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On the Derivative of a Polynomial

Vinay Kumar Jain

Abstract: For an arbitrary polynomial $P(z)$, let $M(P,r) = \max_{|z|=r} |P(z)|$ and $m(P,r) = \min_{|z|=r} |P(z)|$, $(r > 0)$. For a polynomial $p(z) = \sum_{j=0}^{n} a_j z^j = a_n \prod_{\nu=1}^{n}(z - z_\nu)$, of degree n, having all its zeros in $|z| \leq k, (k \geq 1)$, with a zero of order $s, (s \geq 0)$, at 0 and

$F_0, F_1, F_2, G_{n-s}, F_3, F_4, H_{n-s}, F_{n-s}, B_0, B_1, E_{n-1}, B_2, B_3, D_{n-1}$ and B_{n-1},

as in Theorem, we have obtained a refinement

$$\begin{aligned} M(p^{'},1) \quad \geq \quad & \frac{2}{1+k^{n-s}}\Big(\sum_{\nu=1}^{n}\frac{k}{k+|z_\nu|}\Big)M(p,1) \\ & +\frac{k^{n-s}-1}{k^n(1+k^{n-s})}\Big(\sum_{\nu=1}^{n}\frac{k}{k+|z_\nu|}\Big)m(p,k) \\ & +\frac{2}{k^{n-s}(1+k^{n-s})}\Big(\sum_{\nu=1}^{n}\frac{k}{k+|z_\nu|}\Big)F_{n-s}+\frac{B_{n-1}}{k^{n-1}}, \end{aligned}$$

of our old result (1997), thereby obtaining a new refinement of known results

$$M(p',1) \quad \geq \quad \frac{n}{1+k^n}M(p,1), (1973)$$

and

$$M(p',1) \quad \geq \quad \frac{2}{1+k^n}\Big(\sum_{\nu=1}^{n}\frac{k}{k+|z_\nu|}\Big)M(p,1), (1983).$$

AMS Subject Classification: 30C10, 30A10.
Keywords and Phrases: Polynomial; Derivative; Lower bound for $M(p',1)$; Zero of order s at 0; Refinement.

1. Introduction and statement of result

For an arbitrary polynomial $P(z)$, let $M(P,r) = \max_{|z|=r} |P(z)|$ and $m(P,r) = \min_{|z|=r} |P(z)|, (r > 0)$. For a given polynomial $p(z)$, concerning the estimate of $|p'(z)|$ on $|z| \leq 1$, we have the following well-known result due to Turán [9], suggesting a lower bound for $M(p',1)$.

Theorem A. *If $p(z)$ is a polynomial of degree n, having all its zeros in $|z| \leq 1$ then*

$$M(p',1) \geq \frac{n}{2} M(p,1).$$

The result is sharp with equality for the polynomial $p(z)$ having all its zeros on $|z| = 1$.

Malik [8] obtained a generalization of Theorem A, namely

Theorem B. *If $p(z)$ is a polynomial of degree n, having all its zeros in $|z| \leq k, (k \leq 1)$ then*

$$M(p',1) \geq \frac{n}{1+k} M(p,1).$$

The result is sharp with equality for the polynomial $p(z) = (z+k)^n$,

and Govil [4] obtained the generalization

Theorem C. *If $p(z)$ is a polynomial of degree n, having all its zeros in $|z| \leq k, (k \geq 1)$ then*

$$M(p',1) \geq \frac{n}{1+k^n} M(p,1).$$

The result is sharp with equality for the polynomial $p(z) = z^n + k^n$.

Aziz [1] obtained a refinement of Theorem C in the form

Theorem D. *If all the zeros of the polynomial $p(z) = a_n \prod_{j=1}^{n}(z - z_j)$, of degree n lie in $|z| \leq k, (k \geq 1)$ then*

$$M(p',1) \geq \frac{2}{1+k^n} \Big(\sum_{j=1}^{n} \frac{k}{k + |z_j|} \Big) M(p,1).$$

The result is best possible with equality for the polynomial $p(z) = z^n + k^n$,

which was further refined by Govil [5] to give

Theorem E. *Let $p(z) = \sum_{j=0}^{n} a_j z^j = a_n \prod_{t=1}^{n}(z - z_t)$, be a polynomial of degree $n \geq 2, |z_t| \leq K_t$,*
$1 \leq t \leq n$ and let $K = \max(K_1, K_2, \dots, K_n) \geq 1$. Then

$$\begin{aligned} M(p',1) \quad \geq \quad & \frac{2}{1+K^n} \Big(\sum_{t=1}^{n} \frac{K}{K+K_t} \Big) M(p,1) + \\ & \frac{2|a_{n-1}|}{1+K^n} \Big(\sum_{t=1}^{n} \frac{1}{K+K_t} \Big) \Big(\frac{K^n - 1}{n} - \frac{K^{n-2}-1}{n-2} \Big) + |a_1| \Big(1 - \frac{1}{K^2} \Big), n > 2 \end{aligned}$$

and

$$\begin{aligned} M(p',1) \quad &\geq \quad \frac{2}{1+K^n}\Big(\sum_{t=1}^{n}\frac{K}{K+K_t}\Big)M(p,1)+\frac{(K-1)^n}{1+K^n}|a_1|\Big(\sum_{t=1}^{n}\frac{1}{K+K_t}\Big) \\ &\quad +|a_1|\Big(1-\frac{1}{K}\Big),\ \ n=2. \end{aligned}$$

The result is best possible with equality for the polynomial $p(z)=z^n+K^n$.

We, in our old result [6], had considered the polynomial having all its zeros in $|z|\leq k, (k\geq 1)$, with a possible zero of order $m, (m\geq 0)$, at 0 and had obtained the following refinement of both Theorem C and Theorem D.

Theorem F. *Let $p(z)=\sum_{s=0}^{n}a_sz^s=a_n\prod_{\gamma=1}^{n}(z-z_\gamma)$ be a polynomial of degree n, having all its zeros in $|z|\leq k, (k\geq 1)$. Then*

$$M(p',1)\geq \frac{2}{1+k^{n-m}}\Big(\sum_{\gamma=1}^{n}\frac{k}{k+|z_\gamma|}\Big)M(p,1)+\frac{C}{k(1+k^{n-m})}\Big(\sum_{\gamma=1}^{n}\frac{1}{k+|z_\gamma|}\Big)+D, \tag{1.1}$$

where

$$p(z)=z^mp_1(z),\ \ \textit{with } p_1(0)\neq 0,\ \ \textit{for some non-negative integer } m,$$

non-negative real number

$$C=\begin{cases} 4|a_{n-2}|\left\{c_{n-m-2}(k)-c_{n-m-4}(k)-\right. & \\ \left.\left(\frac{k^{n-m-1}-1}{n-m-1}-\frac{k^{n-m-3}-1}{n-m-3}\right)\right\} & ,n>4\ \&\ 0\leq m<n-4, \\ 4|a_{n-2}|\left\{D_k-\left(\frac{k^3-1}{3}-\frac{k^2-1}{2}\right)\right\} & ,n\geq 4\ \&\ m=n-4, \\ 4|a_{n-2}|\left\{F_k-\frac{k^2-1}{2}\right\} & ,n\geq 3\ \&\ m=n-3, \\ |a_{n-1}|k(k-1)^2 & ,n>2\ \&\ m=n-2, \\ (|a_n|k-|a_{n-1}|)k(k-1) & ,n\geq 1\ \&\ m=n-1, \\ 0 & ,n\geq 1\ \&\ m=n, \end{cases}$$

non-negative real number

$$D=\begin{cases} 2|a_2|\left(\frac{1}{k}-\frac{1}{k^3}\right)(\sqrt{k^2+1}-1) & ,n>4\ \&\ m\leq n-1, \\ 2|a_2|\left(\frac{1}{k}-\frac{1}{k^2}\right)(\sqrt{k^2+k+1}-1) & ,n=4\ \&\ m\leq n-1, \\ \frac{2|a_2|}{k}\left(\sqrt{\frac{k^2+1}{2}}-1\right) & ,n=3\ \&\ m\leq n-1, \\ |a_1|\left(1-\frac{1}{k}\right) & ,n=2\ \&\ 0<m\leq n-1, \\ 0 & ,n>1\ \&\ m=n, \\ 0 & ,n=1, \end{cases}$$

$$\begin{aligned} c_t(k) &= \int_1^k r^t\sqrt{r^2+1}dr, t>0,\\ D_k &= \int_1^k (r^2-r)\sqrt{r^2+r+1}dr \end{aligned}$$

and

$$F_k = \int_1^k r\sqrt{\frac{r^2+1}{2}}dr.$$

In (1.1) *equality holds for the polynomial* $p(z) = z^n + k^n$.

In this paper we have obtained a refinement of our old result, namely Theorem F, thereby obtaining a new refinement of Theorem C and Theorem D. More precisely we have proved

Theorem. *Let* $p(z) = \sum_{j=0}^n a_j z^j = a_n \prod_{\nu=1}^n (z - z_\nu)$ *be a polynomial of degree* n*, having all its zeros in* $|z| \le k, (k \ge 1)$*, with a zero of order* $s, (s \ge 0)$*, at* 0*. Then*

$$\begin{aligned} M(p',1) \geq\ & \frac{2}{1+k^{n-s}}\Big(\sum_{\nu=1}^n \frac{k}{k+|z_\nu|}\Big)M(p,1) + \frac{k^{n-s}-1}{k^n(1+k^{n-s})}\Big(\sum_{\nu=1}^n \frac{k}{k+|z_\nu|}\Big)m(p,k)\\ & + \frac{2}{k^{n-s}(k^{n-s}+1)}\Big(\sum_{\nu=1}^n \frac{k}{k+|z_\nu|}\Big)F_{n-s} + \frac{B_{n-1}}{k^{n-1}}, \qquad (1.2) \end{aligned}$$

where

$$\begin{aligned} B_0 &= 0,\\ B_1 &= (k-1)|a_1|,\\ B_2 &= \max\Big(E_2|a_1|, 2|a_2|k\Big(\sqrt{\frac{k^2+1}{2}}-1\Big)\Big),\\ B_3 &= \max\Big(E_3|a_1|, 2|a_2|(k^2-k)(\sqrt{k^2+k+1}-1)\Big),\\ B_{n-1} &= \max\Big(E_{n-1}|a_1|, 2|a_2|D_{n-1}\Big), n-1 \ge 4,\\ E_{n-1} &= k^{n-1}-k^{n-3}, n-1 \ge 2,\\ D_{n-1} &= \Big(k^{n-2}-k^{n-4}\Big)\Big(\sqrt{k^2+1}-1\Big), n-1 \ge 4,\\ F_0 &= 0,\\ F_1 &= 0,\\ F_2 &= |a_{n-1}|k\frac{(k-1)^2}{2}, \end{aligned}$$

$$
\begin{aligned}
F_3 &= \max\Big(k^2|a_{n-1}|G_3, 2k|a_{n-2}|\Big(\int_1^k r\sqrt{\frac{r^2+1}{2}}dr - \frac{k^2-1}{2}\Big)\Big),\\
F_4 &= \max\Big(k^3|a_{n-1}|G_4, 2k^2|a_{n-2}|\Big(\int_1^k (r^2-r)\sqrt{r^2+r+1}dr - \Big(\frac{k^3-1}{3}-\frac{k^2-1}{2}\Big)\Big)\Big),\\
F_{n-s} &= \max\Big(k^{n-s-1}|a_{n-1}|G_{n-s}, 2k^{n-s-2}|a_{n-2}|H_{n-s}\Big), \quad n-s\geq 5,\\
G_{n-s} &= \frac{k^{n-s}-1}{n-s} - \frac{k^{n-s-2}-1}{n-s-2}, \quad n-s \geq 3
\end{aligned}
$$

and

$$
\begin{aligned}
H_{n-s} = {} & \int_1^k r^{n-s-2}\sqrt{r^2+1}dr - \int_1^k r^{n-s-4}\sqrt{r^2+1}dr - \Big(\frac{k^{n-s-1}-1}{n-s-1} - \frac{k^{n-s-3}-1}{n-s-3}\Big),\\
& n-s \geq 5.
\end{aligned}
$$

In (1.2) equality holds for the polynomial $p(z) = z^n + k^n$.

Remark 1. In many cases, our Theorem gives a better lower bound for $M(p',1)$ than those given by other known results, as for the polynomial $p(z) = z(z^3+8)(z+3)$, having all its zeros in $|z| \leq 3$, we get

$$
\begin{aligned}
M(p',1) &\geq 25.5, \text{ by Theorem},\\
M(p',1) &\geq 13.1, \text{ by Theorem F},\\
M(p',1) &\geq 23.4, \text{ by Theorem E}
\end{aligned}
$$

and

$$
M(p',1) \geq 5.8, \text{by result [7, Theorem 1.7]}.
$$

2. Lemmas

For the proof of Theorem we require the following lemmas.

Lemma 1. *If $p(z)$ is a polynomial of degree $n(\geq 2)$ then for all $R>1$*

$$
M(p,R) \leq R^n M(p,1) - (R^n - R^{n-2})|p(0)|.
$$

Lemma 1 is due to Frappier et al. [3, Theorem 2].

Lemma 2. *Let $p(z)$ be a polynomial of degree $n(\geq 2)$ and let $R \geq 1$. Then*

$$
\begin{aligned}
M(p,R) &\leq R^n M(p,1) - |p'(0)|(R^{n-1}-R^{n-3})(\sqrt{R^2+1}-1), \ n \geq 4,\\
M(p,R) &\leq R^n M(p,1) - |p'(0)|(R^2-R)(\sqrt{R^2+R+1}-1), \ n = 3
\end{aligned}
$$

and

$$
M(p,R) \leq R^n M(p,1) - |p'(0)|R\Big(\sqrt{\frac{R^2+1}{2}}-1\Big), \ n=2.
$$

Lemma 2 is due to Frappier et al. [3, Theorem 4].

Using Lemma 1 and Lemma 2 one easily obtains

Lemma 3. *If $p(z)$ is a polynomial of degree n then for $R \geq 1$*

$$M(p,R) \leq R^n M(p,1) - B_n(p,R),$$

where

$$\begin{aligned}
B_1(p,R) &= (R-1)|p(0)|, \\
B_2(p,R) &= \max\Big(E_2(R)|p(0)|, R\Big(\sqrt{\frac{R^2+1}{2}}-1\Big)|p'(0)|\Big), \\
B_3(p,R) &= \max\Big(E_3(R)|p(0)|, (R^2-R)\Big(\sqrt{R^2+R+1}-1\Big)|p'(0)|\Big), \\
B_n(p,R) &= \max(E_n(R)|p(0)|, D_n(R)|p'(0)|),\ n \geq 4, \\
E_n(R) &= R^n - R^{n-2},\ n \geq 2
\end{aligned}$$

and

$$D_n(R) = (R^{n-1} - R^{n-3})(\sqrt{R^2+1}-1),\ n \geq 4.$$

Remark 2. One can note that Lemma 3 is trivially true for $n = 0$, with $B_0(p,R) = 0$.

Lemma 4. *If $p(z)$ is a polynomial of degree n, having no zeros in $|z| < 1$ then*

$$M(p',1) \leq \frac{n}{2}\left\{M(p,1) - m(p,1)\right\}. \tag{2.1}$$

There is equality in (2.1) for $p(z) = \alpha + \beta z^n, |\alpha| = |\beta|$.

Lemma 4 is due to Aziz and Dawood [2].

Lemma 5. *If $p(z) = \sum_{j=0}^{n} a_j z^j$ is a polynomial of degree $n > 2$, having no zeros in $|z| < 1$ then for $R \geq 1$*

$$M(p,R) \leq \frac{R^n+1}{2}M(p,1) - m(p,1)\frac{R^n-1}{2} - |a_1|\Big(\frac{R^n-1}{n} - \frac{R^{n-2}-1}{n-2}\Big). \tag{2.2}$$

Equality holds in (2.2) for $p(z) = z^n + 1$.

Proof of Lemma 5. It is similar to the proof of Lemma 4 [5] with one change: Lemma 4 instead of Lemma 2 [5].

Lemma 6. *If $p(z)$ is a polynomial of degree $n > 4$, having no zeros in $|z| < 1$ then for $R \geq 1$*

$$\begin{aligned}
M(p,R) \leq\ & \frac{R^n+1}{2}M(p,1) - \frac{R^n-1}{2}m(p,1) - |p''(0)|\{c_{n-2}(R) - c_{n-4}(R) \\
& -\Big(\frac{R^{n-1}-1}{n-1} - \frac{R^{n-3}-1}{n-3}\Big)\Big\},
\end{aligned} \tag{2.3}$$

where

$$c_t(R) = \int_1^R r^t \sqrt{r^2+1}dr, t > 0.$$

There is equality in (2.3) *for* $p(z) = \alpha + \beta z^n,\ |\alpha| = |\beta|$.

Proof of Lemma 6. It is similar to the proof of Lemma 4 [6] with one change: Lemma 4 instead of Lemma 2 [6].

Lemma 7. *If* $p(z)$ *is a polynomial of degree* $n = 4$, *having no zeros in* $|z| < 1$ *then for* $R \geq 1$

$$M(p,R) \leq \frac{R^n+1}{2}M(p,1) - \frac{R^n-1}{2}m(p,1) - |p''(0)|\left\{D_R - \Big(\frac{R^3-1}{3} - \frac{R^2-1}{2}\Big)\right\}, \tag{2.4}$$

where

$$D_R = \int_1^R (r^2-r)\sqrt{r^2+r+1}dr.$$

There is equality in (2.4) *for* $p(z) = \alpha + \beta z^n, |\alpha| = |\beta|$.

Proof of Lemma 7. It is similar to the proof of Lemma 5 [6] with one change: Lemma 4 instead of Lemma 2 [6].

Lemma 8. *If* $p(z)$ *is a polynomial of degree* $n = 3$, *having no zeros in* $|z| < 1$ *then for* $R \geq 1$

$$M(p,R) \leq \frac{R^n+1}{2}M(p,1) - \frac{R^n-1}{2}m(p,1) - |p''(0)|\Big(F_R - \frac{R^2-1}{2}\Big), \tag{2.5}$$

where

$$F_R = \int_1^R r\sqrt{\frac{r^2+1}{2}}dr.$$

There is equality in (2.5) *for* $p(z) = \alpha + \beta z^n, |\alpha| = |\beta|$.

Proof of Lemma 8. It is similar to the proof of Lemma 6 [6] with one change: Lemma 4 instead of Lemma 2 [6].

Lemma 9. *If* $p(z)$ *is a polynomial of degree* $n = 2$, *having no zeros in* $|z| < 1$ *then for* $R \geq 1$

$$M(p,R) \leq \frac{R^n+1}{2}M(p,1) - \frac{R^n-1}{2}m(p,1) - |p'(0)|\frac{(R-1)^2}{2}. \tag{2.6}$$

There is equality in (2.6) *for* $p(z) = \alpha + \beta z^n, |\alpha| = |\beta|$.

Proof of Lemma 9. It is similar to the proof of Lemma 8 [6] with one change: Lemma 4 instead of Lemma 2 [6].

Using Lemma 5, Lemma 6, Lemma 7, Lemma 8 and Lemma 9 one easily obtains

Lemma 10. *If $p(z)$ is a polynomial of degree n, having no zeros in $|z| < 1$ then for $R \geq 1$*

$$M(p,R) \leq \frac{R^n+1}{2}M(p,1) - \frac{R^n-1}{2}m(p,1) - F_n(p,R), \tag{2.7}$$

where

$$F_1(p,R) = 0,$$

$$F_2(p,R) = |p'(0)|\frac{(R-1)^2}{2},$$

$$F_3(p,R) = \max\Big(G_3(R)|p'(0)|, \Big(\int_1^R r\sqrt{\frac{r^2+1}{2}}dr - \frac{R^2-1}{2}\Big)|p''(0)|\Big),$$

$$F_4(p,R) = \max\Big(G_4(R)|p'(0)|, \Big(\int_1^R (r^2-r)\sqrt{r^2+r+1}dr - \Big(\frac{R^3-1}{3} - \frac{R^2-1}{2}\Big)\Big)|p''(0)|\Big),$$

$$F_n(p,R) = \max\Big(G_n(R)|p'(0)|, H_n(R)|p''(0)|\Big),\ n \geq 5,$$

$$G_n(R) = \Big(\frac{R^n-1}{n} - \frac{R^{n-2}-1}{n-2}\Big),\ n \geq 3$$

and

$$H_n(R) = \int_1^R r^{n-2}\sqrt{r^2+1}dr - \int_1^R r^{n-4}\sqrt{r^2+1}dr - \Big(\frac{R^{n-1}-1}{n-1} - \frac{R^{n-3}-1}{n-3}\Big),\ n \geq 5.$$

There is equality in (2.7) *for* $p(z) = \alpha + \beta z^n,\ |\alpha| = |\beta|$.

Remark 3. One can note that Lemma 10 is trivially true for $n = 0$, with $F_0(p,R) = 0$.

3. Proof of Theorem

It is similar to the main part of Proof of Theorem [6] with two changes:
Lemma 3 along with Remark 2 instead of Lemma 3 [6],
Lemma 10 along with Remark 3 instead of Lemma 4 [6].

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Pontryagin's Maximum Principle for Optimal Control Problems Governed by Nonlinear Impulsive Differential Equations

Hugo Leiva

Abstract: In this paper, we derive the Pontryagin's maximum principle for optimal control problems governed by nonlinear impulsive differential equations. Our method is based on Dubovitskii-Milyutin theory, but in doing so, we assumed that the linear variational impulsive differential equation around the optimal solution is exactly controllable, which can be satisfied in many cases. Then, we consider an example as an application of the main result. After that, we study the case when the differential equation is of neutral type. Finally, several possible problems are proposed for future research where the differential equation, the constraints, the time scale, the impulses, etc. are changed.

In honor to Dr. Zoltan Varga

AMS Subject Classification: 49K20, 35K2.
Keywords and Phrases: Pontryagin maximum principle; Optimal control problem; Nonlinear impulsive differential equations; Dubovitskii-Milyutin theory; Variational impulsive differential equation.

1. Introduction

Pontriaguin's maximum (minimum) principle is used to optimize a functional depending on the state of the system and the best possible control that takes a dynamical system from one state to another, especially in the presence of constraints on state or input controls. It was formulated in 1956 by the Russian mathematician Lev Pontriaguin and his students(see [41]). It has as a special case the Euler-Lagrange equation of the calculus of variations. The result was first successfully applied to minimal time problems when input control is constrained, but it can also be useful in studying state constrained problems. In the following decades several abstract theories have been published to give a synthesis that would include different chapters of optimization, such as mathematical programming, classical variational calculus, and optimal control. The two most prominent theories are: Dubovitskii–Milyutin [16] and Iofee-Tihomirov [22]. In [22], the main result is a first order necessary condition for problems called "soft convex". The condition is formulated in terms of Lagrange's multipliers. In the Dubovitskii–Milyutin theory (which is applied in the present work) the fundamental idea is the following: Conic approximations are constructed to the data of an optimization problem with constrains, and in terms of duals elements of these cones, the optimality condition is expressed in the abstract Euler–Lagrange equation form. Given a class of optimization problems, the application of this theory consists in specifying the cones and their dual to express the Abstract Euler-Lagrange equation in terms of the problem in question. In the book of I. V. Girsanov [18] this method is carried out for several cases, such as the optimal control problem with a *finite number* of constraints on the state of the system. The main goal of this paper is to derive a general optimal condition (Pontryagin's maximum principle) for optimal control problems governed by impulsive differential equations. More specifically, we shall study the following problem

Problem 1.1.

$$\int_0^T \Phi(x(t),\, u(t),\, t)dt \longrightarrow \min \text{ loc.} \tag{1.1}$$

$$(x,\, u) \in E := \mathcal{PW}([0,T];\mathbb{R}^n) \times L_\infty^r([0,T];\mathbb{R}^r), \tag{1.2}$$

$$\dot{x}(t) = \varphi(x(t),\, u(t),\, t), \qquad x(0) = x_0 \tag{1.3}$$

$$x(T) = x_1;\ x_1,\, x_0 \in I\!R^n, \tag{1.4}$$

$$x(t_k^+) = x(t_k^-) + \mathcal{J}_k(x(t_k)), \quad k = 1, 2, 3, \ldots, p. \tag{1.5}$$

$$u(t) \in M, \quad t \in [0,\, T], \quad a.e., \tag{1.6}$$

where $0 < t_1 < t_2 < \cdots < t_p < T$, are fixed real numbers, $x \in \mathcal{PW}([0,T];\mathbb{R}^n)$, the control function u belongs to L_∞^r, $M \subset I\!R^r$ and the functions

$$\begin{aligned}
\varphi &: I\!R^n \times I\!R^r \times [0,\, T] \longrightarrow I\!R,\\
\Phi &: I\!R^n \times I\!R^r \times [0,\, T] \longrightarrow I\!R^n,\\
\mathcal{J}_k &: I\!R^n \longrightarrow I\!R^n,
\end{aligned}$$

where $\mathcal{PW}([0,T];\mathbb{R}^n)$ and L_∞^r are define by

$$\begin{aligned}\mathcal{PW}([0,T];\mathbb{R}^n) \quad &= \quad \{z:[0,T]\to\mathbb{R}^n : z\in C(J';\mathbb{R}^n), \exists z(t_k^+), z(t_k^-)\\ &\qquad \text{and} \quad z(t_k)=z(t_k^-), \quad k=1,2,\dots,p\},\end{aligned}$$

where $J=[0,T]$ and $J'=J\backslash\{t_1,t_2,\dots,t_p\}$, endowed with the norm

$$\|z\|_0 = \sup_{t\in[0,T]} \|z(t)\|_{\mathbb{R}^n},$$

and $L_\infty^r = L_\infty^r([0,T];\mathbb{R}^r)$ is the space of measurable function essentially bounded with the essential supremum norm.

For now these functions are smooth enough, so to prove the main results we will impose some additional conditions on the terms involved in the problem 1.1. The study of the controllability of differential equations with impulses is in effervescence at the moment, we can mention the following recent works on the controllability of such equations (see [8, 10, 28, 29, 30, 31, 32, 33, 36]), this in finite dimension, whereas in infinite dimension we can cite the following works ([2, 3, 5, 20, 38]). The Dubovitskii-Milyutin theory has been used to study optimal control problems for a long time, but not for impulsive differential equations, in this sense it is worthwhile to mention the work done in [9, 14, 15, 19, 21, 16, 24, 34]. Furthermore, we know there are a lot of works on optimal control problems using different techniques, for which one can see the research done in [23, 35, 37, 40] But, as far as we know, the optimal control problems for impulsive differential equations have not been studied much, only some particular works can be found in the literature, to mention some of them, we have the works carried out by ([1, 4, 6, 11, 26, 39, 42]).

Outline of the work: Section 2 contains preliminary results, here we summarizes the fundamental concepts and results of Dubovitskii-Milyutin theory that will be applied later; the intersection of cones lemma is presented. Then, the optimality condition in the abstract Euler–Lagrange equation form for a general optimization problem with constraints is formulated. To apply the general scheme of this theory to a specific class of problems, we must first compute the approximation cones. To do this, in subsections 2.3-2.5, we summarize and develop the methods to calculate the decay, admissible and tangent cones that appear in [18]. In addition, several extensions of these results are demonstrated, which facilitate the treatment for impulsive differential equations. In subsection 2.6, we present and prove some modifications of Minkowski-Farkas's Theorem, which simplify the explicit calculation of dual cones. Results of this subsection are useful to express the corresponding Euler-Lagrange equation to many problems in future investigations. In section 3, an optimal control problem governed by a nonlinear impulsive differential equation is considered. The main objective is to see that under certain conditions the impulses do not affect the optimality condition obtained by Pontryagin; roughly speaking, if the pulses are small enough, the maximum principle remains the same. In section 4, we prove that the necessary condition of optimality presented in Theorem 3.1 (maximum principle), under certain additional conditions, is also sufficient. To do this, we must assume

conditions that allow us to apply the general theorem of sufficient condition of optimality from the Dubovitskii-Milyutin theory, Theorem 2.17.
In section 5 we modify the optimal control problem by changing the boundary condition in its final state by placing a finite number of nonlinear constraints, and under certain conditions we again prove that the maximum principle persists.
In section 6 an example is presented as an application of these results obtained here.
In this section 7, we will show how Dubovitskii–Milyutin theory can be applied to generalize the Maximum principle of [18] to the case of optimal control problems governed by impulsive nonlinear neutral differential equations.
Finally, in section 8, we present several problems that could be solved in a similar way, which are part of future research.

2. Preliminaries Results

In this section, we summarize some fundamental results of the Dubovitskii- Milyutin theory. We formulate the general optimization problem with constraints and construct the approximation cones to the problem data (the objective function and restrictions), and the optimality condition in terms of the approximation cones dual is expressed by the Euler-Lagrange equation. The proof of these results can be refereed in [18].

2.1. Cones, Dual Cones and Dubovitskii–Milyutin Lemma

Let E be a locally convex topological linear space, and denote its dual space by E^*, the space of continuous linear functionals.

Definition 2.1. $K \subset E$ is a *cone with apex at zero* , if

$$\lambda K = K, \qquad (\lambda > 0).$$

Definition 2.2.

$$K^+ = \{f \in E^* \,/\, f(x) \geq 0, \quad \forall\, x \in K\},$$

is called the *dual cone* of K.

Proposition 2.3.

a) K^+ *is a* w^*- *closed and convex cone.*

b) $K^+ = (\overline{K})^+$, *(*$\overline{K}$ *is the* $w-$ *closure of* K*).*

c) $\left(\bigcup_{\alpha\in A} K_\alpha\right)^+ = \bigcap_{\alpha\in A} K_\alpha^+$ *where* A *is an index–set.*

d) If $K_1 \subset K_2$, *then* $K_2^+ \subset K_1^+$.

Definition 2.4. Let A be an arbitrary set and $K_\alpha \subset E, \quad \alpha \in A$, be cones with apex at zero. Then, we define the following set

$$\sum_{\alpha \in A} K_\alpha^+ = \{f_{\alpha_1} + f_{\alpha_2} + \cdots + f_{\alpha_n}, \quad f_{\alpha_i} \in K_{\alpha_i}^+, \quad n \in \mathbb{N}, \quad \alpha_i \in A \quad (i = 1, \ldots, n)\}.$$

Lemma 2.5. *Let $K_\alpha \subset E \quad (\alpha \in A)$ be convex cones $w-$closed, then*

$$\left(\bigcap_{\alpha \in A} K_\alpha \right)^+ = \overline{\sum_{\alpha \in A} K_\alpha^+} \qquad (w^* - closure\ ).$$

Lemma 2.6. *Let $K \subset E$ be a convex cone with apex at zero, $L \subset E$ a linear subspace such that $\overset{\circ}{K} \cap L \neq \emptyset$. Then $(K \cap L)^+ = K^+ + L^+$.*

Lemma 2.7. *Let $K_1, K_2, \ldots, K_n \subset E$ be open convex cones such that*

$$\bigcap_{i=1}^n K_i \neq \emptyset.$$

Then

$$\left(\bigcap_{i=1}^n K_i \right)^+ = \sum_{i=1}^n K_i^+.$$

Lemma 2.8. *(Dubovitskii–Milyutin). Let $K_1, K_2, \ldots, K_{n+1} \subset E$ be convex cones with apex at zero, with $K_1, K_2, \ldots, K_n$ open. Then*

$$\bigcap_{i=1}^{n+1} K_i = \emptyset$$

if and only if there are $f_i \in K_i^+ \quad (i = 1, 2, \ldots, n+1)$, not all zero such that

$$f_1 + f_2 + \cdots + f_n + f_{n+1} = 0.$$

2.2. The Abstract Euler–Lagrange Equation

Let us consider $F : E \longrightarrow I\!R$, and
$Q_i \subset E \quad (i = 1, 2, \ldots, n+1)$ such that the interior $\overset{\circ}{Q}_i \neq \emptyset \ \ (i = 1, 2, \ldots, n)$. Consider the following problem

$$F(x) \longrightarrow \min\ loc \tag{2.1}$$

$$x \in Q_i \qquad (i = 1, 2, \ldots, n+1). \tag{2.2}$$

Remark 2.9. The sets Q_i, $(i = 1, 2, \dots, n)$ usually are given by constraints inequality type, and Q_{n+1} by constraints equality type, and in general the interior $\overset{\circ}{Q}_{n+1} = \emptyset$.

To study the above problem, we give some previous definitions and lemmas.

Definition 2.10. The vector $h \in E$ is a *vector of decay direction* of $F : E \longrightarrow I\!R$ at the point $x^\circ \in E$, if there exists a neighborhood U of the point x°, numbers $\alpha = \alpha(F, x^\circ, h) < 0$ and $\varepsilon_0 \in I\!R_+$, such that for all $\varepsilon \in (0, \varepsilon_0)$ and all $\overline{h} \in U$ the following inequality holds

$$F(x^\circ + \varepsilon\overline{h}) \leq F(x^\circ) + \varepsilon\,\alpha.$$

Lemma 2.11. *The decay vectors of F in x° generate an open cone with apex at zero which will be denoted by $K_d = K_d(F, x^\circ)$, and it will be called as decay cone.*

Next, we introduce similar definitions for different constraints of the problem. For a constraint of inequality–type, we give the following definition.

Definition 2.12. The vector $h \in E$ is an *admissible vector to $Q \subset E$ in the point $x^\circ \in Q$*, if there is a neighborhood U of the point x° and $\varepsilon_0 \in I\!R_+$, such that for all $\varepsilon \in (0, \varepsilon_0)$ and all $\overline{h} \in U$, we have that

$$x^\circ + \varepsilon\,\overline{h} \in Q.$$

Lemma 2.13. *The admissible vectors to Q in x° generate an open cone with apex at zero, which will be denoted by $K_a := K_a(Q, x^\circ)$, and will be called admissible cone to Q in x°.*

To constraints of equality–type, we introduce the following definition.

Definition 2.14. The vector $h \in E$ is called a *tangent vector* to $Q \subset E$ at the point x°, if there are $\varepsilon_0 \in I\!R_+$ and a function $\theta : [0, \varepsilon] \longrightarrow E$, such that

$$\lim_{\varepsilon \to 0^+} \frac{\theta(\varepsilon)}{\varepsilon} = 0,$$

and

$$x^\circ + \varepsilon\,h + \theta(\varepsilon) \in Q \qquad (\varepsilon \in (0, \varepsilon_0)).$$

The set of all tangent vectors to Q in x° is a cone with apex at zero, which will be denoted by $K_T := K_T(Q, x^\circ)$; and will be called *tangent cone*.

Theorem 2.15. *(Dubovitskii–Milyutin). Let us consider the following problem*

$$\begin{cases} F(x) \longrightarrow \min\ loc \\ \\ x \in Q_i, \quad (i = 1, 2, \dots, n+1). \end{cases} \tag{2.3}$$

Let $x^\circ \in E$ be a solution of problem (2.3), and suppose that:

a) K_0 is the decay cone of F in x°.

b) K_i are the admissible cones to Q_i in $x^\circ \in Q_i$ $(i = 1, 2, \ldots, n)$.

c) K_{n+1} is the tangent cone to Q_{n+1} in x°.

Then, if K_i $\quad (i = 0, 1, 2, \ldots, n+1)$ are convex, there exist functions $f_i \in K_i^+$, $(i = 0, 1, \ldots, n+1)$ not all zero such that

$$f_0 + f_1 + \cdots + f_{n+1} = 0 \tag{2.4}$$

Equation (2.4) is called the **Abstract Euler-Lagrange equation**.

Remark 2.16. Sometimes it is important to ensure that $f_0 \neq 0$; an examination of the proof of Theorem 2.15 shows that a sufficient condition for this is that

$$\bigcap_{i=1}^{n+1} K_i = \emptyset.$$

To apply the Dubovitskii–Milyutin theorem to specific problems, we must follow the following scheme:

i) Determine the decay vectors.

ii) Determine the admissible vectors.

iii) Determine the tangent vectors.

iv) Build the dual cones.

Next, we will face problems (i) - (iv). The necessary optimality condition stated in Theorem 2.15, under certain conditions, is also sufficient:

Theorem 2.17. *Suppose that the following conditions hold:*

α) F is continuous and convex,

β) Q_i is convex $(i = 1, 2, \ldots, n+1)$,

γ) $\left(\bigcap_{i=1}^{n} \mathring{Q}_i \right) \cap Q_{n+1} \neq \emptyset,$

δ) $x^\circ \in \bigcap_{i=1}^{n+1} Q_i,$

ε) K_i $\quad (i = 0, 1, \ldots, n+1)$ are defined as in Theorem 2.15.

Then, x° is a solution of the problem (2.3) if and only if there exist $f_i \in K_i^+$ $\quad (i = 0, 1, 2, \ldots, n+1)$ not all zero such that

$$f_0 + f_1 + f_2 + \cdots + f_{n+1} = 0.$$

2.3. Cones of Decay Vectors

In this subsection we explicitly compute the cones of decay vectors for several functions.

Definition 2.18. Let E be a linear space and $F : E \longrightarrow I\!R$ a function. Then, we shall say that F has directional derivative in $x^\circ \in E$ on the direction of $h \in E$ if the following limit there exists:

$$\lim_{\varepsilon \to 0^+} \frac{F(x^\circ + \varepsilon\, h) - F(x^\circ)}{\varepsilon} =: F'(x^\circ,\, h). \tag{2.5}$$

For $x^\circ \in E$.

Theorem 2.19. *If $h \in K_d$ and there exists $F'(x^\circ,\, h)$, then $F'(x^\circ,\, h) < 0$.*

Theorem 2.20. *If E is a Banach space, F is locally Lipschitzian in x°, and $F'(x^\circ,\, h) < 0$, then $h \in K_d(F,\, x^\circ)$.*

Theorem 2.21. *(See [18, pg 45]). Let $F : E \longrightarrow I\!R$ be a continuous and convex function in a topological linear space E and $x^\circ \in E$, then F has directional derivative in all directions at x° and also we have that*

a) $F'(x^\circ,\, h) = \inf \left\{ \dfrac{F(x^\circ + \varepsilon\, h) - F(x^\circ)}{\varepsilon} / \varepsilon \in I\!R_+ \right\}$,

b) $K_d(F,\, x^\circ) = \{h \in E / F'(x^\circ,\, h) < 0\}$.

Theorem 2.22. *(See [18, pg 48]). If E is a Banach space and F is Fréchet–differentiable in $x^\circ \in E$, then*

$$K_d(F,\, x^\circ) = \{h \in E / F'(x^\circ)h < 0\}$$

where $F'(x^\circ)$ is the Fréchet's derivative of F in x°.

Example 2.23. In the same way as the example 7.3 of (See [18, pg 50]) we obtain the following result:

Let $E = \mathcal{PW}([0,T]; \mathbb{R}^n) \times L^r_\infty[0,\, T]$ and $F : E \longrightarrow I\!R$ defined as follows

$$F(x,\, u) := \int_0^T \Phi(x(t),\, u(t),\, t)dt,$$

$\Phi : I\!R^n \times I\!R^r \times I\!R \longrightarrow I\!R$ is a continuous function in its first two variables and measurable in the third variable, and has a derivative in its first and second variables Φ_x and Φ_u respectively bounded. Then, we have that

$$\begin{aligned} F'(x^\circ,\, u^\circ)(x,\, u) &= \int_0^T [\Phi_x(x^\circ,\, u^\circ,\, t)x(t) + \Phi_u(x^\circ,\, u^\circ,\, t)u(t)]dt, \\ \text{and} \quad K_d(F,\, (x^\circ,\, u^\circ)) &= \{(x,\, u) \in E / F'(x^\circ,\, u^\circ)(x,\, u) < 0\}. \end{aligned}$$

2.4. Cones of Admissible Vectors

Let E be a topological linear space, $F : E \longrightarrow I\!R$ a continuous function, $x^\circ \in E$ and

$$Q = \{x \in E / F(x) \leq F(x^\circ)\}.$$

Lemma 2.24. *Let $K_a = K_a(Q, x^\circ)$ and $K_d = K_d(F, x^\circ)$, then $K_d \subset K_a$.*

The proof of above Lemma is trivial. There are cases in which $K_d = K_a$.

Theorem 2.25. *(See [18, pg 58]) Suppose that*

i) There exists $F'(x^\circ, h) \quad (h \in E)$.

ii) There exists $\overline{h} \in E$ such that $F'(x^\circ, \overline{h}) < 0$.

iii) $F'(x^\circ, \cdot)$ is convex.

Then

$$K_a \subset \{h \in E \,/\, F'(x^\circ, h) < 0\} = K_d.$$

Theorem 2.26. *(See [18, pg 59]). If Q is an arbitrary convex set with $\overset{\circ}{Q} \neq \emptyset$, then*

$$K_a = \{h \in E / h = \lambda(x^\circ - x),\ x \in \overset{\circ}{Q},\ \lambda \in I\!R_+\}.$$

2.5. Cones of Tangent Vectors

In this section we basically mention the so-called Lusternik Theorem, which is a powerful tool for calculating the cone of tangent vectors.

Theorem 2.27. *(Lusternik). Let E_1, E_2 Banach spaces, and suppose that*

a) $x^\circ \in E_1,\ P : E_1 \longrightarrow E_2$ is Fréchet's differentiable in x° and $P(x^\circ) = 0$.

b) $P'(x^\circ) : E_1 \longrightarrow E_2$ is surjective.

Then the cone of tangent vectors K_T to the set $Q := \{x \in E_1 / P(x) = 0\}$ in the point $x^\circ \in Q$, is given by

$$K_T = Ker\ P'(x^\circ).$$

The proof of above theorem (which is not trivial) can be found in [22, pg 30].

2.6. Relationship Between Approximation Cones and Their Dual

In this subsection, we present results that establishes a closed relationship between approximation cones and their dual.

Theorem 2.28. *If K is a linear subspace of a topological linear space E, then*

$$K^{+} = \{f \in E^{*} \,/\, f(x) = 0, \quad \forall\, x \in K\} =: K^{\perp},$$

where $K^{\perp}$ is called the annihilator of K.

Theorem 2.29. *Let $f \in E^{*}$ and $K_1 := \{x \in E \,/\, f(x) = 0\}$,
$K_2 := \{x \in E \,/\, f(x) \geq 0\}, K_3 := \{x \in E \,/\, f(x) > 0\}$. Then:*

i) If $f \neq 0$, then $K_1^{+} = \{\lambda f \,/\, \lambda \in \mathbb{R}\}, \quad K_2^{+} = K_3^{+} = \{\lambda f \,/\, \lambda \in \mathbb{R}_{+0}\}$.

ii) If $f = 0$, then $K_1^{+} = \{0\}, \quad K_2^{+} = \{0\}$ and $K_3^{+} = E^{*}$.

The proof of Theorems 2.28 and 2.29 is trivial.

Theorem 2.30. *Let E be a topological linear space and $F : E \longrightarrow \mathbb{R}$ continuous and convex. For $x^{\circ} \in E$, let us consider the following set*

$$Q := \{x \in E \,/\, F(x) \leq F(x^{\circ})\}.$$

Now, we define

$$Q^{*} := \{f \in E^{*} \,/\, f(x) \geq f(x^{\circ}), \quad (x \in Q)\}.$$

Then

i) $K_T^{+}(Q, x^{\circ}) = Q^{*}$,

ii) If there exists $\overline{x} \in E$ such that $F(\overline{x}) < F(x^{\circ})$, then

$$K_d^{+} = K_a^{+} = K_T^{+} = Q^{*}.$$

Proof. Let $f \in Q^{*}$ and $h \in K_T$; then, by definition of K_T, there are $\epsilon_0 \in \mathbb{R}_{+}$, and $\theta : [0, \varepsilon_0] \to E$ such that

$$\lim_{\varepsilon \to 0^{+}} \frac{\theta(\varepsilon)}{\varepsilon} = 0,$$

and

$$x^{\circ} + \varepsilon\, h + \theta(\varepsilon) \in Q, \quad (\varepsilon \in (0, \varepsilon_0)).$$

Therefore

$$f(x^{\circ} + \varepsilon\, h + \theta(\varepsilon)) \geq f(x^{\circ}), \quad (\varepsilon \in (0, \varepsilon_0)).$$

Then $f(h) \geq 0$. Hence $f \in K_T^{+}$, that is to say

$$Q^{*} \subset K_T^{+}.$$

Let $f \in K_T^{+}$ and $x \in Q$, then by the convexity of Q, we have that $x - x^{\circ}$ is a tangent vector to Q in the point x°, then it follows

$$f(x - x^{\circ}) \geq 0,$$

or equivalently $f \in Q^*$.

Therefore

$$K_T^+ = Q^*.$$

Suppose (ii) hods, i.e., there exists $\overline{x} \in E$ such that $F(\overline{x}) < F(x^\circ)$, this implies that there exists $\overline{h} \in E$ such that $F'(x^\circ, \overline{h}) < 0$. In fact

Let $\overline{h} = \overline{x} - x^\circ$. Then, since F is continuous and convex, it follows

$$\begin{aligned} F'(x^\circ, \overline{h}) &\leq F(x^\circ + \overline{h}) - F(x^\circ) \\ &= F(\overline{x}) - F(x^\circ) < 0. \end{aligned}$$

Now, let us see that $K_a \subset K_d$; by Theorem 2.21, we have that

$$K_d = \{h \in E \,/\, F'(x^\circ, h) < 0\}.$$

Let $h \in K_a$, then there is $\varepsilon_0 \in I\!R_+$ such that $x_0 + \varepsilon\, h \in Q$ for all $\varepsilon \in (0, \varepsilon_0)$, therefore $F(x^\circ + \varepsilon\, h) \leq F(x^\circ)$, $\quad (\varepsilon \in (0, \varepsilon_0))$, which implies that $F'(x^\circ, h) \leq 0$. Since K_a is open, there is a neighbourhood U of h such that $U \subset K_a$. Then, for $\gamma \in I\!R_+$ small enough, we have that

$$h_\gamma := h + \gamma(h - \overline{h}) \in U.$$

Then

$$F'(x^\circ, h_\gamma) \leq 0 \quad \text{and} \quad h = \frac{1}{1+\gamma} h_\gamma + \frac{\gamma}{1+\gamma}\overline{h}.$$

Due to the fact that $F'(x^\circ, \cdot)$ is convex, we obtain that

$$F'(x^\circ, h) \quad \leq \quad \frac{1}{1+\gamma} F'(x^\circ, h_\gamma) + \frac{\gamma}{1+\gamma} F'(x^\circ, \overline{h}) < 0.$$

By Theorem 2.25, we have that $K_a \subset K_d$.

Let us prove that $K_a^+ = K_T^+$. In fact, condition (ii) implies that $\overset{\circ}{Q} \neq \emptyset$. Thus, by Theorem 2.26, it follows that

$$K_a = \{h \in E \,/\, h = \lambda(x - x^\circ), \quad x \in \overset{\circ}{Q}, \quad \lambda \in I\!R_+\}.$$

Let $f \in K_a^+$ and $x \in \overset{\circ}{Q}$, then $x - x^\circ \in K_a$, thus, we have that

$$f(x) \geq f(x^\circ) \qquad (x \in \overset{\circ}{Q}),$$

Given that F is continuous and convex, $\overline{\overset{\circ}{Q}} = \overline{Q} = Q$, we have that

$$f(x) \geq f(x^\circ) \qquad (x \in Q).$$

Therefore $f \in Q^*$, that is
$$K_a^+ \subset Q^* = K_T^+,$$
but $K_a \subset K_T$. Then
$$K_a^+ = K_T^+ = Q^*.$$
□

From the above proof, we have the following consequence

Corollary 2.31. *If F is convex and continuous, and there is $\overline{x} \in E$ such that $F(\overline{x}) < F(x^\circ)$, then*

$F'(x^\circ, h) < 0$ if and only if, there exists $\lambda \in I\!R_+,\ x \in E$ such that $F(x) < F(x^\circ)$ and $h = \lambda(x - x^\circ)$.

Theorem 2.32. *Let E_1, E_2 be topological linear spaces and $A : E_1 \longrightarrow E_2$ a linear operator. Let $E = E_1 \times E_2$ be the product space and consider*
$$K := G_A = \{x \in E \,/\, x = (x_1,\, x_2), \quad Ax_1 = x_2\}.$$
Then
$$K^+ = \{f \in E^*,\ f = (f_1,\, f_2) \,/\, f_1 = -A^* f_2\}.$$

The proof of above Theorem is trivial.

2.6.1. Minkowski-Farkas's Theorem and its Aplications

Theorem 2.33. *(Minkowski-Farkas see [18, pg 70]). Let E_1 and E_2 be topological linear spaces, and $K_2 \subset E_2$ a convex cone with apex at zero, and consider $A : E_1 \longrightarrow E_2$ a continuous linear operator. If we define*
$$K_1 := \{x_1 \in E_1 \,/\, Ax_1 \in K_2\},$$
and suppose that there exists $\overline{x}_1 \in E_1$ such that $A\overline{x}_1 \in \overset{\circ}{K}_2$, then
$$K_1^+ = A^* K_2^+.$$

Remark 2.34. Below we will give different versions of Minkowski-Farkas's Theorem. Before, we shall prove a known Lemma since part of its proof given here will be applied in the proof of Theorem 2.36.

Lemma 2.35. *Let E be a locally convex topological linear space, and A, B linear subspaces such that A is finite-dimensional, and B is closed. Then $A + B$ is also closed.*

Proof. Assume, without loss of generality, that $A \cap B = \{0\}$. Let $\{e_1, e_2, \ldots, e_n\}$ be a basis of A, then, since the space E is locally convex and $e_i \notin B,\ i = 1, 2, \ldots, n$, by Separation Theorem there are $g_i \in E^*$ $(i = 1, 2, \ldots, n)$ such that

$$\begin{aligned} g_i(e_i) &> 0 \qquad (i = 1, 2, \ldots, n) \\ g_i(x) &= 0 \qquad (i = 1, 2, \ldots, n;\ x \in B). \end{aligned}$$

We consider the following functionals

$$f_i = \frac{g_i}{g_i(e_i)} \qquad (i = 1, 2, \ldots, n).$$

Let's introduce the following operator

$$\begin{aligned} P &: E \longrightarrow I\!R^n \cong A, \\ P &= (f_1, f_2, \ldots, f_n). \end{aligned}$$

Then we have that

$$\begin{aligned} P(x) &= x \qquad (x \in A) \\ P(x) &= 0 \qquad (x \in B). \end{aligned}$$

Let $a_s + b_s \in A + B$ $(s \in S)$ be a generalized sequence such that $(a_s + b_s)$ converges to $z \in E$. The fact that P is continuous implies that $P(a_s + b_s)$ converges to $P(z)$, which implies that $(a_s) \to P(z)$, and given that A is closed $P(z) \in A$, and by the same reason $(b_s) \to z - P(z) \in B$, that is

$$z = P(z) + z - P(z), \quad P(z) \in A, \quad z - P(z) \in B.$$

□

Theorem 2.36. *Let E_1, E_2 topological linear spaces and $A : E_1 \longrightarrow E_2$ a continuous linear operator. Let $K_2 \subset E_2$ be a convex cone with apex at zero such that K_2^+ is finitely generated, and define*

$$K_1 := \{x_1 \in E_1 \,/\, Ax_1 \in K_2\}.$$

Then

$$(K \cap L)^+ = K^+ + L^+ \quad \textit{and} \quad K_1^+ = A^* K_2^+.$$

Proof. Let $E := E_1 \times E_2$, $K := E_1 \times K_2$ and $L = G_A$. By the hypotheses $K^+ = \{0\} \times K_2^+$ is closed and finite-dimensional, and since L^+ is a subspace, which is w^*- closed, then we claim that $K^+ + L^+$ is w^*-closed. In fact, since E_1^*, E_2^* are locally convex topological linear spaces with respect to the w^*-topology, it follows that

$E_1^* \times E_2^*$ is a linear locally convex topological linear space with the product-topology, then we can apply Lemma 2.35 by taking A the subspace generated by $K^+,\ B = L^+$ and $P : E^* \longrightarrow A$ in the same way as in lemma 2.35.

Let $a_s + b_s \in K^+ + L^+ = A + B$ $(s \in I)$ be a generalized convergent sequence to $z \in E^*$, then

$$a_s \to P(z) \quad \text{and} \quad b_s \to z - P(z).$$

Since $K^+,\ L^+$ are closed, we have that $P(z) \in K^+$ and $z - P(z) \in L^+$.

Now, by applying Lemma 2.5, we obtain that

$$(K \cap L)^+ = K^+ + L^+.$$

On the other hand, we have

$$\begin{aligned} K^+ &= \{(f_1,\, f_2) \in E^* \,/\, f_1 = 0,\ f_2 \in K_2^+\}, \\ L^+ &= \{(g_1,\, g_2) \in E^* \,/\, g_1 = -A^* g_2\}, \end{aligned}$$

by Theorem 2.32. Let $f_1 \in K_1^+$ and put $f := (f_1,\, 0)$. Then, $f \in (K \cap L)^+$. In fact, let $x \in (K \cap L)$. Hence $x = (x_1,\, Ax_1)$ and $Ax_1 \in K_2$, this implies that $x_1 \in K_1$ by definition of K_1, thus $f(x) = f_1(x_1) \geq 0$ for all $x \in (K \cap L)$, that is $f \in (K \cap L)^+$. Then, there exist

$$(0,\, h) \in K^+,\ h \in K_2^+,\ (g_1,\, g_2) \in L^+,\ g_1 = -A^* g_2,$$

such that

$$(f_1,\, 0) = (0,\, h) + (g_1,\, g_2),$$

which implies that $f_1 = g_1$ and $h + g_2 = 0$, and therefore $f_1 = A^* h,\ h \in K_2^+$. Thus

$$K_1^+ \subset A^* K_2^+.$$

This claim $A^* K_2^+ \subset K_1^+$ is trivial. □

Theorem 2.37. *Let E_1, E_2 be topological linear spaces and $A : E_1 \longrightarrow E_2$ a continuous linear operator, and let $K_2 \subset E_2$ be a convex cone with apex at point zero. Let us define the following cone*

$$K_1 := \{x_1 \in E_1 \,/\, Ax_1 \in K_2\}.$$

Suppose that there are $g \in E_1^*$ and $h \in K_2^+$ such that

$$A^* h \neq 0,\ K_1 = \{x_1 \in E_1 \,/\, g(x_1) \geq 0\}.$$

Then

$$K_1^+ = A^* K_2^+.$$

Proof. The proof that $A^*K_2^+ \subset K_1^+$ is trivial. Let us see that $K_1^+ \subset A^*K_2^+$. In fact, since $A^*K_2^+ \subset K_1^+$, by Theorem 2.28 there is $\beta \in I\!R_+$ such that $A^*h = \beta\, g$. Now, let $f_1 \in K_1^+$, then there exists $\lambda_1 \in I\!R_+$, such that $f_1 = \lambda_1\, g$. Therefore

$$f_1 = A^* \left(\frac{\lambda_1}{\beta} h \right), \quad \frac{\lambda_1}{\beta} h \in K_2^+,$$

which implies that $K_1^+ \subset A^*K_2^+$. $\square$

Proposition 2.38. *Let E_1, E_2 be Banach spaces and $A : E_1 \longrightarrow E_2$ a continuous linear operator such that Im $A = E_2$, and a convex cone $K_2 \subset E_2$ with apex at zero. Now, we define K_1 as follows*

$$K_1 := \{x_1 \in E_1 \,/\, Ax_1 \in K_2\}.$$

Then

$$K_1^+ \subset Im\ A^*.$$

Proof. From the fact that $K_2^+ = \overline{K}_2^+$, we can assume without loss of generality that $0 \in K_2$; which implies that Ker $A \subset K_1$, then by item $d)$ from Proposition 2.3, we have that $K_1^+ \subset (\text{Ker}A)^+$. But $(\text{Ker } A)^+ = (\text{Ker } A)^\perp$, then from the factorization lemma from [22, pg 16], we get that $(\text{Ker } A)^\perp = \text{Im } A^*$. $\square$

Proposition 2.39. *Let E_1, E_2 be topological linear spaces and $A_i : E_1 \longrightarrow E_2$ $(i = 1, 2, \ldots, h)$ continuous linear operators, and consider $K_2 \subset E_2$ a convex cone with apex at zero. Let us define the following cones*

$$K_1 := \{x_1 \in E_1 \,/\, A_i x_1 \in K_2 \quad i = 1, 2, \ldots, n\}.$$

Suppose that there exists $\overline{x}_1 \in E_1$ such that $A_i\overline{x}_1 \in \overset{\circ}{K}_2$ $\quad (i = 1, 2, \ldots, n)$. Then

$$K_1^+ = \left(\sum_{i=1}^n A_i \right)^* K_2^+.$$

Proof. Let us define the following cones

$$K_{1i} := \{x_1 \in E_1 \,/\, A_i x_1 \in K_2\}; \qquad i = 1, 2, \ldots, n.$$

Then by the continuity of A_i $(i = 1, 2, \ldots, n)$, we have that $\overline{x}_1 \in \overset{\circ}{K}_{1i}$, $(i = 1, 2, \ldots, n)$, which implies that $\left(\bigcap_{i=1}^n \overset{\circ}{K}_{1i} \right) =: K_3 \neq \emptyset$. So, by Lemma 2.7 it follows that

$$K_3^+ = \sum_{i=1}^n (\overset{\circ}{K}_{1i})^+.$$

We have that $K_3 \subset K_1$, which implies

$$K_1^+ \subset K_3^+ = \sum_{i=1}^{n} (\overset{\circ}{K}_{1i})^+ = \sum_{i=1}^{n} K_{1i}^+$$

Therefore $K_1^+ \subset \sum_{i=1}^{n} K_{1i}^+$ and $\sum_{i=1}^{n} K_{1i}^+ \subset K_1^+$, which implies that

$$K_1^+ = \sum_{i=1}^{n} K_{1i}^+.$$

But, from Theorem 2.33, we have that $K_{1i}^+ = A_i^* K_2^+$ $(i = 1, 2, \dots, n)$, then

$$K_1^+ = \sum_{i=1}^{n} A_i^* \, K_2^+.$$

□

To conclude this section, below we will see some applications of the Minkowski-Farkas's Theorem and its versions.

Proposition 2.40. *Let us consider* $E = \mathcal{PW}([0,T], I\!R^n)$ *and the following cone*

$$K = \{x \in E : x(T) = 0\}.$$

Then $f \in K^+$ *if, an only if, there is* $a \in I\!R^n$ *such that*

$$f(x) = \langle a, \, x(T) \rangle \qquad (x \in E).$$

Proof. The sufficiency is trivial. Let us prove the necessity. Define the operator $L : E \longrightarrow I\!R^n$, $L(x) := x(T)$, $(x \in E)$, and consider $f \in K^+$. Then, Im $L = I\!R^n$ and Ker $L \subset$ Ker f, hence by The Factorization Lemma from (see [22, pg 15]), there is a linear-continuous function $g : I\!R^n \longrightarrow I\!R$ such that

$$f = g \circ L.$$

But it is well known that g has the following form

$$g(x) = \langle a, \, x \rangle, \qquad (x \in I\!R^n),$$

for some fixed $a \in I\!R^n$. Therefore

$$f(x) = \langle a, \, x(T) \rangle, \qquad (x \in E).$$

□

Example 2.41. Let $A : [0,\, T] \longrightarrow I\!R^{n\times n}$ and $B : [0,\, T] \longrightarrow I\!R^{n\times r}$ be measurable and bounded functions; and consider the following linear control system

$$\begin{aligned} \dot{x}(t) &= A(t)x(t) + B(t)u(t), \quad t \in [0,T], a.e, && (2.6)\\ x(T) &= 0 && (2.7) \end{aligned}$$

where $(x,\, u) \in E_1 := \mathcal{PW}([0,T];\mathbb{R}^n) \times L^r_\infty\,[0,\, T]$. Let us define the following cone

$$K_1 := \{(x,\, u) \in E_1 \,/\, (2.6) - (2.7) \quad \text{hold}\}\,.$$

Proposition 2.42. *If (2.6) is controllable, then dim $K_1^+ = n$, and also for all $f \in K_1^+$ there is $a \in I\!R^n$ such that*

$$f(x,\, u) = \left\langle a,\, \int_0^T [A(t)x(t) + B(t)u(t)]dt \right\rangle, \qquad ((x,\, u) \in E_1).$$

Proof. Let $E_2 := \mathcal{PW}([0,T];\mathbb{R}^n)$ and $K_2 := \{x \in E_2 \,/\, x(T) = 0\}$, and define the following operator $\Lambda : E_1 \longrightarrow E_2$ as follows

$$\Lambda(x,\, u)(t) := \int_0^t [A(\tau)x(\tau) + B(\tau)u(\tau)]d\tau \qquad ((x,\, u) \in E_1,\ t \in [0,\, T]).$$

Then

$$K_1 = \{(x,\, u) \in E_1 \,/\, \Lambda(x,\, u) \in K_2\}.$$

Λ is a continuous linear operator and dim $K_2^+ = n$. In fact, the assertion for Λ is trivial. Let us see that dim $K_2^+ = n$; for which it is enough to see the following: $f_2 \in K_2^+$ if, an only if, there is $a \in I\!R^n$ such that

$$f_2(x) = \langle a,\, x(T)\rangle \qquad (x \in E_2).$$

This follows from Proposition 2.40.
Now. let $\{e_1,\, e_2, \ldots,\, e_n\}$ be the canonic basis of $I\!R^n$, and define the following linear functionals $\overline{f}_i : E_2 \to I\!R, \quad i = 1,\, 2, \ldots,\, n$

$$\overline{f}_i(x) = \langle e_i,\, x(T)\rangle\,, \qquad (x \in E_2).$$

Then, given $f_2 \in K_2^+$ there exists $a \in I\!R^n$ such that $f_2(x) = \langle a,\, x(T)\rangle$. On the other hand, we now that $a = \sum_{i=}^n a_i e_i$. Then,

$$f_2 = \sum_{i=1}^n a_i \overline{f}_i \qquad (a_i \in I\!R,\ i = 1,\, 2, \ldots,\, n).$$

Let us see that $\{\overline{f}_1,\, \overline{f}_2, \ldots,\, \overline{f}_n\}$ is a linearly independent family, for which we consider $\alpha_i \in I\!R$ $(i = 1,\, 2, \ldots,\, n)$ such that

$$\alpha_1\overline{f}_1 + \alpha_2\overline{f}_2 + \cdots + \alpha_n\overline{f}_n = 0.$$

Next, since (2.6) is controllable, then for each $e_i,\ i = 1, 2, \ldots, n$, there is $(x_i, u_i) \in E_1 \quad (i = 1, \ldots, n)$ such that

$$x(T) = \alpha_i e_i \quad (i = 1, 2, \ldots, n).$$

Thus

$$\alpha_1 \overline{f}_1(x_i) + \cdots + \alpha_n \overline{f}_n(x_i) = \sum_{i=}^{n} \alpha_i^2 = 0,$$

which proves that $\{\overline{f}_1, \overline{f}_2, \ldots, \overline{f}_n\}$ is linearly independent; therefore $\dim K_2^+ = n$. Then, by Theorem 2.36 (Minkowski-Farkas´s theorem version), we have that

$$K_1^+ = \Lambda^* K_2^+.$$

That is to say, for all $f_1 \in K_1^+$, there is $a \in I\!R^n$ such that

$$\begin{aligned} f_1(x, u) &= \langle a, \Lambda(x, u)(T) \rangle \\ &= \left\langle a \int_0^T [A(t)x(t) + B(t)u(t)]dt \right\rangle \qquad ((x, u) \in E_1). \end{aligned}$$

□

Proposition 2.43. *Let $A, \mathcal{J}_k : [0, T] \longrightarrow I\!R^{n \times n}, k = 1, 2, 3, \ldots, p$ and $B : [0, T] \longrightarrow I\!R^{n \times r}$ be measurable and bounded matrix functions. Suppose the following impulsive linear system is controllable on $[0, T]$ for any $\bar{b} = (\bar{b}_1, \bar{b}_2, \ldots, \bar{b}_p) \in I\!R^{np} = (I\!R^n)^p$*

$$\begin{cases} \dot{x}(t) = A(t)x(t) + B(t)u(t), & t \in (0, \tau], \quad t \neq t_k \\ x(t_k) = \mathcal{J}_k(t_k)x(t_k^-) + \bar{b}_k, & k = 1, 2, 3, \ldots, p., \end{cases} \tag{2.8}$$

where $(x, u) \in E_1 := \mathcal{PW}([0, T]; \mathbb{R}^n) \times L_\infty^r [0, T]$. Let us define the following cone

$$K_2 := \left\{ (x, u) \in E_1 \,/\, x(T) = 0, \quad x(t_k^+) - \mathcal{J}_k(t_k)x(t_k^-) = 0, \quad k = 1, 2, 3, \ldots, p \right\}.$$

Then $\dim K_2^+ = n(p + 1)$, and also for all $f \in K_2^+$ there is $a \in I\!R^{n(p+1)}$ such that

$$f(x, u) = \left\langle a, \quad (x(T), x(t_1) - \mathcal{J}_k(t_1)x(t_1^-), \ldots, x(t_p) - \mathcal{J}_k(t_p)x(t_p^-)) \right\rangle, \quad (x, u) \in E_1.$$

Proof. Consider the following linear and continuous operator $\Lambda : E_1 \to I\!R^{n(1+p)}$ defined as follows

$$\Lambda(x, u) = \left(x(T), x(t_1) - \mathcal{J}_k(t_1)x(t_1^-), \ldots, x(t_p) - \mathcal{J}_k(t_p)x(t_p^-) \right).$$

Since system (2.8) is controllable, then $\mathrm{Im}\, \Lambda = I\!R^{n(1+p)}$. Now, let $f \in K_2^+$, then $\ker \Lambda \subset \ker f$, and by the factorization lemma from (see [22, pg 15]), there is a linear-continuous function $g : I\!R^{n(1+p)} \longrightarrow I\!R$ such that

$$f = g \circ \Lambda.$$

But, it is well known that g has the following form

$$g(x) = \langle a,\, x\rangle, \qquad (x \in I\!R^{n(1+p)}),$$

for some fixed $a \in I\!R^{n(1+p)}$. Therefore

$$f(x,u) = \langle a, \quad (x(T), x(t_1) - \mathcal{J}_k(t_1)x(t_1^-), \dots, x(t_p) - \mathcal{J}_k(t_p)x(t_p^-))\rangle, \quad ((x,u) \in E_1).$$

Now. let $\{e_1, e_2, \dots, e_{n(p+1)}\}$ be the canonic basis of $I\!R^{n(p+1)}$, where $e_i = (e_{i,1}, e_{i,2}, \dots, e_{i,(p+1)})$, with $e_{i,k} \in I\!R^n$, and define the following linear functionals $\overline{f}_i : E_1 \to I\!R, \quad i = 1, 2, \dots, n(p+1)$,

$$\overline{f}_i(x) = \left\langle e_i,\, (x(T), x(t_1) - \mathcal{J}_1(t_1)x(t_1^-), \dots, x(t_p) - \mathcal{J}_p(t_p)x(t_p^-))\right\rangle, \, ((x,u) \in E_1).$$

Then, given $f_2 \in K_2^+$ there exists $a \in I\!R^{n(p+1)}$ such that

$$f_2(x,u) = \left\langle a,\, (x(T), x(t_1) - \mathcal{J}_1(t_1)x(t_1^-), \dots, x(t_p) - \mathcal{J}_p(t_p)x(t_p^-))\right\rangle.$$

On the other hand, we know that $a = \sum_{i=1}^{p+1} a_i e_i$. Then,

$$f_2 = \sum_{i=1}^{p+1} a_i \overline{f}_i \qquad (a_i \in I\!R,\; i = 1,\, 2, \dots,\, n(p+1)).$$

Let us see that $\{\overline{f}_1,\, \overline{f}_2, \dots,\, \overline{f}_{(p+1)}\}$ is a linearly independent family, for which we consider $\alpha_i \in I\!R$ $(i = 1,\, 2, \dots,\, p+1)$ such that

$$\alpha_1 \overline{f}_1 + \alpha_2 \overline{f}_2 + \cdots + \alpha_{n(p+1)} \overline{f}_{(p+1)} = 0.$$

Since, for any $\bar{b} = (\bar{b}_1, \bar{b}_2, \dots, \bar{b}_p) \in I\!R^{np}$ the impulsive system (2.8) is controllable, then for each $e_i,\; i = 1,\, 2, \dots,\, n(p+1)$, with $e_i = (e_{i1}, e_{i2}, \cdots, e_{ip})$, there is $(x_i,\, u_i) \in E_1 \quad (i = 1, \dots,\, p+1)$ such that

$$x(T) = \alpha_i e_{i1}, \quad \text{and} \quad x(t_k) - \mathcal{J}_k(t_k)x(t_k^-) = \alpha_i e_{ik} \;\; (k = 2,\, 3, \dots,\, p+1).$$

Thus

$$\alpha_1 \overline{f}_1(x_1, u_1) + \cdots + \alpha_n \overline{f}_{(p+1}(x_{(p+1},\, u_{(p+1)}) = \sum_{i=}^{p+1} \alpha_i^2 = 0,$$

which proves that $\{\overline{f}_1,\, \overline{f}_2, \dots,\, \overline{f}_{(p+1)}\}$ is linearly independent; therefore dim $K_2^+ = n(p+1)$.

□

Now, we will give an important example related with support functionals.

Example 2.44. Let $M \subset I\!R^r$ and $Q := \{u \in L^r_\infty[0,\,T]\,/\,u(t) \in M,\ t \in [0,\,T], a.e.\}$ and consider $u^\circ \in Q,\ a \in L^r_1[0,\,T]$ and $f : L^r_\infty \longrightarrow I\!R$ defined as follows

$$f(u) := \int_0^T \langle a(t),\, u(t)\rangle dt, \qquad (u \in L^r_\infty).$$

Let us suppose that $f(u) \geq f(u^\circ) \quad (u \in Q)$, then for all $U \in M$ and almost all $t \in [0,\,T]$

$$\langle a(t),\, U - u^\circ(t)\rangle \geq 0.$$

For details of this example see [18, pg 76].

Finally, we have the well known formula for integrating by part in the Lebesgue Integral

Proposition 2.45. *(Integration by parts for Lebesgues integral)*
Let $f, g : [\alpha, \beta] \to I\!R$ be two differentiable function almost every well, such that $f'g, fg' \in L^1([\alpha,\beta], I\!R)$. The the following formula holds

$$\int_{[\alpha,\beta]} f'g d\mu = \lim_{t\to\beta} f(t)g(t) - \lim_{t\to\alpha} f(t)g(t) - \int_{[\alpha,\beta]} fg' d\mu.$$

3. Optimal Control Problem for Impulsive Differential Equations

In this section we will show how Dubovitskii–Milyutin theory can be applied to generalize the maximum principle of [18]. The generalization consists in admitting an finite number of impulses in the differential equation presented in the problem. We will also see that in a linear dynamics case, under certain additional conditions, the maximum principle is a sufficient condition for optimality. After that, we shall give an example that illustrates the applicability of the main result of this section.

3.1. Maximum Principle in the Space $\mathcal{PW}([0,T];\mathbb{R}^n) \times L^r_\infty$

Let $n,\, r \in \mathbb{N}$ and $T \in I\!R_+$, and consider the functions $\Phi,\, \varphi, \mathcal{J}_k$:

$$\begin{aligned}
\varphi &: I\!R^n \times I\!R^r \times [0,\,T] \longrightarrow I\!R, \\
\Phi &: I\!R^n \times I\!R^r \times [0,\,T] \longrightarrow I\!R^n, \\
\mathcal{J}_k &: I\!R^n \longrightarrow I\!R^n,
\end{aligned}$$

where $\mathcal{PW}([0,T];\mathbb{R}^n)$ and L^r_∞ are define by

$$\begin{aligned}
\mathcal{PW}([0,T];\mathbb{R}^n) &= \{z : [0,T] \to \mathbb{R}^n : z \in C(J';\mathbb{R}^n), \exists z(t_k^+), z(t_k^-) \\
&\quad \text{and} \quad z(t_k) = z(t_k^-), \quad k = 1,2,3\ldots,p\},
\end{aligned}$$

where $J = [0,T]$ and $J' = J\backslash\{t_1, t_2, \dots, t_p\}$, endowed with the norm

$$\|z\|_0 = \sup_{t\in[0,T]} \|z(t)\|_{\mathbb{R}^n},$$

and $L^r_\infty = L^r_\infty([0,T];\mathbb{R}^r)$ be the space of measurable function essentially bounded with essential norm.

Let us suppose that the following conditions are fulfilled

a) Φ, φ and $\mathcal{J}_k$ are continuous functions, with derivatives Φ_x, Φ_u, φ_x, φ_u, $\mathcal{J}'_k$ are bounded functions on compact sets of $I\!R^n \times I\!R^r \times [0,\, T]$.

b) $M \subset I\!R^r$ is convex and closed with $\overset{\circ}{M} \neq \emptyset$.

c) The following linear system is controllable

$$\dot{x}(t) = \varphi_x(x^\circ(t),\, u^\circ(t),\, t)x(t) + \varphi_u(x^\circ(t),\, u^\circ(t),\, t)u(t), \quad t \in (0,\tau],\ a.e. \tag{3.1}$$

d) The corresponding impulsive linear variational equations around the point $(x^\circ,\, u^\circ) \in E$ is controllable on $[0,T]$ for any $\bar{b} = (\bar{b}_1, \bar{b}_2, \dots, \bar{b}_p) \in (I\!R^n)^p$

$$\begin{cases} \dot{x}(t) = \varphi_x(x^\circ(t),\, u^\circ(t),\, t)x(t) + \varphi_u(x^\circ(t),\, u^\circ(t),\, t)u(t), & t \in (0,T], \quad t \neq t_k \\ x(t_k^+) = \mathcal{J}'_k(x^0(t_k^-))x(t_k^-) + \bar{b}_k, \quad k = 1,2,3,\dots,p., \end{cases} \tag{3.2}$$

Remark 3.1. *According to the results presented in the references [10, 28, 29, 30, 31, 32, 33]) on the controllability of control systems governed by impulsive differential equations, a sufficient condition for system (3.2) to be controllable is that system (3.1) is controllable and the following condition holds for the impulses.*

$$\|\mathcal{J}'_k(x^0(t_k^-))\| < \frac{1}{p}, \quad k = 1,2,3,\dots,p. \tag{3.3}$$

Theorem 3.1. *Suppose that conditions a) - d) are fulfilled. Let $(x^\circ,\, u^\circ) \in E$ be a solution of Problem 1.1:*
Then, there exists $\lambda_0 \in I\!R_{+0}$ and a function $\psi \in \mathcal{PW}([0,T];\mathbb{R}^n)$ such that λ_0 and ψ both are different from zero, and ψ is solution of the following differential equation

$$\begin{cases} \dot{\psi}(\tau) = -\varphi_x^*(x^\circ(\tau),\, u^\circ(\tau),\, \tau)\psi(\tau) + \lambda_0\Phi_x(x^\circ(\tau),\, u^\circ(\tau),\, \tau), \\ \psi(T) = a. \end{cases} \tag{3.4}$$

Moreover, for all $U \in M$ and almost all $t \in [0,\, T]$ the following inequality hols

$$\langle -\varphi_u^*(x^\circ(t),\, u^\circ(t),\, t)\psi(t) + \lambda_0\, \Phi_u(x^\circ(t),\, u^\circ(t),\, t),\, U - u^\circ(t)\rangle \geq 0 \tag{3.5}$$

Proof. Let $\overline{F} : E \longrightarrow I\!R$ be a function defined as follows

$$\overline{F}(x,\, u) = \int_0^T \Phi(x(t),\, u(t),\, t)dt,$$

and let $Q := Q_1 \cap Q_2$ where Q_2, Q_1 are given by points $(x,\, u) \in E$, which satisfy (1.3)-(1.5) and (1.6) respectively.
Then, Problem 1.1 is equivalent to

$$\begin{cases} \overline{F}(x,\, u) \longrightarrow \min \\ (x,\, u) \in Q. \end{cases}$$

a) Analysis of the function $\overline{F}$.

Let $K_0 := K_d(F,\, (x^\circ,\, u^\circ))$ be the decay cone of $\overline{F}$ in the point $(x^\circ,\, u^\circ)$. Then, by Theorem 2.22, we have that

$$K_0 = \{(x,\, u) \in E \,/\, \overline{F}(x^\circ,\, u^\circ)(x,\, u) < 0\}.$$

Suppose for a moment that $K_0 \neq \emptyset$, then by Theorem 2.29 we obtain

$$K_0^+ = \{-\lambda_0\, \overline{F}(x^\circ,\, u^\circ) \,/\, \lambda_0 \in I\!R_{+0}\}.$$

By example 2.23, we obtain that

$$\overline{F}'(x^\circ,\, u^\circ)(x,\, u) = \int_0^T [\Phi_x(x^\circ,\, u^\circ,\, t)x(t) + \Phi_u(x^\circ,\, u^\circ,\, t)u(t)]dt, \qquad ((x,\, u) \in E).$$

Therefore, for all $f_0 \in K_0^+$, there exists $\lambda_0 \in I\!R_{+0}$ such that

$$f_0(x,\, u) = -\lambda_0 \int_0^T [\Phi_x(x^\circ,\, u^\circ,\, t)x(t) + \Phi_u(x^\circ,\, u^\circ,\, t)u(t)]dt, \qquad ((x,\, u) \in E).$$

b) Analysis of constraint Q_1.

Let us consider the set

$$Q_1' := \{u \in L_\infty^r[0,\, T] \,/\, u(t) \in M, \quad t \in [0,\, T], \quad a.e.\}.$$

Then $Q_1 = \mathcal{PW}([0,T];\mathbb{R}^n) \times Q_1'$. Moreover, by the hypothesis M is convex and closed, with $\overset{\circ}{M} = \emptyset$. So, the following statements hold

i) Q_1, Q_1' are convex and closed.

ii) $\overset{\circ}{Q}_1 \neq \emptyset, \quad \overset{\circ}{Q'}_1 \neq \emptyset$.

If we call K_1 the admissible cone to Q_1 in $(x^\circ,\, u^\circ) \in Q_1$, then

$$K_1 = \mathcal{PW}([0,T];\mathbb{R}^n) \times K_1',$$

where K_1' is the admissible cone Q_1' in $u^\circ \in Q_1'$.

Therefore, for all $f_1 \in K_1^+$ there is $f_1' \in K_1'^+$ such that $f_1 = (0,\, f_1')$.

By Theorem 2.26, it follows that f_1' is a support of Q_1' at u°.

c) Analysis of the constraint Q_2.

Let us find the tangent cone to Q_2 at the point $(x^\circ,\, u^\circ)$

$$K_2 := K_T(Q_2,\, (x^\circ,\, u^\circ)).$$

Consider the space $E_1 = \mathcal{PW}([0,T];\mathbb{R}^n) \times I\!R^{n(1+p)} = E_2$ and the operator: $P : E_1 \to E_2$ defined by

$$P(x,u)(t) = \left(x(t) - x_0 - \int_0^t \varphi(x(l),u(l),l)dl, \quad S(x,u), \quad x(T) - x_1 \right),$$

where

$$S(x,u) = \left(x(t_1) - \mathcal{J}_1(x(t_1^-)), x(t_2) - \mathcal{J}_2(x(t_2^-)) \cdots, x(t_p) - \mathcal{J}_p(x(t_p^-)) \right).$$

Then

$$P'(x^0,u^0)(\overline{x},\overline{u}) =$$
$$\left(\overline{x}(t) - \int_0^t (\varphi_x(x^\circ(l),\, u^\circ(l),\, l)\overline{x}(l) + \varphi_u(x^\circ(l),\, u^\circ(l),\, l)\overline{u}(l))dl, \quad S'(\overline{x},\overline{u}), \quad \overline{x}(T) \right),$$

with

$$S'(\overline{x},\overline{u}) = \left(\overline{x}(t_1) - \mathcal{J}_1'(x^0(t_1^-))\overline{x}(t_1^-), \cdots\cdots, \overline{x}(t_p) - \mathcal{J}_p'(x^0(t_p^-))\overline{x}(t_p^-) \right).$$

We want to find conditions under which the operator $P'(x^0,u^0)$ is onto in order to apply Lustenik theorem 2.27. So, for $(a(\cdot), b_1, b_2, \ldots, b_p, x_1) \in E_2$, we want to solve the equation

$$P'(x^0,u^0)(\overline{x},\overline{u}) = (a(\cdot), b_1, b_2, \ldots, b_p, x_1).$$

Now, suppose that $\overline{u} = 0$. Then, from ([25], pg 89), we know that the following Volterra integral equation

$$z(t) = a(t) + \int_0^t (\varphi_x(x^\circ(l),\, u^\circ(l),\, l)z(l)dl,$$

has a solution $z \in \mathcal{PW}([0,T];\mathbb{R}^n)$.
Next, since the impulsive linear variational equation (3.2) is controllable, for a point $(\bar{b}_1, \bar{b}_2, \ldots, \bar{b}_p) \in I\!R^{np}$ with

$$\bar{b}_k = b_k - z(t_k) + \mathcal{J}_k'(x^0(t_k^-))z(t_k^-), \quad k = 1,2,3,\ldots,p,$$

there exists a control $\overline{u} \in L_\infty^r$ such that the corresponding solution $y(t)$ of (3.2) satisfies $y(T) = x_1 - z(T)$ and

$$y(t_k) = \mathcal{J}_k'(x^0(t_k^-))y(t_k^-) + \bar{b}_k, \quad k = 1,2,3,\ldots,p.$$

Let us make the following change of variable $\overline{x} = y + z$. then

$$\begin{aligned}
&P'(x^0,u^0)(\overline{x},\overline{u})(t) = \Big((y+z)(t) - \\
&\int_0^t (\varphi_x(x^\circ,u^\circ,l)(y+z)(l) + \varphi_u(x^\circ,u^\circ,l)\overline{u}(l))dl, \quad S'(\overline{x},\overline{u}), \quad (y+z)(T)\Big) \\
&= \Big(y(t) + a(t) - \\
&\int_0^t (\varphi_x(x^\circ,\, u^\circ,\, l)y(l) + \varphi_u(x^\circ,\, u^\circ,l)\overline{u}(l))dl, \quad S'(\overline{x},\overline{u}), \quad (y+z)(T)\Big) \\
&= (a(t), \quad S'(\overline{x},\overline{u}), \quad x_1)\,.
\end{aligned}$$

Now, we shall see that $S'(\overline{x},\overline{u}) = (b_1, b_2, \ldots, b_p)$. In fact,

$$\begin{aligned}
&S'(\overline{x},\overline{u}) = \\
&\big(\overline{x}(t_1) - \mathcal{J}_1'(x^0(t_1^-))\overline{x}(t_1^-), \cdots\cdots, \overline{x}(t_p) - \mathcal{J}_p'(x^0(t_p^-))\overline{x}(t_p^-)\big) = \\
&\big((y+z)(t_1) - \mathcal{J}_1'(x^0(t_1^-))(y+z)(t_1^-), \cdots, (y+z)(t_p) - \mathcal{J}_p'(x^0(t_p^-))(y+z)(t_p^-)\big) \\
&= \big(\bar{b}_1 + z(t_1) - \mathcal{J}_1'(x^0(t_1^-))z(t_1^-), \cdots\cdots, \bar{b}_p + z(t_p) - \mathcal{J}_p'(x^0(t_p^-))z(t_p^-)\big) \\
&= (b_1, b_2, \ldots, b_p)\,.
\end{aligned}$$

Therefore, the operator $P'(x^0,u^0)$ is onto. Then, applying Lusternik's Theorem 2.27, we get that tangent cone K_2 is given by

$$K_2 = \{(x,\, u) \in E_1 \,/\, P'(x^\circ,\, u^\circ)(\overline{x},\overline{u}) = 0\}.$$

i.e., K_2 is the set of points $(\overline{x},\overline{u}) \in E_1$ such that

$$\begin{aligned}
\dot{\overline{x}}(t) &= \varphi_x(x^\circ(t),\, u^\circ(t),\, t)\overline{x}(t) + \varphi_u(x^\circ(t),\, u^\circ(t),\, t)u(t), \quad t \neq t_k && (3.6)\\
\overline{x}(t_k^+) &= \mathcal{J}_k'(x^0(t_k^-))\overline{x}(t_k^-), \quad k = 1,2,3,\ldots,p. && (3.7)\\
\overline{x}(T) &= 0 && (3.8)
\end{aligned}$$

Consider the following linear subspaces

$$L_1 = \{(\overline{x},\overline{u}) \in E_1/(3.6) - (3.7) \quad \text{hold}\}, \quad L_2 = \{(\overline{x},\overline{u}) \in E_1/ \quad \overline{x}(T) = 0\}.$$

Then, $K_2 = L_1 \cap L_2$. Now, let us compute K_2^+. By Proposition 2.40, we have that $f_{22} \in L_2^+$ if, and only if, there exists $a \in I\!R^n$ such that

$$f_{22}(x,\, u) = \langle a,\, x(T)\rangle \qquad ((x,\, u) \in E).$$

Moreover, the controllability of systems (3.1) - (3.2) implies that $L_1 + L_2$ is closed, then it follows that $L_1^+ + L_2^+$ is w^*- closed; hence by Lemma 2.5 we obtain that

$$K_2^+ = L_1^+ + L_2^+.$$

Since L_1 is a linear subspace, it follows from Theorem 2.28 that, for any

$f_{21} \in L_1^+$, $f_{21}(\overline{x}, \overline{u}) = 0$ for all $(\overline{x}, \overline{u})$ satisfying (3.6)-(3.7).

e) Euler-Lagrange equation.

It is easy to see that K_0, K_1, K_2, are convex cones. Hence, by Theorem 2.15 there are functionals $f_i \in K_i^+ \quad (i = 0,\, 1,\, 2,)$ not all zero such that

$$f_0 + f_1 + f_2 = f_0 + f_1 + f_{21} + f_{22} = 0. \tag{3.9}$$

Equation (3.9) can be written in the following form

$$\begin{gathered}
-\lambda_0 \int_0^T [\Phi_x(x^\circ,\, u^\circ,\, t)x(t) + \Phi_u(x^\circ,\, u^\circ,\, t)u(t)]dt + \\
f_1'(x,\, u) + f_{21}(x,\, u) + \langle a,\, x(T)\rangle = 0, \quad ((x,\, u) \in E).
\end{gathered}$$

Now, for all $u \in L_\infty^r$ there exists $\overline{x}$, solution of system (3.6)-(3.7) with $\overline{x}(0) = 0$. Then $(\overline{x},\, u) \in L_1$. Therefore $f_{21}(\overline{x},\, u) = 0$.
Let ψ be the solution of equation (3.4), that is

$$\begin{cases} \dot{\psi}(t) = -\varphi_x^*(x^\circ(\tau),\, u^\circ(\tau),\, \tau)\psi(\tau) + \lambda_0 \Phi_x(x^\circ(\tau),\, u^\circ(\tau),\, \tau) \\ \psi(T) = a. \end{cases}$$

Multiplying both sides of this equation by $\overline{x}$ and integrating from 0 to T, we get

$$\begin{aligned}
&\lambda_0 \int_0^T \Phi_x(x^\circ,\, u^\circ,\, t)\overline{x}(t)dt - \langle a,\, x(T)\rangle = \\
&\int_0^T \langle \dot{\psi}(t),\, \overline{x}(t)\rangle dt + \int_0^T \langle \varphi_x^*(x^\circ,\, u^\circ,\, t)\psi(t), \overline{x}(t)\rangle dt \\
&-\langle a,\, \overline{x}(T)\rangle = \langle \psi(t),\, \overline{x}(t)\rangle]_0^T - \int_0^T \langle \psi(t),\, \dot{\overline{x}}(t)\rangle dt \\
&+ \int_0^T \langle \varphi_x^*(x^\circ,\, u^\circ,\, t)\psi(t), \overline{x}(t)\rangle dt - \langle a,\, \overline{x}(T)\rangle = \langle \psi(T),\, \overline{x}(T)\rangle - \langle \psi(0),\, \overline{x}(0)\rangle \\
&-\langle a,\, \overline{x}(T)\rangle + \int_0^T \langle \psi(t), \varphi_x(x^\circ,\, u^\circ,\, t)\overline{x}(t) - \dot{\overline{x}}(t)\rangle dt = \\
&-\int_0^T \langle \psi(t), \varphi_u(x^\circ,\, u^\circ,\, t)\overline{u}(t)\rangle dt = -\int_0^T \langle \varphi_u^*(x^\circ,\, u^\circ,\, t)\psi(t), \overline{u}(t)\rangle dt.
\end{aligned}$$

Then, from Euler-Lagrange equation (3.9), we obtain for ($u \in L_\infty^r[0, T]$), that

$$f_1'(t) = \int_0^T \langle -\varphi_u^*(x^\circ(t), u^\circ(t), t)\psi(t) + \lambda_0 \, \Phi_u(x^\circ(t), u^\circ(t), t)u(t)\rangle dt. \quad (3.10)$$

Since f_1' is a support of Q_1' at the point $u^\circ \in Q_1'$, from example 2.44, it follows that

$$\langle -\varphi_u^*(x^\circ(t), u^\circ(t), t)\psi(t) + \lambda_0 \, \Phi_u(x^\circ(t), u^\circ(t), t), U - u^\circ(t)\rangle \geq 0,$$

for all $U \in M$ and almost all $t \in [0, T]$.
Now, we will see that the case $\lambda_0 = 0, \ \psi = 0$, is not possible. In fact

If $\psi = 0$, then $\psi(T) = a = 0$. Thus

$$f_{22}(x, u) = \langle a, x(T)\rangle = 0 \qquad ((x, u) \in E),$$

that is $f_{22} \equiv 0$. So, from the fact that $\lambda_0 = 0$, we get that $f_0 = 0$. Also, from (3.10), we have that $f_1'(u) = 0 \quad (u \in L_\infty^r[0, T])$; then from Euler– Lagrange equation it follows that $f_{21} = 0$, where

$$f_2 = f_{21} + f_{22} = 0,$$

which contradicts Theorem 2.15.
So far, we have two additional assumptions:
Firstly, we assumed that $K_0 \neq \emptyset$, and secondly, we assumed that system

$$\dot{x} = \varphi_x(x^\circ, u^\circ, t)x(t) + \varphi_u(x^\circ, u^\circ, \ , t)u(t)$$

is controllable.
Now, we will prove, that these assumptions are superfluous. In fact, if $K_0 = \emptyset$, then by definition of K_0, we have that

$$\int_0^T [\Phi_x(x^\circ(t), u^\circ(t), t)x(t) + \Phi_u(x^\circ(t), u^\circ(t), t)u(t)]dt = 0 \quad ((x, u) \in E).$$

Let us put $\lambda_0 = 1, \ \psi(T) = a = 0$, then, from last computation, we have that

$$\int_0^T \langle \Phi_x^*(x^\circ, u^\circ, t)\psi(t), x(t)\rangle dt = -\int_0^T \langle \varphi_u^*(x^\circ, u^\circ, t)\psi(t), u(t)\rangle dt,$$

for all (x, u) such that x is solution of equation the (3.6)-(3.7). Then

$$\int_0^T \langle \varphi_u^*(x^\circ(t), u^\circ(t), t)\psi(t) + \Phi_u(x^\circ(t), u^\circ(t), t)u(t)\rangle dt = 0 \quad (u \in L_\infty^r[0, T])$$

which implies that

$$\langle -\varphi_u^*(x^\circ,\, u^\circ,\, t)\psi(t) + \Phi_u(x^\circ,\, u^\circ,\, t),\, U - u^\circ(t)\rangle = 0,$$

for all $U \in M$ and almost all $t \in [0,\, T]$.

Now, suppose that system (3.1) is not controllable, then there is a non-trivial function $\psi \in C([0,\, T], I\!R^n)$ that is solution of

$$\dot{\psi}(t) = \varphi_x^*(x^\circ(t),\, u^\circ(t),\, t)\psi(t),$$

such that, for all $t \in [0,\, T]$ it follows that

$$\varphi_u^*(x^\circ(t),\, u^\circ(t),\, t)\psi(t) = 0.$$

By taking $\lambda_0 = 0$, we get that ψ is solution of (3.4), and therefore

$$\langle -\varphi_u^*(x^\circ(t),\, u^\circ(t),\, t)\psi(t),\, U - u^\circ(t)\rangle \geq 0,$$

for all $U \in M$ and almost all $t \in [0,\, T]$.

Thus, the proof of Theorem 3.1 is completed.

□

4. Sufficient Condition of Optimality

The necessary condition of optimality proved in Theorem 3.1 (Maximum Principle), under certain additional conditions, is also sufficient. In fact, let us consider the particular case of Problem 1.1 in which the differential equation is linear.

Problem 4.1.

$$\int_0^T \Phi(x(t),\, u(t),\, t)dt \longrightarrow \min$$
$$(x,\, u) \in E = \mathcal{PW}([0, T]; \mathbb{R}^n) \times L_\infty^r([0, T]; \mathbb{R}^r), \tag{4.1}$$

$$\dot{x}(t) = A(t)x(t) + B(t)u(t), \tag{4.2}$$

$$x(0) = x_0, \quad x(T) = x_1; \qquad x_1,\, x_0 \in I\!R^n, \tag{4.3}$$

$$x(t_k^+) = x(t_k^-) + \mathcal{J}_k(x(t_k)), \quad k = 1, 2, 3, \ldots, p. \tag{4.4}$$

$$u(t) \in M, \qquad t \in [0,\, T],\ a.e. \tag{4.5}$$

where $A(\cdot) : [0,\, T] \longrightarrow I\!R^{n\times n}$, $\quad B(\cdot) : [0,\, T] \longrightarrow I\!R^{n\times r}$ are measurable and bounded matrix functions and $\mathcal{J}_k - n \times n$ matrix, $k = 1, 2, 3, \ldots, p$. Let $(x^\circ,\, u^\circ) \in E$ be a point satisfying conditions (4.2)– (4.5).

Theorem 4.1. *Let us suppose that the conditions a) - d) from Theorem 3.1 are satisfied.*

Besides, let us assume the following hypotheses:

I) The system (4.2) and the impulsive system (4.2)-(4.4) are controllable.

II) There exists $\widetilde{u} \in L^r_\infty[0, T]$ such that $\widetilde{u}(t) \in \overset{\circ}{M}$, for almost all $t \in [0, T]$.

III) Φ is a convex function in its two first variables.

Then (x°, u°) is global solution of Problem 4.1.

Proof. Let us define the function $\overline{F} : E \longrightarrow I\!R$ as follows

$$\overline{F}(x, u) = \int_0^T \Phi(x(t), u(t), t)dt,$$

and the set $Q := Q_1 \cap Q_2$, where Q_2 is given by (4.2)-(4.4) and Q_1 by (4.5).
Then, Problem 4.1 is equivalent to:

$$\begin{cases} \overline{F}(x, u) \longrightarrow \min \\ (x, u) \in Q. \end{cases}$$

It is clear that $Q_i \quad (i = 1, 2)$ are convex sets, and from the condition $III)$ we have that $\overline{F}$ is convex, and from condition II) we have that $(\widetilde{x}, \widetilde{u}) \in \overset{\circ}{Q}_1 \cap Q_2$.
Thus, by Theorem 2.17 it follows:

(x°, u°) is a minimum point of F in Q if, and only if, there are $f_i \in K_i^+ \quad (i = 0, 1, 2)$, not all zero such that

$$f_0 + f_1 + f_2 = 0.$$

Here, $K_i \quad (i = 0, 1, 2)$ are cones defined as in Theorem 3.1. Now, suppose that the Maximum Principle of Theorem 3.1 holds. That is to say, there exist $\lambda_0 \in I\!R_{+0}$ and a function $\psi \in \mathcal{PW}([0, T]; \mathbb{R}^n)$ such that λ_0 and ψ are not both zero, and ψ is a solution of the following differential equation

$$\begin{cases} \dot{\psi}(t) = -A^*(t)\psi(t) + \lambda_0 \Phi_x(x^\circ(t), u^\circ(t), t) \\ \psi(T) = a \end{cases} \tag{4.6}$$

Moreover, for all $U \in M$ and almost all $t \in [0, T]$, we have that

$$\langle -B^*(t)\psi(t) + \lambda_0\, \Phi_u(x^\circ(t), u^\circ(t), t), U - u^\circ(t)\rangle \geq 0. \tag{4.7}$$

Then, to prove the theorem, it is enough to see that there are $f_i \in K_i^+ \quad (i = 0, 1, 2)$ not all zero, such that $f_0 + f_1 + f_2 = 0$. To do so, we define the following functionals:

$$\begin{aligned} f_1' &: L_\infty^r \longrightarrow I\!R,\ f_1 : E \longrightarrow I\!R \\ f_1'(u) &:= \int_0^T \langle -B^*(t)\psi(t) + \lambda_0\, \Phi_u(x^\circ(t),\, u^\circ(t),\, t),\, u(t)\rangle dt, \\ f_1 &= (0,\, f_1'). \end{aligned}$$

Let

$$Q_1' = \{u \in L_\infty^r \,/\, u(t) \in M, \quad t \in [0,\, T], \quad a.e.\}.$$

Then, from (4.7), we obtain

$$f_1'(u) \geq f_1'(u^\circ) \qquad (u \in Q_1').$$

Therefore f_1' is a support of Q_1' at u°. Hence $f_1 = (0,\, f_1') \in K_1^+$. Let us define the functional $f_{21} : E \longrightarrow I\!R$ as follows

$$f_{21}(x,\, u) := \lambda_0 \int_0^T [\Phi_x(x^\circ(t),\, u^\circ(t),\, t)x(t) + \Phi_u(x^\circ(t),\, u^\circ(t),\, t)u(t)]dt \\ -f_1'(u) - \langle a,\, x(T)\rangle.$$

Now, we will see that $f_{21} \in L_1^+$, where

$$L_1 = \{(x,\, u) \,/\, (4.2), (4.4) \quad \text{hold}\},$$

as in the Theorem 3.1. In fact, suppose that $(x,\, u) \in L_1$, then multiplying both sides of the equation (4.6) by $\dot{x}$ and integrating by parts from 0 to T, we obtain that

$$\lambda_0 \int_0^T \langle \Phi_x(x^\circ(t),\, u^\circ(t),\, t)\psi(t), x(t)\rangle dt \\ -\langle a,\, x(T)\rangle = -\int_0^T \langle B^*(t)\psi(t),\, u(t)\rangle dt.$$

Then

$$f_{21}(x,\, u) = -f_1'(u) - \int_0^T \langle B^*(t)\psi(t),\, u(t)\rangle dt + \lambda_0 \int_0^T \Phi_u(x^\circ(t),\, u^\circ(t),\, t)u(t)dt.$$

Therefore

$$f_{21}(x,\, u) = -f_1'(u) + f_1'(u) = 0,$$

Thus $f_{21} \in L_1^+$.
Next, we shall define the following functionals

$$f_0,\, f_1,\, f_2;\, E \longrightarrow I\!R,$$

by

$$\begin{aligned} f_0(x,\, u) &:= \lambda_0 \int_0^T [\Phi_x(x^\circ(t),\, u^\circ(t),\, t)x(t) + \Phi_u(x^\circ(t),\, u^\circ(t),\, t)u(t)]dt \\ f_2(x,\, u) &:= f_{21}(x,\, u) + \langle a,\, x(T)\rangle = f_{21}(x,\, u) + f_{22}(x,\, u). \end{aligned}$$

Then $f_0 \in K_0^+,\ f_1 \in K_1^+,\ f_2 \in K_2^+$, and also

$$f_0 + f_1 + f_2 = 0,$$

not all these functionals are zero, because by hypothesis λ_0 and ψ are not both zero. From the convexity conditions, it follows the global-minimality of $(x^\circ,\, u^\circ)$. □

5. Modification of Boundary Conditions

We now discuss problem 1.1 with modified boundary condition. We replace the end condition of (1.4) by a more general condition, in other word, we consider the following optimal control problem

Problem 5.1.

$$\int_0^T \Phi(x(t),\, u(t),\, t)dt \longrightarrow \min \text{ loc.} \tag{5.1}$$

$$(x,\, u) \in E := \mathcal{PW}([0,T];\mathbb{R}^n) \times L^r_\infty([0,T];\mathbb{R}^r), \tag{5.2}$$

$$\dot{x}(t) = \varphi(x(t),\, u(t),\, t), \qquad x(0) = x_0 \tag{5.3}$$

$$x_0 \in I\!R^n;\ G_i(x(T)) = 0, \quad i = 1, 2, \ldots, q. \tag{5.4}$$

$$x(t_k^+) = x(t_k^-) + \mathcal{J}_k(x(t_k)), \quad k = 1, 2, 3, \ldots, p. \tag{5.5}$$

$$u(t) \in M, \quad t \in [0,\, T], \quad a.e., \tag{5.6}$$

where $G_i(x)$ are differentiable scalar functions on $I\!R^n$. So arguing exactly as in the previous problem 1.1, under certain conditions that we will present immediately, the cone of tangent vectors K_2 is the set of points $(\overline{x}, \overline{u}) \in E$ such that

$$\begin{aligned} \dot{\overline{x}}(t) &= \varphi_x(x^\circ(t),\, u^\circ(t),\, t)\overline{x}(t) + \varphi_u(x^\circ(t),\, u^\circ(t),\, t)u(t), \quad t \neq t_k && (5.7) \\ \overline{x}(t_k^+) &= \mathcal{J}_k'(x^0(t_k^-))\overline{x}(t_k^-), \quad k = 1, 2, 3, \ldots, p. && (5.8) \\ & \langle G_i'(x^0(T)), \overline{x}(T)\rangle = 0, \quad i = 1, 2, 3, \ldots, q. && (5.9) \end{aligned}$$

But, in order to compute the tangent cone K_2 we have to assume the following condition on $G_i'(x^\circ(T))$. Consider the jacobian matrix of

$$G(x) = (G_1(x), G_2(x), G_3(x), \cdots, G_q(x)) \tag{5.10}$$

around the point $x^\circ(T)$

$$\Xi = G'(x^\circ(T)) = \begin{pmatrix} G'_{11}(x^\circ(T)) & G'_{12}(x^\circ(T)) & \cdots & G'_{1n}(x^\circ(T)) \\ G'_{21}(x^\circ(T)) & G'_{22}(x^\circ(T)) & \cdots & G'_{2n}(x^\circ(T)) \\ \vdots & \vdots & \vdots & \vdots \\ G'_{q1}(x^\circ(T)) & G'_{q2}(x^\circ(T)) & \cdots & G'_{qn}(x^\circ(T)) \end{pmatrix}. \tag{5.11}$$

Additional Hypothesis

H) Rank$(\Xi) = q$.

REMARK 5.1. *Condition H) is equivalent to say that the operator $\Xi : I\!R^n \to I\!R^q$ is onto (Range$(\Xi) = I\!R^q$), which is equivalent that $(\Xi\Xi^*)^{-1}$ exists. Therefore $\Xi^+ = \Xi^*(\Xi\Xi^*)^{-1}$ is a right inverse of Ξ. So, the equation $\Xi x(T) = \hat{a}$ admits the solution $x(T) = \Xi^*(\Xi\Xi^*)^{-1}\hat{a}$.*

In order to compute the tangent cone, we have to modify the operator P defined in problem 1.1, Let us find the tangent cone to Q_2 at the point $(x^\circ,\, u^\circ)$

$$K_2 := K_T(Q_2,\, (x^\circ,\, u^\circ)).$$

Consider the space $E_1 = \mathcal{PW}([0,T];\mathbb{R}^n) \times I\!R^{n(1+p)} \times I\!R^q = E_2$ and the operator: $P : E_1 \to E_2$ defined by

$$P(x,u)(t) = \left(x(t) - x_0 - \int_0^t \varphi(x(l), u(l), l)dl, \quad S(x,u), \quad G(x(T)) \right),$$

where

$$S(x,u) = \left(x(t_1) - \mathcal{J}_1(x(t_1^-)), x(t_2) - \mathcal{J}_2(x(t_2^-)) \cdots, x(t_p) - \mathcal{J}_p(x(t_p^-))\right),$$

and G is given by (5.10). Then

$$P'(x^0,u^0)(\overline{x},\overline{u}) =$$
$$\left(\overline{x}(t) - \int_0^t (\varphi_x(x^\circ(l),\, u^\circ(l),\, l)\overline{x}(l) + \varphi_u(x^\circ(l),\, u^\circ(l),\, l)\overline{u}(l))dl, \quad S'(\overline{x},\overline{u}), \quad \Xi\overline{x}(T) \right)$$

where

$$S'(\overline{x},\overline{u}) = \left(\overline{x}(t_1) - \mathcal{J}_1'(x^0(t_1^-))\overline{x}(t_1^-), \cdots\cdots, \overline{x}(t_p) - \mathcal{J}_p'(x^0(t_p^-))\overline{x}(t_p^-)\right)$$

and Ξ is given by (5.11). We want to find conditions under which the operator $P'(x^0, u^0)$ is onto in order to apply Lustenik Theorem 2.27. So, for $(a(\cdot), b_1, b_2, \ldots, b_p, \hat{a}) \in E_2$, we want to solve the equation

$$P'(x^0, u^0)(\overline{x}, \overline{u}) = (a(\cdot), b_1, b_2, \ldots, b_p, \hat{a}).$$

Now, suppose that $\overline{u} = 0$. Then, from ([25], pg 89), we know that the following Volterra integral equation

$$z(t) = a(t) + \int_0^t (\varphi_x(x^\circ(l),\, u^\circ(l),\, l)z(l)dl,$$

has a solution $z \in \mathcal{PW}([0, T]; \mathbb{R}^n)$.
Next, since the impulsive linear variational equation (3.2) is controllable, for a point $(\bar{b}_1, \bar{b}_2, \ldots, \bar{b}_p) \in I\!R^{np}$ such that

$$\bar{b}_k = b_k - z(t_k) + \mathcal{J}_k'(x^0(t_k^-))z(t_k^-), \quad k = 1, 2, 3, \ldots, p.$$

Then, there exists a control $\overline{u} \in L_\infty^r$ such that the corresponding solution $y(t)$ of (3.2) satisfies

$$y(T) = \Xi^*(\Xi\Xi^*)^{-1}\hat{a} - z(T).$$

Let us make the following change of variable $\overline{x} = y + z$. Then

$$\begin{aligned}
&P'(x^0, u^0)(\overline{x}, \overline{u})(t) = ((y+z)(t) - \\
&\left.\int_0^t (\varphi_x(x^\circ, u^\circ, l)(y+z)(l) + \varphi_u(x^\circ, u^\circ, l)\overline{u}(l))dl, \quad S'(\overline{x}, \overline{u}), \quad \Xi(y+z)(T)\right) \\
&= (y(t) + a(t) - \\
&\left.\int_0^t (\varphi_x(x^\circ,\, u^\circ,\, l)y(l) + \varphi_u(x^\circ,\, u^\circ, l)\overline{u}(l))dl, \quad S'(\overline{x}, \overline{u}), \quad \Xi\Xi^*(\Xi\Xi^*)^{-1}\hat{a}\right) \\
&= (a(t), \quad S'(\overline{x}, \overline{u}), \hat{a})\,.
\end{aligned}$$

Now, we shall see that $S'(\overline{x}, \overline{u}) = (b_1, b_2, \ldots, b_p)$. In fact,

$$\begin{aligned}
&S'(\overline{x}, \overline{u}) = \\
&\big(\overline{x}(t_1) - \mathcal{J}_1'(x^0(t_1^-))\overline{x}(t_1^-), \cdots\cdots, \overline{x}(t_p) - \mathcal{J}_p'(x^0(t_p^-))\overline{x}(t_p^-)\big) = \\
&\big((y+z)(t_1) - \mathcal{J}_1'(x^0(t_1^-))(y+z)(t_1^-), \cdots, (y+z)(t_p) - \mathcal{J}_p'(x^0(t_p^-))(y+z)(t_p^-)\big) \\
&= \big(\bar{b}_1 + z(t_1) - \mathcal{J}_1'(x^0(t_1^-))z(t_1^-), \cdots\cdots, \bar{b}_p + z(t_p) - \mathcal{J}_p'(x^0(t_p^-))z(t_p^-)\big) \\
&= (b_1, b_2, \ldots, b_p)\,.
\end{aligned}$$

Therefore, the operator $P'(x^0, u^0)$ is onto. Then, applying Lusternik's Theorem 2.27, we get that tangent cone K_2 is given by

$$K_2 = \{(x,\, u) \in E_1 \,/\, P'(x^\circ,\, u^\circ)(\overline{x}, \overline{u}) = 0\}.$$

i.e., K_2 is the set of points $(\overline{x}, \overline{u}) \in E_1$ such that

$$\begin{aligned}
\dot{\overline{x}}(t) &= \varphi_x(x^\circ(t),\, u^\circ(t),\, t)\overline{x}(t) + \varphi_u(x^\circ(t),\, u^\circ(t),\, t)u(t), t \neq t_k && (5.12)\\
\overline{x}(t_k^+) &= \mathcal{J}_k'(x^0(t_k^-))\overline{x}(t_k^-), \quad k = 1, 2, 3, \ldots, p. && (5.13)\\
\Xi\overline{x}(T) &= 0. && (5.14)
\end{aligned}$$

Consider the following linear subspaces

$$L_1 = \{(\overline{x}, \overline{u}) \in E_1/(5.12) - (5.13) \quad \text{hold}\}, \quad L_2 = \{(\overline{x}, \overline{u}) \in E_1/ \quad \Xi\overline{x}(T) = 0\}.$$

Then, $K_2 = L_1 \cap L_2$. Now, let us compute K_2^+. By Proposition 2.40, we have that $f_{22} \in L_2^+$ if, and only if, there exists $a \in I\!\!R^k$ such that

$$f_{22}(x,\, u) = \langle a,\, \Xi x(T)\rangle \qquad ((x,\, u) \in E).$$

Moreover, the controllability of systems (3.1)- (3.2) implies that $L_1 + L_2$ is closed, then it follows that $L_1^+ + L_2^+$ is w^*- closed; hence by Lemma 2.5 we obtain that

$$K_2^+ = L_1^+ + L_2^+.$$

Since L_1 is a linear subspace, it follows from Theorem 10.1 of (See [18, pg 59]) that,

for any $f_{21} \in L_1^*$, $f_{21}(\overline{x}, \overline{u}) = 0$ for all $(\overline{x}, \overline{u})$ satisfying (5.12)-(5.13).

Euler-Lagrange Equation.

Clearly that K_0, K_1, K_2, are convex cones. Hence, by Theorem 2.15 there are functionals $f_i \in K_i^+ \quad (i = 0,\, 1,\, 2,)$ not all zero such that

$$f_0 + f_1 + f_2 = f_0 + f_1 + f_{21} + f_{22} = 0. \tag{5.15}$$

Equation (5.15) takes the following form

$$\begin{cases} -\lambda_0 \displaystyle\int_0^T [\Phi_x(x^\circ,\, u^\circ,\, t)x(t) + \Phi_u(x^\circ,\, u^\circ,\, t)u(t)]dt + \\ \\ +f_1'(x,\, u) + f_{21}(x,\, u) + \langle a,\, \Xi x(T)\rangle = 0, \quad ((x,\, u) \in E). \end{cases} \tag{5.16}$$

Now, for all $u \in L_\infty^r$ there exists x, solution of equation (3.2) with $x(0) = 0$, then $(x,\, u) \in L_1$. Therefore $f_{21}(x,\, u) = 0$.
Let ψ be a solution of the system

$$\begin{cases} \dot{\psi}(\tau) = -\varphi_x^*(x^\circ(\tau),\, u^\circ(\tau),\, \tau)\psi(\tau) + \lambda_0\Phi_x(x^\circ(\tau),\, u^\circ(\tau),\, \tau) \\ \psi(T) = \Xi^* a \end{cases}$$

Multiplying both sides of this equation by $\overline{x}$ and integrating by parts from 0 to T, we get

$$
\begin{aligned}
&\lambda_0 \int_0^T \Phi_x(x^\circ,\, u^\circ,\, t)\overline{x}(t)dt - \langle a,\, \Xi\overline{x}(T)\rangle = \\
&\int_0^T \langle \dot{\psi}(t),\, \overline{x}(t)\rangle dt + \int_0^T \langle \varphi_x^*(x^\circ,\, u^\circ,\, t)\psi(t), \overline{x}(t)\rangle dt - \langle a,\, \Xi\overline{x}(T)\rangle = \\
&\langle \psi(t),\, \overline{x}(t)\rangle]_0^T - \int_0^T \langle \psi(t),\, \dot{\overline{x}}(t)\rangle dt + \int_0^T \langle \varphi_x^*(x^\circ,\, u^\circ,\, t)\psi(t), \overline{x}(t)\rangle dt - \langle a,\, E\overline{x}(T)\rangle = \\
&\langle \Xi^* a,\, \overline{x}(T)\rangle - \langle \psi(0),\, \overline{x}(0)\rangle - \langle a,\, \Xi\overline{x}(T)\rangle + \int_0^T \langle \psi(t), \varphi_x(x^\circ,\, u^\circ,\, t)\overline{x}(t) - \dot{\overline{x}}(t)\rangle dt = \\
&-\int_0^T \langle \psi(t), \varphi_u(x^\circ,\, u^\circ,\, t)\overline{u}(t)\rangle dt = -\int_0^T \langle \varphi_u^*(x^\circ,\, u^\circ,\, t)\psi(t), \overline{u}(t)\rangle dt.
\end{aligned}
$$

Then from Euler–Lagrange equation (5.15), we obtain for ($u \in L_\infty^r[0,\, T]$), that

$$
f_1'(u) = \int_0^T \langle -\varphi_u^*(x^\circ(t),\, u^\circ(t),\, t)\psi(t) + \lambda_0\, \Phi_u(x^\circ(t),\, u^\circ(t),\, t)u(t)\rangle dt. \tag{5.17}
$$

Since f_1' is a support of Q_1' at the point $u^\circ \in Q_1'$, from example 2.44, it follows that

$$
\langle -\varphi_u^*(x^\circ(t),\, u^\circ(t),\, t)\psi(t) + \lambda_0\, \Phi_u(x^\circ(t),\, u^\circ(t),\, t),\, U - u^\circ(t)\rangle \geq 0,
$$

for all $U \in M$ and almost all $t \in [0,\, T]$.

□

Remark 5.2. *Now, we will see that under these assumptions, the case $\lambda_0 = 0,\ \psi = 0$, can not occurs. If $\psi = 0$, then $\psi(T) = \Xi^* a = 0$. Thus*

$$
f_{22}(x,\, u) = \langle a,\, \Xi x(T)\rangle = 0 \qquad ((x,\, u) \in E),
$$

that is $f_{22} \equiv 0$. So, from equation (5.16), and the fact that $\lambda_0 = 0$, which implies that $f_0 = 0$. Also, from (5.17), we have that $f_1'(u) = 0 \quad (u \in L_\infty^r[0,\, T])$; then from Euler–Lagrange Equation it follows that $f_{21} = 0$, hence

$$
f_2 = f_{21} + f_{22} = 0,
$$

which contradicts Theorem 2.15.

Remark 5.3. **Analysis of the exceptional cases.** *In the course of the proof we have to made two additional assumptions: Firstly, we assumed that $K_0 \neq \emptyset$, and secondly, we assumed that system*

$$
\dot{x}(t) = \varphi_x(x^\circ(t),\, u^\circ(t),\, t)x(t) + \varphi_u(x^\circ(t),\, u^\circ(t),\, t)u(t), \quad t \in (0, \tau], \tag{5.18}
$$

is controllable.

Now, we will prove, that these assumptions are superfluous. In fact, if $K_0 = \emptyset$, then by definition of K_0, we have that

$$\int_0^T [\Phi_x(x^\circ(t),\, u^\circ(t),\, t)x(t) + \Phi_u(x^\circ(t),\, u^\circ(t),\, t)u(t)]dt = 0 \quad ((x,\, u) \in E).$$

Let us put $\lambda_0 = 1,\ \psi(T) = \Xi^* a = 0$, then, from last computation, we have that

$$\int_0^T \langle \Phi_x^*(x^\circ,\, u^\circ,\, t)\psi(t), x(t)\rangle dt = -\int_0^T \langle \varphi_u^\star(x^\circ,\, u^\circ,\, t)\psi(t), u(t)\rangle dt,$$

for all $(x,\, u)$ such that x is a solution of equation the (5.18). Then

$$\int_0^T \langle -\varphi_u^*(x^\circ(t),\, u^\circ(t),\, t)\psi(t) + \Phi_u(x^\circ(t),\, u^\circ(t),\, t)u(t)\rangle dt = 0, \quad (u \in L_\infty^r[0,\, T]),$$

which implies that

$$\langle -\varphi_u^*(x^\circ,\, u^\circ,\, t)\psi(t) + \Phi_u(x^\circ,\, u^\circ,\, t),\, U - u^\circ(t)\rangle = 0,$$

for all $U \in M$ and almost all $t \in [0,\, T]$.

Remark 5.4. *The controllability of the linear system,*

$$\dot{x}(t) = A(t)x(t) + B(t)u(t), \quad t \in (0, \tau], \tag{5.19}$$

where $A(t) = \varphi_x(x^\circ(t),\, u^\circ(t),\, t)$ and $B(t) = \varphi_u(x^\circ(t),\, u^\circ(t),\, t)$, is equivalent to:

$$B^*(t)[\Psi^*]^{-1}(t)z = 0, \quad \forall t \in [0, T] \Rightarrow z = 0.$$

Here $\Psi(t)$ is the fundamental matrix of the uncontrolled system $\dot{z} = A(t)z$ and $\psi(t) = [\Psi^*]^{-1}(t)z_0$ is a solution of the adjoint initial value problem

$$\dot{z} = -A^*(t)z, \quad z(0) = z_0.$$

Now, suppose that system (5.19) is not controllable, then there is a non-trivial function $\psi \in \mathcal{PW}([0, T]; \mathbb{R}^n)$ that is a solution of

$$\dot{\psi}(t) = -\varphi_x^*(x^\circ(t),\, u^\circ(t),\, t)\psi(t),$$

such that, for almost all $t \in [0,\, T]$ it follows that

$$-\varphi_u^*(x^\circ(t),\, u^\circ(t),\, t)\psi(t) = 0.$$

By taking $\lambda_0 = 0$, we get that ψ is a solution of (3.4), and therefore

$$\langle -\varphi_u^*(x^\circ(t),\, u^\circ(t),\, t)\psi(t),\, U - u^\circ(t)\rangle \geq 0,$$

for all $U \in M$ and almost all $t \in [0,\, T]$.

Throughout this reasoning, we have proved the following theorem:

Theorem 5.1. *Under conditions of Theorem 3.1. Let assume that Rank*$(\Xi) = q$ *and* $(x^\circ, u^\circ) \in E$ *be a solution of Problem 5.1:*
Then, there exists $\lambda_0 \in I\!R_{+0}$ *and a function* $\psi \in \mathcal{PW}([0,T];\mathbb{R}^n)$ *such that* λ_0 *and* ψ *both are different from zero, and* ψ *is a solution of the following differential equation*

$$\begin{cases} \dot{\psi}(t) = -\varphi_x^*(x^\circ(\tau),\, u^\circ(\tau),\, \tau)\psi(\tau) + \lambda_0\Phi_x(x^\circ(\tau),\, u^\circ(\tau),\, \tau) \\ \psi(T) = \Xi^* a. \end{cases} \tag{5.20}$$

Moreover, for all $U \in M$ *and almost all* $t \in [0,\, T]$ *the following inequality holds*

$$\langle -\varphi_u^*(x^\circ(t),\, u^\circ(t),\, t)\psi(t) + \lambda_0\, \Phi_u(x^\circ(t),\, u^\circ(t),\, t),\, U - u^\circ(t)\rangle \geq 0. \tag{5.21}$$

Remark 5.5. *Consider the function*

$$H(x,u,\psi,t) = \langle \varphi^*(x,\, u,\, t), \psi(t)\rangle - \lambda_0\Phi(x,u,t).$$

Then

$$H_u(x^\circ,u^\circ,\psi,t) = \varphi_u^*(x^\circ,\, u^\circ,\, t)\psi(t) - \lambda_0\Phi_u(x^\circ,u^\circ,t).$$

Since a necessary condition for $H(x^\circ,u,\psi,t)$ *to have a maximum on* M*, as a function of* u*, is that* $-H_u(x^\circ,u^\circ,\psi,t)$ *be a support ot* M *at the point* $u^\circ(t)$*, it follows that (3.5) may be paraphrased as follows. If* (x°,u°) *is a solution of Problem (1.1) and the assumptions of Theorem 3.1 hold, then* $H(x^\circ,u,\psi,t)$ *as a function of* u *on* M*, satisfies the necessary conditions for a maximum for almost all* $0 \leq t \leq T$ *at the point* $u = u^\circ(t)$*. A comparison of this statement with the classic maximum principle justifies the designation "local maximum principle". Specifically we have the following:*

$$\begin{aligned} \langle -H_u(x^\circ,u^\circ,\psi,t), U - u^\circ(t)\rangle &\geq 0 \iff \\ H_u(x^\circ,u^\circ,\psi,t)u^\circ(t) &\geq H_u(x^\circ,u^\circ,\psi,t)U. \end{aligned}$$

Hence,

$$H_u(x^\circ,u^\circ,\psi,t)u^\circ(t) = \max_{U\in M} H_u(x^\circ,u^\circ,\psi,t)U, \quad t \in [0,T], \quad a.e.$$

Since the linear system (3.1) is controllable, then slight modification of the proof of Theorem 3.1 allows us to assume that $\lambda_0 = 1$*.*

6. Example

Now, we shall give an example as an applications of the main result of this work. In this regard, we will give below two previous propositions.

Proposition 6.1. *Let* $x_0 \in I\!R^n_+$ *and* $A = (a_{ij})_{n\times n}$ *be a real matrix, such that* $a_{ij} > 0 \quad (i \neq j,\, i,j = 1,2,\ldots,n)$*. Then*

$$e^{At}\, x_0 \in I\!R^n_+, \quad (t \in I\!R).$$

The proof of above proposition is trivial.
Let $M \subset I\!R^r$ be a set, then we define the set Q_M as follows:

$$Q_M := \{u \in L^r_\infty\,[0,\,T]\,/\,u(t) \in M,\ t \in [0,\,T], \quad a.e.\}.$$

Proposition 6.2. *Let $x_0 \in I\!R^n_+$, and $B = (b_{ij})_{n\times r}$ a real matrix. Then there exists $M \subset I\!R^r$ convex and closed, with $\overset{\circ}{M} \neq \emptyset$ such that*

$$\left(e^{At}\,x_0 + \int_0^t e^{A(t-s)}B\,u(s)ds\right) \in I\!R^n_+, \qquad (u \in Q_M,\ t \in [0,\,T],\ a.e.).$$

Proof. Let $\{e_1,\,e_2,\ldots,\,e_n\}$ be the canonical basis of $I\!R^n$, and define

$$\begin{aligned}\alpha_i &:= \min_{t\in[0,\,T]} \langle e_i,\, e^{At}\,x_0\rangle, \quad (i = 1,\,2,\ldots,\,n),\\ V &:= (\alpha_1,\,\alpha_2,\ldots,\,\alpha_n).\end{aligned}$$

Then, by proposition 6.1 it follows that $V \in I\!R^n_+$.
Let $\delta := \min\{\alpha_i\,/\,i = 1,\,2,\ldots,\,n\}$; then for all $x \in I\!R^n$ such that $|x| < \delta$, we have that $V + x \in I\!R^n_+$.
Let us consider

$$K_1 := \max_{t\in[0,\,T]} \|e^{At}\|, \quad K_2 := \max_{t\in[0,\,T]} \|e^{-At}\|.$$

Then

$$\left|\int_0^T e^{A(t-s)}\,B\,u(s)ds\right| < T\,K_1\,K_2\|B\|\|u\|_\infty,$$

and taking

$$M := \left\{U \in I\!R^r\,/\,|U| \leq \frac{\delta}{TK_1K_2\|B\|}\right\},$$

we finish the proof. $\square$

Next, we shall consider the following example where Theorem 3.1 can be applied:

Example 6.3. Let $n = 2,\ r = 1$ and suppose that Φ satisfies the same conditions as in the Problem 1.1, furthermore let us consider

$$\begin{aligned}B &= \begin{pmatrix} b_{11} \\ b_{12} \end{pmatrix}, \qquad A = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}; \quad a_{12} > 0, \quad a_{21} > 0\\ M &:= \left\{U \in R\,/\,|U| \leq \frac{\delta}{TK_1K_2\|B\|}\right\},\end{aligned}$$

where $\delta,\,K_1,\,K_2$ are defined as in Proposition 6.2.
Let us consider the following problem

$$\int_0^T \Phi(x(t),\,u(t),\,t)dt \longrightarrow \min -loc. \tag{6.1}$$

$$(x,\,u)\in\mathcal{PW}([0,T];\mathbb{R}^n)\times L^r_\infty([0,T];\mathbb{R}^r)$$

$$\dot{x}(t)=Ax(t)+Bu(t) \tag{6.2}$$

$$x(0)=x_0,\ x(T)=x_1;\quad x_0,\,x_1\in I\!R^2_+. \tag{6.3}$$

$$x(t_k^+)=x(t_k^-)+\mathcal{J}_k(x(t_k)),\quad k=1,2,3,\dots,p. \tag{6.4}$$

$$u(t)\in M,\ t\in[0,\,T],\quad a.e. \tag{6.5}$$

Let $(x^\circ,\,u^\circ)\in E=\mathcal{PW}([0,T];\mathbb{R}^n)\times L^r_\infty([0,T];\mathbb{R}^r)$ be a solution of the above problem, then conditions of Theorem 3.1 are fulfilled. In fact, clearly condition a) is satisfied. Also, M closed and convex set with $M^\circ\neq\emptyset$.
c) The linear system (6.2) is controllable. Since this is an autonomous system, we assume that Kalman's Rank condition is satisfied (see [12, 13, 27]). i.e.,

$$\text{Rank }[B\vdots AB]=2.$$

d) The linear system (6.2) with impulses (6.4) is controllable if the following condition is assumed:

$$p\max\|\mathcal{J}_k\|<1,\quad k=1,2,\dots,p.$$

(see [8, 29, 31, 33]). Hence, there exist $\lambda_0\in I\!R_+$, $a\in I\!R^2$, and a function $\psi\in C([0,\,T],I\!R^2)$, which is a solution of the equation

$$\dot{\psi}(t)=-A^*(t)\psi(t)+\lambda_0\Phi_x(x^\circ(t),\,u^\circ(t),\,t), \tag{6.6}$$

such that λ_0 and ψ are not both zero, and for all $U\in M$ and almost all $t\in[0,\,T]$, we have that

$$\langle -B^*\,\psi(t)+\lambda_0\,\Phi_u(x^\circ(t),\,u^\circ(t),\,t),\,U-u^\circ(t)\rangle\geq 0$$

or equivalently

$$\max_{U\in M}\langle B^*\,\psi(t)-\lambda_0\,\Phi_u(x^\circ(t),\,u^\circ(t),\,t),\,U\rangle=\langle B^*\,\psi(t)-\lambda_0\,\Phi_u(x^\circ(t),\,u^\circ(t),\,t),\,u^\circ(t)\rangle \tag{6.7}$$

for almost all $t\in[0,\,T]$.
Let us consider the particular case, in which

$$\Phi(x,\,u)=Cu\qquad((x,\,u,\,t)\in I\!R^2\times I\!R\times[0,\,T]),$$

and let us see how should be the controls $u\in L_\infty[0,\,T]$ that solve the problem:

$$\begin{aligned}\dot{\psi}(t)&=-A^*(t)\psi(t)+\lambda_0\Phi_x(x^\circ(t),\,u^\circ(t),\,t),\\ \max(B^*\psi(t)-\lambda_0\,C)U&=(B^*\psi(t)-\lambda_0\,C)u^\circ(t),\quad U\in[-\rho,\,\rho]\end{aligned}$$

for almost all $t\in[0,\,T]$, where $\rho=\delta/K_1\,K_2\,\|B\|T$.

Let

$$\begin{aligned} N_{B^*} &:= \{x \in I\!R^2 \,/\, B^* x - \lambda_0 \, C = 0\}, \\ S &:= \{t \in [0,\, T] \,/\, \psi(t) \notin N_{B^*}\}, \end{aligned}$$

then $u^\circ(t) := \rho \text{ sig } (B^*\psi(t) - \lambda_0 \, C)$ if $t \in S$.

This means that the optimal control should be of the "bang–bang" type over the set S.

7. Optimal Control Problem for Impulsive Neutral Differential Equations

In this section we will show how Dubovitskii–Milyutin theory can be applied to generalize the Maximum Principle of [18] to the case of optimal control problems governed by impulsive nonlinear neutral differential equations. We will also see that in a linear dynamics case, under certain additional conditions, the Maximum Principle is a sufficient condition for optimality.

7.1. Maximum Principle for Neutral Differential Equations in the Space $\mathcal{PW}([0,T];\mathbb{R}^n) \times L^r_\infty$.

Let n, $r \in \mathbb{N}$ and $T \in I\!R_+$, and consider the functions Φ, $\varphi, \mathcal{J}_k$:

$$\begin{aligned} \varphi &: I\!R^n \times I\!R^r \times [0,\, T] \longrightarrow I\!R, \\ \Phi &: I\!R^n \times I\!R^r \times [0,\, T] \longrightarrow I\!R^n, \\ \mathcal{J}_k &: I\!R^n \longrightarrow I\!R^n, \\ f &: I\!R^n \longrightarrow I\!R^n \end{aligned}$$

where $\mathcal{PW}([0,T];\mathbb{R}^n)$ and L^r_∞ are define by

$$\begin{aligned} \mathcal{PW}([0,T];\mathbb{R}^n) = \; & \{z : [0,T] \to \mathbb{R}^n : z \in C(J';\mathbb{R}^n), \exists z(t_k^-), z(t_k^-) \\ & \text{and} \quad z(t_k) = z(t_k^-), \quad k = 1, 2, \dots, p\}, \end{aligned}$$

where $J = [0,T]$ and $J' = J \backslash \{t_1, t_2, \dots, t_p\}$, endowed with the norm

$$\|z\|_0 = \sup_{t \in [0,T]} \|z(t)\|_{\mathbb{R}^n},$$

and $L^r_\infty = L^r_\infty([0,T];\mathbb{R}^r)$ is the space of measurable function essentially bounded with essential norm.

Let us suppose the following conditions are fulfilled

a) Φ, φ, f and $\mathcal{J}_k$ are continuous functions, with derivatives Φ_x, Φ_u, φ_x, φ_u, $\mathcal{J}_k'$, f' are bounded functions on compact sets of $I\!R^n \times I\!R^r \times [0, T]$.

b) $M \subset I\!R^r$ is convex and closed with $\overset{\circ}{M} \neq \emptyset$.

c) The following linear neutral system is controllable on $[0, T]$,

$$\frac{d}{dt}[x(t)+f'(x^\circ(t))x(t)] = \varphi_x(x^\circ(t), u^\circ(t), t)x(t)+\varphi_u(x^\circ(t), u^\circ(t), t)u(t). \quad (7.1)$$

d) The corresponding impulsive linear variational equations around the point $(x^\circ, u^\circ) \in E$ is controllable on $[0, T]$ for any $\bar{b} = (\bar{b}_1, \bar{b}_2, \ldots, \bar{b}_p) \in (I\!R^n)^p$

$$\begin{cases} \dfrac{d}{dt}[(I + f'(x^\circ(t)))x(t)] = \varphi_x(x^\circ(t), u^\circ(t), t)x(t) + \varphi_u(x^\circ(t), u^\circ(t), t)u(t), \\ x(t_k^+) = \mathcal{J}_k'(x^0(t_k^-))x(t_k^-) + \bar{b}_k, \quad k = 1, 2, 3, \ldots, p. \end{cases} \quad (7.2)$$

e) The following conditions hold for all $k = 1, 2, 3, \ldots, p$

$$\sup_{t \in [0,T]} \|f'(x^\circ(t))\| < 1, \quad f'(x^\circ(t_k))\mathcal{J}_k'(x^0(t_k^-)) = \mathcal{J}_k'(x^0(t_k^-))f'(x^\circ(t_k)). \quad (7.3)$$

Let us consider the following optimal control problem governed by a nonlinear neutral differential equation:

Problem 7.1.

$$\int_0^T \Phi(x(t), u(t), t)dt \longrightarrow \min \text{ loc.} \quad (7.4)$$

$$(x, u) \in E := \mathcal{PW}([0, T]; \mathbb{R}^n) \times L_\infty^r([0, T]; \mathbb{R}^r), \quad (7.5)$$

$$\frac{d}{dt}[x(t) + f(x(t))] = \varphi(x(t), u(t), t), \quad x(0) = x_0 \quad (7.6)$$

$$x(T) = x_1; \; x_1, x_0 \in I\!R^n, \quad (7.7)$$

$$x(t_k^+) = x(t_k^-) + \mathcal{J}_k(x(t_k)), \quad k = 1, 2, 3, \ldots, p. \quad (7.8)$$

$$u(t) \in M, \quad t \in [0, T], \quad a.e., \quad (7.9)$$

Theorem 7.1. *Let us suppose that conditions a) - e) are fulfilled, and* $(x^\circ, u^\circ) \in E$ *is a solutions of the Problem 7.1.*
Then, there exists $\lambda_0 \in I\!R_{+0}$ *and a function* $\psi \in \mathcal{PW}([0,T];\mathbb{R}^n)$ *such that* λ_0 *and* ψ *are not both zero.*
Moreover, ψ *is a solution of the following differential equation*

$$\begin{cases} \dot{\psi}(\tau) = -\left(\varphi_x(x^\circ(\tau),\, u^\circ(\tau),\, \tau)\Gamma^{-1}(\tau)\right)^* \psi(\tau) + \lambda_0 \Phi_x(x^\circ(\tau),\, u^\circ(\tau),\, \tau) \\ \psi(T) = a \end{cases} \tag{7.10}$$

where $\Gamma(\tau) = I + f'(x^\circ(\tau))$*, and also, for all* $U \in M$ *and almost all* $t \in [0,\, T]$ *it follows*

$$\langle -\varphi_u^*(x^\circ(t),\, u^\circ(t),\, t)\psi(t) + \lambda_0\, \Phi_u(x^\circ(t),\, u^\circ(t),\, t),\, U - u^\circ(t)\rangle \geq 0. \tag{7.11}$$

Proof. Let $\overline{F} : E \longrightarrow I\!R$ be a function defined as follows

$$\overline{F}(x,\, u) = \int_0^T \Phi(x(t),\, u(t),\, t)dt,$$

and let $Q := Q_1 \cap Q_2$ where Q_2, Q_1 are given by pairs sets $(x,\, u) \in E$, which satisfy (7.6)-(7.8) and (7.9) respectively.
Then, Problem 7.1 is equivalent to

$$\begin{cases} \overline{F}(x,\, u) \longrightarrow \min \ \ \text{loc} \\ \\ (x,\, u) \in Q. \end{cases}$$

a) Analysis of the function $\overline{F}$.
Let $K_0 := K_d(F,\, (x^\circ,\, u^\circ))$ be the decay cone of $\overline{F}$ in the point $(x^\circ,\, u^\circ)$. Then, by Theorem 2.22, we have that

$$K_0 = \{(x,\, u) \in E \,/\, \overline{F}(x^\circ,\, u^\circ)(x,\, u) < 0\}.$$

Suppose for a moment that $K_0 \neq \emptyset$, then by Theorem 2.29 we obtain

$$K_0^+ = \{-\lambda_0\, \overline{F}(x^\circ,\, u^\circ) \,/\, \lambda_0 \in I\!R_{+0}\}.$$

By example 2.23, we obtain that

$$\overline{F}'(x^\circ,\, u^\circ)(x,\, u) = \int_0^T [\Phi_x(x^\circ,\, u^\circ,\, t)x(t) + \Phi_u(x^\circ,\, u^\circ,\, t)u(t)]dt, \qquad ((x,\, u) \in E).$$

Therefore, for all $f_0 \in K_0^+$, there exists $\lambda_0 \in I\!R_{+0}$ such that

$$f_0(x,\, u) = -\lambda_0 \int_0^T [\Phi_x(x^\circ,\, u^\circ,\, t)x(t) + \Phi_u(x^\circ,\, u^\circ,\, t)u(t)]dt, \qquad ((x,\, u) \in E).$$

b) Analysis of constraint Q_1.
Let us consider the set

$$Q_1' := \{u \in L_\infty^r[0,\, T] \,/\, u(t) \in M, \quad t \in [0,\, T], \quad a.e.\},$$

and $Q_1 = \mathcal{PW}([0,T];\mathbb{R}^n) \times Q_1'$. Moreover, by hypothesis M is convex and closed, with $\overset{\circ}{M} = \emptyset$. So, the following statements hold

i) Q_1, Q_1' are convex and closed.

ii) $\overset{\circ}{Q}_1 \neq \emptyset, \quad \overset{\circ}{Q'}_1 \neq \emptyset.$

If we call K_1 the admissible cone to Q_1 in $(x^\circ,\, u^\circ) \in Q_1$, then

$$K_1 = \mathcal{PW}([0,T];\mathbb{R}^n) \times K_1',$$

where K_1' is the admissible cone to Q_1' in $u^\circ \in Q_1'$.

Therefore, for all $f_1 \in K_1^+$ there is $f_1' \in K_1'^+$ such that $f_1 = (0,\, f_1')$.

By Theorem 2.26, it follows that f_1' is a support of Q_1' at u°.

c) Analysis of the constraint Q_2.

Let us find the tangent cone to Q_2 at the point $(x^\circ,\, u^\circ)$

$$K_2 := K_T(Q_2,\, (x^\circ,\, u^\circ)).$$

Consider the space $E_1 = \mathcal{PW}([0,T];\mathbb{R}^n) \times I\!R^{n(1+p)} = E_2$ and the operator: $P : E_1 \to E_2$ defined by

$$P(x,u)(t) = (L(x,u)(t), \quad S(x,u), \quad x(T) - x_1)\,,$$

where

$$L(x,u)(t) = x(t) - x_0 - f(x_0) + f(x(t)) - \int_0^t \varphi(x(l), u(l), l)dl,$$

$$S(x,u) = \big(x(t_1) - \mathcal{J}_1(x(t_1^-)), \quad x(t_2) - \mathcal{J}_2(x(t_2^-)) \cdots, \quad x(t_p) - \mathcal{J}_p(x(t_p^-))\big)\,.$$

Then

$$P'(x^0,u^0)(\overline{x},\overline{u}) = (L'(\overline{x},\overline{u}), \quad S'(\overline{x},\overline{u}), \quad \overline{x}(T))\,,$$

where

$$L'(\overline{x},\overline{u})(t) = \overline{x}(t) + f'(x^\circ(t))\overline{x}(t) - \int_0^t (\varphi_x(x^\circ,\, u^\circ, l)\overline{x}(l) + \varphi_u(x^\circ,\, u^\circ, l)\overline{u}(l))dl$$

$$S'(\overline{x},\overline{u}) = \left(\overline{x}(t_1) - \mathcal{J}_1'(x^0(t_1^-))\overline{x}(t_1^-), \cdots\cdots, \overline{x}(t_p) - \mathcal{J}_p'(x^0(t_p^-))\overline{x}(t_p^-)\right).$$

We want to find conditions under which the operator $P'(x^0,u^0)$ is onto in order to apply Lustenik theorem 2.27. So, for $(a(\cdot), b_1, b_2, \ldots, b_p, x_1) \in E_2$, we want to solve the equation

$$P'(x^0,u^0)(\overline{x},\overline{u}) = (a(\cdot), b_1, b_2, \ldots, b_p, x_1).$$

Now, suppose that $\overline{u} = 0$. Then, we want to solve the following integral differential equation

$$\Gamma(t)z(t) = a(t) + \int_0^t (\varphi_x(x^\circ(l),\, u^\circ(l),\, l)z(l)dl,$$

which is equivalent to the integral equation

$$z(t) = \Gamma^{-1}(t)a(t) + \int_0^t \Gamma^{-1}(t)\varphi_x(x^\circ(l),\, u^\circ(l),\, l)z(l)dl.$$

From ([25], pg 89), we know that this is a Volterra integral equation, which has a solution $z \in \mathcal{PW}([0,T];\mathbb{R}^n)$.
Next, since the impulsive linear variational equation (7.2) is controllable for all points $b \in I\!R^{np}$. In particular, for a point $(\overline{b}_1, \overline{b}_2, \ldots, \overline{b}_p) \in I\!R^{np}$ such that

$$\overline{b}_k = b_k - z(t_k) + \mathcal{J}_k'(x^0(t_k^-))z(t_k^-), \quad k = 1,2,3,\ldots,p,$$

there exists a control $\overline{u} \in L_\infty^r$ such that the corresponding solution $y(t)$ of (7.2) satisfies $y(T) = x_1 - z(T)$.
Therefore,

$$\Gamma(t)y(t) = \int_0^t (\varphi_x(x^\circ,u^\circ,l)y(l) + \varphi_u(x^\circ,u^\circ,l)\overline{u}(l))dl, \quad t \in [0,T].$$

Let us make the following change of variable $\overline{x} = y + z$. Then

$$\begin{aligned}&L'(x^\circ,u^\circ)(y+z,\overline{u})(t) = \Gamma(t)y(t) + \Gamma(t)z(t) - \\ &\int_0^t (\varphi_x(x^\circ,u^\circ,l)(y+z)(l) + \varphi_u(x^\circ,u^\circ,l)\overline{u}(l))dl \\ &= \Gamma(t)y(t) + a(t) - \int_0^t (\varphi_x(x^\circ,\, u^\circ,\, l)y(l) + \varphi_u(x^\circ,\, u^\circ, l)\overline{u}(l))dl = a(t).\end{aligned}$$

Clearly that $x(T) = x_1$. Now, we shall see that $S'(\overline{x},\overline{u}) = (b_1, b_2, \ldots, b_p)$. In fact,

$$\begin{aligned}&S'(\overline{x},\overline{u}) = \\ &\left(\overline{x}(t_1) - \mathcal{J}_1'(x^0(t_1^-))\overline{x}(t_1^-), \cdots\cdots, \overline{x}(t_p) - \mathcal{J}_p'(x^0(t_p^-))\overline{x}(t_p^-)\right) = \\ &\left((y+z)(t_1) - \mathcal{J}_1'(x^0(t_1^-))(y+z)(t_1^-), \cdots, (y+z)(t_p) - \mathcal{J}_p'(x^0(t_p^-))(y+z)(t_p^-)\right) \\ &= \left(\overline{b}_1 + z(t_1) - \mathcal{J}_1'(x^0(t_1^-))z(t_1^-), \cdots\cdots, \overline{b}_p + z(t_p) - \mathcal{J}_p'(x^0(t_p^-))z(t_p^-)\right) \\ &= (b_1, b_2, \ldots, b_p).\end{aligned}$$

Thus

$$P'(x^0, u^0)(\overline{x}, \overline{u})(t) = (L'(x^\circ, u^\circ)(x, \overline{u}),\, S'(\overline{x}, \overline{u}),\, x(T)) = (a(\cdot), b_1, b_2, \ldots, b_p, x_1).$$

Therefore, the operator $P'(x^0, u^0)$ is onto. Then, applying Lusternik's theorem 2.27, we get that tangent cone K_2 is given by

$$K_2 = \{(x,\, u) \in E_1 \,/\, P'(x^\circ,\, u^\circ)(\overline{x}, \overline{u}) = 0\}.$$

i.e., K_2 is the set of points $(\overline{x}, \overline{u}) \in E_1$ such that

$$\begin{aligned} [\Gamma(t)\overline{x}(t)]' &= \varphi_x(x^\circ(t),\, u^\circ(t),\, t)\overline{x}(t) + \varphi_u(x^\circ(t),\, u^\circ(t),\, t)u(t), \\ \overline{x}(t_k^+) &= \mathcal{J}_k'(x^0(t_k^-))\overline{x}(t_k^-), \quad k = 1, 2, 3, \ldots, p. \\ \overline{x}(T) &= 0 \end{aligned}$$

From condition (7.3), we can see that this system is equivalent to the following

$$\begin{aligned} [\Gamma(t)\overline{x}(t)]' &= \left(\varphi_x(x^\circ(t),\, u^\circ(t),\, t)\Gamma^{-1}(t)\right)\Gamma(t)\overline{x}(t) + \varphi_u(x^\circ(t),\, u^\circ(t),\, t)u(t), \\ \Gamma(t_k^+)\overline{x}(t_k^+) &= \mathcal{J}_k'(x^0(t_k^-))\Gamma(t_k^-)\overline{x}(t_k^-), \quad k = 1, 2, 3, \ldots, p. \\ \Gamma(T)\overline{x}(T) &= 0 \end{aligned}$$

Making the change of variable $z(t) = \Gamma(t)\overline{x}(t)$, we get the following equivalent controllable system

$$z(t)' = \left(\varphi_x(x^\circ(t),\, u^\circ(t),\, t)\Gamma^{-1}(t)\right) z(t) + \varphi_u(x^\circ(t),\, u^\circ(t),\, t)u(t), \tag{7.12}$$

$$z(t_k^+) = \mathcal{J}_k'(x^0(t_k^-))z(t_k^-), \quad k = 1, 2, 3, \ldots, p. \tag{7.13}$$

$$z(T) = 0 \tag{7.14}$$

Consider the following linear subspaces

$$L_1 = \{(\overline{z}, \overline{u}) \in E_1/(7.12) - (7.13) \quad \text{hold}\}, \quad L_2 = \{(\overline{z}, \overline{u}) \in E_1/ \quad \overline{z}(T) = 0\}.$$

Then, $K_2 = L_1 \cap L_2$. Now, let us compute K_2^+. By Proposition 2.40, we have that $f_{22} \in L_2^+$ if, and only if, there exists $a \in I\!R^n$ such that

$$f_{22}(x,\, u) = \langle a,\, z(T)\rangle \qquad ((x,\, u) \in E).$$

Moreover, the controllability of systems (7.1)- (7.2) implies that $L_1 + L_2$ is closed, then it follows that $L_1^+ + L_2^+$ is w^*- closed; then by Lemma 2.5, we obtain that

$$K_2^+ = L_1^+ + L_2^+.$$

Since L_1 is a linear subspace, it follows from Theorem 2.28 that, for any

$f_{21} \in L_1^+$, $f_{21}(\overline{z}, \overline{u}) = 0$ for all $(\overline{x}, \overline{u})$ satisfying (7.12)-(7.13).

e) Euler-Lagrange equation.

It is easy to see that K_0, K_1, K_2, are convex cones. Hence, by Theorem 2.15 there are functionals $f_i \in K_i^+$ $(i = 0, 1, 2,)$ not all zero such that

$$f_0 + f_1 + f_2 = f_0 + f_1 + f_{21} + f_{22} = 0. \tag{7.15}$$

Equation (7.15) can be written in the following form

$$\begin{aligned}&-\lambda_0 \int_0^T [\Phi_x(x^\circ, u^\circ, t)x(t) + \Phi_u(x^\circ, u^\circ, t)u(t)]dt+ \\ &f_1'(x, u) + f_{21}(x, u) + \langle a, x(T)\rangle = 0, \quad ((x, u) \in E).\end{aligned}$$

Now, for all $\overline{u} \in L^r_\infty$ there exist $\overline{z}$, solution of system (7.12)-(7.13) with $\overline{z}(0) = 0$. Then $(\overline{z}, u) \in L_1$. Therefore $f_{21}(\overline{z}, \overline{u}) = 0$.

Let ψ be the solution of equation (7.10), that is

$$\begin{cases} \dot\psi(\tau) = -\left(\varphi_x(x^\circ(\tau), u^\circ(\tau), \tau)\Gamma^{-1}(\tau)\right)^* \psi(\tau) + \lambda_0\Phi_x(x^\circ(\tau), u^\circ(\tau), \tau) \\ \psi(T) = a \end{cases}$$

Multiplying both sides of this equation by $\overline{z} = \Gamma(\tau)\overline{x}$ and integrating from 0 to T, we get

$$\begin{aligned}&\lambda_0 \int_0^T \Phi_x(x^\circ, u^\circ, t)\overline{z}(t)dt - \langle a, \overline{z}(T)\rangle = \int_0^T \langle\dot\psi(t), \overline{z}(t)\rangle dt \\
&+ \int_0^T \langle\left(\varphi_x(x^\circ(t), u^\circ(t), t)\Gamma^{-1}(\tau)\right)^* \psi(t), \overline{z}(t)\rangle dt - \langle a, \overline{z}(T)\rangle = \\
&\langle\psi(t), \overline{z}(t)\rangle]_0^T - \int_0^T \langle\psi(t), \dot{\overline{z}}(t)\rangle dt \\
&+ \int_0^T \langle\left(\varphi_x(x^\circ(t), u^\circ(t), t)\Gamma^{-1}(\tau)\right)^* \psi(t), \overline{z}(t)\rangle dt - \langle a, \overline{z}(T)\rangle = \\
&\langle\psi(T), \overline{z}(T)\rangle - \langle\psi(0), \overline{z}(0)\rangle - \langle a, \overline{z}(T)\rangle \\
&+ \int_0^T \langle\psi(t), \varphi_x(x^\circ, u^\circ, t)\Gamma^{-1}(\tau)\overline{z}(t) - \dot{\overline{z}}(t)\rangle dt = \\
&\int_0^T \langle\psi(t), \varphi_x(x^\circ, u^\circ, t)\overline{x}(t) - [\Gamma(\tau)\overline{x}(t)]'\rangle dt = \\
&- \int_0^T \langle\psi(t), \varphi_u(x^\circ, u^\circ, t)\overline{u}(t)\rangle dt = - \int_0^T \langle\varphi_u^*(x^\circ, u^\circ, t)\psi(t), \overline{u}(t)\rangle dt.\end{aligned}$$

Then, from Euler–Lagrange equation (7.15), we obtain for $(\overline{u} \in L^r_\infty[0, T])$, that

$$f_1'(t) = \int_0^T \langle-\varphi_u^*(x^\circ(t), u^\circ(t), t)\psi(t) + \lambda_0\, \Phi_u(x^\circ(t), u^\circ(t), t)u(t)\rangle dt. \tag{7.16}$$

Since f_1' is a support of Q_1' at the point $u^\circ \in Q_1'$, from example 2.44, it follows that

$$\langle -\varphi_u^*(x^\circ(t),\, u^\circ(t),\, t)\psi(t) + \lambda_0\, \Phi_u(x^\circ(t),\, u^\circ(t),\, t),\, U - u^\circ(t)\rangle \geq 0,$$

for all $U \in M$ and almost all $t \in [0,\, T]$.
Now, we will see that the case $\lambda_0 = 0,\ \psi = 0$, is not possible. In fact

If $\psi = 0$, then $\psi(T) = a = 0$. Thus

$$f_{22}(x,\, u) = \langle a,\, x(T)\rangle = 0 \qquad ((x,\, u) \in E),$$

that is $f_{22} \equiv 0$. So, from the fact that $\lambda_0 = 0$, we get that $f_0 = 0$. Also, from (7.16), we have that $f_1'(u) = 0 \quad (u \in L_\infty^r[0,\, T])$; then from Euler– Lagrange equation it follows that $f_{21} = 0$, where

$$f_2 = f_{21} + f_{22} = 0,$$

which contradicts Theorem 2.15.

So far, we have two additional assumptions:

Firstly, we assumed that $K_0 \neq \emptyset$, and secondly, we assumed that the system

$$[\Gamma(t)x(t)]' = \varphi_x(x^\circ,\, u^\circ,\, t)x(t) + \varphi_u(x^\circ,\, u^\circ,\ ,t)u(t)$$

is controllable.

Now, we will prove, that these assumptions are superfluous. In fact, if $K_0 = \emptyset$, then by definition of K_0, we have that

$$\int_0^T [\Phi_x(x^\circ(t),\, u^\circ(t),\, t)x(t) + \Phi_u(x^\circ(t),\, u^\circ(t),\, t)u(t)]dt = 0 \quad ((x,\, u) \in E).$$

Let us put $\lambda_0 = 1,\ \psi(T) = a = 0$, then, from last computation, we have that

$$\int_0^T \Phi_x(x^\circ,\, u^\circ,\, t)x(t)dt = -\int_0^T \langle \varphi_u^*(x^\circ,\, u^\circ,\, t)\psi(t), u(t)\rangle dt,$$

for all $(x,\, u)$ such that x is a solution of equation the (7.12)-(7.13). Then

$$\int_0^T \langle \varphi_u^*(x^\circ(t),\, u^\circ(t),\, t)\psi(t) + \Phi_u(x^\circ(t),\, u^\circ(t),\, t), u(t)\rangle dt = 0 \quad (u \in L_\infty^r[0,\, T])$$

which implies that

$$\langle -\varphi_u^*(x^\circ,\, u^\circ,\, t)\psi(t) + \Phi_u(x^\circ,\, u^\circ,\, t),\, U - u^\circ(t)\rangle = 0,$$

for all $U \in M$ and almost all $t \in [0,\, T]$.

Now, suppose that system (7.1) is not controllable, then there is a non-trivial function $\psi \in \mathcal{PW}([0,T];\mathbb{R}^n)$ that is a solution of

$$\dot{\psi}(t) = (\varphi_x(x^\circ(t),\, u^\circ(t),\, t)\Gamma^{-1}(t))^*\psi(t),$$

such that, for all $t \in [0,\, T]$ it follows that

$$-\varphi_u^*(x^\circ(t),\, u^\circ(t),\, t)\psi(t) = 0.$$

By taking $\lambda_0 = 0$, we get that ψ is a solution of (7.10), and therefore

$$\langle -\varphi_u^*(x^\circ(t),\, u^\circ(t),\, t)\psi(t),\, U - u^\circ(t)\rangle \geq 0,$$

for all $U \in M$ and almost all $t \in [0,\, T]$.

Thus, the proof of Theorem 7.1 is totally completed.

$\square$

8. Open Problems

Our first open problem concerns with optimal control problems for impulsive nonlinear neutral differential equations with modified boundary condition. In other word, we want to propose the following optimal control problem for future research

8.1. Open Problem

Problem 8.1.

$$\int_0^T \Phi(x(t),\, u(t),\, t)dt \longrightarrow \min \text{ loc.} \tag{8.1}$$

$$(x,\, u) \in E := \mathcal{PW}([0,T];\mathbb{R}^n) \times L_\infty^r([0,T];\mathbb{R}^r), \tag{8.2}$$

$$\frac{d}{dt}[x(t) + f(x(t))] = \varphi(x(t),\, u(t),\, t), \qquad x(0) = x_0 \tag{8.3}$$

$$x_0 \in I\!R^n;\ \mathbf{G_i(x(T)) = 0}, \quad i = 1, 2, \ldots, q. \tag{8.4}$$

$$x(t_k^+) = x(t_k^-) + \mathcal{J}_k(x(t_k)), \quad k = 1, 2, 3, \ldots, p. \tag{8.5}$$

$$u(t) \in M, \quad t \in [0,\, T], \quad a.e., \tag{8.6}$$

8.2. Open Problem

Second open problem is about optimal control problem on time scales. Basically, we want to analyze the following optimal control problem on time scales for our future investigation:

PROBLEM 8.2.

$$\int_0^T \Phi(x(t),\, u(t),\, t)\Delta t \longrightarrow \min \text{ loc.} \tag{8.7}$$

$$(x,\, u) \in E := PC([0,T]_{\mathbb{T}}; I\!R^n) \times C_{rd}([0,\tau]_{\mathbb{T}}, I\!R^r), \tag{8.8}$$

$$x^{\Delta}(t) = \varphi(x(t),\, u(t),\, t), \qquad x(0) = x_0 \tag{8.9}$$

$$x_0 \in I\!R^n;\ \mathbf{G_i(x(T)) = 0}, \quad i = 1, 2, \ldots, q. \tag{8.10}$$

$$x(t_k^+) = x(t_k^-) + \mathcal{J}_k(x(t_k)), \quad k = 1, 2, 3, \ldots, p. \tag{8.11}$$

$$u(t) \in M, \quad t \in [0,\, T]_{\mathbb{T}}, \quad a.e., \tag{8.12}$$

where the state function $x(t) \in I\!R^n$, the control u belongs to $C_{rd}([0,\tau]_{\mathbb{T}}, I\!R^r)$, the points $t_k \in \mathbb{T}$ are right dense for $k = 1, \ldots, p$ with $0 \leq t_1 < \cdots < t_p < \tau$, $x(t_k^+) = \lim\limits_{h\to 0^+} x(t_k + h)$, $x(t_k^-) = \lim\limits_{h\to 0^+} x(t_k - h)$ denotes the left and right limits of $x(t)$ at $t = t_k$ in terms of time scales. Also, we consider the Banach space:

$$\begin{aligned} PC([0,T]_{\mathbb{T}}; I\!R^n) \;=\; & \{x : [0,\tau]_{\mathbb{T}} \longrightarrow I\!R^n : x \in C(J'; I\!R^n), \text{there exist} \quad x(t_k^+), x(t_k^-) \\ & \text{and } x(t_k) = x(t_k^-), k = 1, 2, \ldots, p\} \end{aligned}$$

where $J' = [0,T]_{\mathbb{T}} \setminus \{t_1, \ldots, t_p\}$, is endowed with the norm

$$\|x\|_{PC} = \sup\{\|x(t)\| : t \in [0,T]_{\mathbb{T}}\}.$$

8.3. Open Problem

In the third problem we will study an optimal control problem governed by differential equations of the neutral type on time scales:

PROBLEM 8.3.

$$\int_0^T \Phi(x(t),\, u(t),\, t)\Delta t \longrightarrow \min \text{ loc.} \tag{8.13}$$

$$(x,\, u) \in E := PC([0,T]_{\mathbb{T}}; I\!R^n) \times C_{rd}([0,\tau]_{\mathbb{T}}, I\!R^r), \tag{8.14}$$

$$[x(t) + f(t, x(t))]^{\Delta} = \varphi(x(t),\, u(t),\, t), \qquad x(0) = x_0 \tag{8.15}$$

$$x_0 \in I\!R^n;\ \mathbf{G_i}(\mathbf{x}(\mathbf{T})) = \mathbf{0}, \quad i = 1, 2, \ldots, q. \tag{8.16}$$

$$x(t_k^+) = x(t_k^-) + \mathcal{J}_k(x(t_k)), \quad k = 1, 2, 3, \ldots, p. \tag{8.17}$$

$$u(t) \in M, \quad t \in [0,\, T]_{\mathbb{T}}, \quad a.e. \tag{8.18}$$

8.4. Open Problem

Our fourth open problem can be an optimal control system governed by an impulsive equation of the neutral type and nonlocal conditions. It can also be formulated in time scale .

PROBLEM 8.4.

$$\int_0^T \Phi(x(t),\, u(t),\, t)dt \longrightarrow \min \text{ loc.} \tag{8.19}$$

$$(x,\, u) \in E := \mathcal{PW}([0, T]; \mathbb{R}^n) \times L^r_\infty([0, T]; \mathbb{R}^r), \tag{8.20}$$

$$\frac{d}{dt}[x(t) + f(t, x(t))] = \varphi(x(t),\, u(t),\, t), \qquad \mathbf{x}(\mathbf{0}) = \mathbf{g}(\mathbf{x}) + \mathbf{x_0} \tag{8.21}$$

$$x_0 \in I\!R^n;\ \mathbf{G_i}(\mathbf{x}(\mathbf{T})) = \mathbf{0}, \quad i = 1, 2, \ldots, q. \tag{8.22}$$

$$x(t_k^+) = x(t_k^-) + \mathcal{J}_k(x(t_k)), \quad k = 1, 2, 3, \ldots, p. \tag{8.23}$$

$$u(t) \in M, \quad t \in [0,\, T], \quad a.e. \tag{8.24}$$

8.5. Open Problem

Our fifth open problem deals with an optimal control problem for non-autonomous semilinear neutral differential equations with unbounded delay, non-instantaneous impulses, and nonlocal conditions. Specifically, we are interested in finding a maximal principle for the following problem.

PROBLEM 8.5.

$$\int_0^T \Phi(x(t),\, u(t),\, t)dt \longrightarrow \min \text{ loc.} \tag{8.25}$$

$$(x,\, u) \in E := \mathcal{PW}((-\infty, T]; \mathbb{R}^n) \times L^r_\infty([0, T]; \mathbb{R}^r), \tag{8.26}$$

$$\frac{d}{dt}[x(t) - g(t, x_t)] = \mathcal{A}(t)x(t) + \mathsf{B}(t)u(t) + f(t, x_t, u(t)), \quad t \in \bigcup_{k=0}^{N} J_k^1, \tag{8.27}$$

$$x(t) = \Gamma_k(t, x(t_k^-), u(t_k^-)), \quad t \in J_k^2, k = 1, \ldots, N, \tag{8.28}$$

$$x(s) + \zeta(x_{\lambda_1}, \ldots, x_{\lambda_q})(s) = \phi(s) \quad s \in (-\infty, 0]. \tag{8.29}$$

$$x(T) = x_1; \; x_1 \in I\!R^n, \quad \phi \in \mathfrak{L}, \tag{8.30}$$

$$u(t) \in M, \quad t \in [0, T]_{\mathbb{T}}, \quad a.e., \tag{8.31}$$

where the state function $x(t)$ takes values in $I\!R^n$, meanwhile the control $u(\cdot)$ belongs to $L^r_\infty([0,T];\mathbb{R}^r)$, the space of admissible control functions. The matrices $\mathcal{A}(t)$ and $\mathsf{B}(t)$ are continuous of order $n \times n$ and $n \times m$, separately. The functions $x_t : (-\infty, 0] \longrightarrow I\!R^n$ given by $x_t(\theta) = x(t+\theta), \theta \leq 0$, belong to the phase space $\mathfrak{L}$ and represent the history of x up to t. Here $0 \leq \lambda_1 < \lambda_2 < \cdots < \lambda_q < T$ are prefixed numbers selected conveniently according the phenomenon to be modelled. Similarly, $s_0 = 0 < t_1 < s_1 < t_2 < \cdots < t_N < s_N < t_{N+1} = T$, $J_0 = [0, t_1]$, $J^1_k = (s_k, t_{k+1}]$ and $J^2_k = (t_k, s_k]$. The functions $g : [0,T] \times \mathfrak{L} \to I\!R^n$, $f : [0,T] \times \mathfrak{L} \times I\!R^m \to I\!R^n$, $x_t \in \mathfrak{L}$, $\phi \in \mathfrak{L}$, $\Gamma_k : (t_k, s_k] \times I\!R^n \times I\!R^m \to I\!R^n$ and $\zeta : \mathfrak{L}^q \to \mathfrak{L}$ are appropiate functions. In particular, $\Gamma_k, k = 1, 2, ...$, describes the non-instantaneous impulses in the model and ζ denotes the nonlocal conditions. For more information about the controllability of differential equations with noninstantaneous pulses, nonlocal conditions, and infinite delay, one can review the following references [11, 17, 28, 30].

9. Conclusion and Final Remark

As we have seen in this work, Pontryaguin's maximum principle is still valid for optimal control problems governed by differential equations with impulses as long as the impulses are small in some sense; that is, the linear variational equation around the optimal point is controllable. In many articles, of which we can mention ([7, 8, 10, 28, 29, 30, 31, 32, 33, 36]), it has already been verified that the controllability of the linear system is robust if we add impulses to the differential equation, delays and the non-local conditions as disturbances of the system. So, here we have seen that the maximum principle remains invariant under certain conditions on the impulses, so we believe that we can also say something if we add non-local conditions, and also consider dynamical equations on time scales.

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On Some Classes of Block Repetition Codes with Covering Radius of the Codes $\mathbb{Z}_{3^2}$

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ABSTRACT: In this paper, to obtain the bounds for some classes of repetition codes with covering radius by using various weight and also the same size and different size of length in repetition codes over a finite ring $\mathbb{Z}_{3^2}$.

AMS Subject Classification: 94B75, 16P10.
Keywords and Phrases: Finite ring; Linear Code; Covering radius; Generator matrix; Different distance.

1. Introduction

In coding theory for the last five decades, many researchers has been attraction in codes over finite rings and the special types of the rings $\mathbb{Z}_{2n}$, where $2n$ is the ring of integers modulo.

The authors was discovered the best well known non-linear binary codes can be constructed by cyclic codes and gray map over a finite ring $\mathbb{Z}_4$ in [19] and many research articles has indicated codes over a finite ring $\mathbb{Z}_4$ received much attention [1, 5–9]. Coding theory, the covering radius is one of the important parameter to find the maximum error-correcting capability of codes. In Binary code, [3, 4, 13–15], the covering radius of codes are studied for linear and non-linear codes can be received from codes over a finite ring $\mathbb{Z}_4$ via the gray map. In [10–12], the author to find lower bound and upper bound of covering radius in a suitable of different types repetition codes by using some finite rings with respect to various weight.

In this paper, to determine the covering radius of some attraction classes of repetition codes over a finite commutative ring $\mathbb{Z}_{3^2}$ of interger modulo 3^2 by using to different weight(distance).

2. Preliminaries

Let $\mathbb{Z}_{3^2}$ be a finite set with nine elements $\{0, 1, 2, 3, 4, 5, 6, 7, 8\}$ with two operation $\oplus_{3^2}, \odot_{3^2}$ is said to be a finite commutative ring. It is denoted by $(\mathbb{Z}_{3^2}, \oplus_{3^2}, \odot_{3^2}) = \mathbb{Z}$ with a characteristic 3^2. Let $C \subseteq \mathbb{Z}$, then C is say that a *code*. A code C is called the *linear code*, if the ring $\mathbb{Z}$ is an $\mathbb{Z}$-submodule of $\mathbb{Z}^l$, where l is the length of a code(that is, $C = (11111)$, $l(C) = 5$, $C_1 = (3333)$, $l(C_1) = 4$). The elements of C is called a *codeword* of C.

A Gray Map $h : \mathbb{Z}_{3^2} \to (\mathbb{Z}_3 \times \mathbb{Z}_3)$ is defined by

$$h(0) = 00, h(1) = 01, h(2) = 02, h(3) = 10, h(4) = 11, h(5) = 12,$$

$$h(6) = 20, h(7) = 21, h(8) = 22,$$

then the Gray map $h_1 : \mathbb{Z}^l_{3^2} \to (\mathbb{Z}_3 \times \mathbb{Z}_3)^l$ is define $h_1(y) = (h(y_1), h(y_2), \cdots, h(y_n))$, where $y = (y_1, y_2, \cdots, y_n)$ in [17].

Let $y \in \mathbb{Z}^l$ be a codeword of code, that is $y = (y_1, y_2, \cdots, y_n)$ and in [20], the Lee weight of y as given

$$w_L(y) = \begin{cases} 0 & \text{if} \quad y = 0 \\ 1 & \text{if} \quad y = 1, 8 \\ 2 & \text{if} \quad y = 2, 7 \\ 3 & \text{if} \quad y = 3, 4, 5, 6. \end{cases}$$

Let $y_i \in \mathbb{Z}$ be the codeword of Lee weight of y_i is defined as $\sum\limits_i w_L(y_i)_{,i=0 \text{ to } 8}$. If $c_1, c_2 \in C$, be any two distinct codewords of Lee distance is defined as $d_L(C) = \{d_L(c_1, c_2) | c_1 - c_2 \neq 0 \text{ and } c_1, c_2 \in C\}$. The minimum Lee weight of C is $d_L(C) = \min\{d_L(c_1, c_2) | c_1 - c_2 \neq 0 \text{ and } c_1, c_2 \in C\}$. In C is a *linear code* C, thus $d_L(C) = \min\{w_L(c) | c \neq 0 \in C\}$. Therefore, $d_L(c_1, c_2) = w_L(c_1 - c_2)$. If C is a linear code of length l over $\mathbb{Z}$ with the number of codewords W and the minimum Lee distance d_L, is said to be an (l, W, d_L) code in $\mathbb{Z}$. In C is a linear code of length l over $\mathbb{Z}$, therefore the Lee distance between z and C is defined as $d_L(z, C) = \min\{d_L(z, c) | \forall c \in C\}$, for any $z \in \mathbb{Z}^l$.

The Chinese Euclidean weight of x is

$$w_{CE}(y) = \begin{cases} 0 & \text{if} \quad y = 0 \\ 1 & \text{if} \quad y = 1, 8 \\ 2 & \text{if} \quad y = 2, 7 \\ 3 & \text{if} \quad y = 3, 6 \\ 4 & \text{if} \quad y = 4, 5 \end{cases}$$

in [18], where $y = (y_1, y_2, \cdots, y_n)$ be a codeword of code over $\mathbb{Z}^l$.

The parameters of Chinese Euclidean weight code is an (l, W, d_{CE}). In Chinese Euclidean distance(weight), let $c_1, c_2 \in \mathbb{Z}^l$ be any two different codewords is defined as $d_{CE}(c_1, c_2) = wt_{CE}(c_1 - c_2)$. Let C be a linear code of length l over $\mathbb{Z}$. Then $d_{CE}(z, C) = \min\{d_{CE}(z, c) | \forall c \in C\}$, for any $z \in \mathbb{Z}^l$.

In Gray weight, let $y \in \mathbb{Z}^l$ be a codeword of code, is define as

$$w_G(y) = \begin{cases} 0 & \text{if} \quad y = 0 \\ 1 & \text{if} \quad y = 1, 2, 3 \text{ and } 6 \\ 2 & \text{if} \quad \text{otherwise} \end{cases}$$

in [17].

In C is a linear code with Gray weight(distance), is an (l, W, d_G) code. Define, $d_G(c_1, c_2) = wt_G(c_1 - c_2)$, where $c_1, c_2 \in \mathbb{Z}^l$ and $d_G(z, C) = \min\{d_G(z, c) | \forall c \in C\}$, for any $z \in \mathbb{Z}^l$.

In [2], Let $y \in \mathbb{Z}^l$. The Bachoc weight of x is defined as

$$w_B(y) = \begin{cases} 0 & \text{if} \quad y = 0 \\ 1 & \text{if} \quad y = 1, 2, 4, 5, 7, 8, \\ 3 & \text{if} \quad y = 3, 6. \end{cases}$$

In C is a linear code with Bachoc weight(distance) is an (l, W, d_B) code. Define, $d_B(c_1, c_2) = wt_B(c_1 - c_2)$, where $c_1, c_2 \in \mathbb{Z}^n$ and $d_B(z, C) = \min\{d_B(z, c) | \forall c \in C\}$, for any $z \in \mathbb{Z}^n$.

Example 2.1. *Let* $y = 1\ 3\ 4\ 7\ 2 \in \mathbb{Z}^5$. *Then,*

$$w_L(y) = w_L(1) + w_L(3) + w_L(4) + w_L(7) + w_L(2) = 11,$$

$$w_{CE}(y) = w_{CE}(1) + w_{CE}(3) + w_{CE}(4) + w_{CE}(7) + w_{CE}(2) = 12,$$

$$w_G(y) = w_G(1) + w_G(3) + w_G(4) + w_G(7) + w_G(2) = 8 \text{ and}$$

$$w_B(y) = w_B(1) + w_B(3) + w_B(4) + w_B(7) + w_B(2) = 10.$$

3. Repetition code with Covering radius of code in $\mathbb{Z}$

Let d be the distance of a code C in $\mathbb{Z}$ with respect to different distance(weight), such as Lee weight, Chinese Euclidean weight, Gray weight and Bachoc weight. The *covering radius* of a code C is

$$R_d(C) = \max_{w \in \mathbb{Z}^n} \left\{ \min_{c \in C} \{d(w, c)\} \right\},$$

where C is a code and $R_d(C)$ is a covering radius of the code C with distance d.

In $F_q = \{0, 1, \gamma_2, \cdots, \gamma_{q-1}\}$ is a finite field. Let C be a q-ary *repetition code* C over F_q. That is $C = \{\bar{\gamma} = (\gamma\gamma \cdots \gamma) | \gamma \in F_q\}$ and the repetition code C is an $[l, 1, l]$ code. Therefore, the covering radius of the code C is $\lfloor \frac{l(q-1)}{q} \rfloor$ by using in [16].

Let C be a block repetition code of size l, the parameter of C is an $[l(q-1), 1, l(q-1)]$ be a generated by $G = [\overbrace{11\cdots1}^{l}\overbrace{\gamma_2\gamma_2\cdots\gamma_2}^{l}\cdots\overbrace{\gamma_{q-1}\gamma_{q-1}\cdots\gamma_{q-1}}^{l}]$. In [16], thus the covering radius of the code C is $\lfloor\frac{l(q-1)^2}{q}\rfloor$, since it will be equivalent to a repetition code of length $(q-1)l$.

A code $C \subseteq \mathbb{Z}$ is also linear code and is called the Generator matrix(G), if the basis elements in a row of matrix.

In repetition code over $\mathbb{Z}$, there are two type of repetition codes of length l viz.

1. Type A-(A Generator matrix(G_A) with unit element in $\mathbb{Z}$ and its generated by the code C_A)
2. Type B-(A Generator matrix(G_B) with zero divisor element in $\mathbb{Z}$ and its generated by the code C_B)

Type A (G_A) $\rightarrow$	$[\overbrace{11\cdots1}^{l}], [\overbrace{22\cdots2}^{l}], [\overbrace{44\cdots4}^{l}]$, $[\overbrace{55\cdots5}^{l}], [\overbrace{77\cdots7}^{l}], [\overbrace{88\cdots8}^{l}]$	$[l, k=1, d_i=l]$, $i=\{L, CE, G, B\}$
Type B (G_B) $\rightarrow$	$[\overbrace{33\cdots3}^{l}], [\overbrace{66\cdots6}^{l}]$, $[\overbrace{36\ 36\cdots36}^{l}], [\overbrace{63\ 63\cdots63}^{l}]$	$(l, W=3, d_j=3l)$, $j=\{L, CE, G, B\}$

Theorem 3.1.

- $R_L(C_A) = 2l$,
- $R_L(C_B) = 2l$, *here* $R_L(C_{A(B)})$ *is a covering radius of codes* $C_{A(B)}$ *for generator matrix* $G_{A(B)}$ *by using Lee weight and* l *is a length of code in Type A and Type B.*

Proof. Let $y \in \mathbb{Z}^l$ by ϱ_0 times $0'$s, ϱ_1 times $1'$s, ϱ_2 times $2'$s, ϱ_3 times $3'$s, ϱ_4 times $4'$s, ϱ_5 times $5'$s, ϱ_6 times $6'$s, ϱ_7 times $7'$s, ϱ_8 times $8'$s in y and $\sum_i \varrho_i = l$ and the code $c_i \in \{\gamma(C_A)|\gamma \in \mathbb{Z}^l\}$, where $i=0$ to 8. Then

$$\begin{aligned}
d_L(y, c_0) &= wt_L(y - 00\cdots0)\\
&= 0\varrho_0 + 1\varrho_1 + 2\varrho_2 + 3\varrho_3 + 4\varrho_4 + 5\varrho_5 + 6\varrho_6 + 7\varrho_7 + 8\varrho_8\\
&= \varrho_1 + 2\varrho_2 + 3\varrho_3 + 3\varrho_4 + 3\varrho_5 + 3\varrho_6 + 2\varrho_7 + \varrho_8\\
d_L(y, c_0) &= l - \varrho_0 + \varrho_2 + 2\varrho_3 + 2\varrho_4 + 2\varrho_5 + 2\varrho_6 + \varrho_7.
\end{aligned}$$

Alike,

$$d_L(y, c_1) = l - \varrho_1 + \varrho_3 + 2\varrho_4 + 2\varrho_5 + 2\varrho_6 + 2\varrho_7 + \varrho_8,$$

$$d_L(y,c_2) = l - \varrho_2 + \varrho_0 + \varrho_4 + 2\varrho_5 + 2\varrho_6 + 2\varrho_7 + 2\varrho_8,$$
$$d_L(y,c_3) = l - \varrho_3 + 2\varrho_0 + \varrho_1 + \varrho_5 + 2\varrho_6 + 2\varrho_7 + 2\varrho_8,$$
$$d_L(y,c_4) = l - \varrho_4 + 2\varrho_0 + 2\varrho_1 + \varrho_2 + \varrho_6 + 2\varrho_7 + 2\varrho_8,$$
$$d_L(y,c_5) = l - \varrho_5 + 2\varrho_0 + 2\varrho_1 + 2\varrho_2 + \varrho_3 + \varrho_7 + 2\varrho_8,$$
$$d_L(y,c_6) = l - \varrho_6 + 2\varrho_0 + 2\varrho_1 + 2\varrho_2 + 2\varrho_3 + \varrho_4 + \varrho_8,$$
$$d_L(y,c_7) = l - \varrho_7 + \varrho_0 + 2\varrho_1 + 2\varrho_2 + 2\varrho_3 + 2\varrho_4 + \varrho_5,$$
$$d_L(y,c_8) = l - \varrho_8 + \varrho_1 + 2\varrho_2 + 2\varrho_3 + 2\varrho_4 + 2\varrho_5 + \varrho_6.$$

Then, $d_L(y,C_A) = \min\{d_L(x,c_i) | i = 0 \text{ to } 8\} \leq 2l$ and $r_L(C_A) \leq 2l$.

If $y_1 \in \mathbb{Z}^l$, whereas $y_1 = \overbrace{00\cdots0}^{k}\overbrace{11\cdots1}^{k}\overbrace{22\cdots2}^{k}\overbrace{33\cdots3}^{k}\overbrace{44\cdots4}^{k}\overbrace{55\cdots5}^{k}$ $\overbrace{66\cdots6}^{k}\overbrace{77\cdots7}^{k}\overbrace{88\cdots8}^{l-8k}$, here $k = \lfloor \frac{l}{3^2} \rfloor$. Thus, $d_L(y_1,c_i) = 12k,\ i = 0$ to 8 and $r_L(C_A) \geq \min\{d_L(y_1,c_i)|\ i = 0 \text{ to } 8\} \geq 2l$ and hence, $r_L(C_A) = 2l$.

Let $y = \overbrace{33\cdots3}^{\frac{l}{2}}\overbrace{000\cdots0}^{\frac{l}{2}} \in \mathbb{Z}^l$. The code $C_B = \{\gamma(33\cdots3) \mid \gamma \in \mathbb{Z}^l\}$ and it is generated by Type-B. Thus, $r_L(C_B) \geq 2l$.

If $y \in \mathbb{Z}^l$ be any codeword and take y has ϱ_i links i's, with $\sum_i \varrho_i = l$, where $i = 0$ to 8. Then, $r_L(C_B) \leq 2l$.

$\square$

Theorem 3.2. *For* $R_d(C) = \max_{w\in\mathbb{Z}^n}\{\min_{c\in C}\{d(w,c)\}\}$, *where* $d = \{$ *Chinese Euclidean weight, Gray weight and Bachoc weight* $\}$.

1. $R_{CE}(C_A) = \frac{20l}{9}, \frac{3n}{2} \leq R_{CE}(C_B) \leq 2l,$
2. $R_G(C_A) = \frac{4l}{3}, R_G(C_B) = l$ *and*
3. $R_B(C_A) = \frac{4l}{3}, \frac{3l}{2} \leq R_B(C_{B^*}) \leq 2l,$ *where* $B^* =$ *Type-B and* l *is a length of code in Type A and Type B.*

Proof. The methods of proof is follows from Theorem 3.1, by using the Type A and Type B with different weight, such as $w_{CE}(x), w_G(x)$, and $w_B(x)$. $\square$

4. Same size of length in Block repetition code

Let BRC^{2l} be a Block Repetition Code with length $2l$ and its generated by $G_{AB} = [\overbrace{11\cdots1}^{l}\overbrace{33\cdots3}^{l}]$ is size of length(l) for each block and the parameters of BRC^{2l} code is an $[2l,1,3l,3l,3l,3l]$.

Theorem 4.1.

1. $R_L(BRC^{2l}) = 4l$,
2. $R_{CE}(BRC^{2l}) = \frac{38l}{9}$,
3. $R_G(BRC^{2l}) = \frac{7l}{3}$ *and*
4. $R_B(BRC^{2l}) = \frac{8l}{3}$.

Proof. Generator matrix G_{AB} and [13] and by using theorem 3.1, then

$$R_L(BRC^{2l}) \geq 4l. \tag{4.1}$$

Consider $y = (y_1 \mid y_2) \in \mathbb{Z}^{2l}$, where $y_1, y_2 \in \mathbb{Z}^{2l}$ and also take in y_1, ϱ_j appears j's, and in y_2, ϱ_j appears j's, with $\sum\limits_j r_j = \sum\limits_j s_j = l$ and $c_j \in \{\gamma(G_{AB}) | \gamma \in \mathbb{Z}^{2l}\}, j = 0$ to 8.

Then, $d_L(y, BRC^{2l}) = \min\{d_L(y, c_j) | j = 0 \text{ to } 8\}$ is less than or equal to $2l + 2l = 4l$. Thus, $d_L(y, BRC^{2l}) \leq 4l$. Hence,

$$R_L(BRC^{2l}) \leq 4l \tag{4.2}$$

By (4.1) and (4.2), thus $R_L(BRC^{2l}) = 4l$.

The remaining Proof of the Theorem 4.1 is pursue from first part. □

Corollary 4.2. *Let*

$$G_A = [\overbrace{11\cdots1}^{l}\overbrace{22\cdots2}^{l}\overbrace{44\cdots4}^{l}\overbrace{55\cdots5}^{l}\overbrace{77\cdots7}^{l}\overbrace{88\cdots8}^{l}] \tag{4.3}$$

is a Type A with unit element in $\mathbb{Z}$. *Then,*

- $R_L(BRC^{6l}) = 12l$,
- $R_{CE}(BRC^{6l}) = \frac{40l}{3}$,
- $R_G(BRC^{6l}) = 8l$ *and*
- $R_B(BRC^{6l}) = 8l$.

Proof. From (4.3) and use to Theorem 3.1, 3.2 and 4.1. □

Corollary 4.3. *Let*

$$G_B = [\overbrace{33\cdots3}^{l}\overbrace{66\cdots6}^{l}] \tag{4.4}$$

is a Type B with zero divisor element in $\mathbb{Z}$. *Then,*

- $R_L(BRC^{2l}) = 4l$,
- $3l \leq R_{CE}(BRC^{2l}) \leq 4l$,
- $R_G(BRC^{2l}) = 2l$ *and*
- $3l \leq R_B(BRC^{2l}) \leq 4l$.

Proof. In (4.4) is apply to Theorem 3.1, 3.2 and 4.1. □

5. Different size of the length for Block repetition code

Let

$$G_{AB} = [\overbrace{11\cdots1}^{k_1}\overbrace{33\cdots3}^{k_2}] \tag{5.1}$$

be the generated matrix for the two various block repetition code for a size of length is k_1, k_2 and it is denoted by $BRC^{k_1+k_2}$. The parameters of $BRCp^{k_1+k_2}$ code is an $[k_1+k_2, 1, \min\{3k_1, k_1+3k_2\}, \min\{k_1, k_1+k_2\}, \min\{3k_1, k_1+3k_2\}, \min\{3k_1, k_1+3k_2\}, \min\{3k_1, 2k_1+3_2\}]$.

Theorem 5.1.

- $R_L(BRC^k) = 2k$,
- $R_{CE}(BRC^k) = \frac{20k_1}{9} + 2k_2$,
- $R_G(BRC^k) = \frac{4k}{3}$ *and*
- $R_B(BRC^k) = \frac{4k}{3}$, *there with* $k = \sum_{i=1}^{2} k_i$.

Proof. A generator matrix (5.1), use to Theorem 4.1 and apply the two different size of length(k_1, k_2). □

Corollary 5.2. *Let*

$$G_B = [\overbrace{33\cdots3}^{k_1}\overbrace{66\cdots6}^{k_2}] \tag{5.2}$$

is a Type B with zero divisor element and two distinct length(k_1, k_2) *in* $\mathbb{Z}$. *Then*

- $R_L(BRC^k) = 2k$,
- $\frac{3k}{2} \leq R_{CE}(BRC^k) \leq 2k$,
- $R_G(BRC^k) = k$ *and*
- $\frac{4k}{3} \leq R_B(BRC^k) \leq 2k$, *here* $k = \sum_{i=1}^{2} k_i$.

Proof. In (5.2) by two distinct length(k_1, k_2) and different weights in put to Theorem 5.1. □

Corollary 5.3. *Let*

$$G_A = [\overbrace{11\cdots1}^{k_1}\overbrace{22\cdots2}^{k_2}\overbrace{44\cdots4}^{k_3}\overbrace{55\cdots5}^{k_4}\overbrace{77\cdots7}^{k_5}\overbrace{88\cdots8}^{k_6}]. \tag{5.3}$$

be a Type A with unit element and alternate size of length in $\mathbb{Z}$. *Then*

- $R_L(BRC^k) = 2k$,
- $R_{CE}(BRC^k) = \frac{20k}{9}$,
- $R_G(BRC^k) = \frac{4k}{3}$ *and*
- $R_B(BRC^k) = \frac{4k}{3}$, *where* $k = \sum\limits_{i=1}^{6} k_i$.

Proof. In (5.3) with alternate size of length and also weight is apply to Theorem 5.1. □

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Equilibrium Stacks for a Non-Cooperative Game Defined on a Product of Staircase-Function Continuous and Finite Strategy Spaces

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Abstract: A method of solving a non-cooperative game defined on a product of staircase-function strategy spaces is presented. The spaces can be finite and continuous as well. The method is based on stacking equilibria of "short" non-cooperative games, each defined on an interval where the pure strategy value is constant. In the case of finite non-cooperative games, which factually are multidimensional-matrix games, the equilibria are considered in general terms, so they can be in mixed strategies as well. The stack is any combination (succession) of the respective equilibria of the "short" multidimensional-matrix games. Apart from the stack, there are no other equilibria in this "long" (staircase-function) multidimensional-matrix game. An example of staircase-function quadmatrix game is presented to show how the stacking is fulfilled for a case of when every "short" quadmatrix game has a single pure-strategy equilibrium. The presented method, further "breaking" the initial staircase-function game into a succession of "short" games, is far more tractable than a straightforward approach to solving directly the "long" non-cooperative game would be.

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1. Introduction

Non-cooperative games are applied for rationalizing the distribution of limited resources (e. g., see [23, 4, 26, 15]). A simple case of the non-cooperative game is a finite non-cooperative game, which always has an equilibrium, either in pure or mixed strategies [22, 23, 10, 9]. An infinite or continuous non-cooperative game is far more complicated as, opposed to a finite game, an equilibrium is not always determinable. Moreover, a solution of an infinite game, in which a strategy has an infinite support, is not practically realizable [8, 23, 21, 12, 15]. This is due to a finite number of factual actions of a player. Therefore, any game is approximated to a finite one, which always has an equilibrium [22].

A finite non-cooperative game is easily rendered to a multidimensional-matrix game [13, 17], wherein the pure strategy can be a complex action through time rather than an elementary action [4, 26, 3, 1, 17, 18]. Although the game rendering can be fulfilled regardless of the pure strategy complexity [22, 13], such rendering is impossible if the set of the player's strategies is either infinite or continuous. If the player's pure strategy is a function (commonly, it is a function of time), and every player possesses a finite set of such function-strategies, the rendering results in huge multidimensional payoff matrices. This is a far more complicated finite game, in which the player's payoff is a functional [25, 16, 17, 18]. Regardless of the function-strategy set finiteness, each player's functional maps every set of functions (pure strategies of the players defined on a time interval) into a real value. However, a finite game is not obtained by just breaking (sampling) a time interval, on which the pure strategy is defined, into a set of subintervals, on which the strategy could be approximately considered constant. This is so because of the continuity of possible values of the strategy on a subinterval. The continuity is removed by sampling along the strategy value axis [13, 16]. Then the set of function-strategies becomes finite, and that results in a finite non-cooperative game. The size and properties of such a game strongly depend on both samplings [13, 17].

2. Motivation

The number of factual actions of a player in any game has a natural limit, whichever the form of the pure strategy is [23, 10, 9, 12]. Nevertheless, if the rules of a system which is game-modeled are defined and administered beforehand, the administrator is likely to define (or constrain) the form of the strategies players will use [21, 26, 24, 16]. The most trivial case is when the player's pure strategy is an elementary action whose duration is negligibly short and thus is represented as just a time point. This case is exhaustively studied as bimatrix, trimatrix, and dyadic games [6, 22, 23, 15]. In a more complicated case, the player's pure strategy is a function of time [25, 16], so the player's action is a complex process whose duration cannot be reduced to a time point. A way to appropriately administer the players' actions is to constrain them to staircase functions whose points of discontinuities (breakpoints) have to be the same for all the players [20, 24, 16]. Along with the discrete time, possible values of the player's pure strategy should be discrete as well. Then the game can be represented as a multidimensional-matrix game, in which the player's selection of a pure strategy means using a staircase function on a time interval whereon every pure strategy is defined.

It is easy to get convinced of that the number of the player's pure strategies in the multidimensional-matrix staircase-function game grows immensely as the number of breakpoints ("stair" intervals) or/and the number of possible values of the player's pure strategy increases. For instance, if the number of intervals is 4, and the number of possible values of the player's pure strategy is 5, then there are $5^4 = 625$ possible pure strategies at this player, where every strategy is a 4-interval 5-staircased function of time. Whereas the respective bimatrix 625×625 game still may be solved in a reasonable time, the respective trimatrix $625 \times 625 \times 625$ game appears to be big enough (having 244140625 situations), let alone $625\times625\times625\times625$ quadmatrix game whose number of situations is 152587890625 (more than 152 billion). This trivialized example shows that a finite staircase-function game becomes practically intractable to solve it when there are more than two players. An exclusion is the ultimately trivialized instance, when every player has 2-interval 2-staircased function-strategies. Then the respective 4×4, $4\times4\times4$, $4\times4\times4\times4$, ..., games can be solved fast enough even for 10 players, although the $\bigtimes_{n=1}^{10} 4$ game has 1048576 situations. It is worth noting that it may take no less than 0.4 seconds to solve a $\bigtimes_{n=1}^{6} 4$ game on a laptop with an Intel Core i7 processor, whereas a $10\times10\times10\times10$ game is solved at least in 1.1 seconds. When every strategy, say, is a 6-interval 10-staircased function of time, even the respective bimatrix $10^6 \times 10^6$ staircase-function game appears to be intractably gigantic (there is a trillion situations in this game!). This is a simple example of the intractability even for a bimatrix game, let alone finite staircase-function games with three or more players. This means that, instead of rendering a non-cooperative game defined on a product of staircase-function finite spaces to a multidimensional-matrix game, a tractable method of solving it should be suggested.

3. Objective and tasks to be fulfilled

Issuing from the impracticability of rendering a finite non-cooperative game with staircase-function strategies to a multidimensional-matrix game, the objective is to develop a tractable method of solving non-cooperative games defined on a product of staircase-function finite spaces. For achieving the objective, the following five tasks are to be fulfilled:

1. To formalize a non-cooperative game (of any number of players), in which the players' strategies are staircase functions. In such a game, the set of the player's pure strategies is a continuum of staircase functions of time. Herein, the time can be thought of as it is discrete.

2. To discretize the set of possible values of the player's pure strategy so that the game be defined on a product of staircase-function finite spaces.

3. To formalize a method of solving non-cooperative games defined on a product of staircase-function finite spaces.
4. To consider an example of solving a finite game defined on a product of staircase-function spaces. A special attention should be paid to the computation time.
5. To discuss and conclude on applicability and significance of the method, as well as its possible drawbacks and limitations.

4. A non-cooperative game with staircase-function strategies

Consider a non-cooperative game of N players, $N \in \mathbb{N} \backslash \{1\}$. In this game the player's pure strategy is a function of time. Let each of the players use time-varying strategies defined almost everywhere on interval $[t_1;\ t_2]$ by $t_2 > t_1$. Denote a strategy of the n-th player by $x_n(t)$, $n = \overline{1,\ N}$. These functions are presumed to be bounded, i. e.

$$x_n^{(\min)} \leqslant x_n(t) \leqslant x_n^{(\max)} \text{ by } x_n^{(\min)} < x_n^{(\max)} \tag{1}$$

defined almost everywhere on $[t_1;\ t_2]$. Besides, the square of the function-strategy is presumed to be Lebesgue-integrable. Thus, pure strategies of the player belong to a rectangular functional space of functions of time:

$$X_n = \\ = \left\{ x_n(t),\ t \in [t_1;\ t_2],\ t_1 < t_2 :\ x_n^{(\min)} \leqslant x(t) \leqslant x_n^{(\max)} \text{ by } x_n^{(\min)} < x_n^{(\max)} \right\} \subset \\ \subset \mathbb{L}_2[t_1;\ t_2] \tag{2}$$

is the set of the n-th player's pure strategies, $n = \overline{1,\ N}$.

The player's payoff in situation

$$\{x_n(t)\}_{n=1}^{N} \tag{3}$$

is presumed to be an integral functional [2, 11, 18, 19]. Thus, the n-th player's payoff in situation (3) is

$$K_n\left(\{x_i(t)\}_{i=1}^{N}\right) = \int\limits_{[t_1;\ t_2]} f_n\left(\{x_i(t)\}_{i=1}^{N},\ t\right) d\mu(t) \tag{4}$$

by a function

$$f_n\left(\{x_i(t)\}_{i=1}^{N},\ t\right) \tag{5}$$

of time functions (3) explicitly including time t. Therefore, the continuous N-person game

$$\left\langle \{X_n\}_{n=1}^{N},\ \left\{K_n\left(\{x_i(t)\}_{i=1}^{N}\right)\right\}_{n=1}^{N} \right\rangle \tag{6}$$

is defined on product

$$\mathop{\times}_{n=1}^{N} X_n \subset \mathop{\times}_{n=1}^{N} \mathbb{L}_2 \left[t_1;\ t_2\right] \tag{7}$$

of rectangular functional spaces (2) of players' pure strategies.

First, it is presumed that game (6) is administered so that the players are forced to use pure strategies $\{x_i(t)\}_{i=1}^{N}$ such that they change their values for a finite number of times. Denote by M the number of intervals at which the player's pure strategy is constant, where $M \in \mathbb{N} \backslash \{1\}$. Then the player's pure strategy is a staircase function having only M different values. If $\left\{\tau^{(l)}\right\}_{l=1}^{M-1}$ are time points at which the staircase-function strategy changes its value, where

$$t_1 = \tau^{(0)} < \tau^{(1)} < \tau^{(2)} < \ldots < \tau^{(M-1)} < \tau^{(M)} = t_2, \tag{8}$$

then

$$\left\{x_n\left(\tau^{(l)}\right)\right\}_{l=0}^{M} \tag{9}$$

are the values of the n-th player's strategy in a play-off of game (6), $n = \overline{1,\ N}$. The staircase-function strategies are right-continuous [2]:

$$\lim_{\substack{\varepsilon>0\\ \varepsilon\to 0}} x_n\left(\tau^{(l)} + \varepsilon\right) = x_n\left(\tau^{(l)}\right) \quad \text{for}\ \ l = \overline{1,\ M-1}\ \ \text{by}\ \ n = \overline{1,\ N}, \tag{10}$$

whereas

$$\lim_{\substack{\varepsilon>0\\ \varepsilon\to 0}} x_n\left(\tau^{(l)} - \varepsilon\right) \neq x_n\left(\tau^{(l)}\right) \quad \text{for}\ \ l = \overline{1,\ M-1}\ \ \text{by}\ \ n = \overline{1,\ N}. \tag{11}$$

As an exception,

$$\lim_{\substack{\varepsilon>0\\ \varepsilon\to 0}} x_n\left(\tau^{(M)} - \varepsilon\right) = x_n\left(\tau^{(M)}\right), \tag{12}$$

so

$$x_n\left(\tau^{(M-1)}\right) = x_n\left(\tau^{(M)}\right) \quad \forall\, n = \overline{1,\ N}.$$

Then constant values (9) by (8) mean that game (6) can be thought of as it is a succession of M continuous games

$$\left\langle \left\{\left[x_n^{(\min)};\ x_n^{(\max)}\right]\right\}_{n=1}^{N},\ \left\{K_n\left(\{\alpha_{il}\}_{i=1}^{N}\right)\right\}_{n=1}^{N} \right\rangle \tag{13}$$

defined on hyperparallelepiped

$$\mathop{\times}_{n=1}^{N} \left[x_n^{(\min)};\ x_n^{(\max)}\right] \tag{14}$$

by

$$\alpha_{nl} = x_n(t) \in \left[x_n^{(\min)};\ x_n^{(\max)}\right] \quad \text{by}\ \ n = \overline{1,\ N}$$

$$\forall\, t \in \left[\tau^{(l-1)};\ \tau^{(l)}\right) \text{ for } l = \overline{1,\ M-1} \text{ and } \forall\, t \in \left[\tau^{(M-1)};\ \tau^{(M)}\right], \tag{15}$$

where the factual players' payoffs in situation $\{\alpha_{il}\}_{i=1}^{N}$ are

$$K_n\left(\{\alpha_{il}\}_{i=1}^{N}\right) = \int\limits_{\left[\tau^{(l-1)};\ \tau^{(l)}\right)} f_n\left(\{\alpha_{il}\}_{i=1}^{N},\ t\right) d\mu\,(t)\ \ \forall\, l = \overline{1,\ M-1} \tag{16}$$

by

$$K_n\left(\{\alpha_{iM}\}_{i=1}^{N}\right) = \int\limits_{\left[\tau^{(M-1)};\ \tau^{(M)}\right]} f_n\left(\{\alpha_{iM}\}_{i=1}^{N},\ t\right) d\mu\,(t) \tag{17}$$

for $n = \overline{1,\ N}$. So, let such game (6) be called staircase [18, 19]. A pure-strategy situation in staircase game (6) is a succession of M situations $\{\alpha_{il}\}_{i=1}^{N}$ in games (13).

Theorem 1. *In a pure-strategy situation of staircase game (6), represented as a succession of M games (13), functional (4) is re-written as an interval-wise sum*

$$K_n\left(\{x_i\,(t)\}_{i=1}^{N}\right) = $$
$$= \sum_{l=1}^{M-1} \int\limits_{\left[\tau^{(l-1)};\ \tau^{(l)}\right)} f_n\left(\{\alpha_{il}\}_{i=1}^{N},\ t\right) d\mu\,(t) +$$
$$+ \int\limits_{\left[\tau^{(M-1)};\ \tau^{(M)}\right]} f_n\left(\{\alpha_{iM}\}_{i=1}^{N},\ t\right) d\mu\,(t). \tag{18}$$

Proof. Situation $\{\alpha_{il}\}_{i=1}^{N}$ is tied to half-interval $\left[\tau^{(l-1)};\ \tau^{(l)}\right)$ by $l = \overline{1,\ M-1}$ and to interval $\left[\tau^{(M-1)};\ \tau^{(M)}\right]$ by $l = M$. Function (5) in this situation is some function of time t. Denote this function by $\psi_{nl}\,(t)$. For situation $\{\alpha_{il}\}_{i=1}^{N}$ function

$$\psi_{nl}\,(t) = 0\ \ \forall\, t \notin \left[\tau^{(l-1)};\ \tau^{(l)}\right), \tag{19}$$

and for situation $\{\alpha_{iM}\}_{i=1}^{N}$ function

$$\psi_{nM}\,(t) = 0\ \ \forall\, t \notin \left[\tau^{(M-1)};\ \tau^{(M)}\right]. \tag{20}$$

Therefore,

$$f_n\left(\{x_i\,(t)\}_{i=1}^{N},\ t\right) = \sum_{l=1}^{M} \psi_{nl}\,(t) \tag{21}$$

in a pure-strategy situation $\{x_i\,(t)\}_{i=1}^{N}$ of staircase game (6), by using (19) and (20). Consequently,

$$K_n\left(\{x_i\,(t)\}_{i=1}^{N}\right) = \int\limits_{[t_1;\ t_2]} f_n\left(\{x_i\,(t)\}_{i=1}^{N},\ t\right) d\mu\,(t) =$$

$$= \sum_{l=1}^{M-1} \int\limits_{\left[\tau^{(l-1)};\, \tau^{(l)}\right)} \psi_{nl}(t)\, d\mu(t) + \int\limits_{\left[\tau^{(M-1)};\, \tau^{(M)}\right]} \psi_{nM}(t)\, d\mu(t) =$$

$$= \sum_{l=1}^{M-1} \int\limits_{\left[\tau^{(l-1)};\, \tau^{(l)}\right)} f_n\left(\{\alpha_{il}\}_{i=1}^{N},\, t\right) d\mu(t) +$$

$$+ \int\limits_{\left[\tau^{(M-1)};\, \tau^{(M)}\right]} f_n\left(\{\alpha_{iM}\}_{i=1}^{N},\, t\right) d\mu(t) \tag{22}$$

in a pure-strategy situation $\{x_i(t)\}_{i=1}^{N}$ of staircase game (6). □

In other words, if every equilibrium situation in pure strategies in game (6) on product (7) by conditions (1) — (5) is (or forced to be) of staircase functions satisfying conditions (8) — (12), then this game is equivalent to the succession of M games (13) defined on parallelepiped (14) by (8) — (12) and (15) — (18). In this case game (6) can be represented by the succession of games (13).

Theorem 2. *If each of M games (13) by (8) — (12) and (15) — (18) has a single equilibrium situation in pure strategies, and game (6) on product (7) by conditions (1) — (5) is equivalent to the succession of these games, then the equilibrium situation in pure strategies in game (6) is determined by independently finding pure-strategy equilibria in M games (13), whereupon these equilibria are successively stacked.*

Proof. First, the equivalency means that game (6) has only staircase pure-strategy equilibria. Next, it should be proved that game (6) has a pure-strategy equilibrium situation, which is a successive stack of the M "short" games (13). Let $\left\{\{\alpha_{il}^{*}\}_{i=1}^{N}\right\}_{l=1}^{M}$ be pure-strategy equilibria in games (13) by (8) — (12) and (15) — (18). Then

$$K_n\left(\left\{\{\alpha_{il}^{*}\}_{i=1}^{N} \setminus \{\alpha_{nl}^{*}\}\right\} \cup \{\alpha_{nl}\}\right) \leqslant K_n\left(\{\alpha_{il}^{*}\}_{i=1}^{N}\right)$$

$$\forall\, \alpha_{nl} \in \left[x_n^{(\min)};\, x_n^{(\max)}\right] \text{ and } \forall\, n = \overline{1,\, N} \text{ and } \forall\, l = \overline{1,\, M}. \tag{23}$$

Inequalities (23) are re-written using statements (15) — (18):

$$K_n\left(\left\{\{\alpha_{il}^{*}\}_{i=1}^{N} \setminus \{\alpha_{nl}^{*}\}\right\} \cup \{\alpha_{nl}\}\right) =$$

$$= \int\limits_{\left[\tau^{(l-1)};\, \tau^{(l)}\right)} f_n\left(\left\{\{\alpha_{il}^{*}\}_{i=1}^{N} \setminus \{\alpha_{nl}^{*}\}\right\} \cup \{\alpha_{nl}\},\, t\right) d\mu(t) \leqslant$$

$$\leqslant \int\limits_{\left[\tau^{(l-1)};\, \tau^{(l)}\right)} f_n\left(\{\alpha_{il}^{*}\}_{i=1}^{N},\, t\right) d\mu(t) = K_n\left(\{\alpha_{il}^{*}\}_{i=1}^{N}\right)$$

$$\forall\, \alpha_{nl} \in \left[x_n^{(\min)};\, x_n^{(\max)}\right] \text{ and } \forall\, n = \overline{1,\, N} \text{ and } \forall\, l = \overline{1,\, M-1}, \tag{24}$$

$$
\begin{aligned}
&K_n\left(\left\{\{\alpha_{iM}^*\}_{i=1}^{N} \setminus \{\alpha_{nM}^*\}\right\} \cup \{\alpha_{nM}\}\right) = \\
&= \int\limits_{\left[\tau^{(M-1)};\ \tau^{(M)}\right]} f_n\left(\left\{\{\alpha_{iM}^*\}_{i=1}^{N} \setminus \{\alpha_{nM}^*\}\right\} \cup \{\alpha_{nM}\},\ t\right) d\mu(t) \leqslant \\
&\leqslant \int\limits_{\left[\tau^{(M-1)};\ \tau^{(M)}\right]} f_n\left(\{\alpha_{iM}^*\}_{i=1}^{N},\ t\right) d\mu(t) = \\
&= K_n\left(\{\alpha_{iM}^*\}_{i=1}^{N}\right)\ \ \forall\, \alpha_{nM} \in \left[x_n^{(\min)};\ x_n^{(\max)}\right] \ \text{ and } \ \forall\, n = \overline{1,\ N}. \qquad (25)
\end{aligned}
$$

Owing to Theorem 1,

$$
\sum_{l=1}^{M} K_n\left(\left\{\{\alpha_{il}^*\}_{i=1}^{N} \setminus \{\alpha_{nl}^*\}\right\} \cup \{\alpha_{nl}\}\right) \leqslant \sum_{l=1}^{M} K_n\left(\{\alpha_{il}^*\}_{i=1}^{N}\right)\ \ \forall\, n = \overline{1,\ N}. \qquad (26)
$$

Therefore, the successive stack of pure-strategy equilibria $\left\{\{\alpha_{il}^*\}_{i=1}^{N}\right\}_{l=1}^{M}$ is a pure-strategy equilibrium in game (6). Obviously, games (13) can be solved independently, whose equilibria are stacked afterwards to form the pure-strategy equilibrium in game (6). □

In fact, Theorem 2 claims that if each of N "short" games (13) has a single pure-strategy equilibrium, then the solution of N-person game (6) can be determined in a simpler way, by solving games (13) and successively stacking their equilibria. They are solved in parallel (independently), without caring of the succession. However, Theorem 2 does not determine a probability (likelihood) of the case when every "short" game has a single pure-strategy equilibrium. Obviously, the likelihood decays as the number of intervals increases.

Besides, Theorem 2 does not directly imply that the stacked equilibrium in game (6) is single. The question of whether the stacked equilibrium in game (6) is single or not is answered by the following assertion.

Theorem 3. *If each of M games (13) by (8) — (12) and (15) — (18) has a single equilibrium situation in pure strategies, and game (6) on product (7) by conditions (1) — (5) is equivalent to the succession of these games, then the equilibrium situation in pure strategies in game (6) is single being the successive stack of the "short" games equilibria.*

Proof. The pure-strategy equilibrium in game (6) is constructed according to Theorem 2, i. e., it is the successive stack of pure-strategy equilibria $\left\{\{\alpha_{il}^*\}_{i=1}^{N}\right\}_{l=1}^{M}$. Let this equilibrium be referred to as the

$$
\left\{\{\alpha_{il}^*\}_{i=1}^{N}\right\}_{l=1}^{M}\text{-stack equilibrium.} \qquad (27)
$$

Suppose that there is another pure-strategy equilibrium in game (6). Without losing generality, let this equilibrium differ from (27) in just that the first player uses some $\alpha_{1k}^{(0)} \in \left[x_1^{(\min)};\ x_1^{(\max)}\right]$ instead of α_{1k}^{*} by some $k \in \left\{\overline{1,\ M}\right\}$. So, this is the

$$\left\{\left\{\{\alpha_{il}^{*}\}_{i=1}^{N}\right\}_{l \in \left\{\overline{1,\ M}\right\} \setminus \{k\}} \cup \left\{\alpha_{1k}^{(0)},\ \{\alpha_{ik}^{*}\}_{i=2}^{N}\right\}\right\}\text{-stack equilibrium,}$$

which means that

$$\begin{aligned} \sum_{l \in \left\{\overline{1,\ M}\right\} \setminus \{k\}} K_1\left(\alpha_{1l},\ \{\alpha_{il}^{*}\}_{i=2}^{N}\right) + K_1\left(\alpha_{1k},\ \{\alpha_{ik}^{*}\}_{i=2}^{N}\right) \leqslant \\ \leqslant \sum_{l \in \left\{\overline{1,\ M}\right\} \setminus \{k\}} K_1\left(\{\alpha_{il}^{*}\}_{i=1}^{N}\right) + K_1\left(\alpha_{1k}^{(0)},\ \{\alpha_{ik}^{*}\}_{i=2}^{N}\right), \end{aligned} \tag{28}$$

$$\begin{aligned} \sum_{l \in \left\{\overline{1,\ M}\right\} \setminus \{k\}} K_n\left(\left\{\{\alpha_{il}^{*}\}_{i=1}^{N} \setminus \{\alpha_{nl}^{*}\}\right\} \cup \{\alpha_{nl}\}\right) + \\ + K_n\left(\left\{\left\{\alpha_{1k}^{(0)},\ \{\alpha_{ik}^{*}\}_{i=2}^{N}\right\} \setminus \{\alpha_{nk}^{*}\}\right\} \cup \{\alpha_{nk}\}\right) \leqslant \\ \leqslant \sum_{l \in \left\{\overline{1,\ M}\right\} \setminus \{k\}} K_n\left(\{\alpha_{il}^{*}\}_{i=1}^{N}\right) + K_n\left(\alpha_{1k}^{(0)},\ \{\alpha_{ik}^{*}\}_{i=2}^{N}\right) \quad \forall\, n = \overline{2,\ N}, \end{aligned} \tag{29}$$

i. e.,

$$\begin{aligned} K_1\left(\alpha_{1l},\ \{\alpha_{il}^{*}\}_{i=2}^{N}\right) \leqslant K_1\left(\{\alpha_{il}^{*}\}_{i=1}^{N}\right) \\ \forall\, \alpha_{1l} \in \left[x_1^{(\min)};\ x_1^{(\max)}\right] \text{ and } \forall\, l \in \left\{\overline{1,\ M}\right\} \setminus \{k\} \end{aligned} \tag{30}$$

by

$$K_1\left(\alpha_{1k},\ \{\alpha_{ik}^{*}\}_{i=2}^{N}\right) \leqslant K_1\left(\alpha_{1k}^{(0)},\ \{\alpha_{ik}^{*}\}_{i=2}^{N}\right) \quad \forall\, \alpha_{1k} \in \left[x_1^{(\min)};\ x_1^{(\max)}\right], \tag{31}$$

and

$$\begin{aligned} K_n\left(\left\{\{\alpha_{il}^{*}\}_{i=1}^{N} \setminus \{\alpha_{nl}^{*}\}\right\} \cup \{\alpha_{nl}\}\right) \leqslant K_n\left(\{\alpha_{il}^{*}\}_{i=1}^{N}\right) \\ \forall\, \alpha_{nl} \in \left[x_n^{(\min)};\ x_n^{(\max)}\right] \text{ and } \forall\, l \in \left\{\overline{1,\ M}\right\} \setminus \{k\} \text{ and } \forall\, n = \overline{2,\ N} \end{aligned} \tag{32}$$

by

$$\begin{aligned} K_n\left(\left\{\left\{\alpha_{1k}^{(0)},\ \{\alpha_{ik}^{*}\}_{i=2}^{N}\right\} \setminus \{\alpha_{nk}^{*}\}\right\} \cup \{\alpha_{nk}\}\right) \leqslant K_n\left(\alpha_{1k}^{(0)},\ \{\alpha_{ik}^{*}\}_{i=2}^{N}\right) \\ \forall\, \alpha_{nk} \in \left[x_n^{(\min)};\ x_n^{(\max)}\right] \text{ and } \forall\, n = \overline{2,\ N}. \end{aligned} \tag{33}$$

Inequalities (31) and (33) imply that $\left\{\alpha_{1k}^{(0)},\ \{\alpha_{ik}^*\}_{i=2}^{N}\right\}$ is a pure-strategy equilibrium at the k-th interval (in the k-th game), which is impossible due to every interval has a single pure-strategy equilibrium. The impossibility of the other pure-strategy equilibrium for the remaining players in such a case is proved symmetrically.

Suppose that the other pure-strategy equilibrium differs from (27) in that the first player uses some $\alpha_{1k_1}^{(0)} \in \left[x_1^{(\min)};\ x_1^{(\max)}\right]$ instead of $\alpha_{1k_1}^*$ by some $k_1 \in \left\{\overline{1,\ M}\right\}$ and the second player uses some $\alpha_{2k_2}^{(0)} \in \left[x_2^{(\min)};\ x_2^{(\max)}\right]$ instead of $\alpha_{2k_2}^*$ by some $k_2 \in \left\{\overline{1,\ M}\right\}$. So, this is the

$$\left\{\left\{\{\alpha_{il}^*\}_{i=1}^{N}\right\}_{l\in\left\{\overline{1,\ M}\right\}\backslash\{k_1\}} \cup \left\{\alpha_{1k_1}^{(0)},\ \alpha_{2k_1}^{(0)},\ \left\{\alpha_{ik_1}^*\right\}_{i=3}^{N}\right\}\right\}\text{-stack equilibrium} \tag{34}$$

if $k_1 = k_2$, and is the

$$\left\{\left\{\{\alpha_{il}^*\}_{i=1}^{N}\right\}_{l\in\left\{\overline{1,\ M}\right\}\backslash\{k_1,\ k_2\}} \cup \right.$$
$$\left. \cup \left\{\alpha_{1k_1}^{(0)},\ \left\{\alpha_{ik_1}^*\right\}_{i=2}^{N}\right\} \cup \left\{\alpha_{1k_2}^*,\ \alpha_{2k_2}^{(0)},\ \left\{\alpha_{ik_2}^*\right\}_{i=3}^{N}\right\}\right\}\text{-stack equilibrium} \tag{35}$$

if $k_1 \neq k_2$. Thus, (34) means that

$$\sum_{l\in\left\{\overline{1,\ M}\right\}\backslash\{k_1\}} K_1\left(\alpha_{1l},\ \{\alpha_{il}^*\}_{i=2}^{N}\right) + K_1\left(\alpha_{1k_1},\ \alpha_{2k_1}^{(0)},\ \left\{\alpha_{ik_1}^*\right\}_{i=3}^{N}\right) \leqslant$$
$$\leqslant \sum_{l\in\left\{\overline{1,\ M}\right\}\backslash\{k_1\}} K_1\left(\{\alpha_{il}^*\}_{i=1}^{N}\right) + K_1\left(\alpha_{1k_1}^{(0)},\ \alpha_{2k_1}^{(0)},\ \left\{\alpha_{ik_1}^*\right\}_{i=3}^{N}\right) \tag{36}$$

and

$$\sum_{l\in\left\{\overline{1,\ M}\right\}\backslash\{k_1\}} K_2\left(\alpha_{1l}^*,\ \alpha_{2l},\ \{\alpha_{il}^*\}_{i=3}^{N}\right) + K_2\left(\alpha_{1k_1}^{(0)},\ \alpha_{2k_1},\ \left\{\alpha_{ik_1}^*\right\}_{i=3}^{N}\right) \leqslant$$
$$\leqslant \sum_{l\in\left\{\overline{1,\ M}\right\}\backslash\{k_1\}} K_2\left(\{\alpha_{il}^*\}_{i=1}^{N}\right) + K_2\left(\alpha_{1k_1}^{(0)},\ \alpha_{2k_1}^{(0)},\ \left\{\alpha_{ik_1}^*\right\}_{i=3}^{N}\right) \tag{37}$$

and

$$\sum_{l\in\left\{\overline{1,\ M}\right\}\backslash\{k_1\}} K_n\left(\left\{\{\alpha_{il}^*\}_{i=1}^{N} \setminus \{\alpha_{nl}^*\}\right\} \cup \{\alpha_{nl}\}\right) +$$
$$+K_n\left(\left\{\left\{\alpha_{1k_1}^{(0)},\ \alpha_{2k_1}^{(0)},\ \left\{\alpha_{ik_1}^*\right\}_{i=3}^{N}\right\} \setminus \left\{\alpha_{nk_1}^*\right\}\right\} \cup \{\alpha_{nk_1}\}\right) \leqslant$$
$$\leqslant \sum_{l\in\left\{\overline{1,\ M}\right\}\backslash\{k_1\}} K_n\left(\{\alpha_{il}^*\}_{i=1}^{N}\right) + K_n\left(\alpha_{1k_1}^{(0)},\ \alpha_{2k_1}^{(0)},\ \left\{\alpha_{ik_1}^*\right\}_{i=3}^{N}\right) \quad \forall\, n = \overline{3,\ N}, \tag{38}$$

i. e., inequalities (30) by $k = k_1$ and inequality

$$K_1\left(\alpha_{1k_1},\ \alpha_{2k_1}^{(0)},\ \left\{\alpha_{ik_1}^{*}\right\}_{i=3}^{N}\right) \leqslant K_1\left(\alpha_{1k_1}^{(0)},\ \alpha_{2k_1}^{(0)},\ \left\{\alpha_{ik_1}^{*}\right\}_{i=3}^{N}\right)$$
$$\forall\, \alpha_{1k_1} \in \left[x_1^{(\min)};\ x_1^{(\max)}\right] \tag{39}$$

hold along with (23) for $n = 1$, inequalities

$$K_2\left(\alpha_{1l}^{*},\ \alpha_{2l},\ \{\alpha_{il}^{*}\}_{i=3}^{N}\right) \leqslant K_2\left(\{\alpha_{il}^{*}\}_{i=1}^{N}\right)$$
$$\forall\, \alpha_{2l} \in \left[x_2^{(\min)};\ x_2^{(\max)}\right] \text{ and } \forall\, l \in \left\{\overline{1,\ M}\right\} \setminus \{k_1\} \tag{40}$$

and inequality

$$K_2\left(\alpha_{1k_1}^{(0)},\ \alpha_{2k_1},\ \left\{\alpha_{ik_1}^{*}\right\}_{i=3}^{N}\right) \leqslant K_2\left(\alpha_{1k_1}^{(0)},\ \alpha_{2k_1}^{(0)},\ \left\{\alpha_{ik_1}^{*}\right\}_{i=3}^{N}\right)$$
$$\forall\, \alpha_{2k_1} \in \left[x_2^{(\min)};\ x_2^{(\max)}\right] \tag{41}$$

hold along with (23) for $n = 2$, inequalities (32) by $k = k_1$ and inequality

$$K_n\left(\left\{\left\{\alpha_{1k_1}^{(0)},\ \alpha_{2k_1}^{(0)},\ \left\{\alpha_{ik_1}^{*}\right\}_{i=3}^{N}\right\} \setminus \left\{\alpha_{nk_1}^{*}\right\}\right\} \cup \{\alpha_{nk_1}\}\right) \leqslant$$
$$\leqslant K_n\left(\alpha_{1k_1}^{(0)},\ \alpha_{2k_1}^{(0)},\ \left\{\alpha_{ik_1}^{*}\right\}_{i=3}^{N}\right)$$
$$\forall\, \alpha_{nk_1} \in \left[x_n^{(\min)};\ x_n^{(\max)}\right] \text{ and } \forall\, n = \overline{3,\ N} \tag{42}$$

hold along with (23). Inequalities (39) — (42) imply that $\left\{\alpha_{1k_1}^{(0)},\ \alpha_{2k_1}^{(0)},\ \left\{\alpha_{ik_1}^{*}\right\}_{i=3}^{N}\right\}$ is a pure-strategy equilibrium at the k_1-th interval (in the k_1-th game), which is impossible. The same conclusion is valid for a two-person non-cooperative game, where (34), (36), (37), (39), (41) are written by retaining $\left\{\alpha_{ik_1}^{*}\right\}_{i=3}^{N} = \emptyset$, $\{\alpha_{il}^{*}\}_{i=3}^{N} = \emptyset$, and (38), (42) are omitted. If (35) is true, then

$$\sum_{l \in \left\{\overline{1,\ M}\right\} \setminus \{k_1,\ k_2\}} K_1\left(\alpha_{1l},\ \{\alpha_{il}^{*}\}_{i=2}^{N}\right) +$$
$$+ K_1\left(\alpha_{1k_1},\ \left\{\alpha_{ik_1}^{*}\right\}_{i=2}^{N}\right) + K_1\left(\alpha_{1k_2},\ \alpha_{2k_2}^{(0)},\ \left\{\alpha_{ik_2}^{*}\right\}_{i=3}^{N}\right) \leqslant$$
$$\leqslant \sum_{l \in \left\{\overline{1,\ M}\right\} \setminus \{k_1,\ k_2\}} K_1\left(\{\alpha_{il}^{*}\}_{i=1}^{N}\right) +$$
$$+ K_1\left(\alpha_{1k_1}^{(0)},\ \left\{\alpha_{ik_1}^{*}\right\}_{i=2}^{N}\right) + K_1\left(\alpha_{1k_2}^{*},\ \alpha_{2k_2}^{(0)},\ \left\{\alpha_{ik_2}^{*}\right\}_{i=3}^{N}\right) \tag{43}$$

and

$$\sum_{l \in \left\{\overline{1,\ M}\right\} \setminus \{k_1,\ k_2\}} K_2\left(\alpha_{1l}^{*},\ \alpha_{2l},\ \{\alpha_{il}^{*}\}_{i=3}^{N}\right) +$$

$$+K_2\left(\alpha_{1k_1}^{(0)},\ \alpha_{2k_1},\ \left\{\alpha_{ik_1}^{*}\right\}_{i=3}^{N}\right)+K_2\left(\alpha_{1k_2}^{*},\ \alpha_{2k_2},\ \left\{\alpha_{ik_2}^{*}\right\}_{i=3}^{N}\right)\leqslant$$
$$\leqslant\sum_{l\in\left\{\overline{1,\ M}\right\}\setminus\{k_1,\ k_2\}}K_2\left(\{\alpha_{il}^{*}\}_{i=1}^{N}\right)+$$
$$+K_2\left(\alpha_{1k_1}^{(0)},\ \left\{\alpha_{ik_1}^{*}\right\}_{i=2}^{N}\right)+K_2\left(\alpha_{1k_2}^{*},\ \alpha_{2k_2}^{(0)},\ \left\{\alpha_{ik_2}^{*}\right\}_{i=3}^{N}\right)\tag{44}$$

and

$$\sum_{l\in\left\{\overline{1,\ M}\right\}\setminus\{k_1,\ k_2\}}K_n\left(\left\{\{\alpha_{il}^{*}\}_{i=1}^{N}\setminus\{\alpha_{nl}^{*}\}\right\}\cup\{\alpha_{nl}\}\right)+$$
$$+K_n\left(\left\{\left\{\alpha_{1k_1}^{(0)},\ \left\{\alpha_{ik_1}^{*}\right\}_{i=2}^{N}\right\}\setminus\left\{\alpha_{nk_1}^{*}\right\}\right\}\cup\{\alpha_{nk_1}\}\right)+$$
$$+K_n\left(\left\{\left\{\alpha_{1k_2}^{*},\ \alpha_{2k_2}^{(0)},\ \left\{\alpha_{ik_2}^{*}\right\}_{i=3}^{N}\right\}\setminus\left\{\alpha_{nk_2}^{*}\right\}\right\}\cup\{\alpha_{nk_2}\}\right)\leqslant$$
$$\leqslant\sum_{l\in\left\{\overline{1,\ M}\right\}\setminus\{k_1,\ k_2\}}K_n\left(\{\alpha_{il}^{*}\}_{i=1}^{N}\right)+K_n\left(\alpha_{1k_1}^{(0)},\ \left\{\alpha_{ik_1}^{*}\right\}_{i=2}^{N}\right)+$$
$$+K_n\left(\alpha_{1k_2}^{*},\ \alpha_{2k_2}^{(0)},\ \left\{\alpha_{ik_2}^{*}\right\}_{i=3}^{N}\right)\quad\forall\, n=\overline{3,\ N},\tag{45}$$

i. e., inequalities

$$K_1\left(\alpha_{1l},\ \{\alpha_{il}^{*}\}_{i=2}^{N}\right)\leqslant K_1\left(\{\alpha_{il}^{*}\}_{i=1}^{N}\right)$$
$$\forall\,\alpha_{1l}\in\left[x_1^{(\min)};\ x_1^{(\max)}\right]\ \text{ and }\ \forall\, l\in\left\{\overline{1,\ M}\right\}\setminus\{k_1,\ k_2\}\tag{46}$$

and inequality

$$K_1\left(\alpha_{1k_1},\ \left\{\alpha_{ik_1}^{*}\right\}_{i=2}^{N}\right)+K_1\left(\alpha_{1k_2},\ \alpha_{2k_2}^{(0)},\ \left\{\alpha_{ik_2}^{*}\right\}_{i=3}^{N}\right)\leqslant$$
$$\leqslant K_1\left(\alpha_{1k_1}^{(0)},\ \left\{\alpha_{ik_1}^{*}\right\}_{i=2}^{N}\right)+K_1\left(\alpha_{1k_2}^{*},\ \alpha_{2k_2}^{(0)},\ \left\{\alpha_{ik_2}^{*}\right\}_{i=3}^{N}\right)$$
$$\forall\,\alpha_{1k_1}\in\left[x_1^{(\min)};\ x_1^{(\max)}\right]\ \text{ and }\ \forall\,\alpha_{1k_2}\in\left[x_1^{(\min)};\ x_1^{(\max)}\right]\tag{47}$$

hold along with (23) for $n=1$, inequalities

$$K_2\left(\alpha_{1l}^{*},\ \alpha_{2l},\ \{\alpha_{il}^{*}\}_{i=3}^{N}\right)\leqslant K_2\left(\{\alpha_{il}^{*}\}_{i=1}^{N}\right)$$
$$\forall\,\alpha_{2l}\in\left[x_2^{(\min)};\ x_2^{(\max)}\right]\ \text{ and }\ \forall\, l\in\left\{\overline{1,\ M}\right\}\setminus\{k_1,\ k_2\}\tag{48}$$

and inequality

$$K_2\left(\alpha_{1k_1}^{(0)},\ \alpha_{2k_1},\ \left\{\alpha_{ik_1}^{*}\right\}_{i=3}^{N}\right)+K_2\left(\alpha_{1k_2}^{*},\ \alpha_{2k_2},\ \left\{\alpha_{ik_2}^{*}\right\}_{i=3}^{N}\right)\leqslant$$
$$\leqslant K_2\left(\alpha_{1k_1}^{(0)},\ \left\{\alpha_{ik_1}^{*}\right\}_{i=2}^{N}\right)+K_2\left(\alpha_{1k_2}^{*},\ \alpha_{2k_2}^{(0)},\ \left\{\alpha_{ik_2}^{*}\right\}_{i=3}^{N}\right)$$
$$\forall\,\alpha_{2k_1}\in\left[x_2^{(\min)};\ x_2^{(\max)}\right]\ \text{ and }\ \forall\,\alpha_{2k_2}\in\left[x_2^{(\min)};\ x_2^{(\max)}\right]\tag{49}$$

hold along with (23) for $n = 2$, inequalities

$$K_n\left(\left\{\{\alpha_{il}^*\}_{i=1}^N \setminus \{\alpha_{nl}^*\}\right\} \cup \{\alpha_{nl}\}\right) \leqslant K_n\left(\{\alpha_{il}^*\}_{i=1}^N\right)$$
$$\forall\, \alpha_{nl} \in \left[x_n^{(\min)};\; x_n^{(\max)}\right] \text{ and } \forall\, l \in \left\{\overline{1,\; M}\right\} \setminus \{k_1,\; k_2\} \text{ and } \forall\, n = \overline{3,\; N} \quad (50)$$

and inequality

$$K_n\left(\left\{\left\{\alpha_{1k_1}^{(0)},\; \left\{\alpha_{ik_1}^*\right\}_{i=2}^N\right\} \setminus \left\{\alpha_{nk_1}^*\right\}\right\} \cup \{\alpha_{nk_1}\}\right) +$$
$$+K_n\left(\left\{\left\{\alpha_{1k_2}^*,\; \alpha_{2k_2}^{(0)},\; \left\{\alpha_{ik_2}^*\right\}_{i=3}^N\right\} \setminus \left\{\alpha_{nk_2}^*\right\}\right\} \cup \{\alpha_{nk_2}\}\right) \leqslant$$
$$\leqslant K_n\left(\alpha_{1k_1}^{(0)},\; \left\{\alpha_{ik_1}^*\right\}_{i=2}^N\right) + K_n\left(\alpha_{1k_2}^*,\; \alpha_{2k_2}^{(0)},\; \left\{\alpha_{ik_2}^*\right\}_{i=3}^N\right)$$
$$\forall\, \alpha_{nk_1} \in \left[x_n^{(\min)};\; x_n^{(\max)}\right] \text{ and } \forall\, \alpha_{nk_2} \in \left[x_n^{(\min)};\; x_n^{(\max)}\right] \text{ and } \forall\, n = \overline{3,\; N} \quad (51)$$

hold along with (23). Plugging $\alpha_{1k_2} = \alpha_{1k_2}^*$ in the left side of inequality (47) and plugging $\alpha_{2k_2} = \alpha_{2k_2}^{(0)}$ in the left side of inequality (49) and plugging $\alpha_{nk_2} = \alpha_{nk_2}^*$ in the left side of inequality (51) for $n = \overline{3,\; N}$ gives inequalities

$$K_1\left(\alpha_{1k_1},\; \left\{\alpha_{ik_1}^*\right\}_{i=2}^N\right) \leqslant K_1\left(\alpha_{1k_1}^{(0)},\; \left\{\alpha_{ik_1}^*\right\}_{i=2}^N\right) \;\; \forall\, \alpha_{1k_1} \in \left[x_1^{(\min)};\; x_1^{(\max)}\right], \quad (52)$$

$$K_2\left(\alpha_{1k_1}^{(0)},\; \alpha_{2k_1},\; \left\{\alpha_{ik_1}^*\right\}_{i=3}^N\right) \leqslant K_2\left(\alpha_{1k_1}^{(0)},\; \left\{\alpha_{ik_1}^*\right\}_{i=2}^N\right)$$
$$\forall\, \alpha_{2k_1} \in \left[x_2^{(\min)};\; x_2^{(\max)}\right], \quad (53)$$

$$K_n\left(\left\{\left\{\alpha_{1k_1}^{(0)},\; \left\{\alpha_{ik_1}^*\right\}_{i=2}^N\right\} \setminus \left\{\alpha_{nk_1}^*\right\}\right\} \cup \{\alpha_{nk_1}\}\right) \leqslant K_n\left(\alpha_{1k_1}^{(0)},\; \left\{\alpha_{ik_1}^*\right\}_{i=2}^N\right)$$
$$\forall\, \alpha_{nk_1} \in \left[x_n^{(\min)};\; x_n^{(\max)}\right] \text{ and } \forall\, n = \overline{3,\; N}, \quad (54)$$

which are impossible due to $\left\{\alpha_{1k_1}^{(0)},\; \left\{\alpha_{ik_1}^*\right\}_{i=2}^N\right\}$ is not a pure-strategy equilibrium. Therefore, the supposition about (34) and (35) are true is contradictory. The same conclusion is valid for a two-person non-cooperative game, where (35), (43), (44), (47) — (49), (53) are written by retaining $\left\{\alpha_{ik_2}^*\right\}_{i=3}^N = \emptyset$, $\{\alpha_{il}^*\}_{i=3}^N = \emptyset$, $\left\{\alpha_{ik_1}^*\right\}_{i=3}^N = \emptyset$, and (45), (50), (51), (54) are omitted.

Now, for the case of $N \geqslant 3$, suppose that the other pure-strategy equilibrium differs from (27) in that the first player uses some $\alpha_{1k_1}^{(0)} \in \left[x_1^{(\min)};\; x_1^{(\max)}\right]$ instead of $\alpha_{1k_1}^*$ by some $k_1 \in \left\{\overline{1,\; M}\right\}$, the second player uses some $\alpha_{2k_2}^{(0)} \in \left[x_2^{(\min)};\; x_2^{(\max)}\right]$ instead of $\alpha_{2k_2}^*$ by some $k_2 \in \left\{\overline{1,\; M}\right\}$, and the third player uses some $\alpha_{3k_3}^{(0)} \in \left[x_3^{(\min)};\; x_3^{(\max)}\right]$ instead of $\alpha_{3k_3}^*$ by some $k_3 \in \left\{\overline{1,\; M}\right\}$. So, this is the

$$\left\{\left\{\{\alpha_{il}^*\}_{i=1}^N\right\}_{l \in \left\{\overline{1,\; M}\right\} \setminus \{k_1,\; k_2,\; k_3\}} \cup \left\{\alpha_{1k_1}^{(0)},\; \left\{\alpha_{ik_1}^*\right\}_{i=2}^N\right\} \cup \left\{\alpha_{1k_2}^*,\; \alpha_{2k_2}^{(0)},\; \left\{\alpha_{ik_2}^*\right\}_{i=3}^N\right\} \cup\right.$$

$$\cup\left\{\left\{\left\{\alpha^*_{ik_3}\right\}_{i=1}^{N}\setminus\left\{\alpha^*_{3k_3}\right\}\right\}\cup\left\{\alpha^{(0)}_{3k_3}\right\}\right\}\text{-stack equilibrium.} \tag{55}$$

Thus, (55) means that

$$\sum_{l\in\left\{\overline{1,\,M}\right\}\setminus\{k_1,\,k_2,\,k_3\}} K_1\left(\alpha_{1l},\ \{\alpha^*_{il}\}_{i=2}^{N}\right)+$$
$$+K_1\left(\alpha_{1k_1},\ \left\{\alpha^*_{ik_1}\right\}_{i=2}^{N}\right)+K_1\left(\alpha_{1k_2},\ \alpha^{(0)}_{2k_2},\ \left\{\alpha^*_{ik_2}\right\}_{i=3}^{N}\right)+$$
$$+K_1\left(\alpha_{1k_3},\ \left\{\left\{\alpha^*_{ik_3}\right\}_{i=2}^{N}\setminus\left\{\alpha^*_{3k_3}\right\}\right\}\cup\left\{\alpha^{(0)}_{3k_3}\right\}\right)\leqslant$$
$$\leqslant\sum_{l\in\left\{\overline{1,\,M}\right\}\setminus\{k_1,\,k_2,\,k_3\}} K_1\left(\{\alpha^*_{il}\}_{i=1}^{N}\right)+$$
$$+K_1\left(\alpha^{(0)}_{1k_1},\ \left\{\alpha^*_{ik_1}\right\}_{i=2}^{N}\right)+K_1\left(\left\{\left\{\alpha^*_{ik_2}\right\}_{i=1}^{N}\setminus\left\{\alpha^*_{2k_2}\right\}\right\}\cup\left\{\alpha^{(0)}_{2k_2}\right\}\right)+$$
$$+K_1\left(\left\{\left\{\alpha^*_{ik_3}\right\}_{i=1}^{N}\setminus\left\{\alpha^*_{3k_3}\right\}\right\}\cup\left\{\alpha^{(0)}_{3k_3}\right\}\right) \tag{56}$$

and

$$\sum_{l\in\left\{\overline{1,\,M}\right\}\setminus\{k_1,\,k_2,\,k_3\}} K_2\left(\left\{\{\alpha^*_{il}\}_{i=1}^{N}\setminus\{\alpha^*_{2l}\}\right\}\cup\{\alpha_{2l}\}\right)+$$
$$+K_2\left(\alpha^{(0)}_{1k_1},\ \alpha_{2k_1},\ \left\{\alpha^*_{ik_1}\right\}_{i=3}^{N}\right)+$$
$$+K_2\left(\left\{\left\{\alpha^*_{ik_2}\right\}_{i=1}^{N}\setminus\left\{\alpha^*_{2k_2}\right\}\right\}\cup\{\alpha_{2k_2}\}\right)+K_2\left(\alpha^*_{1k_3},\ \alpha_{2k_3},\ \alpha^{(0)}_{3k_3},\ \left\{\alpha^*_{ik_3}\right\}_{i=4}^{N}\right)\leqslant$$
$$\leqslant\sum_{l\in\left\{\overline{1,\,M}\right\}\setminus\{k_1,\,k_2,\,k_3\}} K_2\left(\{\alpha^*_{il}\}_{i=1}^{N}\right)+$$
$$+K_2\left(\alpha^{(0)}_{1k_1},\ \left\{\alpha^*_{ik_1}\right\}_{i=2}^{N}\right)+K_2\left(\left\{\left\{\alpha^*_{ik_2}\right\}_{i=1}^{N}\setminus\left\{\alpha^*_{2k_2}\right\}\right\}\cup\left\{\alpha^{(0)}_{2k_2}\right\}\right)+$$
$$+K_2\left(\left\{\left\{\alpha^*_{ik_3}\right\}_{i=1}^{N}\setminus\left\{\alpha^*_{3k_3}\right\}\right\}\cup\left\{\alpha^{(0)}_{3k_3}\right\}\right) \tag{57}$$

and

$$\sum_{l\in\left\{\overline{1,\,M}\right\}\setminus\{k_1,\,k_2,\,k_3\}} K_3\left(\left\{\{\alpha^*_{il}\}_{i=1}^{N}\setminus\{\alpha^*_{3l}\}\right\}\cup\{\alpha_{3l}\}\right)+$$
$$+K_3\left(\alpha^{(0)}_{1k_1},\ \left\{\left\{\alpha^*_{ik_1}\right\}_{i=2}^{N}\setminus\left\{\alpha^*_{3k_1}\right\}\right\}\cup\{\alpha_{3k_1}\}\right)+$$
$$+K_3\left(\alpha^*_{1k_2},\ \alpha^{(0)}_{2k_2},\ \alpha_{3k_2},\ \left\{\alpha^*_{ik_2}\right\}_{i=4}^{N}\right)+K_3\left(\left\{\left\{\alpha^*_{ik_3}\right\}_{i=1}^{N}\setminus\left\{\alpha^*_{3k_3}\right\}\right\}\cup\{\alpha_{3k_3}\}\right)\leqslant$$
$$\leqslant\sum_{l\in\left\{\overline{1,\,M}\right\}\setminus\{k_1,\,k_2,\,k_3\}} K_3\left(\{\alpha^*_{il}\}_{i=1}^{N}\right)+$$
$$+K_3\left(\alpha^{(0)}_{1k_1},\ \left\{\alpha^*_{ik_1}\right\}_{i=2}^{N}\right)+K_3\left(\left\{\left\{\alpha^*_{ik_2}\right\}_{i=1}^{N}\setminus\left\{\alpha^*_{2k_2}\right\}\right\}\cup\left\{\alpha^{(0)}_{2k_2}\right\}\right)+$$

$$+K_3\left(\left\{\left\{\alpha^*_{ik_3}\right\}_{i=1}^N \setminus \left\{\alpha^*_{3k_3}\right\}\right\} \cup \left\{\alpha^{(0)}_{3k_3}\right\}\right) \tag{58}$$

and

$$\begin{gathered}
\sum_{l\in\left\{\overline{1,\,M}\right\}\setminus\{k_1,\,k_2,\,k_3\}} K_n\left(\left\{\left\{\alpha^*_{il}\right\}_{i=1}^N \setminus \left\{\alpha^*_{nl}\right\}\right\} \cup \left\{\alpha_{nl}\right\}\right)+\\
+K_n\left(\alpha^{(0)}_{1k_1},\ \left\{\left\{\alpha^*_{ik_1}\right\}_{i=2}^N \setminus \left\{\alpha^*_{nk_1}\right\}\right\} \cup \left\{\alpha_{nk_1}\right\}\right)+\\
+K_n\left(\alpha^*_{1k_2},\ \alpha^{(0)}_{2k_2},\ \left\{\left\{\alpha^*_{ik_2}\right\}_{i=3}^N \setminus \left\{\alpha^*_{nk_2}\right\}\right\} \cup \left\{\alpha_{nk_2}\right\}\right)+\\
+K_n\left(\left\{\left\{\alpha^*_{ik_3}\right\}_{i=1}^N \setminus \left\{\alpha^*_{3k_3},\ \alpha^*_{nk_3}\right\}\right\} \cup \left\{\alpha^{(0)}_{3k_3},\ \alpha_{nk_3}\right\}\right) \leqslant\\
\leqslant \sum_{l\in\left\{\overline{1,\,M}\right\}\setminus\{k_1,\,k_2,\,k_3\}} K_n\left(\left\{\alpha^*_{il}\right\}_{i=1}^N\right) + K_n\left(\alpha^{(0)}_{1k_1},\ \left\{\alpha^*_{ik_1}\right\}_{i=2}^N\right)+\\
+K_n\left(\left\{\left\{\alpha^*_{ik_2}\right\}_{i=1}^N \setminus \left\{\alpha^*_{2k_2}\right\}\right\} \cup \left\{\alpha^{(0)}_{2k_2}\right\}\right)+\\
+K_n\left(\left\{\left\{\alpha^*_{ik_3}\right\}_{i=1}^N \setminus \left\{\alpha^*_{3k_3}\right\}\right\} \cup \left\{\alpha^{(0)}_{3k_3}\right\}\right)\quad \forall\, n=\overline{4,\ N},
\end{gathered} \tag{59}$$

i. e., inequalities

$$\begin{gathered}
K_1\left(\alpha_{1l},\ \left\{\alpha^*_{il}\right\}_{i=2}^N\right) \leqslant K_1\left(\left\{\alpha^*_{il}\right\}_{i=1}^N\right)\\
\forall\, \alpha_{1l} \in \left[x_1^{(\min)};\ x_1^{(\max)}\right] \ \text{ and } \ \forall\, l \in \left\{\overline{1,\ M}\right\} \setminus \{k_1,\ k_2,\ k_3\}
\end{gathered} \tag{60}$$

and inequality

$$\begin{gathered}
K_1\left(\alpha_{1k_1},\ \left\{\alpha^*_{ik_1}\right\}_{i=2}^N\right) + K_1\left(\alpha_{1k_2},\ \alpha^{(0)}_{2k_2},\ \left\{\alpha^*_{ik_2}\right\}_{i=3}^N\right)+\\
+K_1\left(\alpha_{1k_3},\ \left\{\left\{\alpha^*_{ik_3}\right\}_{i=2}^N \setminus \left\{\alpha^*_{3k_3}\right\}\right\} \cup \left\{\alpha^{(0)}_{3k_3}\right\}\right) \leqslant\\
\leqslant K_1\left(\alpha^{(0)}_{1k_1},\ \left\{\alpha^*_{ik_1}\right\}_{i=2}^N\right) + K_1\left(\left\{\left\{\alpha^*_{ik_2}\right\}_{i=1}^N \setminus \left\{\alpha^*_{2k_2}\right\}\right\} \cup \left\{\alpha^{(0)}_{2k_2}\right\}\right)+\\
+K_1\left(\left\{\left\{\alpha^*_{ik_3}\right\}_{i=1}^N \setminus \left\{\alpha^*_{3k_3}\right\}\right\} \cup \left\{\alpha^{(0)}_{3k_3}\right\}\right) \forall\, \alpha_{1k_1} \in \left[x_1^{(\min)};\ x_1^{(\max)}\right]\\
\text{and } \ \forall\, \alpha_{1k_2} \in \left[x_1^{(\min)};\ x_1^{(\max)}\right] \ \text{ and } \ \forall\, \alpha_{1k_3} \in \left[x_1^{(\min)};\ x_1^{(\max)}\right]
\end{gathered} \tag{61}$$

hold along with (23) for $n=1$, inequalities

$$\begin{gathered}
K_2\left(\left\{\left\{\alpha^*_{il}\right\}_{i=1}^N \setminus \left\{\alpha^*_{2l}\right\}\right\} \cup \left\{\alpha_{2l}\right\}\right) \leqslant K_2\left(\left\{\alpha^*_{il}\right\}_{i=1}^N\right)\\
\forall\, \alpha_{2l} \in \left[x_2^{(\min)};\ x_2^{(\max)}\right] \ \text{ and } \ \forall\, l \in \left\{\overline{1,\ M}\right\} \setminus \{k_1,\ k_2,\ k_3\}
\end{gathered} \tag{62}$$

and inequality

$$K_2\left(\alpha^{(0)}_{1k_1},\ \alpha_{2k_1},\ \left\{\alpha^*_{ik_1}\right\}_{i=3}^N\right) + K_2\left(\left\{\left\{\alpha^*_{ik_2}\right\}_{i=1}^N \setminus \left\{\alpha^*_{2k_2}\right\}\right\} \cup \left\{\alpha_{2k_2}\right\}\right)+$$

$$+K_2\left(\alpha^*_{1k_3},\ \alpha_{2k_3},\ \alpha^{(0)}_{3k_3},\ \left\{\alpha^*_{ik_3}\right\}_{i=4}^{N}\right)\leqslant$$
$$\leqslant K_2\left(\alpha^{(0)}_{1k_1},\ \left\{\alpha^*_{ik_1}\right\}_{i=2}^{N}\right)+K_2\left(\left\{\left\{\alpha^*_{ik_2}\right\}_{i=1}^{N}\setminus\left\{\alpha^*_{2k_2}\right\}\right\}\cup\left\{\alpha^{(0)}_{2k_2}\right\}\right)+$$
$$+K_2\left(\left\{\left\{\alpha^*_{ik_3}\right\}_{i=1}^{N}\setminus\left\{\alpha^*_{3k_3}\right\}\right\}\cup\left\{\alpha^{(0)}_{3k_3}\right\}\right)\forall\,\alpha_{2k_1}\in\left[x_2^{(\min)};\ x_2^{(\max)}\right]$$
$$\text{and }\ \forall\,\alpha_{2k_2}\in\left[x_2^{(\min)};\ x_2^{(\max)}\right]\ \text{ and }\ \forall\,\alpha_{2k_3}\in\left[x_2^{(\min)};\ x_2^{(\max)}\right] \tag{63}$$

hold along with (23) for $n=2$, inequalities

$$K_3\left(\left\{\left\{\alpha^*_{il}\right\}_{i=1}^{N}\setminus\left\{\alpha^*_{3l}\right\}\right\}\cup\left\{\alpha_{3l}\right\}\right)\leqslant K_3\left(\left\{\alpha^*_{il}\right\}_{i=1}^{N}\right)$$
$$\forall\,\alpha_{3l}\in\left[x_3^{(\min)};\ x_3^{(\max)}\right]\ \text{ and }\ \forall\,l\in\left\{\overline{1,\ M}\right\}\setminus\left\{k_1,\ k_2,\ k_3\right\} \tag{64}$$

and inequality

$$K_3\left(\alpha^{(0)}_{1k_1},\ \left\{\left\{\alpha^*_{ik_1}\right\}_{i=2}^{N}\setminus\left\{\alpha^*_{3k_1}\right\}\right\}\cup\left\{\alpha_{3k_1}\right\}\right)+$$
$$+K_3\left(\alpha^*_{1k_2},\ \alpha^{(0)}_{2k_2},\ \alpha_{3k_2},\ \left\{\alpha^*_{ik_2}\right\}_{i=4}^{N}\right)+$$
$$+K_3\left(\left\{\left\{\alpha^*_{ik_3}\right\}_{i=1}^{N}\setminus\left\{\alpha^*_{3k_3}\right\}\right\}\cup\left\{\alpha_{3k_3}\right\}\right)\leqslant$$
$$\leqslant K_3\left(\alpha^{(0)}_{1k_1},\ \left\{\alpha^*_{ik_1}\right\}_{i=2}^{N}\right)+K_3\left(\left\{\left\{\alpha^*_{ik_2}\right\}_{i=1}^{N}\setminus\left\{\alpha^*_{2k_2}\right\}\right\}\cup\left\{\alpha^{(0)}_{2k_2}\right\}\right)+$$
$$+K_3\left(\left\{\left\{\alpha^*_{ik_3}\right\}_{i=1}^{N}\setminus\left\{\alpha^*_{3k_3}\right\}\right\}\cup\left\{\alpha^{(0)}_{3k_3}\right\}\right)\forall\,\alpha_{3k_1}\in\left[x_3^{(\min)};\ x_3^{(\max)}\right]$$
$$\text{and }\ \forall\,\alpha_{3k_2}\in\left[x_3^{(\min)};\ x_3^{(\max)}\right]\ \text{ and }\ \forall\,\alpha_{3k_3}\in\left[x_3^{(\min)};\ x_3^{(\max)}\right] \tag{65}$$

hold along with (23) for $n=3$, inequalities

$$K_n\left(\left\{\left\{\alpha^*_{il}\right\}_{i=1}^{N}\setminus\left\{\alpha^*_{nl}\right\}\right\}\cup\left\{\alpha_{nl}\right\}\right)\leqslant K_n\left(\left\{\alpha^*_{il}\right\}_{i=1}^{N}\right)$$
$$\forall\,\alpha_{nl}\in\left[x_n^{(\min)};\ x_n^{(\max)}\right]\ \text{ and }\ \forall\,l\in\left\{\overline{1,\ M}\right\}\setminus\left\{k_1,\ k_2,\ k_3\right\}\ \text{ and }\ \forall\,n=\overline{4,\ N} \tag{66}$$

and inequality

$$K_n\left(\alpha^{(0)}_{1k_1},\ \left\{\left\{\alpha^*_{ik_1}\right\}_{i=2}^{N}\setminus\left\{\alpha^*_{nk_1}\right\}\right\}\cup\left\{\alpha_{nk_1}\right\}\right)+$$
$$+K_n\left(\alpha^*_{1k_2},\ \alpha^{(0)}_{2k_2},\ \left\{\left\{\alpha^*_{ik_2}\right\}_{i=3}^{N}\setminus\left\{\alpha^*_{nk_2}\right\}\right\}\cup\left\{\alpha_{nk_2}\right\}\right)+$$
$$+K_n\left(\left\{\left\{\alpha^*_{ik_3}\right\}_{i=1}^{N}\setminus\left\{\alpha^*_{3k_3},\ \alpha^*_{nk_3}\right\}\right\}\cup\left\{\alpha^{(0)}_{3k_3},\ \alpha_{nk_3}\right\}\right)\leqslant$$
$$\leqslant K_n\left(\alpha^{(0)}_{1k_1},\ \left\{\alpha^*_{ik_1}\right\}_{i=2}^{N}\right)+K_n\left(\left\{\left\{\alpha^*_{ik_2}\right\}_{i=1}^{N}\setminus\left\{\alpha^*_{2k_2}\right\}\right\}\cup\left\{\alpha^{(0)}_{2k_2}\right\}\right)+$$
$$+K_n\left(\left\{\left\{\alpha^*_{ik_3}\right\}_{i=1}^{N}\setminus\left\{\alpha^*_{3k_3}\right\}\right\}\cup\left\{\alpha^{(0)}_{3k_3}\right\}\right)\ \ \forall\,n=\overline{4,\ N} \tag{67}$$

hold along with (23). Plugging $\alpha_{1k_2}=\alpha^*_{1k_2}$ and $\alpha_{1k_3}=\alpha^*_{1k_3}$ in the left side of inequality (61) gives inequality

$$K_1\left(\alpha_{1k_1},\ \left\{\alpha^*_{ik_1}\right\}_{i=2}^{N}\right)\leqslant K_1\left(\alpha^{(0)}_{1k_1},\ \left\{\alpha^*_{ik_1}\right\}_{i=2}^{N}\right)\ \ \forall\,\alpha_{1k_1}\in\left[x_1^{(\min)};\ x_1^{(\max)}\right], \tag{68}$$

plugging $\alpha_{2k_2} = \alpha_{2k_2}^{(0)}$ and $\alpha_{2k_3} = \alpha_{2k_3}^{*}$ in the left side of inequality (63) gives inequality

$$K_2\left(\alpha_{1k_1}^{(0)},\ \alpha_{2k_1},\ \left\{\alpha_{ik_1}^{*}\right\}_{i=3}^{N}\right) \leqslant K_2\left(\alpha_{1k_1}^{(0)},\ \left\{\alpha_{ik_1}^{*}\right\}_{i=2}^{N}\right)$$
$$\forall\ \alpha_{2k_1} \in \left[x_2^{(\min)};\ x_2^{(\max)}\right], \tag{69}$$

plugging $\alpha_{3k_2} = \alpha_{3k_2}^{*}$ and $\alpha_{3k_3} = \alpha_{3k_3}^{(0)}$ in the left side of inequality (65) gives inequality

$$K_3\left(\alpha_{1k_1}^{(0)},\ \left\{\left\{\alpha_{ik_1}^{*}\right\}_{i=2}^{N} \setminus \left\{\alpha_{3k_1}^{*}\right\}\right\} \cup \left\{\alpha_{3k_1}\right\}\right) \leqslant K_3\left(\alpha_{1k_1}^{(0)},\ \left\{\alpha_{ik_1}^{*}\right\}_{i=2}^{N}\right)$$
$$\forall\ \alpha_{3k_1} \in \left[x_3^{(\min)};\ x_3^{(\max)}\right], \tag{70}$$

and plugging $\alpha_{nk_2} = \alpha_{nk_2}^{*}$ and $\alpha_{nk_3} = \alpha_{nk_3}^{*}$ in the left side of inequality (67) for $n = \overline{4,\ N}$ gives inequality

$$K_n\left(\alpha_{1k_1}^{(0)},\ \left\{\left\{\alpha_{ik_1}^{*}\right\}_{i=2}^{N} \setminus \left\{\alpha_{nk_1}^{*}\right\}\right\} \cup \left\{\alpha_{nk_1}\right\}\right) \leqslant K_n\left(\alpha_{1k_1}^{(0)},\ \left\{\alpha_{ik_1}^{*}\right\}_{i=2}^{N}\right)$$
$$\forall\ \alpha_{nk_1} \in \left[x_n^{(\min)};\ x_n^{(\max)}\right] \text{ and } \forall\, n = \overline{4,\ N}. \tag{71}$$

Inequalities (68) — (71) imply that $\left\{\alpha_{1k_1}^{(0)},\ \left\{\alpha_{ik_1}^{*}\right\}_{i=2}^{N}\right\}$ is a pure-strategy equilibrium, which is impossible. Therefore, (55) is false. The same conclusion is valid for a three-person non-cooperative game, where (57), (58), (63), (65) are written by retaining $\left\{\alpha_{ik_3}^{*}\right\}_{i=4}^{N} = \emptyset$, $\left\{\alpha_{ik_2}^{*}\right\}_{i=4}^{N} = \emptyset$, and (59), (66), (67), (71) are omitted. The impossibility of the other pure-strategy equilibrium for the remaining players' subsets in the case of three different strategies at three players is proved symmetrically.

Finally, suppose that the other pure-strategy equilibrium differs from (27) in that the first player uses some $\alpha_{1k_1}^{(0)} \in \left[x_1^{(\min)};\ x_1^{(\max)}\right]$ instead of $\alpha_{1k_1}^{*}$ by some $k_1 \in \left\{\overline{1,\ M}\right\}$ and some $\alpha_{1k_2}^{(0)} \in \left[x_1^{(\min)};\ x_1^{(\max)}\right]$ instead of $\alpha_{1k_2}^{*}$ by some $k_2 \in \left\{\overline{1,\ M}\right\}$. The respective

$$\left\{\left\{\left\{\alpha_{il}^{*}\right\}_{i=1}^{N}\right\}_{l \in \left\{\overline{1,\ M}\right\} \setminus \{k_1,\ k_2\}} \cup\right.$$
$$\left. \cup \left\{\alpha_{1k_1}^{(0)},\ \left\{\alpha_{ik_1}^{*}\right\}_{i=2}^{N}\right\} \cup \left\{\alpha_{1k_2}^{(0)},\ \left\{\alpha_{ik_2}^{*}\right\}_{i=2}^{N}\right\}\right\}\text{-stack equilibrium} \tag{72}$$

means that

$$\sum_{l \in \left\{\overline{1,\ M}\right\} \setminus \{k_1,\ k_2\}} K_1\left(\alpha_{1l},\ \left\{\alpha_{il}^{*}\right\}_{i=2}^{N}\right) +$$
$$+K_1\left(\alpha_{1k_1},\ \left\{\alpha_{ik_1}^{*}\right\}_{i=2}^{N}\right) + K_1\left(\alpha_{1k_2},\ \left\{\alpha_{ik_2}^{*}\right\}_{i=2}^{N}\right) \leqslant$$
$$\leqslant \sum_{l \in \left\{\overline{1,\ M}\right\} \setminus \{k_1,\ k_2\}} K_1\left(\left\{\alpha_{il}^{*}\right\}_{i=1}^{N}\right) +$$

$$+K_1\left(\alpha_{1k_1}^{(0)},\ \left\{\alpha_{ik_1}^*\right\}_{i=2}^N\right)+K_1\left(\alpha_{1k_2}^{(0)},\ \left\{\alpha_{ik_2}^*\right\}_{i=2}^N\right) \tag{73}$$

and

$$\sum_{l\in\left\{\overline{1,\ M}\right\}\setminus\{k_1,\ k_2\}} K_n\left(\left\{\{\alpha_{il}^*\}_{i=1}^N\setminus\{\alpha_{nl}^*\}\right\}\cup\{\alpha_{nl}\}\right)+$$
$$+K_n\left(\alpha_{1k_1}^{(0)},\ \left\{\left\{\alpha_{ik_1}^*\right\}_{i=2}^N\setminus\left\{\alpha_{nk_1}^*\right\}\right\}\cup\{\alpha_{nk_1}\}\right)+$$
$$+K_n\left(\alpha_{1k_2}^{(0)},\ \left\{\left\{\alpha_{ik_2}^*\right\}_{i=2}^N\setminus\left\{\alpha_{nk_2}^*\right\}\right\}\cup\{\alpha_{nk_2}\}\right)\leqslant$$
$$\leqslant\sum_{l\in\left\{\overline{1,\ M}\right\}\setminus\{k_1,\ k_2\}} K_n\left(\{\alpha_{il}^*\}_{i=1}^N\right)+$$
$$+K_n\left(\alpha_{1k_1}^{(0)},\ \left\{\alpha_{ik_1}^*\right\}_{i=2}^N\right)+K_n\left(\alpha_{1k_2}^{(0)},\ \left\{\alpha_{ik_2}^*\right\}_{i=2}^N\right)\quad\forall\,n=\overline{2,\ N}, \tag{74}$$

i. e., inequalities (46) and inequality

$$K_1\left(\alpha_{1k_1},\ \left\{\alpha_{ik_1}^*\right\}_{i=2}^N\right)+K_1\left(\alpha_{1k_2},\ \left\{\alpha_{ik_2}^*\right\}_{i=2}^N\right)\leqslant$$
$$\leqslant K_1\left(\alpha_{1k_1}^{(0)},\ \left\{\alpha_{ik_1}^*\right\}_{i=2}^N\right)+K_1\left(\alpha_{1k_2}^{(0)},\ \left\{\alpha_{ik_2}^*\right\}_{i=2}^N\right)$$
$$\forall\,\alpha_{1k_1}\in\left[x_1^{(\min)};\ x_1^{(\max)}\right]\ \text{ and }\ \forall\,\alpha_{1k_2}\in\left[x_1^{(\min)};\ x_1^{(\max)}\right] \tag{75}$$

hold along with (23) for $n=1$, inequalities

$$K_n\left(\left\{\{\alpha_{il}^*\}_{i=1}^N\setminus\{\alpha_{nl}^*\}\right\}\cup\{\alpha_{nl}\}\right)\leqslant K_n\left(\{\alpha_{il}^*\}_{i=1}^N\right)$$
$$\forall\,\alpha_{nl}\in\left[x_n^{(\min)};\ x_n^{(\max)}\right]\ \text{ and }\ \forall\,l\in\left\{\overline{1,\ M}\right\}\setminus\{k_1,\ k_2\}\ \text{ and }\ \forall\,n=\overline{2,\ N} \tag{76}$$

and inequality

$$K_n\left(\alpha_{1k_1}^{(0)},\ \left\{\left\{\alpha_{ik_1}^*\right\}_{i=2}^N\setminus\left\{\alpha_{nk_1}^*\right\}\right\}\cup\{\alpha_{nk_1}\}\right)+$$
$$+K_n\left(\alpha_{1k_2}^{(0)},\ \left\{\left\{\alpha_{ik_2}^*\right\}_{i=2}^N\setminus\left\{\alpha_{nk_2}^*\right\}\right\}\cup\{\alpha_{nk_2}\}\right)\leqslant$$
$$\leqslant K_n\left(\alpha_{1k_1}^{(0)},\ \left\{\alpha_{ik_1}^*\right\}_{i=2}^N\right)+K_n\left(\alpha_{1k_2}^{(0)},\ \left\{\alpha_{ik_2}^*\right\}_{i=2}^N\right)\quad\forall\,n=\overline{2,\ N} \tag{77}$$

hold along with (23). Plugging $\alpha_{1k_2}=\alpha_{1k_2}^{(0)}$ in the left side of inequality (75) and plugging $\alpha_{nk_2}=\alpha_{nk_2}^*$ in the left side of inequality (77) for $n=\overline{2,\ N}$ gives inequalities (68) — (71), which are impossible due to $\left\{\alpha_{1k_1}^{(0)},\ \left\{\alpha_{ik_1}^*\right\}_{i=2}^N\right\}$ is not a pure-strategy equilibrium. So, the supposition about (72) is contradictory. The same conclusion is valid for a two-person non-cooperative game, where (70), (71) are omitted, and it is valid for a three-person non-cooperative game, where (71) is omitted. The impossibility of the other pure-strategy equilibrium for the remaining players in such a case (of two intervals) is proved symmetrically. The impossibility of other pure-strategy equilibria differing from (27) in that the players use some other values at intervals is proved symmetrically as well. □

Therefore, Theorem 3 along with Theorem 2 allows obtaining the single pure-strategy solution of game (6) directly from equilibria in games (13). The application of these assertions significantly simplifies the solving of game (6). Under conditions of the assertions, game (6) is "discretized" or "broken" into simpler N-person games, whereupon their equilibria are stacked [18, 19].

But what if the conditions are inverted? Does the equilibrium singularity in games (13) change when the single pure-strategy equilibrium of game (6) is already known? This question is answered by the following assertion.

Theorem 4. *If game (6) on product (7) by conditions (1) — (5) and (8) — (12) has a single equilibrium situation in pure strategies, then each of M games (13) by (8) — (12) and (15) — (18) has a single pure-strategy equilibrium, which is the respective interval part of the game (6) equilibrium.*

Proof. Let game (6) have (27) which is single. This implies that inequalities (26) hold. Plugging

$$\alpha_{nl} = \alpha^*_{nl} \ \ \forall\, l \in \left\{\overline{1,\ M}\right\} \setminus \{k_*\}$$

in the left side of inequalities (26) gives inequalities

$$K_n \left(\left\{ \left\{\alpha^*_{ik_*}\right\}_{i=1}^{N} \setminus \left\{\alpha^*_{nk_*}\right\} \right\} \cup \left\{\alpha_{nk_*}\right\} \right) \leqslant K_n \left(\left\{\alpha^*_{ik_*}\right\}_{i=1}^{N} \right)$$
$$\forall\ \alpha_{nk_*} \in \left[x_n^{(\min)};\ x_n^{(\max)} \right] \ \text{ and } \ \forall\, n = \overline{1,\ N} \tag{78}$$

whence $\left\{\alpha^*_{ik_*}\right\}_{i=1}^{N}$ is a pure-strategy equilibrium at the k_*-th interval (in the k_*-th game) for every $k_* \in \left\{\overline{1,\ M}\right\}$.

Suppose that $\exists\ k_0 \in \left\{\overline{1,\ M}\right\}$ such that $\left\{\alpha^{(0)}_{1k_0},\ \left\{\alpha^*_{ik_0}\right\}_{i=2}^{N}\right\}$ is an equilibrium by $\alpha^{(0)}_{1k_0} \neq \alpha^*_{1k_0}$. Then inequalities

$$K_1 \left(\alpha_{1k_0},\ \left\{\alpha^*_{ik_0}\right\}_{i=2}^{N} \right) \leqslant K_1 \left(\alpha^{(0)}_{1k_0},\ \left\{\alpha^*_{ik_0}\right\}_{i=2}^{N} \right) \ \ \forall\ \alpha_{1k_0} \in \left[x_1^{(\min)};\ x_1^{(\max)} \right] \tag{79}$$

and

$$K_n \left(\left\{ \left\{ \alpha^{(0)}_{1k_0},\ \left\{\alpha^*_{ik_0}\right\}_{i=2}^{N} \right\} \setminus \left\{\alpha^*_{nk_0}\right\} \right\} \cup \left\{\alpha_{nk_0}\right\} \right) \leqslant K_n \left(\alpha^{(0)}_{1k_0},\ \left\{\alpha^*_{ik_0}\right\}_{i=2}^{N} \right)$$
$$\forall\ \alpha_{nk_0} \in \left[x_n^{(\min)};\ x_n^{(\max)} \right] \ \text{ and } \ \forall\, n = \overline{2,\ N} \tag{80}$$

hold, whence inequalities

$$\sum_{k_* \in \left\{\overline{1,\ M}\right\} \setminus \{k_0\}} K_1 \left(\left\{ \left\{\alpha^*_{ik_*}\right\}_{i=1}^{N} \setminus \left\{\alpha^*_{1k_*}\right\} \right\} \cup \left\{\alpha_{1k_*}\right\} \right) + K_1 \left(\alpha_{1k_0},\ \left\{\alpha^*_{ik_0}\right\}_{i=2}^{N} \right) \leqslant$$
$$\leqslant \sum_{k_* \in \left\{\overline{1,\ M}\right\} \setminus \{k_0\}} K_1 \left(\left\{\alpha^*_{ik_*}\right\}_{i=1}^{N} \right) + K_1 \left(\alpha^{(0)}_{1k_0},\ \left\{\alpha^*_{ik_0}\right\}_{i=2}^{N} \right) \tag{81}$$

and

$$\sum_{k_* \in \left\{\overline{1,\ M}\right\} \backslash \{k_0\}} K_n \left(\left\{ \left\{ \alpha_{ik_*}^{*} \right\}_{i=1}^{N} \backslash \left\{ \alpha_{nk_*}^{*} \right\} \right\} \cup \left\{ \alpha_{nk_*} \right\} \right) +$$
$$+ K_n \left(\left\{ \left\{ \alpha_{1k_0}^{(0)},\ \left\{ \alpha_{ik_0}^{*} \right\}_{i=2}^{N} \right\} \backslash \left\{ \alpha_{nk_0}^{*} \right\} \right\} \cup \left\{ \alpha_{nk_0} \right\} \right) \leqslant$$
$$\leqslant \sum_{k_* \in \left\{\overline{1,\ M}\right\} \backslash \{k_0\}} K_n \left(\left\{ \alpha_{ik_*}^{*} \right\}_{i=1}^{N} \right) + K_n \left(\alpha_{1k_0}^{(0)},\ \left\{ \alpha_{ik_0}^{*} \right\}_{i=2}^{N} \right) \quad \forall\, n = \overline{2,\ N} \tag{82}$$

must hold as well. However, inequalities (81) and (82) imply that there is the

$$\left\{ \left\{ \left\{ \alpha_{il}^{*} \right\}_{i=1}^{N} \right\}_{l \in \left\{\overline{1,\ M}\right\} \backslash \{k_0\}} \cup \left\{ \alpha_{1k_0}^{(0)},\ \left\{ \alpha_{ik_0}^{*} \right\}_{i=2}^{N} \right\} \right\}\text{-stack equilibrium,}$$

which is impossible. Supposing that $\left\{ \alpha_{1k_0}^{(0)},\ \alpha_{2k_0}^{(0)},\ \left\{ \alpha_{ik_0}^{*} \right\}_{i=3}^{N} \right\}$ is an equilibrium by $\alpha_{1k_0}^{(0)} \neq \alpha_{1k_0}^{*}$ and $\alpha_{2k_0}^{(0)} \neq \alpha_{2k_0}^{*}$ leads to inequalities

$$K_1 \left(\alpha_{1k_0},\ \alpha_{2k_0}^{(0)},\ \left\{ \alpha_{ik_0}^{*} \right\}_{i=3}^{N} \right) \leqslant K_1 \left(\alpha_{1k_0}^{(0)},\ \alpha_{2k_0}^{(0)},\ \left\{ \alpha_{ik_0}^{*} \right\}_{i=3}^{N} \right)$$
$$\forall\, \alpha_{1k_0} \in \left[x_1^{(\min)};\ x_1^{(\max)} \right] \tag{83}$$

and

$$K_2 \left(\alpha_{1k_0}^{(0)},\ \alpha_{2k_0},\ \left\{ \alpha_{ik_0}^{*} \right\}_{i=3}^{N} \right) \leqslant K_2 \left(\alpha_{1k_0}^{(0)},\ \alpha_{2k_0}^{(0)},\ \left\{ \alpha_{ik_0}^{*} \right\}_{i=3}^{N} \right)$$
$$\forall\, \alpha_{2k_0} \in \left[x_2^{(\min)};\ x_2^{(\max)} \right] \tag{84}$$

and

$$K_n \left(\left\{ \left\{ \alpha_{1k_0}^{(0)},\ \alpha_{2k_0}^{(0)},\ \left\{ \alpha_{ik_0}^{*} \right\}_{i=3}^{N} \right\} \backslash \left\{ \alpha_{nk_0}^{*} \right\} \right\} \cup \left\{ \alpha_{nk_0} \right\} \right) \leqslant$$
$$\leqslant K_n \left(\alpha_{1k_0}^{(0)},\ \alpha_{2k_0}^{(0)},\ \left\{ \alpha_{ik_0}^{*} \right\}_{i=3}^{N} \right)$$
$$\forall\, \alpha_{nk_0} \in \left[x_n^{(\min)};\ x_n^{(\max)} \right] \ \text{ and } \ \forall\, n = \overline{3,\ N}. \tag{85}$$

Inequalities (83) — (85) imply that inequalities

$$\sum_{k_* \in \left\{\overline{1,\ M}\right\} \backslash \{k_0\}} K_1 \left(\left\{ \left\{ \alpha_{ik_*}^{*} \right\}_{i=1}^{N} \backslash \left\{ \alpha_{1k_*}^{*} \right\} \right\} \cup \left\{ \alpha_{1k_*} \right\} \right) +$$
$$+ K_1 \left(\alpha_{1k_0},\ \alpha_{2k_0}^{(0)},\ \left\{ \alpha_{ik_0}^{*} \right\}_{i=3}^{N} \right) \leqslant$$
$$\leqslant \sum_{k_* \in \left\{\overline{1,\ M}\right\} \backslash \{k_0\}} K_1 \left(\left\{ \alpha_{ik_*}^{*} \right\}_{i=1}^{N} \right) + K_1 \left(\alpha_{1k_0}^{(0)},\ \alpha_{2k_0}^{(0)},\ \left\{ \alpha_{ik_0}^{*} \right\}_{i=3}^{N} \right) \tag{86}$$

and

$$\sum_{k_* \in \left\{\overline{1,\ M}\right\} \setminus \{k_0\}} K_2 \left(\left\{ \left\{ \alpha^*_{ik_*} \right\}_{i=1}^{N} \setminus \left\{ \alpha^*_{2k_*} \right\} \right\} \cup \left\{ \alpha_{2k_*} \right\} \right) +$$
$$+ K_2 \left(\alpha^{(0)}_{1k_0},\ \alpha_{2k_0},\ \left\{ \alpha^*_{ik_0} \right\}_{i=3}^{N} \right) \leqslant$$
$$\leqslant \sum_{k_* \in \left\{\overline{1,\ M}\right\} \setminus \{k_0\}} K_2 \left(\left\{ \alpha^*_{ik_*} \right\}_{i=1}^{N} \right) + K_2 \left(\alpha^{(0)}_{1k_0},\ \alpha^{(0)}_{2k_0},\ \left\{ \alpha^*_{ik_0} \right\}_{i=3}^{N} \right) \tag{87}$$

and

$$\sum_{k_* \in \left\{\overline{1,\ M}\right\} \setminus \{k_0\}} K_n \left(\left\{ \left\{ \alpha^*_{ik_*} \right\}_{i=1}^{N} \setminus \left\{ \alpha^*_{nk_*} \right\} \right\} \cup \left\{ \alpha_{nk_*} \right\} \right) +$$
$$+ K_n \left(\left\{ \left\{ \alpha^{(0)}_{1k_0},\ \alpha^{(0)}_{2k_0},\ \left\{ \alpha^*_{ik_0} \right\}_{i=3}^{N} \right\} \setminus \left\{ \alpha^*_{nk_0} \right\} \right\} \cup \left\{ \alpha_{nk_0} \right\} \right) \leqslant$$
$$\leqslant \sum_{k_* \in \left\{\overline{1,\ M}\right\} \setminus \{k_0\}} K_n \left(\left\{ \alpha^*_{ik_*} \right\}_{i=1}^{N} \right) + K_n \left(\alpha^{(0)}_{1k_0},\ \alpha^{(0)}_{2k_0},\ \left\{ \alpha^*_{ik_0} \right\}_{i=3}^{N} \right)$$
$$\forall\, n = \overline{3,\ N} \tag{88}$$

must hold as well. Then inequalities (83) — (85) imply that there is the

$$\left\{ \left\{ \left\{ \alpha^*_{il} \right\}_{i=1}^{N} \right\}_{l \in \left\{\overline{1,\ M}\right\} \setminus \{k_0\}} \cup \left\{ \alpha^{(0)}_{1k_0},\ \alpha^{(0)}_{2k_0},\ \left\{ \alpha^*_{ik_0} \right\}_{i=3}^{N} \right\} \right\}\text{-stack equilibrium,}$$

which is impossible again. The same conclusion is valid for a two-person non-cooperative game, where (83), (84), (86), (87) are written by retaining $\left\{ \alpha^*_{ik_0} \right\}_{i=3}^{N} = \emptyset$, $\left\{ \alpha^*_{ik_0} \right\}_{i=3}^{N} = \emptyset$, and (85), (88) are omitted. The impossibility of other pure-strategy equilibrium cases in “short” games (13) is proved symmetrically. □

In finite games of three players and more, which are a partial case of non-cooperative games, the case when every “short” game has just a single pure-strategy equilibrium seems to be rarer than the case with multiple equilibria. Obviously, the equilibrium singleness likelihood expectedly decays as the number of players increases. This, however, does not diminish the importance of Theorem 2 along with Theorem 3 and Theorem 4. These assertions allow to build a simpler proof of a more generalized assertion.

Theorem 5. *If each of M games (13) by (8) — (12) and (15) — (18) has a nonempty set of equilibrium situations in pure strategies, and game (6) on product (7) by conditions (1) — (5) is equivalent to the succession of these games, then every pure-strategy equilibrium in game (6) is a stack of any respective M equilibria in games (13). Apart from the stack, there are no other pure-strategy equilibria in game (6).*

Proof. Let the l-th game have J_l equilibria $\left\{ \left\{ \alpha^*_{ilj_l} \right\}_{i=1}^{N} \right\}_{j_l=1}^{J_l}$ by $J_l \in \mathbb{N}$, where

$$\alpha^*_{nlj_l} \in \left[x_n^{(\min)};\ x_n^{(\max)} \right] \quad \forall\, n = \overline{1,\ N}.$$

Then

$$K_n\left(\left\{\left\{\alpha^*_{ilj_l}\right\}_{i=1}^{N}\setminus\left\{\alpha^*_{nlj_l}\right\}\right\}\cup\{\alpha_{nl}\}\right)\leqslant K_n\left(\left\{\alpha^*_{ilj_l}\right\}_{i=1}^{N}\right)$$
$$\forall\,\alpha_{nl}\in\left[x_n^{(\min)};\ x_n^{(\max)}\right]\ \text{ and }\ \forall\,n=\overline{1,\ N}\tag{89}$$

whence

$$\sum_{l=1}^{M}K_n\left(\left\{\left\{\alpha^*_{ilj_l}\right\}_{i=1}^{N}\setminus\left\{\alpha^*_{nlj_l}\right\}\right\}\cup\{\alpha_{nl}\}\right)\leqslant\sum_{l=1}^{M}K_n\left(\left\{\alpha^*_{ilj_l}\right\}_{i=1}^{N}\right)\ \forall\,n=\overline{1,\ N}.\tag{90}$$

Inequalities (90) directly imply the

$$\left\{\left\{\alpha^*_{ilj_l}\right\}_{i=1}^{N}\right\}_{l=1}^{M}\text{-stack equilibrium}\tag{91}$$

for every $j_l\in\left\{\overline{1,\ J_l}\right\}$ by $l=\overline{1,\ M}$. Apart from stacks (91), there are no other pure-strategy equilibria in game (6) owing to Theorem 4 along with Theorem 3. $\square$

It is quite obvious that Theorems 2 — 5 are valid for any non-cooperative games whose players are constrained (forced) to use staircase-function strategies, i. e., they are valid for finite non-cooperative games (with staircase-function strategies) as well. It remains only to study a possibility of equilibria in mixed strategies in such finite games.

5. Representation by a succession of finite games

Along with discrete time intervals, players may be forced to act within a finite subset of possible values of their pure strategies. That is, these values are

$$x_n^{(\min)}=x_n^{(0)}<x_n^{(1)}<x_n^{(2)}<\ldots<x_n^{(Q_n-1)}<x_n^{(Q_n)}=x_n^{(\max)}\tag{92}$$

for the n-th player, $Q_n\in\mathbb{N}\ \forall\,n=\overline{1,\ N}$. Then the succession of M continuous games (13) by (8) — (12) and (15) — (18) becomes a succession of M finite games

$$\left\langle\left\{\left\{x_i^{(m_i-1)}\right\}_{m_i=1}^{Q_i+1}\right\}_{i=1}^{N},\ \{\mathbf{H}_{il}\}_{i=1}^{N}\right\rangle\tag{93}$$

with the n-th player's payoff matrix

$$\mathbf{H}_{nl}=\left[h_{nl\boldsymbol{\Omega}}\right]_{\mathscr{F}}\tag{94}$$

whose format is

$$\mathscr{F}=\bigtimes_{n=1}^{N}(Q_n+1)\tag{95}$$

and elements are

$$h_{nl\mathbf{\Omega}} = \int\limits_{\left[\tau^{(l-1)};\ \tau^{(l)}\right)} f_n\left(\left\{x_i^{(m_i-1)}\right\}_{i=1}^{N},\ t\right) d\mu(t) \ \text{for}\ l = \overline{1,\ M-1} \tag{96}$$

and

$$h_{nM\mathbf{\Omega}} = \int\limits_{\left[\tau^{(M-1)};\ \tau^{(M)}\right]} f_n\left(\left\{x_i^{(m_i-1)}\right\}_{i=1}^{N},\ t\right) d\mu(t) \tag{97}$$

by indexing

$$\mathbf{\Omega} = \{\omega_k\}_{k=1}^{N},\ \ \omega_k \in \left\{\overline{1,\ Q_k+1}\right\}\ \ \forall\, k = \overline{1,\ N}. \tag{98}$$

It is well-known that a finite non-cooperative game always has an equilibrium either in pure or mixed strategies. So, if game (6) is made equivalent to a series of finite games (or, in other words, is represented by a succession of finite games), then it is easy to see that, unlike the representation with continuous games (13) by (8) — (12) and (15) — (18), the game always has a solution (at least, in mixed strategies).

Theorem 6. *If game (6) on product (7) by conditions (1) — (5) is equivalent to the succession of M finite games (93) by (94) — (98), then the game is always solved as a stack of respective equilibria in these finite games. Apart from the stack, there are no other equilibria in game (6).*

Proof. An equilibrium situation in the finite game always exists, either in pure or mixed strategies. Denote by

$$\mathbf{U}_{nl} = \left[u_{nl}^{(m_n)}\right]_{1\times(Q_n+1)}$$

a mixed strategy of the n-th player in finite game (93). The respective set of mixed strategies of this player is

$$\mathcal{U}_n = \left\{\mathbf{U}_{nl} \in \mathbb{R}^{Q_n+1} : u_{nl}^{(m_n)} \geqslant 0,\ \sum_{m_n=1}^{Q_n+1} u_{nl}^{(m_n)} = 1\right\}, \tag{99}$$

so $\mathbf{U}_{nl} \in \mathcal{U}_n$, and $\{\mathbf{U}_{il}\}_{i=1}^{N}$ is a situation in game (93), where J_l equilibria exist, $J_l \in \mathbb{N}$. Let $\left\{\left\{\mathbf{U}_{ilj_l}^{*}\right\}_{i=1}^{N}\right\}_{l=1}^{M}$ be equilibria in M games (93) by (94) — (98), where

$$\mathbf{U}_{nlj_l}^{*} = \left[u_{nlj_l}^{(m_n)*}\right]_{1\times(Q_n+1)} \in \mathcal{U}_n. \tag{100}$$

Henceforward, the proof is similar to that in Theorem 5. For equilibria $\left\{\left\{\mathbf{U}_{ilj_l}^{*}\right\}_{i=1}^{N}\right\}_{l=1}^{M}$ by (100), inequalities

$$\sum_{\substack{m_k=\overline{1,\ Q_k+1} \\ k=\overline{1,\ N}}} \left(h_{nl\mathbf{\Omega}} u_{nl}^{(m_n)} \prod_{\substack{k=\overline{1,\ N} \\ k\neq n}} u_{klj_l}^{(m_k)*}\right) =$$

$$= \sum_{\substack{m_k=\overline{1,\, Q_k+1} \\ k=\overline{1,\, N}}} \left(\left(u_{nl}^{(m_n)} \prod_{\substack{k=\overline{1,\, N} \\ k \neq n}} u_{klj_l}^{(m_k)*} \right) \int\limits_{\left[\tau^{(l-1)};\, \tau^{(l)}\right)} f_n \left(\left\{ x_i^{(m_i-1)} \right\}_{i=1}^{N},\, t \right) d\mu\left(t\right) \right) \leqslant$$

$$\leqslant \sum_{\substack{m_k=\overline{1,\, Q_k+1} \\ k=\overline{1,\, N}}} \left(\left(\prod_{k=\overline{1,\, N}} u_{klj_l}^{(m_k)*} \right) \int\limits_{\left[\tau^{(l-1)};\, \tau^{(l)}\right)} f_n \left(\left\{ x_i^{(m_i-1)} \right\}_{i=1}^{N},\, t \right) d\mu\left(t\right) \right) =$$

$$= \sum_{\substack{m_k=\overline{1,\, Q_k+1} \\ k=\overline{1,\, N}}} \left(h_{nl\boldsymbol{\Omega}} \prod_{k=\overline{1,\, N}} u_{klj_l}^{(m_k)*} \right)$$

$$\forall\, \mathbf{U}_{nl} = \left[u_{nl}^{(m_n)} \right]_{1 \times (Q_n+1)} \in \mathcal{U}_n \ \text{ for } \ l = \overline{1,\, M-1} \ \ \forall\, n = \overline{1,\, N}, \tag{101}$$

$$\sum_{\substack{m_k=\overline{1,\, Q_k+1} \\ k=\overline{1,\, N}}} \left(h_{nM\boldsymbol{\Omega}} u_{nM}^{(m_n)} \prod_{\substack{k=\overline{1,\, N} \\ k \neq n}} u_{kMj_M}^{(m_k)*} \right) =$$

$$= \sum_{\substack{m_k=\overline{1,\, Q_k+1} \\ k=\overline{1,\, N}}} \left(\left(u_{nM}^{(m_n)} \prod_{\substack{k=\overline{1,\, N} \\ k \neq n}} u_{kMj_M}^{(m_k)*} \right) \int\limits_{\left[\tau^{(M-1)};\, \tau^{(M)}\right]} f_n \left(\left\{ x_i^{(m_i-1)} \right\}_{i=1}^{N},\, t \right) d\mu\left(t\right) \right) \leqslant$$

$$\leqslant \sum_{\substack{m_k=\overline{1,\, Q_k+1} \\ k=\overline{1,\, N}}} \left(\left(\prod_{k=\overline{1,\, N}} u_{kMj_M}^{(m_k)*} \right) \int\limits_{\left[\tau^{(M-1)};\, \tau^{(M)}\right]} f_n \left(\left\{ x_i^{(m_i-1)} \right\}_{i=1}^{N},\, t \right) d\mu\left(t\right) \right) =$$

$$= \sum_{\substack{m_k=\overline{1,\, Q_k+1} \\ k=\overline{1,\, N}}} \left(h_{nM\boldsymbol{\Omega}} \prod_{k=\overline{1,\, N}} u_{kMj_M}^{(m_k)*} \right)$$

$$\forall\, \mathbf{U}_{nM} = \left[u_{nM}^{(m_n)} \right]_{1 \times (Q_n+1)} \in \mathcal{U}_n \ \text{ and } \ \forall\, n = \overline{1,\, N} \tag{102}$$

hold. So, inequalities

$$\sum_{l=1}^{M-1} \sum_{\substack{m_k=\overline{1,\, Q_k+1} \\ k=\overline{1,\, N}}} \left(h_{nl\boldsymbol{\Omega}} u_{nl}^{(m_n)} \prod_{\substack{k=\overline{1,\, N} \\ k \neq n}} u_{klj_l}^{(m_k)*} \right) +$$

$$+\sum_{\substack{m_k=\overline{1,\ Q_k+1}\\ k=\overline{1,\ N}}}\left(h_{nM\boldsymbol{\Omega}}u_{nM}^{(m_n)}\prod_{\substack{k=\overline{1,\ N}\\ k\neq n}}u_{kMj_M}^{(m_k)*}\right)=$$

$$=\sum_{l=1}^{M-1}\left(\sum_{\substack{m_k=\overline{1,\ Q_k+1}\\ k=\overline{1,\ N}}}\left(\left(u_{nl}^{(m_n)}\prod_{\substack{k=\overline{1,\ N}\\ k\neq n}}u_{klj_l}^{(m_k)*}\right)\int\limits_{\left[\tau^{(l-1)};\ \tau^{(l)}\right)}f_n\left(\left\{x_i^{(m_i-1)}\right\}_{i=1}^{N},\ t\right)d\mu\left(t\right)\right)\right)+$$

$$+\sum_{\substack{m_k=\overline{1,\ Q_k+1}\\ k=\overline{1,\ N}}}\left(\left(u_{nM}^{(m_n)}\prod_{\substack{k=\overline{1,\ N}\\ k\neq n}}u_{kMj_M}^{(m_k)*}\right)\int\limits_{\left[\tau^{(M-1)};\ \tau^{(M)}\right]}f_n\left(\left\{x_i^{(m_i-1)}\right\}_{i=1}^{N},\ t\right)d\mu\left(t\right)\right)\leqslant$$

$$\leqslant\sum_{l=1}^{M-1}\left(\sum_{\substack{m_k=\overline{1,\ Q_k+1}\\ k=\overline{1,\ N}}}\left(\left(\prod_{k=\overline{1,\ N}}u_{klj_l}^{(m_k)*}\right)\int\limits_{\left[\tau^{(l-1)};\ \tau^{(l)}\right)}f_n\left(\left\{x_i^{(m_i-1)}\right\}_{i=1}^{N},\ t\right)d\mu\left(t\right)\right)\right)+$$

$$+\sum_{\substack{m_k=\overline{1,\ Q_k+1}\\ k=\overline{1,\ N}}}\left(\left(\prod_{k=\overline{1,\ N}}u_{kMj_M}^{(m_k)*}\right)\int\limits_{\left[\tau^{(M-1)};\ \tau^{(M)}\right]}f_n\left(\left\{x_i^{(m_i-1)}\right\}_{i=1}^{N},\ t\right)d\mu\left(t\right)\right)=$$

$$=\sum_{l=1}^{M-1}\sum_{\substack{m_k=\overline{1,\ Q_k+1}\\ k=\overline{1,\ N}}}\left(h_{nl\boldsymbol{\Omega}}\prod_{k=\overline{1,\ N}}u_{klj_l}^{(m_k)*}\right)+$$

$$+\sum_{\substack{m_k=\overline{1,\ Q_k+1}\\ k=\overline{1,\ N}}}\left(h_{nM\boldsymbol{\Omega}}\prod_{k=\overline{1,\ N}}u_{kMj_M}^{(m_k)*}\right)\ \forall\,n=\overline{1,\ N} \tag{103}$$

hold as well. Therefore, the stack of successive equilibria $\left\{\left\{\mathbf{U}_{ilj_l}^{*}\right\}_{i=1}^{N}\right\}_{l=1}^{M}$ is an equilibrium in game (6). The sub-assertion of that, apart from such stacks, there are no other equilibria in game (6) is proved similarly to Theorem 4 along with Theorem 3. □

Clearly, inequalities (89) by $l=\overline{1,\ M}$ are a partial case of inequalities (101), (102). Inequalities (90) are a partial case of inequalities (103). In a way, Theorem 6 is a generalization of Theorem 5 for the case of finite game (6), which is correspondingly defined on a product of staircase-function finite spaces. Nevertheless, stacking up pure-strategy equilibria and mixed-strategy equilibria of $\underset{n=1}{\overset{N}{\times}}(Q_n+1)$ finite games (93) can be cumbersome. The best case is when every "short" game has a single pure-strategy equilibrium, although the likelihood of the best case is low.

The likeliest case is when those M finite games have multiple pure-strategy equilibria and mixed-strategy equilibria. To hit on a series of single-pure-strategy-equilibrium finite games, plainly speaking, many tries should be done. For instance, $5\times5\times5\times5$ games, in which payoffs are generated by a $5\times5\times5\times5$ standard-normally-distributed array multiplied by 10 and rounded to the nearest integers towards $-\infty$, have roughly 27.5 % mixed-strategy equilibria only. The percentage rate of the case when the game has one pure-strategy equilibrium is at least 36 %. Meanwhile, these rates for $5 \times 5 \times 5$ games are 28 % and 37 %, respectively.

6. An example of solving a finite game

To exemplify how the suggested method solves finite games defined on a product of staircase-function spaces (which are obviously finite), consider a case in which $t \in [0;\ 0.16\pi]$, the set of pure strategies of the first player is

$$X_1 = \{x_1(t),\ t \in [0;\ 0.16\pi] :\ 2 \leqslant x_1(t) \leqslant 3\} \subset \mathbb{L}_2[0;\ 0.16\pi], \tag{104}$$

the set of pure strategies of the second player is

$$X_2 = \{x_2(t),\ t \in [0;\ 0.16\pi] :\ 4 \leqslant x_2(t) \leqslant 4.75\} \subset \mathbb{L}_2[0;\ 0.16\pi], \tag{105}$$

and the set of pure strategies of the third player is

$$X_3 = \{x_3(t),\ t \in [0;\ 0.16\pi] :\ 1 \leqslant x_3(t) \leqslant 1.5\} \subset \mathbb{L}_2[0;\ 0.16\pi], \tag{106}$$

and the set of pure strategies of the fourth player is

$$X_4 = \{x_4(t),\ t \in [0;\ 0.16\pi] :\ 3 \leqslant x_4(t) \leqslant 3.4\} \subset \mathbb{L}_2[0;\ 0.16\pi]. \tag{107}$$

The players' payoff functionals (4) are

$$K_1(x_1(t),\ x_2(t),\ x_3(t),\ x_4(t)) = \\ = \int\limits_{[0;\ 0.16\pi]} \sin(0.2x_1x_2x_3x_4t)\, d\mu(t), \tag{108}$$

$$K_2(x_1(t),\ x_2(t),\ x_3(t),\ x_4(t)) = \\ = \int\limits_{[0;\ 0.16\pi]} \sin\left(0.3x_1x_2x_3x_4t - \frac{\pi}{6}\right) d\mu(t), \tag{109}$$

$$K_3(x_1(t),\ x_2(t),\ x_3(t),\ x_4(t)) = \\ = \int\limits_{[0;\ 0.16\pi]} \sin\left(0.15x_1x_2x_3x_4t - \frac{\pi}{5}\right) d\mu(t), \tag{110}$$

$$K_4 \left(x_1 \left(t \right), \; x_2 \left(t \right), \; x_3 \left(t \right), \; x_4 \left(t \right) \right) =$$
$$= \int\limits_{[0;\; 0.16\pi]} \sin \left(0.54 x_1 x_2 x_3 x_4 t - \frac{\pi}{4} \right) d\mu \left(t \right). \tag{111}$$

The players are forced to use pure strategies $\{x_i \left(t \right)\}_{i=1}^{4}$ such that

$$x_1 \left(t \right) \in \{2 + 0.5 \cdot (m_1 - 1)\}_{m_1=1}^{3} \subset [2;\; 3] \tag{112}$$

and

$$x_2 \left(t \right) \in \{4 + 0.25 \cdot (m_2 - 1)\}_{m_2=1}^{4} \subset [4;\; 4.75] \tag{113}$$

and

$$x_3 \left(t \right) \in \{1 + 0.5 \cdot (m_3 - 1)\}_{m_3=1}^{2} \subset [1;\; 1.5] \tag{114}$$

and

$$x_4 \left(t \right) \in \{3 + 0.2 \cdot (m_4 - 1)\}_{m_4=1}^{3} \subset [3;\; 3.4] \,, \tag{115}$$

and they can change their values only at time points

$$\left\{ \tau^{(l)} \right\}_{l=1}^{7} = \{0.02 l \pi\}_{l=1}^{7} \,. \tag{116}$$

Consequently, this game can be thought of as it is defined on parallelepiped lattice

$$\{2 + 0.5 \cdot (m_1 - 1)\}_{m_1=1}^{3} \times \{4 + 0.25 \cdot (m_2 - 1)\}_{m_2=1}^{4} \times$$
$$\times \{1 + 0.5 \cdot (m_3 - 1)\}_{m_3=1}^{2} \times \{3 + 0.2 \cdot (m_4 - 1)\}_{m_4=1}^{3} \subset$$
$$\subset [2;\; 3] \times [4;\; 4.75] \times [1;\; 1.5] \times [3;\; 3.4] \,, \tag{117}$$

that is this game is a succession of 8 finite $3 \times 4 \times 2 \times 3$ (quadmatrix) games

$$\left\langle \left\{ \left\{ x_1^{(m_1 - 1)} \right\}_{m_1=1}^{3}, \; \left\{ x_2^{(m_2 - 1)} \right\}_{m_2=1}^{4}, \; \left\{ x_3^{(m_3 - 1)} \right\}_{m_3=1}^{2}, \; \left\{ x_4^{(m_4 - 1)} \right\}_{m_4=1}^{3} \right\}, \right.$$
$$\left. \{\mathbf{H}_{1l}, \; \mathbf{H}_{2l}, \; \mathbf{H}_{3l}, \; \mathbf{H}_{4l}\} \right\rangle =$$
$$= \left\langle \left\{ \{2 + 0.5 \cdot (m_1 - 1)\}_{m_1=1}^{3}, \; \{4 + 0.25 \cdot (m_2 - 1)\}_{m_2=1}^{4}, \; \{1 + 0.5 \cdot (m_3 - 1)\}_{m_3=1}^{2}, \right. \right.$$
$$\left. \left. \{3 + 0.2 \cdot (m_4 - 1)\}_{m_4=1}^{3} \right\}, \; \{\mathbf{H}_{1l}, \; \mathbf{H}_{2l}, \; \mathbf{H}_{3l}, \; \mathbf{H}_{4l}\} \right\rangle \tag{118}$$

with first player's payoff matrices

$$\left\{ \mathbf{H}_{1l} = \left[h_{1l\omega_1\omega_2\omega_3\omega_4} \right]_{3 \times 4 \times 2 \times 3} \right\}_{l=1}^{8}$$

whose elements are

$$h_{1lm_1m_2m_3m_4} =$$

$$= \int\limits_{[0.02\cdot(l-1)\pi;\ 0.02l\pi)} f_1\left(x_1^{(m_1-1)},\ x_2^{(m_2-1)},\ x_3^{(m_3-1)},\ x_4^{(m_4-1)},\ t\right) d\mu\,(t) =$$

$$= \int\limits_{[0.02\cdot(l-1)\pi;\ 0.02l\pi)} f_1\big(2+0.5\cdot(m_1-1)\,,\ 4+0.25\cdot(m_2-1)\,,\ 1+$$

$$+0.5\cdot(m_3-1)\,,\ 3+0.2\cdot(m_4-1)\,,\ t\big)d\mu\,(t) =$$

$$= \int\limits_{[0.02\cdot(l-1)\pi;\ 0.02l\pi)} \sin\big(0.2\cdot(2+0.5\cdot(m_1-1))\,(4+0.25\cdot(m_2-1))\,(1+$$

$$+0.5\cdot(m_3-1))\,(3+0.2\cdot(m_4-1))\,t\big)d\mu\,(t) =$$

$$= \int\limits_{[0.02\cdot(l-1)\pi;\ 0.02l\pi)} \sin\left(0.0025t\cdot(3+m_1)\,(15+m_2)\,(1+m_3)\,(14+m_4)\right)d\mu\,(t)$$

$$\text{for}\ \ l=\overline{1,\ 7} \tag{119}$$

and

$$h_{1,8m_1m_2m_3m_4} =$$

$$= \int\limits_{[0.14\pi;\ 0.16\pi]} \sin\left(0.0025t\cdot(3+m_1)\,(15+m_2)\,(1+m_3)\,(14+m_4)\right)d\mu\,(t), \tag{120}$$

with second player's payoff matrices

$$\left\{\mathbf{H}_{2l}=\left[h_{2l\omega_1\omega_2\omega_3\omega_4}\right]_{3\times4\times2\times3}\right\}_{l=1}^{8}$$

whose elements are

$$h_{2lm_1m_2m_3m_4} =$$

$$= \int\limits_{[0.02\cdot(l-1)\pi;\ 0.02l\pi)} f_2\left(x_1^{(m_1-1)},\ x_2^{(m_2-1)},\ x_3^{(m_3-1)},\ x_4^{(m_4-1)},\ t\right) d\mu\,(t) =$$

$$= \int\limits_{[0.02\cdot(l-1)\pi;\ 0.02l\pi)} f_2\big(2+0.5\cdot(m_1-1)\,,\ 4+0.25\cdot(m_2-1)\,,\ 1+$$

$$+0.5\cdot(m_3-1)\,,\ 3+0.2\cdot(m_4-1)\,,\ t\big)d\mu\,(t) =$$

$$= \int\limits_{[0.02\cdot(l-1)\pi;\ 0.02l\pi)} \sin\big(0.3\cdot(2+0.5\cdot(m_1-1))\,(4+0.25\cdot(m_2-1))\,(1+$$

$$+0.5\cdot(m_3-1))\,(3+0.2\cdot(m_4-1))\,t-\tfrac{\pi}{6}\big)d\mu\,(t) =$$

$$= \int\limits_{[0.02\cdot(l-1)\pi;\ 0.02l\pi)} \sin\left(0.00375t\cdot(3+m_1)\,(15+m_2)\,(1+m_3)\,(14+m_4)-\frac{\pi}{6}\right)d\mu\,(t)$$

$$\text{for}\ \ i=\overline{1,\ 7} \tag{121}$$

and

$$h_{2,8m_1m_2m_3m_4} = = \int\limits_{[0.14\pi;\ 0.16\pi]} \sin\left(0.00375t \cdot (3+m_1)(15+m_2)(1+m_3)(14+m_4) - \frac{\pi}{6}\right) d\mu(t), \tag{122}$$

with third player's payoff matrices

$$\left\{\mathbf{H}_{3l} = [h_{3l\omega_1\omega_2\omega_3\omega_4}]_{3\times4\times2\times3}\right\}_{l=1}^{8}$$

whose elements are

$$\begin{aligned} h_{3lm_1m_2m_3m_4} &= \\ = \int\limits_{[0.02\cdot(l-1)\pi;\ 0.02l\pi)} & f_3\left(x_1^{(m_1-1)},\ x_2^{(m_2-1)},\ x_3^{(m_3-1)},\ x_4^{(m_4-1)},\ t\right) d\mu(t) = \\ = \int\limits_{[0.02\cdot(l-1)\pi;\ 0.02l\pi)} & f_3\big(2+0.5\cdot(m_1-1),\ 4+0.25\cdot(m_2-1),\ 1+ \\ & +0.5\cdot(m_3-1),\ 3+0.2\cdot(m_4-1),\ t\big) d\mu(t) = \\ = \int\limits_{[0.02\cdot(l-1)\pi;\ 0.02l\pi)} & \sin\big(0.15\cdot(2+0.5\cdot(m_1-1))(4+0.25\cdot(m_2-1))(1+ \\ & +0.5\cdot(m_3-1))(3+0.2\cdot(m_4-1))t - \tfrac{\pi}{5}\big) d\mu(t) = \\ = \int\limits_{[0.02\cdot(l-1)\pi;\ 0.02l\pi)} & \sin\left(0.001875t\cdot(3+m_1)(15+m_2)(1+m_3)(14+m_4) - \frac{\pi}{5}\right) d\mu(t) \\ & \text{for } \ i = \overline{1,\ 7} \end{aligned} \tag{123}$$

and

$$h_{3,8m_1m_2m_3m_4} = = \int\limits_{[0.14\pi;\ 0.16\pi]} \sin\left(0.001875t \cdot (3+m_1)(15+m_2)(1+m_3)(14+m_4) - \frac{\pi}{5}\right) d\mu(t), \tag{124}$$

and with fourth player's payoff matrices

$$\left\{\mathbf{H}_{4l} = [h_{4l\omega_1\omega_2\omega_3\omega_4}]_{3\times4\times2\times3}\right\}_{l=1}^{8}$$

whose elements are

$$h_{4lm_1m_2m_3m_4} = = \int\limits_{[0.02\cdot(l-1)\pi;\ 0.02l\pi)} f_4\left(x_1^{(m_1-1)},\ x_2^{(m_2-1)},\ x_3^{(m_3-1)},\ x_4^{(m_4-1)},\ t\right) d\mu(t) =$$

$$= \int\limits_{[0.02\cdot(l-1)\pi;\ 0.02l\pi)} f_4\big(2+0.5\cdot(m_1-1)\,,\ 4+0.25\cdot(m_2-1)\,,\ 1+$$

$$+0.5\cdot(m_3-1)\,,\ 3+0.2\cdot(m_4-1)\,,\ t\big)d\mu\,(t) =$$

$$= \int\limits_{[0.02\cdot(l-1)\pi;\ 0.02l\pi)} \sin\big(0.54\cdot(2+0.5\cdot(m_1-1))\,(4+0.25\cdot(m_2-1))\,(1+$$

$$+0.5\cdot(m_3-1))\,(3+0.2\cdot(m_4-1))\,t-\tfrac{\pi}{4}\big)d\mu\,(t) =$$

$$= \int\limits_{[0.02\cdot(l-1)\pi;\ 0.02l\pi)} \sin\left(0.00675t\cdot(3+m_1)\,(15+m_2)\,(1+m_3)\,(14+m_4)-\frac{\pi}{4}\right)d\mu\,(t)$$

$$\text{for}\ \ i=\overline{1,\ 7} \tag{125}$$

and

$$h_{4,8m_1m_2m_3m_4} =$$

$$= \int\limits_{[0.14\pi;\ 0.16\pi]} \sin\left(0.00675t\cdot(3+m_1)\,(15+m_2)\,(1+m_3)\,(14+m_4)-\frac{\pi}{4}\right)d\mu\,(t). \tag{126}$$

Each of the $3\times4\times2\times3$ quadmatrix games (118) with (119) — (126) is solved in pure strategies. It takes no longer than 1.2 seconds to obtain all the 8 interval solutions with an Intel Core i7 processor. Besides, each of the games has a single pure-strategy equilibrium on intervals

$$\{[0.02\cdot(l-1)\,\pi;\ 0.02l\pi)\}_{l=1}^{7}\,,\ \ [0.14\pi;\ 0.16\pi]\,.$$

Consequently, there is a single equilibrium stack $x_n^*\,(t) \in X_n$ for the n-th player, where $x_n^*\,(t)$ takes on values $\{\alpha_{nl}^*\}_{l=1}^{8}$ only. It is shown player-wise in Figure 1. The respective players' payoffs

$$\Big\{K_1\,(\alpha_{1l}^*,\ \alpha_{2l}^*,\ \alpha_{3l}^*,\ \alpha_{4l}^*)\,,\ K_2\,(\alpha_{1l}^*,\ \alpha_{2l}^*,\ \alpha_{3l}^*,\ \alpha_{4l}^*)\,,\ K_3\,(\alpha_{1l}^*,\ \alpha_{2l}^*,\ \alpha_{3l}^*,\ \alpha_{4l}^*)\,,$$

$$K_4\,(\alpha_{1l}^*,\ \alpha_{2l}^*,\ \alpha_{3l}^*,\ \alpha_{4l}^*)\Big\}_{l=1}^{8} = \{h_{1l}^*,\ h_{2l}^*,\ h_{3l}^*,\ h_{4l}^*\}_{l=1}^{8} \tag{127}$$

are presented in Figure 2 along with the polylines of payoff cumulative sums

$$\left\{\sum_{q=1}^{l} h_{1q}^*,\ \sum_{k=1}^{l} h_{2q}^*,\ \sum_{k=1}^{l} h_{3q}^*,\ \sum_{k=1}^{l} h_{4q}^*\right\}_{l=1}^{8} = \left\{h_{1\sum}^{(l)*},\ h_{2\sum}^{(l)*},\ h_{3\sum}^{(l)*},\ h_{4\sum}^{(l)*}\right\}_{l=1}^{8}. \tag{128}$$

The final payoffs of the players

$$\left\{\sum_{q=1}^{8} h_{1q}^*,\ \sum_{k=1}^{8} h_{2q}^*,\ \sum_{k=1}^{8} h_{3q}^*,\ \sum_{k=1}^{8} h_{4q}^*\right\} = \left\{h_{1\sum}^{(8)*},\ h_{2\sum}^{(8)*},\ h_{3\sum}^{(8)*},\ h_{4\sum}^{(8)*}\right\} \tag{129}$$

are highlighted in Figure 2 with circles. Note that payoff cumulative sums $h_{2\sum}^{(l)*}$ and $h_{4\sum}^{(l)*}$ are not increasing polylines. Contrary to this, cumulative sums $h_{1\sum}^{(l)*}$ and $h_{3\sum}^{(l)*}$

are increasing polylines. Generally speaking, payoff cumulative sums

$$\left\{ \left\{ \sum_{q=1}^{l} h_{iq}^{*} \right\}_{i=1}^{N} \right\}_{l=1}^{M} = \left\{ \left\{ h_{i\sum}^{(l)*} \right\}_{i=1}^{N} \right\}_{l=1}^{M} \tag{130}$$

do not have to be non-decreasing polylines.

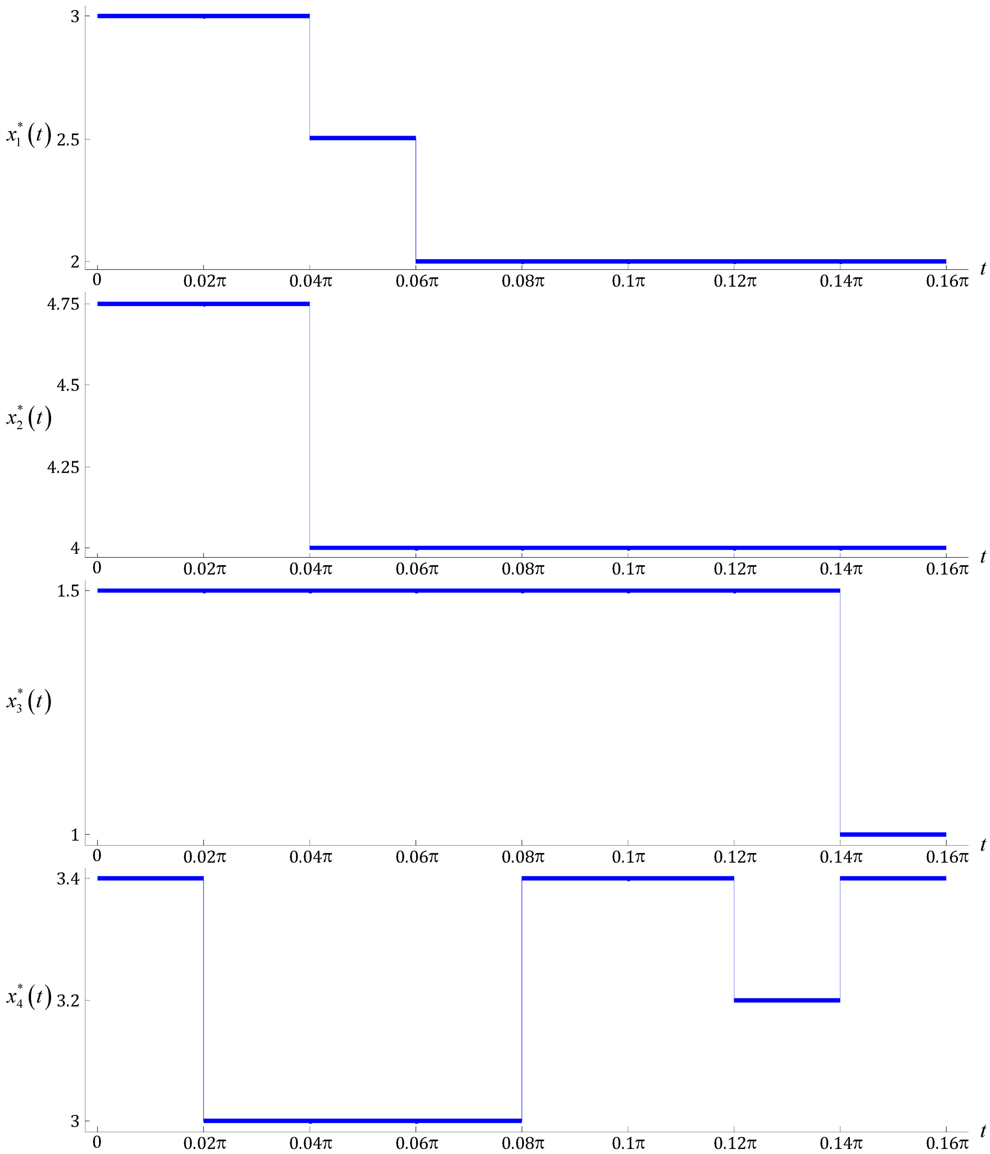

Figure 1: The players' pure-strategy equilibrium stacks in the game by (104) — (116)

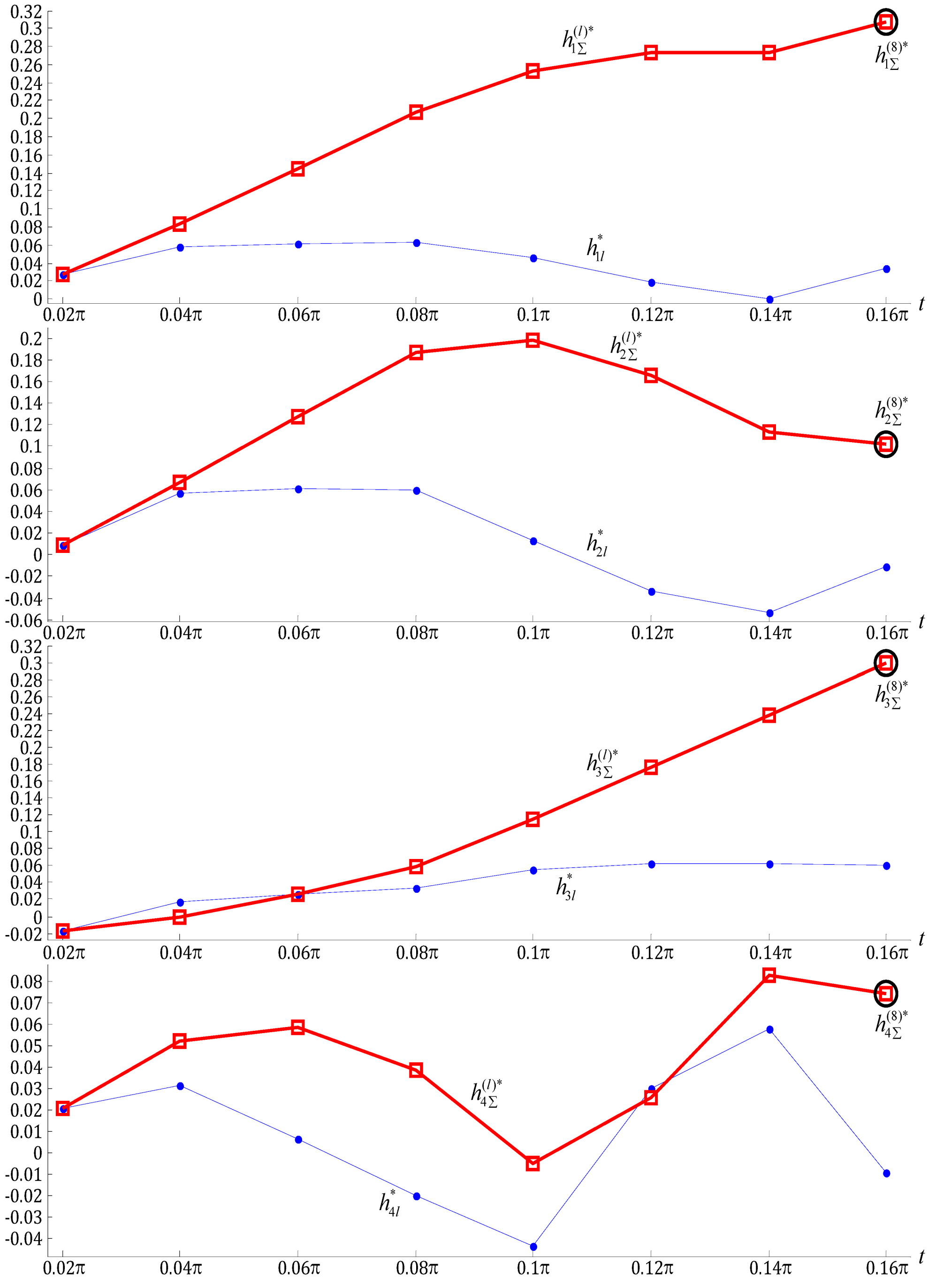

Figure 2: Interval-wise payoffs (127) and payoff cumulative sums (128) in the game by (104) — (116)

7. Discussion

The example clearly shows that solving a succession of multidimensional-matrix (quadmatrix in the considered example) games is far easier than tackling games whose players' pure strategies look like those staircase functions in Figure 1. Indeed, without solving the succession, the respective finite game by (104) — (116) defined on parallelepiped lattice (117) is rendered to a

$$6561 \times 65536 \times 256 \times 6561 \text{ staircase-function game.}$$

This quadmatrix game has 722204136308736 (more than 722 trillion) situations in pure strategies, which can hardly be handled in a reasonable computation time. By the way, the computation time has an exponential growth pattern as the size of the (hypercubic lattice) matrix increases.

Even if not every multidimensional-matrix game has a single equilibrium, a solution of the initial staircase-function game is built in the same way as (104) — (126). The only difference is that then there will be multiple stacked equilibria, which commonly induce instability of the players' behavior [23, 5, 14]. The time spent on computation of a stack depends on both the number of the player's pure strategies (on an interval) and the number of intervals. Stacking the "short" games' pure-strategy equilibria (by Theorem 5) is fulfilled trivially. When there is at least an equilibrium in mixed strategies for an interval (that actually falls within conditions of Theorem 6), the stacking is fulfilled as well implying that the resulting pure-mixed-strategy equilibrium in game (6) is realized successively, interval by interval, spending the same amount of time to implement both pure strategy and mixed strategy equilibria [18, 19].

The abovementioned behavior instability is a serious problem in non-cooperative games having multiple equilibria differing in the player's payoffs [22, 23, 15]. It is particularly solved by equilibria refinement with using domination efficiency along with maximin and the superoptimality rule [14]. The necessary condition is to have an asymmetry in the payoffs. The asymmetry allows distinguish more profitable (and thus stable) equilibria, whereupon the best equilibrium (equilibria) or equilibrium stacks are selected. Otherwise, they are not distinguishable.

Continuous games are ever struggled to be approximated or rendered to finite games so that their solutions could be easily implemented and practiced [10, 9, 11, 12, 15]. However, even a finite (that is, multidimensional-matrix) game may be not tractable due to gigantic number of situations in game. The presented method further "breaks" the initial staircase-function game with a purpose to obtain an equilibrium in a more reasonable time. So, the method is far more tractable than a straightforward approach to solving directly the staircase-function multidimensional-matrix game would be.

Here, the tractability does not depend on the number of (time) intervals. Unless the sets of possible values of players' pure strategies are of order of hundreds or thousands (when searching for equilibria in a "short" multidimensional-matrix game may take a few seconds and more), the method is entirely applicable. Moreover, the presented method is a significant contribution to the mathematical game theory and practice for avoiding too complicated solution approaches resulting from game continuities and functional spaces of pure strategies. This is similar to preventing Einstellung effect in modeling [16, 7]. The "breaking" of the staircase-function finite game into a succession of "short" multidimensional-matrix games herein "deeinstellungizes" such non-cooperative games.

A drawback is that a "short" multidimensional-matrix game may be intractable itself if its size is too big or there is a large number of players. The size limitation depends on requirements from the administrator, which, say, can limit the number of players to 3 or 4. If the interval breaking is over-thick, the "long" staircase-function multidimensional-matrix game may be solved in an unreasonable amount of time (although every "short" game is tractable and solved relatively fast). Consequently, the size of the "short" multidimensional-matrix game should be made as small as possible. The number of players should be necessarily limited.

8. Conclusion

A non-cooperative game defined on a product of staircase-function finite spaces is equivalent to a multidimensional-matrix game. In this game, a (pure) strategy is a complex set of simple actions ordinarily represented as a function of time. Players' payoff matrices in this game are built very slowly, so it is impracticable to find any equilibria (as well as the other solution types) in such games using straightforwardly methods to solve a finite non-cooperative game. However, the multidimensional-matrix staircase-function game is equivalent to the succession of "short" multidimensional-matrix games, each defined on an interval where the pure strategy value is constant.

Owing to Theorem 6 along with Theorem 4 and Theorem 5 the equilibrium of the initial staircase-function game can be obtained by stacking the equilibria of the "short" multidimensional-matrix games. The stack is always possible, even when only time is discrete (and the set of pure strategy possible values is continuous). Any combination of the respective equilibria of the "short" multidimensional-matrix games is an equilibrium of the initial staircase-function game. Moreover, Theorem 5 allows finding a pure-strategy equilibrium of the initial (infinite or continuous) game by stacking the pure-strategy equilibria of the "short" (infinite or continuous) games.

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Bayer Noise Symmetric Functions and Some Combinatorial Algebraic Structures

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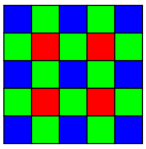
Abstract: Symmetric functions play a crucial role in classifying representations of symmetric groups, and they are largely involved with combinatorial algebras and graph theory. Bayer filter technique is largely applied in most of the professional digital cameras due to the fact that it is a low-cost, and it allows photosensors not only to capture the intensity of light, but also to record the wavelength of light as well. Using Bayer Pattern, we introduce the Bayer Noise symmetric functions and the Bayer Noise Schur functions, and we study some combinatorial structures on the Bayer Noise modules. We study the connection between Bayer Noise symmetric functions and other bases for the algebra of symmetric functions, and we explicitly calculate special cases over a fixed commutative ring $\mathbf{k}$. We also study the compatibility of such algebraic and coalgebraic structures.

AMS Subject Classification: 05E05, 05E40, 05E18, 05E15, 16T15.
Keywords and Phrases: Symmetric functions; Schur functions; Algebra; Coalgebra; Noise.

1. Introduction

A Bayer filter mosaic is a color filter array by which RGB color filters are arranged on a square grid of photosensors. This approach is very common and applied in most single-chip digital image sensors and extensively in professional equipment. Half (50%) of the filter elements are green and the rest are composed of blue and red (25% red and 25% blue). This gives an approximation for human photopic vision where the M and L cones amalgamate to produce a bias in the green spectral region [1, p. 124].



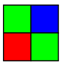
Bayer Filter Mosaic (in terms of colors)

B	G	B	G	B
G	R	G	R	G
B	G	B	G	B
G	R	G	R	G
B	G	B	G	B

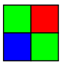
Bayer Filter Mosaic (in terms of letters).

Basically, there are four patterns of this filter: GBRG, GRBG, BGGR and RGGB. A Bayer pattern array can be shown in the following figure.

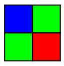
There are basically four patterns of this filter: GBRG, GRBG, BGGR and RGGB.



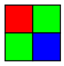
GBRG Pattern



GRBG Pattern



BGGR Pattern



RGGB Pattern

Every BGGR-Bayer Young diagram of shape λ corresponds to a unique symmetric monomial function whose degree equals to the number of its pixels. This monomial function (which we call the Bayer Noise monomial function) can be seen as splitting an image into three parts GB-part, G-part and R-part. The GB-part can be thought of as a full-size (free color (G, B)) image (the original image) while the other parts can be seen as full-sizes (free color G) and (free color R) images respectively (see Figure 5: *Block diagram of the proposed restoration technique* in [6]). Such monomial functions allow us to define and study some interesting modules over a fixed commutative ring $\mathbf{k}$. More importantly, we study some combinatorial algebraic and coalgebraic structures on such modules. The order and color of the cells in the Bayer filter mosaic play a crucial role in defining such algebraic and coalgebraic structures.

This paper is basically an application of combinatorial algebra in image processing. To see the connection more clearly, we refer the reader to [6]. The paper is organized as follows. In section 2, we recall some basic concepts of symmetric functions. In section 3, Bayer Young diagrams and Bayer Noise monomials have been introduced. In section 4, we study some algebraic structures on Bayer Noise modules while section 5 is devoted for studying some coalgebraic structures on such modules. In section 6, we introduce Bayer Noise Schur functions, and we prove that the set of all Bayer Noise Schur functions forms another basis for the Bayer Noise module Γ.

2. Preliminaries

Throughout this paper, $\mathbf{k}$ is a commutative ring, and all unadorned tensor products are over $\mathbf{k}$. Following [2], we recall some basic concepts of symmetric functions. For the basic notions of symmetric functions, the reader is referred to [2], [3], [8], [5], [11], [10], [4] or [9]. Given an infinite variable set $\mathbf{x} = (x_1, x_2, \ldots)$, a monomial

$\mathbf{x}^\alpha := x_1^{\alpha_1} x_2^{\alpha_2} \cdots$ is indexed by a sequence $\alpha = (\alpha_1, \alpha_2, \ldots)$ in $\mathbb{N}^\infty$ having finite support; such sequences α are called *weak compositions*. The nonzero entries of the sequence $\alpha = (\alpha_1, \alpha_2, \ldots)$ are called the *parts* of the weak composition α.

The sum $\alpha_1 + \alpha_2 + \alpha_3 + \cdots$ of all entries of a weak composition $\alpha = (\alpha_1, \alpha_2, \alpha_3, \ldots)$ (or, equivalently, the sum of all parts of α) is called the *size* of α and denoted by $|\alpha|$.

Consider the $\mathbf{k}$-algebra $\mathbf{k}[[\mathbf{x}]] := \mathbf{k}[[x_1, x_2, x_3, \ldots]]$ of all formal power series in the indeterminates $x_1, x_2, x_3, \ldots$ over $\mathbf{k}$; these series are infinite $\mathbf{k}$-linear combinations $\sum_\alpha c_\alpha \mathbf{x}^\alpha$ (with c_α in $\mathbf{k}$) of the monomials $\mathbf{x}^\alpha$ where α ranges over all weak compositions. The product of two such formal power series is well-defined by the usual multiplication rule.

The *degree* of a monomial $\mathbf{x}^\alpha$ is defined to be the number $\deg(\mathbf{x}^\alpha) := \sum_i \alpha_i \in \mathbb{N}$. Given a number $d \in \mathbb{N}$, we say that a formal power series $f(\mathbf{x}) = \sum_\alpha c_\alpha \mathbf{x}^\alpha \in \mathbf{k}[[\mathbf{x}]]$ (with c_α in $\mathbf{k}$) is *homogeneous of degree* d if every weak composition α satisfying $\deg(\mathbf{x}^\alpha) \neq d$ must satisfy $c_\alpha = 0$. In other words, a formal power series is homogeneous of degree d if it is an infinite $\mathbf{k}$-linear combination of monomials of degree d. Every formal power series $f \in \mathbf{k}[[\mathbf{x}]]$ can be uniquely represented as an infinite sum $f_0 + f_1 + f_2 + \cdots$, where each f_d is homogeneous of degree d; in this case, we refer to each f_d as the *d-th homogeneous component of f*. Note that this does not make $\mathbf{k}[[\mathbf{x}]]$ into a graded $\mathbf{k}$-module, since these sums $f_0 + f_1 + f_2 + \cdots$ can have infinitely many nonzero addends. Nevertheless, if f and g are homogeneous power series of degrees d and e, then fg is homogeneous of degree $d + e$.

A formal power series $f(\mathbf{x}) = \sum_\alpha c_\alpha \mathbf{x}^\alpha \in \mathbf{k}[[\mathbf{x}]]$ (with c_α in $\mathbf{k}$) is said to be *of bounded degree* if there exists some bound $d = d(f) \in \mathbb{N}$ such that every weak composition $\alpha = (\alpha_1, \alpha_2, \alpha_3, \ldots)$ satisfying $\deg(\mathbf{x}^\alpha) > d$ must satisfy $c_\alpha = 0$. Equivalently, a formal power series $f \in \mathbf{k}[[\mathbf{x}]]$ is of bounded degree if all but finitely many of its homogeneous components are zero. (For example, $x_1^2 + x_2^2 + x_3^2 + \cdots$ and $1 + x_1 + x_2 + x_3 + \cdots$ are of bounded degree, while $x_1 + x_1 x_2 + x_1 x_2 x_3 + \cdots$ and $1 + x_1 + x_1^2 + x_1^3 + \cdots$ are not.) It is easy to see that the sum and the product of two power series of bounded degree also have bounded degree. Thus, the formal power series of bounded degree form a $\mathbf{k}$-subalgebra of $\mathbf{k}[[\mathbf{x}]]$, which we call $R(\mathbf{x})$. This subalgebra $R(\mathbf{x})$ is graded (by degree). The symmetric group $\mathfrak{S}_n$ permuting the first n variables $x_1, \ldots, x_n$ acts as a group of automorphisms on $R(\mathbf{x})$, as does the union $\mathfrak{S}_{(\infty)} = \bigcup_{n \geq 0} \mathfrak{S}_n$ of the infinite ascending chain $\mathfrak{S}_0 \subset \mathfrak{S}_1 \subset \mathfrak{S}_2 \subset \cdots$ of symmetric groups. This group $\mathfrak{S}_{(\infty)}$ can also be described as the group of all permutations of the set $\{1, 2, 3, \ldots\}$ which leave all but finitely many elements invariant. It is known as the *finitary symmetric group* on $\{1, 2, 3, \ldots\}$. The group $\mathfrak{S}_{(\infty)}$ also acts on the set of all weak compositions by permuting their entries:

$$\sigma(\alpha_1, \alpha_2, \alpha_3, \ldots) = \left(\alpha_{\sigma^{-1}(1)}, \alpha_{\sigma^{-1}(2)}, \alpha_{\sigma^{-1}(3)}, \ldots\right)$$

for any weak composition $(\alpha_1, \alpha_2, \alpha_3, \ldots)$ and any $\sigma \in \mathfrak{S}_{(\infty)}$. These two actions are connected by the equality $\sigma(\mathbf{x}^\alpha) = \mathbf{x}^{\sigma\alpha}$ for any weak composition α and any $\sigma \in \mathfrak{S}_{(\infty)}$. The *ring of symmetric functions in* $\mathbf{x}$ *with coefficients in* $\mathbf{k}$, denoted $\Lambda = \Lambda(\mathbf{k}) = \Lambda(\mathbf{x}) = \Lambda(\mathbf{k})(\mathbf{x})$, is the $\mathfrak{S}_{(\infty)}$-invariant subalgebra $R(\mathbf{x})^{\mathfrak{S}_{(\infty)}}$ of $R(\mathbf{x})$:

$$\begin{aligned}\Lambda &:= \left\{f \in R(\mathbf{x}) : \sigma(f) = f \text{ for all } \sigma \in \mathfrak{S}_{(\infty)}\right\} \\ &= \left\{f = \sum_{\alpha} c_\alpha \mathbf{x}^\alpha \in R(\mathbf{x}) : c_\alpha = c_\beta \text{ if } \alpha, \beta \text{ lie in the same } \mathfrak{S}_{(\infty)}\text{-orbit}\right\}.\end{aligned}$$

We refer to the elements of Λ as *symmetric functions* (over $\mathbf{k}$); however, despite this terminology, they are not functions in the usual sense.

Note that Λ is a graded $\mathbf{k}$-algebra, since $\Lambda = \bigoplus_{n \geq 0} \Lambda_n$ where Λ_n are the symmetric functions $f = \sum_\alpha c_\alpha \mathbf{x}^\alpha$ which are *homogeneous of degree* n, meaning $\deg(\mathbf{x}^\alpha) = n$ for all $c_\alpha \neq 0$. A *partition* $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_\ell, 0, 0, \ldots)$ is a weak composition whose entries weakly decrease: $\lambda_1 \geq \cdots \geq \lambda_\ell > 0$. The (uniquely defined) ℓ is said to be the *length* of the partition λ and denoted by $\ell(\lambda)$. Thus, $\ell(\lambda)$ is the number of parts of λ. One sometimes omits trailing zeroes from a partition: e.g., one can write the partition $(3, 1, 0, 0, 0, \ldots)$ as $(3, 1)$. We will often (but not always) write λ_i for the i-th entry of the partition λ (for instance, if $\lambda = (5, 3, 1, 1)$, then $\lambda_2 = 3$ and $\lambda_5 = 0$). If λ_i is nonzero, we will also call it the *i-th part* of λ. The sum $\lambda_1 + \lambda_2 + \cdots + \lambda_\ell = \lambda_1 + \lambda_2 + \cdots$ (where $\ell = \ell(\lambda)$) of all entries of λ (or, equivalently, of all parts of λ) is the size $|\lambda|$ of λ. For a given integer n, the partitions of size n are referred to as the *partitions of* n. The empty partition $() = (0, 0, 0, \ldots)$ is denoted by $\varnothing$. Every weak composition α lies in the $\mathfrak{S}_{(\infty)}$-orbit of a unique partition $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_\ell, 0, 0, \ldots)$ with $\lambda_1 \geq \cdots \geq \lambda_\ell > 0$. For any partition λ, define the *monomial symmetric function*

$$m_\lambda := \sum_{\alpha \in \mathfrak{S}_{(\infty)} \lambda} \mathbf{x}^\alpha. \tag{2.1}$$

Letting λ run through the set *Par* of all partitions, this gives the *monomial* $\mathbf{k}$-*basis* $\{m_\lambda\}$ of Λ. Letting λ run only through the set Par_n of partitions of n gives the monomial $\mathbf{k}$-basis for Λ_n.

It is straightforward to check that $(\Lambda, \underline{m}, \underline{u}, \underline{\Delta}, \underline{\epsilon})$ is a connected graded $\mathbf{k}$-bialgebra of finite type, and hence also a Hopf algebra, where

- The multiplication is the map

$$\Lambda \otimes \Lambda \xrightarrow{\underline{m}} \Lambda, \; m_\mu \otimes m_\nu \mapsto m_\mu m_\nu.$$

- The unit is the inclusion map

$$\mathbf{k} = \Lambda_0 \xrightarrow{\underline{u}} \Lambda.$$

- The comultiplication is the map

$$\Lambda \xrightarrow{\underline{\Delta}} \Lambda \otimes \Lambda, \; m_\lambda \mapsto \sum_{\substack{(\mu,\nu): \\ \mu \sqcup \nu = \lambda}} m_\mu \otimes m_\nu,$$

  in which $\mu \sqcup \nu$ is the partition obtained by taking the multiset union of the parts of μ and ν, and then reordering them to make them weakly decreasing.

- The counit is the $\mathbf{k}$-linear map

$$\mathbf{k} = \Lambda_0 \xrightarrow{\epsilon} \Lambda$$

with $\underline{\epsilon}|_{\Lambda_0=\mathbf{k}} = id_{\mathbf{k}}$ and $\underline{\epsilon}|_{I=\bigoplus_{n>0}\Lambda_n} = 0$.

3. Bayer Young Diagram

Definition 3.1. Let λ be a partition.

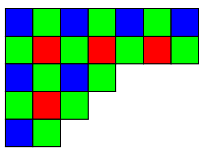
(1) A *colored Young diagram of shape* λ is a Young diagram of shape λ whose cells are colored with green, blue or red.

(2) A Young diagram of shape λ is called a *BGGR-Bayer Young diagram* of shape λ if the corresponding Young diagram of λ has a BGGR pattern. If a tableau does not have enough cells for BGGR pattern (it takes 4 cells to have BGGR), then do it whenever possible. Similarly, *GBRG-Bayer Young diagram*, *GRBG-Bayer Young diagram* and *RGGB-Bayer Young diagram* can be defined.

(3) By *Bayer Young Diagrams*, we will simply mean BGGR-Bayer Young Diagrams (since the other Bayer Young Diagrams can be characterized similarly). Clearly, Bayer Young diagrams are colored Young Diagrams. The converse, however, needs not be true.

(4) Let $\mathcal{YD}$ be the set of all Young diagrams. Let $T : Par \to \mathcal{YD}$ be the bijective map that takes any partition λ to its corresponding Young diagram $T(\lambda)$. Let $\mathcal{BYD}$ be the set of all Bayer Young diagrams. There is a bijective map $\mathcal{B} : Par \to \mathcal{BYD}, \lambda \mapsto \mathcal{B}(\lambda)$.

Example 3.2. Let $\lambda = (7, 7, 4, 3, 2)$. We have



$\mathcal{B}(7, 7, 4, 3, 2)$

Definition 3.3.

(1) Let $\mathcal{B}(\lambda)$ be a Bayer Young diagram of shape λ. Then its corresponding *Bayer Noise Young diagram*, denoted by $\mathcal{C}(\lambda, GBR)$, is the (colored) Young diagram obtained by rearranging the colored cells of $\mathcal{B}(\lambda)$ using the order $G < B < R$ as

follows. First, we rearrange the colored cells of $\mathcal{B}(\lambda)$ to be weakly increasing left-to-right in rows, and then we rearrange the colored cells of the resulting colored Young diagram to be weakly increasing top-to-bottom in columns. One might note that the green part of $\mathcal{C}(\lambda, GBR)$ forms a colored Young subdiagram, denoted by $\mathcal{C}(\lambda, GBR, G)$, of $\mathcal{C}(\lambda, GBR)$ (of shape λ_G) while the region of both the green part and the blue part of $\mathcal{C}(\lambda, GBR)$ forms a colored Young subdiagram, denoted by $\mathcal{C}(\lambda, GBR, GB)$, of $\mathcal{C}(\lambda, GBR)$ (of shape λ_{GB}). Here, λ_G and λ_{GB} are the shapes of the colored Young diagrams $\mathcal{C}(\lambda, GBR, G)$ and $\mathcal{C}(\lambda, GBR, GB)$ respectively. Analogously, one could define $\mathcal{C}(\lambda, GRB)$, $\mathcal{C}(\lambda, RBG)$, $\mathcal{C}(\lambda, RGB)$, $\mathcal{C}(\lambda, BRG)$, $\mathcal{C}(\lambda, BGR)$ and $\mathcal{C}(\lambda, BRG)$. Unless confusion is possible, λ_R always denotes the partition corresponding to the Young subdiagram $\mathcal{C}(\lambda, RGB, R)$ of $\mathcal{C}(\lambda, RGB)$.

(2) Let $\mathfrak{C}$ be the set of all colored Young diagrams and $\mathfrak{A} = \{GBR, GRB, BGR, BRG, RGB, RBG\}$. then $\mathcal{C}$ can be thought of as a map

$$\mathcal{C} : Par \times \mathfrak{A} \to \mathfrak{C},\ (\lambda, E) \mapsto \mathcal{C}(\lambda, E),$$

for any $(\lambda, E) \in Par \times \mathfrak{A}$.

(3) Define a map $\mathfrak{D}_{GBR} : Par \to Par \times Par \times Par,\ \lambda \mapsto (\lambda_{GB}, \lambda_G, \lambda_R)$. If $\lambda, \lambda' \in Par$ with $(\lambda_{GB}, \lambda_G, \lambda_R) = (\lambda'_{GB}, \lambda'_G, \lambda'_R)$, then $\lambda_G = \lambda'_G$, $\lambda_R = \lambda'_R$ and $\lambda_{GB} = \lambda'_{GB}$. This implies that $\mathcal{C}(\lambda, GBR) = \mathcal{C}(\lambda', GBR)$. Since $\lambda_G = \lambda'_G$, $\lambda_R = \lambda'_R$ and $\lambda_{GB} = \lambda'_{GB}$, we have $\lambda_B = \lambda'_B$. Thus, $\mathfrak{D}_{GBR}$ is injective (but not surjective). Composing this map with the projections maps $\pi_{GB} : Par \times Par \times Par \to Par,\ (\lambda_{GB}, \lambda_G, \lambda_R) \mapsto \lambda_{GB}$, $\pi_G : Par \times Par \times Par \to Par,\ (\lambda_{GB}, \lambda_G, \lambda_R) \mapsto \lambda_G$ and $\pi_R : Par \times Par \to Par,\ (\lambda_{GB}, \lambda_G, \lambda_R) \mapsto \lambda_R$, we respectively obtain the maps

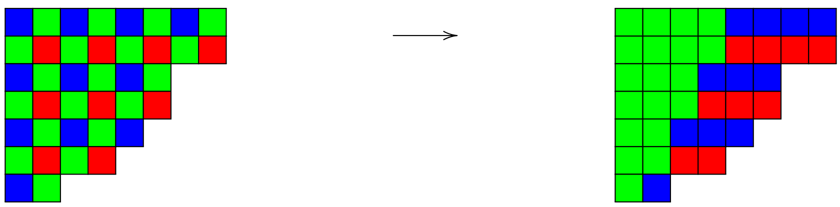
$$\mathfrak{D}_{GB} : Par \to Par,\ \lambda \mapsto \lambda_{GB},$$
$$\mathfrak{D}_G : Par \to Par,\ \lambda \mapsto \lambda_G,$$

and

$$\mathfrak{D}_R : Par \to Par,\ \lambda \mapsto \lambda_R.$$

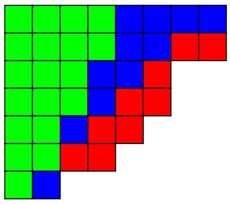
One might note that the maps $\mathfrak{D}_{GB}$ and $\mathfrak{D}_R$ are neither injective nor surjective maps.

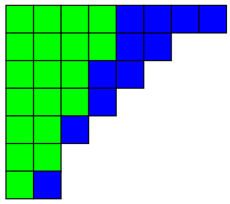
Example 3.4. Consider $\lambda = (8, 8, 6, 6, 5, 4, 2)$. To get $\mathcal{C}_{(\lambda, GBR)}$, we first use the order $G < B < R$ to rearrange $\mathcal{B}(\lambda)$ to be weakly increasing left-to-right in rows. So, we have

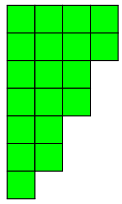


$\mathcal{B}(8, 8, 6, 6, 5, 4, 2)$

Then we rearrange the resulting colored Young diagram to be weakly increasing top-to-bottom in columns to obtain $\mathcal{C}(\lambda, GBR)$. Explicitly, $\mathcal{C}(\lambda, GBR)$ and its corresponding Young subdiagrams $\mathcal{C}(\lambda, GBR, GB)$ and $\mathcal{C}(\lambda, GBR, G)$ are given respectively by the following

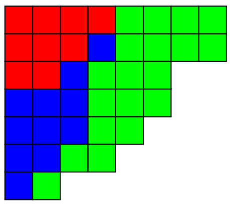


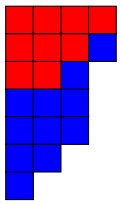


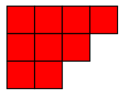


$\mathcal{C}((8,8,6,6,5,4,2), GBR)$ $\mathcal{C}((8,8,6,6,5,4,2), GB)$
$\mathcal{C}((8,8,6,6,5,4,2), G)$

Similarly, one might check that using the order $R < B < G$ gives the following





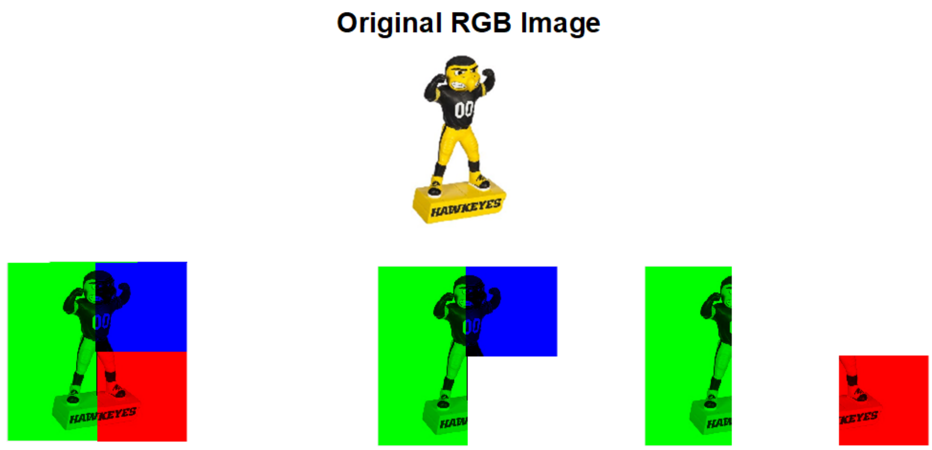


$\mathcal{C}((8,8,6,6,5,4,2), RBG)$ $\mathcal{C}((8,8,6,6,5,4,2), RB)$
$\mathcal{C}((8,8,6,6,5,4,2), R)$

Remark 3.5.

(i) It is well-known that the color channels for a color image are represented by three distinct $2D$ arrays with dimension $m \times n$ for an image with m rows and n columns, with one array for each color, red (color channel 1), green (color channel 2), blue (color channel 3). A pixel color is modeled as 1×3 array [7]. It is also well-known that the spatial domain of each RGB image can be represented as a 3D vector of 2D arrays. The Bayer Noise Young machinery, however, provides us with a new approach by which every Bayer Young diagram can be represented by three special types of colored (noise) diagrams RG, G and R diagrams. This can be depicted in the following example:

3 RGB Channels

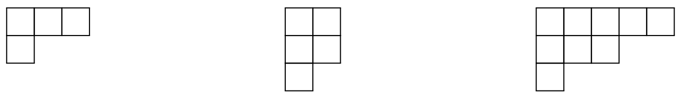


Bayer Noise Young Channels

(ii) Let $\lambda, \lambda' \in Par$ and write $\lambda = (\lambda_1, \cdots, \lambda_m)$ and $\lambda' = (\lambda'_1, \cdots, \lambda'_n)$. Using the convention $\lambda_i = 0$ and $\lambda'_j = 0$ for any $i > m$ and $j > n$, we recall that $\lambda + \lambda'$ is defined as follows:

$$\lambda + \lambda' = (\lambda_1 + \lambda'_1, \cdots, \lambda_k + \lambda'_k),$$

where $k = max\{m, n\}$. For example, if $\lambda = (3, 1)$ and $\lambda' = (2, 2, 1)$, then $\lambda + \lambda' = (5, 3, 1)$. This can be depicted as follows:



$T(\lambda)$ $\quad$ $T(\lambda')$ $\quad$ $T(\lambda+\lambda')$

The following proposition is an obvious consequence.

Proposition 3.6.

(1) *The colored Young diagrams* $\mathcal{C}(\lambda, BGR, B)$ *and* $\mathcal{C}(\lambda, BRG, B)$ *have the same shape* λ_B. *Similarly,* $\mathcal{C}(\lambda, GBR, G)$ *and* $\mathcal{C}(\lambda, GRB, G)$ *have the same shape* λ_G *while* $\mathcal{C}(\lambda, RGB, R)$ *and* $\mathcal{C}(\lambda, RBG, R)$ *have the same shape* λ_R.

(2) *We have* $\lambda_{GB} = \lambda_{BG}$, $\lambda_{GR} = \lambda_{RG}$ *and* $\lambda_{BR} = \lambda_{RB}$.

Definition 3.7. Let $\lambda = (\lambda_1, \cdots, \lambda_{\ell(\lambda)}) \in Par$, where $\ell(\lambda)$ is the length of λ.

(1) Define $\lambda^{GB} = (\mu_1, \cdots, \mu_{\ell(\lambda)})$, where

$$\mu_i = \begin{cases} \lambda_i & \text{if } i \text{ is odd} \\ \frac{\lambda_i}{2} & \text{if } i \text{ and } \lambda_i \text{ are both even} \\ \frac{\lambda_i+1}{2} & \text{if } i \text{ is even and } \lambda_i \text{ is odd} \end{cases}$$

(2) We define λ^{G} to be the sequence of nonzero integers $\lambda^{G} = (\mu'_1, \cdots, \mu'_{\ell(\lambda)})$, where

$$\mu'_i = \begin{cases} \frac{\lambda_i}{2} & \text{if } \lambda_i \text{ is even} \\ \frac{\lambda_i-1}{2} & \text{if } i \text{ and } \lambda_i \text{ are both odd} \\ \frac{\lambda_i+1}{2} & \text{if } i \text{ is even and } \lambda_i \text{ is odd} \end{cases}$$

(3) We define λ^{R} to be the sequence of nonzero integers $\lambda^{R} = (\mu''_1, \cdots, \mu''_m)$, where

$$\mu''_i = \begin{cases} \frac{\lambda_{2i}}{2} & \text{if } \lambda_{2i} \text{ is even} \\ \frac{\lambda_{2i}-1}{2} & \text{if } \lambda_{2i} \text{ is odd} \end{cases}$$

and

$$m = \begin{cases} \frac{\ell(\lambda)-1}{2} & \text{if } \ell(\lambda) \text{ is odd} \\ \frac{\ell(\lambda)}{2} & \text{if } \ell(\lambda) \text{ is even} \end{cases}$$

Let $\lambda = (\lambda_1, \cdots, \lambda_{\ell(\lambda)}) \in Par$. Note that λ^{GB}, λ^{G} and λ^{R} need not be in Par. More explicitly, write $\lambda^{G} = (\mu'_1, \cdots, \mu'_{\ell(\lambda)})$ and $\lambda^{R} = (\mu''_1, \cdots, \mu''_m)$. Then if i is odd and $\lambda_i = 1$, then $\mu'_i = 0$. Similarly, if i is even and $\lambda_i = 1$, then $\mu''_i = 0$. The following proposition gives an equivalent setting for Definition (3.3), and the proof is straightforward.

Proposition 3.8. *Let $\lambda = (\lambda_1, \cdots, \lambda_{\ell(\lambda)}) \in Par$, where $\ell(\lambda)$ is the length of λ.*

(1) Let $\tilde{\lambda}^{GB}$ be the partition obtained by reordering the parts of λ^{GB} to make them weakly decreasing. Then we have $\tilde{\lambda}^{GB} = \lambda_{GB}$.

(2) Let $\tilde{\lambda}^{G}$ be the partition obtained by taking the multiset union of the parts of λ^{G}, reordering them to make them weakly decreasing and removing all zero parts of them. Then $\tilde{\lambda}^{G} = \lambda_{G}$.

(3) Let $\tilde{\lambda}^{R}$ be the partition obtained by taking the multiset union of the parts of λ^{R}, reordering them to make them weakly decreasing and removing all zero parts of them. Then $\tilde{\lambda}^{R} = \lambda_{R}$.

Example 3.9. Let $\lambda = (8, 8, 6, 6, 5, 4, 1) \in Par$.

(1) We have $\lambda^{GB} = (8, 4, 6, 3, 5, 2, 1)$ and $\tilde{\lambda}^{GB} = (8, 6, 5, 4, 3, 2, 1) = \lambda_{GB}$.

(2) We have $\lambda^{G} = (4, 4, 3, 3, 2, 2, 0)$ and $\tilde{\lambda}^{G} = (4, 4, 3, 3, 2, 2) = \lambda_{G}$.

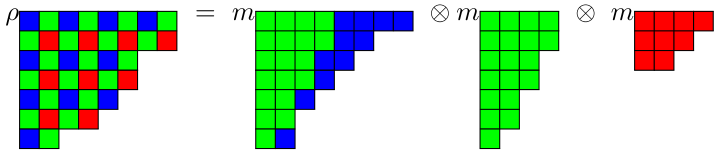
(3) We have $\lambda^{R} = (4, 3, 2) = \tilde{\lambda}^{R} = \lambda_{R}$.

Definition 3.10. For any partition $\lambda \in Par$, the *Bayer Noise monomial* is defined to be the monomial

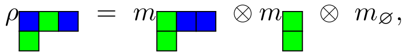
$$\rho_\lambda(x, y, z) = m_{\lambda_{GB}}(x) \otimes m_{\lambda_R}(y) \otimes m_{\lambda_R}(z).$$

We will simply write it as $\rho_\lambda = m_{\lambda_{GB}} \otimes m_{\lambda_G} \otimes m_{\lambda_R}$.

Example 3.11. We have



Remark 3.12. If $\lambda \in Par$, then the partition λ_R could be the empty partition, for example, we have



where $\varnothing$ here is the correspondent empty Young diagram $\mathcal{B}((0))$ of the empty partition (0).

Definition 3.13. Let $\Gamma_n(\mathbf{k})$ be the free $\mathbf{k}$-module with the basis $\{\rho_\lambda\}_{\lambda \in Par_n}$, where Par_n is the set of partitions of n. Note that $dim(\Gamma_n(\mathbf{k})) = |Par_n|$, where $|Par_n|$ is the number of elements of Par_n. Let $\Gamma(\mathbf{k}) = \bigoplus_{n \geq 0} \Gamma_n(\mathbf{k})$. Then the set $\{\rho_\lambda\}_{\lambda \in Par}$ forms a basis for $\Gamma(\mathbf{k})$ over $\mathbf{k}$, and $\Gamma(\mathbf{k})$ is called the *Bayer Noise module*. Obviously, as modules, $\Gamma_n(\mathbf{k}) \cong \Lambda_n(\mathbf{k})$ for every $n \in \mathbb{N}$ and hence $\Gamma(\mathbf{k}) \cong \Lambda(\mathbf{k})$.

Remark 3.14.

(1) When no confusion is possible, we will simply write Γ_n and Γ instead of $\Gamma_n(\mathbf{k})$ and $\Gamma(\mathbf{k})$ respectively.

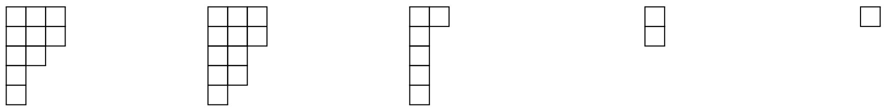
(2) Let $\mu, \nu \in Par$. Then, in general, $(\mu_{GB} + \nu_{GB}, \mu_G + \nu_G, \mu_R + \nu_R)$ need not be in $\mathfrak{D}_{GBR}(Par)$, and hence $\rho_\mu \, \rho_\nu$ need not be in Γ, where $\rho_\mu \, \rho_\nu$ is the regular multiplication of the monomials ρ_μ and ρ_ν. For example, if $\mu = (1,1) = \nu$, then $\mu_{GB} = \nu_{GB} = (1,1)$, $\mu_G = \nu_G = (1)$ and $\mu_R = \nu_R = (0)$ (the empty partition). However, $(\mu_{GB} + \nu_{GB}, \mu_G + \nu_G, \mu_R + \nu_R) = ((2,2),(2),(0))$ which is clearly not in $\mathfrak{D}_{GBR}(Par)$. It turns out that the operation $(\rho_\mu, \rho_\nu) \mapsto \rho_\mu \, \rho_\nu$ does not define an algebra structure on Γ.

(3) One might notice that in general if $(\mu, \nu) \in Par \times Par$, then $((\mu \sqcup \nu)_{GB}, (\mu \sqcup \nu)_G, (\mu \sqcup \nu)_R) \neq (\mu_{GB} \sqcup \nu_{GB}, \mu_R \sqcup \nu_R)$ and $(\mu_{GB} \sqcup \nu_{GB}, \mu_G \sqcup \nu_G, \mu_R \sqcup \nu_R)$ need not be in $\mathfrak{D}_{GBR}(Par)$. For example, if $\mu = (3,3,2)$, $\nu = (3,1)$, then we have

$\mathcal{B}(\mu)$ $\mathcal{C}(\mu, GB)$ $\mathcal{C}(\mu, G)$ $\mathcal{C}(\mu, R)$

$\mathcal{B}(\nu)$ $\mathcal{C}(\nu, GB)$ $\mathcal{C}(\nu, G)$ $\mathcal{C}(\nu, R)$

$\mathcal{B}((\mu \sqcup \nu))$ $\mathcal{C}((\mu \sqcup \nu), GB)$ $\mathcal{C}((\mu \sqcup \nu), G)$ $\mathcal{C}((\mu \sqcup \nu), R)$



$T((\mu \sqcup \nu)_{GB})$ $\quad T(\mu_{GB} \sqcup \nu_{GB})$ $\quad T(\mu_G \sqcup \nu_G)$ $\quad T((\mu \sqcup \nu)_R)$ $\quad T(\mu_R \sqcup \nu_R)$

Thus, $((\mu \sqcup \nu)_{GB}, (\mu \sqcup \nu)_G), (\mu \sqcup \nu)_R) \neq (\mu_{GB} \sqcup \nu_{GB}, \mu_G \sqcup \nu_G, \mu_R \sqcup \nu_R)$ and $(\mu_{GB} \sqcup \nu_{GB}, \mu_G \sqcup \nu_G, \mu_R \sqcup \nu_R) \notin \mathfrak{D}_{GBR}(Par)$.

4. Algebraic Structures

Recall that for any $n, m_1, \cdots, m_t \in \mathbb{N}$ with $\sum_{i=1}^{t} m_i = n$ and $t \geq 2$, the multinomial coefficient, denoted by $\binom{n}{m_1, \cdots, m_t}$, is defined by

$$\binom{n}{m_1, \cdots, m_t} = \frac{(\sum_{i=1}^{t} m_i)!}{(m_1)! \cdots (m_t)!}$$

Let η be the map

$$\eta : \Gamma \otimes \Gamma \to \Gamma, \quad \rho_\lambda \otimes \rho_{\lambda'} \mapsto \binom{|\lambda| + |\lambda'|}{|\lambda|, |\lambda'|} \rho_{\lambda \sqcup \lambda'},$$

and let

$$\mathbf{k} = \Gamma_0 \xrightarrow{u} \Gamma$$

be the inclusion map. We have the following proposition.

Proposition 4.1. *The triple* (Γ, η, u) *is a* $\mathbf{k}$*-algebra.*

Proof. Consider the following diagrams:

$$\begin{tikzcd}
\Gamma \otimes \Gamma \otimes \Gamma \arrow[r, "id\otimes\eta"] \arrow[d, "\eta\otimes id"'] & \Gamma \otimes \Gamma \arrow[d, "\eta"] \\
\Gamma \otimes \Gamma \arrow[r, "\eta"'] & \Gamma
\end{tikzcd}
\qquad
\begin{tikzcd}
\Gamma \otimes \mathbf{k} \arrow[d, "id\otimes u"'] & \Gamma \arrow[l, "\sim"'] \arrow[r, "\sim"] \arrow[d, "id"'] & \mathbf{k} \otimes \Gamma \arrow[d, "u\otimes id"] \\
\Gamma \otimes \Gamma \arrow[r, "\eta"] & \Gamma & \Gamma \otimes \Gamma \arrow[l, "\eta"']
\end{tikzcd} \tag{4.1}$$

We have to show that they are commutative. Write $\rho_\lambda \odot \rho_{\lambda'} \; = \; \eta(\rho_\lambda \otimes \rho_{\lambda'})$.

$$\begin{aligned}
(\rho_\lambda \odot \rho_{\lambda'}) \odot \rho_{\lambda''} &= (\binom{|\lambda| + |\lambda'|}{|\lambda|, |\lambda'|} \rho_{\lambda \sqcup \lambda'}) \odot \rho_{\lambda''} \\
&= \binom{|\lambda| + |\lambda'|}{|\lambda|, |\lambda'|} (\rho_{\lambda \sqcup \lambda'} \odot \rho_{\lambda''}) \\
&= \binom{|\lambda| + |\lambda'|}{|\lambda|, |\lambda'|} \binom{|\lambda \sqcup \lambda'| + |\lambda''|}{|\lambda \sqcup \lambda'|, |\lambda''|} \rho_{\lambda \sqcup \lambda' \sqcup \lambda''} \\
&= \binom{|\lambda| + |\lambda'|}{|\lambda|, |\lambda'|} \binom{|\lambda| + |\lambda'| + |\lambda''|}{|\lambda| + |\lambda'|, |\lambda''|} \rho_{\lambda \sqcup \lambda' \sqcup \lambda''} \\
&= \frac{(|\lambda| + |\lambda'| + |\lambda''|)!}{(|\lambda|)!\,(|\lambda'|)!\,(|\lambda''|)!} \rho_{\lambda \sqcup \lambda' \sqcup \lambda''} \\
&= \binom{|\lambda| + |\lambda'| + |\lambda''|}{|\lambda|, |\lambda'|, |\lambda''|} \rho_{\lambda \sqcup \lambda' \sqcup \lambda''}
\end{aligned}$$

On the other hand, we have

$$\begin{aligned}
\rho_\lambda \odot (\rho_{\lambda'} \odot \rho_{\lambda''}) &= \rho_\lambda \odot (\binom{|\lambda'| + |\lambda''|}{|\lambda'|, |\lambda''|} \rho_{\lambda' \sqcup \lambda''}) \\
&= \binom{|\lambda'| + |\lambda''|}{|\lambda'|, |\lambda''|} (\rho_\lambda \odot \rho_{\lambda' \sqcup \lambda''}) \\
&= \binom{|\lambda'| + |\lambda''|}{|\lambda'|, |\lambda''|} (\binom{|\lambda| + |\lambda' \sqcup \lambda''|}{|\lambda|, |\lambda' \sqcup \lambda''|} \rho_{\lambda \sqcup (\lambda' \sqcup \lambda'')} \\
&= \binom{|\lambda'| + |\lambda''|}{|\lambda'|, |\lambda''|} (\binom{|\lambda| + |\lambda'| + |\lambda''|}{|\lambda|, |\lambda'| + |\lambda''|} \rho_{\lambda \sqcup \lambda' \sqcup \lambda''} \\
&= \binom{|\lambda| + |\lambda'| + |\lambda''|}{|\lambda|, |\lambda'|, |\lambda''|} \rho_{\lambda \sqcup \lambda' \sqcup \lambda''}.
\end{aligned}$$

Accordingly, we have $(\rho_\lambda \odot \rho_{\lambda'}) \odot \rho_{\lambda''} = \rho_\lambda \odot (\rho_{\lambda'} \odot \rho_{\lambda''})$ for any $\lambda, \lambda', \lambda'' \in Par$, and hence the commutativity of the first diagram (the associativity diagram) of (4.1) follows. For the other diagram, we note that

$$\begin{aligned}
\eta(u \otimes id)(\rho_\lambda \otimes 1) &= \eta(\rho_\lambda \otimes 1) \\
&= \binom{|\lambda| + |\emptyset|}{|\lambda|, |\emptyset|} \rho_{\lambda \sqcup \emptyset} \\
&= \binom{|\lambda| + 0}{|\lambda|, 0} \rho_\lambda \\
&= \binom{|\lambda|}{|\lambda|, 0} \rho_\lambda
\end{aligned}$$

$$\begin{aligned}
&= \rho_\lambda \\
&= id(\rho_\lambda) \\
&= \binom{0+|\lambda|}{0,|\lambda|}\rho_\lambda \\
&= \binom{|\emptyset|+|\lambda|}{|\emptyset|,|\lambda|}\rho_{\emptyset\sqcup\lambda} \\
&= \eta(1\otimes\rho_\lambda) \\
&= \eta(id\otimes u)(1\otimes\rho_\lambda)
\end{aligned}$$

As a consequence, (Γ,η,ϵ) is a **k**-algebra. □

Definition 4.2. Let $Par^e = \{\lambda \in Par \ : \ \text{all } \lambda\text{- parts are even}\}$, and let $\Gamma^{(e,n)}(\mathbf{k})$ be the free **k**-module with the basis $\{\rho_\lambda\}_{\lambda\in Par^{(e,n)}}$, where $Par^{(e,n)} = Par_n \bigcap Par^e$. Let $\Gamma^e(\mathbf{k}) = \bigoplus_{n\geq 0}\Gamma^{(e,n)}(\mathbf{k})$. Then the set $\{\rho_\lambda\}_{\lambda\in Par^e}$ forms a basis for $\Gamma^e(\mathbf{k})$ over **k**.

The proof of the following lemma is straightforward and left to the reader.

Lemma 4.3. *We have $(\lambda+\lambda')_{GB} = \lambda_{GB}+\lambda'_{GB}$ and $(\lambda+\lambda')_R = \lambda_R+\lambda'_R$ for every $\lambda,\lambda'\in Par^e$.*

The following theorem emphasizes the importance of Definition (4.2)

Theorem 4.4. *We have the following:*

(1) $\binom{|(\lambda+\lambda')_{GB}|}{|\lambda_{GB}|} = \binom{|\lambda_{GB}|+|\lambda'_{GB}|}{|\lambda_{GB}|} = \binom{|\lambda_{GB}|+|\lambda'_{GB}|}{|\lambda_{GB}|,|\lambda'_{GB}|}$ *for any* $\lambda,\lambda'\in Par^e$.

(2) $\binom{|(2(\lambda+\lambda'))_{GB}|}{|2\lambda_{GB}|} = \binom{2(|\lambda_{GB}|+|\lambda'_{GB}|)}{2|\lambda_{GB}|} = \binom{2(|\lambda_{GB}|+|\lambda'_{GB}|)}{2|\lambda'_{GB}|} = \binom{2(|\lambda_{GB}|+|\lambda'_{GB}|)}{2|\lambda_{GB}|,2|\lambda'_{GB}|}$ *for any* $\lambda,\lambda'\in Par$.

(3) $\binom{|(\lambda+\lambda')_{GB}|}{|\lambda_{GB}|}\binom{|(\lambda+\lambda'+\lambda'')_{GB}|}{|(\lambda+\lambda')_{GB}|} = \binom{|\lambda_{GB}|+|\lambda'_{GB}|+|\lambda''_{GB}|}{|\lambda_{GB}|,|\lambda'_{GB}|,|\lambda''_{GB}|}$ *for any* $\lambda,\lambda',\lambda''\in Par^e$.

(4) *In general, we have*

$$\binom{|(\lambda^{(1)}+\lambda^{(2)}+\ldots+\lambda^{(t)})_{GB}|}{|\lambda^{(1)}_{GB}|,|\lambda^{(2)}_{GB}|,\ldots,|\lambda^{(t)}_{GB}|} = \binom{|\lambda^{(1)}_{GB}|+|\lambda^{(2)}_{GB}|+\ldots+|\lambda^{(t)}_{GB}|}{|\lambda^{(1)}_{GB}|,|\lambda^{(2)}_{GB}|,\ldots,|\lambda^{(t)}_{GB}|}$$

for every $(\lambda^{(1)},\lambda^{(2)},...,\lambda^{(t)})\in Par^e$, *and*

$$\binom{|(2(\lambda^{(1)}+\lambda^{(2)}+\ldots+\lambda^{(t)}))_{GB}|}{2|\lambda^{(1)}_{GB}|,2|\lambda^{(2)}_{GB}|,\ldots,2|\lambda^{(t)}_{GB}|} = \binom{2|\lambda^{(1)}_{GB}|+2|\lambda^{(2)}_{GB}|+\ldots+2|\lambda^{(t)}_{GB}|}{2|\lambda^{(1)}_{GB}|,2|\lambda^{(2)}_{GB}|,\ldots,2|\lambda^{(t)}_{GB}|}$$

for every $(\lambda^{(1)},\lambda^{(2)},...,\lambda^{(t)})\in Par$.

(5) *The triple* $(\Gamma^e(\mathbf{k}), \eta_e, u_e)$ *is a* $\mathbf{k}$*-algebra, where* η_e *is the map*

$$\eta_e \,:\, \Gamma^e(\mathbf{k}) \otimes \Gamma^e(\mathbf{k}) \to \Gamma^e(\mathbf{k}), \;\; \rho_\lambda \otimes \rho_{\lambda'} \;\mapsto\; \binom{|(\lambda+\lambda')_{GB}|}{|\lambda_{GB}|}\rho_{\lambda\sqcup\lambda'}$$

and

$$\mathbf{k} = \Gamma^{(e,0)}(\mathbf{k}) \xrightarrow{u_e} \Gamma^e(\mathbf{k})$$

is the inclusion map.

Proof. (1) We have

$$\begin{aligned}
\binom{|(\lambda+\lambda')_{GB}|}{|\lambda_{GB}|} &= \binom{|\lambda_{GB}+\lambda'_{GB}|}{|\lambda_{GB}|} \text{ (by using Lemma (4.3))}\\
&= \binom{|\lambda_{GB}|+|\lambda'_{GB}|}{|\lambda_{GB}|}\\
&= \binom{|\lambda_{GB}|+|\lambda'_{GB}|}{|\lambda_{GB}|,\, |\lambda'_{GB}|}.
\end{aligned}$$

(2) An easy calculation gives the following:

$$\begin{aligned}
\binom{|(2(\lambda+\lambda'))_{GB}|}{|2\lambda_{GB}|} &= \binom{2|(\lambda+\lambda')_{GB}|}{2|\lambda_{GB}|} \text{ (since } |2\lambda| = 2|\lambda|,\ \forall \lambda \in Par)\\
&= \binom{2(|\lambda_{GB}|+|\lambda'_{GB}|)}{2|\lambda_{GB}|} \text{ (by using Lemma (4.3))}\\
&= \binom{2(|\lambda_{GB}|+|\lambda'_{GB}|)}{2|\lambda'_{GB}|}\\
&= \binom{2(|\lambda_{GB}|+|\lambda'_{GB}|)}{2|\lambda_{GB}|,\, 2|\lambda'_{GB}|}.
\end{aligned}$$

(3) We calculate

$$\begin{aligned}
\binom{|(\lambda+\lambda')_{GB}|}{|\lambda_{GB}|}\binom{|(\lambda+\lambda'+\lambda'')_{GB}|}{|(\lambda+\lambda')_{GB}|} &= \binom{|\lambda_{GB}|+|\lambda'_{GB}|}{|\lambda_{GB}|}\binom{|\lambda_{GB}|+|\lambda'_{GB}|+|\lambda''_{GB}|}{|\lambda_{GB}|+|\lambda'_{GB}|}\\
&= \frac{(|\lambda_{GB}|+|\lambda'_{GB}|)!}{(|\lambda_{GB}|)!(|\lambda'_{GB}|)!}\;\frac{(|\lambda_{GB}|+|\lambda'_{GB}|+|\lambda''_{GB}|)!}{(|\lambda_{GB}|+|\lambda'_{GB}|)!(|\lambda''_{GB}|)!}\\
&= \frac{(|\lambda_{GB}|+|\lambda'_{GB}|+|\lambda''_{GB}|)!}{(|\lambda_{GB}|)!(|\lambda'_{GB}|)!(|\lambda''_{GB}|)!}\\
&= \binom{|\lambda_{GB}|+|\lambda'_{GB}|+|\lambda''_{GB}|}{|\lambda_{GB}|, |\lambda'_{GB}|, |\lambda''_{GB}|}.
\end{aligned}$$

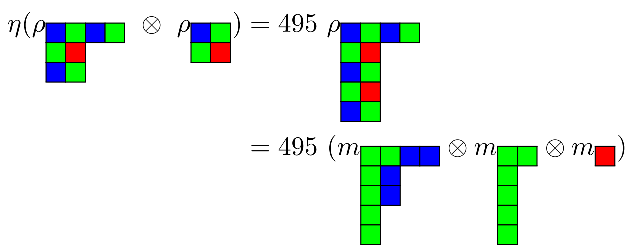
(4) This follows immediately from the proof of the previous part.

(5) This can be easily proved using parts (ii) and (iii) of the proposition.

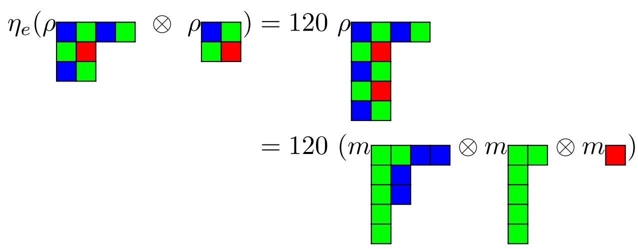
$\square$

Example 4.5.

(1) Let $\lambda = (4,2,2)$ and $\lambda' = (2,2)$. Then we have



(2) A direct calculation gives the following:



5. Coalgebraic Structures

Consider the map

$$\Delta\rho_\lambda = \sum_{\substack{(\mu,\nu)\in Par\times Par:\\ \mu\sqcup\nu=\lambda}} \rho_\mu \otimes \rho_\nu, \tag{5.1}$$

in which $\mu \sqcup \nu$ is the partition obtained by taking the multiset union of the parts of μ and ν, and then reordering them to make them weakly decreasing. Interestingly, one might define the map $\tilde{\Delta} : \Gamma \to \Gamma \otimes \Gamma$ defined $\mathbf{k}$-linearly by

$$\tilde{\Delta}\rho_\lambda = \sum_{\substack{(\mu,\nu)\in Par\times Par:\\ \mu_U\sqcup\nu_U=\lambda_U, \forall U\in\{GB,R\}}} \binom{|\mu|+|\nu|}{|\mu|,|\nu|} \rho_\mu \otimes \rho_\nu, \tag{5.2}$$

in which $\mu_U \sqcup \nu_U$ is the partition obtained by taking the multiset union of the parts of μ_U and ν_U, and then reordering them to make them weakly decreasing. From image processing point of view, we find the image noise corresponding to the Bayer Noise Young diagram of λ, and then we split the resulting one into pieces: one with less noise having only two color sensors (G and B), and one with more noise having only one color sensor R. We have the following theorem.

Theorem 5.1. *Let* $\Gamma \xrightarrow{\epsilon} \mathbf{k}$ *be the map defined* $\mathbf{k}$*-linearly by*

$$\epsilon|_{\Gamma_0 = \mathbf{k}} = id_{\mathbf{k}} \text{ and } \epsilon|_{I = \bigoplus_{n>0} \Gamma_n} = 0.$$

Then

(i) The triple $(\Gamma, \Delta, \epsilon)$ *is a* $\mathbf{k}$*-coalgebra.*

(ii) The triple $(\Gamma, \tilde{\Delta}, \epsilon)$ *is a* $\mathbf{k}$*-coalgebra.*

Proof. The proof of (i) is obvious. To prove part (ii), we have to show the following diagrams are commutative.

$$\begin{tikzcd} & \Gamma \otimes \Gamma \otimes \Gamma & \\ \Gamma \otimes \Gamma \arrow[ur, "\tilde{\Delta} \otimes id"] & & \Gamma \otimes \Gamma \arrow[ul, "id \otimes \tilde{\Delta}"'] \\ & \Gamma \arrow[ul, "\tilde{\Delta}"] \arrow[ur, "\tilde{\Delta}"'] & \end{tikzcd} \qquad \begin{tikzcd} \Gamma \otimes \mathbf{k} \arrow[r, "\Phi"] & \Gamma & \mathbf{k} \otimes \Gamma \arrow[l, "\Psi"'] \\ \Gamma \otimes \Gamma \arrow[u, "id \otimes \epsilon"] & \Gamma \arrow[l, "\tilde{\Delta}"] \arrow[r, "\tilde{\Delta}"'] \arrow[u, "id"] & \Gamma \otimes \Gamma \arrow[u, "\epsilon \otimes id"'] \end{tikzcd} \tag{5.3}$$

Here Φ and Ψ are the isomorphisms $\Phi : \Gamma \otimes \mathbf{k} \to \Gamma,\ \rho_\lambda \otimes 1 \mapsto \rho_\lambda$ and $\Psi : \mathbf{k} \otimes \Gamma \to \Gamma,\ 1 \otimes \rho_\lambda \mapsto \rho_\lambda$. For any $\lambda \in Par$, we have

$$\begin{aligned}
(\tilde{\Delta} \otimes id)\tilde{\Delta}\rho_\lambda &= (\tilde{\Delta} \otimes id)\Big(\sum_{\substack{(\mu,\mu') \in Par \times Par: \\ \mu_U \sqcup \mu'_U = \lambda_U, \forall U \in \{GB,R\}}} \binom{|\mu| + |\mu'|}{|\mu|, |\mu'|} \rho_\mu \otimes \rho_{\mu'} \Big) \\
&= \sum_{\substack{(\mu,\mu') \in Par \times Par: \\ \mu_U \sqcup \mu'_U = \lambda_U, \forall U \in \{GB,R\}}} \binom{|\mu| + |\mu'|}{|\mu|, |\mu'|} \tilde{\Delta}\rho_\mu \otimes \rho_{\mu'} \\
&= \sum_{\substack{(\mu,\mu') \in Par \times Par: \\ \mu_U \sqcup \mu'_U = \lambda_U, \forall U \in \{GB,R\}}} \binom{|\mu| + |\mu'|}{|\mu|, |\mu'|} \sum_{\substack{(\nu,\nu') \in Par \times Par: \\ \nu_W \sqcup \nu'_W = \mu_W, \forall W \in \{GB,R\}}} \binom{|\nu| + |\nu'|}{|\nu|, |\nu'|} (\rho_\nu \otimes \rho_{\nu'}) \otimes \rho_{\mu'} \\
&= \sum_{\substack{(\nu,\nu',\mu') \in Par \times Par \times Par: \\ \nu_U \sqcup \nu'_U \sqcup \mu'_U = \lambda_U, \forall U \in \{GB,R\}}} \binom{|\mu| + |\mu'|}{|\mu|, |\mu'|} \binom{|\nu| + |\nu'|}{|\nu|, |\nu'|} \rho_\nu \otimes \rho_{\nu'} \otimes \rho_{\mu'}
\end{aligned}$$

One can easily check the following:

$$|\lambda| = |\lambda_{GB}| + |\lambda_R| = |\mu_{GB}| + |\mu'_{GB}| + |\mu_R| + |\mu'_R| = |\mu| + |\mu'|$$

and

$$|\mu| = |\mu_{GB}| + |\mu_R| = |\nu_{GB}| + |\nu'_{GB}| + |\nu_R| + |\nu'_R| = |\nu| + |\nu'|.$$

As a result, we have

$$\begin{aligned}\binom{|\mu| + |\mu'|}{|\mu|, |\mu'|}\binom{|\nu| + |\nu'|}{|\nu|, |\nu'|} &= \binom{|\nu| + |\nu'| + |\mu'|}{|\mu|, |\mu'|}\binom{|\mu|}{|\nu|, |\nu'|} \\ &= \binom{|\nu| + |\nu'| + |\mu'|}{|\nu|, |\nu'|, |\mu'|} \text{ (by part (2) of Proposition (4.1)).}\end{aligned}$$

Thus, we have

$$\begin{aligned}(\tilde{\Delta} \otimes id)\tilde{\Delta}\rho_\lambda &= \sum_{\substack{(\nu,\nu',\mu') \in Par \times Par \times Par: \\ \nu_U \sqcup \nu'_U \sqcup \mu'_U = \lambda_U, \forall U \in \{GB,R\}}} \binom{|\nu| + |\nu'| + |\mu'|}{|\nu|, |\nu'|, |\mu'|} \rho_\nu \otimes \rho_{\nu'} \otimes \rho_{\mu'} \\ &= \sum_{\substack{(\mu,\mu') \in Par \times Par: \\ \mu_U \sqcup \mu'_U = \lambda_U, \forall U \in \{GB,R\}}} \binom{|\mu| + |\mu'|}{|\mu|, |\mu'|} \sum_{\substack{(\nu,\nu') \in Par \times Par: \\ \nu_W \sqcup \nu'_W = \mu'_W, \forall W \in \{GB,R\}}} \binom{|\nu| + |\nu'|}{|\nu|, |\nu'|} \rho_\mu \otimes (\rho_\nu \otimes \rho_{\nu'}) \\ &= \sum_{\substack{(\mu,\mu') \in Par \times Par: \\ \mu_U \sqcup \mu'_U = \lambda_U, \forall U \in \{GB,R\}}} \binom{|\mu| + |\mu'|}{|\mu|, |\mu'|} \rho_\mu \otimes \sum_{\substack{(\nu,\nu') \in Par \times Par: \\ \nu_W \sqcup \nu'_W = \mu'_W, \forall W \in \{GB,R\}}} \binom{|\nu| + |\nu'|}{|\nu|, |\nu'|} (\rho_\nu \otimes \rho_{\nu'}) \\ &= \sum_{\substack{(\mu,\mu') \in Par \times Par: \\ \mu_U \sqcup \mu'_U = \lambda_U, \forall U \in \{GB,R\}}} \binom{|\mu| + |\mu'|}{|\mu|, |\mu'|} \rho_\mu \otimes \tilde{\Delta}\rho_{\mu'} \\ &= (id \otimes \tilde{\Delta})\Big(\sum_{\substack{(\mu,\mu') \in Par \times Par: \\ \mu_U \sqcup \mu'_U = \lambda_U, \forall U \in \{GB,R\}}} \binom{|\mu| + |\mu'|}{|\mu|, |\mu'|} \rho_\mu \otimes \rho_{\mu'}\Big) \\ &= (id \otimes \tilde{\Delta})\tilde{\Delta}\rho_\lambda.\end{aligned}$$

Therefore, the commutativity of the associativity diagram follows. Checking the commutativity of the unity diagram can be done as follows:

$$\begin{aligned}
\Psi(\epsilon \otimes id)\tilde{\Delta}\rho_\lambda &= \Psi(\epsilon \otimes id)\Big(\sum_{\substack{(\mu,\mu')\in Par\times Par:\\ \mu_U \sqcup \mu'_U = \lambda_U, \forall U\in\{GB,R\}}} \binom{|\mu|+|\mu'|}{|\mu|,|\mu'|} \rho_\mu \otimes \rho_{\mu'}\Big) \\
&= \Psi\Big(\sum_{\substack{(\mu,\mu')\in Par\times Par:\\ \mu_U \sqcup \mu'_U = \lambda_U, \forall U\in\{GB,R\}}} \binom{|\mu|+|\mu'|}{|\mu|,|\mu'|} \epsilon(\rho_\mu) \otimes \rho_{\mu'}\Big) \\
&= \sum_{\substack{(\mu,\mu')\in Par\times Par:\\ \mu_U \sqcup \mu'_U = \lambda_U, \forall U\in\{GB,R\}}} \binom{|\mu|+|\mu'|}{|\mu|,|\mu'|} \epsilon(\rho_\mu)\, \rho_{\mu'} \\
&= \rho_\lambda \text{ (since } \epsilon|_{\mathbf{k}} = id_{\mathbf{k}} \text{ and } \epsilon|_{I=\bigoplus_{n>0}\Gamma_n} = 0). \\
&= id(\rho_\lambda) \\
&= \sum_{\substack{(\mu,\mu')\in Par\times Par:\\ \mu_U \sqcup \mu'_U = \lambda_U, \forall U\in\{GB,R\}}} \binom{|\mu|+|\mu'|}{|\mu|,|\mu'|} \rho_\mu\, \epsilon(\rho_{\mu'}) \\
&= \Phi\Big(\sum_{\substack{(\mu,\mu')\in Par\times Par:\\ \mu_U \sqcup \mu'_U = \lambda_U, \forall U\in\{GB,R\}}} \binom{|\mu|+|\mu'|}{|\mu|,|\mu'|} \rho_\mu \otimes \epsilon(\rho_{\mu'})\Big) \\
&= \Phi(id \otimes \epsilon)\Big(\sum_{\substack{(\mu,\mu')\in Par\times Par:\\ \mu_U \sqcup \mu'_U = \lambda_U, \forall U\in\{GB,R\}}} \binom{|\mu|+|\mu'|}{|\mu|,|\mu'|} \rho_\mu \otimes \rho_{\mu'}\Big) \\
&= \Phi(id \otimes \epsilon)\tilde{\Delta}\rho_\lambda.
\end{aligned}$$

It follows that $(\Gamma, \tilde{\Delta}, \epsilon)$ is a **k**-coalgebra. ☐

We call the **k**-coalgebra $(\Gamma, \tilde{\Delta}, \epsilon)$ as the *Bayer Noise coalgebra* over **k**. The following proposition gives an explicit description for primitive with respect to the comultiplication $\tilde{\Delta}$.

Proposition 5.2. *Let $\lambda \in Par$. The element ρ_λ is primitive (with respect to $\tilde{\Delta}$) if and only if $\lambda = (m)$ for some non-negative integer m.*

Proof. It is straightforward to prove that $\tilde{\Delta}\rho_\lambda = \rho_\lambda \otimes 1 + 1 \otimes \rho_\lambda$ if and only if $\lambda = (m)$ for some non-negative integer m. This completes the proof. ☐

Let $\widehat{\Delta} : \Gamma^e(\mathbf{k}) \to \Gamma^e(\mathbf{k}) \otimes \Gamma^e(\mathbf{k})$ be the map defined **k**-linearly by

$$\widehat{\Delta}\rho_\lambda = \sum_{\substack{(\mu,\nu)\in Par^e\times Par^e:\\ \mu_U \sqcup \nu_U = \lambda_U, \forall U\in\{GB,R\}}} \binom{|\mu|+|\nu|}{|\mu|,|\nu|} \rho_\mu \otimes \rho_\nu, \tag{5.4}$$

in which $\mu_U \sqcup \nu_U$ is the partition obtained by taking the multiset union of the parts of μ_U and ν_U, and then reordering them to make them weakly decreasing.

Using part (3) of Theorem (4.4), the following theorem can be proved similarly to the proof of Theorem (5.1).

Theorem 5.3. *The triple* $(\Gamma^e(\mathbf{k}), \widehat{\Delta}, \widehat{\epsilon})$ *is a* $\mathbf{k}$*-coalgebra, where* $\Gamma \overset{\widehat{\epsilon}}{\to} \mathbf{k}$ *is the map defined* $\mathbf{k}$*-linearly by*

$$\widehat{\epsilon}|_{\Gamma^{(e,0)}=\mathbf{k}} = id_{\mathbf{k}} \text{ and } \widehat{\epsilon}|_{I=\bigoplus_{n>0}\Gamma^{(e,n)}} = 0.$$

The primitive elements in $\Gamma^e(\mathbf{k})$ (with respect to the comultiplication $\widehat{\Delta}$) can be explicitly described as follows:

Proposition 5.4. *The primitive basis elements for* $\Gamma^e(\mathbf{k})$ *(with respect to the comultiplication* $\widehat{\Delta}$*) are precisely of the form* ρ_λ*, where* $\lambda = (m)$ *for some* $m \in 2\mathbb{N} = \{0, 2, 4, \ldots\}$.

Proof. The proof is very similar to the proof of Proposition (5.2). □

Similarly, we define the map $\Delta^{(e)} : \Gamma \to \Gamma \otimes \Gamma$ defined $\mathbf{k}$-linearly by

$$\Delta^{(e)}\rho_\lambda = \sum_{\substack{(\mu,\nu)\in Par\times Par: \\ \mu_U \sqcup \nu_U = \lambda_U, \forall U \in \{GB,R\}}} \binom{2|\mu| + 2|\nu|}{2|\mu|, 2|\nu|} \rho_\mu \otimes \rho_\nu, \tag{5.5}$$

in which $\mu_U \sqcup \nu_U$ is the partition obtained by taking the multiset union of the parts of μ_U and ν_U, and then reordering them to make them weakly decreasing.
The following are analogous consequences to those of Theorem (5.3) and Proposition (5.4) respectively.

Theorem 5.5. *The triple* $(\Gamma, \Delta^{(e)}, \epsilon^{(e)})$ *is a* $\mathbf{k}$*-coalgebra, where* $\Gamma \overset{\epsilon^{(e)}}{\to} \mathbf{k}$ *is the map defined* $\mathbf{k}$*-linearly by*

$$\epsilon^{(e)}|_{\Gamma_0=\mathbf{k}} = id_{\mathbf{k}} \text{ and } \epsilon^{(e)}|_{I=\bigoplus_{n>0}\Gamma_n} = 0.$$

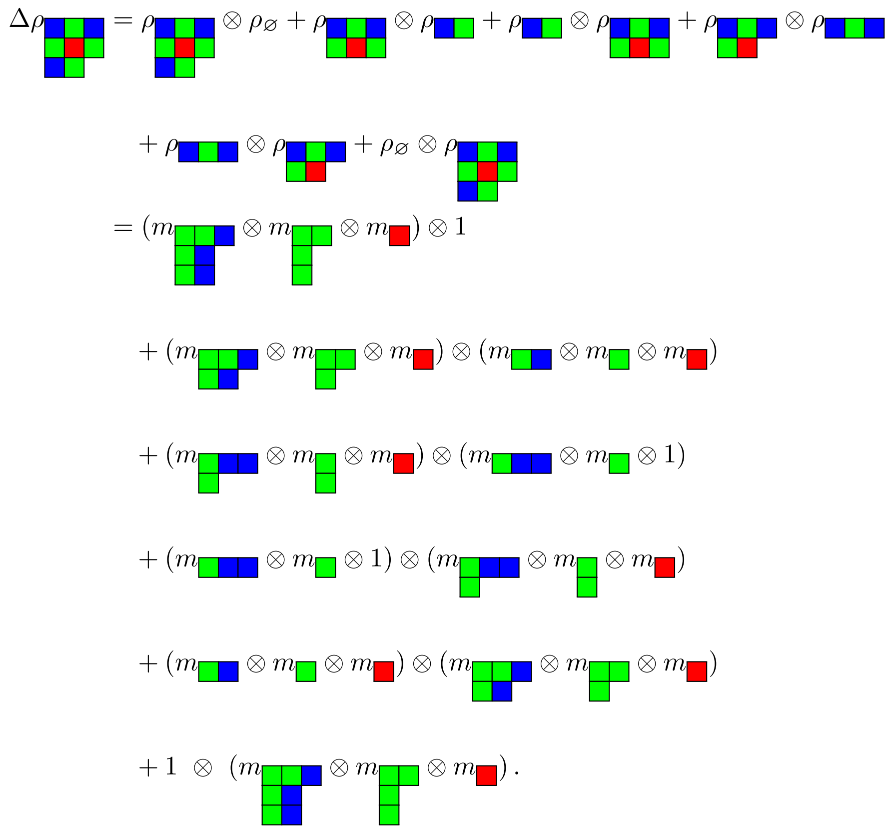
Proposition 5.6. *The primitive basis elements for* Γ *(with respect to the comultiplication* $\Delta^{(e)}$*) are precisely of the form* ρ_λ*, where* $\lambda = (m)$ *for some non-negative integer* m.

Example 5.7.

(1) Let $\lambda = (3,3,2)$. Then we have

$$\begin{aligned}
\Delta\rho_{\lambda} = &\Delta\rho_{(3,3,2)}\\
&= \rho_{(3,3,2)} \otimes \rho_{\varnothing} + \rho_{(3,3)} \otimes \rho_{(2)} + \rho_{(2)} \otimes \rho_{(3,3)} + \rho_{(3,2)} \otimes \rho_{(3)} + \rho_{(3)} \otimes \rho_{(3,2)} + \rho_{\varnothing} \otimes \rho_{(3,3,2)}\\
&= \rho_{(3,3,2)} \otimes 1 + \rho_{(3,3)} \otimes \rho_{(2)} + \rho_{(2)} \otimes \rho_{(3,3)} + \rho_{(3,2)} \otimes \rho_{(3)} + \rho_{(3)} \otimes \rho_{(3,2)} + 1 \otimes \rho_{(3,3,2)}\\
&= (m_{(3,2,2)} \otimes m_{(2,1,1)} \otimes m_{(1)}) \otimes 1 + (m_{(3,2)} \otimes m_{(2,1)} \otimes m_{(1)}) \otimes (m_{(2)} \otimes m_{(1)} \otimes 1)\\
&+ (m_{(2)} \otimes m_{(1)} \otimes 1) \otimes (m_{(3,2)} \otimes m_{(2,1)} \otimes m_{(1)}) + (m_{(3,1)} \otimes m_{(1,1)} \otimes 1) \otimes (m_{(3)} \otimes m_{(1)} \otimes 1)\\
&= (m_{(3,2,2)} \otimes m_{(2,1,1)} \otimes m_{(1)}) \otimes 1 + (m_{(3,2)} \otimes m_{(2,1)} \otimes m_{(1)}) \otimes (m_{(2)} \otimes m_{(1)} \otimes 1)\\
&+ (m_{(2)} \otimes m_{(1)} \otimes 1) \otimes (m_{(3,2)} \otimes m_{(2,1)} \otimes m_{(1)}) + (m_{(3,1)} \otimes m_{(1,1)} \otimes 1) \otimes (m_{(3)} \otimes m_{(1)} \otimes 1)\\
&+ (m_{(3)} \otimes m_{(1)} \otimes 1) \otimes (m_{(3,1)} \otimes m_{(1,1)} \otimes m_{(1)}) + 1 \otimes (m_{(3,2,2)} \otimes m_{(2,1,1)} \otimes m_{(1)}).
\end{aligned}$$

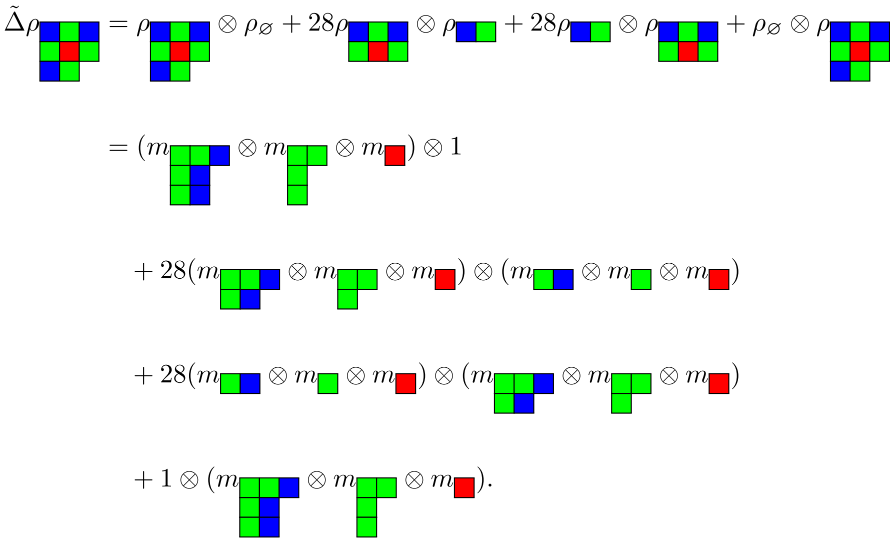
This can be pictured as



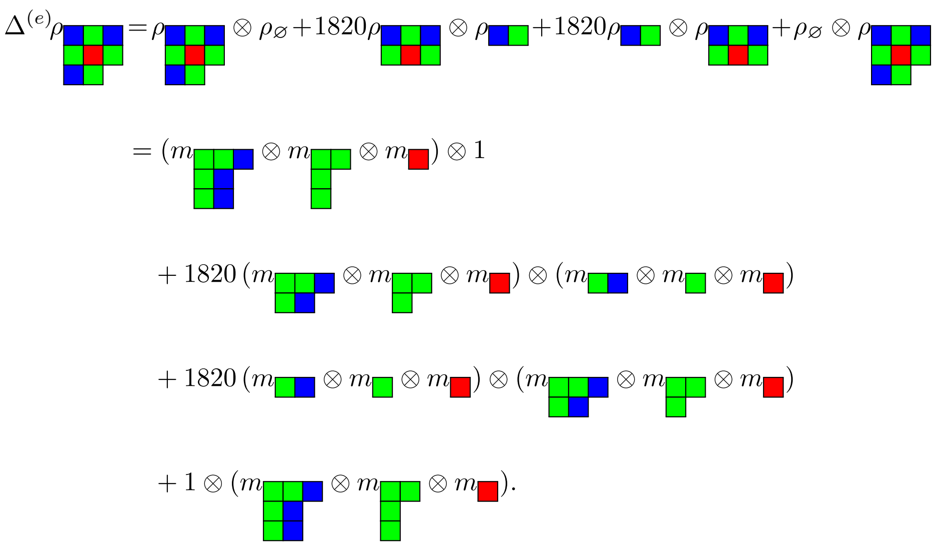
On the other hand, we have

$$
\begin{aligned}
\tilde{\Delta}\rho_{\lambda} &= \tilde{\Delta}\rho_{(3,3,2)} \\
&= \rho_{(3,3,2)} \otimes \rho_{\varnothing} + 28\rho_{(3,3)} \otimes \rho_{(2)} + 28\rho_{(2)} \otimes \rho_{(3,3)} + \rho_{\varnothing} \otimes \rho_{(3,3,2)} \\
&= \rho_{(3,3,2)} \otimes 1 + \rho_{(3,3)} \otimes \rho_{(2)} \quad + \rho_{(2)} \otimes \rho_{(3,3)} + 1 \otimes \rho_{(3,3,2)} \\
&= (m_{(3,2,2)} \otimes m_{(2,1,1)} \otimes m_{(1)}) \otimes 1 + (m_{(3,2)} \otimes m_{(2,1)} \otimes m_{(1)}) \otimes (m_{(2)} \otimes m_{(1)} \otimes m_{(1)}) \\
&\quad + (m_{(2)} \otimes m_{(1)} \otimes m_{(1)}) \otimes (m_{(3,2)} \otimes m_{(2,1)} \otimes m_{(1)}) + 1 \otimes (m_{(3,2,2)} \otimes m_{(2,1,1)} \otimes m_{(1)}).
\end{aligned}
$$

This can be visualized as



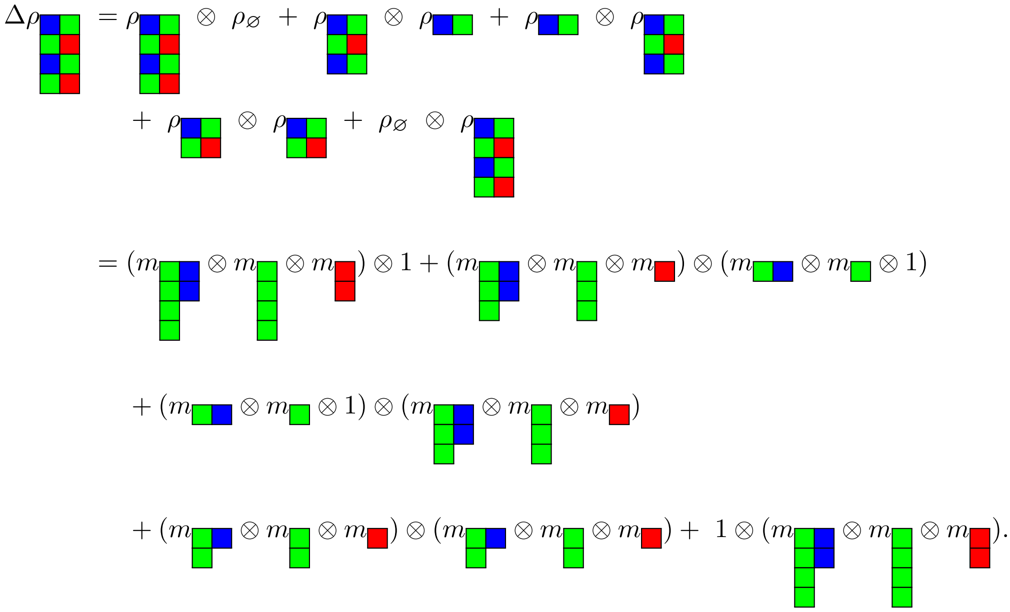
Similarly, one could visualize $\Delta^{(e)}\rho_{(3,3,2)}$ as follows:



(2) To see the difference between Δ, $\tilde{\Delta}$ and $\widehat{\Delta}$ more clearly, let $\lambda = (2,2,2,2)$. Clearly, we have

$$\begin{aligned}
\Delta\rho_\lambda &= \Delta\rho_{(2,2,2,2)} \\
&= \rho_{(2,2,2,2)} \otimes \rho_\varnothing + \rho_{(2,2,2)} \otimes \rho_{(2)} + \rho_{(2)} \otimes \rho_{(2,2,2)} + \rho_{(2,2)} \otimes \rho_{(2,2)} + \rho_\varnothing \otimes \rho_{(2,2,2,2)} \\
&= \rho_{(2,2,2,2)} \otimes 1 + \rho_{(2,2,2)} \otimes \rho_{(2)} + \rho_{(2)} \otimes \rho_{(2,2,2)} + \rho_{(2,2)} \otimes \rho_{(2,2)} + 1 \otimes \rho_{(2,2,2,2)} \\
&= (m_{(2,2,1,1)} \otimes m_{(1,1,1,1)} \otimes m_{(1,1)}) \otimes 1 + (m_{(2,2,1)} \otimes m_{(1,1,1)} \otimes m_{(1)}) \otimes (m_{(2)} \otimes m_{(1)} \otimes 1) \\
&\quad + (m_{(2)} \otimes m_{(1)} \otimes 1) \otimes (m_{(2,2,1)} \otimes m_{(1,1,1)} \otimes m_{(1)}) \\
&\quad + (m_{(2,1)} \otimes m_{(1,1)} \otimes m_{(1)}) \otimes (m_{(2,1)} \otimes m_{(1,1)} \otimes m_{(1)}) \\
&\quad + 1 \otimes (m_{(2,2,1,1)} \otimes m_{(1,1,1,1)} \otimes m_{(1,1)}),
\end{aligned}$$

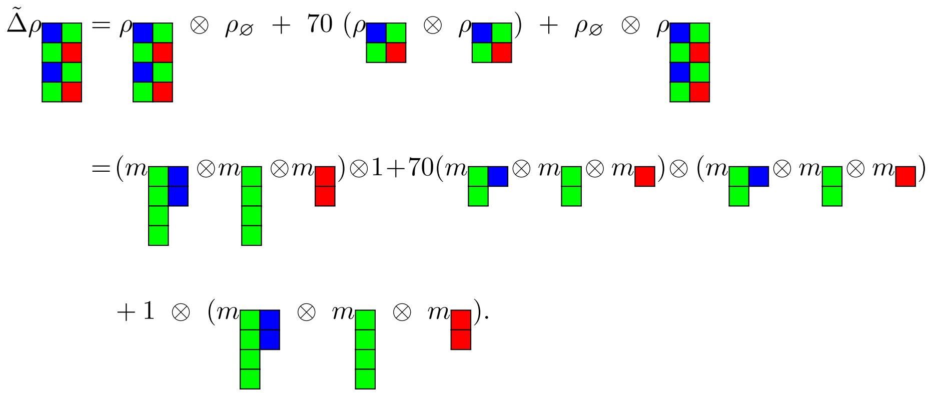
which can be visualized as the following.



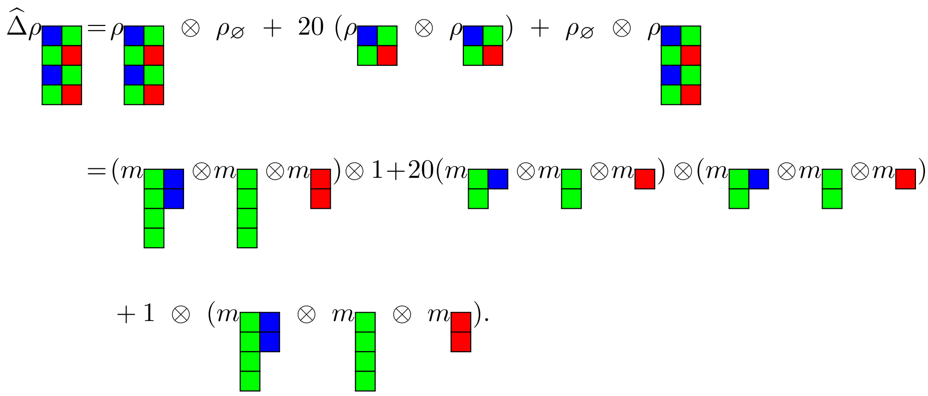
It is easy to check that $\tilde{\Delta}\rho_\lambda$ is given by

$$\begin{aligned}
\tilde{\Delta}\rho_\lambda &= \tilde{\Delta}\rho_{(2,2,2,2)} \\
&= \rho_{(2,2,2,2)} \otimes \rho_\varnothing + \rho_{(2,2)} \otimes \rho_{(2,2)} + \rho_\varnothing \otimes \rho_{(2,2,2,2)} \\
&= \rho_{(2,2,2,2)} \otimes 1 + 70\ (\rho_{(2,2)} \otimes \rho_{(2,2)}) + 1 \otimes \rho_{(2,2,2,2)} \\
&= (m_{(2,2,1,1)} \otimes m_{(1,1,1,1)} \otimes m_{(1,1)}) \otimes 1 \\
&\quad + 70\ (m_{(2,1)} \otimes m_{(1,1)} \otimes m_{(1)}) \otimes (m_{(2,1)} \otimes m_{(1,1)} \otimes m_{(1)}) \\
&\quad + 1 \otimes (m_{(2,2,1,1)} \otimes m_{(1,1,1,1)} \otimes m_{(1,1)})
\end{aligned}$$

One might visualize $\tilde{\Delta}\rho_\lambda$ as follows:



Notably, $\widehat{\Delta}\rho_{(2,2,2,2)}$ looks very similar to $\tilde{\Delta}\rho_{(2,2,2,2)}$. Indeed, the only difference between them is their coefficients. Explicitly, we have



Consider the diagrams:

$$
\begin{tikzcd}
\Gamma\otimes\Gamma \arrow[r, "\eta"] \arrow[d, "\Delta\otimes\Delta"'] & \Gamma \arrow[dd, "\Delta"'] \\
\Gamma\otimes\Gamma\otimes\Gamma\otimes\Gamma \arrow[d, "id\otimes\theta\otimes id"'] & \\
\Gamma\otimes\Gamma\otimes\Gamma\otimes\Gamma \arrow[r, "\eta\otimes\eta"'] & \Gamma\otimes\Gamma
\end{tikzcd}
\quad
\begin{tikzcd}
\Gamma\otimes\Gamma \arrow[r, "\epsilon\otimes\epsilon"] \arrow[d, "\eta"'] & \mathbf{k}\otimes\mathbf{k} \arrow[d, "\eta"] \\
\Gamma \arrow[r, "\epsilon"'] & \mathbf{k}
\end{tikzcd}
\quad
\begin{tikzcd}
\mathbf{k} \arrow[r, "u"] \arrow[d, "\Delta"'] & \Gamma \arrow[d, "\Delta"] \\
\mathbf{k}\otimes\mathbf{k} \arrow[r, "u\otimes u"'] & \Gamma\otimes\Gamma
\end{tikzcd}
\quad
\begin{tikzcd}
\mathbf{k} \arrow[rr, "id"] \arrow[dr, "u"'] & & \mathbf{k} \\
& \Gamma \arrow[ur, "\epsilon"'] &
\end{tikzcd}
\tag{5.6}
$$

where $\theta : \Gamma \otimes \Gamma \to \Gamma \otimes \Gamma$ is the twist map. The proof of the following consequence is obvious.

Proposition 5.8.

(1) *The last three diagrams of (5.6) commute while the map η needs not be a* $\mathbf{k}$*-algebra morphism.*

(2) *The last three diagrams of (5.6) are still commutative if one replaces Δ by $\tilde{\Delta}$.*

(3) *Let $CAlg_{\mathbf{k}}$ be the category of commutative* $\mathbf{k}$*-algebras. Then the assignment*

$$\mathscr{G} : CAlg_{\mathbf{k}} \to CAlg_{\mathbf{k}},\ R \mapsto \Gamma(R),\ R \stackrel{f}{\to} R' \mapsto (\Gamma(R) \stackrel{\epsilon_{\Gamma(R)}}{\to} R \stackrel{f}{\to} R' \stackrel{u_{\Gamma(R')}}{\to} \Gamma(R'))$$

defines a semiendofunctor of $CAlg_{\mathbf{k}}$. Furthermore, we have

$$\mathscr{G}(R \stackrel{id_R}{\to} R) = (\Gamma(R) \stackrel{\epsilon_{\Gamma(R)}}{\to} R \stackrel{id_R}{\to} R \stackrel{u_{\Gamma(R)}}{\to} \Gamma(R)) = u_{\Gamma(R)}\epsilon_{\Gamma(R)}$$

the convolutional identity element in $End(\Gamma(R))$.

Let $\lambda \in Par$. Write $\lambda^{(GB,0)} = \lambda$, $\lambda^{(GB,1)} = \lambda_{GB}$ and $\lambda^{(GB,2)} = (\lambda_{GB})_{GB} = (\lambda^{(GB,1)})_{GB}$. Inductively, we have $\lambda^{(GB,t)} = (\lambda^{(GB,t-1)})_{GB}$ for any $t \in \mathbb{N}$ with $t \geq 1$. Similarly, one could define $\lambda^{(R,t)}$.

Definition 5.9. Let $\lambda \in Par$.

(1) The GB-order of λ, denoted by $|\lambda|^{GB}$, is the least positive integer t with $\lambda^{(GB,t)} = (\lambda^{(GB,t-1)})_{GB}$. Note that $|\lambda|^{GB} \geq 1$.

(2) Let $t = |\lambda|^{GB}$. Define the sets

$$Par^{(GB)^t} = \{\lambda \in Par \,:\, |\lambda|^{GB} \leq t\}$$

and

$$Par^{(GB,n)^t} = \{\lambda \in Par_n \,:\, |\lambda|^{GB} \leq t\}.$$

(3) For any $n \in \mathbb{N}$, let $\boxplus(n)$ denote the partition defined by

$$\boxplus(n) = (\underbrace{n, n, \ldots, n}_{n \text{ times}}).$$

Example 5.10.

(1) To find $|\boxplus(4)|^{GB}$ and $|\boxplus(8)|^{GB}$, one might easily calculate

and

Thus, $|\boxplus(4)|^{GB} = 5$ and $|\boxplus(8)|^{GB} = 8$.

$\boxplus(4)^{(GB,0)}$	$\boxplus(4)^{(GB,1)}$	$\boxplus(4)^{(GB,2)}$	$\boxplus(4)^{(GB,3)}$	$\boxplus(4)^{(GB,4)}$
$\boxplus(4)$	$(4,4,2,2)$	$(4,2,2,1)$	$(4,2,1,1)$	$(4,1,1,1)$

$\boxplus(8)^{(GB,0)}$	$\boxplus(8)$
$\boxplus(8)^{(GB,1)}$	$(8,8,8,8,4,4,4,4)$
$\boxplus(8)^{(GB,2)}$	$(8,8,4,4,4,4,2,2)$
$\boxplus(8)^{(GB,3)}$	$(8,4,4,4,2,2,2,1)$
$\boxplus(8)^{(GB,4)}$	$(8,4,2,2,2,2,1,1)$
$\boxplus(8)^{(GB,5)}$	$(8,2,2,2,1,1,1,1)$
$\boxplus(8)^{(GB,6)}$	$(8,2,1,1,1,1,1,1)$
$\boxplus(8)^{(GB,7)}$	$(8,1,1,1,1,1,1,1)$

(2) Similarly, one could check that $|\boxplus(3)|^{GB} = 5$, $|\boxplus(5)|^{GB} = |\boxplus(6)|^{GB} = 8$ and $|\boxplus(10)|^{GB} = 11$.

Remark 5.11. Let $\lambda \in Par$ and $t \in \mathbb{N}$ with $t \geq 2$.

(1) Clearly, $|\lambda|^{GB} \leq |\lambda|$ for $\lambda \in Par$ with $|\lambda| \geq 1$.

(2) If $\lambda \in Par^{(GB)^t}$, then $\lambda_{GB} \in Par^{(GB)^{(t-1)}}$.

(3) If $\lambda \in Par^{(GB)^t}$, then $(\lambda^{(GB,t-1)})_R = \emptyset$. In particular, if $\lambda \in Par^{(GB)^2}$, then $(\lambda^{(GB,2)})_R = \emptyset$, $\lambda^{(R,2)} = \emptyset$ and $(\lambda^{(GB,1)})_R = \emptyset$.

Consider the map

$$\Gamma(\mathbf{k}) \xrightarrow{\Delta^{GB}} \Gamma(\mathbf{k}) \otimes \Gamma(\mathbf{k}) \tag{5.7}$$

defined $\mathbf{k}$-linearly by

$$\Delta^{GB}\rho_\lambda = \begin{cases} 1 \otimes 1 & \text{if } \lambda = \emptyset \\ \rho_{\lambda^{(GB,|\lambda|^{GB}-1)}} \otimes 1 + 1 \otimes \rho_{\lambda^{(GB,|\lambda|^{GB}-1)}} & \text{if } \lambda \neq \emptyset \end{cases}$$

We have the following proposition.

Proposition 5.12. $(\Gamma(\mathbf{k}), \Delta^{GB})$ *is a nonunital* $\mathbf{k}$*-coalgebra.*

Proof. We have to show that the following diagram is satisfied.

$$\begin{tikzcd} & (\Gamma(\mathbf{k}))^{\otimes 3} & \\ (\Gamma(\mathbf{k}))^{\otimes 2} \arrow[ur, "\Delta^{GB}\otimes id"] & & (\Gamma(\mathbf{k}))^{\otimes 2} \arrow[ul, "id\otimes\Delta^{GB}"'] \\ & \Gamma(\mathbf{k}) \arrow[ul, "\Delta^{GB}"] \arrow[ur, "\Delta^{GB}"'] & \end{tikzcd} \tag{5.8}$$

$$\begin{aligned}(\Delta^{GB} \otimes id)\Delta^{GB}\rho_\lambda &= (\Delta^{GB} \otimes id)(\rho_{\lambda^{(GB,|\lambda|^{GB}-1)}} \otimes 1 + 1 \otimes \rho_{\lambda^{(GB,|\lambda|^{GB}-1)}}) \\ &= \Delta^{GB}(\rho_{\lambda^{(GB,|\lambda|^{GB}-1)}}) \otimes 1 + \Delta^{GB}(1) \otimes \rho_{\lambda^{(GB,|\lambda|^{GB}-1)}} \\ &= (\rho_{\lambda^{(GB,|\lambda|^{GB})}} \otimes 1 + 1 \otimes \rho_{\lambda^{(GB,|\lambda|^{GB})}}) \otimes 1 + 1 \otimes 1 \otimes \rho_{\lambda^{(GB,|\lambda|^{GB}-1)}} \\ &= \rho_{\lambda^{(GB,|\lambda|^{GB})}} \otimes 1 \otimes 1 + 1 \otimes \rho_{\lambda^{(GB,|\lambda|^{GB})}} \otimes 1 + 1 \otimes 1 \otimes \rho_{\lambda^{(GB,|\lambda|^{GB}-1)}}\end{aligned}$$

$$\begin{aligned}(id \otimes \Delta^{GB})\Delta^{GB}\rho_\lambda &= (id \otimes \Delta^{GB})(\rho_{\lambda^{(GB,|\lambda|^{GB}-1)}} \otimes 1 + 1 \otimes \rho_{\lambda^{(GB,|\lambda|^{GB}-1)}}) \\ &= \rho_{\lambda^{(GB,|\lambda|^{GB}-1)}} \otimes \Delta^{GB}(1) + 1 \otimes \Delta^{GB}(\rho_{\lambda^{(GB,|\lambda|^{GB}-1)}}) \\ &= \rho_{\lambda^{(GB,|\lambda|^{GB}-1)}} \otimes (1 \otimes 1) + 1 \otimes (\rho_{\lambda^{(GB,|\lambda|^{GB})}} \otimes 1 \otimes 1 + 1 \otimes \rho_{\lambda^{(GB,|\lambda|^{GB})}}) \\ &= \rho_{\lambda^{(GB,|\lambda|^{GB})}} \otimes 1 \otimes 1 + 1 \otimes \rho_{\lambda^{(GB,|\lambda|^{GB})}} \otimes 1 + 1 \otimes 1 \otimes \rho_{\lambda^{(GB,|\lambda|^{GB})}}\end{aligned}$$

For any $\lambda \in Par$, $\lambda^{(GB,|\lambda|^{GB})} = \lambda^{(GB,|\lambda|^{GB}-1)}$ (by the definition of $|\lambda|^{GB}$). It follows that $(\Delta^{GB} \otimes id)\Delta^{GB} = (id \otimes \Delta^{GB})\Delta^{GB}$. Thus, the diagram (5.8) is commutative, and hence $(\Gamma(\mathbf{k}), \Delta^{GB})$ is a nonunital $\mathbf{k}$-coalgebra. $\square$

Definition 5.13. Fix $t \in \mathbb{N}$ with $t \geq 1$. Let $\Gamma^{(GB,n)^t}(\mathbf{k})$ be the free $\mathbf{k}$-module with the basis $\{\rho_\lambda\}_{\lambda \in Par^{(GB,n)^t}}$. Let $\Gamma^{(GB)^t}(\mathbf{k}) = \bigoplus_{n\geq 0} \Gamma^{(GB,n)^t}(\mathbf{k})$. Then the set $\{\rho_\lambda\}_{\lambda \in Par^{(GB)^t}}$ forms a basis for $\Gamma^{(GB)^t}(\mathbf{k})$ over $\mathbf{k}$, and $\Gamma^{(GB)^t}(\mathbf{k})$ is called the (GB,t)-*Bayer Noise module* over $\mathbf{k}$.

Now consider the map

$$\Gamma^{(GB)^t}(\mathbf{k}) \xrightarrow{\Delta^{(GB)^t}} \Gamma^{(GB)^t}(\mathbf{k}) \otimes \Gamma^{(GB)^t}(\mathbf{k}) \tag{5.9}$$

defined $\mathbf{k}$-linearly by

$$\Delta^{(GB)^t}\rho_\lambda = \begin{cases} 1 \otimes 1 & \text{if } \lambda = \emptyset \\ \rho_{\lambda^{(GB,t-1)}} \otimes 1 + 1 \otimes \rho_{\lambda^{(GB,t-1)}} & \text{if } \lambda \neq \emptyset. \end{cases}$$

We have the following proposition.

Proposition 5.14. *$(\Gamma^{(GB)^t}(\mathbf{k}), \Delta^{(GB)^t})$ is a nonunital $\mathbf{k}$-coalgebra.*

Proof. The proof is very similar to the proof of Proposition (5.12). $\square$

It is well known that the nonunital $\mathbf{k}$-coalgebras $(\Gamma(\mathbf{k}), \Delta^{GB})$ can be extended for a unital $\mathbf{k}$-coalgebra $(\overline{\Gamma(\mathbf{k})}, \overline{\Delta^{GB}}, \overline{\epsilon^{GB}})$, where $\overline{\Gamma(\mathbf{k})} = \Gamma(\mathbf{k}) \oplus \mathbf{k}$, and $\overline{\epsilon^{GB}} : \overline{\Gamma(\mathbf{k})} = \Gamma(\mathbf{k}) \oplus \mathbf{k} \to \mathbf{k}$ is the projection map, and $\overline{\Delta^{GB}}$ is the map

$$\overline{\Gamma(\mathbf{k})} \xrightarrow{\overline{\Delta^{GB}}} \overline{\Gamma(\mathbf{k})} \otimes \overline{\Gamma(\mathbf{k})} \tag{5.10}$$

defined by

$$\overline{\Delta^{GB}}(f+a) = \Delta^{GB}(f) + f \otimes 1 + 1 \otimes f + a(1 \otimes 1)$$

for any $f \in \overline{\Gamma(\mathbf{k})}$ and $a \in \mathbf{k}$. Similarly, the nonunital $(\Gamma^{(GB)^t}(\mathbf{k}), \Delta^{(GB)^t})$ can be extended for a unital $\mathbf{k}$-coalgebra $(\overline{\Gamma^{(GB)^t}(\mathbf{k})}, \overline{\Delta^{(GB)^t}}, \overline{\epsilon^{(GB)^t}})$, where $\overline{\Gamma^{(GB)^t}(\mathbf{k})} = \Gamma^{(GB)^t}(\mathbf{k}) \oplus \mathbf{k}$, and $\overline{\epsilon^{(GB)^t}} : \overline{\Gamma^{(GB)^t}(\mathbf{k})} = \Gamma^{(GB)^t}(\mathbf{k}) \oplus \mathbf{k} \to \mathbf{k}$ is the projection map, and $\overline{\Delta^{(GB)^t}}$ is the map

$$\overline{\Gamma^{(GB)^t}(\mathbf{k})} \xrightarrow{\overline{\Delta^{(GB)^t}}} \overline{\Gamma^{(GB)^t}(\mathbf{k})} \otimes \overline{\Gamma^{(GB)^t}(\mathbf{k})} \tag{5.11}$$

defined by

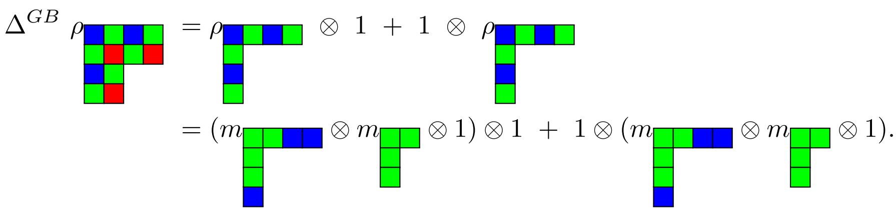
$$\overline{\Delta^{(GB)^t}}(f+a) = \Delta^{(GB)^t}(f) + f \otimes 1 + 1 \otimes f + a(1 \otimes 1)$$

for any $f \in \overline{\Gamma^{(GB)^t}(\mathbf{k})}$ and $a \in \mathbf{k}$. Consequently, we have the following.

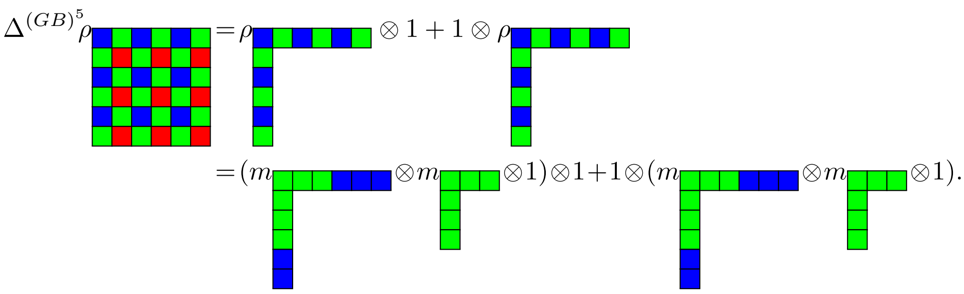
Proposition 5.15. $(\overline{\Gamma(\mathbf{k})}, \overline{\Delta^{GB}}, \overline{\epsilon^{GB}})$ *and* $(\overline{\Gamma^{(GB)^t}(\mathbf{k})}, \overline{\Delta^{(GB)^t}}, \overline{\epsilon^{(GB)^t}})$ *are (unital)* $\mathbf{k}$-*coalgebras.*

Example 5.16.

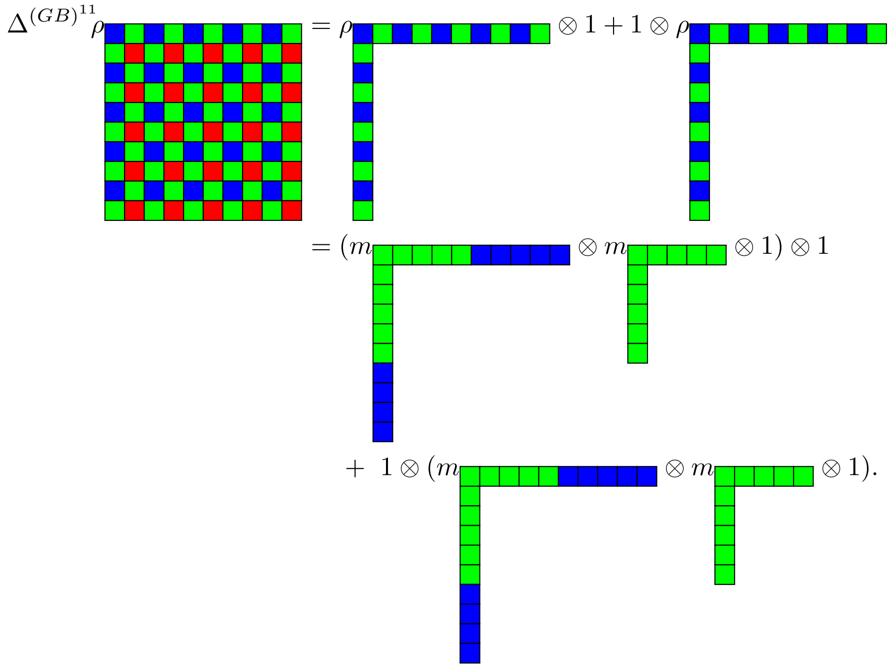
(1) A direct calculation for $\Delta^{GB}\, \rho_{(4,4,2,2)}$ gives the following:



(2) Calculating $\Delta^{(GB)^8}\, \rho_{\boxplus(6)} = \Delta^{(GB)^8}\, \rho_{(6,6,6,6,6,6)}$ gives the following:



(3) One could calculate $\Delta^{(GB)^{11}}\rho_{\boxplus(10)} = \Delta^{(GB)^{11}}\rho_{(10,10,10,10,10,10,10,10,10,10)}$ as follows:



6. Bayer Noise Functions and Other Bases

Recall that for any $\lambda \in Par$, the *Schur function* is defined to be

$$s_\lambda := \sum_{\mathcal{T}} \mathbf{x}^{cont(\mathcal{T})} \tag{6.1}$$

where $\mathcal{T}$ runs through all semistandard tableaux of shape λ, that is, $\mathcal{T}$ is an assignment of entries in $\{1, 2, 3, \ldots\}$ to the cells of the Young diagram for λ, weakly increasing left-to-right in rows, and strictly increasing top-to-bottom in columns. Here $cont(\mathcal{T})$ denotes the weak composition $\big(|\mathcal{T}^{-1}(1)|, |\mathcal{T}^{-1}(2)|, |\mathcal{T}^{-1}(3)|, \ldots\big)$, so that $\mathbf{x}^{cont(\mathcal{T})} = \prod_i x_i^{|\mathcal{T}^{-1}(i)|}$ [2]. For example,

$$\mathcal{T} = \begin{array}{ccccc} 1 & 1 & 1 & 2 & 7 \\ 2 & 3 & 4 & & \\ 3 & 4 & 4 & & \\ 6 & 7 & & & \end{array}$$

is a semistandard tableaux of shape $\lambda = (5, 3, 3, 2)$ with $\mathbf{x}^{cont(T)} = x_1^3 x_2^2 x_3^2 x_4^3 x_5^0 x_6 x_7^2$. It is well-known that the set $\{s_\lambda\}_{\lambda \in Par}$ forms a $\mathbf{k}$-basis for Λ for any commutative ring $\mathbf{k}$, and for any $\lambda \in Par$, one has

$$s_\lambda = \sum_{\nu \in Par} K_{\lambda,\nu}\; m_\nu,$$

where $K_{\lambda,\nu}$ is the *Kostka number* (a non-negative integer that is equal to the number of semistandard Young tableaux of shape λ and weight ν). This can be used as a inspiration for the following definition.

Definition 6.1. For any $\lambda \in Par$, define the *Bayer Noise Schur function*

$$\delta_\lambda = \sum_{\nu \in Par} K_{\lambda,\nu}\; \rho_\nu \;=\; \sum_{\nu \in Par} K_{\lambda,\nu}\; m_{\nu_{GB}} \otimes m_{\nu_G} \otimes m_{\nu_R}.$$

Example 6.2. For $\lambda = (2,2)$, one has

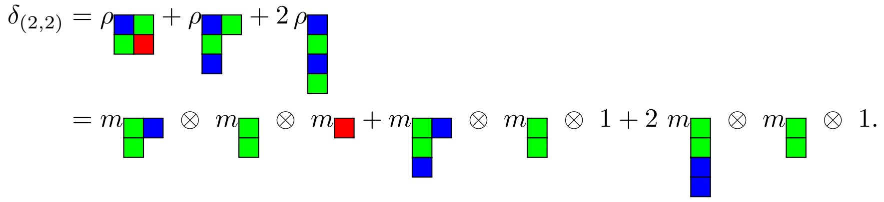
$$\begin{array}{llllllll} s_{(2,2)} = x_1^2x_2^2 & +x_1^2x_3^2 & +x_1^2x_2x_3 & +x_1^2x_2x_4 & +x_1x_2^2x_3 & +x_1x_2x_3x_4 & +x_1x_2x_3x_4 & +\cdots \\ \quad 11 & \quad 11 & \quad 11 & \quad 11 & \quad 12 & \quad 12 & \quad 13 & \\ \quad 22 & \quad 33 & \quad 23 & \quad 24 & \quad 23 & \quad 34 & \quad 24 & \end{array}$$

$$= m_{(2,2)} + m_{(2,1,1)} + 2m_{(1,1,1,1)}.$$

The Bayer-Schur function $\delta_{(2,2)}$, however, is given by

$$\begin{aligned} \delta_{(2,2)} &= \rho_{(2,2)} + \rho_{(2,1,1)} + 2\rho_{(1,1,1,1)} \\ &= (m_{(2,1)} \otimes m_{(1,1)} \otimes m_{(1)}) + (m_{(2,1,1)} \otimes m_{(1,1)} \otimes m_\emptyset) + 2(m_{(1,1,1,1)} \otimes m_{(1,1)} \otimes m_\emptyset). \end{aligned}$$

This can be visualized as follows:



Recall that the *dominance* or *majorization* order on Par_n is the partial order on the set Par_n whose greater-or-equal relation $\triangleright$ is defined as follows: For two partitions λ and μ of n, we set $\lambda \triangleright \mu$ (and say that λ *dominates*, or *majorizes*, μ) if and only if

$$\lambda_1 + \lambda_2 + \cdots + \lambda_k \geq \mu_1 + \mu_2 + \cdots + \mu_k \quad \text{for } k = 1, 2, \ldots, n \text{ [2]}.$$

It is well-known that the Kostka numbers are triangular with respect to the dominance order. The following is an immediate consequence of the triangularity of the transition matrix of $\{\delta_\lambda\}_{\lambda \in Par}$.

Proposition 6.3. *The set $\{\delta_\lambda\}_{\lambda \in Par}$ forms a $\mathbf{k}$-basis for Γ for any commutative ring $\mathbf{k}$.*

Following [3], define of the families of *power sum symmetric functions* p_n and *elementary symmetric functions* e_n, for $n = 1, 2, 3, \ldots$ by

$$p_n := x_1^n + x_2^n + \cdots = m_{(n)}, \tag{6.2}$$

$$e_n := \sum_{i_1 < \cdots < i_n} x_{i_1} \cdots x_{i_n} = m_{(1^n)}. \tag{6.3}$$

Here,

$$(1^n) = \left(\underbrace{1, 1, \ldots, 1}_{n \text{ ones}} \right).$$

We have the following consequence.

Proposition 6.4. *Let $n \in \mathbb{N}$ and $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_\ell) \in Par$ with $\lambda_1 \geq \cdots \geq \lambda_\ell > 0$.*

(1) We have

$$\rho_{(n)} = \begin{cases} p_n \otimes p_{\frac{n}{2}} \otimes 1 & \text{if } n \text{ is even} \\ p_n \otimes p_{\frac{n-1}{2}} \otimes 1 & \text{if } n \text{ is odd} \end{cases}$$

(2) We have

$$\rho_{(1^n)} = \begin{cases} e_n \otimes e_{\frac{n}{2}} \otimes 1 & \text{if } n \text{ is even} \\ e_n \otimes e_{\frac{n-1}{2}} \otimes 1 & \text{if } n \text{ is odd} \end{cases} = \begin{cases} s_{(1^n)} \otimes s_{(1^{\frac{n}{2}})} \otimes 1 & \text{if } n \text{ is even} \\ s_{(1^n)} \otimes s_{(1^{\frac{n-1}{2}})} \otimes 1 & \text{if } n \text{ is odd} \end{cases}$$

Proof. (1) We have $\rho_{(n)} = m_{(n)_{GB}} \otimes m_{(n)_G} \otimes m_{(n)_R}$. Since $(n)_{GB} = (n)$, $(n)_R = \emptyset$ and

$$(n)_G = \begin{cases} (\frac{n}{2}) & \text{if } n \text{ is even} \\ (\frac{n-1}{2}) & \text{if } n \text{ is odd} \end{cases}$$

Thus,

$$\rho_{(n)} = \begin{cases} p_n \otimes p_{\frac{n}{2}} \otimes 1 & \text{if } n \text{ is even} \\ p_n \otimes p_{\frac{n-1}{2}} \otimes 1 & \text{if } n \text{ is odd} \end{cases}$$

(2) We have $\rho_{(1^n)} = m_{(1^n)_{GB}} \otimes m_{(1^n)_G} \otimes m_{(1^n)_R}$. Since $(1^n)_{GB} = (1^n)$, $(1^n)_R = \emptyset$ and

$$(1^n)_G = \begin{cases} (1^{\frac{n}{2}}) & \text{if } n \text{ is even} \\ (1^{\frac{n-1}{2}}) & \text{if } n \text{ is odd} \end{cases}$$

Thus,

$$\rho_{(1^n)} = \begin{cases} e_n \otimes e_{\frac{n}{2}} \otimes 1 & \text{if } n \text{ is even} \\ e_n \otimes e_{\frac{n-1}{2}} \otimes 1 & \text{if } n \text{ is odd} \end{cases} = \begin{cases} s_{(1^n)} \otimes s_{(1^{\frac{n}{2}})} \otimes 1 & \text{if } n \text{ is even} \\ s_{(1^n)} \otimes s_{(1^{\frac{n-1}{2}})} \otimes 1 & \text{if } n \text{ is odd} \end{cases}$$

□

Recall that a sequence $\alpha = (\alpha_1, \alpha_2, \ldots)$ in $\mathbb{N}^\infty$ that have finite support is called a *weak composition*. The nonzero entries of the sequence $\alpha = (\alpha_1, \alpha_2, \ldots)$ are called the *parts* of the weak composition α. The sum $\alpha_1 + \alpha_2 + \alpha_3 + \cdots$ of all entries of a weak composition $\alpha = (\alpha_1, \alpha_2, \alpha_3, \ldots)$ (or, equivalently, the sum of all parts of α) is called the *size* of α and denoted by $|\alpha|$. A *composition* is a finite tuple $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_m)$ of positive integers. In other words, it is a weak composition with no zero entries. We write $\varnothing$ or (0) for the empty composition $()$. Its *length* is defined to be m and denoted by $\ell(\alpha)$; its *size* is defined to be $\alpha_1 + \alpha_2 + \cdots + \alpha_m$ and denoted by $|\alpha|$; its *parts* are its entries $\alpha_1, \alpha_2, \ldots, \alpha_m$. The compositions of size n are called the *compositions of* n. Clearly, any partition of n is a composition of n. Let $Comp_n$ denote the set of all compositions of n, and let $Comp$ denote the set of all compositions. An expansion of a composition α is a weak composition $\bar{\alpha}$ such that removing the zeros from $\bar{\alpha}$ one obtains α. If $\alpha, \beta, \gamma \in Comp$, then we say γ is a shuffle sum of the other two compositions if there are expansions $\bar{\alpha}$ and $\bar{\beta}$ of α and β, respectively, which have length $\ell(\gamma)$ such that $\gamma = \bar{\alpha} + \bar{\beta}$. Here, addition is componentwise [8].

It is well-known that for any $\lambda, \lambda' \in Par$, we have

$$\underline{m}(m_{\lambda} \otimes m_{\lambda'}) = m_{\lambda} m_{\lambda'} = \sum_{\substack{\nu \in Par: \\ \nu \vdash |\lambda| + |\lambda'|}} c^{\nu}_{\lambda,\lambda'} \; m_{\nu}, \tag{6.4}$$

where $c^{\nu}_{\lambda,\lambda'}$ is the number of ways of writing ν as a shuffle sum of λ and λ'.

Let $\vartheta \,:\, \Gamma^e(\mathbf{k}) \otimes \Gamma^e(\mathbf{k}) \to \Gamma^e(\mathbf{k})$ be a map defined by

$$\begin{aligned}
\vartheta(\rho_{\lambda} \otimes \rho_{\lambda'}) &= \sum_{\substack{\nu \in Par^e : \, \nu_U \vdash |(\lambda+\lambda')_U|, \\ \forall U \in \{GB, R\}}} c^{\nu}_{\lambda,\lambda'} \; \rho_{\nu} \\
&= \sum_{\substack{\nu \in Par^e : \, \nu_U \vdash |\lambda_U| + |\lambda'_U|, \\ \forall U \in \{GB, R\}}} c^{\nu}_{\lambda,\lambda'} \; \rho_{\nu} \;\; (\text{by } 4.3) \,,
\end{aligned}$$

where $c^{\nu}_{\lambda,\lambda'}$ is the number of ways of writing ν as a shuffle sum of λ and λ', and the map

$$\mathbf{k} = \Gamma^{(e,0)}(\mathbf{k}) \xrightarrow{u_e} \Gamma^e(\mathbf{k})$$

is the inclusion map.

Let $\delta : \Gamma^e(\mathbf{k}) \to \Gamma^e(\mathbf{k}) \otimes \Gamma^e(\mathbf{k})$ be the map defined $\mathbf{k}$-linearly by

$$\delta \rho_{\lambda} = \sum_{\substack{(\mu,\nu) \in Par^e \times Par^e : \\ \mu_U \sqcup \nu_U = \lambda_U, \forall U \in \{GB, R\}}} \rho_{\mu} \otimes \rho_{\nu}, \tag{6.5}$$

in which $\mu_U \sqcup \nu_U$ is the partition obtained by taking the multiset union of the parts of μ_U and ν_U, and then reordering them to make them weakly decreasing. We end up this paper with the following theorem.

Theorem 6.5.

(1) The triple $(\Gamma^e(\mathbf{k}), \vartheta, u_e)$ *is a* $\mathbf{k}$*-algebra, where*

$$\mathbf{k} = \Gamma^{(e,0)}(\mathbf{k}) \xrightarrow{u_e} \Gamma^e(\mathbf{k})$$

is the inclusion map.

(2) The triple $(\Gamma^e(\mathbf{k}), \delta, \widehat{\epsilon})$ *is a* $\mathbf{k}$*-coalgebra, where* $\Gamma \overset{\widehat{\epsilon}}{\to} \mathbf{k}$ *is the map defined* $\mathbf{k}$*-linearly by*

$$\widehat{\epsilon}|_{\Gamma^{(e,0)}=\mathbf{k}} = id_{\mathbf{k}} \text{ and } \widehat{\epsilon}|_{I=\bigoplus_{n>0}\Gamma^{(e,n)}} = 0.$$

(3) $(\Gamma^e(\mathbf{k}), \vartheta, u_e, \delta, \widehat{\epsilon})$ *is a* $\mathbf{k}$*-bialgebra, and hence* $(\Gamma^e(\mathbf{k}), \vartheta, u_e, \delta, \widehat{\epsilon})$ *is* $\mathbf{k}$*-Hopf algebra.*

Fix a commutative ring $\mathbf{k}$. We end this paper with the following few things as suggestions to the reader who might be interested in.

- Finding a connection between Hall algebras and Bayer Young diagrams.
- Establishing another bases for the Bayer Noise module over $\mathbf{k}$.
- Defining noise symmetric functions using other filters.
- Defining symmetric functions based on the denoising concept.

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Existence of Traveling Profiles Solutions to Porous Medium Equation

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Abstract: In this paper, we shall study the existence and uniqueness of solutions called "traveling profiles solutions" to the porous medium equation in one dimension. By these solutions, we generalize the results obtained by Gilding and Peletier who proved the existence of self similar solutions of type I, II and III to the same equation. The principal idea of our work is to convert the porous media equation in to an equivalent nonlinear differential equation, and to prove the existence and uniqueness of these new solutions under certain conditions.

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1. Introduction:

Consider the one dimensionel porous media equation, which is written as:

$$\left\{ \frac{\partial u}{\partial t} = \frac{\partial^2}{\partial x^2} (u^m) \right\}, \tag{1.1}$$

where $u > 0$, $x \in \mathbb{R}$, $t > 0$, and $m > 1$, is a fixed real number. Equation (1.1) is parabolic at any point (x, t) at which $u > 0$. However, at points where $u = 0$, it is degenerate parabolic. Equation (1.1) arises in many other applications, e.g, in the theory of ionized gases at high temperature [21] for values of $m > 1$, and in various models in plasma physics [5] for values of $m < 1$. Of course, for $m = 1$, equation (1.1) is the classical equation of heat conduction. In this paper we will focus on the case where $m > 1$.

Classes of weak solutions for the Cauchy problem and the Cauchy-Dirichlet problem of Eq. (1.1) were introduced by Oleinik, Kalashnikov and Yui-Lin [15], they proved existence and uniqueness of such solutions and they showed that if at some instant t_0, a weak solution $U(x,t_0)$ has compact support, then $u(x,t)$ has also compact support for any $t \geqslant t_0$. Gilding and Peletier [12], has study a class of similarity solutions of (1.1) for $0 < x < \infty$ and $0 < t \leqslant T$, where T is some positive constant. These solutions has three following form:

$$\begin{aligned} 1.\ u_1(x,t) &= (t+\tau)^{\alpha} f_1(\eta),\ \eta = x(t+\tau)^{\frac{-1}{2}(1+(m-1)\alpha)},\ \text{for } \tau > 0, && (1.2)\\ 2.\ u_2(x,t) &= (t-\tau)^{\alpha} f_2(\eta),\ \eta = x(t-\tau)^{\frac{-1}{2}(1+(m-1)\alpha)},\ \text{for } \tau > T, && (1.3)\\ 3.\ u_3(x,t) &= e^{\alpha(t+\tau)} f_3(\eta),\ \eta = x\exp(\frac{-1}{2}\alpha\,(m-1)\,(t+\tau)),\ \text{for any } \tau. && (1.4) \end{aligned}$$

After substitution of u_1, u_2 and u_3 into (1.1), they have obtained the following equations for the functions f_1, f_2 and f_3:

$$\begin{aligned} I.\quad (f_1^m)^{''} &= \alpha f_1 - \frac{1}{2}\{1+(m-1)\alpha\}\,\eta f_1^{'},\quad 0<\eta<\infty, && (1.5)\\ II.\quad (f_2^m)^{''} &= -\alpha f_2 + \frac{1}{2}(\{1+(m-1)\alpha\}\,\eta f_2^{'},\quad 0<\eta<\infty, && (1.6)\\ III.\quad (f_3^m)^{''} &= \alpha f_3 - \frac{1}{2}\alpha(m-1)\eta f_3^{'},\quad 0<\eta<\infty. && (1.7) \end{aligned}$$

At the boundaries, the following conditions are imposed:

$$f_i\,(0) = U\ \ (\geq 0)\,,\ f_i\,(\infty) = 0,\ i = 1,2,3.$$

Thus the solutions $u_i(x,t)$ satisfy the lateral boundary conditions

$$u_1(0,t) = (t+\tau)^{\alpha} U,\ u_2(0,t) = (t-\tau)^{\alpha} U,\ u_3(0,t) = e^{\alpha(t+\tau)} U,$$

and

$$u_i(x,t) \to 0 \text{ as } t \to \infty,\ i = 1,2,3,$$

for fixed $t \in [0,T]$.

It was Barenblatt [4], who first discussed the similarity solution u_1; he did this for $\alpha \geq 0$. In a subsequent paper [5] he also investigated the solution u_3 for $\alpha > 0$ and $m = 2$. Later Marshak [14] also discussed solution u_3; in addition he made a detailed, and partly numerical, study of solution u_1 for $\alpha = \frac{1}{5}$. For a number of values of α, explicit solutions were found by various authors [1,4,5,13–15,21].

In this work we discuss the existence and uniqueness of a most general form of solutions (1.2)–(1.4) to equation (1.1), which are written in the form:

$$u\,(x,t) = c\,(t)\,f(\eta),\ \text{with } \eta = \frac{x-b(t)}{a\,(t)},\ a,\ c,\ b \in \mathbb{R}^+, \tag{1.8}$$

where $a(t)$, $c(t), b(t)$ and the profile f are to be determined. By replacing this form of solution in this equation, we obtain a general form of nonlinear differential equation which we prove the existence of their solutions under certain conditions. These solutions are called "traveling profiles solutions" [7,8].

2. Traveling profiles solutions to porous medium equation:

If we replace this form of solutions in equation (1.1) we find,

$$\frac{\dot{c}}{c}f - \frac{\dot{a}}{a}\eta f' - \frac{\dot{b}}{a}f' = \frac{c^{m-1}}{a^2}(f^m)'', \tag{2.1}$$

this equation depends on many unknown parameters, our aim is to determine the coefficients $a(t), c(t), b(t)$ and to prove the existence of the profile f.
In that case, a simple separation of variables argument implies that the following three conditions must hold:

$$\begin{cases} \frac{\dot{c}}{c} &= \frac{c^{m-1}}{a^2}\alpha \\ \frac{\dot{a}}{a} &= -\frac{c^{m-1}}{a^2}\beta \\ \frac{\dot{b}}{a} &= -\frac{c^{m-1}}{a^2}\gamma \end{cases} \tag{2.2}$$

with parameters $\alpha, \beta, \gamma \in \mathbb{R}$, and the profile f must satisfy the equation

$$(f^m)''(\eta) = \alpha f(\eta) + \beta\eta f'(\eta) + \gamma f'(\eta). \tag{2.3}$$

2.1. Resolution of the differential system:

At the boundaries, we impose the lateral boundary conditions

$$a(0) = 1,\ c(0) = 1,\ b(0) = 0, \tag{2.4}$$

we can see that from (2.2), we have

$$\begin{cases} c(t) = a(t)^{\frac{-\alpha}{\beta}} \\ b(t) = \frac{\gamma}{\beta}a(t) + K_2 \end{cases}, \tag{2.5}$$

if we replace (2.5) in (2.2), then we deduct

$$\begin{cases} a(t) = (1 - A\beta t)^{\frac{1}{A}} \\ c(t) = (1 - A\beta t)^{\frac{-\alpha}{\beta A}} \\ b(t) = \frac{\gamma}{\beta}(1 - A\beta t)^{\frac{1}{A}} - \frac{\gamma}{\beta} \end{cases}, \text{ for } 0 < t < T, \tag{2.6}$$

with

$$2\beta + (m-1)\alpha > 0, \tag{2.7}$$

and

$$A = 2 + \frac{\alpha}{\beta}(m-1), \tag{2.8}$$

the finite time T is given by

$$T = \frac{1}{2\beta + (m-1)\alpha}. \tag{2.9}$$

In other hand, we have

$$\begin{cases} a(t) = \exp(-\beta t) \\ c(t) = \exp(\alpha t) \\ b(t) = \frac{\gamma}{\beta}\exp(-\beta t) - \frac{\gamma}{\beta} \end{cases}, \text{ for } 0 < t < \infty, \tag{2.10}$$

with

$$2\beta + (m-1)\alpha = 0. \tag{2.11}$$

Now we want to prove the existence of the profile f of equation (2.3).

3. Existence and uniqueness of the "based profile":

In this section, we discuss the existence and uniqueness of weak solutions with compact support for the boundary value problem

$$(f^m)''_{\eta\eta} = \alpha f + \beta\eta f'_\eta + \gamma f'_\eta, \quad 0 < \eta < \infty, \text{ where } \alpha, \beta, \gamma \in \mathbb{R}, \tag{3.1}$$

with $\eta = \frac{x-b(t)}{a(t)}$, and

$$f(0) = V \quad \text{and} \quad f(\infty) = 0, \tag{3.2}$$

where $V > 0$ are arbitrary real constants. With this equation for $\gamma = 0$, we recover the forms (1.5)-(1.7) which has been investigated in detail in a series papers (Gilding and Peletier, 1976,1977; Gilding 1980, [12]).
Thus the solution $u(x,t)$ satisfy the lateral boundary condition

$$u(b(t), t) = c(t)V, \text{ with } V \in \mathbb{R}^+, \tag{3.3}$$

to the porous medium equation 1.1 in the domain $b(t) < x < \infty, \ t > 0$

$$\frac{\partial u}{\partial t} = \frac{\partial^2}{\partial x^2}(u^m), \quad b(t) < x < \infty, \ t > 0. \tag{3.4}$$

Our aim is to generalize the results of [12] for $\gamma \neq 0$, we follow definition.

Definition 3.1. A function f is a weak solution of (3.1) if it satisfies the following conditions:
a) f is bounded, continuous, and nonnegative on $[0,\infty)$.
b) $(f^m)(\eta)$ has a continuous derivative with respect to η on $(0,\infty)$.
c) f satisfies the equation

$$\int_0^\infty \phi' \left\{ (f^m)' - (\beta\eta + \gamma) f \right\} d\xi + (\alpha - \beta)\int_0^\infty \phi f d\eta = 0,$$

for all $\phi \in C_0^1(0,\infty)$.

We will prove the following theorem:

Theorem 3.2. *Suppose that $V > 0$. Then the boundary value problem (3.1)-(3.2) has a weak solution with compact support if and only if $\beta \leq 0,\ \gamma \leq 0$ and $\alpha - 2\beta > 0$. Furthermore, this weak solution is unique.*

To prove this theorem, we impose the following boundary value for equation (3.1):

$$f(0) = V, \tag{3.5}$$

and

$$f(\lambda) = 0, \quad (f^m)^{'}(\lambda) = 0, \tag{3.6}$$

where $\lambda > 0$ is a real number. Using a shooting argument with $\lambda > 0$ as the shooting parameter, we first prove the following theorem for the existence and uniqueness of classical solutions for (3.1) with the boundary conditions (3.5)-(3.6).

Theorem 3.3. *Suppose that $V > 0$. Then the boundary value problem (3.1), (3.5) and (3.6) has a unique solution and there exists a unique $\lambda(V) > 0$ such that $f(\eta; \lambda(V))$ is positive on $(0, \lambda)$ if and only if $\beta \leq 0,\ \gamma \leq 0$ and $\alpha - 2\beta > 0$.*

We first determine necessary conditions on the parameters $\alpha,\ \beta$ and γ for the existence of a nontrivial weak solution of (3.1) with compact support.

Lemma 3.4. *There exists a nontrivial weak solution of (3.1) with a compact support only when $\beta = \gamma = 0$ and $\alpha > 0$ or $\beta < 0$ and $\gamma < 0$.*

Proof. Suppose that $f(\eta; \lambda)$ is a nontrivial weak solution of (3.1) with compact support. Then $f > 0$ in $(\lambda - \varepsilon, \lambda)$ and $f = 0$ in $[\lambda, \infty)$ for some $\lambda > 0$ and $\varepsilon > 0$.
It follows that f is a classical solution of (3.1) on $(\lambda - \varepsilon, \lambda)$ and satisfies (3.6) at $\eta = \lambda$, that is, $f(\lambda) = 0$, $(f^m)^{'}(\lambda) = 0$. Integrating (3.1) from η to λ, where $\lambda - \varepsilon < \eta < \lambda$, we get:

$$-(f^m)^{'}(\eta) = -(\beta\eta + \gamma) f(\eta) + (\alpha - \beta) \int_\eta^\lambda f(\xi)\, d\xi. \tag{3.7}$$

The continuity of f and $(f^m)^{'}$ ensures the existence of $\eta_0 \in (\lambda - \varepsilon, \lambda)$ such that $f^{'}(\eta_0) < 0$. This implies that the LHS of (3.7) is positive at $\eta = \eta_0$, and, therefore, $-(\beta\eta_0 + \gamma)$ and $\alpha - \beta$ cannot both be less than zero. Thus, $\beta = \gamma = 0$ implies that $\alpha > 0$.

Now consider the case $\beta > 0$ and $\gamma > 0$. This requires that $\alpha - \beta > 0$, and hence $\alpha > 0$. We easily check from (3.1) that f cannot have a maximum as long as f is positive. Therefore, f does not assume a maximum at any point in $(\lambda - \varepsilon, \lambda)$, thus, $f^{'}(\eta_0) < 0$ on $(\lambda - \varepsilon, \lambda)$. It follows from (3.7) that

$$-mf^{m-2}(\eta) f^{'}(\eta) + (\beta\eta + \gamma)\eta \leq (\alpha - \beta)(\lambda - \eta), \tag{3.8}$$

where we have used the fact that $f(\xi) \leq f(\eta)$ for $\xi \in (\eta, \lambda),\ \lambda - \varepsilon < \eta < \lambda$. As $\eta \to \lambda$ in (3.8), LHS becomes positive, and the RHS tends to zero, a contradiction.
Thus we have shown that $\beta = \gamma = 0$ and $\alpha > 0$ or $\beta < 0$ and $\gamma < 0$ are the only cases for which a nontrivial weak solution of (3.1) exists with a compact support. $\square$

3.1. The case when $\beta = \gamma = 0$ and $\alpha > 0$

With $\beta = \gamma = 0$ and $\alpha > 0$, the solution of (3.1), were obtained by Gilding and Peletier, see [12]), and are given by

$$f(\eta;\lambda) = \left[\frac{\alpha\,(m-1)^2}{2m\,(m+1)}\,(\lambda-\eta)^2 \right]^{\frac{1}{m-1}}, \; 0 < \eta < \lambda, \tag{3.9}$$

which is the unique solution of the problem (3.1) satisfying (3.6).
We observe that

$$f(0;\lambda) = \left[\frac{\alpha\,(m-1)^2}{2m\,(m+1)}\lambda^2 \right]^{\frac{1}{m-1}}.$$

Because $m > 1$, $f(0;\lambda)$ is a continuous function of λ with $f(0;0) = 0$ and $f(0;\infty) = \infty$; furthermore, f is a continuous and monotonically increasing function of a. This implies that, for a given $V > 0$, there exists a unique $\lambda(V)$ such that $f(0;\lambda(V)) = V$. Therefore, $f(\eta;\lambda(V))$ is the unique solution of (3.1) satisfying (3.5) and (3.6). An easy calculation shows that

$$\lambda(V) = \left[\frac{2m\,(m+1)}{\alpha\,(m-1)^2} V^{m-1} \right]^{\frac{1}{2}}.$$

3.2. The case when $\beta < 0$ and $\gamma < 0$

We give below an elementary lemma for the case $\beta < 0$ and $\gamma < 0$.

Lemma 3.5. *Suppose that $0 < \mu < \lambda$ and f is a positive solution of (3.1) on $[\mu,\lambda)$ satisfying (3.6). Then the following results hold.:*
(i) *$f'(\eta) < 0$ on $[\mu,\lambda)$ provided that $\alpha - \beta \geq 0$.*
(ii) *Suppose that $\alpha - \beta < 0$ and $f'(\eta_0) = 0$ for some $\eta_0 \in [\mu,\lambda)$. Then f has a maximum at η_0 for $\eta_0 < \frac{\lambda(\alpha-\beta)-\gamma}{\alpha}$.*
Suppose that f is a positive solution of (3.1) and (3.6) on $[0,\lambda)$. Then

$$f'(0) < 0, \; \textit{for } \alpha - \beta \geq 0.$$

Proof.
(i) By integration of (3.1) from $\mu < \eta < \lambda$, we obtain

$$-\left(f^m\right)'(\eta) = -\left(\beta\eta+\gamma\right) f(\eta) + (\alpha-\beta)\int_\eta^\lambda f(\xi)\,d\xi. \tag{3.10}$$

Because $\beta < 0$ and $\gamma < 0$, the RHS of (3.10) is positive when $\alpha - \beta \geq 0$ and hence $(f^m)'(\eta) < 0$. This implies that $f'(\eta) < 0$ on $[\mu,\lambda)$.
(ii) if $\alpha - \beta < 0$ then $\alpha < 0$ (because $\beta < 0$), by (3.1), $f''(\eta_0) < 0$ when $f'(\eta_0) = 0$,

thus f has a maximum at $\eta = \eta_0$ and is strictly decreasing on (η_0, λ); that is, $f^{'}(\eta) < 0$ on (η_0, λ). Putting $\eta = \eta_0$ in (3.10), we have:

$$0 = -(\beta\eta_0 + \gamma) f(\eta_0) + (\alpha - \beta) \int_{\eta_0}^{\lambda} f(\xi)\, d\xi > -(\beta\eta_0 + \gamma) f(\eta_0) \\ + (\alpha - \beta)(\lambda - \eta_0) f(\eta_0),$$

therefore,

$$-(\beta\eta_0 + \gamma) + (\alpha - \beta)(\lambda - \eta_0) < 0 \text{ or } \eta_0 < \frac{\lambda(\alpha - \beta) - \gamma}{\alpha},$$

With $\eta = 0$, (3.10) becomes

$$-(f^m)^{'}(0) = -\gamma f(0) + (\alpha - \beta) \int_{\eta}^{\lambda} f(\xi)\, d\xi. \tag{3.11}$$

The result for $f^{'}(0)$ follows immediately from (3.11). □

In the next lemma, we prove the local existence and uniqueness of a solution of (3.1) satisfying (3.6). This is accomplished by formulating an equivalent integral equation following the work of Atkinson and Peletier [3].

Lemma 3.6. *Suppose that $\beta < 0$, $\gamma < 0$ and α is any real number. Then, for any $\lambda > 0$, equation (3.1) with initial condition (3.6) at $\eta = \lambda$, has a unique positive solution in a neighborhood $(\lambda - \varepsilon, \lambda)$ of λ, here, $\varepsilon > 0$ is a constant.*

Proof. Suppose that f is a positive solution in a left neighborhood of $\eta = \lambda$. By lemma 3.5, $f^{'}(\eta) < 0$ for $\eta \in (\lambda - \varepsilon, \lambda)$ for some $\varepsilon > 0$.
Let $\eta = G(f)$ where G is the inverse of f on $(\lambda - \varepsilon, \lambda)$. Rewriting (3.10), we have:

$$(f^m)^{'}(\eta) = (\alpha\eta + \gamma) f(\eta) + (\alpha - \beta) \int_{\eta}^{\lambda} \xi f^{'}(\xi)\, d\xi. \tag{3.12}$$

With $G(f) = \eta$ in (3.12) we have:

$$\frac{dG}{df} = \frac{m f^{m-1}}{(\alpha G + \gamma) f - (\alpha - \beta) \int_0^f G(\varphi) d\varphi}, \tag{3.13}$$

equation (3.13) is an integro-differential equation for $G = G(f)$. Integrating (3.13) from zero to f , we obtain

$$G(f) - \lambda = m \int_0^f \frac{\phi^{m-1} d\phi}{(\alpha G + \gamma)\phi - (\alpha - \beta) \int_0^{\phi} G(\psi) d\psi}. \tag{3.14}$$

Let

$$H(f) = 1 - \lambda^{-1} G(f), \tag{3.15}$$

Then, equation (3.14) becomes

$$H(f) = \frac{m}{\lambda^2} \int_0^f \frac{\phi^{m-1} d\phi}{(-\beta-\gamma)\,\phi + \alpha\phi H(\phi) - (\alpha-\beta)\int_0^\phi H(\psi)d\psi}. \tag{3.16}$$

By using the Banach–Cacciopoli contraction mapping principle, we now show that equation (3.16) admits a unique positive solution in a right neighborhood of $f = 0$. Let X be the set of all bounded functions $H(f)$ on $[0,h], h > 0$, satisfying

$$0 \leq H(f) \leq \rho = \frac{|\beta+\gamma|}{2(|\alpha| + |\alpha-\beta|)}. \tag{3.17}$$

Let $\|..\|$ be the sup norm defined on X. Then X is a complete metric space.

$$M\,(H)\,(f) = \frac{m}{\lambda^2} \int_0^f \frac{\phi^{m-1} d\phi}{-\left(\beta+\gamma\right)\phi + \alpha\phi H(\phi) - (\alpha-\beta)\int_0^\phi H(\psi)d\psi},\ \ H(f) \in X. \tag{3.18}$$

First we show that M maps X into X over $[0,h_0]$, $h \leq h_0$. Let $H \in X$. Clearly,

$$\begin{aligned} -\left(\beta+\gamma\right)\phi + \alpha\phi H(\phi) - (\alpha-\beta)\int_0^\phi H(\psi) &\geq -\left(\beta+\gamma\right)\phi - |\alpha|\,\phi H(\phi) && (3.19)\\ &- |\alpha-\beta|\,\|H\|\,\phi \\ &\geq -\left(\beta+\gamma\right)\phi && (3.20)\\ &- (|\alpha| + |\alpha-\beta|)\,\|H\|\,\phi \\ &\geq \frac{-\left(\beta+\gamma\right)\phi}{2}, && (3.21)\end{aligned}$$

where we have used (3.17). Therefore, from (3.18), we have

$$\begin{aligned} M\,(H)\,(f) &\leq \frac{2m}{-\left(\beta+\gamma\right)\lambda^2}\int_0^f \phi^{m-2} d\phi \\ &= \frac{2mf^{m-2}}{-\left(\beta+\gamma\right)\lambda^2\,(m-1)} \\ &\leq \frac{2mh^{m-2}}{-\left(\beta+\gamma\right)\lambda^2\,(m-1)}. && (3.22)\end{aligned}$$

Thus, $M(H)$ is well defined on X and $M(H) : [0,h] \to \mathbb{R}$ is nonnegative and continuous. The RHS of (3.22) suggests that we may find h_0, $h \leq h_0$ such that $\|M(H)\| \leq \rho$, $H \in X$. Thus M maps X into X for $h \leq h_0$. In the next step, we show that M is a contraction map on X. Let $H_1, H_2 \in X$, and $h \leq h_0$.

Then

$$\begin{aligned}
\|M(H_1)-M(H_2)\| &\le \frac{4m}{(\beta+\gamma)^2\lambda^2}\int_0^f \phi^{m-3}\Big(|\alpha|\,\phi\,\|H_1-H_2\| \\
&+ |\alpha-\beta|\int_0^{\phi}\|H_1-H_2\|\,d\psi\Big)d\phi \\
&\le \frac{4m}{(m-1)(\beta+\gamma)^2\lambda^2}\left(|\alpha|+|\alpha-\beta|\right)h^{m-1}\,\|H_1-H_2\|\,.
\end{aligned}$$

Therefore, there exists $h_1 \in (0, h_0]$ such that if $h \le h_1$, M is a contraction on X. By the Banach–Cacciopoli contraction principle, M has a unique fixed point in X and hence equation (3.16) has a unique solution. This, in turn, implies that there exists a unique positive solution of (3.1)-(3.6) in an interval $(\lambda-\varepsilon, \lambda)$ for some $\varepsilon > 0$. □

In the next lemma, we prove that a positive solution $f(\eta;\lambda)$ of (3.1) and (3.6) cannot be unbounded.

Lemma 3.7. *Suppose that $\beta < 0$, $\gamma < 0$ and $\mu \in [0,\lambda)$. Furthermore, let f be a positive solution of (3.1) and (3.6) on (μ,λ). Then f is bounded on (μ,λ) and*

$$\sup f(\eta) \le \left[\frac{(m-1)\lambda}{2m}\max\left\{-(\beta\lambda+2\gamma), [(\alpha-2\beta)\lambda-2\gamma]\right\}\right]^{\frac{1}{m-1}}.$$

Proof. We prove this lemma for the two following cases: (i) $\alpha-\beta \ge 0$, (ii) $\alpha-\beta < 0$.
Case $\alpha-\beta \ge 0$:
In this case, $f'(\eta) < 0$ on (μ,λ) by Lemma 3.5, $f(\eta) \ge f(\xi)$, $\xi \in (\eta,\lambda)$. By (3.10), we have

$$-(f^m)'(\eta) \le -(\beta\eta+\gamma)f(\eta) + (\alpha-\beta)f(\eta)(\lambda-\eta),\ \mu \le \eta < \lambda,$$

or

$$-mf^{m-2}f' \le -\alpha\eta-\gamma+\lambda(\alpha-\beta) \le -\lambda\beta-\gamma+\alpha(\lambda-\eta),\ \mu\le\eta<\lambda. \tag{3.23}$$

Integrating (3.23) from η to λ, we obtain

$$\frac{m}{m-1}f^{m-1}(\eta) \le \left[-\lambda\beta-\gamma+\frac{1}{2}\alpha(\lambda-\eta)\right](\lambda-\eta),\ \mu\le\eta\le\lambda. \tag{3.24}$$

Thus

$$\frac{m}{m-1}\sup_{(\mu,\lambda)} f^{m-1}(\eta) \le \frac{1}{2}\left[(\alpha-2\beta)\lambda-2\gamma\right]\lambda. \tag{3.25}$$

Case $\alpha-\beta < 0$:
By equation (3.10),

$$-(f^m)'(\eta) \le -(\beta\eta+\gamma)f(\eta),\ \mu\le\eta<\lambda,$$

or

$$-mf^{m-2}f^{'} \leq -\left(\beta\eta+\gamma\right), \; \mu \leq \eta < \lambda. \tag{3.26}$$

Integrating (3.26) from η to λ, we obtain

$$\frac{m}{m-1}f^{m-1}\left(\eta\right) \leq -\left[\frac{\beta}{2}\left(\lambda^{2}-\eta^{2}\right)+\gamma\left(\lambda-\eta\right)\right], \; \mu \leq \eta \leq \lambda. \tag{3.27}$$

This in turn, implies that

$$\frac{m}{m-1}\sup_{(\mu,\lambda)}f^{m-1}\left(\eta\right) \leq -\frac{\lambda}{2}\left(\beta\lambda+2\gamma\right). \tag{3.28}$$

Observe that the bounds in (3.25) and (3.28) are independent of μ and, therefore, $f\left(\eta\right)$ cannot be unbounded as η decreases from $\eta=\lambda$. □

Lemma 3.8. *Suppose that f is a positive solution of (3.1) and (3.6) in a left neighborhood of $\eta=\lambda$, and $\beta<0$, $\gamma<0$. Then $f(\eta)>0$ on $[0,\lambda)$ when $\alpha-2\beta>0$.*

Proof. Integrating (3.10) from η to λ we have

$$f^{m}\left(\eta\right)=-\left(\beta\eta+\gamma\right)\int_{\eta}^{\lambda}f\left(\xi\right)d\xi+\left(\alpha-2\beta\right)\int_{\eta}^{\lambda}\left(\xi-\eta\right)f\left(\xi\right)d\xi. \tag{3.29}$$

It is easy to see from (3.29) that, if $\alpha-2\beta>0$, then $f(\eta)>0$ on $(0,\lambda)$. □

Prove of Theorem 3.3. Now we proceed to prove Theorem 3.3. We have already proved in Lemma 3.6 the local existence of a solution about $\eta=\lambda$ for (3.1) and (3.6). This unique local solution may be extended back to $\eta=0$ as a positive solution with $f(0)>0$ if and only if when $\alpha-2\beta>0$ (see Lemma 3.8). Now if we can prove that there exists $\lambda(V)$ such that $f(0;\lambda(V))=V$, then Theorem 3.3 is proved. To that end, we use the following result due to Barenblatt (see [5]). Suppose that $f(\eta;\lambda)$ is a solution of (3.1) and (3.6) on $(0,\lambda)$; then $\omega^{-\frac{2}{m-1}}f(\omega\eta;\omega\lambda)$ is a solution of (3.1) and (3.6) on $(0,\omega\lambda)$ for any $\omega>0$. Let $\omega=\lambda^{-1}$, then,

$$f(0;\lambda)=\lambda^{\frac{2}{m-1}}f(0;1)=V. \tag{3.30}$$

Because $f(0;1)>0$ for $\alpha-2\beta>0,\ \beta<0,\ \gamma<0$, we get a unique root $\lambda=\lambda(V)$ of (3.30). Thus, $f(\eta;\lambda(V))$ is the unique solution of (3.1), (3.5) and (3.6).

Theorem 3.3 follows if we add that, for $\beta=\gamma=0$, we have already constructed the explicit solution (3.10):

$$f(\eta;\lambda)=\left[\frac{\alpha\left(m-1\right)^{2}}{2m\left(m+1\right)}\left(\lambda-\eta\right)^{2}\right]^{\frac{1}{m-1}}, \; 0<\eta<\lambda.$$

□

Prove of Theorem 3.2. We observe that

$$f(\eta) = \begin{cases} f(\eta;\lambda), & 0 < \eta < \lambda \\ 0, & \lambda < \eta < \infty \end{cases}, \tag{3.31}$$

is a weak solution of (3.1) and (3.6). Now we must show that, given $V > 0$, (3.31) is the only solution of (3.1), (3.5) and (3.6) with compact support.
Suppose that $f(\eta)$ is a weak solution of the problem (3.1) and (3.2) with compact support. By Lemma 3.8, this is possible only if $\alpha - 2\beta > 0$. Moreover,

$$f(\eta) \begin{cases} > 0, & \text{on} \quad \eta \in [0, \lambda), \\ = 0, & \text{on} \quad \eta \in [\lambda, \infty), \ \lambda > 0. \end{cases}$$

By Theorem 3.3, this is also the unique solution. Thus, we have proved Theorem 3.2. □

We conclude with a discussion of the implications of Theorems 3.2 and 3.3 for general form of self similar solutions to equation (1.1).

Theorem 3.9. *If* $\beta < 0, \ \gamma < 0$ *and* $\alpha \geq \frac{2\beta}{1-m}$,
the problem (3.4), (3.3) has a weak solution with compact support in the form

$$u(x,t) = c(t)\, f(\eta), \ \textit{with}\ \eta = \frac{x - b(t)}{a(t)},$$

where the "based profile" f is a solution of following differential equation

$$(f^m)''_{\eta\eta} = \alpha f + \beta\eta f'_\eta + \gamma f'_\eta, \quad 0 < \eta < \infty.$$

and the coefficients $c(t)$, $a(t)$ and $b(t)$ are given by
1)

$$\begin{cases} a(t) = (1 - A\beta t)^{\frac{1}{A}} \\ c(t) = (1 - A\beta t)^{\frac{-\alpha}{\beta A}} \\ b(t) = \frac{\gamma}{\beta}(1 - A\beta t)^{\frac{1}{A}} - \frac{\gamma}{\beta} \end{cases}, \ 0 < t < T,$$

if $\alpha > \frac{2\beta}{1-m}$, $\ A = 2 + \frac{\alpha}{\beta}(m-1)$ *with*

$$T = \frac{1}{2\beta + (m-1)\alpha},$$

and by
2)

$$\begin{cases} a(t) = \exp(-\beta t) \\ c(t) = \exp(\alpha t) \\ b(t) = \frac{\gamma}{\beta}\exp(-\beta t) - \frac{\gamma}{\beta} \end{cases}, \ 0 < t < \infty.$$

if $\alpha = \frac{2\beta}{1-m}$.

Proof. We have already proved in Theorem 3.3 the existence of "based profile" f with compact support if and only if $\beta < 0,\ \gamma < 0$ and $\alpha - 2\beta > 0$.
The coefficients $c\,(t)\,, a\,(t)$ and $b(t)$ are given by (2.6)

$$\begin{cases} a(t) = (1 - A\beta t)^{\frac{1}{A}} \\ c(t) = (1 - A\beta t)^{\frac{-\alpha}{\beta A}} \\ b(t) = \frac{\gamma}{\beta}\,(1 - A\beta t)^{\frac{1}{A}} - \frac{\gamma}{\beta} \end{cases}, \text{ for } 0 < t < T$$

with $A = 2 + \frac{\alpha}{\beta}(m-1)$ and $T = \frac{1}{2\beta+(m-1)\alpha} > 0$, then $2\beta + (m-1)\alpha > 0$, ie $\alpha > \frac{2\beta}{1-m}$.
Clearly the coefficients $c(t), a(t)$ and $b(t)$ are defined if $1 - A\beta t > 0$, this implies

$$t < \frac{1}{A\beta} = \frac{1}{2\beta + (m-1)\alpha} = T.$$

We see that the solution $u\,(x,t)$ blows up at $t = T$. and $T = \frac{1}{2\beta+(m-1)\alpha}$, is the blow-up time, such that the solution is well defined for all $0 < t < T$, while $u\,(x,t) \to \infty$ as $t = T$.
We recover also (2.10)

$$\begin{cases} a(t) = \exp\,(-\beta t) \\ c(t) = \exp\,(\alpha t) \\ b(t) = \frac{\gamma}{\beta}\exp\,(-\beta t) - \frac{\gamma}{\beta} \end{cases}, \text{ for } 0 < t < \infty.$$

if $2\beta + (m-1)\alpha = 0$, ie $\alpha = \frac{2\beta}{1-m}$.
Finally, we have proved that solution $u\,(x,t) = c\,(t)\,f\left[\frac{x-b(t)}{a(t)}\right]$ exists for $\beta < 0,\ \gamma < 0$ and $\alpha \geq \frac{2\beta}{1-m}$. □

Conclusion

In this work we have proved the existence of some class of solutions called "traveling profiles solutions" to the porous medium equation in one dimension. We have generalized the results obtained by Gilding who proved the existence of weak solutions with compact support under self similar form. we have also found new exact solutions of porous media equation in our general form of self similar solutions.

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Asymptotic Analysis for Coupled Parabolic Problem With Dirichlet-Fourier Boundary Conditions in a Thin Domain

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Abstract: This paper concerns the asymptotic behaviour of the initial boundary value problem of a class of reaction-diffusion systems (coupled parabolic systems) posed in a thin domain with Dirichlet-Fourier boundary conditions. We first prove the existence and uniqueness of the solution to the problem for fixed $\varepsilon > 0$ by the Galerkin method. Then, we give the characterization of the limiting behaviour of these solution as the thinness tends to zero.

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1. Introduction

Let Ω^ε be a bounded open subset of $\mathbb{R}^2$ with a sufficiently regular boundary $\partial\Omega^\varepsilon$. We define the thin domain as follows

$$\Omega^\varepsilon = \left\{x = (x_1, x_2) \in \mathbb{R}^2,\ 0 < x_1 < L,\, 0 < x_2 < \varepsilon h\left(x_1\right)\right\},$$

where $\varepsilon > 0$ is a small parameter that will tend to zero and $h\left(.\right)$ is a function of class C^1 defined on $[0, L]$ such that

$$0 < \underline{h} = \min_{x_1\in[0,L]} h(x_1) \leq h(x_1) \leq \overline{h} = \max_{x_1\in[0,L]} h(x_1),\, \forall x_1 \in [0, L]\,.$$

The boundary of Ω^ε consists of three parts: $\partial\Omega^\varepsilon = \partial\Omega_1^\varepsilon \cup \partial\Omega_2^\varepsilon \cup \partial\Omega_3^\varepsilon$, where $\partial\Omega_1^\varepsilon = \{x \in \partial\Omega^\varepsilon : x_2 = \varepsilon h\left(x_1\right)\}$ is the upper boundary, $\partial\Omega_3 = \left]0, L\right[$ is the bottom boundary and $\partial\Omega_2^\varepsilon = (\{x_1 = 0\} \cup \{x_1 = L\}) \times]0, \varepsilon h(x_1)[$ is the lateral part of the boundary of Ω^ε.

In the thin domain Ω^ε, we are interested in analyzing the behaviour of the solutions, as the parameter $\varepsilon \longrightarrow 0$, of the following coupled parabolic problem with Dirichlet-Fourier boundary conditions

$$\partial_t u^\varepsilon - \mathcal{A}_{\alpha^\varepsilon}(u^\varepsilon) + \lambda^\varepsilon v^\varepsilon = f^\varepsilon \text{ on } \Omega^\varepsilon \times (0,T), \tag{1.1}$$

$$\partial_t v^\varepsilon - \mathcal{A}_{\beta^\varepsilon}(v^\varepsilon) + \lambda^\varepsilon u^\varepsilon = g^\varepsilon \text{ on } \Omega^\varepsilon \times (0,T), \tag{1.2}$$

$$\left.\begin{array}{l} u^\varepsilon = 0 \\ v^\varepsilon = 0 \end{array}\right\} \text{ on } (\partial\Omega_1^\varepsilon \cup \partial\Omega_2^\varepsilon) \times (0,T), \tag{1.3}$$

$$\left.\begin{array}{l} \exists l_1^\varepsilon, r^\varepsilon \in \mathbb{R}_+^* : \partial_{n,\alpha^\varepsilon}(u^\varepsilon) + l_1^\varepsilon u^\varepsilon - r^\varepsilon v^\varepsilon = 0 \\ \exists l_2^\varepsilon, r^\varepsilon \in \mathbb{R}_+^* : \partial_{n,\beta^\varepsilon}(v^\varepsilon) + l_2^\varepsilon v^\varepsilon + r^\varepsilon u^\varepsilon = 0 \end{array}\right\} \text{ on }]0,L[\times (0,T), \tag{1.4}$$

where $\mathcal{A}_{c^\varepsilon}(\cdot)$ is the differential operator given by

$$\mathcal{A}_{c^\varepsilon}(\cdot) = \sum_{i,j=1}^{2} \partial_{x_i}\left[c_{ij}^\varepsilon(x)\,\partial_{x_j}(\cdot)\right],$$

λ^ε is a positive constant, $f^\varepsilon(\cdot)$, $g^\varepsilon(\cdot)$, $c_{ij}^\varepsilon(\cdot)$ are given functions and $\partial_{n,c^\varepsilon}(\cdot) = \sum_{i,j=1}^{2} c_{ij}^\varepsilon(x)\,\partial_{x_j}(\cdot).n_j$ indicate the derivative compared to the external normal on the boundary $]0,L[$, such that $n = (0,-1)$ is the unit outward normal to $]0,L[$. We complete the problem $(1.1)-(1.4)$ with the following initial conditions

$$(u^\varepsilon(x,0), v^\varepsilon(x,0)) = (0,0), \quad \forall x \in \Omega^\varepsilon. \tag{1.5}$$

We will deal with the problem $(1.1)-(1.5)$ under the following conditions:

$$c_{ij}^\varepsilon \in L_+^\infty(\Omega^\varepsilon),\; c_{ij}^\varepsilon(\cdot) = c_{ji}^\varepsilon(\cdot),\; 1 \le i,j \le 2,$$

also $\exists \mu_c > 0$, such that $\forall \eta \in \mathbb{R}^2$

$$\sum_{i,j=1}^{2} c_{ij}^\varepsilon(x)\,\eta_i\eta_j \ge \mu_c \sum_{i=1}^{2} (\eta_i)^2.$$

The study of thin structures with coarse features, fluids filling fine spheres, or even the process of chemical diffusion in the presence of narrow grains is very common in engineering and applied sciences. Recently, the study of the problems of thin structures has been extended to include many problems arising in applications such as mechanics of solids (thin rods, plates, shells), fluid dynamics (lubrication, meteorological problems, ocean dynamics). We refer to ([17], [11]) for some concrete applied problems.

Analyzing the properties of thin structures and the processes that take place on them and understanding how the micro-geometry of a thinner structure affects the overall properties of a material is a very important issue in engineering and materials design. In this regard, obtaining the specific equations of primitive models allows analysis of how different micro-scales affect primitive problems and allows for study and understanding in more complex situations.

Mathematically, the behaviour of the solutions of partial differential equations dealing with the problems of thin domains is a subject that has been addressed in the literature by different authors, we may mention; In [1], they studied the behaviour of the solutions of nonlinear parabolic problems posed in a domain that degenerates into a line segment (thin domain) which has an oscillating boundary. In the paper [15], the authors investigated the asymptotic behavior of the solutions to the p-Laplacian equation posed in a 2-dimensional open set which degenerates into a line segment when a positive parameter ε goes to zero. In [3], they studied the asymptotic behaviour of the solution of a boundary-value problem for the second-order elliptic equation in the bounded domain $\Omega^{\varepsilon} \subset \mathbb{R}^3$ with Robin type boundary conditions in the oscillating part of the boundary. The authors in [12], examined the limiting behaviour of dynamics for stochastic reaction-diffusion equations driven by an additive noise and a deterministic non-autonomous forcing on an $(n+1)$-dimensional thin region. A nonuniform Neumann boundary-value problem was considered for the Poisson equation in a thin domain Ω^{ε} coinciding with two thin rectangles connected through a joint of diameter $O(\varepsilon)$ in [10]. For the Stokes system in a thin domain with slip boundary conditions, we mention the works ([2], [7]). For the case of thin elastic structures, there are many works of literature, we mention for example; The authors in [4], addressed the problem of the junction between 3-dimensional and 2-dimensional linearly elastic structures and various asymptotic developments for the junction between plates. The asymptotic analysis of a dynamical problem of elasticity with non-linear dissipative term and non-linear friction of Tresca type was studied in [5]. Along the same lines, the authors in [6], have proved the asymptotic analysis of the solutions of a linear viscoelastic problem with a dissipative and source terms in a three-dimensional thin domain Ω^{ε}, with non-linear boundary conditions. The authors in [8], were interested in studying the asymptotic analysis of a mathematical model involving a frictionless contact between an quasi-static electro-viscoelastic and a conductive foundation in a three-dimensional thin domain Ω^{ε}.

On the other hand, a lot of mathematical systems models have been recently used to study pattern formation in population ecology, morphogenesis, neurobiology, chemical reactor theory, and in other fields, see for example ([18], [16], [9]). These phenomena are usually described by the coupled parabolic systems similar to (1.1)-(1.5).

The main purpose of the paper is to prove the existence and uniqueness of the weak solution for the boundary value problem $(1.1)-(1.5)$, and study the asymptotic behaviour of the solution when ε tends to zero.

The rest of the paper is organized as follows. In Section 2, we derive the weak formulation of the problem and prove the theorem of the existence and uniqueness of the weak solution by the classic Faedo-Galerkin method. In Section 3, we seek to know the behaviour of the solution when the small parameter ε tend to zero. For this purpose, we use the technique of the change of the variable to establish some estimates independent of the parameter ε. These estimates will be useful in order to prove the convergence results and the limit problem.

2. Weak formulation of the problem

For obtain the weak formulation of the problem, we introduce some spaces: let $L^2(\Omega^\varepsilon)$ be the usual Lebesgue space with the norm denoted by $\|\cdot\|_{L^2(\Omega^\varepsilon)}$ and $H^1(\Omega^\varepsilon)$ be the Sobolev space

$$H^1(\Omega^\varepsilon) = \left\{ u \in L^2(\Omega^\varepsilon) : \partial_{x_j} u \in L^2(\Omega^\varepsilon),\ j = 1, 2 \right\}.$$

We denote by $H_0^1(\Omega^\varepsilon)$ the closure of $D(\Omega^\varepsilon)$ in $H^1(\Omega^\varepsilon)$, and $H^{-1}(\Omega^\varepsilon)$ the dual space of $H_0^1(\Omega^\varepsilon)$. Let X a Banach space endowed with the norm $\|\cdot\|_X$, we denotes by $L^2(0, T; X)$ the space of functions $u : (0, T) \to X$ such that $u(t)$ is measurable for dt. This space is a Banach space endowed with the norm

$$\|u\|_{L^2(0.T;X)} = \left(\int_0^T \|u(s)\|_X^2 \, ds \right)^{\frac{1}{2}}.$$

We multiply the equation (1.1) by φ and the equation (1.2) by ψ where $(\varphi, \psi) \in H^1(\Omega^\varepsilon)^2$, then we integrate over Ω^ε and applying Green's formula, we obtain the following weak formulation of the problem

$$\text{Find } (u^\varepsilon, v^\varepsilon) \in (K^\varepsilon)^2 \text{ such that} \tag{2.1}$$

$$(\partial_t u^\varepsilon, \varphi) + a_{\alpha^\varepsilon}(u^\varepsilon, \varphi) + (\lambda^\varepsilon v^\varepsilon, \varphi) + \int_0^L (l_1^\varepsilon u^\varepsilon - r^\varepsilon v) .\varphi dx_1 = (f^\varepsilon, \varphi),\ \forall \varphi \in K^\varepsilon,$$

$$(\partial_t v^\varepsilon, \psi) + a_{\beta^\varepsilon}(v^\varepsilon, \psi) + (\lambda^\varepsilon u^\varepsilon, \psi) + \int_0^L (l_2^\varepsilon v^\varepsilon + r^\varepsilon u^\varepsilon) .\psi dx_1 = (g^\varepsilon, \psi),\ \forall \psi \in K^\varepsilon,$$

$$(u^\varepsilon(x, 0), v^\varepsilon(x, 0)) = (0, 0),$$

where

$$K^\varepsilon = \left\{ \zeta \in H^1(\Omega^\varepsilon) : \zeta = 0 \text{ on } \partial\Omega_1^\varepsilon \cup \partial\Omega_2^\varepsilon \right\},$$

and

$$a_{c^\varepsilon}(\cdot, \cdot) = \sum_{i,j=1}^{2} \int_{\Omega^\varepsilon} c_{ij}^\varepsilon(x) \, \partial_{x_i}(\cdot) \, \partial_{x_j}(\cdot) \, dx.$$

Theorem 1. *Assume that*

$$(f^\varepsilon, g^\varepsilon) \in L^2\left(0, T, L^2(\Omega^\varepsilon)\right)^2.$$

Then, there exists a unique solution $(u^\varepsilon, v^\varepsilon)$ *to problem* (2.1) *such that*

$$(u^\varepsilon, v^\varepsilon) \in L^2\left(0, T, H^1(\Omega^\varepsilon)\right)^2,$$

$$(\partial_t u^\varepsilon, \partial_t v^\varepsilon) \in L^2\left(0, T, L^2(\Omega^\varepsilon)\right)^2.$$

Proof.
A) Uniqueness.

Let $(u_1^\varepsilon, v_1^\varepsilon)$ and $(u_2^\varepsilon, v_2^\varepsilon)$ are two possible solutions. Taking in (2.1) $(\varphi, \psi) = (u_2^\varepsilon - u_1^\varepsilon, v_2^\varepsilon - v_1^\varepsilon)$ (respectively $(\varphi, \psi) = (u_1^\varepsilon - u_2^\varepsilon, v_1^\varepsilon - v_2^\varepsilon)$) in the equation relating to $(u_1^\varepsilon, v_1^\varepsilon)$ (respectively $(u_2^\varepsilon, v_2^\varepsilon)$), we find

$$\begin{aligned}&(\partial_t u_1^\varepsilon, u_2^\varepsilon - u_1^\varepsilon) + a_{\alpha^\varepsilon}(u_1^\varepsilon, u_2^\varepsilon - u_1^\varepsilon) + \lambda^\varepsilon (v_1^\varepsilon, u_2^\varepsilon - u_1^\varepsilon)\\ &+ \int_0^L (l_1^\varepsilon u_1^\varepsilon - r^\varepsilon v_1^\varepsilon)(u_2^\varepsilon - u_1^\varepsilon)\, dx_1\\ &= (f^\varepsilon, u_2^\varepsilon - u_1^\varepsilon),\end{aligned} \tag{2.2}$$

$$\begin{aligned}&(\partial_t u_2^\varepsilon, u_1^\varepsilon - u_2^\varepsilon) + a_{\alpha^\varepsilon}(u_2^\varepsilon, u_1^\varepsilon - u_2^\varepsilon) + \lambda^\varepsilon (v_2^\varepsilon, u_1^\varepsilon - u_2^\varepsilon)\\ &+ \int_0^L (l_1^\varepsilon u_2^\varepsilon - r^\varepsilon v_2)(u_1^\varepsilon - u_2^\varepsilon)\, dx_1\\ &= (f^\varepsilon, u_1^\varepsilon - u_2^\varepsilon),\end{aligned} \tag{2.3}$$

and

$$\begin{aligned}&(\partial_t v_1^\varepsilon, v_2^\varepsilon - v_1^\varepsilon) + a_{\beta^\varepsilon}(v_1^\varepsilon, v_2^\varepsilon - v_1^\varepsilon) + \lambda^\varepsilon (u_1^\varepsilon, v_2^\varepsilon - v_1^\varepsilon)\\ &+ \int_0^L (l_2^\varepsilon v_1^\varepsilon + r u_1)(v_2^\varepsilon - v_1^\varepsilon)\, dx_1\\ &= (g^\varepsilon, v_2^\varepsilon - v_1^\varepsilon),\end{aligned} \tag{2.4}$$

$$\begin{aligned}&(\partial_t v_2^\varepsilon, v_1^\varepsilon - v_2^\varepsilon) + a_{\beta^\varepsilon}(v_2^\varepsilon, v_1^\varepsilon - v_2^\varepsilon) + \lambda^\varepsilon (u_2^\varepsilon, v_1^\varepsilon - v_2^\varepsilon)\\ &+ \int_0^L (l_2^\varepsilon v_2^\varepsilon + r^\varepsilon u_2^\varepsilon)(v_1^\varepsilon - v_2^\varepsilon)\, dx_1\\ &= (g^\varepsilon, v_1^\varepsilon - v_2^\varepsilon),\end{aligned} \tag{2.5}$$

we put $\mathcal{U}^\varepsilon = u_1^\varepsilon - u_2^\varepsilon$, and $\mathcal{V}^\varepsilon = v_1^\varepsilon - v_2^\varepsilon$, thus the sum of (2.2) with (2.3), and (2.4) with (2.5) gives

$$-(\partial_t \mathcal{U}^\varepsilon, \mathcal{U}^\varepsilon) - a_{\alpha^\varepsilon}(\mathcal{U}^\varepsilon, \mathcal{U}^\varepsilon) - \lambda^\varepsilon (\mathcal{V}^\varepsilon, \mathcal{U}^\varepsilon) - l_1^\varepsilon \int_0^L \mathcal{U}^\varepsilon . \mathcal{U}^\varepsilon dx_1 + \int_0^L r^\varepsilon \mathcal{V}^\varepsilon . \mathcal{U}^\varepsilon dx_1 = 0,$$

and

$$-(\partial_t \mathcal{V}^\varepsilon, \mathcal{V}^\varepsilon) - a_{\beta^\varepsilon}(\mathcal{V}^\varepsilon, \mathcal{V}^\varepsilon) - \lambda^\varepsilon (\mathcal{U}^\varepsilon, \mathcal{V}^\varepsilon) - l_2^\varepsilon \int_0^L \mathcal{V}^\varepsilon . \mathcal{V}^\varepsilon dx_1 - \int_0^L r^\varepsilon \mathcal{U}^\varepsilon . \mathcal{V}^\varepsilon dx_1 = 0.$$

Now, adding the two equations above, we find

$$\begin{aligned}&(\partial_t \mathcal{U}^\varepsilon(s), \mathcal{U}^\varepsilon(s)) + a_{\alpha^\varepsilon}(\mathcal{U}^\varepsilon(s), \mathcal{U}^\varepsilon(s)) + a_{\beta^\varepsilon}(\mathcal{V}^\varepsilon(s), \mathcal{V}^\varepsilon(s))\\ &\le -2\lambda^\varepsilon (\mathcal{V}^\varepsilon(s), \mathcal{U}^\varepsilon(s)).\end{aligned} \tag{2.6}$$

On the other hand, we have

$$\int_0^t a_{c^\varepsilon}\left(\Psi\left(s\right),\Psi\left(s\right)\right)ds\geq\mu_c\int_0^t\left\|\Psi\left(s\right)\right\|^2_{H^1(\Omega^\varepsilon)}ds,$$

then, integrating the inequality (2.6) over $(0,t)$, we get

$$\left(\left\|\mathcal{U}^\varepsilon\left(s\right)\right\|^2_{L^2(\Omega^\varepsilon)}+\left\|\mathcal{V}^\varepsilon\left(s\right)\right\|^2_{L^2(\Omega^\varepsilon)}\right)+\int_0^t\left(\mu_\alpha\left\|\mathcal{U}^\varepsilon\left(s\right)\right\|^2_{H^1(\Omega^\varepsilon)}+\mu_\beta\left\|\mathcal{V}^\varepsilon\left(s\right)\right\|^2_{H^1(\Omega^\varepsilon)}\right)ds$$
$$\leq 2\lambda^\varepsilon\int_0^t\left(\left\|\mathcal{U}^\varepsilon\left(s\right)\right\|^2_{L^2(\Omega^\varepsilon)}+\left\|\mathcal{V}^\varepsilon\left(s\right)\right\|^2_{L^2(\Omega^\varepsilon)}\right)ds,$$

now, using Gronwall's lemma, we find

$$\left(\mathcal{U}^\varepsilon\left(s\right),\mathcal{V}^\varepsilon\left(s\right)\right)=\left(0,0\right),\ \ \forall s\in\left(0,T\right).$$

Thus, we obtain the uniqueness of the solution.

B) Existence.

To show the existence of the solution, we use the Faedo-Galerkin approximation.

Let $\{K_m^\varepsilon\}$ be a family of finite dimensional spaces. It introduces a sequence $\left(w_j^\varepsilon\right)$ of functions having the following properties:

* $w_j^\varepsilon\in K^\varepsilon$, $\forall j=1,...,m$.

* The family $\{w_1^\varepsilon,w_2^\varepsilon,...,w_m^\varepsilon\}$ is linearly independent.

* The $K_m^\varepsilon=[w_1^\varepsilon,w_2^\varepsilon,...,w_m^\varepsilon]$ generated by $\{w_1^\varepsilon,w_2^\varepsilon,...,w_m^\varepsilon\}$ is dense in K^ε.

Let $(u_m^\varepsilon,v_m^\varepsilon)=\left(u_m^\varepsilon\left(t\right),v_m^\varepsilon\left(t\right)\right)$ be an approximate solution such that

$$u_m^\varepsilon\left(t\right)=\sum_{j=1}^m R_{jm}\left(t\right)w_j^\varepsilon,\quad v_m^\varepsilon\left(t\right)=\sum_{j=1}^m P_{jm}\left(t\right)w_j^\varepsilon,$$

where $R_{jm}\left(t\right)$ and $P_{jm}\left(t\right)$ are determined by the following ordinary differential equations:

$$\left(\partial_t u_m^\varepsilon,w_j^\varepsilon\right)+a_{\alpha^\varepsilon}(u_m^\varepsilon,w_j^\varepsilon)+\lambda^\varepsilon(v_m^\varepsilon,w_j^\varepsilon)+\int_0^L\left(l_1^\varepsilon u_m^\varepsilon-r^\varepsilon v_m^\varepsilon\right).\left(w_j^\varepsilon\right)dx_1 \tag{2.7}$$
$$=\left(f^\varepsilon,w_j^\varepsilon\right),\ 1\leq j\leq m,$$
$$\left(\partial_t v_m^\varepsilon,w_j^\varepsilon\right)+a_{\beta^\varepsilon}(v_m^\varepsilon,w_j^\varepsilon)+\lambda^\varepsilon(u_m^\varepsilon,w_j^\varepsilon)+\int_0^L\left(l_2^\varepsilon v_m^\varepsilon+r^\varepsilon u_m^\varepsilon\right).\left(w_j^\varepsilon\right)dx_1$$
$$=\left(g^\varepsilon,w_j^\varepsilon\right),\ 1\leq j\leq m,$$

with the initial conditions

$$u_m(x,0)=0,$$
$$u_m\left(0\right)=\sum_{j=1}^m\gamma_{jm}\left(0\right)w_j\overset{m\to\infty}{\longrightarrow}0\text{ in }K^\varepsilon,$$

$$v_m(x,0)=0,$$
$$v_m\left(0\right)=\sum_{j=1}^m\eta_{jm}\left(0\right)w_j\overset{m\to\infty}{\longrightarrow}0\text{ in }K^\varepsilon.$$

Now, we will establish some estimates independent on m.

The first estimate.

By multiplying the first and the second equation of (2.7) by $R_{jm}(t)$ and $P_{jm}(t)$ respectively, then sum over j from 1 to m, we obtain

$$(\partial_t u_m^\varepsilon, u_m^\varepsilon) + a_{\alpha^\varepsilon}(u_m^\varepsilon, u_m^\varepsilon) + \lambda^\varepsilon (v_m^\varepsilon, u_m^\varepsilon) + \int_0^L (l_1^\varepsilon u_m^\varepsilon - r^\varepsilon v_m^\varepsilon) u_m^\varepsilon dx_1 = (f^\varepsilon, u_m^\varepsilon),$$

and

$$(\partial_t v_m^\varepsilon, v_m^\varepsilon) + a_{\beta^\varepsilon}(v_m^\varepsilon, v_m^\varepsilon) + \lambda^\varepsilon (u_m^\varepsilon, v_m^\varepsilon) + \int_0^L (l_2^\varepsilon v_m^\varepsilon + r^\varepsilon u_m^\varepsilon) v_m^\varepsilon dx_1 = (g^\varepsilon, v_m^\varepsilon),$$

by integrating over $(0,t)$ the two equations above, and summing the result, we deduce that

$$\begin{aligned}
&\frac{1}{2}\|u_m^\varepsilon(s)\|^2_{L^2(\Omega^\varepsilon)} + \frac{1}{2}\|v_m^\varepsilon(s)\|^2_{L^2(\Omega^\varepsilon)} ds + \mu_\alpha \int_0^t \|u_m^\varepsilon(s)\|^2_{H^1(\Omega^\varepsilon)} ds \\
&+ \mu_\beta \int_0^t \|v_m^\varepsilon(s)\|^2_{H^1(\Omega^\varepsilon)} ds + l_1^\varepsilon \int_0^t \int_0^L |u_m^\varepsilon(s)|^2 dx_1 ds + l_2^\varepsilon \int_0^t \int_0^L |v_m^\varepsilon(s)|^2 dx_1 ds \\
&\leq \int_0^t (f^\varepsilon(s), u_m^\varepsilon(s)) ds + \int_0^t (g^\varepsilon(s), v_m(s)) ds - 2\lambda^\varepsilon \int_0^t (v_m^\varepsilon(s), u_m^\varepsilon(s)) ds.
\end{aligned}$$

Now, using the fact that

$$\int_0^t |(f^\varepsilon(s), u_m^\varepsilon(s))| ds \leq \int_0^t \|f^\varepsilon(s)\|^2_{L^2(\Omega^\varepsilon)} ds + \int_0^t \|u_m^\varepsilon(s)\|^2_{L^2(\Omega^\varepsilon)} ds,$$

$$\int_0^t |(g^\varepsilon(s), v_m^\varepsilon(s))| ds \leq \int_0^t \|g^\varepsilon(s)\|^2_{L^2(\Omega^\varepsilon)} ds + \int_0^t \|v_m^\varepsilon(s)\|^2_{L^2(\Omega^\varepsilon)} ds,$$

and

$$2\lambda^\varepsilon \int_0^t |(v_m^\varepsilon(s), u_m^\varepsilon(s))| ds \leq 2\lambda^\varepsilon \int_0^t \|u_m^\varepsilon(s)\|^2_{L^2(\Omega^\varepsilon)} ds + 2\lambda^\varepsilon \int_0^t \|v_m^\varepsilon(s)\|^2_{L^2(\Omega^\varepsilon)} ds,$$

we find the following estimate

$$\|u_m^\varepsilon(s)\|^2_{L^2(\Omega^\varepsilon)} + \|v_m^\varepsilon(s)\|^2_{L^2(\Omega^\varepsilon)} + 2\mu_\alpha \int_0^t \|u_m^\varepsilon(s)\|^2_{H^1(\Omega^\varepsilon)} ds$$

$$+ 2\mu_\beta \int_0^t \|v_m^\varepsilon(s)\|^2_{H^1(\Omega^\varepsilon)} ds + 2l_1^\varepsilon \int_0^t \|u_m^\varepsilon(s)\|^2_{L^2(]0,L[)} ds + 2l_2^\varepsilon \int_0^t \|v_m^\varepsilon(s)\|^2_{L^2(]0,L[)}$$

$$\leq (2+4\lambda^\varepsilon) \int_0^t \left(\|u_m^\varepsilon(s)\|^2_{L^2(\Omega^\varepsilon)} + \|v_m^\varepsilon(s)\|^2_{L^2(\Omega^\varepsilon)} \right) ds$$

$$+ 2\int_0^t \|f^\varepsilon(s)\|^2_{L^2(\Omega^\varepsilon)} ds + 2\int_0^t \|g^\varepsilon(s)\|^2_{L^2(\Omega^\varepsilon)} ds.$$

After applying Gronwall's lemma in the above inequality, we get

$$\begin{aligned} &\|u_m^\varepsilon(s)\|^2_{L^2(\Omega^\varepsilon)} + \|v_m^\varepsilon(s)\|^2_{L^2(\Omega^\varepsilon)} + \|u_m^\varepsilon(s)\|^2_{L^2(0,T,H^1(\Omega^\varepsilon))} \\ &+ \|v_m^\varepsilon(s)\|^2_{L^2(0,T,H^1(\Omega^\varepsilon))} + \|u_m^\varepsilon(s)\|^2_{L^2(0,T,L^2(]0,L[))} + \|v_m^\varepsilon(s)\|^2_{L^2(0,T,L^2(]0,L[))} \\ &\leq c_T^\varepsilon, \end{aligned} \tag{2.8}$$

where c_T^ε is a constant independent on m.

The second estimate.

We multiply the first and the second equation of (2.7) by $R'_{jm}(\tau)$ and $P'_{jm}(\tau)$ respectively, then sum over j from 1 to m, we have

$$(\partial_t u_m^\varepsilon, \partial_t u_m^\varepsilon) + a_{\alpha^\varepsilon}(u_m^\varepsilon, \partial_t u_m^\varepsilon) + \lambda^\varepsilon(v_m^\varepsilon, \partial_t u_m^\varepsilon) + \int_0^L (l_1^\varepsilon u_m^\varepsilon - r^\varepsilon v_m^\varepsilon)\, \partial_t u_m^\varepsilon dx_1$$

$$= (f^\varepsilon, \partial_t u_m^\varepsilon),$$

$$(\partial_t v_m^\varepsilon, \partial_t v_m^\varepsilon) + a_{\beta^\varepsilon}(v_m^\varepsilon, \partial_t v_m^\varepsilon) + \lambda^\varepsilon(u_m^\varepsilon, \partial_t v_m^\varepsilon) + \int_0^L (l_2^\varepsilon v_m^\varepsilon + r^\varepsilon u_m^\varepsilon)\, \partial_t v_m^\varepsilon dx_1$$

$$= (g^\varepsilon, \partial_t v_m^\varepsilon),$$

as $\int_0^t a_{\alpha^\varepsilon}(u_m^\varepsilon(s), \partial_t u_m^\varepsilon(s))\, ds$, $\int_0^t \int_0^L (u_m^\varepsilon(s) . \partial_t u_m^\varepsilon(s))\, dx_1 ds$, $\int_0^t \int_0^L (v_m^\varepsilon . \partial_t v_m^\varepsilon)\, dx_1 ds$ and $\int_0^t a_{\beta^\varepsilon}(v_m^\varepsilon(s), \partial_t v_m^\varepsilon(s))\, ds$ are positive terms, integrating from 0 to t, we find

$$\int_0^t \|\partial_t u_m^\varepsilon(s)\|^2_{L^2(\Omega^\varepsilon)} ds + \int_0^t \|\partial_t v_m^\varepsilon(s)\|^2_{L^2(\Omega^\varepsilon)} ds + \lambda^\varepsilon \int_0^t (v_m^\varepsilon(s), \partial_t u_m^\varepsilon(s))\, ds$$

$$+ \lambda^\varepsilon \int_0^t (u_m^\varepsilon(s), \partial_t v_m^\varepsilon(s))$$

$$\leq \int_0^t (f^\varepsilon(s), \partial_t u_m^\varepsilon(s))\, ds + \int_0^t (g^\varepsilon(s), \partial_t v_m^\varepsilon(s))$$

$$- r^\varepsilon \int_0^t \int_0^L u_m^\varepsilon(s) . \partial_t v_m^\varepsilon(s)\, dx_1 ds + r^\varepsilon \int_0^t \int_0^L v_m^\varepsilon(s) . (\partial_t u_m^\varepsilon(s))\, dx_1 ds,$$

Next, by using Cauchy-Schwarz inequality, trace theorem and Young's inequality, we obtain

$$\begin{aligned}
&\frac{1}{4}\int_0^t \|\partial_t u_m^\varepsilon(s)\|^2_{L^2(\Omega^\varepsilon)} ds + \frac{1}{4}\int_0^t \|\partial_t v_m^\varepsilon(s)\|^2_{L^2(\Omega^\varepsilon)} ds \qquad (2.9)\\
&\le 4\lambda^\varepsilon \int_0^t \|v_m^\varepsilon(s)\|_{L^2(\Omega^\varepsilon)} ds + 4\lambda^\varepsilon \int_0^t \|u_m^\varepsilon(s)\|_{L^2(\Omega^\varepsilon)} ds + 4\int_0^t \|f^\varepsilon(s)\|_{L^2(\Omega^\varepsilon)} ds\\
&+ 4r^\varepsilon C(\Omega^\varepsilon)\int_0^t \left(\|u_m^\varepsilon(s)\|_{L^2(]0,L[)} + \|v_m^\varepsilon(s)\|_{L^2(]0,L[)}\right) ds + 4\int_0^t \|g^\varepsilon(s)\|_{L^2(\Omega^\varepsilon)} ds,
\end{aligned}$$

on the other hand, by the estimate (2.8), we have

$$\begin{aligned}
&\int_0^t \|v_m^\varepsilon(s)\|_{L^2(]0,L[)} ds + \int_0^t \|u_m^\varepsilon(s)\|_{L^2(]0,L[)} ds\\
&+ \int_0^t \|u_m^\varepsilon(s)\|_{L^2(\Omega^\varepsilon)} ds + \int_0^t \|v_m^\varepsilon(s)\|_{L^2(\Omega^\varepsilon)} ds\\
&\le c_T^\varepsilon.
\end{aligned}$$

So, from (2.9), we deduce that there exists $c_1^\varepsilon > 0$ which does not depend to m such that

$$\|\partial_t u_m^\varepsilon(s)\|^2_{L^2(0,T,L^2(\Omega^\varepsilon))} + \|\partial_t v_m^\varepsilon(s)\|^2_{L^2(0,T,L^2(\Omega^\varepsilon))} \le c_1^\varepsilon. \qquad (2.10)$$

C) Limit procedure.

From (2.8) and (2.10), we conclude that there exists a subsequence of the sequence $(u_m^\varepsilon, v_m^\varepsilon)$, with the same notation, such that

$$\begin{aligned}
(u_m^\varepsilon, v_m^\varepsilon) &\rightharpoonup (u^\varepsilon, v^\varepsilon) \text{ weakly in } L^2\left(0,T,H^1(\Omega^\varepsilon)\right)^2,\\
(\partial_t u_m^\varepsilon, \partial_t v_m^\varepsilon) &\rightharpoonup (\partial_t u^\varepsilon, \partial_t v^\varepsilon) \text{ weakly in } L^2\left(0,T,L^2(\Omega^\varepsilon)\right)^2.
\end{aligned}$$

Finally, using the arguments in reference [13] and the fact that the space K_m^ε is dense in K^ε, we pass to the limit as m→ 0 in (2.7), we find that u^ε and v^ε satisfy

$$\begin{aligned}
&(\partial_t u^\varepsilon, \varphi) + a_{\alpha^\varepsilon}(u^\varepsilon, \varphi) + \lambda^\varepsilon(v^\varepsilon, \varphi) + \int_0^L (l_1^\varepsilon u^\varepsilon - r^\varepsilon v^\varepsilon).(\varphi)\, dx_1\\
=\;& (f^\varepsilon, \varphi), \forall \varphi \in K^\varepsilon,
\end{aligned}$$

and

$$\begin{aligned}
&(\partial_t v^\varepsilon, \psi) + a_{\beta^\varepsilon}(v^\varepsilon, \psi) + \lambda^\varepsilon(u^\varepsilon, \psi) + \int_0^L (l_2^\varepsilon v_m^\varepsilon + r^\varepsilon u_m^\varepsilon).(\psi)\, dx_1\\
=\;& (g^\varepsilon, \psi), \forall \psi \in K^\varepsilon,
\end{aligned}$$

this imply that

$$\left.\begin{array}{l}
\partial_t u^\varepsilon + \mathcal{A}_{\alpha^\varepsilon}(u^\varepsilon) + \lambda^\varepsilon v^\varepsilon = f^\varepsilon,\\
\partial_t v^\varepsilon + \mathcal{A}_{\beta^\varepsilon}(v^\varepsilon) + \lambda^\varepsilon u^\varepsilon = g^\varepsilon,
\end{array}\right\} \text{ a.e in } \Omega^\varepsilon \times (0,T).$$

Theorem **1** is proved. □

3. Asymptotic analysis of the problem

For the asymptotic analysis of the problem (2.1), we use the approach which consists in transposing the problem initially posed in the domain which depends on a small parameter ε in an equivalent problem posed in the fixed domain which is independent on ε.

3.1. The problem in a fixed domain and some estimates

By introducing the change of variables $z = \frac{x_2}{\varepsilon}$, we get the fixed domain

$$\Omega = \left\{ (x_1, z) \in \mathbb{R}^2,\ 0 < x_1 < L,\ 0 < z < h(x_1) \right\},$$

we denote by $\partial\Omega = \partial\Omega_1 \cup \partial\Omega_2 \cup \partial\Omega_3$ its boundary, where $\partial\Omega_1 = \{x \in \partial\Omega : x_2 = h(x_1)\}$, $\partial\Omega_2 = (\{x_1 = 0\} \cup \{x_1 = L\}) \times]0, h(x_1)[$ and $\partial\Omega_3 =]0, L[$.

Now, we define the following functions in Ω

$$u^\varepsilon(x_1, x_2, t) = \hat{u}^\varepsilon(x_1, z, t), \quad v^\varepsilon(x_1, x_2, t) = \hat{v}^\varepsilon(x_1, z, t).$$

Let us assume the following dependence (with respect of ε) of the data

$$\begin{aligned}
\alpha_{ij}^\varepsilon(x_1, x_2) &= \hat{\alpha}_{ij}(x_1, z),\ 1 \le i, j \le 2, \\
\beta_{ij}^\varepsilon(x_1, x_2) &= \hat{\beta}_{ij}(x_1, z),\ 1 \le i, j \le 2, \\
\varepsilon^2 f^\varepsilon(x_1, x_2, t) &= \hat{f}(x_1, z, t), \quad \varepsilon^2 g^\varepsilon(x_1, x_2, t) = \hat{g}(x_1, z, t), \\
\varepsilon^2 \lambda^\varepsilon &= \hat{\lambda},\ \ \varepsilon l_1^\varepsilon = \hat{l}_1, \quad \varepsilon l_2^\varepsilon = \hat{l}_2, \quad \varepsilon r^\varepsilon = \hat{r}.
\end{aligned} \tag{3.1}$$

Assuming (3.1), the problem (2.1) leads to the following form

$$\text{Find } (\hat{u}^\varepsilon, \hat{v}^\varepsilon) \in K, \text{ such that} \tag{3.2}$$

$$\int_\Omega \varepsilon^2 \partial_t \hat{u}^\varepsilon \varphi dx_1 dz + \varepsilon^2 \int_\Omega \hat{\alpha}_{11}(x_1, z)(\partial_{x_1}\hat{u}^\varepsilon)(\partial_{x_1}\varphi) dx_1 dz$$

$$+\varepsilon \int_\Omega \hat{\alpha}_{12}(x_1, z)(\partial_{x_1}\hat{u}^\varepsilon)(\partial_z\varphi) dx_1 dz + \varepsilon \int_\Omega \hat{\alpha}_{21}(x_1, z)(\partial_z\hat{u}^\varepsilon)(\partial_{x_1}\varphi) dx_1 dz$$

$$+\int_\Omega \hat{\alpha}_{22}(x_1, z)(\partial_z\hat{u}^\varepsilon)(\partial_z\varphi) dx_1 dz + \hat{\lambda} \int_\Omega \hat{v}^\varepsilon.\varphi dx_1 dz + \hat{l}_1 \int_0^L \hat{u}^\varepsilon.\varphi dx_1 - \hat{r} \int_0^L \hat{v}^\varepsilon.\varphi dx_1$$

$$= \int_\Omega \hat{f} \varphi dx_1 dz, \quad \forall \varphi \in K,$$

$$\int_{\Omega} \varepsilon^2 \left(\partial_t \hat{v}^\varepsilon\right) \psi dx_1 dz + \varepsilon^2 \int_{\Omega} \hat{\beta}_{11}\left(x_1, z\right)\left(\partial_{x_1} \hat{v}^\varepsilon\right)\left(\partial_{x_1} \psi\right) dx_1 dz \tag{3.3}$$
$$+\varepsilon \int_{\Omega} \hat{\beta}_{12}\left(x_1, z\right)\left(\partial_{x_1} \hat{v}^\varepsilon\right)\left(\partial_z \psi\right) dx_1 dz + \varepsilon \int_{\Omega} \hat{\beta}_{21}\left(x_1, z\right)\left(\partial_z \hat{v}^\varepsilon\right)\left(\partial_{x_1} \psi\right) dx_1 dz$$
$$+\int_{\Omega} \hat{\beta}_{22}\left(x_1, z\right)\left(\partial_z \hat{v}^\varepsilon\right)\left(\partial_z \psi\right) dx_1 dz + \hat{\lambda} \int_{\Omega} \hat{u}^\varepsilon . \varphi dx_1 dz + \hat{l}_2 \int_0^L \hat{v}^\varepsilon . \psi dx_1 + \hat{r} \int_0^L \hat{v}^\varepsilon . \psi dx_1$$
$$= \int_{\Omega} \hat{g} \psi dx_1 dz, \ \forall \psi \in K,$$

$$\left(\hat{u}^\varepsilon(x_1, z, 0), \hat{v}^\varepsilon(x_1, z, 0)\right) = (0, 0), \tag{3.4}$$

where

$$K = \left\{\zeta \in H^1(\Omega) : \zeta = 0 \text{ on } \partial\Omega_1 \cup \partial\Omega_2\right\}.$$

Now, we will obtain estimates on $\hat{u}^\varepsilon$, $\hat{v}^\varepsilon$, $\partial_t \hat{u}^\varepsilon$ and $\partial_t \hat{v}^\varepsilon$. These estimates will be useful in order for obtaining the convergence results and the limit problem.

Theorem 2. *Assume that f^ε, $g^\varepsilon \in L^2\left(0, T, L^2\left(\Omega^\varepsilon\right)\right)$ and $4\hat{\lambda}\bar{h}^2 < \min\left(\mu_\alpha, \mu_\beta\right)$. Then there exists a constant c independent on ε such that*

$$\begin{aligned} &\|\varepsilon \hat{u}^\varepsilon\|_{L^2(\Omega)}^2 + \|\varepsilon \hat{v}^\varepsilon\|_{L^2(\Omega)}^2 + \|\varepsilon \partial_{x_1} \hat{u}^\varepsilon\|_{L^2(0,T,L^2(\Omega))}^2 + \|\partial_z \hat{u}^\varepsilon\|_{L^2(0,T,L^2(\Omega))}^2 \\ &+ \|\varepsilon \partial_{x_1} \hat{v}^\varepsilon\|_{L^2(0,T,L^2(\Omega))}^2 + \|\partial_z \hat{v}^\varepsilon\|_{L^2(0,T,L^2(\Omega))}^2 + \|u^\varepsilon(s)\|_{L^2(0,T,L^2(]0,L[))}^2 \\ &+ \|v^\varepsilon(s)\|_{L^2(0,T,L^2(]0,L[))}^2 \\ &\le c, \end{aligned} \tag{3.5}$$

$$\left\|\varepsilon^2 \partial_t \hat{u}^\varepsilon\right\|_{L^2(0,T,L^2(\Omega))}^2 + \left\|\varepsilon^2 \partial_t \hat{v}^\varepsilon\right\|_{L^2(0,T,L^2(\Omega))}^2 \le c. \tag{3.6}$$

Proof. Let $(u^\varepsilon, v^\varepsilon)$ be the solution of the problem $(1.1) - (1.2)$. Putting $(\varphi, \psi) = (u^\varepsilon, v^\varepsilon)$ in (2.1), leads to

$$\left(\partial_t u^\varepsilon, u^\varepsilon\right) + a_{\alpha^\varepsilon}\left(u^\varepsilon, u^\varepsilon\right) + \lambda^\varepsilon\left(v^\varepsilon, u^\varepsilon\right) + \int_0^L \left(l_1^\varepsilon u^\varepsilon - r^\varepsilon v^\varepsilon\right) . u^\varepsilon dx_1 = \left(f^\varepsilon, u^\varepsilon\right),$$

and

$$\left(\partial_t v^\varepsilon, v^\varepsilon\right) + a_{\beta^\varepsilon}\left(v^\varepsilon, v^\varepsilon\right) + \lambda^\varepsilon\left(u^\varepsilon, v^\varepsilon\right) + \int_0^L \left(l_2^\varepsilon v^\varepsilon + r^\varepsilon u^\varepsilon\right) . v^\varepsilon dx_1 = \left(g^\varepsilon, v^\varepsilon\right),$$

by integrating the two equalities over $(0,t)$ and summing the result, we get

$$\frac{1}{2}\left(\|u^{\varepsilon}(s)\|^2_{L^2(\Omega^{\varepsilon})}+\|v^{\varepsilon}(s)\|^2_{L^2(\Omega^{\varepsilon})}\right)+\mu_{\alpha}\int_0^t\|\nabla u^{\varepsilon}(s)\|^2_{L^2(\Omega^{\varepsilon})^2}\,ds \tag{3.7}$$

$$+\mu_{\beta}\int_0^t\|\nabla v^{\varepsilon}(s)\|^2_{L^2(\Omega^{\varepsilon})^2}\,ds+l_1^{\varepsilon}\int_0^t\|u^{\varepsilon}(s)\|^2_{L^2(]0,L[)}\,ds+l_2^{\varepsilon}\int_0^t\|v^{\varepsilon}(s)\|^2_{L^2(]0,L[)}\,ds$$

$$\leq\int_0^t\int_{\Omega^{\varepsilon}}|f^{\varepsilon}(s).u^{\varepsilon}(s)|\,dxds+\int_0^t\int_{\Omega^{\varepsilon}}|g^{\varepsilon}(s).v^{\varepsilon}(s)|\,dxds$$

$$+2\lambda^{\varepsilon}\int_0^t\int_{\Omega^{\varepsilon}}|u^{\varepsilon}(s).v^{\varepsilon}(s)|\,dxds.$$

Now, we estimate the right-hand side of the inequality (3.7). Using the Cauchy-Schwarz inequality, Poincaré's inequality

$$\|\varphi\|_{L^2(\Omega^{\varepsilon})}\leq\varepsilon\bar{h}\,\|\nabla\varphi\|_{L^2(\Omega^{\varepsilon})^2},\ \forall\varphi\in K^{\varepsilon},$$

and Young's inequality, we have

$$\begin{aligned}\int_0^t\int_{\Omega^{\varepsilon}}|f^{\varepsilon}(s).u^{\varepsilon}(s)|\,dxds \quad &\leq \quad \frac{2\varepsilon^2\bar{h}^2}{\mu_{\alpha}}\int_0^t\|f^{\varepsilon}(s)\|^2_{L^2(\Omega^{\varepsilon})}\,ds+ \\ &\frac{\mu_{\alpha}}{2}\int_0^t\|\nabla u^{\varepsilon}(s)\|^2_{L^2(\Omega^{\varepsilon})^2}\,ds,\end{aligned} \tag{3.8}$$

and

$$\begin{aligned}\int_0^t\int_{\Omega^{\varepsilon}}|g^{\varepsilon}(s).v^{\varepsilon}(s)|\,dxds \quad &\leq \tfrac{2\varepsilon^2\bar{h}^2}{\mu_{\beta}}\int_0^t\|g^{\varepsilon}(s)\|^2_{L^2(\Omega^{\varepsilon})}\,ds+ \\ &\tfrac{\mu_{\beta}}{2}\int_0^t\|\nabla v^{\varepsilon}(s)\|^2_{L^2(\Omega^{\varepsilon})^2}\,ds,\end{aligned} \tag{3.9}$$

also, we have

$$\lambda^{\varepsilon}\int_0^t\int_{\Omega^{\varepsilon}}|v^{\varepsilon}(s).u^{\varepsilon}(s)|\,dxds\leq\lambda^{\varepsilon}\int_0^t\|v^{\varepsilon}(s)\|_{L^2(\Omega^{\varepsilon})}.\|u^{\varepsilon}(s)\|_{L^2(\Omega^{\varepsilon})}\,ds \tag{3.10}$$

$$\leq\hat{\lambda}\bar{h}^2\int_0^t\|\nabla v^{\varepsilon}(s)\|^2_{L^2(\Omega^{\varepsilon})^2}\,ds+\hat{\lambda}\bar{h}^2\int_0^t\|\nabla u^{\varepsilon}(s)\|^2_{L^2(\Omega^{\varepsilon})^2}\,ds.$$

Injecting the inequalities (3.8), (3.9) and (3.10) in (3.7), we obtain

$$\begin{aligned}
&\|u^\varepsilon(s)\|^2_{L^2(\Omega^\varepsilon)} + \|v^\varepsilon(s)\|^2_{L^2(\Omega^\varepsilon)} + \left(\frac{\mu_\alpha}{2} - 2\hat{\lambda}\bar{h}^2\right)\int_0^t \|\nabla u^\varepsilon(s)\|^2_{L^2(\Omega^\varepsilon)^2}\,ds \\
&+ \left(\frac{\mu_\beta}{2} - 2\hat{\lambda}\bar{h}^2\right)\int_0^t \|\nabla v^\varepsilon(s)\|^2_{L^2(\Omega^\varepsilon)^2}\,ds + l_1^\varepsilon \int_0^t \|u^\varepsilon(s)\|^2_{L^2(]0,L[)}\,ds \\
&+ l_2^\varepsilon \int_0^t \|v^\varepsilon(s)\|^2_{L^2(]0,L[)}\,ds \\
&\le \frac{2\varepsilon^2\bar{h}^2}{\mu_\alpha}\int_0^t \|f^\varepsilon(s)\|^2_{L^2(\Omega^\varepsilon)}\,ds + \frac{2\varepsilon^2\bar{h}^2}{\mu_\beta}\int_0^t \|g^\varepsilon(s)\|^2_{L^2(\Omega^\varepsilon)}\,ds,
\end{aligned} \tag{3.11}$$

as

$$\varepsilon^2\|f^\varepsilon\|^2_{L^2(\Omega^\varepsilon)} = \varepsilon^{-1}\left\|\hat{f}\right\|^2_{L^2(\Omega)}, \quad \varepsilon^2\|g^\varepsilon\|^2_{L^2(\Omega^\varepsilon)} = \varepsilon^{-1}\|\hat{g}\|^2_{L^2(\Omega)},$$

we multiply the inequality (3.11) by ε. Then we obtain

$$\begin{aligned}
&\varepsilon\|u^\varepsilon(s)\|^2_{L^2(\Omega^\varepsilon)} + \varepsilon\|v^\varepsilon(s)\|^2_{L^2(\Omega^\varepsilon)} + \left(\frac{\mu_\alpha}{2} - 2\hat{\lambda}\bar{h}^2\right)\int_0^t \varepsilon\|\nabla u^\varepsilon(s)\|^2_{L^2(\Omega^\varepsilon)^2}\,ds \\
&+ \left(\frac{\mu_\beta}{2} - 2\hat{\lambda}\bar{h}^2\right)\int_0^t \varepsilon\|\nabla v^\varepsilon(s)\|^2_{L^2(\Omega^\varepsilon)^2}\,ds + \hat{l}_1 \int_0^t \|u^\varepsilon(s)\|^2_{L^2(]0,L[)}\,ds \\
&+ \hat{l}_2 \int_0^t \|v^\varepsilon(s)\|^2_{L^2(]0,L[)}\,ds \\
&\le A,
\end{aligned} \tag{3.12}$$

where $A = \frac{2\bar{h}^2}{\mu_\alpha}\left\|\hat{f}(t)\right\|^2_{L^2(0,T,L^2(\Omega))} + \frac{2\bar{h}^2}{\mu_\beta}\|\hat{g}(t)\|^2_{L^2(0,T,L^2(\Omega))}$ is a constant independent on ε.

From (3.12), we deduce (3.5).

To show the estimate (3.6), we choose $\hat{\varphi} = \partial_t \hat{u}^\varepsilon$ in (3.2), we find

$$\begin{aligned}
&\int\limits_\Omega \varepsilon^2 \partial_t \hat{u}^\varepsilon \partial_t \hat{u}^\varepsilon dx_1 dz + \varepsilon^2 \int\limits_\Omega \hat{a}_{11}(x_1,z)(\partial_{x_1}\hat{u}^\varepsilon)(\partial_{x_1}\partial_t \hat{u}^\varepsilon) dx_1 dz \qquad (3.13)\\
&+ \varepsilon \int\limits_\Omega \hat{a}_{12}(x_1,z)(\partial_{x_1}\hat{u}^\varepsilon)(\partial_z \partial_t \hat{u}^\varepsilon) dx_1 dz \\
&+ \varepsilon \int\limits_\Omega \hat{a}_{21}(x_1,z)(\partial_z \hat{u}^\varepsilon)(\partial_{x_1}\partial_t \hat{u}^\varepsilon) dx_1 dz \\
&+ \int\limits_\Omega \hat{a}_{22}(x_1,z)(\partial_z \hat{u}^\varepsilon)(\partial_z \partial_t \hat{u}^\varepsilon) dx_1 dz + \hat{\lambda} \int_\Omega \hat{v}^\varepsilon . \partial_t \hat{u}^\varepsilon dx_1 dz \\
&+ \hat{l}_1 \int_0^L \hat{u}^\varepsilon . \varepsilon \partial_t \hat{u}^\varepsilon dx_1 - \hat{r} \int_0^L \hat{v}^\varepsilon . \partial_t \hat{u}^\varepsilon dx_1 \\
&= \int\limits_\Omega \hat{f} . \partial_t \hat{u}^\varepsilon dx_1 dz,
\end{aligned}$$

integrating this equalit over $(0,t)$, we deduce that

$$\begin{aligned}
&\int\limits_0^t \|\varepsilon \partial_t \hat{u}^\varepsilon(s)\|^2_{L^2(\Omega)} ds - \hat{\lambda} \int\limits_0^t \int_\Omega \partial_t \hat{v}^\varepsilon(s) . \hat{u}^\varepsilon(s) dx_1 dz ds \\
\leq \; & \int\limits_0^t \int\limits_\Omega \hat{f}(s) . \partial_t \hat{u}^\varepsilon(s) dx_1 dz ds - \hat{\lambda} \int_\Omega \hat{v}^\varepsilon(t) . \hat{u}^\varepsilon(t) dx_1 dz + \hat{r} \int_0^L \hat{v}^\varepsilon(t) . \partial_t \hat{u}^\varepsilon(t) dx_1,
\end{aligned}$$

this leads to

$$\begin{aligned}
&\int\limits_0^t \|\varepsilon \partial_t \hat{u}^\varepsilon(s)\|^2_{L^2(\Omega)} ds - \hat{\lambda} \int\limits_0^t \int_\Omega \partial_t \hat{v}^\varepsilon(s) . \hat{u}^\varepsilon(s) dx_1 dz ds \\
&\leq \frac{1}{\varepsilon} \int\limits_0^t \int\limits_\Omega \hat{f}(s) . (\varepsilon \partial_t \hat{u}^\varepsilon(s)) dx_1 dz ds + \frac{\hat{r}}{\varepsilon} \int_0^L (\hat{v}^\varepsilon(t)) . (\varepsilon \partial_t \hat{u}^\varepsilon(t)) dx_1 \\
&+ \hat{\lambda} \int_\Omega \hat{v}^\varepsilon(t) . \hat{u}^\varepsilon(t) dx_1 dz,
\end{aligned}$$

by using Cauchy-Schwarz inequality, trace theorem and Young's inequality, we obtain

$$\int_0^t \|\varepsilon\partial_t\hat{u}^\varepsilon(s)\|^2_{L^2(\Omega)}\,ds - \hat{\lambda}\int_0^t\int_\Omega \partial_t\hat{v}^\varepsilon(s)\,.\hat{u}^\varepsilon(s)\,dx_1dzds \tag{3.14}$$

$$\leq \frac{4}{\varepsilon^2}\int_0^t \left\|\hat{f}(s)\right\|^2_{L^2(\Omega)} ds + \frac{1}{4}\int_0^t \|\varepsilon\partial_t\hat{u}^\varepsilon(s)\|^2_{L^2(\Omega)^2}\,ds + \hat{\lambda}\,\|\hat{v}^\varepsilon(s)\|^2_{L^2(\Omega)}$$

$$+\,\hat{\lambda}\,\|\hat{u}^\varepsilon(s)\|^2_{L^2(\Omega)} + \frac{4}{\varepsilon^2}\hat{r}^2\left(C_{(\Omega)}\right)^2\int_0^t \|\hat{v}^\varepsilon(s)\|^2_{L^2(]0,L[)}\,ds + \frac{1}{4}\int_0^t \|\varepsilon\partial_t\hat{u}^\varepsilon(s)\|^2_{L^2(\Omega)^2}\,ds.$$

On the other hand, we choose $\psi = \partial_t\hat{v}^\varepsilon$ in (3.3) and we use the same techniques as before that we applied to equality (3.13), we find the following inequality

$$\int_0^t \|\varepsilon\partial_t\hat{v}^\varepsilon(s)\|^2_{L^2(\Omega)}\,ds + \hat{\lambda}\int_0^t\int_\Omega \partial_t\hat{v}^\varepsilon(s)\,.\hat{u}^\varepsilon(s)\,dx_1dzds \tag{3.15}$$

$$\leq \quad \frac{4}{\varepsilon^2}\int_0^t \|\hat{g}(s)\|^2_{L^2(\Omega)}\,ds + \frac{1}{4}\int_0^t \|\varepsilon\partial_t\hat{v}^\varepsilon(s)\|^2_{L^2(\Omega)^2}\,ds$$

$$+\frac{4}{\varepsilon^2}\hat{r}^2\left(C_{(\Omega)}\right)^2\int_0^t \|\hat{u}^\varepsilon(s)\|^2_{L^2(]0;L[)}\,ds + \frac{1}{4}\int_0^t \|\varepsilon\partial_t\hat{v}^\varepsilon(s)\|^2_{L^2(\Omega)^2}\,ds.$$

Now, we add the two inequalities (3.14) and (3.15), then we multiply the result by ε^2. Then we get

$$\int_0^t \left\|\varepsilon^2\partial_t\hat{u}^\varepsilon(s)\right\|^2_{L^2(\Omega)} ds + \int_0^t \left\|\varepsilon^2\partial_t\hat{v}^\varepsilon(s)\right\|^2_{L^2(\Omega)} ds$$

$$\leq 8\int_0^t \left\|\hat{f}(s)\right\|^2_{L^2(\Omega)} ds + 8\int_0^t \|\hat{g}(s)\|^2_{L^2(\Omega)}\,ds + \hat{\lambda}\,\|\varepsilon\hat{v}^\varepsilon\|^2_{L^2(\Omega)} + \hat{\lambda}\,\|\varepsilon\hat{u}^\varepsilon\|^2_{L^2(\Omega)}$$

$$+\,8\hat{r}^2\left(C_{(\Omega)}\right)^2\left(\int_0^t \|\hat{u}^\varepsilon(s)\|^2_{L^2(]0;L[)}\,ds + \int_0^t \|\hat{v}^\varepsilon(s)\|^2_{L^2(]0;L[)}\,ds\right),$$

using the fact that

$$\|\varepsilon\hat{v}^\varepsilon\|^2_{L^2(\Omega)} + \|\varepsilon\hat{u}^\varepsilon\|^2_{L^2(\Omega)} + \|\hat{u}^\varepsilon\|^2_{L^2(0,T,L^2(]0;L[))} + \|\hat{v}^\varepsilon\|^2_{L^2(0,T,L^2(]0;L[))} \leq c,$$

we find, that there is a constant c independent on ε, such that

$$\left\|\varepsilon^2\partial_t\hat{u}^\varepsilon\right\|^2_{L^2(0,T,L^2(\Omega))} + \left\|\varepsilon^2\partial_t\hat{v}^\varepsilon\right\|^2_{L^2(0,T,L^2(\Omega))} \leq c.$$

□

3.2. Study of the limit problem as $\varepsilon \to 0$

In this section we give the system satisfied by the limit of the sequences $(\hat{u}^\varepsilon, \hat{v}^\varepsilon)$ on Ω and the two equations describing the boundary conditions on $]0, L[$, for this purpose we introduce the Banach space

$$V_z = \left\{ \zeta \in L^2(\Omega) : \frac{\partial \zeta}{\partial z} \in L^2(\Omega),\, \zeta = 0 \text{ on } \partial\Omega_1 \right\},$$

with norm

$$\|\zeta\|_{V_z} = \left(\|\zeta\|^2_{L^2(\Omega)} + \left\| \frac{\partial \zeta}{\partial z} \right\|^2_{L^2(\Omega)} \right)^{\frac{1}{2}}.$$

We recall that the Poincaré inequality in the fixed domain Ω gives

$$\|\zeta\|_{L^2(\Omega)} \leq \bar{h} \left\| \frac{\partial \zeta}{\partial z} \right\|_{L^2(\Omega)}, \text{ for all } \zeta \in V_z.$$

Theorem 3. *Under the hypotheses of the Theorem 2, there exists $u^*, v^* \in L^2(0, T; V_z)$ such that*

$$(\hat{u}^\varepsilon, \hat{v}^\varepsilon) \rightharpoonup (u^*, v^*) \qquad \textit{weakly in } L^2(0, T; V_z)^2, \tag{3.16}$$

$$\begin{array}{rcl} (\varepsilon\partial_{x_1}\hat{u}^\varepsilon, \varepsilon\partial_{x_1}\hat{v}^\varepsilon) & \rightharpoonup & (0,0) \quad \textit{weakly in } L^2\left(0, T; L^2(\Omega)\right)^2, \\ (\varepsilon\partial_z\hat{u}^\varepsilon, \varepsilon\partial_z\hat{v}^\varepsilon) & \rightharpoonup & (0,0) \quad \textit{weakly in } L^2\left(0, T; L^2(\Omega)\right)^2, \end{array} \tag{3.17}$$

$$\left(\varepsilon^2\partial_t\hat{u}^\varepsilon, \varepsilon^2\partial_t\hat{v}^\varepsilon\right) \rightharpoonup (0,0) \quad \textit{weakly in } L^2\left(0, T; L^2(\Omega)\right)^2. \tag{3.18}$$

Where (u^*, v^*) is the weak solution to the limit problem

$$\left| \begin{array}{l} -\frac{\partial}{\partial z}\left[\hat{\alpha}_{22}(x_1, z)\frac{\partial u^*(x_1,z,t)}{\partial z}\right] + \hat{\lambda} v^*(x_1, z, t) = \hat{f}(x_1, z, t), \\ -\frac{\partial}{\partial z}\left[\hat{\beta}_{22}(x_1, z)\frac{\partial v^*(x_1,z,t)}{\partial z}\right] + \hat{\lambda} u^*(x_1, z, t) = \hat{g}(x_1, z, t), \end{array} \right\} a.e \textit{ in } \Omega \times (0, T), \tag{3.19}$$

$$\left| \begin{array}{l} -\hat{\alpha}_{22}(x_1, 0)\,\partial_z u^*(x_1, 0, t) + \hat{l}_1 u^*(x_1, 0, t) - \hat{r} v^*(x_1, 0, t) = 0, \\ -\hat{\beta}_{22}(x_1, 0)\,\partial_z v^*(x_1, 0, t) + \hat{l}_1 v^*(x_1, 0, t) + \hat{r} u^*(x_1, 0, t) = 0, \end{array} \right\} a.e \textit{ on}]0, L[\times(0, T), \tag{3.20}$$

$$(u^*(x, 0), v^*(x, 0)) = (0, 0).$$

Proof.
By the Theorem 2, there exists a constant c independent of ε such that

$$\int_0^t \|\partial_z \hat{u}^\varepsilon(s)\|^2_{L^2(\Omega)}\, ds \leq c, \qquad \int_0^t \|\partial_z \hat{v}^\varepsilon(s)\|^2_{L^2(\Omega)}\, ds \leq c.$$

Using these estimates with the Poincaré inequality in the domain Ω, we get

$$\|\hat{u}^{\varepsilon}(s)\|^2_{L^2(0,T,V_z)} \leq c,$$

and

$$\|\hat{v}^{\varepsilon}(s)\|^2_{L^2(0,T,V_z)} \leq c.$$

So $(\hat{u}^{\varepsilon}, \hat{v}^{\varepsilon})_{\varepsilon}$ is bounded in $L^2(0,T,V_z)^2$, which implies the existence of an element (u^*, v^*) in $L^2(0,T,V_z)^2$ such that $(\hat{u}^{\varepsilon}, \hat{v}^{\varepsilon})_{\varepsilon}$ converges weakly to (u^*, v^*) in $L^2(0,T,V_z)^2$, thus we obtain (3.16). For (3.17) through (3.5) and (3.16). As $(\hat{u}^{\varepsilon}, \hat{v}^{\varepsilon})_{\varepsilon}$ converges weakly to (u^*, v^*) in $L^2(0,T,V_z)^2$ and $\left(\varepsilon^2\partial_t\hat{u}^{\varepsilon}, \varepsilon^2\partial_t\hat{v}^{\varepsilon}\right)$ converges weakly to (χ, ζ) in $L^2\left(0,T,L^2(\Omega)\right)^2$, we deduce $(\chi,\zeta) = (0,0)$.

Now, by passage to the limit when ε tends to zero in the variational problem $(3.3)-(3.4)$, and using the convergence results, we deduce

$$\int\limits_{\Omega} \hat{\alpha}_{22}(x_1,z)\,\partial_z u^*\partial_z\varphi dx_1dz + \hat{\lambda}\int\limits_{\Omega} v^*\varphi dx_1dz + \int_0^L \left(\hat{l}_1u^* - \hat{r}v^*\right)\varphi dx_1 \tag{3.21}$$
$$= \int\limits_{\Omega} \hat{f}.\varphi dx_1dz, \quad \forall\varphi\in K,$$

and

$$\int\limits_{\Omega} \hat{\beta}_{22}(x_1,z)\,\partial_z v^*\partial_z\psi dx_1dz + \hat{\lambda}\int\limits_{\Omega} u^*\psi dx_1dz + \int_0^L \left(\hat{l}_2v^* + \hat{r}\right)\psi dx_1 \tag{3.22}$$
$$= \int\limits_{\Omega} \hat{g}.\psi dx_1dz, \quad \forall\psi\in K,$$

we choice φ and ψ in $H_0^1(\Omega)$, then using Green's formula, we obtain

$$\begin{aligned}
-\int\limits_{\Omega} \partial_z\left[\hat{\alpha}_{22}(x_1,z)\,\partial_z u^*\right]\varphi dx_1dz + \hat{\lambda}\int\limits_{\Omega} v^*\varphi dx_1dz &= \int\limits_{\Omega}\hat{f}.\varphi dx_1dz,\\
-\int\limits_{\Omega} \partial_z\left[\hat{\beta}_{22}(x_1,z)\,\partial_z u^*\right]\psi dx_1dz + \hat{\lambda}\int\limits_{\Omega} u^*\psi dx_1dz &= \int\limits_{\Omega}\hat{g}.\psi dx_1dz,
\end{aligned}$$

thus

$$\left.\begin{aligned}
-\partial_z\left[\hat{\alpha}_{22}(x_1,z)\,\partial_z u^*(x_1,z,t)\right] + \hat{\lambda}v^*(x_1,z,t) = \hat{f}(x_1,z,t)\\
-\partial_z\left[\hat{\beta}_{22}(x_1,z)\,\partial_z v^*(x_1,z,t)\right] + \hat{\lambda}u^*(x_1,z,t) = \hat{g}(x_1,z,t)
\end{aligned}\right\}\text{ in } H^{-1}(\Omega), \tag{3.23}$$

as $\hat{f}, \hat{g} \in L^2\left(0,T;L^2(\Omega)\right)$, then (3.23), is valid a.e in $\Omega\times(0,T)$.

Now, let's go back to the two formulas (3.21) and (3.22), using Green's formula and the fact that $(\varphi,\psi)=(0,0)$ on $\partial\Omega_1\cap\partial\Omega_L$, we deduce

$$\begin{aligned}&\int_\Omega \left(-\partial_z\left[\hat{\alpha}_{22}(x_1,z)\,\partial_z u^*\right]+\hat{\lambda}v^*-\hat{f}\right)\varphi dx_1dz-\int_0^L \hat{\alpha}_{22}(x_1,0)\,\partial_z u^*\varphi dx_1\\&+\hat{l}_1\int_0^L u^*.\varphi dx_1-\hat{r}\int_0^L v^*.\varphi dx_1\\&=0,\quad \forall\varphi\in K,\end{aligned}$$

and

$$\begin{aligned}&\int_\Omega \left(-\partial_z\left[\hat{\beta}_{22}(x_1,z)\,\partial_z u^*\right]+\hat{\lambda}u-\hat{g}\right)\psi dx_1dz-\int_0^L \hat{b}_{22}(x_1,0)\,\partial_z v^*\psi dx_1\\&+\hat{l}_2\int_0^L v^*.\psi dx_1+\hat{r}\int_0^L u^*.\psi dx_1\\&=0,\ \forall\psi\in K,\end{aligned}$$

this leads to

$$\left.\begin{array}{l}\int_0^L \left(-\hat{\alpha}_{22}(x_1,0)\,\partial_z u^*+\hat{l}_1u^*-\hat{r}v^*\right)\varphi dx_1=0,\\ \int_0^L \left(-\hat{\beta}_{22}(x_1,0)\,\partial_z v^*+\hat{l}_2v^*-\hat{r}u^*\right)\psi dx_1=0,\end{array}\right\},\ \forall(\varphi,\psi)\in D\left(]0,L[\right)^2,$$

by the density of $D\left(]0,L[\right)^2$ in $L^2\left(]0,L[\right)^2$, we get (3.20). □

Theorem 4. *Assume that* $\min\left(\min_{(x_1,z)\in\Omega}\left(\hat{\alpha}_{22}(x_1,z)\right),\min_{(x_1,z)\in\Omega}\left(\hat{\beta}_{22}(x_1,z)\right)\right)>2\hat{\lambda}$. *Then the weak solution (u^*,v^*) of the limit problem is unique and satisfies the following two weak formulas*

$$\begin{aligned}&\int_0^L\left(-\int_0^h\int_0^y \alpha_{22}(x_1,\varsigma)\,\partial_\varsigma u^*(x_1,\varsigma,t)\,d\varsigma dy+\hat{\lambda}\int_0^h\int_0^y\int_0^\eta v^*(x_1,\varsigma,t)\,d\varsigma d\eta dy\right.\\&+\frac{h(x_1)}{3}\int_0^h \alpha_{22}(x_1,\varsigma)\,\partial_\varsigma u^*(x_1,\varsigma,t)\,d\varsigma+\tilde{F}\\&\left.-\frac{h(x_1)}{3}\hat{\lambda}\int_0^h\int_0^\eta v^*(x_1,\varsigma,t)\,d\varsigma d\eta\right)\Phi_1'(x_1)\,dx_1\\&=0,\ \forall\Phi_1\in H^1\left(]0,L[\right),\end{aligned}\tag{3.24}$$

and

$$\int_0^L \left(-\int_0^h \int_0^y \beta_{22}(x_1,\varsigma)\,\partial_\varsigma v^*(x_1,\varsigma,t)\,d\varsigma dy + \hat{\lambda}\int_0^h \int_0^y \int_0^\eta u^*(x_1,\varsigma,t)\,d\varsigma d\eta dy \right. \tag{3.25}$$
$$+ \frac{h(x_1)}{3}\int_0^h \beta_{22}(x_1,\varsigma)\,\partial_\varsigma v^*(x_1,\varsigma,t)\,d\zeta + \tilde{G}$$
$$\left. - \frac{h(x_1)}{3}\hat{\lambda}\int_0^h \int_0^\eta u^*(x_1,\varsigma,t)\,d\varsigma d\eta \right) \Phi_2'(x_1)\,dx_1$$
$$= 0,\ \ \forall \Phi_2 \in H^1(]0,L[),$$

with

$$\begin{aligned} \tilde{F} &= \int_0^h \int_0^y \int_0^\eta \hat{f}(x_1,\varsigma,t)\,d\varsigma d\eta dy - \frac{h(x_1)}{3}\int_0^h \int_0^\eta \hat{f}(x_1,\varsigma,t)\,d\varsigma d\eta, \\ \tilde{G} &= \int_0^h \int_0^y \int_0^\eta \hat{g}(x_1,\varsigma,t)\,d\varsigma d\eta dy - \frac{h(x_1)}{3}\int_0^h \int_0^\eta \hat{g}(x_1,\varsigma,t)\,d\varsigma d\eta. \end{aligned}$$

Proof. To prove the uniqueness result, we suppose that there exist two solutions (u^*, v^*) and (u^{**}, v^{**}) of the variational problem $(3.21) - (3.22)$, we have

$$\int_\Omega \hat{\alpha}_{22}(x_1,z)\,\partial_z u^* \partial_z \varphi dx_1 dz + \hat{\lambda}\int_\Omega v^* \varphi dx_1 dz + \int_0^L \left(\hat{l}_1 u^* - \hat{r} v^*\right).\varphi dx_1 \tag{3.26}$$
$$= \int_\Omega \hat{f}.\varphi dx_1 dz, \forall \varphi \in K,$$

$$\int_\Omega \hat{\alpha}_{22}(x_1,z)\,\partial_z u^{**} \partial_z \varphi dx_1 dz + \hat{\lambda}\int_\Omega v^{**} \varphi dx_1 dz + \int_0^L \left(\hat{l}_1 u^{**} - \hat{r} v^{**}\right).\varphi dx_1 \tag{3.27}$$
$$= \int_\Omega \hat{f}.\varphi dx_1 dz, \forall \varphi \in K,$$

and

$$\int_\Omega \hat{\beta}_{22}(x_1,z)\,\partial_z v^* \partial_z \psi dx_1 dz + \hat{\lambda}\int_\Omega u^* \psi dx_1 dz + \int_0^L \left(\hat{l}_2 v^* + \hat{r} u^*\right).\psi dx_1 \tag{3.28}$$
$$= \int_\Omega \hat{g}.\psi dx_1 dz, \forall \psi \in K,$$

$$\int_{\Omega} \hat{\beta}_{22}(x_1,z)\,\partial_z v^{**}\partial_z\psi dx_1dz + \hat{\lambda}\int_{\Omega} u^{**}\psi dx_1dz + \int_0^L \left(\hat{l}_2 v^{**} + \hat{r}u^{**}\right).\psi dx_1 \quad (3.29)$$
$$= \int_{\Omega} \hat{g}.\psi dx_1dz, \forall\psi \in K.$$

By subtracting the equations (3.26) with (3.27) and (3.28) with (3.29), then we take $\varphi = u^* - u^{**}$ and $\psi = v^* - v^{**}$, we get

$$\int_{\Omega} \hat{\alpha}_{22}(x_1,z)\,|\partial_z u^* - \partial_z u^{**}|^2 dx_1dz + \hat{\lambda}\int_{\Omega}(v^* - v^{**})(u^* - u^{**})\,dx_1dz \quad (3.30)$$
$$+\,\hat{l}_1\int_0^L |u^* - u^{**}|^2 dx_1 - \hat{r}\int_0^L (v^* - v^{**}).(u^* - u^{**})\,dx_1$$
$$= 0,$$

and

$$\int_{\Omega} \hat{\beta}_{22}(x_1,z)\,|\partial_z v^* - \partial_z v^{**}|^2 dx_1dz + \hat{\lambda}\int_{\Omega}(u^* - u^{**})(v^* - v^{**})\,dx_1dz \quad (3.31)$$
$$+\,\hat{l}_1\int_0^L |v^* - v^{**}|^2 dx_1 + \hat{r}\int_0^L (u^* - u^{**}).(v^* - v^{**})\,dx_1$$
$$= 0.$$

Now, by summing the two equations and applying Young's and Poincare's inequalities, we conclude

$$\left(\min(\hat{\alpha}_{22}) - 2\hat{\lambda}\right)\|u^* - u^{**}\|^2_{L^2(0,T;V_z)} + \left(\min\left(\hat{\beta}_{22}\right) - 2\hat{\lambda}\right)\|v^* - v^{**}\|^2_{L^2(0,T;V_z)} \leq 0,$$

then, we obtain

$$(u^*, v^*) = (u^{**}, v^{**}).$$

For prove the two weak formulas, we integrate twice the first and the second equation of (3.19) between 0 and z, we obtain

$$-\int_0^z \alpha_{22}(x_1,\varsigma)\,\partial_\varsigma u^*(x_1,\varsigma,t)\,d\varsigma + \frac{z^2}{2}\alpha_{22}(x_1,0)\,\partial_z u^*(x_1,0,t) \quad (3.32)$$
$$+\,\hat{\lambda}\int_0^z\int_0^\eta v^*(x_1,\varsigma,t)\,d\varsigma d\eta$$
$$= \int_0^z\int_0^\eta \hat{f}(x_1,\varsigma,t)\,d\varsigma d\eta,$$

and

$$
\begin{aligned}
&-\int_0^z \beta_{22}(x_1,\varsigma)\,\partial_\varsigma v^*(x_1,\varsigma,t)\,d\zeta+\frac{z^2}{2}\beta_{22}(x_1,0)\,\partial_z v_i^*(x_1,0,t) \\
&+\hat{\lambda}\int_0^z\int_0^\eta u^*(x_1,\varsigma,t)\,d\varsigma d\eta \\
&=\int_0^z\int_0^\eta \hat{g}(x_1,\varsigma,t)\,d\varsigma d\eta,
\end{aligned}
\tag{3.33}
$$

in particular for $z=h(x_1)$, we obtain

$$
\begin{aligned}
&-\int_0^h \alpha_{22}(x_1,\varsigma)\,\partial_\varsigma u^*(x_1,\varsigma,t)\,d\varsigma+\frac{h(x_1)^2}{2}\alpha_{22}(x_1,0)\,\partial_z u_i^*(x_1,0,t) \\
&+\hat{\lambda}\int_0^h\int_0^\eta v^*(x_1,\varsigma,t)\,d\varsigma d\eta \\
&=\int_0^h\int_0^\eta \hat{f}(x_1,\varsigma,t)\,d\varsigma d\eta,
\end{aligned}
$$

and

$$
\begin{aligned}
&-\int_0^h \beta_{22}(x_1,\varsigma)\,\partial_\zeta v^*(x_1,\varsigma,t)\,d\zeta+\frac{h(x_1)^2}{2}\beta_{22}(x_1,0)\,\partial_z v_i^*(x_1,0,t) \\
&+\hat{\lambda}\int_0^h\int_0^\eta u^*(x_1,\varsigma,t)\,d\varsigma d\eta \\
&=\int_0^h\int_0^\eta \hat{g}(x_1,\varsigma,t)\,d\varsigma d\eta.
\end{aligned}
$$

Thus, by integrating (3.32) and (3.33) between 0 and $h(x_1)$, we get (3.24) and (3.25). □

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