 1 + (n^2 - 1)\cos \alpha}}.$$

**Proof.** Let us denote  $|a_n| = r$  and  $a_n z^{n-1} = r e^{i\varphi}$ . For a function  $f(z) = z + a_n z^n$  and for  $|z| = 1$  we have

$$\begin{aligned} \left| f'(z) + \frac{1 + e^{i\alpha}}{2} z f''(z) - b \right| &= \left| 1 - b + na_n z^{n-1} \left[ 1 + \frac{1}{2}(n-1)(1 + e^{i\alpha}) \right] \right| \leq \\ &\leq b - 1 + nr \left| 1 + \frac{1}{2}(n-1)(1 + e^{i\alpha}) \right|. \end{aligned}$$

Thus the condition  $\left| f'(z) + \frac{1+e^{i\alpha}}{2} z f''(z) - b \right| < b$  will be satisfied if

$$\frac{\sqrt{2}}{2} nr \sqrt{n^2 + 1 + (n^2 - 1)\cos \alpha} < 1.$$

A simple calculation leads to the required result.

**Theorem 3.3** Let  $b \geq 1$ . A function  $f(z) = \frac{z}{1-Bz} \in \mathcal{L}_\alpha(b)$  if  $|B| < r_0$ , where  $r_0$  is the unique real root of the equation

$$2br^3 + (6b + 1)r - 1 = 0.$$

**Proof.** For a function  $f(z) = \frac{z}{1-Bz}$  we denote  $|B| = r$ ,  $Bz = r e^{i\varphi}$ . Note that for  $|z| = 1$  we have

$$\left| f'(z) + \frac{1 + e^{i\alpha}}{2} z f''(z) - b \right| \leq \frac{br^3 + 3br^2 + (3b + 1)r + b - 1}{|1 - r|^3}.$$

Therefore  $f(z) = \frac{z}{1-Bz} \in \mathcal{L}_\alpha(b)$  if

$$\frac{br^3 + 3br^2 + (3b+1)r + b - 1}{|1-r|^3} < b$$

or equivalently if

$$br^3 + 3br^2 + (3b+1)r + b - 1 < b|1-r|^3, \quad r \neq 1.$$

The above inequality has no solution for  $r > 1$ . For  $r < 1$  it takes a form

$$w(r) = 2br^3 + (6b+1)r - 1 < 0.$$

The polynomial  $w(r)$  takes negative values for  $r < r_0$ , where  $0 < r_0 < 1$  is the unique real root of  $w(r)$ , so the result yields.

## 4 Coefficient bounds

First, we give a sufficient condition for  $f \in \mathcal{A}$  to be in the class  $\mathcal{L}_\alpha(b)$ .

**Theorem 4.1** *If a function  $f \in \mathcal{A}$  satisfies the condition*

$$\sum_{n=2}^{\infty} n \left[ 1 + \frac{\sqrt{2}}{2}(n-1)\sqrt{1+\cos\alpha} \right] |a_n| \leq 1,$$

then  $f \in \mathcal{L}_\alpha(b)$ .

**Proof.** Note that

$$\begin{aligned} \left| f'(z) + \frac{1+e^{i\alpha}}{2} z f''(z) - b \right| &= \left| 1 - b + \sum_{n=2}^{\infty} n \left[ 1 + \frac{n-1}{2}(1+e^{i\alpha}) \right] a_n z^{n-1} \right| \leq \\ &\leq b - 1 + \sum_{n=2}^{\infty} n \left[ 1 + \frac{n-1}{2} |1+e^{i\alpha}| \right] |a_n| = \\ &= b - 1 + \sum_{n=2}^{\infty} n \left[ 1 + \frac{\sqrt{2}}{2}(n-1)\sqrt{1+\cos\alpha} \right] |a_n|. \end{aligned}$$

Therefore the inequality

$$\left| f'(z) + \frac{1+e^{i\alpha}}{2} z f''(z) - b \right| \leq b$$

will be satisfied if

$$\sum_{n=2}^{\infty} n \left[ 1 + \frac{\sqrt{2}}{2}(n-1)\sqrt{1+\cos\alpha} \right] |a_n| \leq 1.$$

The proof is thus completed.

In order to develop next coefficient result for classes  $\mathcal{L}_\alpha(b)$  we need

**Lemma 4.1 (Rogosinski theorem [3])** *Let  $h(z) = 1 + \sum_{k=1}^{\infty} c_k z^k$  be subordinate to  $H(z) = \sum_{k=1}^{\infty} C_k z^k$  in  $\Delta$ . If  $H(z)$  is univalent in  $\Delta$  and  $H(\Delta)$  is convex, then  $|c_n| \leq C_1$ .*

Now we present upper bounds on coefficients in  $\mathcal{L}_\alpha(b)$ . Unfortunately, they are not sharp, except in the case  $n = 2$ .

**Theorem 4.2** *Let  $f \in \mathcal{L}_\alpha(b)$ . Then*

$$(4) \quad |a_2| \leq \frac{2b - 1}{b\sqrt{10 + 6 \cos \alpha}},$$

and

$$(5) \quad |a_n| \leq \frac{\sqrt{2}(2b - 1)}{bn\sqrt{n^2 + 1 + (n^2 - 1) \cos \alpha}}, \quad n \geq 3.$$

Equality in (4) holds for the function  $f_{\alpha,b}$  given by (3).

**Proof.** Let  $f(z) = z + \sum_{k=2}^{\infty} a_k z^k \in \mathcal{L}_\alpha(b)$  for  $\alpha \in (-\pi, \pi]$  and  $b \geq 1$ . Let us define  $q(z) = f'(z) + \frac{1+e^{i\alpha}}{2} z f''(z) = 1 + \sum_{k=1}^{\infty} b_k z^k$ . Then from the definition of  $\mathcal{L}_\alpha(b)$  we get  $q(z) \prec q_b(z)$ . The function  $q_b$  is univalent in  $\Delta$  and  $q_b(\Delta)$  is a convex region, so Rogosinski theorem applies. Since  $q_b(z) = 1 + \frac{2b-1}{b} z + \dots$ , so we obtain  $|b_n| \leq \frac{2b-1}{b}$ . Comparing coefficients of  $z^n$  on both sides of equality

$$q(z) = f'(z) + \frac{1 + e^{i\alpha}}{2} z f''(z)$$

we get

$$b_1 = (3 + e^{i\alpha}) a_2$$

and

$$b_{n-1} = na_n \left[ 1 + \frac{1}{2}(n-1)(1 + e^{i\alpha}) \right] \text{ for } n \geq 3.$$

Thus on account of Lemma 2.1 we obtain

$$|b_1| = |3 + e^{i\alpha}| |a_2| \leq \frac{2b - 1}{b},$$

which is equivalent to the inequality

$$|a_2| \leq \frac{2b - 1}{b|3 + e^{i\alpha}|} = \frac{2b - 1}{b\sqrt{10 + 6 \cos \alpha}}.$$

Note that for a function  $f_{\alpha,b}$  belonging to  $\mathcal{L}_\alpha(b)$  the second coefficient in its Taylor series expansion has the form  $a_2 = \frac{2b-1}{b(3+e^{i\alpha})}$ , which shows that result (4) is sharp. Further we get for  $n \geq 3$

$$|a_n| \leq \frac{2b-1}{bn \left| 1 + \frac{1}{2}(n-1)(1+e^{i\alpha}) \right|},$$

which is equivalent to (5). A sharp bound of  $|a_n|$  for  $n \geq 3$  is still an open problem.

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## Some model of stochastic prediction

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ABSTRACT: This article presents a method of the stochastic prediction, more exactly the method of estimating of the covariance function of the stochastic process that is stationary in a wider sense. The reason for doing this is that the covariance function is necessary for prediction.

AMS Subject Classification: *60G10, 60G25, 93E24*

Key Words and Phrases: *stationary stochastic processes, stochastic prediction, least squares method*

### 1 Introduction

The paper includes the solution of the stochastic prediction problem using the property of the stationary in a wider sense for the stochastic processes and also the method of the least squares (see also [3]-[5]). The method can be used firstly in the situation in which observations are "double-valued" and when a random variable which is equal to a number of changes of values of the double-valued process has the Poisson distribution ([2]). Then we generalize the model and we indicate some economical applications.

### 2 Definitions and notation

Let  $\{\xi_t, t \in [0, \infty)\}$  be the stochastic process defined on the probability space  $(\Omega, M, P)$  in which for every  $\omega \in \Omega$ :

$$\xi_t(\omega) = \begin{cases} 1 & \text{when at time } t \text{ an event occurred,} \\ -1 & \text{when at time } t \text{ an event did not occur.} \end{cases} \quad (1)$$

Let us assume that at time  $t = 0$ , of an initial observation of an event, the probability that the event occurred is the same as the probability that an event did not occur. So let:

$$P(\xi_0 = 1) = P(\xi_0 = -1) = 0,5. \quad (2)$$

Further let  $\{\eta_t, t \in [0, +\infty)\}$  be another stochastic process for which

1.  $\eta_0 = 0$  with probability 1,
2.  $\eta_t$  adopts, for each  $t$ , a value which is equal to a number of changes of the signs of the process  $\{\xi_t, t \in [0, +\infty)\}$  in the interval  $[0, t)$ .

It is easy to notice that for each  $t \in [0, +\infty)$  there is the following property:

$$\xi_t = \xi_0(-1)^{\eta_t} . \quad (3)$$

### 3 The main problem

We shall prove the following theorem

**Theorem 1.** *Assume that the process  $\xi_t$  is given by (1)-(3). Moreover the processes  $\xi_0$  and  $\eta_t$  are independent for each  $t \in [0, +\infty)$  and the processes  $\xi_t$  and  $\eta_{t+k} - \eta_k$  are independent for each  $t \in [0, +\infty)$  and  $k > 0$ .*

Then, the process  $\{\xi_t, t \in [0, +\infty)\}$  is stationary in a wider sense.

**Proof.** In order to prove the above theorem, it is advisable to show that the expected value is - for all random variables of the process - a constant value. Moreover, it is advisable to show that the variance is finite and the covariance function depends on only one variable, namely the difference of indexes  $t + k$  and  $t$ .

Since  $\xi_t = \xi_0(-1)^{\eta_t}$ , so the expected value  $m_t = E\xi_t$  can be written as:

$$\begin{aligned} m_t &= 1P(\xi_t = 1) + (-1)P(\xi_t = -1) \\ &= P(\xi_0 = 1)P(\eta_t \text{ has even values}) + P(\xi_0 = -1)P(\eta_t \text{ has odd values}) \\ &\quad - P(\xi_0 = -1)P(\eta_t \text{ has even values}) - P(\xi_0 = 1)P(\eta_t \text{ has odd values}) \\ &= P(\eta_t \text{ has odd values}) \{P(\xi_0 = -1) - P(\xi_0 = 1)\} \\ &\quad - P(\eta_t \text{ has even values}) \{P(\xi_0 = -1) - P(\xi_0 = 1)\} \\ &= P(\eta_t \text{ has odd values}) \cdot 0 - P(\eta_t \text{ has even values}) \cdot 0 = 0 . \end{aligned}$$

So  $m_t = 0 = \text{const.}$

In order to calculate the covariance function  $B(t, t+k)$  let us assume that  $t < t+k$ . Then, because of the stationarity of the process, we have:

$$\begin{aligned} B(t, t+k) &= E(\xi_t - m_t)(\xi_{t+k} - m_{t+k}) = E\xi_t\xi_{t+k} = E[\xi_0 \cdot (-1)^{\eta_t} \cdot \xi_0 \cdot (-1)^{\eta_{t+k}}] \\ &= \xi_0^2 E[(-1)^{\eta_t} \cdot (-1)^{\eta_{t+k}}] = 1 \cdot E[(-1)^{\eta_t} \cdot (-1)^{\eta_{t+k} - \eta_t} \cdot (-1)^{\eta_t}] \\ &= E[(-1)^{2\eta_t} \cdot (-1)^{\eta_{t+k} - \eta_t}] \quad (4) \\ &= E[(-1)^{\eta_k}] = 1 \cdot P(\eta_k \text{ has even values}) + (-1) \cdot P(\eta_k \text{ has odd values}) \\ &= 1 \cdot \left[ \frac{1}{2} + e^{-2\lambda k} \right] + (-1) \cdot \left[ 1 - \frac{1}{2} - e^{-2\lambda k} \right] = \frac{e^{-2\lambda k}}{2} + \frac{e^{-2\lambda k}}{2} = e^{-2\lambda k} . \end{aligned}$$

Indeed, we have used the fact that

$$\begin{aligned}
 P(\eta_k \text{ has even values}) &= \sum_{n=0}^{\infty} e^{-\lambda k} \cdot \frac{(2k)^{2n}}{(2n)!} = e^{-\lambda k} \cdot \cos h(\lambda k) \\
 &= e^{-\lambda k} \cdot \frac{e^{\lambda k} + e^{-\lambda k}}{2} = e^{-\lambda k} \left[ \frac{e^{\lambda k} + e^{-\lambda k}}{2} \right] \\
 &= \frac{e^0 + e^{-2\lambda k}}{2} = \frac{1}{2} + \frac{e^{-2\lambda k}}{2} .
 \end{aligned}$$

Thus, the covariance function  $B(t, t+k)$  depends only on the argument  $k = t+k-1$ . For the covariance function, we shall use the symbol  $r(k)$ .

The variance of the stochastic process  $\{\xi_t, t \in [0, +\infty)\}$  is equal to the value  $r(0)$ . Since here  $r(0) = 1$ , we can state that the variance is finite.

It completes the proof that the process  $\{\xi_t, t \in [0, +\infty)\}$  is stationary in a wider sense.  $\square$

Thanks to the properties of stationary in a wider sense of the process  $\{\xi_t, t \in [0, +\infty)\}$  one can specify such value  $\xi_{t+m}^*$ , which will constitute the forecast of the stochastic process for  $m$ -steps in the future.

The assumptions that we will make are:

1. There is a time series  $x_1 = \xi_{t-1}, x_2 = \xi_{t-2}, \dots, x_n = \xi_{t-n}$  of the stochastic process  $\{\xi_t, t \in [0, +\infty)\}$ ,
2. We are looking for the prediction by the linear method of the least squares. The aim is to specify such a random variable  $\xi_{t+m}^*$  for which the mean square error of the prediction

$$d^2 = E|\xi_{t+m} - \xi_{t+m}^*|^2$$

reaches the minimum in the class of all the linear forms of random variables  $\xi_{t-1}, \dots, \xi_{t-n}$ .

The quickest answer to the problem of the stochastic prediction is by making the assumption that the stochastic process  $\{\xi_t, t \in [0, +\infty)\}$  is an element of a Hilbert space. So, let  $H$  be the space of all the random variables which are defined within the common probability space  $(\Omega, M, P)$  and are square-integrable:

$$\int_{\Omega} |\xi(\omega)|^2 dP(\omega) < +\infty .$$

The space  $H$ , together with the operations of the summation of the random variables and the scalar multiplication of the random variables, is a vector space. The formula

$$(\xi|\eta) = \int_{\Omega} \xi(\omega)\bar{\eta}(\omega)dP(\omega); \quad \xi, \eta \in H \quad (5)$$





We come up to the following formula for the covariance function of the process  $\{\xi_t, t \in [0, +\infty)\}$

$$r(k) = e^{-2\lambda|k|}, \quad k \in R. \quad (9)$$

So then the system of equations (8) gives us

$$a_1 e^{-2\lambda|j-1|} + a_2 e^{-2\lambda|j-2|} + \dots + a_n e^{-2\lambda|j-n|} = e^{-2\lambda|j+m|}, \quad j = 1, \dots, n \quad (10)$$

and

$$\begin{aligned} r(0) &= 1, \quad r(1) = e^{-2\lambda}, \quad r(2) = e^{-4\lambda}, \dots, r(n-1) \\ &= e^{-2\lambda(n-1)}, \quad r(m+1) = e^{-2\lambda(m+1)} \end{aligned} \quad (11)$$

The appropriate system of equations takes the form

$$\begin{cases} a_1 + a_2 e^{-2\lambda} + \dots + a_n e^{-2\lambda(n-1)} = e^{-2\lambda(m+1)} \\ a_1 e^{-2\lambda} + a_2 + \dots + a_n e^{-2\lambda(n-2)} = e^{-2\lambda(m+2)} \\ \dots \\ a_1 e^{-2\lambda(n-1)} + a_2 e^{-2\lambda(n-2)} + \dots + a_n = e^{-2\lambda(m+n)} \end{cases} \quad (12)$$

The solution  $a_1, a_2, \dots, a_n$  of this system of equations helps us to come up to the prediction. Namely, the expression:

$$\xi_{t+m}^* = a_1 \xi_{t-1} + a_2 \xi_{t-2} + \dots + a_n \xi_{t-n}, \quad (13)$$

as some estimate of the predicted quantity  $\xi_{t+m}$  - after taking into consideration the conditions of our example - helps us finally to treat it as the prediction of the process  $\xi_t, t \in [0, +\infty)$ , for  $m$  - steps forward.

In practice, in most cases, the covariance function is not known and it should not be estimated on the basis of the experimental data. Our analysis is concerned with the necessity of finding an estimator for the parameter  $\lambda$ .

Considering that we know exactly the analytical formula for the quantity of the covariance function, the suggestion for an estimator is the following. Let the estimator of the function  $r(k)$  be

$$r^\star(k) = \exp(-2\lambda^\star|k|),$$

where  $\lambda^\star$  is the estimator of the parameter  $\lambda$  in Poisson distribution obtained on the basis of a random sample through the method of maximum likelihood. It is well known that this estimator is the expected value (or mean value of samples) of the Poisson process.

In such case, the statistic  $\lambda^\star$  are the consistent and asymptotically unbiased estimator of the parameter  $\lambda$ .

The method of prediction that has been used here, guarantees that the prediction obtained in this way carries the smallest mean squared error.

## 4 Some generalizations

The process is not useful for many economical applications ([1]) because it has only values 1 and  $-1$ . We shall generalize it.

Let

$$X_t = |the\ value\ of\ changing\ of\ an\ economical\ indicator\ at\ time\ t|. \quad (14)$$

Assume that they are positive independent random variable of the same distribution (i.i.d) and assume that  $X_t$  is uniformly distributed on the interval  $[a, b]$ ,  $a < b$ , and  $E(X_t) = m = const$ . It is known that  $E(X_t) = \frac{a+b}{2}$ .

**Definition 1.** Let

$$\xi_{t+1} = X_{t+1} \cdot (-1)^{\eta_{t+1}}, \quad (15)$$

where  $\eta_t$  is the Poisson process. If for example  $X_t$  is the value of changing of share price at time  $t$  then we have

$$S_{t+1} = S_t \cdot [1 + X_{t+1} \cdot (-1)^{\eta_{t+1}}] \quad (16)$$

which is equivalent to the following notation

$$\frac{S_{t+1} - S_t}{S_t} = \xi_{t+1}.$$

This is a so called the percentage change of a share price. It can also be interpreted as a value of a firm.

Assume that  $\eta_t$ , is defined as typical Poisson process that is,  $\eta_1$  is the time of waiting for the first event. Let  $\eta_2$  be the time of waiting between the first and second event. We obtain the sequence  $\eta_1, \eta_2, \dots$ . Now  $S_n = \eta_1 + \eta_2 + \dots + \eta_n$  is the moment of occurring of the  $n$ -th events. Let  $S_0 = 0$ . The amount  $\eta_t$  of events that occurred in the interval  $[0, t]$  is defined as

$$\eta_t = \max\{n : S_n \leq t\}.$$

Notice that the amount of event ocured in the interval  $(s, t]$  is equal to  $\eta_t - \eta_s$ . It is proved in [2], §23, that  $\eta_t$  is a random variable with the Poisson distribution. It is easy to show the following theorem.

**Theorem 2.** Assume that the process  $\{\xi_t, t \in [0, +\infty)\}$  is given by (14) i (15). Moreover, the processes  $\xi_t$  and  $\eta_{t+k} - \eta_t$  are independent for  $k > 0$ . Then, the process  $\xi_t$  is stationary in a wider sense.

*Proof.* In order to prove the above theorem we shall firstly show that the expected value of  $\xi_t$  is constant. Indeed, we assume that  $E(X_t) = 0$  (it is not changing the generality of our considerations).

We have

$$m_t = E\xi_t = E[X_t \cdot (-1)^{\eta_t}] = E[X_t] \cdot E[(-1)^{\eta_t}] = 0.$$

Further, we compute the covariance function using (4):

$$\begin{aligned} B(t, t+k) &= E[(\xi_t - m_t)(\xi_{t+k} - m_{t+k})] = E[X_t \cdot (-1)^{\eta_t} \cdot X_{t+k} \cdot (-1)^{\eta_{t+k}}] \\ &= E[X_t] \cdot E[X_{t+k}] \cdot E[(-1)^{\eta_t} \cdot (-1)^{\eta_{t+k}}] = \left(\frac{a+b}{2}\right)^2 \cdot e^{-2\lambda k} = r_1(k) . \end{aligned}$$

Finally, the variance is finite, because

$$r_1(0) = \left(\frac{a+b}{2}\right)^2 < \infty .$$

□

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## On quasiconformal extensions of an automorphism of the real axis II

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Submitted by: *Leopold Koczan*

ABSTRACT: In this paper some extension operator (defined by a simple geometric condition) for automorphism of a real axis are investigated. The sufficient and necessary conditions for  $K$ -quasiconformality of this extension are given.

AMS Subject Classification: *30C75, 30C85*

Key Words and Phrases: *quasiconformal mapping, extension operator, cross-ratio*

### 1 Introduction

Numerous extension operators acting on homeomorphic self-mapping of a Jordan curve were investigated by many mathematicians; cf e.g. [3], [7] [5], [6].

Let  $\mathbb{R}$  be the real axis and  $\mathbb{H} := \{z : \text{Im}z \geq 0\}$  be the upper half plane. By  $\text{Aut}^+(\mathbb{R})$  we denote the set of all increasing homeomorphism of the real axis  $\mathbb{R}$  onto itself and  $\text{Aut}^+(\mathbb{H})$  be the set of all sense-preserving homeomorphic self-mapping of the upper half plane  $\mathbb{H}$  onto itself.

In this paper we will discuss some properties of the extension operator  $G$  defined on  $\text{Aut}^+(\mathbb{R})$  with values in  $\text{Aut}^+(\mathbb{H})$  given by the simple geometric and analytic condition.

In a paper [6] titled *On quasiconformal extensions of an automorphism of the real axis* similar extension operator  $H : \text{Aut}^+(\mathbb{R}) \rightarrow \text{Aut}^+(\mathbb{H})$  was investigated.

Recall that a cross-ratio of points  $a, b, c, d \in \mathbb{C}$  is defined by

$$[a, b; c, d] := \frac{c-a}{c-b} : \frac{d-a}{d-b}.$$

If one of this four points is in the infinity we define (for example)

$$[a, b, \infty, d] := \lim_{t \rightarrow \infty} [a, b, t, c] = \frac{d-b}{d-a}.$$

Let  $\mathcal{T}$  stand for the family of all triangular  $\Delta(x_1, x_2, z)$  which are rectangular and isosceles and such that  $x_1, x_2 \in \mathbb{R}$ ,  $x_1 < x_2$ ,  $z \in \mathbb{H}$  and  $\angle x_1 z x_2 = \pi/2$ .

Simple calculations show that

**Lemma 1.1** *For all  $x_1, x_2 \in \mathbb{R}$ ,  $x_1 < x_2$  and  $z = x + iy \in \mathbb{H}$  the following properties are equivalent:*

- i)  $\Delta(x_1, x_2, z) \in \mathcal{T}$ ;
- ii)  $\lim_{t \rightarrow \infty} [x_1, x_2, t, z] = i$ ;
- iii)  $z = \frac{1}{2}(1 - i)(x_1 + ix_2)$ ;
- iv)  $x_1 = x - y$  and  $x_2 = x + y$ .

**Definition 1.1** *Let  $f \in \text{Aut}^+(\mathbb{R})$  we define  $F = G[f]$  by the formula*

$$G[f](x + iy) := \frac{1}{2}(1 - i)(f(x - y) + if(x + y)), \quad z = x + iy \in \mathbb{H}. \quad (1.1)$$

**Remark 1.1** *We can also write*

$$G[f](x + iy) = \frac{1}{2}(f(x + y) + f(x - y)) + \frac{1}{2}i(f(x + y) - f(x - y)), \quad z = x + iy \in \mathbb{H}. \quad (1.2)$$

or equivalent if we denote

$$\alpha := \frac{1}{2}(1 + i) \quad (1.3)$$

we get

$$G[f](z) = \alpha f(\bar{\alpha}z + \alpha\bar{z}) + \bar{\alpha}f(\alpha z + \bar{\alpha}\bar{z}), \quad z \in \mathbb{H}. \quad (1.4)$$

By Lemma 1.1 we get

**Theorem 1.1** *Let  $f \in \text{Aut}^+(\mathbb{R})$  and  $F = G[f]$ , if for arbitrary point  $z = x + iy \in \mathbb{H}$  we put  $x_1 = x - y$ ,  $x_2 = x + y$  then the following properties are equivalent:*

- i) *The point  $F(z)$  is the unique point in  $\mathbb{H}$  such that the cross-ratios  $[x_1, x_2, \infty, z]$  and  $[f(x_1), f(x_2), \infty, F(z)]$  are equal  $i$ , precisely*

$$\lim_{t \rightarrow \infty} [x_1, x_2, t, z] = \lim_{t \rightarrow \infty} [f(x_1), f(x_2), t, F(z)] = i.$$

- ii) *The point  $F(z)$  is the unique point in  $\mathbb{H}$  such that*

$$\frac{z - x_2}{z - x_1} = \frac{F(z) - f(x_2)}{F(z) - f(x_1)} = i.$$

- iii) *The point  $F(z)$  is the unique point in  $\mathbb{H}$  such that the triangle*

$$\Delta(f(x_1), f(x_2), F(z)) \in \mathcal{T}$$

*is rectangular and isosceles.*

**Remark 1.2** The operator  $H : \text{Aut}^+(\mathbb{R}) \rightarrow \text{Aut}^+(\mathbb{H})$  which was investigated in the paper [6] has a similar property, exactly: If  $f \in \text{Aut}^+(\mathbb{R})$  and  $\tilde{F} = H[f]$  and if for arbitrary  $z = x + iy$  we put  $\tilde{x}_1 = x - y/\sqrt{3}$ ,  $\tilde{x}_2 = x + y/\sqrt{3}$ , then the following conditions are equivalent

- i) The point  $\tilde{F}(z)$  is the unique point in  $\mathbb{H}$  such that the cross-ratios  $[x_1, x_2, \infty, z]$  and  $[f(x_1), f(x_2), \infty, \tilde{F}(z)]$  are equal  $p := \frac{1}{2}(1 + i\sqrt{3})$ , precisely

$$\lim_{t \rightarrow \infty} [\tilde{x}_1, \tilde{x}_2, t, z] = \lim_{t \rightarrow \infty} [f(\tilde{x}_1), f(\tilde{x}_2), t, \tilde{F}(z)] = p.$$

- ii) The point  $\tilde{F}(z)$  is the unique point in  $\mathbb{H}$  such that

$$\frac{z - \tilde{x}_2}{z - \tilde{x}_1} = \frac{\tilde{F}(z) - f(\tilde{x}_2)}{\tilde{F}(z) - f(\tilde{x}_1)} = p.$$

- iii) The point  $\tilde{F}(z)$  is the unique point in  $\mathbb{H}$  such that the triangle

$$\Delta(f(\tilde{x}_1), f(\tilde{x}_2), \tilde{F}(z))$$

is an equilateral triangle.

- iv) The function  $\tilde{F}$  has the form

$$\tilde{F}(x + iy) = \bar{p}f(\tilde{x}_1) + pf(\tilde{x}_2).$$

**Theorem 1.2** If  $f \in \text{Aut}^+(\mathbb{R})$  and  $F = G[f]$ , then  $F$  has a continuous extension to the closure  $\overline{\mathbb{H}}$ ,  $\hat{F} : \overline{\mathbb{H}} \rightarrow \overline{\mathbb{H}}$  such that  $\hat{F}|_{\mathbb{R}} = f$  and  $\hat{F}|_{\mathbb{H}} = F$ .

**Proof.** We can calculate, using (1.1), for  $\xi \in \mathbb{R}$

$$\begin{aligned} \hat{F}(\xi) &:= \lim_{\mathbb{H} \ni z \rightarrow \xi \in \mathbb{R}} F(z) = \lim_{\mathbb{H} \ni x+iy \rightarrow \xi \in \mathbb{R}} \frac{(1-i)}{2} (f(x-y) - if(x+y)) \\ &= f(\xi) = \frac{1}{2}f(\xi)(1-i)(1+i) = f(\xi). \end{aligned}$$

■

**Theorem 1.3** If  $f \in \text{Aut}^+(\mathbb{R})$  and  $F = G[f]$ , then  $F \in \text{Aut}^+(\mathbb{H})$ .

**Proof.** It is enough to show that:  $F(\mathbb{H}) = \mathbb{H}$ ,  $F = G[f]$  is injective,  $F = G[f]$  is a sense-preserving mapping.

From (1.2) it is obviously that  $F(\mathbb{H}) \subset \mathbb{H}$ . Now let  $w = u + iv \in \mathbb{H}$  be an arbitrary point on the upper half-plane, because  $f \in \text{Aut}^+(\mathbb{R})$  then exists such points  $x_1, x_2 \in \mathbb{R}$ ,  $x_2 > x_1$  that  $f(x_1) = u - v$  and  $f(x_2) = u + v$ . Then for  $z = \frac{x_1+x_2}{2} + i\frac{x_2-x_1}{2}$  we have  $F(z) = w$  and  $\mathbb{H} \subset F(\mathbb{H})$ . Therefore  $F(\mathbb{H}) = \mathbb{H}$ .

Next by the simply calculation it is easy to verify that the mapping  $G[F]$  is an injective mapping.

Finally, we note that for a sense-preserving mapping  $f$  by Theorem 1.2 the mapping  $F$  is a sense-preserving also. ■

## 2 Properties of the operator $G$

Lets denote

$$\mathcal{M} = \{h : \mathbb{C} \rightarrow \mathbb{C} : h(z) = az + b \text{ for some } a, b \in \mathbb{R}\}.$$

Note that  $\mathcal{M}$  is a set of all conformal mappings fixed a point in infinity and  $h(\mathbb{R}) = \mathbb{R}$ .

**Theorem 2.1** *If  $h \in \text{Aut}^+(\mathbb{R})$  and  $h \in \mathcal{M}$ , then  $G[h] = h$ .*

**Proof.** Because  $h$  has the form  $h(x) = ax+b$  for some  $a, b \in \mathbb{R}$ , then for  $z = x+iy \in \mathbb{H}$  using (1.2) we can calculate

$$\begin{aligned} G[h](x+iy) &= \frac{1}{2}(f(x+y) + f(x-y)) + \frac{1}{2}i(f(x+y) - f(x-y)) \\ &= \frac{1}{2}(a(x+y) + b + a(x-y) + b) + \frac{1}{2}i(a(x+y) + b - a(x-y) - b) \\ &= a(x+iy) + b = h(x+iy). \end{aligned}$$

■

**Theorem 2.2** *If  $f, g \in \text{Aut}^+(\mathbb{R})$ , then*

$$G[f_1 \circ f_2] = G[f_1] \circ G[f_2].$$

**Proof.** Let  $x+iy \in \mathbb{H}$ , using (1.2) and (1.1) we can calculate that

$$\begin{aligned} G[f_1] \circ G[f_2](x+iy) &= G[f_1] \left( \frac{1}{2}(f_2(x+y) + f_2(x-y)) + \frac{1}{2}i(f_2(x+y) - f_2(x-y)) \right) \\ &= \frac{1}{2}(1-i)(f_1(f_2(x-y)) + if_1(f_2(x+y))) \\ &= G[f_1 \circ f_2](x+iy). \end{aligned}$$

■

From Theorem 2.1 and Theorem 2.2 follows:

**Corollary 2.1** *Extension operator  $G$  is conformally natural, i.e.*

$$G[h_1 \circ f \circ h_2] = h_1 \circ G[f] \circ h_2$$

for all  $f \in \text{Aut}^+(\mathbb{R})$ ,  $h_1, h_2 \in \mathcal{M}$ .

**Corollary 2.2** *If  $f \in \text{Aut}^+(\mathbb{R})$ , then  $G[f^{-1}] = (G[f])^{-1}$ .*

### 3 Bilipschitz property of the operator $G$

For arbitrary  $D \subset \mathbb{C}$ ,  $a > 0$ ,  $m \geq 1$  we denote

$$\mathcal{L}_D(a, m) := \left\{ f : D \rightarrow D : \frac{a}{m} |t_2 - t_1| \leq |f(t_2) - f(t_1)| \leq am |t_2 - t_1|, t_1, t_2 \in D \right\}, \quad (3.1)$$

$$\mathcal{L}_D(m) := \mathcal{L}_D(1, m). \quad (3.2)$$

The set  $\mathcal{L}_D(m)$  is the set of  $m$ -bilipschitz mapping.

Of course

$$\mathcal{L}_D(a, m) \subset \mathcal{L}_D(\mu), \quad \text{for } \mu = \max \left\{ \frac{m}{a}, am \right\}. \quad (3.3)$$

Note that from Theorem 1.2 we have immediately

**Theorem 3.1** *If  $f \in \text{Aut}^+(\mathbb{R})$  and  $F = G[f] \in \mathcal{L}_{\mathbb{H}}(a, m)$  for some  $m \geq 1$  and  $a > 0$ , then also  $f \in \mathcal{L}_{\mathbb{R}}(a, m)$ .*

**Lemma 3.1** *If  $f \in \mathcal{L}_{\mathbb{R}}(a, m) \cap \text{Aut}^+(\mathbb{R})$ , then*

$$\frac{a}{m} \leq f'(t) \leq am \quad (3.4)$$

$$\frac{1}{m^2} \leq \frac{f'(t_2)}{f'(t_1)} \leq m^2 \quad (3.5)$$

for all  $t, t_1, t_2 \in \mathcal{D}_{f'}$ , where  $\mathcal{D}_{f'}$  is the set of differentiability for the function  $f$ .

**Theorem 3.2** *If  $f \in \mathcal{L}_{\mathbb{R}}(a, m) \cap \text{Aut}^+(\mathbb{R})$  for some  $a > 0$  and  $m \geq 1$ , then*

$$F = G[f] \in \mathcal{L}_{\mathbb{H}}(a, m).$$

**Proof.** Let  $z_k = x_k + iy_k$ ,  $k = 1, 2$ , we can calculate

$$\begin{aligned} |F(z_1) - F(z_2)| &= \left| \frac{1}{2} (1 - i) (f(x_1 - y_1) + if(x_1 + y_1)) \right. \\ &\quad \left. - \frac{1}{2} (1 - i) (f(x_2 - y_2) + if(x_2 + y_2)) \right| \\ &= \frac{\sqrt{2}}{2} \left| f(x_1 - y_1) - f(x_2 - y_2) + i(f(x_1 + y_1) - f(x_2 + y_2)) \right| \\ &= \frac{\sqrt{2}}{2} \sqrt{(f(x_1 - y_1) - f(x_2 - y_2))^2 + (f(x_1 + y_1) - f(x_2 + y_2))^2} \\ &\leq am \frac{\sqrt{2}}{2} \sqrt{(x_1 - y_1 - x_2 + y_2)^2 + (x_1 + y_1 - x_2 - y_2)^2} \\ &= am \frac{\sqrt{2}}{2} \sqrt{2} |z_1 - z_2| = am |z_1 - z_2|. \end{aligned}$$



and

$$\begin{aligned}
|F(z_1) - F(z_2)| &= \left| \frac{1}{2} (1-i) (f(x_1 - y_1) + if(x_1 + y_1)) \right. \\
&\quad \left. - \frac{1}{2} (1-i) (f(x_2 - y_2) + if(x_2 + y_2)) \right| \\
&\geq \frac{a}{m} \frac{\sqrt{2}}{2} \sqrt{(x_1 - y_1 - x_2 + y_2)^2 + (x_1 + y_1 - x_2 - y_2)^2} \\
&= \frac{a}{m} \frac{\sqrt{2}}{2} \sqrt{2(x_1 - x_2)^2 + 2(y_1 - y_2)^2} = \frac{a}{m} |z_1 - z_2|.
\end{aligned}$$

■

**Corollary 3.1** *If  $f \in \mathcal{L}_{\mathbb{R}}(m) \cap \text{Aut}^+(\mathbb{R})$  and  $m \geq 1$ , then  $F = G[f] \in \mathcal{L}_{\mathbb{H}}(m)$ .*

**Proof.** From (3.2) we have

$$f \in \mathcal{L}_{\mathbb{R}}(1, m).$$

Using Theorem 3.2 and (3.3) we obtain

$$F \in \mathcal{L}_{\mathbb{H}}(1, m) = \mathcal{L}_{\mathbb{H}}(m).$$

■

## 4 Quasiconformal property of the operator $G$

Recall that we say that a mapping  $F : D \rightarrow D_1$ , where  $D, D_1 \subset \widehat{\mathbb{C}}$ , is  $K$ -quasiconformal if it satisfies two conditions:

1. The map  $F$  has ACL property, that means that  $f$  is absolutely continuous on a.e. horizontal and a.e. vertical segments in every rectangle

$$P = \{(x, y) : a_1 \leq x \leq a_2, b_1 \leq y \leq b_2\} \subset D.$$

2. There exists a constant  $K < \infty$  such that

$$\frac{1}{K} \leq \frac{|\partial F(z)| - |\bar{\partial} F(z)|}{|\partial F(z)| + |\bar{\partial} F(z)|} \leq K \quad \text{a.e. } z \in D. \quad (4.1)$$

Family of  $K$ -quasiconformal mappings of the domain  $D$  onto  $D_1$  we denote by  $Q_{D, D_1}(K)$ . If  $D = D_1$  we write  $Q_D(K) := Q_{D, D}(K)$ .

The condition (4.1) is called the dilatation condition for  $K$ -quasiconformal mapping. We can replace the dilatation condition by the other condition, see [4]

$$|\kappa_F(z)| \leq k, \quad \text{where } \kappa_F(z) := \frac{|\bar{\partial} F(z)|}{|\partial F(z)|}, \quad \text{and } k = \frac{K-1}{K+1}. \quad (4.2)$$

**Theorem 4.1** *Let  $f \in \mathcal{L}_{\mathbb{R}}(a, m) \cap \text{Aut}^+(\mathbb{R})$  for some  $a > 0$  and  $m \geq 1$ . Then  $F = G[f]$  is  $K$ -quasiconformal mapping on  $\mathbb{H}$  for  $K = m^2$ , this means that  $F \in Q_{\mathbb{H}}(m^2)$ .*

**Proof.** First note that  $f'(t)$  exist for  $t \in \mathcal{D}_{f'} = \mathbb{R} \setminus I$  and one-dimensional Euclidean measure if  $I$  is equal zero,  $|I|_1 = 0$  so  $F(z)$  is differentiable for  $z = x + iy \in \mathcal{D}_F = \mathbb{H} \setminus I^*$ , where

$$I^* = \{x + iy \in \mathbb{H}, x + y \in E \text{ or } x - y \in E\}$$

By  $|I|_1 = 0$  we have  $|I^*|_2 = 0$ .

It is enough to prove that

$$\left| \frac{\bar{\partial}F(z)}{\partial F(z)} \right| \leq \frac{m^2 - 1}{m^2 + 1} \quad (4.3)$$

for all  $z = x + iy \in \mathcal{D}_F$ . We can calculate, using (1.4)

$$\begin{aligned} \bar{\partial}F(z) &= \alpha \alpha f'(\bar{\alpha}z + \alpha\bar{z}) + \bar{\alpha} \bar{\alpha} f'(\alpha z + \bar{\alpha}\bar{z}) \\ &= \frac{1}{2} i f'(\bar{\alpha}z + \alpha\bar{z}) - \frac{1}{2} i f'(\alpha z + \bar{\alpha}\bar{z}) \end{aligned} \quad (4.4)$$

$$\begin{aligned} \bar{\partial}F(z) &= \alpha \bar{\alpha} f'(\bar{\alpha}z + \alpha\bar{z}) + \bar{\alpha} \alpha f'(\alpha z + \bar{\alpha}\bar{z}) \\ &= \frac{1}{2} f'(\bar{\alpha}z + \alpha\bar{z}) + \frac{1}{2} f'(\alpha z + \bar{\alpha}\bar{z}) \end{aligned} \quad (4.5)$$

and

$$\left| \frac{\bar{\partial}F(z)}{\partial F(z)} \right| = \left| \frac{f'(\bar{\alpha}z + \alpha\bar{z}) - f'(\alpha z + \bar{\alpha}\bar{z})}{f'(\bar{\alpha}z + \alpha\bar{z}) + f'(\alpha z + \bar{\alpha}\bar{z})} \right| = \left| \frac{\frac{f'(\bar{\alpha}z + \alpha\bar{z})}{f'(\alpha z + \bar{\alpha}\bar{z})} - 1}{\frac{f'(\bar{\alpha}z + \alpha\bar{z})}{f'(\alpha z + \bar{\alpha}\bar{z})} + 1} \right|.$$

Using (3.5) for  $t_1 = \alpha z + \bar{\alpha}\bar{z} = x - y$ ,  $t_2 = \bar{\alpha}z + \alpha\bar{z} = x + y$  we get (4.3). ■

**Remark 4.1** *If  $f \in \text{Aut}^+(\mathbb{R})$  and  $f$  is continuous on  $\mathbb{R}$  and such that*

$$m \leq f'(t) \leq M \quad \text{for a.e. on } \mathbb{R} \quad (4.6)$$

then

$$m |t_2 - t_1| \leq |f(t_2) - f(t_1)| \leq M |t_2 - t_1|. \quad (4.7)$$

**Proof.** From the fact that  $f \in \text{Aut}^+(\mathbb{R})$  and from continuity of  $f$  we have that  $f'(t)$  exists a.e. in  $\mathbb{R}$  and

$$f(t_2) - f(t_1) = \int_{t_1}^{t_2} f'(t) dt.$$

Using (4.6) we get (4.7). ■

**Theorem 4.2** *If  $f \in \text{Aut}^+(\mathbb{R})$  and  $F = G[f] \in Q_{\mathbb{H}}(K)$  for some  $K \geq 1$ , then*

$$B := \text{ess sup } f'(t) < +\infty, \quad b := \text{ess inf } f'(t) > 0 \quad (4.8)$$

and

$$f \in \mathcal{L}_{\mathbb{R}}(a, m) \quad (4.9)$$

for

$$a = \max \left\{ B, \frac{1}{b} \right\}, \quad m = K.$$

**Proof.** Using (4.4) and (4.5) we can calculate

$$\begin{aligned} J[F](z) &= |\partial F(z)|^2 - |\bar{\partial} F(z)|^2 = f'(\bar{\alpha}z + \alpha\bar{z}) f'(\alpha z + \bar{\alpha}\bar{z}) \\ &= f'(x-y) f'(x+y). \end{aligned} \quad (4.10)$$

By quasiconformality of  $F$  we have

$$J[f](z) > 0 \quad \text{for a. e. on } \mathbb{C}. \quad (4.11)$$

Combining the Jacobian form (4.10) with the condition (4.11) we get

$$f'(t) > 0 \quad \text{for a. e. on } \mathbb{R}. \quad (4.12)$$

From the assumption of quasiconformality of  $F$  we have also

$$\left| \frac{\bar{\partial} F(x+iy)}{\partial F(x+iy)} \right| = \left| \frac{\frac{f'(x-y)}{f'(x+y)} - 1}{\frac{f'(x-y)}{f'(x+y)} + 1} \right| \leq \frac{K-1}{K+1}$$

which is equivalent to

$$\begin{aligned} \frac{1}{K} &\leq \frac{f'(x-y)}{f'(x+y)} \leq K \\ \frac{1}{K} f'(x+y) &\leq f'(x-y) \leq K f'(x+y). \end{aligned}$$

Let us put  $x-y = s$  and  $x+y = t$  then

$$\frac{1}{K} f'(t) \leq |f'(s)| \leq K f'(t). \quad (4.13)$$

Fixing  $t$  in (4.13) and having (4.12) we get (4.8).

Taking into account the determination of  $b$  and  $B$  given in (4.8) we get

$$\frac{b}{K} \leq |f'(s)| \leq BK \quad \text{for a. e. on } \mathbb{R}. \quad (4.14)$$

Now we define

$$v(z) := \alpha z,$$

where  $\alpha$  is given in (1.3). Using (1.4) we see that  $G[f] \circ v$  for  $z = x+iy \in \mathbb{H}$  has the form

$$\begin{aligned} (G[f] \circ v)(z) &= G[f](\alpha z) = \alpha f(\bar{\alpha}\alpha z + \alpha\bar{\alpha}\bar{z}) + \bar{\alpha} f(\alpha\alpha z + \bar{\alpha}\bar{\alpha}\bar{z}) \\ &= \alpha f(x) + \bar{\alpha} f(y). \end{aligned}$$

Because  $v$  is a conformal mapping on a domain  $D = \{u + iv : u, v \in \mathbb{R}, v > u\}$  and  $G[f]$  is a quasiconformal mapping on the domain  $v(D) \subset \mathbb{H}$  so  $G[f] \circ v$  is a quasiconformal mapping on  $D$ . From quasiconformality of the mapping  $G[f] \circ v$  we know that  $G[f] \circ v$  has ACL property and consequently the mapping  $f$  is absolutely continuous. From Remark 4.1 and the condition (4.14) we have

$$\frac{b}{K} |t_2 - t_1| \leq |f(t_2) - f(t_1)| \leq BK |t_2 - t_1|$$

which is equivalent to (4.9). ■

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## Structure of approximate solutions for a class of optimal control systems

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ABSTRACT: We study a turnpike property of approximate solutions of a discrete-time control system with a compact metric space of states which arises in economic dynamics. To have this property means that the approximate solutions of the optimal control problems are determined mainly by an objective function, and are essentially independent of the length of the interval, for all sufficiently large intervals. We show that the turnpike property is stable under perturbations of an objective function.

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### 1 Introduction

Let  $(X, \rho)$  be a compact metric space and  $\Omega$  be a nonempty closed subset of  $X \times X$ .

A sequence  $\{x_t\}_{t=0}^{\infty} \subset X$  is called a program if  $(x_t, x_{t+1}) \in \Omega$  for all integers  $t \geq 0$ . A sequence  $\{x_t\}_{t=T_1}^{T_2} \subset X$  where integers  $T_1, T_2$  satisfy  $0 \leq T_1 < T_2$  is called a program if  $(x_t, x_{t+1}) \in \Omega$  for all integers  $t \in [T_1, T_2 - 1]$ .

In this paper we consider the problem

$$\sum_{i=0}^{T-1} v(x_i, x_{i+1}) \rightarrow \max \quad (\text{P})$$

$$\text{s. t. } \{(x_i, x_{i+1})\}_{i=0}^{T-1} \subset \Omega, \quad x_0 = z_1, \quad x_T = z_2,$$

where  $T$  is a natural number,  $z_1, z_2 \in X$  and  $v : \Omega \rightarrow R^1$  is a bounded function. This discrete-time optimal control system describes a general model of economic dynamics [3, 7, 9, 13-15], where the set  $X$  is the space of states,  $v$  is a utility function and  $v(x_t, x_{t+1})$  evaluates consumption at moment  $t$ . The interest in discrete-time optimal

problems of type (P) also stems from the study of various optimization problems which can be reduced to it, e.g., tracking problems in engineering [5], the study of Frenkel-Kontorova model related to dislocations in one-dimensional crystals [1] and the analysis of a long slender bar of a polymeric material under tension in [6]. Optimization problems of the type (P) with  $\Omega = X \times X$  were considered in [10-12].

We are interested in a turnpike property of the approximate solutions of (P) which is independent of the length of the interval  $T$ , for all sufficiently large intervals. To have this property means, roughly speaking, that the approximate solutions of the optimal control problems are determined mainly by the cost function  $v$ , and are essentially independent of  $T$ ,  $z_1$  and  $z_2$ . Turnpike properties are well known in mathematical economics. The term was first coined by Samuelson in 1948 (see [9]) where he showed that an efficient expanding economy would spend most of the time in the vicinity of a balanced equilibrium path (also called a von Neumann path).

It should be mentioned that the study of the existence and the structure of solutions of optimal control problems defined on infinite intervals and on sufficiently large intervals has recently been a rapidly growing area of research. See, for example, [2, 6, 8, 10-15] and the references mentioned therein.

In the classical turnpike theory [3, 7, 9] the space  $X$  is a compact convex subset of a finite-dimensional Euclidean space, the set  $\Omega$  is convex and the function  $v$  is strictly concave. Under these assumptions the turnpike property can be established and the turnpike  $\bar{x}$  is a unique solution of the maximization problem  $v(x, x) \rightarrow \max, (x, x) \in \Omega$ . In this situation it is shown that for each program  $\{x_t\}_{t=0}^{\infty}$  either the sequence  $\{\sum_{t=0}^{T-1} v(x_t, x_{t+1}) - Tv(\bar{x}, \bar{x})\}_{T=1}^{\infty}$  is bounded (in this case the program  $\{x_t\}_{t=0}^{\infty}$  is called ( $v$ )-good) or it diverges to  $-\infty$ . Moreover, it is also established that any ( $v$ )-good program converges to the turnpike  $\bar{x}$ . In the sequel this property is called as the asymptotic turnpike property.

In [14] we showed that the turnpike property follows from the asymptotic turnpike property. More precisely, we assumed that any ( $v$ )-good program converges to a unique solution  $\bar{x}$  of the problem  $v(x, x) \rightarrow \max, (x, x) \in \Omega$  and showed that the turnpike property holds and  $\bar{x}$  is the turnpike. Note that we do not use convexity (concavity) assumptions. It should be mentioned that in [13] analogous results were established for the problem

$$\sum_{i=0}^{T-1} v(x_i, x_{i+1}) \rightarrow \max, \{(x_i, x_{i+1})\}_{i=0}^{T-1} \subset \Omega, x_0 = z,$$

where  $T$  is a natural number and  $z \in X$ .

In the present paper we improve the turnpike results established in [13, 14] and show that the turnpike property is stable under perturbations of the objective function  $v$ .

Let  $(X, \rho)$  be a compact metric space and  $\Omega$  be a nonempty closed subset of  $X \times X$ . Denote by  $\mathcal{M}$  the set of all bounded functions  $u : \Omega \rightarrow R^1$ . For each  $w \in \mathcal{M}$  set

$$\|w\| = \sup\{|w(x, y)| : (x, y) \in \Omega\}. \quad (1.1)$$

For each  $x, y \in X$ , each integer  $T \geq 1$  and each  $u \in \mathcal{M}$  set

$$\sigma(u, T, x) = \sup\left\{\sum_{i=0}^{T-1} u(x_i, x_{i+1}) : \{x_i\}_{i=0}^T \text{ is a program and } x_0 = x\right\}, \quad (1.2)$$

$$\sigma(u, T, x, y) = \sup\left\{\sum_{i=0}^{T-1} u(x_i, x_{i+1}) : \{x_i\}_{i=0}^T \text{ is a program and } x_0 = x, x_T = y\right\}, \quad (1.3)$$

$$\sigma(u, T) = \sup\left\{\sum_{i=0}^{T-1} u(x_i, x_{i+1}) : \{x_i\}_{i=0}^T \text{ is a program}\right\}. \quad (1.4)$$

(Here we use the convention that the supremum of an empty set is  $-\infty$ ).

For each  $x, y \in X$ , each pair of integers  $T_1, T_2$  satisfying  $0 \leq T_1 < T_2$  and each sequence  $\{u_t\}_{t=T_1}^{T_2-1} \subset \mathcal{M}$  set

$$\begin{aligned} \sigma(\{u_t\}_{t=T_1}^{T_2-1}, T_1, T_2, x) &= \sup\left\{\sum_{t=T_1}^{T_2-1} u_t(x_t, x_{t+1}) : \right. \\ &\left. \{x_t\}_{t=T_1}^{T_2} \text{ is a program and } x_{T_1} = x\right\}, \end{aligned} \quad (1.5)$$

$$\begin{aligned} \sigma(\{u_t\}_{t=T_1}^{T_2-1}, T_1, T_2, x, y) &= \sup\left\{\sum_{t=T_1}^{T_2-1} u_t(x_t, x_{t+1}) : \right. \\ &\left. \{x_t\}_{t=T_1}^{T_2} \text{ is a program and } x_{T_1} = x, x_{T_2} = y\right\}, \end{aligned} \quad (1.6)$$

$$\sigma(\{u_t\}_{t=T_1}^{T_2-1}, T_1, T_2) = \sup\left\{\sum_{t=T_1}^{T_2-1} u_t(x_t, x_{t+1}) : \{x_t\}_{t=T_1}^{T_2} \text{ is a program}\right\}. \quad (1.7)$$

Assume that  $v \in \mathcal{M}$  is an upper semicontinuous function. Since in [13, 14] we assume that objective functions are defined on the set  $X \times X$  in order to apply their results we set  $v(x, y) = -\|v\| - 1$  for all  $(x, y) \in (X \times X) \setminus \Omega$ .

We suppose that there exist  $\bar{x} \in X$  and a constant  $\bar{c} > 0$  such that the following assumptions hold.

(A1)  $(\bar{x}, \bar{x})$  is an interior point of  $\Omega$  (there is  $\epsilon > 0$  such that  $\{(x, y) \in X \times X : \rho(x, \bar{x}), \rho(y, \bar{x}) \leq \epsilon\} \subset \Omega$ ) and  $v$  is continuous at  $(\bar{x}, \bar{x})$ .

(A2)  $\sigma(v, T) \leq Tv(\bar{x}, \bar{x}) + \bar{c}$  for all integers  $T \geq 1$ .

It is easy to see that for each natural number  $T$  and each program  $\{x_t\}_{t=0}^T$

$$\sum_{t=0}^{T-1} v(x_t, x_{t+1}) \leq \sigma(v, T) \leq Tv(\bar{x}, \bar{x}) + \bar{c}. \quad (1.8)$$

Inequality (1.8) implies the following result.

**Proposition 1.1** For each program  $\{x_t\}_{t=0}^\infty$  either the sequence

$$\left\{ \sum_{t=0}^{T-1} v(x_t, x_{t+1}) - Tv(\bar{x}, \bar{x}) \right\}_{T=1}^\infty$$

is bounded or  $\lim_{T \rightarrow \infty} [\sum_{t=0}^{T-1} v(x_t, x_{t+1}) - Tv(\bar{x}, \bar{x})] = -\infty$ .

A program  $\{x_t\}_{t=0}^\infty$  is called  $(v)$ -good if the sequence

$$\left\{ \sum_{t=0}^{T-1} v(x_t, x_{t+1}) - Tv(\bar{x}, \bar{x}) \right\}_{T=1}^\infty$$

is bounded [3, 4, 12].

In this paper we suppose that the following assumption holds.

(A3) (the asymptotic turnpike property) For any  $(v)$ -good program

$$\{x_t\}_{t=0}^\infty, \quad \lim_{t \rightarrow \infty} \rho(x_t, \bar{x}) = 0.$$

Note that (A3) holds for many important infinite horizon optimal control problems. See, for example, [13-15]. In particular, (A3) holds for a general model of economic dynamics.

By (A3)  $\|v\| > 0$ . For each  $M > 0$  denote by  $X_M$  the set of all  $x \in X$  for which there exists a program  $\{x_t\}_{t=0}^\infty$  such that  $x_0 = x$  and that for all integers  $T \geq 1$

$$\sum_{t=0}^{T-1} v(x_t, x_{t+1}) - Tv(\bar{x}, \bar{x}) \geq -M.$$

Clearly  $\cup\{X_M : M \in (0, \infty)\}$  is the set of all  $x \in X$  for which there exists a  $(v)$ -good program  $\{x_t\}_{t=0}^\infty$  such that  $x_0 = x$ .

By (A1) there exists  $\bar{r} \in (0, 1)$  such that

$$\{(x, y) \in X \times X : \rho(x, \bar{x}), \rho(y, \bar{x}) \leq \bar{r}\} \subset \Omega. \quad (1.9)$$

Let  $T$  be a natural number. Denote by  $Y_T$  the set of all  $x \in X$  for which there exists a program  $\{x_t\}_{t=0}^T$  such that  $x_0 = \bar{x}$  and  $x_T = x$ .

Denote by  $\bar{Y}_T$  the set of all  $x \in X$  for which there exists a program  $\{x_t\}_{t=0}^T$  such that  $x_0 = x$  and  $x_T = \bar{x}$ .

It is easy to see that the following result holds.

**Proposition 1.2** Let  $L$  be a natural number. Then  $\bar{Y}_L \subset X_{L\|v\|}$ .

**Proposition 1.3** Let  $M > 0$ . Then there exists a natural number  $L$  such that  $X_M \subset Y_L$ .

For the proof of Proposition 1.3 see Lemma 2.1.

The following three theorems which describe the structure of approximate solutions of our discrete-time control system are our main results.



**Theorem 1.1** *Let  $\epsilon \in (0, 1)$  and  $M > 0$ . Then there exist a natural number  $L_0$  and  $\delta_0 \in (0, \min\{\epsilon, \bar{r}\})$  such that for each integer  $L_1 \geq L_0$  the following assertion holds with  $\delta = \delta_0(4L_1)^{-1}$ .*

Assume that an integer  $T > 2L_1$ ,  $\{u_t\}_{t=0}^{T-1} \subset \mathcal{M}$ , a program  $\{x_t\}_{t=0}^T$  and a finite sequence of integers  $\{S_i\}_{i=0}^q$  satisfy

$$\|u_t - v\| \leq \delta, \quad t = 0, \dots, T-1,$$

$$S_0 = 0, \quad S_{i+1} - S_i \in [L_0, L_1], \quad i = 0, \dots, q-1, \quad S_q > T - L_1,$$

$$\sum_{t=S_i}^{S_{i+1}-1} u_t(x_t, x_{t+1}) \geq \sum_{t=S_i}^{S_{i+1}-1} u_t(\bar{x}, \bar{x}) - M$$

for each integer  $i \in [0, q-1]$ ,

$$\sum_{t=S_i}^{S_{i+2}-1} u_t(x_t, x_{t+1}) \geq \sigma(\{u_t\}_{t=S_i}^{S_{i+2}-1}, S_i, S_{i+2}, x_{S_i}, x_{S_{i+2}}) - \delta_0 \quad (1.10)$$

for each integer  $i \in [0, q-2]$  and

$$\sum_{t=S_{q-2}}^{T-1} u_t(x_t, x_{t+1}) \geq \sigma(\{u_t\}_{t=S_{q-2}}^{T-1}, S_{q-2}, T, x_{S_{q-2}}, x_T) - \delta_0. \quad (1.11)$$

Then there exist integers  $\tau_1, \tau_2 \in [0, T]$  such that  $\tau_1 \leq 2L_0$ ,  $\tau_2 > T - 2L_1$  and

$$\rho(x_t, \bar{x}) \leq \epsilon, \quad t = \tau_1, \dots, \tau_2.$$

Moreover, if  $\rho(x_0, \bar{x}) \leq \delta_0$ , then  $\tau_1 = 0$  and if  $\rho(x_T, \bar{x}) \leq \delta_0$ , then  $\tau_2 = T$ .

**Theorem 1.2** *Let  $\epsilon \in (0, \bar{r})$ ,  $L_0$  be a natural number and  $M_0 > 0$ . Then there exist a natural number  $L$  and  $\delta \in (0, \epsilon)$  such that for each integer  $T > 2L$ , each  $\{u_t\}_{t=0}^{T-1} \subset \mathcal{M}$  satisfying*

$$\|u_t - v\| \leq \delta, \quad t = 0 \dots T-1,$$

and each program  $\{x_t\}_{t=0}^T$  which satisfies

$$x_0 \in \bar{Y}_{L_0}, \quad x_T \in Y_{L_0},$$

$$\sum_{t=0}^{T-1} u_t(x_t, x_{t+1}) \geq \sigma(\{u_t\}_{t=0}^{T-1}, 0, T, x_0, x_T) - M_0$$

and

$$\sum_{t=\tau}^{\tau+L-1} u_t(x_t, x_{t+1}) \geq \sigma(\{u_t\}_{t=\tau}^{\tau+L-1}, \tau, \tau+L, x_\tau, x_{\tau+L}) - \delta \quad (1.12)$$

for each integer  $\tau \in [0, T-L]$  there exist integers  $\tau_1 \in [0, L]$ ,  $\tau_2 \in [T-L, T]$  such that

$$\rho(x_t, \bar{x}) \leq \epsilon, \quad t = \tau_1, \dots, \tau_2.$$

Moreover, if  $\rho(x_0, \bar{x}) \leq \delta$ , then  $\tau_1 = 0$  and if  $\rho(x_T, \bar{x}) \leq \delta$ , then  $\tau_2 = T$ .

**Theorem 1.3** Let  $\epsilon \in (0, \bar{r})$ ,  $L_0$  be a natural number and  $M_0 > 0$ . Then there exist a natural number  $L$  and  $\delta \in (0, \epsilon)$  such that for each integer  $T > 2L$ , each  $\{u_t\}_{t=0}^{T-1} \subset \mathcal{M}$  satisfying

$$\|u_t - v\| \leq \delta, \quad t = 0, \dots, T-1$$

and each program  $\{x_t\}_{t=0}^T$  which satisfies

$$x_0 \in \bar{Y}_{L_0}, \quad \sum_{t=0}^{T-1} u_t(x_t, x_{t+1}) \geq \sigma(\{u_t\}_{t=0}^{T-1}, 0, T, x_0) - M_0$$

and

$$\sum_{t=\tau}^{\tau+L-1} u_t(x_t, x_{t+1}) \geq \sigma(\{u_t\}_{t=\tau}^{\tau+L-1}, \tau, \tau+L, x_\tau, x_{\tau+L}) - \delta \quad (1.13)$$

for each integer  $\tau \in [0, T-L]$  there exist integers  $\tau_1 \in [0, L]$ ,  $\tau_2 \in [T-L, T]$  such that

$$\rho(x_t, \bar{x}) \leq \epsilon, \quad t = \tau_1, \dots, \tau_2.$$

Moreover if  $\rho(x_0, \bar{x}) \leq \delta$ , then  $\tau_1 = 0$ .

Theorems 1.1-1.3 establish the turnpike property for approximate solutions of the optimal control problems with objective functions  $u_t$ ,  $t = 0, \dots, T-1$  which belong to a small neighborhood of  $v$ . They extend the main results of [15] which were obtained when  $u_t = u_0$  for all  $t = 0, \dots, T-1$  and when equations (1.10)-(1.13) were replaced by the stronger equations

$$\sum_{t=0}^{T-1} u_t(x_t, x_{t+1}) \geq \sigma(\{u_t\}_{t=0}^{T-1}, 0, T, x_0, x_T) - \delta$$

and

$$\sum_{t=0}^{T-1} u_t(x_t, x_{t+1}) \geq \sigma(\{u_t\}_{t=0}^{T-1}, 0, T, x_0) - \delta$$

respectively.

Note that examples of pairs  $(v, \Omega)$  for which the assumptions made in this paper hold are presented in [15].

The paper is organized as follows. Section 2 contains auxiliary results. Theorem 1.1 is proved in Section 3 while Section 4 contains the proof of Theorems 1.2 and 1.3.

## 2 Auxiliary results

By (A1) we may assume that

$$|v(x, y) - v(\bar{x}, \bar{x})| \leq 1/8 \text{ for all } x, y \in X \text{ satisfying } \rho(x, \bar{x}), \rho(y, \bar{x}) \leq \bar{r} \quad (2.1)$$

(see (1.9)). Clearly, for each pair of integers  $T_1, T_2$  satisfying  $0 \leq T_1 < T_2$ , each sequence  $\{w_t\}_{t=T_1}^{T_2-1} \subset \mathcal{M}$  and each  $x, y \in X$  satisfying  $\rho(x, \bar{x}), \rho(y, \bar{x}) \leq \bar{r}$  we have that  $\sigma(\{w_t\}_{t=T_1}^{T_2-1}, T_1, T_2, x, y)$  is finite.

In order to prove our main results we need the following lemmas obtained in [14].

**Lemma 2.1** [14, Lemma 3.3] *Let  $M_0, \epsilon$  be positive numbers. Then there exists a natural number  $L_0$  such that for each integer  $T \geq L_0$ , each program  $\{x_t\}_{t=0}^T$  which satisfies*

$$\sum_{t=0}^{T-1} v(x_t, x_{t+1}) \geq Tv(\bar{x}, \bar{x}) - M_0$$

and each integer  $S \in [0, T - L_0]$  the inequality

$$\min\{\rho(x_t, \bar{x}) : t = S + 1, \dots, S + L_0\} \leq \epsilon$$

holds.

Note that Lemma 2.1 implies Proposition 1.3.

**Lemma 2.2** [14, Lemma 3.2]. *Let  $\epsilon > 0$ . Then there exists  $\delta \in (0, \bar{r})$  such that for each integer  $T \geq 1$  and each program  $\{x_t\}_{t=0}^T$  satisfying*

$$\rho(x_0, \bar{x}), \rho(x_T, \bar{x}) \leq \delta, \quad \sum_{t=0}^{T-1} v(x_t, x_{t+1}) \geq \sigma(v, T, x_0, x_T) - \delta$$

the inequality  $\rho(x_t, \bar{x}) \leq \epsilon$  holds for all  $t = 0, \dots, T$ .

### 3 Proof of Theorem 1.1

By Lemma 2.2 there exists a positive number

$$\delta_0 < \min\{\epsilon, \bar{r}\} \tag{3.1}$$

such that the following property holds:

(P1) for each integer  $T \geq 1$  and each program  $\{x_t\}_{t=0}^T$  satisfying

$$\rho(x_0, \bar{x}), \rho(x_T, \bar{x}) \leq \delta_0, \quad \sum_{t=0}^{T-1} v(x_t, x_{t+1}) \geq \sigma(v, T, x_0, x_T) - 2\delta_0$$

the inequality  $\rho(x_t, \bar{x}) \leq \epsilon$  holds for all  $t = 0, \dots, T$ .

By Lemma 2.1 there exists a natural number  $L_0$  such that the following property holds:

(P2) for each integer  $T \geq L_0$ , each program  $\{x_t\}_{t=0}^T$  which satisfies

$$\sum_{t=0}^{T-1} v(x_t, x_{t+1}) \geq Tv(\bar{x}, \bar{x}) - M - 2$$

and each integer  $S \in [0, T - L_0]$  the inequality

$$\min\{\rho(x_t, \bar{x}) : t = S + 1, \dots, S + L_0\} \leq \delta_0$$

holds.

Let an integer

$$L_1 \geq L_0 \quad (3.2)$$

and put

$$\delta = \delta_0(4L_1)^{-1}. \quad (3.3)$$

Assume that an integer  $T > 2L_1$ ,  $\{u_t\}_{t=0}^{T-1} \subset \mathcal{M}$  satisfies

$$\|u_t - v\| \leq \delta, \quad t = 0, \dots, T - 1, \quad (3.4)$$

$\{x_t\}_{t=0}^T$  is a program and that  $\{S_i\}_{i=0}^q$  is a sequence of integers such that

$$S_0 = 0, \quad S_{i+1} - S_i \in [L_0, L_1], \quad i = 0, \dots, q - 1, \quad S_q > T - L_1, \quad (3.5)$$

for each integer  $i \in [0, q - 1]$

$$\sum_{t=S_i}^{S_{i+1}-1} u_t(x_t, x_{t+1}) \geq \sum_{t=S_i}^{S_{i+1}-1} u_t(\bar{x}, \bar{x}) - M, \quad (3.6)$$

for each integer  $i \in [0, q - 2]$

$$\sum_{t=S_i}^{S_{i+2}-1} u_t(x_t, x_{t+1}) \geq \sigma(\{u_t\}_{t=S_i}^{S_{i+2}-1}, S_i, S_{i+2}, x_{S_i}, x_{S_{i+2}}) - \delta_0, \quad (3.7)$$

$$\sum_{t=S_{q-2}}^{T-1} u_t(x_t, x_{t+1}) \geq \sigma(\{u_t\}_{t=S_{q-2}}^{T-1}, S_{q-2}, T, x_{S_{q-2}}, x_T) - \delta_0. \quad (3.8)$$

Let an integer  $i \in [0, q - 1]$ . By (3.3)-(3.6)

$$\begin{aligned} \sum_{t=S_i}^{S_{i+1}-1} v(x_t, x_{t+1}) &\geq \sum_{t=S_i}^{S_{i+1}-1} u_t(x_t, x_{t+1}) - \delta(S_{i+1} - S_i) \\ &\geq \sum_{t=S_i}^{S_{i+1}-1} u_t(\bar{x}, \bar{x}) - M - \delta L_1 \\ &\geq v(\bar{x}, \bar{x})(S_{i+1} - S_i) - \delta L_1 - M - \delta L_1 = v(\bar{x}, \bar{x})(S_{i+1} - S_i) - M - 1. \end{aligned}$$

By the equation above, property (P2) and (3.5) there is an integer  $\tau_i$  such that

$$\tau_i \in [S_i + 1, S_i + L_0], \quad \rho(x_{\tau_i}, \bar{x}) \leq \delta_0. \quad (3.9)$$

Thus for each integer  $i \in [0, q-1]$  there is an integer  $\tau_i$  satisfying (3.9). By (3.9) and (3.5)

$$\tau_0 \leq 2L_0, \tau_{q-1} > T - 2L_1. \quad (3.10)$$

For each integer  $i \in [0, q-2]$

$$0 < \tau_{i+1} - \tau_i \leq 2L_1, \tau_i, \tau_{i+1} \in [S_i, S_{i+2}]. \quad (3.11)$$

By (3.7) for any integer  $i \in [0, q-2]$

$$\sum_{t=\tau_i}^{\tau_{i+1}-1} u_t(x_t, x_{t+1}) \geq \sigma(\{u_t\}_{t=\tau_i}^{\tau_{i+1}-1}, \tau_i, \tau_{i+1}, x_{\tau_i}, x_{\tau_{i+1}}) - \delta_0. \quad (3.12)$$

Thus we have shown that there is a finite sequence of integers  $\{\tau_i\}_{i=0}^p$  such that

$$0 \leq \tau_0 \leq 2L_0, T \geq \tau_p > T - 2L_1,$$

for each integer  $i$  satisfying  $0 \leq i < p$

$$1 \leq \tau_{i+1} - \tau_i \leq 2L_1 \quad (3.13)$$

and that (3.12) holds.

Clearly we may assume without loss of generality that if  $\rho(x_0, \bar{x}) \leq \delta_0$ , then  $\tau_0 = 0$  and if  $\rho(x_T, \bar{x}) \leq \delta_0$ , then  $\tau_p = 0$ .

Let an integer  $i \in \{0, \dots, p-1\}$ . By (3.4), (3.12), (3.13) and (3.3)

$$\begin{aligned} \sum_{t=\tau_i}^{\tau_{i+1}-1} v(x_t, x_{t+1}) &\geq \sum_{t=\tau_i}^{\tau_{i+1}-1} u_t(x_t, x_{t+1}) - \delta(\tau_{i+1} - \tau_i) \\ &\geq \sigma(\{u_t\}_{t=\tau_i}^{\tau_{i+1}-1}, \tau_i, \tau_{i+1}, x_{\tau_i}, x_{\tau_{i+1}}) - \delta_0 - \delta 2L_1 \\ &\geq \sigma(v, \tau_{i+1} - \tau_i, x_{\tau_i}, x_{\tau_{i+1}}) - \delta_0 - \delta 4L_1 \geq \sigma(v, \tau_{i+1} - \tau_i, x_{\tau_i}, x_{\tau_{i+1}}) - 2\delta_0. \end{aligned}$$

By the equation above, (3.9) and (P1),

$$\rho(x_t, \bar{x}) \leq \epsilon, \quad t = \tau_i, \dots, \tau_{i+1}, \quad i = 0, \dots, p-1.$$

Theorem 1.1 is proved.

## 4 Proofs of Theorems 1.2 and 1.3

We prove Theorems 1.2 and 1.3 simultaneously. Fix

$$M_1 > 4. \quad (4.1)$$

By Lemma 2.2 there exists a positive number  $\delta_0 < \epsilon$  such that the following property holds:

(P3) for each integer  $T \geq 1$  and each program  $\{x_t\}_{t=0}^T$  satisfying

$$\rho(x_0, \bar{x}), \rho(x_T, \bar{x}) \leq \delta_0, \quad \sum_{t=0}^{T-1} v(x_t, x_{t+1}) \geq \sigma(v, T, x_0, x_T) - 2\delta_0$$

the inequality  $\rho(x_t, \bar{x}) \leq \epsilon$  holds for all  $t = 0, \dots, T$ .

By Lemma 2.1 there exists a natural number  $L_1 > L_0 + 4$  such that the following property holds:

(P4) for each integer  $T \geq L_1$ , each program  $\{x_t\}_{t=0}^T$  which satisfies

$$\sum_{t=0}^{T-1} v(x_t, x_{t+1}) \geq Tv(\bar{x}, \bar{x}) - M_1 - 2$$

and each integer  $S \in [0, T - L_1]$  the inequality

$$\min\{\rho(x_t, \bar{x}) : t = S + 1, \dots, S + L_1\} \leq \delta_0$$

holds.

Choose a natural number  $k$  such that

$$k > 8L_1(\|v\| + 1) + M_0 + 4, \quad (4.2)$$

set

$$L_2 = kL_1 \quad (4.3)$$

and choose a natural number

$$L > 2L_2 \quad (4.4)$$

and a positive number  $\delta$  for which

$$8L_2\delta < \delta_0. \quad (4.5)$$

Assume that an integer  $T > 2L$ ,  $\{u_t\}_{t=0}^{T-1} \subset \mathcal{M}$  satisfies

$$\|u_t - v\| \leq \delta, \quad t = 0, \dots, T - 1 \quad (4.6)$$

and that a program  $\{x_t\}_{t=0}^T$  satisfies

$$x_0 \in \bar{Y}_{L_0}, \quad (4.7)$$

for each integer  $\tau \in [0, T - L]$

$$\sum_{t=\tau}^{\tau+L-1} u_t(x_t, x_{t+1}) \geq \sigma(\{u_t\}_{t=\tau}^{\tau+L-1}, \tau, \tau + L, x_\tau, x_{\tau+L}) - \delta \quad (4.8)$$

and that

$$x_T \in Y_{L_0}, \quad \sum_{t=0}^{T-1} u_t(x_t, x_{t+1}) \geq \sigma(\{u_t\}_{t=0}^{T-1}, 0, T, x_0, x_T) - M_0 \quad (4.9)$$

in the case of Theorem 1.2 and

$$\sum_{t=0}^{T-1} u_t(x_t, x_{t+1}) \geq \sigma(\{u_t\}_{t=0}^{T-1}, 0, T, x_0) - M_0 \quad (4.10)$$

in the case of Theorem 1.3.

Assume that an integer  $S$  satisfies

$$S \in [0, T - L_2], \quad x_S \in \bar{Y}_{L_0}. \quad (4.11)$$

We show that there is an integer  $t \in [S + 1, S + L_2]$  such that  $\rho(x_t, \bar{x}) \leq \delta_0$ .

Assume the contrary. Then

$$\rho(x_t, \bar{x}) > \delta_0, \quad t = S + 1, \dots, S + L_2. \quad (4.12)$$

There are two cases:

$$\rho(x_t, \bar{x}) > \delta_0 \text{ for all integers } t = S + 1, \dots, T; \quad (4.13)$$

$$\rho(x_t, \bar{x}) \leq \delta_0 \text{ for some integer } t \text{ satisfying } S + L_2 < t \leq T. \quad (4.14)$$

Assume that (4.13) holds. In the case of Theorem 1.3 in view of (4.11) there is a program  $\{y_t\}_{t=0}^T$  such that

$$y_t = x_t, \quad t = 0, \dots, S, \quad y_t = \bar{x} \text{ for all integers } t \in [S + L_0, T]. \quad (4.15)$$

In the case of Theorem 1.2 in view of (4.11) and (4.9) there is a program  $\{y_t\}_{t=0}^T$  such that

$$y_t = x_t, \quad t = 0, \dots, S, \quad y_t = \bar{x} \text{ for all integers } t \in [S + L_0, T - L_0], \quad y_T = x_T. \quad (4.16)$$

By (4.9), (4.10), (4.15) and (4.16)

$$-M_0 \leq \sum_{t=0}^{T-1} u_t(x_t, x_{t+1}) - \sum_{t=0}^{T-1} u_t(y_t, y_{t+1}) = \sum_{t=S}^{T-1} u_t(x_t, x_{t+1}) - \sum_{t=S}^{T-1} u_t(y_t, y_{t+1}). \quad (4.17)$$

By (4.11) there is an integer  $p \geq 0$  such that

$$T - S \in [pL_1, (p + 1)L_1]. \quad (4.18)$$

By (4.3), (4.11) and (4.18)

$$p \geq k. \quad (4.19)$$

By (4.13), (4.18) and (P4) for each integer  $i \in [0, p - 1]$

$$\sum_{t=S+iL_1}^{S+(i+1)L_1-1} v(x_t, x_{t+1}) \leq L_1 v(\bar{x}, \bar{x}) - M_1 - 2.$$

Together with (4.5) and (4.6) this implies that for each integer  $i \in [0, p-1]$

$$\begin{aligned} & \sum_{t=S+iL_1}^{S+(i+1)L_1-1} u_t(x_t, x_{t+1}) \leq \delta L_1 + L_1 v(\bar{x}, \bar{x}) - M_1 - 2 \\ & \leq 2\delta L_1 + \sum_{t=S+iL_1}^{S+(i+1)L_1-1} u_t(\bar{x}, \bar{x}) - M_1 - 2 \leq \sum_{t=S+iL_1}^{S+(i+1)L_1-1} u_t(\bar{x}, \bar{x}) - M_1 - 1. \end{aligned} \quad (4.20)$$

By (4.2), (4.3), (4.6), (4.16), (4.18), (4.19) and (4.20)

$$\begin{aligned} & \sum_{t=S}^{T-1} u_t(x_t, x_{t+1}) - \sum_{t=S}^{T-1} u_t(y_t, y_{t+1}) \\ & \leq \sum_{t=S}^{S+pL_1-1} u_t(\bar{x}, \bar{x}) - (M_1 + 1)p + \sum \{ \|u_t\| : t \text{ is an integer and} \\ & \quad S + pL_1 \leq t \leq T - 1 \} - \sum_{t=S+L_0}^{T-L_0-1} u_t(\bar{x}, \bar{x}) + 2L_0(\|v\| + 1) \\ & \leq \sum_{t=S}^{S+pL_1-1} u_t(\bar{x}, \bar{x}) - p(M_1 + 1) + L_1(\|v\| + 1) + 2L_0(\|v\| + 1) \\ & \quad - \sum_{t=S}^{T-1} u_t(\bar{x}, \bar{x}) + 2L_0(\|v\| + 1) \\ & \leq -p(M_1 + 1) + 6L_1(\|v\| + 1) \leq -k(M_1 + 1) + 6L_1(\|v\| + 1) \leq -M_0 - 4. \end{aligned}$$

This contradicts (4.17). The contradiction we have reached proves that (4.13) does not hold. Thus (4.14) holds.

We may assume without loss of generality that there is an integer  $\tilde{S}$  such that

$$S + L_2 < \tilde{S} \leq T, \quad \rho(x_{\tilde{s}}, \bar{x}) \leq \delta_0, \quad (4.21)$$

$$\rho(x_t, \bar{x}) > \delta_0 \text{ for all integers } t \text{ satisfying } S < t < \tilde{S}. \quad (4.22)$$

By (4.11) and (4.21) there is a program  $\{y_t\}_{t=0}^T$  such that

$$y_t = x_t, \quad t = 0, \dots, S, \quad y_t = \bar{x} \text{ for all integers } t \in [S + L_0, \tilde{S} - 1], \quad (4.23)$$

$$y_t = x_t, \text{ for all integers satisfying } \tilde{S} \leq t \leq T.$$

By (4.23), (4.10), (4.9)

$$-M_0 \leq \sum_{t=0}^{T-1} u_t(x_t, x_{t+1}) - \sum_{t=0}^{T-1} u_t(y_t, y_{t+1}) = \sum_{t=S}^{T-1} u_t(x_t, x_{t+1}) - \sum_{t=S}^{T-1} u_t(y_t, y_{t+1}). \quad (4.24)$$



By (4.21) there is an integer  $p \geq 0$  such that

$$\tilde{S} - S - 1 \in [pL_1, (p+1)L_1]. \quad (4.25)$$

By (4.3), (4.21) and (4.25)

$$p \geq k.$$

By (4.25), (4.22) and (P4) for each integer  $i \in [0, p-1]$

$$\sum_{t=S+iL_1}^{S+(i+1)L_1-1} v(x_t, x_{t+1}) \leq L_1 v(\bar{x}, \bar{x}) - M_1 - 2.$$

Together with (4.5) and (4.6) this implies that for each integer  $i \in [0, p-1]$

$$\begin{aligned} & \sum_{t=S+iL_1}^{S+(i+1)L_1-1} u_t(x_t, x_{t+1}) \leq \delta L_1 + L_1 v(\bar{x}, \bar{x}) - M_1 - 2 \\ & \leq 2\delta L_1 + \sum_{t=S+iL_1}^{S+(i+1)L_1-1} u_t(\bar{x}, \bar{x}) - M_1 - 2 \leq \sum_{t=S+iL_1}^{S+(i+1)L_1-1} u_t(\bar{x}, \bar{x}) - M_1 - 1. \end{aligned} \quad (4.26)$$

By (4.23), (4.21), (4.6), (4.26), (4.25) and the inequality  $p \geq k$

$$\begin{aligned} & \sum_{t=S}^{T-1} u_t(x_t, x_{t+1}) - \sum_{t=S}^{T-1} u_t(y_t, y_{t+1}) \\ & = \sum_{t=S}^{\tilde{S}-1} u_t(x_t, x_{t+1}) - \sum_{t=S}^{\tilde{S}-1} u_t(y_t, y_{t+1}) \\ & \leq \sum_{t=S}^{S+pL_1-1} u_t(x_t, x_{t+1}) + 2L_1(\|v\| + 1) - \sum_{t=S}^{\tilde{S}-1} u_t(\bar{x}, \bar{x}) + 4L_0(\|v\| + 1) \\ & \leq -p(M_1 + 1) + 8L_1(\|v\| + 1) < -k + 8L_1(\|v\| + 1) < -M_0 - 4. \end{aligned}$$

This contradicts (4.24).

The contradiction we have reached proves that there is an integer  $t \in [S+1, S+L_2]$  for which  $\rho(x_t, \bar{x}) \leq \delta_0$ .

Thus we have shown that the following property holds:

(P5) for each integer  $S$  satisfying  $S \in [0, T-L_2]$  and  $x_S \in \bar{Y}_{L_0}$  there is an integer  $t \in [S+1, S+L_2]$  for which  $\rho(x_t, \bar{x}) \leq \delta_0$ .

Using (4.7) and (P5) by induction we construct an increasing sequence of integers  $\{S_i\}_{i=1}^q$  such that

$$\begin{aligned} & S_1 \in [0, L_2], S_q \in (T-L_2, T], S_{i+1} - S_i \in [1, L_2], i = 1, \dots, q-1, \\ & \rho(x_{S_i}, \bar{x}) \leq \delta_0, i = 1, \dots, q. \end{aligned} \quad (4.27)$$

Clearly, we may assume that if  $\rho(x_0, \bar{x}) \leq \delta_0$ , then  $S_1 = 0$  and if  $\rho(x_T, \bar{x}) \leq \delta_0$ , then  $S_q = T$ .

Let an integer  $i \in \{0, \dots, q-1\}$ . By (4.8), (4.9), (4.27) and (4.4)

$$\sum_{t=S_i}^{S_{i+1}-1} u_t(x_t, x_{t+1}) \geq \sigma(\{u_t\}_{t=S_i}^{S_{i+1}-1}, S_i, S_{i+1}, x_{S_i}, x_{S_{i+1}}) - \delta.$$

Together with (4.5), (4.6) and (4.27) this implies that

$$\sum_{t=S_i}^{S_{i+1}-1} v(x_t, x_{t+1}) \geq \sigma(v, S_i, S_{i+1}, x_{S_i}, x_{S_{i+1}}) - \delta_0.$$

By the equation above, (4.27) and (P3),

$$\rho(x_t, \bar{x}) \leq \epsilon, \quad t = S_i, \dots, S_{i+1}, \quad i = 0, \dots, q-1.$$

Theorems 1.2 and 1.3 are proved.

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