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Time-Unit Shifting in 2-Person Games Played in Finite and Uncountably Infinite Staircase-Function Spaces

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ABSTRACT: A computationally efficient and tractable method is presented to find the best equilibrium in a finite 2-person game played with staircase-function strategies. The method is based on stacking equilibria of smaller-sized bimatrix games, each defined on a time unit where the pure strategy value is constant. Every pure strategy is a staircase function defined on a time interval consisting of an integer number of time units (subintervals). If a time-unit shifting happens, where the initial time interval is narrowed by an integer number of time units, the respective equilibrium solution of any "narrower" subgame can be taken from the "wider" game equilibrium. If the game is uncountably infinite, i. e. a set of pure strategy possible values is uncountably infinite, and all time-unit equilibria exist, stacking equilibria of smaller-sized 2-person games defined on a rectangle works as well.

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1. Staircase-function strategies

A noncooperative 2-person game is a model of process where two sides personified and referred to as persons or players interact in struggling for optimizing their own payoffs [24, 25, 7]. The players' payoffs are taken from some limited resources, so the distribution of the limited resources is optimized by the game model [25, 1, 13, 27]. The simplest 2-person game is a bimatrix game [7, 15, 25]. Whereas each of the players in a bimatrix game possesses a finite set (space) of pure strategies, the principles and theory of equilibrium, efficiency, profitability, and eventual optimality of bimatrix game solutions are thoroughly studied [7, 10, 14, 24]. However, the practice of bimatrix game solutions is not that simple. First, a problem may arise with multiplicity of the solutions. Second, a problem may arise with selecting a solution type (regarding equilibrium or profitability, which often are counteractive). Third, another problem does arise when the solution is in mixed strategies but the number of game iterations (moves, actions, plays, etc.) is limited and so a mixed strategy appears to be impracticable (for instance, it is impossible to practically realize a mixed strategy having probability of 7/19 if there are only 10 game iterations) [17, 18, 3, 7]. Furthermore, if at least two solutions are symmetric, they may be quite unstable due to cooperation between the players is excluded [7, 24, 25, 10, 23].

A far more complicated case is a 2-person game, in which the player's (pure) strategy is a function (usually, it is a function of time). In such a game, the player's payoff is a functional mapping every pair of functions (pure strategies of the players defined on a time interval) into a real value [20, 16]. In the case, when each of the players possesses a finite set of such function-strategies, the game might be rendered down to a bimatrix game [19, 13, 15]. The bimatrix game played with function-strategies, apart from those mentioned problems inherent in ordinary bimatrix games, is a far subtler model in the sense of its practicability.

The finiteness of a set of function-strategies is constituted by time interval discretization and discretization of possible values of the strategy. The time interval, on which the pure strategy is defined, is broken into a set of time subintervals (units), on which the strategy is (approximately considered) constant. This is so because there is no natural time continuity — every process is constant on some (usually, short) time period [2, 5, 8, 11, 12]. The continuity of possible values of the strategy on a subinterval is removed also by discretization (or sampling) [22, 18, 9] ruled by laws of the game-modeled system. Then the set of function-strategies becomes finite, where the strategy itself is a staircase function [22] but sometimes it can be conditionally interpreted as a point [30, 16, 18]. Compared to the most trivial strategy, which is a decision corresponding to a one-stage event whose duration through time is (usually, negligibly) short, a staircase-function strategy itself is a multi-stage process defined on a time interval [26, 30, 18, 4, 28, 29]. Nevertheless, the length of the time interval can be varied depending on properties of the process modeled by the game.

2. Multiplicity of equilibria and the time interval length

A 2-person game played in finite staircase-function spaces can be called the bimatrix staircase-function game. It is quite clear that the number of pure-strategy situations in a bimatrix staircase-function game grows immensely as the number of breakpoints ("stair" subintervals) increases, or the number of possible values of the player's pure strategy increases, or they both increase. For instance, if the number of time subintervals is just 5, and the number of possible values of the player's pure strategy is 6, then there is a finite set of

$$6^5 = 7776$$

possible pure strategies (i. e., 5-subinterval staircase functions of time) at this player. If the other player's pure strategy has, say, 8 possible values, then there are

$$8^5 = 32768$$

possible 5-subinterval staircase functions of time at this player, and the respective bimatrix staircase-function game has a size of either

$$7776 \times 32768$$

or

$$32768 \times 7776$$

and there are

$$6^5 \cdot 8^5 = 7776 \cdot 32768 = 254803968$$

pure-strategy situations. If an additional time subinterval is included, there are

$$6^6 \cdot 8^6 = 46656 \cdot 262144 = 12230590464$$

pure-strategy situations (more than 12.23 billion ones!). This is why a tractable method of solving 2-person games defined on a product of staircase-function spaces was presented in [21], where the spaces can be finite and continuous (uncountably infinite) as well. The method is based on stacking equilibria of "short" 2-person games, each defined on a subinterval where the pure strategy value is constant. It is proved in [21] that the bimatrix staircase-function game is solved as a stack of respective equilibria in the "short" (ordinary) bimatrix games (where the pure strategy is a very simple decision corresponding to a one-stage event). The stack is always possible, even when only time is discrete (and the set of pure strategy possible values is infinite or uncountably infinite). However, the problem of multiplicity of equilibria was not raised in [21]. The subinterval equilibrium multiplicity has a dramatic impact on the multiplicity of the equilibrium stack. For instance, if there are two equilibria on each of 5 subintervals, the game has altogether

$$2^5 = 32$$

equilibrium stacks. Then an open question is how to select a single equilibrium stack. Another open question is how to deal with a 2-person game in which the breakpoints of a function-strategy do not change but the time interval length can vary [2, 3, 7, 30, 18, 5].

3. Objective and six tasks to be fulfilled

Due to the above reasons, the objective is to expand and develop the tractable method of solving 2-person games played within players' finite sets of staircase functions [21] for the case when the length of the time interval on which the 2-person game is defined is varied. The case with an uncountably infinite set (space) of staircase functions is to be considered as well. To meet the objective, the following tasks are to be fulfilled:

1. To formalize a 2-person game, in which the players' strategies are staircase functions. In such a game, the set of the player's pure strategies is a continuum of staircase functions of time. Such function-strategies are presumed to be bounded and Lebesgue-integrable, and the time can be thought of as it is discrete.
2. To discretize the set of possible values of the player's pure strategy so that the game be defined on a product of staircase-function finite spaces.
3. To formalize the known method of solving 2-person games (the solution of the equilibrium type) played in staircase-function finite and uncountably infinite spaces by considering a possibility of narrowing the time interval on which the 2-person game is defined.
4. To give an example of how the suggested method is applied. A special attention must be paid to selecting a single equilibrium situation.
5. To discuss practical applicability and scientific significance of the method for the game theory and operations research.
6. To conclude on the study and make an outlook for furthering it.

4. 2-person game played with staircase-function strategies through discrete time

In a 2-person game, in which the player's pure strategy is a function of time, let each of the players use time-varying strategies defined almost everywhere on interval $[t_1;\ t_2]$ by $t_2 > t_1$. Denote a pure strategy of the first player by $x(t)$ and a pure strategy of the second player by $y(t)$. These functions are presumed to be bounded, i. e.

$$a_{\min} \leqslant x(t) \leqslant a_{\max} \ \text{ by } \ a_{\min} < a_{\max} \tag{1}$$

and

$$b_{\min} \leqslant y(t) \leqslant b_{\max} \ \text{ by } \ b_{\min} < b_{\max}, \tag{2}$$

defined almost everywhere on $[t_1;\ t_2]$. Besides, the square of the function-strategy is presumed to be Lebesgue-integrable. Thus, pure strategies of the player belong to a rectangular functional space of functions of time:

$$X = \{x(t),\ t \in [t_1;\ t_2],\ t_1 < t_2 :\ a_{\min} \leqslant x(t) \leqslant a_{\max} \text{ by } a_{\min} < a_{\max}\} \subset \\ \subset \mathbb{L}_2[t_1;\ t_2] \tag{3}$$

and

$$Y = \{y(t),\ t \in [t_1;\ t_2],\ t_1 < t_2 :\ b_{\min} \leqslant y(t) \leqslant b_{\max} \text{ by } b_{\min} < b_{\max}\} \subset$$

$$\subset \mathbb{L}_2 \left[t_1;\ t_2\right] \tag{4}$$

are the sets (sometimes referred to as action spaces) of the players' pure strategies.

The first player's payoff in situation (Figure 1)

$$\{x(t),\ y(t)\} \tag{5}$$

is

$$K\big(x(t),\ y(t)\big) \tag{6}$$

presumed to be an integral functional [21, 22]:

$$K\big(x(t),\ y(t)\big) = \int\limits_{[t_1;\ t_2]} f\big(x(t),\ y(t),\ t\big)d\mu(t), \tag{7}$$

where

$$f\big(x(t),\ y(t),\ t\big) \tag{8}$$

is a function of $x(t)$ and $y(t)$ explicitly including t. The second player's payoff in situation (5) is

$$H\big(x(t),\ y(t)\big) \tag{9}$$

presumed to be an integral functional also:

$$H\big(x(t),\ y(t)\big) = \int\limits_{[t_1;\ t_2]} g\big(x(t),\ y(t),\ t\big)d\mu(t), \tag{10}$$

where

$$g\big(x(t),\ y(t),\ t\big) \tag{11}$$

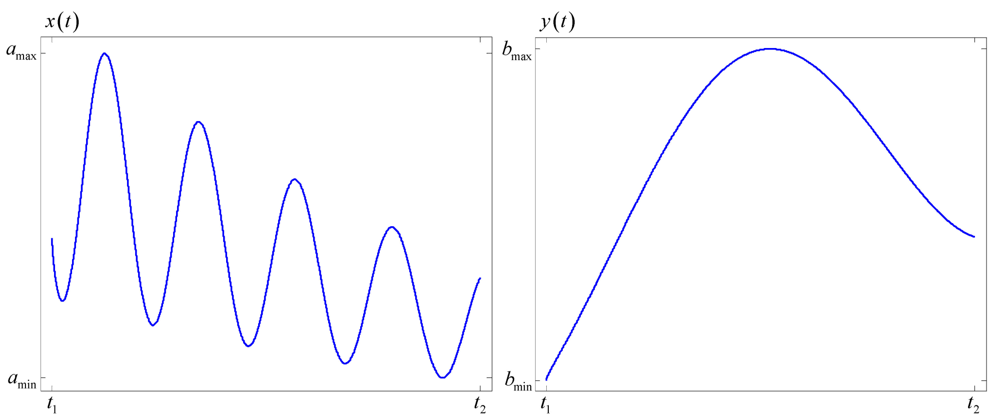

Figure 1: A situation (5) in 2-person game (12) played in uncountably infinite functional spaces (3) and (4)

is a function of $x(t)$ and $y(t)$ explicitly including t also. Therefore, a 2-person game

$$\left\langle \{X,\ Y\},\ \left\{K\big(x(t),\ y(t)\big),\ H\big(x(t),\ y(t)\big)\right\} \right\rangle \tag{12}$$

is uncountably infinite due to it is defined on product

$$X \times Y \subset \mathbb{L}_2\,[t_1;\ t_2] \times \mathbb{L}_2\,[t_1;\ t_2] \tag{13}$$

of uncountably infinite rectangular functional spaces (3) and (4) of players' pure strategies.

Each of sets (3) and (4) is a continuum of functions. It is worth noting that the game continuity is defined by the continuity of spaces (3) and (4), whereas payoff functionals (7) and (10) still can have discontinuities. In general, each of payoff functionals (6) and (9) may have a terminal component like

$$K\big(x(t),\ y(t)\big) = \\ = \int\limits_{[t_1;\ t_2]} f\big(x(t),\ y(t),\ t\big)d\mu(t) + T_f\big(x(t_2),\ y(t_2),\ t_2\big) \tag{14}$$

and

$$H\big(x(t),\ y(t)\big) = \\ = \int\limits_{[t_1;\ t_2]} g\big(x(t),\ y(t),\ t\big)d\mu(t) + T_g\big(x(t_2),\ y(t_2),\ t_2\big) \tag{15}$$

by some terminal functions

$$T_f\big(x(t_2),\ y(t_2),\ t_2\big) \tag{16}$$

and

$$T_g\big(x(t_2),\ y(t_2),\ t_2\big) \tag{17}$$

depending on only the final state of the player's strategy, but this case is not to be considered here.

Presume that the players' pure strategies $x(t)$ and $y(t)$ in game (12) can both change their values only for a finite number of times. Denote by N the number of subintervals at which the player's pure strategy is constant, where $N \in \mathbb{N}\backslash\{1\}$. In other words, when time is discrete, N is a number of time units. Then the player's pure strategy is a staircase function having at most N different values. Let

$$\Theta = \left\{t_1 = \tau^{(0)} < \tau^{(1)} < \tau^{(2)} < \ldots < \tau^{(N-1)} < \tau^{(N)} = t_2\right\}, \tag{18}$$

where $\left\{\tau^{(i)}\right\}_{i=1}^{N-1}$ are time points at which the staircase-function strategy can change its value. Time-interval breaking (18) is not necessarily to be equidistant. The staircase-function strategies are right-continuous [6, 21, 22]:

$$\lim_{\substack{\varepsilon>0 \\ \varepsilon\to 0}} x\left(\tau^{(i)} + \varepsilon\right) = x\left(\tau^{(i)}\right) \tag{19}$$

and

$$\lim_{\substack{\varepsilon>0\\ \varepsilon\to 0}} y\left(\tau^{(i)}+\varepsilon\right) = y\left(\tau^{(i)}\right) \tag{20}$$

for $i=\overline{1,\ N-1}$, whereas (if the strategy value changes)

$$\lim_{\substack{\varepsilon>0\\ \varepsilon\to 0}} x\left(\tau^{(i)}-\varepsilon\right) \neq x\left(\tau^{(i)}\right) \tag{21}$$

and

$$\lim_{\substack{\varepsilon>0\\ \varepsilon\to 0}} y\left(\tau^{(i)}-\varepsilon\right) \neq y\left(\tau^{(i)}\right) \tag{22}$$

for $i=\overline{1,\ N-1}$. As an exception,

$$\lim_{\substack{\varepsilon>0\\ \varepsilon\to 0}} x\left(\tau^{(N)}-\varepsilon\right) = x\left(\tau^{(N)}\right) \tag{23}$$

and

$$\lim_{\substack{\varepsilon>0\\ \varepsilon\to 0}} y\left(\tau^{(N)}-\varepsilon\right) = y\left(\tau^{(N)}\right). \tag{24}$$

A 2-person game played with staircase-function strategies through discrete time can be defined by using (1) — (13), (18) — (24).

Definition 1. 2-person game (12) defined on product (13) of rectangular functional spaces (3) and (4) is called a discrete-time staircase-function 2-person game by time-interval breaking (18), if (19) — (24) hold and

$$\begin{gathered} x(t)=\alpha_i\in[a_{\min};\ a_{\max}] \ \text{ and } \ y(t)=\beta_i\in[b_{\min};\ b_{\max}] \\ \forall\, t\in\left[\tau^{(i-1)};\ \tau^{(i)}\right) \ \text{ for } \ i=\overline{1,\ N-1} \ \text{ and} \\ x(t)=\alpha_N\in[a_{\min};\ a_{\max}] \ \text{ and} \\ y(t)=\beta_N\in[b_{\min};\ b_{\max}] \ \ \forall\, t\in\left[\tau^{(N-1)};\ \tau^{(N)}\right], \end{gathered} \tag{25}$$

where the factual payoff of the first player in situation $\{\alpha_i,\ \beta_i\}$ is

$$K_i(\alpha_i,\ \beta_i) = \int\limits_{\left[\tau^{(i-1)};\ \tau^{(i)}\right)} f(\alpha_i,\ \beta_i,\ t)\, d\mu(t) \ \ \forall\, i=\overline{1,\ N-1} \tag{26}$$

and

$$K_N(\alpha_N,\ \beta_N) = \int\limits_{\left[\tau^{(N-1)};\ \tau^{(N)}\right]} f(\alpha_N,\ \beta_N,\ t)\, d\mu(t), \tag{27}$$

and the factual payoff of the second player in situation $\{\alpha_i,\ \beta_i\}$ is

$$H_i(\alpha_i,\ \beta_i) = \int\limits_{\left[\tau^{(i-1)};\ \tau^{(i)}\right)} g(\alpha_i,\ \beta_i,\ t)\, d\mu(t) \ \ \forall\, i=\overline{1,\ N-1} \tag{28}$$

and

$$H_N\left(\alpha_N,\ \beta_N\right) = \int\limits_{\left[\tau^{(N-1)};\ \tau^{(N)}\right]} g\left(\alpha_N,\ \beta_N,\ t\right) d\mu\left(t\right). \tag{29}$$

Situation (5) in the discrete-time staircase-function 2-person game is a stack of successive situations

$$\left\{\left\{\alpha_i,\ \beta_i\right\}\right\}_{i=1}^{N} \tag{30}$$

in a succession of N (ordinary) 2-person games

$$\left\langle \left\{\left[a_{\min};\ a_{\max}\right],\ \left[b_{\min};\ b_{\max}\right]\right\},\ \left\{K\left(\alpha_i,\ \beta_i\right),\ H\left(\alpha_i,\ \beta_i\right)\right\} \right\rangle \text{ for } i = \overline{1,\ N} \tag{31}$$

defined on product

$$\left[a_{\min};\ a_{\max}\right] \times \left[b_{\min};\ b_{\max}\right] \tag{32}$$

by (25) — (29).

Let a discrete-time staircase-function 2-person game by time-interval breaking (18) be denoted by

$$\left\langle \left\{X\left(\Theta\right),\ Y\left(\Theta\right)\right\},\ \left\{K_i\big(x\left(t\right),\ y\left(t\right)\big),\ H_i\big(x\left(t\right),\ y\left(t\right)\big)\right\} \right\rangle \tag{33}$$

with the players' pure strategy sets

$$\begin{aligned} X\left(\Theta\right) = \Big\{ & x\left(t\right) \in X\left(\left[t_1;\ t_2\right]\right) : x\left(t\right) = \alpha_i \in \left[a_{\min};\ a_{\max}\right] \\ & \forall\, t \in \left[\tau^{(i-1)};\ \tau^{(i)}\right) \text{ for } i = \overline{1,\ N-1} \text{ and} \\ & x\left(t\right) = \alpha_N \in \left[a_{\min};\ a_{\max}\right]\ \forall\, t \in \left[\tau^{(N-1)};\ \tau^{(N)}\right] \Big\} \subset X\left(\left[t_1;\ t_2\right]\right) \end{aligned} \tag{34}$$

and

$$\begin{aligned} Y\left(\Theta\right) = \Big\{ & y\left(t\right) \in Y\left(\left[t_1;\ t_2\right]\right) : y\left(t\right) = \beta_i \in \left[b_{\min};\ b_{\max}\right] \\ & \forall\, t \in \left[\tau^{(i-1)};\ \tau^{(i)}\right) \text{ for } i = \overline{1,\ N-1} \text{ and} \\ & y\left(t\right) = \beta_N \in \left[b_{\min};\ b_{\max}\right]\ \forall\, t \in \left[\tau^{(N-1)};\ \tau^{(N)}\right] \Big\} \subset Y\left(\left[t_1;\ t_2\right]\right). \end{aligned} \tag{35}$$

Obviously, discrete-time staircase-function 2-person game (33) is uncountably infinite as each of sets (34) and (35) contains a continuum of function-strategies. An example of situation (5) in a discrete-time staircase-function 2-person game played through seven time units (subintervals) is given in Figure 2. The exemplified pure-strategy situation of two staircase functions can be also represented as a stack of seven successive situations $\left\{\left\{\alpha_i,\ \beta_i\right\}\right\}_{i=1}^{7}$ of seven ordinary 2-person games (31), where each ordinary pure-strategy situation $\left\{\alpha_i,\ \beta_i\right\}$ for $i = \overline{1,\ 6}$ corresponds to a time unit (subinterval) $\left[\tau^{(i-1)};\ \tau^{(i)}\right)$ and ordinary pure-strategy situation $\left\{\alpha_7,\ \beta_7\right\}$ corresponds to a time unit (subinterval) $\left[\tau^{(6)};\ \tau^{(7)}\right] = \left[\tau^{(6)};\ t_2\right]$.

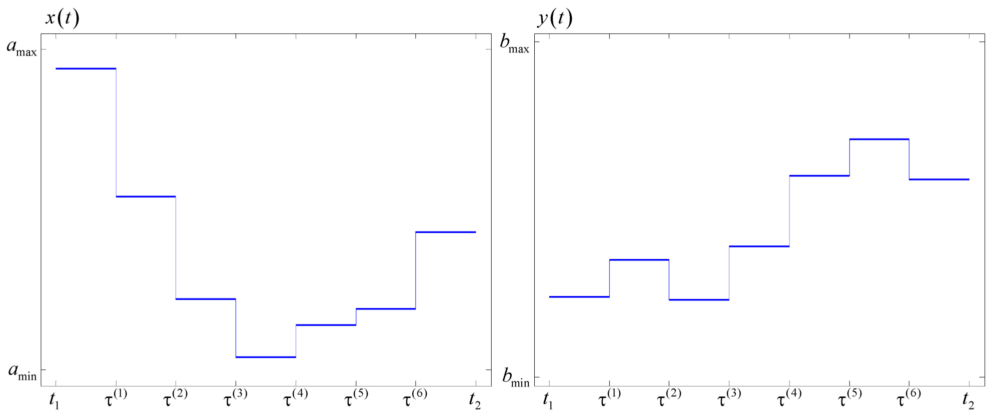

Figure 2: A situation (5) in discrete-time staircase-function 2-person game (33); the game is played in uncountably infinite functional spaces (34) and (35); the exemplified pure-strategy situation is a stack of seven successive situations $\left\{\{\alpha_i,\ \beta_i\}\right\}_{i=1}^{7}$

Time-interval breaking (18) allows considering payoffs (7) and (10) in situation (5) equivalent to the sum of respective payoffs (26) — (29). The proof can be found in [22].

Theorem 1. *In a pure-strategy situation (5) of discrete-time staircase-function 2-person game (33), payoff functionals (7) and (10) are re-written as subinterval-wise sums*

$$K\big(x(t),\ y(t)\big) = \sum_{i=1}^{N} K_i(\alpha_i,\ \beta_i) =$$

$$= \sum_{i=1}^{N-1} \int\limits_{\left[\tau^{(i-1)};\ \tau^{(i)}\right)} f(\alpha_i,\ \beta_i,\ t)\, d\mu(t) + \int\limits_{\left[\tau^{(N-1)};\ \tau^{(N)}\right]} f(\alpha_N,\ \beta_N,\ t)\, d\mu(t) \tag{36}$$

and

$$H\big(x(t),\ y(t)\big) = \sum_{i=1}^{N} H_i(\alpha_i,\ \beta_i) =$$

$$= \sum_{i=1}^{N-1} \int\limits_{\left[\tau^{(i-1)};\ \tau^{(i)}\right)} g(\alpha_i,\ \beta_i,\ t)\, d\mu(t) + \int\limits_{\left[\tau^{(N-1)};\ \tau^{(N)}\right]} g(\alpha_N,\ \beta_N,\ t)\, d\mu(t), \tag{37}$$

where situation (5) is a stack of successive situations (30) in a succession of N 2-person games (31).

Proof. Time interval $[t_1;\ t_2]$ can be re-written as

$$[t_1;\ t_2] = \left\{ \bigcup_{i=1}^{N-1} \left[\tau^{(i-1)};\ \tau^{(i)} \right) \right\} \cup \left[\tau^{(N-1)};\ \tau^{(N)} \right]. \tag{38}$$

Therefore, the property of countable additivity of the Lebesgue integral can be used:

$$K\big(x(t),\ y(t)\big) = \int\limits_{[t_1;\ t_2]} f\big(x(t),\ y(t),\ t\big) d\mu(t) =$$

$$= \int\limits_{\left\{ \bigcup\limits_{i=1}^{N-1} \left[\tau^{(i-1)};\ \tau^{(i)} \right) \right\} \cup \left[\tau^{(N-1)};\ \tau^{(N)} \right]} f\big(x(t),\ y(t),\ t\big) d\mu(t) =$$

$$= \sum_{i=1}^{N-1} \int\limits_{\left[\tau^{(i-1)};\ \tau^{(i)} \right)} f\big(x(t),\ y(t),\ t\big) d\mu(t) + \int\limits_{\left[\tau^{(N-1)};\ \tau^{(N)} \right]} f\big(x(t),\ y(t),\ t\big) d\mu(t). \tag{39}$$

Owing to (25), $x(t) = \alpha_i$ and $y(t) = \beta_i$, so (39) is simplified as

$$\sum_{i=1}^{N-1} \int\limits_{\left[\tau^{(i-1)};\ \tau^{(i)} \right)} f\big(x(t),\ y(t),\ t\big) d\mu(t) + \int\limits_{\left[\tau^{(N-1)};\ \tau^{(N)} \right]} f\big(x(t),\ y(t),\ t\big) d\mu(t) =$$

$$= \sum_{i=1}^{N-1} \int\limits_{\left[\tau^{(i-1)};\ \tau^{(i)} \right)} f(\alpha_i,\ \beta_i,\ t)\, d\mu(t) + \int\limits_{\left[\tau^{(N-1)};\ \tau^{(N)} \right]} f(\alpha_N,\ \beta_N,\ t)\, d\mu(t) =$$

$$= \sum_{i=1}^{N} K_i(\alpha_i,\ \beta_i). \tag{40}$$

Consequently, in discrete-time staircase-function 2-person game (33), subinterval-wise sum (36) holds in any pure-strategy situation (5) consisting of staircase-function strategies $x(t) \in X(\Theta)$ and $y(t) \in Y(\Theta)$. Obviously, subinterval-wise sum (37) is proved similarly to (38) — (40). □

It is noteworthy that Theorem 1 can be proved also by considering function (8) on a time unit (subinterval) as a function of time t, due to $x(t) = \alpha_i$ and $y(t) = \beta_i$ on this subinterval. Denote this function by $\psi_i(t)$. Then this function appears to be zero on any other time unit. Subsequently, function (8) is presented as the sum of those subinterval functions:

$$f\big(x(t),\ y(t),\ t\big) = \sum_{i=1}^{N} \psi_i(t),$$

whereupon (40) is deduced.

Nevertheless, Theorem 1 does not provide a method of solving the discrete-time staircase-function 2-person game, but it hints about how the game might be solved in an easier way [21, 22]. Theorem 1 provides a fundamental decomposition of the staircase game based on the subinterval-wise summing in (36) and (37). This subinterval decomposition allows considering and solving each game (31) separately, whereupon the solutions are stitched (stacked) together, regardless of whether the player's action space is finite or not.

5. Finite discrete-time staircase-function 2-person game

In a discrete-time staircase-function 2-person game (33), let the set of possible values of the first player's pure strategy be discretized as

$$\mathrm{A} = \left\{ a_{\min} = a_i^{(0)} < a_i^{(1)} < a_i^{(2)} < \ldots < a_i^{(M-1)} < a_i^{(M)} = a_{\max} \right\} \tag{41}$$

and the set of possible values of the second player's pure strategy be discretized as

$$\mathrm{B} = \left\{ b_{\min} = b_i^{(0)} < b_i^{(1)} < b_i^{(2)} < \ldots < b_i^{(Q-1)} < b_i^{(Q)} = b_{\max} \right\} \tag{42}$$

by $M \in \mathbb{N}$ and $Q \in \mathbb{N}$, where

$$a_i^{(m-1)} = a^{(m-1)} \;\; \forall\, i = \overline{1,\ N} \;\text{ for }\; m = \overline{1,\ M+1} \tag{43}$$

and

$$b_i^{(q-1)} = b^{(q-1)} \;\; \forall\, i = \overline{1,\ N} \;\text{ for }\; q = \overline{1,\ Q+1}. \tag{44}$$

This means that along with the discrete time units (subintervals), the players are forced (somehow) to act within finite subsets of possible values of their pure strategies

$$A = \left\{ a^{(m-1)} \right\}_{m=1}^{M+1} \tag{45}$$

and

$$B = \left\{ b^{(q-1)} \right\}_{q=1}^{Q+1}. \tag{46}$$

Discretizations (41) — (44) allow defining a finite discrete-time staircase-function 2-person game.

Definition 2. Discrete-time staircase-function 2-person game (33) is called finite if it is played on a product of finite subsets

$$X\left(\Theta,\ A\right) = \left\{ x\left(t\right) \in X\left(\Theta\right) :\ x\left(t\right) \in \left\{ a^{(m-1)} \right\}_{m=1}^{M+1} \right\} \subset$$
$$\subset X\left(\Theta\right) \subset X\left(\left[t_1;\ t_2\right]\right) \tag{47}$$

and

$$Y(\Theta,\ B) = \left\{ y(t) \in Y(\Theta):\ y(t) \in \left\{ b^{(q-1)} \right\}_{q=1}^{Q+1} \right\} \subset$$
$$\subset Y(\Theta) \subset Y([t_1;\ t_2]) \qquad (48)$$

of sets (34) and (35).

So, let a finite discrete-time staircase-function 2-person game be denoted by

$$\left\langle \{X(\Theta,\ A),\ Y(\Theta,\ B)\},\ \left\{ K\big(x(t),\ y(t)\big),\ H\big(x(t),\ y(t)\big) \right\} \right\rangle \qquad (49)$$

with the players' pure strategy sets (47) and (48). In fact, this finite game is a bimatrix staircase-function game (see an example in Figure 3, where every pure strategy as a staircase function of time can be "imagined" as a conditional point pretended to be a simple decision to constitute an 81×256 bimatrix game) that is the succession of N bimatrix games

$$\left\langle \left\{ \left\{ a_i^{(m-1)} \right\}_{m=1}^{M+1},\ \left\{ b_i^{(q-1)} \right\}_{q=1}^{Q+1} \right\},\ \{\mathbf{K}_i,\ \mathbf{H}_i\} \right\rangle \text{ for } i = \overline{1,\ N} \qquad (50)$$

with the first player's payoff matrices

$$\mathbf{K}_i = [k_{imq}]_{(M+1)\times(Q+1)} \qquad (51)$$

whose elements are

$$k_{imq} = \int\limits_{\left[\tau^{(i-1)};\ \tau^{(i)}\right)} f\left(a_i^{(m-1)},\ b_i^{(q-1)},\ t\right) d\mu(t) \text{ for } i = \overline{1,\ N-1} \qquad (52)$$

and

$$k_{Nmq} = \int\limits_{\left[\tau^{(N-1)};\ \tau^{(N)}\right]} f\left(a_N^{(m-1)},\ b_N^{(q-1)},\ t\right) d\mu(t), \qquad (53)$$

and with the second player's payoff matrices

$$\mathbf{H}_i = [h_{imq}]_{(M+1)\times(Q+1)} \qquad (54)$$

whose elements are

$$h_{imq} = \int\limits_{\left[\tau^{(i-1)};\ \tau^{(i)}\right)} g\left(a_i^{(m-1)},\ b_i^{(q-1)},\ t\right) d\mu(t) \text{ for } i = \overline{1,\ N-1} \qquad (55)$$

and

$$h_{Nmq} = \int\limits_{\left[\tau^{(N-1)};\ \tau^{(N)}\right]} g\left(a_N^{(m-1)},\ b_N^{(q-1)},\ t\right) d\mu(t). \qquad (56)$$

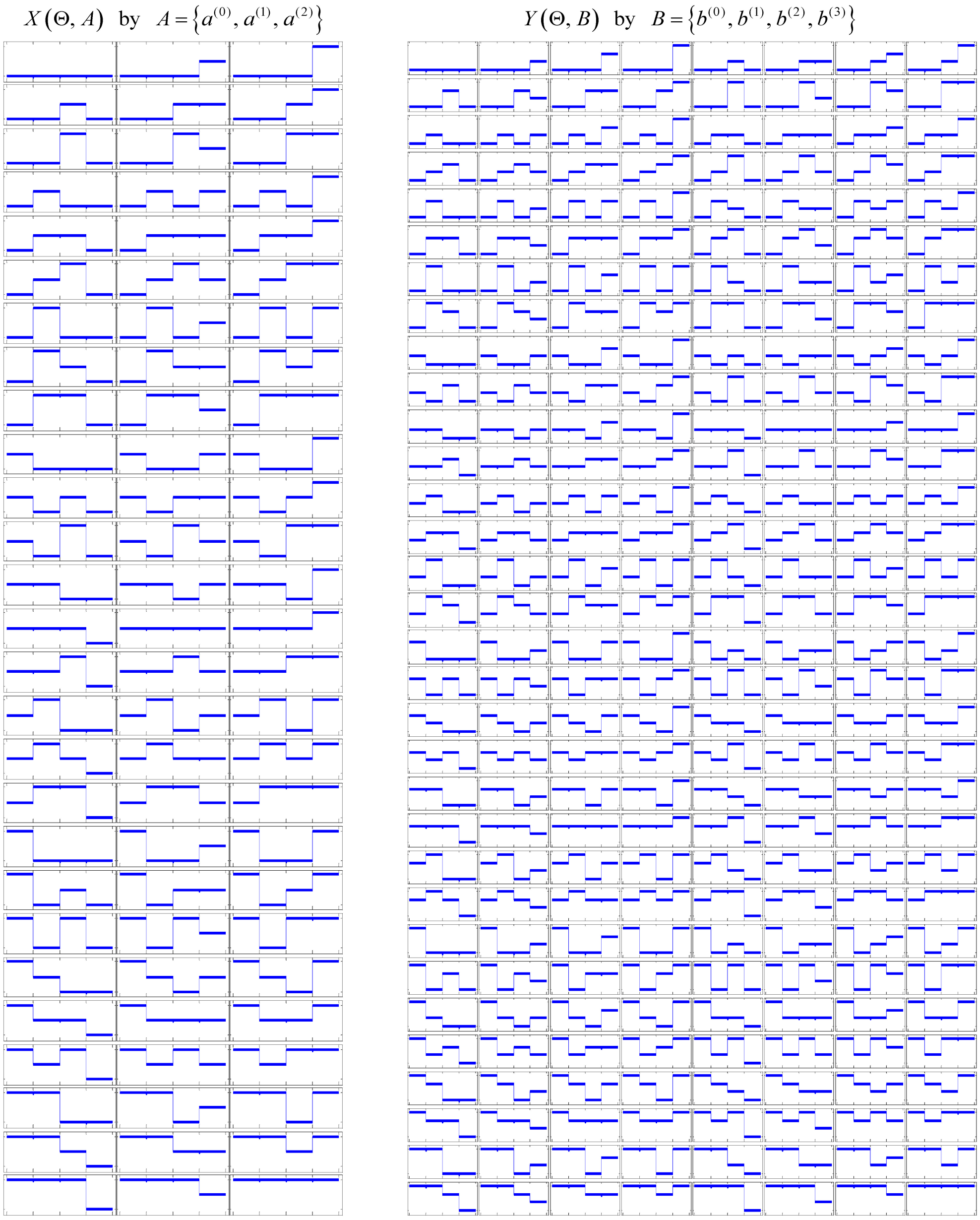

Figure 3: An example of finite pure strategy sets (47) and (48) in a bimatrix staircase-function game; the game is played with 4-subinterval staircase functions of time, where the first and second players have three and four possible values of their pure strategies, respectively

So, according with Definition 1, the first player's payoff in situation $\left\{a_i^{(m-1)},\ b_i^{(q-1)}\right\}$ is (52), (53), for $i = \overline{1,\ N}$, and the second player's payoff in situation $\left\{a_i^{(m-1)},\ b_i^{(q-1)}\right\}$ is (55), (56), for $i = \overline{1,\ N}$. In addition, situation (5) in the bimatrix staircase-function game is a stack of successive situations

$$\left\{\left\{a_i^{(m-1)},\ b_i^{(q-1)}\right\}\right\}_{i=1}^{N} \tag{57}$$

in a succession of N bimatrix games (50). Bimatrix staircase-function game (49) might be rendered down to the ordinary bimatrix game, wherein a pure strategy is a conditional point being in reality a staircase function. This rendering, however, is useless because the much more efficient method exists [21, 22] to consider game (49) as the succession of N bimatrix games (50) by (51) — (56) and find the solution of game (49) by stacking solutions of smaller-sized bimatrix games (50).

6. Time-unit shifting in bimatrix staircase-function games

An equilibrium situation in the bimatrix game always exists, either in pure or mixed strategies. Denote by

$$\mathbf{P}_i = \left[p_i^{(m)}\right]_{1\times(M+1)} \tag{58}$$

and

$$\mathbf{R}_i = \left[r_i^{(q)}\right]_{1\times(Q+1)} \tag{59}$$

the mixed strategies of the first and second players, respectively, in bimatrix game (50). The respective sets of mixed strategies of the first and second players are

$$\mathcal{P} = \left\{\mathbf{P}_i \in \mathbb{R}^{M+1} : p_i^{(m)} \geqslant 0,\ \sum_{m=1}^{M+1} p_i^{(m)} = 1\right\} \tag{60}$$

and

$$\mathcal{R} = \left\{\mathbf{R}_i \in \mathbb{R}^{Q+1} : r_i^{(q)} \geqslant 0,\ \sum_{q=1}^{Q+1} r_i^{(q)} = 1\right\}, \tag{61}$$

so $\mathbf{P}_i \in \mathcal{P}$, $\mathbf{R}_i \in \mathcal{R}$, and $\{\mathbf{P}_i,\ \mathbf{R}_i\}$ is a situation in this game.

Definition 3. A stack

$$\left\{\{\mathbf{P}_i,\ \mathbf{R}_i\}\right\}_{i=1}^{N} \tag{62}$$

of successive situations in bimatrix games (50) is called a (mixed-strategy) situation in bimatrix staircase-function game (49). Stacks $\{\mathbf{P}_i\}_{i=1}^{N}$ and $\{\mathbf{R}_i\}_{i=1}^{N}$ are the respective mixed strategies of the first and second players in this game.

It is clear that an equilibrium situation in a bimatrix staircase-function game is to be sought among stacks (62). The respective assertions can be found in [21, 22]. However, these papers do not directly show how to select the best equilibrium stack in the case of multiple equilibrium stacks.

Theorem 2. *If*

$$\{\mathbf{P}_i^*,\ \mathbf{R}_i^*\} = \left\{ \left[p_i^{(m)*}\right]_{1\times(M+1)},\ \left[r_i^{(q)*}\right]_{1\times(Q+1)} \right\} \tag{63}$$

is an equilibrium situation in bimatrix game (50) for $i = \overline{1,\ N}$, *then a stack*

$$\left\{\{\mathbf{P}_i^*,\ \mathbf{R}_i^*\}\right\}_{i=1}^{N} = \left\{ \left\{ \left[p_i^{(m)*}\right]_{1\times(M+1)},\ \left[r_i^{(q)*}\right]_{1\times(Q+1)} \right\} \right\}_{i=1}^{N} \tag{64}$$

of such successive solutions is an equilibrium situation in bimatrix staircase-function game (49). If multiple equilibria exist (at one or more time units) and the maximum of the players' payoffs sum

$$\mathbf{P}_i^* \cdot \mathbf{K}_i \cdot (\mathbf{R}_i^*)^{\mathrm{T}} + \mathbf{P}_i^* \cdot \mathbf{H}_i \cdot (\mathbf{R}_i^*)^{\mathrm{T}} \tag{65}$$

is reached at $\mathbf{P}_i^* = \mathbf{P}_i^{**}$ *and* $\mathbf{R}_i^* = \mathbf{R}_i^{**}$, *i. e.*

$$\max_{\{\mathbf{P}_i^*,\ \mathbf{R}_i^*\}} \left\{ \mathbf{P}_i^* \cdot \mathbf{K}_i \cdot (\mathbf{R}_i^*)^{\mathrm{T}} + \mathbf{P}_i^* \cdot \mathbf{H}_i \cdot (\mathbf{R}_i^*)^{\mathrm{T}} \right\} =$$
$$= \mathbf{P}_i^{**} \cdot \mathbf{K}_i \cdot (\mathbf{R}_i^{**})^{\mathrm{T}} + \mathbf{P}_i^{**} \cdot \mathbf{H}_i \cdot (\mathbf{R}_i^{**})^{\mathrm{T}}, \tag{66}$$

then the maximum of the players' payoffs sum in an equilibrium stack of bimatrix staircase-function game (49) is reached at stack

$$\left\{\{\mathbf{P}_i^{**},\ \mathbf{R}_i^{**}\}\right\}_{i=1}^{N} \tag{67}$$

and this maximum is

$$s^{**} = \sum_{i=1}^{N} \left(\mathbf{P}_i^{**} \cdot \mathbf{K}_i \cdot (\mathbf{R}_i^{**})^{\mathrm{T}} + \mathbf{P}_i^{**} \cdot \mathbf{H}_i \cdot (\mathbf{R}_i^{**})^{\mathrm{T}} \right). \tag{68}$$

Proof. As (63) is an equilibrium situation, then inequalities

$$\mathbf{P}_i \cdot \mathbf{K}_i \cdot (\mathbf{R}_i^*)^{\mathrm{T}} =$$
$$= \sum_{m=1}^{M+1} \sum_{q=1}^{Q+1} k_{imq} p_i^{(m)} r_i^{(q)*} =$$
$$= \sum_{m=1}^{M+1} \sum_{q=1}^{Q+1} p_i^{(m)} r_i^{(q)*} \int\limits_{\left[\tau^{(i-1)};\ \tau^{(i)}\right)} f\left(a_i^{(m-1)},\ b_i^{(q-1)},\ t\right) d\mu(t) \leqslant$$

$$\leqslant \sum_{m=1}^{M+1} \sum_{q=1}^{Q+1} p_i^{(m)*} r_i^{(q)*} \int\limits_{\left[\tau^{(i-1)};\ \tau^{(i)}\right)} f\left(a_i^{(m-1)},\ b_i^{(q-1)},\ t\right) d\mu\left(t\right) =$$

$$= \sum_{m=1}^{M+1} \sum_{j=1}^{Q+1} k_{imq} p_i^{(m)*} r_i^{(q)*} =$$

$$= \mathbf{P}_i^* \cdot \mathbf{K}_i \cdot \left(\mathbf{R}_i^*\right)^{\mathrm{T}} = v_i^* \;\; \forall\, \mathbf{P}_i \in \mathcal{P} \;\text{ for }\; i = \overline{1,\ N-1}, \tag{69}$$

$$\mathbf{P}_N \cdot \mathbf{K}_N \cdot \left(\mathbf{R}_N^*\right)^{\mathrm{T}} =$$

$$= \sum_{m=1}^{M+1} \sum_{q=1}^{Q+1} k_{Nmq} p_N^{(m)} r_N^{(q)*} =$$

$$= \sum_{m=1}^{M+1} \sum_{q=1}^{Q+1} p_N^{(m)} r_N^{(q)*} \int\limits_{\left[\tau^{(N-1)};\ \tau^{(N)}\right]} f\left(a_N^{(m-1)},\ b_N^{(q-1)},\ t\right) d\mu\left(t\right) \leqslant$$

$$\leqslant \sum_{m=1}^{M+1} \sum_{q=1}^{Q+1} p_N^{(m)*} r_N^{(q)*} \int\limits_{\left[\tau^{(N-1)};\ \tau^{(N)}\right]} f\left(a_N^{(m-1)},\ b_N^{(q-1)},\ t\right) d\mu\left(t\right) =$$

$$= \sum_{m=1}^{M+1} \sum_{q=1}^{Q+1} k_{Nmq} p_N^{(m)*} r_N^{(q)*} =$$

$$= \mathbf{P}_N^* \cdot \mathbf{K}_N \cdot \left(\mathbf{R}_N^*\right)^{\mathrm{T}} = v_N^* \;\; \forall\, \mathbf{P}_N \in \mathcal{P} \tag{70}$$

and inequalities

$$\mathbf{P}_i^* \cdot \mathbf{H}_i \cdot \mathbf{R}_i^{\mathrm{T}} =$$

$$= \sum_{m=1}^{M+1} \sum_{q=1}^{Q+1} h_{imq} p_i^{(m)*} r_i^{(q)} =$$

$$= \sum_{m=1}^{M+1} \sum_{q=1}^{Q+1} p_i^{(m)*} r_i^{(q)} \int\limits_{\left[\tau^{(i-1)};\ \tau^{(i)}\right)} g\left(a_i^{(m-1)},\ b_i^{(q-1)},\ t\right) d\mu\left(t\right) \leqslant$$

$$\leqslant \sum_{m=1}^{M+1} \sum_{q=1}^{Q+1} p_i^{(m)*} r_i^{(q)*} \int\limits_{\left[\tau^{(i-1)};\ \tau^{(i)}\right)} g\left(a_i^{(m-1)},\ b_i^{(q-1)},\ t\right) d\mu\left(t\right) =$$

$$= \sum_{m=1}^{M+1} \sum_{q=1}^{Q+1} h_{imq} p_i^{(m)*} r_i^{(q)*} =$$

$$= \mathbf{P}_i^* \cdot \mathbf{H}_i \cdot \left(\mathbf{R}_i^*\right)^{\mathrm{T}} = z_i^* \;\; \forall\, \mathbf{R}_i \in \mathcal{R} \;\text{ for }\; i = \overline{1,\ N-1}, \tag{71}$$

$$\mathbf{P}_N^* \cdot \mathbf{H}_N \cdot \mathbf{R}_N^{\mathrm{T}} =$$

$$= \sum_{m=1}^{M+1} \sum_{q=1}^{Q+1} h_{Nmq} p_N^{(m)*} r_N^{(q)} =$$

$$= \sum_{m=1}^{M+1} \sum_{q=1}^{Q+1} p_N^{(m)*} r_N^{(q)} \int\limits_{\left[\tau^{(N-1)};\, \tau^{(N)}\right]} g\left(a_N^{(m-1)},\ b_N^{(q-1)},\ t\right) d\mu\left(t\right) \leqslant$$

$$\leqslant \sum_{m=1}^{M+1} \sum_{q=1}^{Q+1} p_N^{(m)*} r_N^{(q)*} \int\limits_{\left[\tau^{(N-1)};\, \tau^{(N)}\right]} g\left(a_N^{(m-1)},\ b_N^{(q-1)},\ t\right) d\mu\left(t\right) =$$

$$= \sum_{m=1}^{M+1} \sum_{q=1}^{Q+1} h_{Nmq} p_N^{(m)*} r_N^{(q)*} =$$

$$= \mathbf{P}_N^* \cdot \mathbf{H}_N \cdot \left(\mathbf{R}_N^*\right)^{\mathrm{T}} = z_N^* \ \ \forall\, \mathbf{R}_N \in \mathcal{R} \tag{72}$$

hold. So, inequalities

$$\sum_{i=1}^{N-1} \mathbf{P}_i \cdot \mathbf{K}_i \cdot \left(\mathbf{R}_i^*\right)^{\mathrm{T}} + \mathbf{P}_N \cdot \mathbf{K}_N \cdot \left(\mathbf{R}_N^*\right)^{\mathrm{T}} =$$

$$= \sum_{i=1}^{N-1} \sum_{m=1}^{M+1} \sum_{q=1}^{Q+1} k_{imq} p_i^{(m)} r_i^{(q)*} + \sum_{m=1}^{M+1} \sum_{q=1}^{Q+1} k_{Nmq} p_N^{(m)} r_N^{(q)*} =$$

$$= \sum_{i=1}^{N-1} \left(\sum_{m=1}^{M+1} \sum_{q=1}^{Q+1} p_i^{(m)} r_i^{(q)*} \int\limits_{\left[\tau^{(i-1)};\, \tau^{(i)}\right)} f\left(a_i^{(m-1)},\ b_i^{(q-1)},\ t\right) d\mu\left(t\right) \right) +$$

$$+ \sum_{m=1}^{M+1} \sum_{q=1}^{Q+1} p_N^{(m)} r_N^{(q)*} \int\limits_{\left[\tau^{(N-1)};\, \tau^{(N)}\right]} f\left(a_N^{(m-1)},\ b_N^{(q-1)},\ t\right) d\mu\left(t\right) \leqslant$$

$$\leqslant \sum_{i=1}^{N-1} \left(\sum_{m=1}^{M+1} \sum_{q=1}^{Q+1} p_i^{(m)*} r_i^{(q)*} \int\limits_{\left[\tau^{(i-1)};\, \tau^{(i)}\right)} f\left(a_i^{(m-1)},\ b_i^{(q-1)},\ t\right) d\mu\left(t\right) \right) +$$

$$+ \sum_{m=1}^{M+1} \sum_{q=1}^{Q+1} p_N^{(m)*} r_N^{(q)*} \int\limits_{\left[\tau^{(N-1)};\, \tau^{(N)}\right]} f\left(a_N^{(m-1)},\ b_N^{(q-1)},\ t\right) d\mu\left(t\right) =$$

$$= \sum_{i=1}^{N-1} \sum_{m=1}^{M+1} \sum_{q=1}^{Q+1} k_{imq} p_i^{(m)*} r_i^{(q)*} + \sum_{m=1}^{M+1} \sum_{q=1}^{Q+1} k_{Nmq} p_N^{(m)*} r_N^{(q)*} =$$

$$= \sum_{i=1}^{N-1} \mathbf{P}_i^* \cdot \mathbf{K}_i \cdot \left(\mathbf{R}_i^*\right)^{\mathrm{T}} + \mathbf{P}_N^* \cdot \mathbf{K}_N \cdot \left(\mathbf{R}_N^*\right)^{\mathrm{T}} =$$

$$= \sum_{i=1}^{N} v_i^* = v^* \;\; \forall\, \mathbf{P}_i \in \mathcal{P} \;\text{ for }\; i = \overline{1,\; N} \tag{73}$$

and

$$\sum_{i=1}^{N-1} \mathbf{P}_i^* \cdot \mathbf{H}_i \cdot \mathbf{R}_i^{\mathrm{T}} + \mathbf{P}_N^* \cdot \mathbf{H}_N \cdot \mathbf{R}_N^{\mathrm{T}} =$$
$$= \sum_{i=1}^{N-1} \sum_{m=1}^{M+1} \sum_{q=1}^{Q+1} h_{imq} p_i^{(m)*} r_i^{(q)} + \sum_{m=1}^{M+1} \sum_{q=1}^{Q+1} h_{Nmq} p_N^{(m)*} r_N^{(q)} =$$
$$= \sum_{i=1}^{N-1} \left(\sum_{m=1}^{M+1} \sum_{q=1}^{Q+1} p_i^{(m)*} r_i^{(q)} \int\limits_{\left[\tau^{(i-1)};\; \tau^{(i)}\right)} g\left(a_i^{(m-1)},\; b_i^{(q-1)},\; t\right) d\mu\left(t\right) \right) +$$
$$+ \sum_{m=1}^{M+1} \sum_{q=1}^{Q+1} p_N^{(m)*} r_N^{(q)} \int\limits_{\left[\tau^{(N-1)};\; \tau^{(N)}\right]} g\left(a_N^{(m-1)},\; b_N^{(q-1)},\; t\right) d\mu\left(t\right) \leqslant$$
$$\leqslant \sum_{i=1}^{N-1} \left(\sum_{m=1}^{M+1} \sum_{q=1}^{Q+1} p_i^{(m)*} r_i^{(q)*} \int\limits_{\left[\tau^{(i-1)};\; \tau^{(i)}\right)} g\left(a_i^{(m-1)},\; b_i^{(q-1)},\; t\right) d\mu\left(t\right) \right) +$$
$$+ \sum_{m=1}^{M+1} \sum_{q=1}^{Q+1} p_N^{(m)*} r_N^{(q)*} \int\limits_{\left[\tau^{(N-1)};\; \tau^{(N)}\right]} g\left(a_N^{(m-1)},\; b_N^{(q-1)},\; t\right) d\mu\left(t\right) =$$
$$= \sum_{i=1}^{N-1} \sum_{m=1}^{M+1} \sum_{q=1}^{Q+1} h_{imq} p_i^{(m)*} r_i^{(q)*} + \sum_{m=1}^{M+1} \sum_{q=1}^{Q+1} h_{Nmq} p_N^{(m)*} r_N^{(q)*} =$$
$$= \sum_{i=1}^{N-1} \mathbf{P}_i^* \cdot \mathbf{H}_i \cdot \left(\mathbf{R}_i^*\right)^{\mathrm{T}} + \mathbf{P}_N^* \cdot \mathbf{H}_N \cdot \left(\mathbf{R}_N^*\right)^{\mathrm{T}} =$$
$$= \sum_{i=1}^{N} z_i^* = z^* \;\; \forall\, \mathbf{R}_i \in \mathcal{R} \;\text{ for }\; i = \overline{1,\; N} \tag{74}$$

hold as well. The assertion of Theorem 1 for bimatrix staircase-function game (49) can be re-written as

$$K\big(x\left(t\right),\; y\left(t\right)\big) = \sum_{i=1}^{N} k_{imq} =$$
$$= \sum_{i=1}^{N-1} \int\limits_{\left[\tau^{(i-1)};\; \tau^{(i)}\right)} f\left(a_i^{(m-1)},\; b_i^{(q-1)},\; t\right) d\mu\left(t\right) +$$

$$+ \int\limits_{\left[\tau^{(N-1)};\ \tau^{(N)}\right]} f\left(a_N^{(m-1)},\ b_N^{(q-1)},\ t\right) d\mu\left(t\right) \tag{75}$$

and

$$H\big(x\left(t\right),\ y\left(t\right)\big) = \sum_{i=1}^{N} h_{imq} =$$

$$= \sum_{i=1}^{N-1} \int\limits_{\left[\tau^{(i-1)};\ \tau^{(i)}\right)} g\left(a_i^{(m-1)},\ b_i^{(q-1)},\ t\right) d\mu\left(t\right) +$$

$$+ \int\limits_{\left[\tau^{(N-1)};\ \tau^{(N)}\right]} g\left(a_N^{(m-1)},\ b_N^{(q-1)},\ t\right) d\mu\left(t\right). \tag{76}$$

Therefore, inequalities (73) and (74) along with using the payoff decomposition by (75) and (76) allow to conclude that the stack of successive equilibria (64) is an equilibrium situation in game (49).

As (66) holds, then

$$\sum_{i=1}^{N} \max_{\left\{\mathbf{P}_i^*,\ \mathbf{R}_i^*\right\}} \left\{\mathbf{P}_i^* \cdot \mathbf{K}_i \cdot \left(\mathbf{R}_i^*\right)^{\mathrm{T}} + \mathbf{P}_i^* \cdot \mathbf{H}_i \cdot \left(\mathbf{R}_i^*\right)^{\mathrm{T}}\right\} =$$

$$= \sum_{i=1}^{N} \left(\mathbf{P}_i^{**} \cdot \mathbf{K}_i \cdot \left(\mathbf{R}_i^{**}\right)^{\mathrm{T}} + \mathbf{P}_i^{**} \cdot \mathbf{H}_i \cdot \left(\mathbf{R}_i^{**}\right)^{\mathrm{T}}\right) =$$

$$= \sum_{i=1}^{N} \mathbf{P}_i^{**} \cdot \mathbf{K}_i \cdot \left(\mathbf{R}_i^{**}\right)^{\mathrm{T}} + \sum_{i=1}^{N} \mathbf{P}_i^{**} \cdot \mathbf{H}_i \cdot \left(\mathbf{R}_i^{**}\right)^{\mathrm{T}} =$$

$$= \sum_{i=1}^{N} v_i^{**} + \sum_{i=1}^{N} z_i^{**} = v^{**} + z^{**} = s^{**}, \tag{77}$$

i. e. the maximum of the players' payoffs sum is (68) reached at stack (67). □

Consider now the case when the bimatrix staircase-function game is played through a lesser number of time units. Thus, instead of time-interval breaking (18), the game is played by a narrower time-interval breaking

$$\Theta_* = \left\{t_1 \leqslant \tau_1 = \tau^{(n)} < \tau^{(n+1)} < \tau^{(n+2)} < \ldots < \tau^{(U-1)} < \tau^{(U)} = \tau_2 \leqslant t_2\right\}, \tag{78}$$

where

$$n \in \left\{\overline{0,\ N-1}\right\},\ \ U \in \left\{\overline{1,\ N}\right\},\ \ n < U, \tag{79}$$

and $\left\{\tau^{(i)}\right\}_{i=n+1}^{U-1}$ are time points at which the staircase-function strategy can change its value. So, $\Theta_* \subset \Theta$ in terms of the interval breaking.

Theorem 3. *In a bimatrix staircase-function game*

$$\Big\langle \{X(\Theta_*,\ A),\ Y(\Theta_*,\ B)\},\ \big\{K\big(x(t),\ y(t)\big),\ H\big(x(t),\ y(t)\big)\big\}\Big\rangle \tag{80}$$

by

$$X(\Theta_*,\ A) = \left\{x(t) \in X(\Theta_*):\ x(t) \in \left\{a^{(m-1)}\right\}_{m=1}^{M+1}\right\} \subset \\ \subset X(\Theta_*) \subset X([\tau_1;\ \tau_2]) \tag{81}$$

and

$$Y(\Theta_*,\ B) = \left\{y(t) \in Y(\Theta_*):\ y(t) \in \left\{b^{(q-1)}\right\}_{q=1}^{Q+1}\right\} \subset \\ \subset Y(\Theta_*) \subset Y([\tau_1;\ \tau_2]) \tag{82}$$

and a time-interval breaking (78) for (79), an equilibrium situation is a stack

$$\left\{\{\mathbf{P}_i^*,\ \mathbf{R}_i^*\}\right\}_{i=n+1}^{U} = \left\{\left\{\left[p_i^{(m)*}\right]_{1\times(M+1)},\ \left[r_i^{(q)*}\right]_{1\times(Q+1)}\right\}\right\}_{i=n+1}^{U} \tag{83}$$

of $U-n$ *successive equilibria (63) of bimatrix game (50) for* $i=\overline{n+1,\ U}$. *If multiple equilibria exist (at one or more time units) and the maximum of the players' payoffs sum (65) is reached at* $\mathbf{P}_i^* = \mathbf{P}_i^{**}$ *and* $\mathbf{R}_i^* = \mathbf{R}_i^{**}$, *i. e. (66) holds, then the maximum of the players' payoffs sum in an equilibrium stack of bimatrix staircase-function game (80) is reached at stack*

$$\left\{\{\mathbf{P}_i^{**},\ \mathbf{R}_i^{**}\}\right\}_{i=n+1}^{U} \tag{84}$$

and this maximum is

$$s^{**(\Theta_*)} = \sum_{i=n+1}^{U} \Big(\mathbf{P}_i^{**} \cdot \mathbf{K}_i \cdot (\mathbf{R}_i^{**})^{\mathrm{T}} + \mathbf{P}_i^{**} \cdot \mathbf{H}_i \cdot (\mathbf{R}_i^{**})^{\mathrm{T}}\Big). \tag{85}$$

Proof. As inequalities (69) — (72) hold $\forall\, i=\overline{1,\ N}$, they hold $\forall\, i=\overline{n+1,\ U}$. For time-interval breaking (78), time interval $[\tau_1;\ \tau_2]$ can be re-written as

$$[\tau_1;\ \tau_2] = \left\{\bigcup_{i=n+1}^{U-1}\left[\tau^{(i-1)};\ \tau^{(i)}\right)\right\} \cup \left[\tau^{(U-1)};\ \tau^{(U)}\right]. \tag{86}$$

So, owing to Theorem 1,

$$K\big(x(t),\ y(t)\big) = \sum_{i=n+1}^{U} k_{imq} = \\ = \sum_{i=n+1}^{U-1} \int\limits_{\left[\tau^{(i-1)};\ \tau^{(i)}\right)} f\left(a_i^{(m-1)},\ b_i^{(q-1)},\ t\right) d\mu(t) +$$

$$+ \int\limits_{\left[\tau^{(U-1)};\, \tau^{(U)}\right]} f\left(a_U^{(m-1)},\, b_U^{(q-1)},\, t\right) d\mu\left(t\right) \tag{87}$$

and

$$H\bigl(x\left(t\right),\, y\left(t\right)\bigr) = \sum_{i=n+1}^{U} h_{imq} =$$

$$= \sum_{i=n+1}^{U-1} \int\limits_{\left[\tau^{(i-1)};\, \tau^{(i)}\right)} g\left(a_i^{(m-1)},\, b_i^{(q-1)},\, t\right) d\mu\left(t\right) +$$

$$+ \int\limits_{\left[\tau^{(U-1)};\, \tau^{(U)}\right]} g\left(a_U^{(m-1)},\, b_U^{(q-1)},\, t\right) d\mu\left(t\right). \tag{88}$$

So, inequalities

$$\sum_{i=n+1}^{U-1} \mathbf{P}_i \cdot \mathbf{K}_i \cdot \left(\mathbf{R}_i^*\right)^{\mathrm{T}} + \mathbf{P}_U \cdot \mathbf{K}_U \cdot \left(\mathbf{R}_U^*\right)^{\mathrm{T}} =$$

$$= \sum_{i=n+1}^{U-1} \sum_{m=1}^{M+1} \sum_{q=1}^{Q+1} k_{imq} p_i^{(m)} r_i^{(q)*} + \sum_{m=1}^{M+1} \sum_{q=1}^{Q+1} k_{Umq} p_U^{(m)} r_U^{(q)*} =$$

$$= \sum_{i=n+1}^{U-1} \left(\sum_{m=1}^{M+1} \sum_{q=1}^{Q+1} p_i^{(m)} r_i^{(q)*} \int\limits_{\left[\tau^{(i-1)};\, \tau^{(i)}\right)} f\left(a_i^{(m-1)},\, b_i^{(q-1)},\, t\right) d\mu\left(t\right) \right) +$$

$$+ \sum_{m=1}^{M+1} \sum_{q=1}^{Q+1} p_U^{(m)} r_U^{(q)*} \int\limits_{\left[\tau^{(U-1)};\, \tau^{(U)}\right]} f\left(a_U^{(m-1)},\, b_U^{(q-1)},\, t\right) d\mu\left(t\right) \leqslant$$

$$\leqslant \sum_{i=n+1}^{U-1} \left(\sum_{m=1}^{M+1} \sum_{q=1}^{Q+1} p_i^{(m)*} r_i^{(q)*} \int\limits_{\left[\tau^{(i-1)};\, \tau^{(i)}\right)} f\left(a_i^{(m-1)},\, b_i^{(q-1)},\, t\right) d\mu\left(t\right) \right) +$$

$$+ \sum_{m=1}^{M+1} \sum_{q=1}^{Q+1} p_U^{(m)*} r_U^{(q)*} \int\limits_{\left[\tau^{(U-1)};\, \tau^{(U)}\right]} f\left(a_U^{(m-1)},\, b_U^{(q-1)},\, t\right) d\mu\left(t\right) =$$

$$= \sum_{i=n+1}^{U-1} \sum_{m=1}^{M+1} \sum_{q=1}^{Q+1} k_{imq} p_i^{(m)*} r_i^{(q)*} + \sum_{m=1}^{M+1} \sum_{q=1}^{Q+1} k_{Umq} p_U^{(m)*} r_U^{(q)*} =$$

$$= \sum_{i=n+1}^{U-1} \mathbf{P}_i^* \cdot \mathbf{K}_i \cdot \left(\mathbf{R}_i^*\right)^{\mathrm{T}} + \mathbf{P}_U^* \cdot \mathbf{K}_U \cdot \left(\mathbf{R}_U^*\right)^{\mathrm{T}} =$$

$$= \sum_{i=n+1}^{U} v_i^{*(\Theta_*)} = v^{*(\Theta_*)} \ \ \forall\, \mathbf{P}_i \in \mathcal{P} \ \text{ for } \ i = \overline{n+1,\ U} \tag{89}$$

and

$$\sum_{i=n+1}^{U-1} \mathbf{P}_i^* \cdot \mathbf{H}_i \cdot \mathbf{R}_i^{\mathrm{T}} + \mathbf{P}_U^* \cdot \mathbf{H}_U \cdot \mathbf{R}_U^{\mathrm{T}} =$$

$$= \sum_{i=n+1}^{U-1} \sum_{m=1}^{M+1} \sum_{q=1}^{Q+1} h_{imq} p_i^{(m)*} r_i^{(q)} + \sum_{m=1}^{M+1} \sum_{q=1}^{Q+1} h_{Umq} p_U^{(m)*} r_U^{(q)} =$$

$$= \sum_{i=n+1}^{U-1} \left(\sum_{m=1}^{M+1} \sum_{q=1}^{Q+1} p_i^{(m)*} r_i^{(q)} \int\limits_{\left[\tau^{(i-1)};\ \tau^{(i)}\right)} g\left(a_i^{(m-1)},\ b_i^{(q-1)},\ t\right) d\mu\left(t\right) \right) +$$

$$+ \sum_{m=1}^{M+1} \sum_{q=1}^{Q+1} p_U^{(m)*} r_U^{(q)} \int\limits_{\left[\tau^{(U-1)};\ \tau^{(U)}\right]} g\left(a_U^{(m-1)},\ b_U^{(q-1)},\ t\right) d\mu\left(t\right) \leqslant$$

$$\leqslant \sum_{i=n+1}^{U-1} \left(\sum_{m=1}^{M+1} \sum_{q=1}^{Q+1} p_i^{(m)*} r_i^{(q)*} \int\limits_{\left[\tau^{(i-1)};\ \tau^{(i)}\right)} g\left(a_i^{(m-1)},\ b_i^{(q-1)},\ t\right) d\mu\left(t\right) \right) +$$

$$+ \sum_{m=1}^{M+1} \sum_{q=1}^{Q+1} p_U^{(m)*} r_U^{(q)*} \int\limits_{\left[\tau^{(U-1)};\ \tau^{(U)}\right]} g\left(a_U^{(m-1)},\ b_U^{(q-1)},\ t\right) d\mu\left(t\right) =$$

$$= \sum_{i=n+1}^{U-1} \sum_{m=1}^{M+1} \sum_{q=1}^{Q+1} h_{imq} p_i^{(m)*} r_i^{(q)*} + \sum_{m=1}^{M+1} \sum_{q=1}^{Q+1} h_{Umq} p_U^{(m)*} r_U^{(q)*} =$$

$$= \sum_{i=n+1}^{U-1} \mathbf{P}_i^* \cdot \mathbf{H}_i \cdot \left(\mathbf{R}_i^*\right)^{\mathrm{T}} + \mathbf{P}_U^* \cdot \mathbf{H}_U \cdot \left(\mathbf{R}_U^*\right)^{\mathrm{T}} =$$

$$= \sum_{i=n+1}^{U} z_i^{*(\Theta_*)} = z^{*(\Theta_*)} \ \ \forall\, \mathbf{R}_i \in \mathcal{R} \ \text{ for } \ i = \overline{n+1,\ U} \tag{90}$$

hold. Therefore, inequalities (89) and (90) along with using the payoff decomposition by (87) and (88) allow to conclude that the stack of successive equilibria (83) is an equilibrium situation in bimatrix staircase-function game (80).

As (66) holds, then

$$\sum_{i=n+1}^{U} \max_{\left\{\mathbf{P}_i^*,\ \mathbf{R}_i^*\right\}} \left\{ \mathbf{P}_i^* \cdot \mathbf{K}_i \cdot \left(\mathbf{R}_i^*\right)^{\mathrm{T}} + \mathbf{P}_i^* \cdot \mathbf{H}_i \cdot \left(\mathbf{R}_i^*\right)^{\mathrm{T}} \right\} =$$

$$= \sum_{i=n+1}^{U} \left(\mathbf{P}_i^{**} \cdot \mathbf{K}_i \cdot \left(\mathbf{R}_i^{**}\right)^{\mathrm{T}} + \mathbf{P}_i^{**} \cdot \mathbf{H}_i \cdot \left(\mathbf{R}_i^{**}\right)^{\mathrm{T}} \right) =$$

$$= \sum_{i=n+1}^{U} \mathbf{P}_i^{**} \cdot \mathbf{K}_i \cdot \left(\mathbf{R}_i^{**}\right)^{\mathrm{T}} + \sum_{i=n+1}^{U} \mathbf{P}_i^{**} \cdot \mathbf{H}_i \cdot \left(\mathbf{R}_i^{**}\right)^{\mathrm{T}} =$$

$$= \sum_{i=n+1}^{U} v_i^{**(\Theta_*)} + \sum_{i=n+1}^{U} z_i^{**(\Theta_*)} = v^{**(\Theta_*)} + z^{**(\Theta_*)} = s^{**(\Theta_*)}, \tag{91}$$

i. e. the maximum of the players' payoffs sum is (85) reached at stack (84). □

It is quite obvious that

$$\left\{ \left\{ \mathbf{P}_i^*,\ \mathbf{R}_i^* \right\} \right\}_{i=n+1}^{U} \subset \left\{ \left\{ \mathbf{P}_i^*,\ \mathbf{R}_i^* \right\} \right\}_{i=1}^{N}. \tag{92}$$

So, Theorem 3 implies that the time-unit shifting does not change the structure and number of equilibria in a bimatrix staircase-function game, nor does it change the structure of the best equilibrium stack determined by the maximum of the players' payoffs sum. In fact, game (80) is a subgame of bimatrix staircase-function game (49). An equilibrium solution of the subgame can be easily taken from the respective equilibrium solution of ("wider") game (49).

7. Time-unit shifting in discrete-time staircase-function 2-person games

See whether the inference above is valid for discrete-time staircase-function 2-person game (33), which, generally speaking, is played within uncountably infinite sets of players' staircase-function strategies. Denote by

$$p_i\left(\alpha_i\right) \tag{93}$$

and

$$r_i\left(\beta_i\right) \tag{94}$$

the mixed strategies of the first and second players, respectively, in (subinterval) infinite 2-person game (31), where

$$P = \left\{ p_i\left(\alpha_i\right) \in \mathbb{L}_2\left[a_{\min};\ a_{\max}\right] : p_i\left(\alpha_i\right) \geqslant 0, \int\limits_{[a_{\min};\, a_{\max}]} p_i\left(\alpha_i\right) d\mu\left(\alpha_i\right) = 1 \right\} \tag{95}$$

and

$$R = \left\{ r_i\left(\beta_i\right) \in \mathbb{L}_2\left[b_{\min};\ b_{\max}\right] : r_i\left(\beta_i\right) \geqslant 0, \int\limits_{[b_{\min};\, b_{\max}]} r_i\left(\beta_i\right) d\mu\left(\beta_i\right) = 1 \right\} \tag{96}$$

are the respective sets of mixed strategies of the players. So, $p_i\left(\alpha_i\right) \in P$, $r_i\left(\beta_i\right) \in R$, and

$$\left\{ p_i\left(\alpha_i\right),\ r_i\left(\beta_i\right) \right\} \tag{97}$$

is a situation in this game.

Definition 4. A stack

$$\left\{\{p_i(\alpha_i),\ r_i(\beta_i)\}\right\}_{i=1}^{N} \tag{98}$$

of successive situations in (ordinary) 2-person games (31) is called a (mixed-strategy) situation in discrete-time staircase-function 2-person game (33). Stacks $\{p_i(\alpha_i)\}_{i=1}^{N}$ and $\{r_i(\beta_i)\}_{i=1}^{N}$ are the respective mixed strategies of the first and second players in this game.

Just like in the case of a finite discrete-time staircase-function 2-person game, it is clear that an equilibrium situation in a discrete-time staircase-function 2-person game is to be sought among stacks (98). The respective assertions in [21], however, concern only the case of equilibrium situations of pure strategies.

Theorem 4. *If $p_i^*(\alpha_i) \in P$, $r_i^*(\beta_i) \in R$, and*

$$\{p_i^*(\alpha_i),\ r_i^*(\beta_i)\} \tag{99}$$

is an equilibrium situation in 2-person game (31) for $i = \overline{1,\ N}$, then a stack

$$\left\{\{p_i^*(\alpha_i),\ r_i^*(\beta_i)\}\right\}_{i=1}^{N} \tag{100}$$

of such successive equilibria is an equilibrium situation in discrete-time staircase-function 2-person game (33).

Proof. As (99) is an equilibrium situation, and all these subinterval equilibria exist, then inequalities

$$\int\limits_{[b_{\min};\, b_{\max}]} r_i^*(\beta_i) \left(\int\limits_{[a_{\min};\, a_{\max}]} p_i(\alpha_i) K_i(\alpha_i,\ \beta_i)\, d\mu(\alpha_i) \right) d\mu(\beta_i) =$$

$$= \int\limits_{[b_{\min};\, b_{\max}]} r_i^*(\beta_i) \left(\int\limits_{[a_{\min};\, a_{\max}]} p_i(\alpha_i) \int\limits_{\left[\tau^{(i-1)};\, \tau^{(i)}\right)} f(\alpha_i,\ \beta_i,\ t)\, d\mu(t) d\mu(\alpha_i) \right) d\mu(\beta_i) \leqslant$$

$$\leqslant \int\limits_{[b_{\min};\, b_{\max}]} r_i^*(\beta_i) \left(\int\limits_{[a_{\min};\, a_{\max}]} p_i^*(\alpha_i) \int\limits_{\left[\tau^{(i-1)};\, \tau^{(i)}\right)} f(\alpha_i,\ \beta_i,\ t)\, d\mu(t) d\mu(\alpha_i) \right) d\mu(\beta_i) =$$

$$= \int\limits_{[b_{\min};\, b_{\max}]} r_i^*(\beta_i) \left(\int\limits_{[a_{\min};\, a_{\max}]} p_i^*(\alpha_i) K_i(\alpha_i,\ \beta_i)\, d\mu(\alpha_i) \right) d\mu(\beta_i)$$

$$\forall\, p_i(\alpha_i) \in P \ \text{ for } \ i = \overline{1,\ N-1}, \tag{101}$$

$$\int\limits_{[b_{\min};\, b_{\max}]} r_N^*(\beta_N) \left(\int\limits_{[a_{\min};\, a_{\max}]} p_N(\alpha_N) K_N(\alpha_N,\, \beta_N)\, d\mu(\alpha_N) \right) d\mu(\beta_N) =$$

$$= \int\limits_{[b_{\min};\, b_{\max}]} r_N^*(\beta_N)$$

$$\left(\int\limits_{[a_{\min};\, a_{\max}]} p_N(\alpha_N) \int\limits_{\left[\tau^{(N-1)};\, \tau^{(N)}\right]} f(\alpha_N,\, \beta_N,\, t)\, d\mu(t) d\mu(\alpha_N) \right) d\mu(\beta_N) \leqslant$$

$$\leqslant \int\limits_{[b_{\min};\, b_{\max}]} r_N^*(\beta_N)$$

$$\left(\int\limits_{[a_{\min};\, a_{\max}]} p_N^*(\alpha_N) \int\limits_{\left[\tau^{(N-1)};\, \tau^{(N)}\right]} f(\alpha_N,\, \beta_N,\, t)\, d\mu(t) d\mu(\alpha_N) \right) d\mu(\beta_N) =$$

$$= \int\limits_{[b_{\min};\, b_{\max}]} r_N^*(\beta_N) \left(\int\limits_{[a_{\min};\, a_{\max}]} p_N^*(\alpha_N) K_N(\alpha_N,\, \beta_N)\, d\mu(\alpha_N) \right) d\mu(\beta_N)$$

$$\forall\, p_N(\alpha_N) \in P \quad (102)$$

and inequalities

$$\int\limits_{[b_{\min};\, b_{\max}]} r_i(\beta_i) \left(\int\limits_{[a_{\min};\, a_{\max}]} p_i^*(\alpha_i) H_i(\alpha_i,\, \beta_i)\, d\mu(\alpha_i) \right) d\mu(\beta_i) =$$

$$= \int\limits_{[b_{\min};\, b_{\max}]} r_i(\beta_i) \left(\int\limits_{[a_{\min};\, a_{\max}]} p_i^*(\alpha_i) \int\limits_{\left[\tau^{(i-1)};\, \tau^{(i)}\right)} g(\alpha_i,\, \beta_i,\, t)\, d\mu(t) d\mu(\alpha_i) \right) d\mu(\beta_i) \leqslant$$

$$\leqslant \int\limits_{[b_{\min};\, b_{\max}]} r_i^*(\beta_i) \left(\int\limits_{[a_{\min};\, a_{\max}]} p_i^*(\alpha_i) \int\limits_{\left[\tau^{(i-1)};\, \tau^{(i)}\right)} g(\alpha_i,\, \beta_i,\, t)\, d\mu(t) d\mu(\alpha_i) \right) d\mu(\beta_i) =$$

$$= \int\limits_{[b_{\min};\, b_{\max}]} r_i^*(\beta_i) \left(\int\limits_{[a_{\min};\, a_{\max}]} p_i^*(\alpha_i) H_i(\alpha_i,\, \beta_i)\, d\mu(\alpha_i) \right) d\mu(\beta_i)$$

$$\forall\, r_i(\beta_i) \in R \;\text{ for }\; i = \overline{1,\, N-1}, \quad (103)$$

$$\int\limits_{[b_{\min};\, b_{\max}]} r_N(\beta_N) \left(\int\limits_{[a_{\min};\, a_{\max}]} p_N^*(\alpha_N)\, H_N(\alpha_N,\ \beta_N)\, d\mu(\alpha_N) \right) d\mu(\beta_N) =$$

$$= \int\limits_{[b_{\min};\, b_{\max}]} r_N(\beta_N)$$

$$\left(\int\limits_{[a_{\min};\, a_{\max}]} p_N^*(\alpha_N) \int\limits_{\left[\tau^{(N-1)};\, \tau^{(N)}\right]} g(\alpha_N,\ \beta_N,\ t)\, d\mu(t) d\mu(\alpha_N) \right) d\mu(\beta_N) \leqslant$$

$$\leqslant \int\limits_{[b_{\min};\, b_{\max}]} r_N^*(\beta_N)$$

$$\left(\int\limits_{[a_{\min};\, a_{\max}]} p_N^*(\alpha_N) \int\limits_{\left[\tau^{(N-1)};\, \tau^{(N)}\right]} g(\alpha_N,\ \beta_N,\ t)\, d\mu(t) d\mu(\alpha_N) \right) d\mu(\beta_N) =$$

$$= \int\limits_{[b_{\min};\, b_{\max}]} r_N^*(\beta_N) \left(\int\limits_{[a_{\min};\, a_{\max}]} p_N^*(\alpha_N)\, H_N(\alpha_N,\ \beta_N)\, d\mu(\alpha_N) \right) d\mu(\beta_N)$$

$$\forall\ r_N(\beta_N) \in R \quad (104)$$

hold. So, inequalities

$$\sum_{i=1}^{N} \int\limits_{[b_{\min};\, b_{\max}]} r_i^*(\beta_i) \left(\int\limits_{[a_{\min};\, a_{\max}]} p_i(\alpha_i)\, K_i(\alpha_i,\ \beta_i)\, d\mu(\alpha_i) \right) d\mu(\beta_i) =$$

$$= \sum_{i=1}^{N-1} \int\limits_{[b_{\min};\, b_{\max}]} r_i^*(\beta_i)$$

$$\left(\int\limits_{[a_{\min};\, a_{\max}]} p_i(\alpha_i) \int\limits_{\left[\tau^{(i-1)};\, \tau^{(i)}\right)} f(\alpha_i,\ \beta_i,\ t)\, d\mu(t) d\mu(\alpha_i) \right) d\mu(\beta_i) +$$

$$+ \int\limits_{[b_{\min};\, b_{\max}]} r_N^*(\beta_N)$$

$$\left(\int\limits_{[a_{\min};\, a_{\max}]} p_N(\alpha_N) \int\limits_{\left[\tau^{(N-1)};\, \tau^{(N)}\right]} f(\alpha_N,\ \beta_N,\ t)\, d\mu(t) d\mu(\alpha_N) \right) d\mu(\beta_N) \leqslant$$

$$\leqslant \sum_{i=1}^{N-1} \int\limits_{[b_{\min};\, b_{\max}]} r_i^*(\beta_i) \left(\int\limits_{[a_{\min};\, a_{\max}]} p_i^*(\alpha_i) \int\limits_{\left[\tau^{(i-1)};\, \tau^{(i)}\right)} f(\alpha_i,\ \beta_i,\ t)\, d\mu(t) d\mu(\alpha_i) \right) d\mu(\beta_i) +$$

$$+ \int\limits_{[b_{\min};\, b_{\max}]} r_N^*(\beta_N) \left(\int\limits_{[a_{\min};\, a_{\max}]} p_N^*(\alpha_N) \int\limits_{\left[\tau^{(N-1)};\, \tau^{(N)}\right]} f(\alpha_N,\ \beta_N,\ t)\, d\mu(t) d\mu(\alpha_N) \right) d\mu(\beta_N) =$$

$$= \sum_{i=1}^{N} \int\limits_{[b_{\min};\, b_{\max}]} r_i^*(\beta_i) \left(\int\limits_{[a_{\min};\, a_{\max}]} p_i^*(\alpha_i)\, K_i(\alpha_i,\ \beta_i)\, d\mu(\alpha_i) \right) d\mu(\beta_i) =$$

$$= \sum_{i=1}^{N} v_i^* = v^* \ \ \forall\, p_i(\alpha_i) \in P \ \text{ for } \ i = \overline{1,\ N} \tag{105}$$

and

$$\sum_{i=1}^{N} \int\limits_{[b_{\min};\, b_{\max}]} r_i(\beta_i) \left(\int\limits_{[a_{\min};\, a_{\max}]} p_i^*(\alpha_i)\, H_i(\alpha_i,\ \beta_i)\, d\mu(\alpha_i) \right) d\mu(\beta_i) =$$

$$= \sum_{i=1}^{N-1} \int\limits_{[b_{\min};\, b_{\max}]} r_i(\beta_i) \left(\int\limits_{[a_{\min};\, a_{\max}]} p_i^*(\alpha_i) \int\limits_{\left[\tau^{(i-1)};\, \tau^{(i)}\right)} g(\alpha_i,\ \beta_i,\ t)\, d\mu(t) d\mu(\alpha_i) \right) d\mu(\beta_i) +$$

$$+ \int\limits_{[b_{\min};\, b_{\max}]} r_N(\beta_N) \left(\int\limits_{[a_{\min};\, a_{\max}]} p_N^*(\alpha_N) \int\limits_{\left[\tau^{(N-1)};\, \tau^{(N)}\right]} g(\alpha_N,\ \beta_N,\ t)\, d\mu(t) d\mu(\alpha_N) \right) d\mu(\beta_N) \leqslant$$

$$\leqslant \sum_{i=1}^{N-1} \int\limits_{[b_{\min};\, b_{\max}]} r_i^*(\beta_i)$$

$$\left(\int\limits_{[a_{\min};\ a_{\max}]} p_i^*(\alpha_i) \int\limits_{\left[\tau^{(i-1)};\ \tau^{(i)}\right)} g(\alpha_i,\ \beta_i,\ t)\, d\mu(t) d\mu(\alpha_i) \right) d\mu(\beta_i) +$$

$$+ \int\limits_{[b_{\min};\ b_{\max}]} r_N^*(\beta_N)$$

$$\left(\int\limits_{[a_{\min};\ a_{\max}]} p_N^*(\alpha_N) \int\limits_{\left[\tau^{(N-1)};\ \tau^{(N)}\right]} g(\alpha_N,\ \beta_N,\ t)\, d\mu(t) d\mu(\alpha_N) \right) d\mu(\beta_N) =$$

$$= \sum_{i=1}^{N} \int\limits_{[b_{\min};\ b_{\max}]} r_i^*(\beta_i) \left(\int\limits_{[a_{\min};\ a_{\max}]} p_i^*(\alpha_i)\, H_i(\alpha_i,\ \beta_i)\, d\mu(\alpha_i) \right) d\mu(\beta_i) =$$

$$= \sum_{i=1}^{N} z_i^* = z^* \ \ \forall\, r_i(\beta_i) \in R \ \text{ for } \ i = \overline{1,\ N} \quad (106)$$

hold as well. Therefore, inequalities (105) and (106) along with using the payoff decomposition by (36) and (37) allow to conclude that the stack of successive equilibria (100) is an equilibrium situation in game (33). □

Theorem 5. *In a discrete-time staircase-function 2-person game*

$$\left\langle \{X(\Theta_*),\ Y(\Theta_*)\},\ \left\{K\big(x(t),\ y(t)\big),\ H\big(x(t),\ y(t)\big)\right\} \right\rangle \quad (107)$$

by

$$X(\Theta_*) = \Big\{ x(t) \in X([\tau_1;\ \tau_2]) : x(t) = \alpha_i \in [a_{\min};\ a_{\max}]$$
$$\forall\, t \in \left[\tau^{(i-1)};\ \tau^{(i)}\right) \ \text{ for } i = \overline{n+1,\ U-1} \ \text{ and} \quad (108)$$
$$x(t) = \alpha_U \in [a_{\min};\ a_{\max}] \ \forall\, t \in \left[\tau^{(U-1)};\ \tau^{(U)}\right] \Big\} \subset X([\tau_1;\ \tau_2])$$

and

$$Y(\Theta_*) = \Big\{ y(t) \in Y([\tau_1;\ \tau_2]) : y(t) = \beta_i \in [b_{\min};\ b_{\max}]$$
$$\forall\, t \in \left[\tau^{(i-1)};\ \tau^{(i)}\right) \ \text{ for } i = \overline{n+1,\ U-1} \ \text{ and} \quad (109)$$
$$y(t) = \beta_U \in [b_{\min};\ b_{\max}] \ \forall\, t \in \left[\tau^{(U-1)};\ \tau^{(U)}\right] \Big\} \subset Y([\tau_1;\ \tau_2])$$

and a time-interval breaking (78) for (79), an equilibrium situation is a stack

$$\left\{ \{p_i^*(\alpha_i),\ r_i^*(\beta_i)\} \right\}_{i=n+1}^{U} \quad (110)$$

of $U - n$ successive equilibria (99) in 2-person game (31) for $i = \overline{n+1,\ U}$.

Proof. As inequalities (101) — (104) hold $\forall\, i = \overline{1,\ N}$, they hold $\forall\, i = \overline{n+1,\ U}$. For time-interval breaking (78), time interval $[\tau_1;\ \tau_2]$ can be re-written as (86), so, owing to Theorem 1,

$$K\bigl(x\,(t)\,,\ y\,(t)\bigr) = \sum_{i=n+1}^{U} K_i\,(\alpha_i,\ \beta_i) =$$

$$= \sum_{i=n+1}^{U-1} \int\limits_{\left[\tau^{(i-1)};\ \tau^{(i)}\right)} f\,(\alpha_i,\ \beta_i,\ t)\,d\mu\,(t) + \int\limits_{\left[\tau^{(U-1)};\ \tau^{(U)}\right]} f\,(\alpha_U,\ \beta_U,\ t)\,d\mu\,(t) \tag{111}$$

and

$$H\bigl(x\,(t)\,,\ y\,(t)\bigr) = \sum_{i=n+1}^{U} H_i\,(\alpha_i,\ \beta_i) =$$

$$= \sum_{i=n+1}^{U-1} \int\limits_{\left[\tau^{(i-1)};\ \tau^{(i)}\right)} g\,(\alpha_i,\ \beta_i,\ t)\,d\mu\,(t) + \int\limits_{\left[\tau^{(U-1)};\ \tau^{(U)}\right]} g\,(\alpha_U,\ \beta_U,\ t)\,d\mu\,(t). \tag{112}$$

So, inequalities

$$\sum_{i=n+1}^{U} \int\limits_{[b_{\min};\ b_{\max}]} r_i^*\,(\beta_i) \left(\int\limits_{[a_{\min};\ a_{\max}]} p_i\,(\alpha_i)\,K_i\,(\alpha_i,\ \beta_i)\,d\mu\,(\alpha_i) \right) d\mu\,(\beta_i) =$$

$$= \sum_{i=n+1}^{U-1} \int\limits_{[b_{\min};\ b_{\max}]} r_i^*\,(\beta_i)$$

$$\left(\int\limits_{[a_{\min};\ a_{\max}]} p_i\,(\alpha_i) \int\limits_{\left[\tau^{(i-1)};\ \tau^{(i)}\right)} f\,(\alpha_i,\ \beta_i,\ t)\,d\mu\,(t) d\mu\,(\alpha_i) \right) d\mu\,(\beta_i) +$$

$$+ \int\limits_{[b_{\min};\ b_{\max}]} r_U^*\,(\beta_U)$$

$$\left(\int\limits_{[a_{\min};\ a_{\max}]} p_U\,(\alpha_U) \int\limits_{\left[\tau^{(U-1)};\ \tau^{(U)}\right]} f\,(\alpha_U,\ \beta_U,\ t)\,d\mu\,(t) d\mu\,(\alpha_U) \right) d\mu\,(\beta_U) \leqslant$$

$$\leqslant \sum_{i=n+1}^{U-1} \int\limits_{[b_{\min};\ b_{\max}]} r_i^*\,(\beta_i)$$

$$\left(\int\limits_{[a_{\min};\, a_{\max}]} p_i^*(\alpha_i) \int\limits_{\left[\tau^{(i-1)};\, \tau^{(i)}\right)} f(\alpha_i,\ \beta_i,\ t)\, d\mu(t) d\mu(\alpha_i) \right) d\mu(\beta_i) +$$

$$+ \int\limits_{[b_{\min};\, b_{\max}]} r_U^*(\beta_U)$$

$$\left(\int\limits_{[a_{\min};\, a_{\max}]} p_U^*(\alpha_U) \int\limits_{\left[\tau^{(U-1)};\, \tau^{(U)}\right]} f(\alpha_U,\ \beta_U,\ t)\, d\mu(t) d\mu(\alpha_U) \right) d\mu(\beta_U) =$$

$$= \sum_{i=n+1}^{U} \int\limits_{[b_{\min};\, b_{\max}]} r_i^*(\beta_i) \left(\int\limits_{[a_{\min};\, a_{\max}]} p_i^*(\alpha_i) K_i(\alpha_i,\ \beta_i)\, d\mu(\alpha_i) \right) d\mu(\beta_i) =$$

$$= \sum_{i=n+1}^{U} v_i^{*(\Theta_*)} = v^{*(\Theta_*)} \ \ \forall\, p_i(\alpha_i) \in P \ \text{ for } \ i = \overline{n+1,\ U} \qquad (113)$$

and

$$\sum_{i=n+1}^{U} \int\limits_{[b_{\min};\, b_{\max}]} r_i(\beta_i) \left(\int\limits_{[a_{\min};\, a_{\max}]} p_i^*(\alpha_i) H_i(\alpha_i,\ \beta_i)\, d\mu(\alpha_i) \right) d\mu(\beta_i) =$$

$$= \sum_{i=n+1}^{U-1} \int\limits_{[b_{\min};\, b_{\max}]} r_i(\beta_i)$$

$$\left(\int\limits_{[a_{\min};\, a_{\max}]} p_i^*(\alpha_i) \int\limits_{\left[\tau^{(i-1)};\, \tau^{(i)}\right)} g(\alpha_i,\ \beta_i,\ t)\, d\mu(t) d\mu(\alpha_i) \right) d\mu(\beta_i) +$$

$$+ \int\limits_{[b_{\min};\, b_{\max}]} r_U(\beta_U)$$

$$\left(\int\limits_{[a_{\min};\, a_{\max}]} p_U^*(\alpha_U) \int\limits_{\left[\tau^{(U-1)};\, \tau^{(U)}\right]} g(\alpha_U,\ \beta_U,\ t)\, d\mu(t) d\mu(\alpha_U) \right) d\mu(\beta_U) \leqslant$$

$$\leqslant \sum_{i=n+1}^{U-1} \int\limits_{[b_{\min};\, b_{\max}]} r_i^*(\beta_i)$$

$$\left(\int\limits_{[a_{\min};\, a_{\max}]} p_i^*(\alpha_i) \int\limits_{\left[\tau^{(i-1)};\, \tau^{(i)}\right)} g(\alpha_i,\ \beta_i,\ t)\, d\mu(t) d\mu(\alpha_i) \right) d\mu(\beta_i) +$$

$$+ \int\limits_{[b_{\min};\, b_{\max}]} r_U^*(\beta_U)$$

$$\left(\int\limits_{[a_{\min};\, a_{\max}]} p_U^*(\alpha_U) \int\limits_{\left[\tau^{(U-1)};\, \tau^{(U)}\right]} g(\alpha_U,\ \beta_U,\ t)\, d\mu(t) d\mu(\alpha_U) \right) d\mu(\beta_U) =$$

$$= \sum_{i=n+1}^{U} \int\limits_{[b_{\min};\, b_{\max}]} r_i^*(\beta_i) \left(\int\limits_{[a_{\min};\, a_{\max}]} p_i^*(\alpha_i)\, H_i(\alpha_i,\ \beta_i)\, d\mu(\alpha_i) \right) d\mu(\beta_i) =$$

$$= \sum_{i=n+1}^{U} z_i^{*(\Theta_*)} = z^{*(\Theta_*)} \ \ \forall\, r_i(\beta_i) \in R \ \text{ for } \ i = \overline{n+1,\ U} \tag{114}$$

hold. Therefore, inequalities (113) and (114) along with using the payoff decomposition by (111) and (112) allow to conclude that the stack of successive equilibria (110) is an equilibrium situation in discrete-time staircase-function 2-person game (107). □

The assertion about the maximum of the players' payoffs sum in an equilibrium stack in game (33) could have been proved in a way similar to that in the proof of Theorem 2. However, this question has far less practical sense compared to that for bimatrix staircase-function games (which always have equilibrium solutions). This is so because discrete-time staircase-function 2-person games are played, generally speaking, within uncountably infinite sets of players' staircase-function strategies (93) and (94), and even the latter may have pretty tricky structure, let alone a subinterval game may have no equilibrium at all.

Similarly to game (80) being a subgame of bimatrix staircase-function game (49), and an inclusion by (92), it is quite obvious that game (107) is a subgame of discrete-time staircase-function 2-person game (33) and

$$\left\{ \left\{ p_i^*(\alpha_i),\ r_i^*(\beta_i) \right\} \right\}_{i=n+1}^{U} \subset \left\{ \left\{ p_i^*(\alpha_i),\ r_i^*(\beta_i) \right\} \right\}_{i=1}^{N}. \tag{115}$$

Theorem 5 being a generalization of Theorem 3 implies that the time-unit shifting does not change the structure of equilibria in a discrete-time staircase-function 2-person game. If an equilibrium solution of ("wider") game (33) exists, the respective equilibrium solution of the ("narrower") subgame can be taken from it.

8. An example of the bimatrix staircase-function game

Consider an example of the bimatrix staircase-function game, in which functions (8) and (11) in integral functionals (7) and (10) are

$$f\big(x(t),\ y(t),\ t\big) = \\ = \sin\left(0.05xt - 0.01yt^2 - \frac{\pi}{4}\right) + \cos(0.04xyt)\, e^{1.3\cos(0.01xyt)} \tag{116}$$

and

$$g\big(x(t),\ y(t),\ t\big) = t\sin\left(0.03xyt - \frac{\pi}{5}\right) e^{-2.44\cos\left(0.02xyt+\frac{\pi}{3}\right)}, \tag{117}$$

where the players are forced (somehow) to act within finite subsets of possible values of their pure strategies (45) and (46):

$$A = \left\{a^{(m-1)}\right\}_{m=1}^{M+1} = \left\{a_i^{(m-1)}\right\}_{m=1}^{7} = \{m+1\}_{m=1}^{7} \tag{118}$$

and

$$B = \left\{b^{(q-1)}\right\}_{q=1}^{Q+1} = \left\{b_i^{(q-1)}\right\}_{q=1}^{6} = \{12+2q\}_{q=1}^{6}. \tag{119}$$

The time unit (or the time subinterval length) is 0.1π, i. e. the players may (synchronously, simultaneously) change their pure strategies values only through this time step. The tasks are to solve such bimatrix staircase-function game (80) for time intervals

$$[\tau_1;\ \tau_2] = [0.7\pi;\ 1.3\pi], \tag{120}$$

$$[\tau_1;\ \tau_2] = [1.8\pi;\ 2.5\pi], \tag{121}$$

$$[\tau_1;\ \tau_2] = [2.8\pi;\ 3.6\pi], \tag{122}$$

where

$$X(\Theta_*,\ A) = \left\{x(t) \in X(\Theta_*):\ x(t) \in \{m+1\}_{m=1}^{7}\right\} \subset \\ \subset X(\Theta_*) \subset X([\tau_1;\ \tau_2]) \tag{123}$$

and

$$Y(\Theta_*,\ B) = \left\{y(t) \in Y(\Theta_*):\ y(t) \in \{12+2q\}_{q=1}^{6}\right\} \subset \\ \subset Y(\Theta_*) \subset Y([\tau_1;\ \tau_2]) \tag{124}$$

by

$$\tau^{(i)} - \tau^{(i-1)} = 0.1\pi \ \text{ for } \ i = \overline{n+1,\ U} \tag{125}$$

in time-interval breaking (78).

According with Theorem 3, it is sufficient to find an equilibrium stack of the bimatrix staircase-function game with (116) — (119) played during time interval

$$[t_1;\ t_2] = [0.7\pi;\ 3.6\pi] \tag{126}$$

where the respective payoff functionals

$$K\big(x(t),\ y(t)\big) = \int\limits_{[0.7\pi;\ 3.6\pi]} f\big(x(t),\ y(t),\ t\big)d\mu(t) \tag{127}$$

and

$$H\big(x(t),\ y(t)\big) = \int\limits_{[0.7\pi;\ 3.6\pi]} g\big(x(t),\ y(t),\ t\big)d\mu(t), \tag{128}$$

due to there are 29 time units in (126), are transformed into 29 payoff 7×6 matrices of the first player and 29 payoff 7×6 matrices of the second player. So, the "wider" bimatrix staircase-function game is the succession of 29 bimatrix games

$$\left\langle \left\{ \left\{ a_i^{(m-1)} \right\}_{m=1}^{7},\ \left\{ b_i^{(q-1)} \right\}_{q=1}^{6} \right\},\ \{\mathbf{K}_i,\ \mathbf{H}_i\} \right\rangle \text{ for } i = \overline{1,\ 29} \tag{129}$$

with the first player's payoff matrices

$$\mathbf{K}_i = [k_{imq}]_{7\times 6} \tag{130}$$

whose elements are

$$k_{imq} = \int\limits_{\left[\tau^{(i-1)};\ \tau^{(i)}\right)} f\left(a_i^{(m-1)},\ b_i^{(q-1)},\ t\right) d\mu(t) =$$

$$= \int\limits_{[0.6\pi+0.1\pi i;\ 0.7\pi+0.1\pi i)} f(m+1,\ 12+2q,\ t)\, d\mu(t) =$$

$$= \int\limits_{[0.6\pi+0.1\pi i;\ 0.7\pi+0.1\pi i)} \left[\sin\left(0.05\cdot(m+1)\,t - 0.01\cdot(12+2q)\,t^2 - \frac{\pi}{4}\right) + \right.$$

$$\left. + \cos\big(0.04\cdot(m+1)\cdot(12+2q)\,t\big)e^{1.3\cos\left(0.01\cdot(m+1)\cdot(12+2q)t\right)}\right] d\mu(t)$$

$$\text{for } i = \overline{1,\ 28} \tag{131}$$

and

$$k_{29mq} = \int\limits_{\left[\tau^{(28)};\ \tau^{(29)}\right]} f\left(a_{29}^{(m-1)},\ b_{29}^{(q-1)},\ t\right) d\mu(t) =$$

$$= \int\limits_{[3.5\pi;\ 3.6\pi]} f(m+1,\ 12+2q,\ t)\, d\mu(t) =$$

$$= \int\limits_{[3.5\pi;\ 3.6\pi]} \left[\sin\left(0.05 \cdot (m+1)\,t - 0.01 \cdot (12+2q)\,t^2 - \frac{\pi}{4}\right) + \right.$$

$$\left. + \cos\bigl(0.04 \cdot (m+1) \cdot (12+2q)\,t\bigr) e^{1.3\cos\left(0.01 \cdot (m+1) \cdot (12+2q)t\right)}\right] d\mu\,(t)\,, \qquad (132)$$

and with the second player's payoff matrices

$$\mathbf{H}_i = [h_{imq}]_{7\times 6} \qquad (133)$$

whose elements are

$$h_{imq} = \int\limits_{\left[\tau^{(i-1)};\ \tau^{(i)}\right)} g\left(a_i^{(m-1)},\ b_i^{(q-1)},\ t\right) d\mu\,(t) =$$

$$= \int\limits_{[0.6\pi+0.1\pi i;\ 0.7\pi+0.1\pi i)} g\,(m+1,\ 12+2q,\ t)\, d\mu\,(t) =$$

$$= \int\limits_{[0.6\pi+0.1\pi i;\ 0.7\pi+0.1\pi i)} t\sin\left(0.03 \cdot (m+1) \cdot (12+2q)\,t - \frac{\pi}{5}\right)$$

$$e^{-2.44\cos\left(0.02 \cdot (m+1) \cdot (12+2q)t + \frac{\pi}{3}\right)} d\mu\,(t) \quad \text{for } i = \overline{1,\ 28} \quad (134)$$

and

$$h_{29mq} = \int\limits_{\left[\tau^{(28)};\ \tau^{(29)}\right]} g\left(a_{29}^{(m-1)},\ b_{29}^{(q-1)},\ t\right) d\mu\,(t) =$$

$$= \int\limits_{[3.5\pi;\ 3.6\pi]} g\,(m+1,\ 12+2q,\ t)\, d\mu\,(t) =$$

$$= \int\limits_{[3.5\pi;\ 3.6\pi]} t\sin\left(0.03 \cdot (m+1) \cdot (12+2q)\,t - \frac{\pi}{5}\right)$$

$$e^{-2.44\cos\left(0.02 \cdot (m+1) \cdot (12+2q)t + \frac{\pi}{3}\right)} d\mu\,(t)\,. \quad (135)$$

In the "wider" bimatrix staircase-function game, each of the players is allowed to change its pure strategy value only at time points

$$\left\{\tau^{(i)}\right\}_{i=1}^{28} = \{0.7\pi + 0.1\pi i\}_{i=1}^{28}\,.$$

Payoff matrix (130) on each subinterval of set

$$\left\{\{[0.6\pi + 0.1\pi i;\ 0.7\pi + 0.1\pi i)\}_{i=1}^{28},\ [3.5\pi;\ 3.6\pi]\right\} \qquad (136)$$

is shown in Figure 4 as a meshed surface, where a close-to-chaotic payoff distribution can be seen. Payoff matrix (133) on each subinterval of set (136) is shown in Figure 5 as a meshed surface also, where a close-to-chaotic payoff distribution is seen

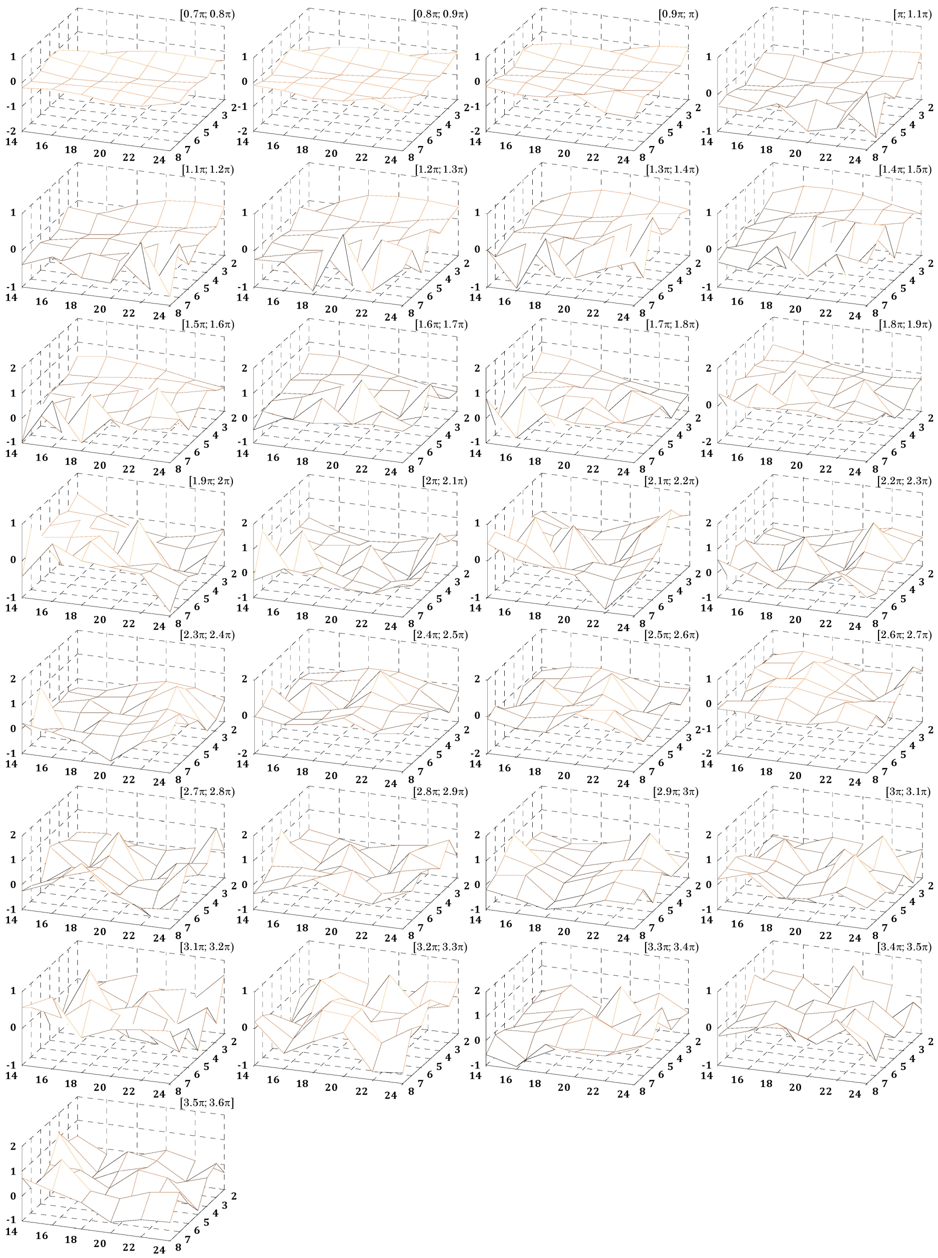

Figure 4: First player's payoffs in matrix (130) as a meshed surface on the 29 subintervals of set (136)

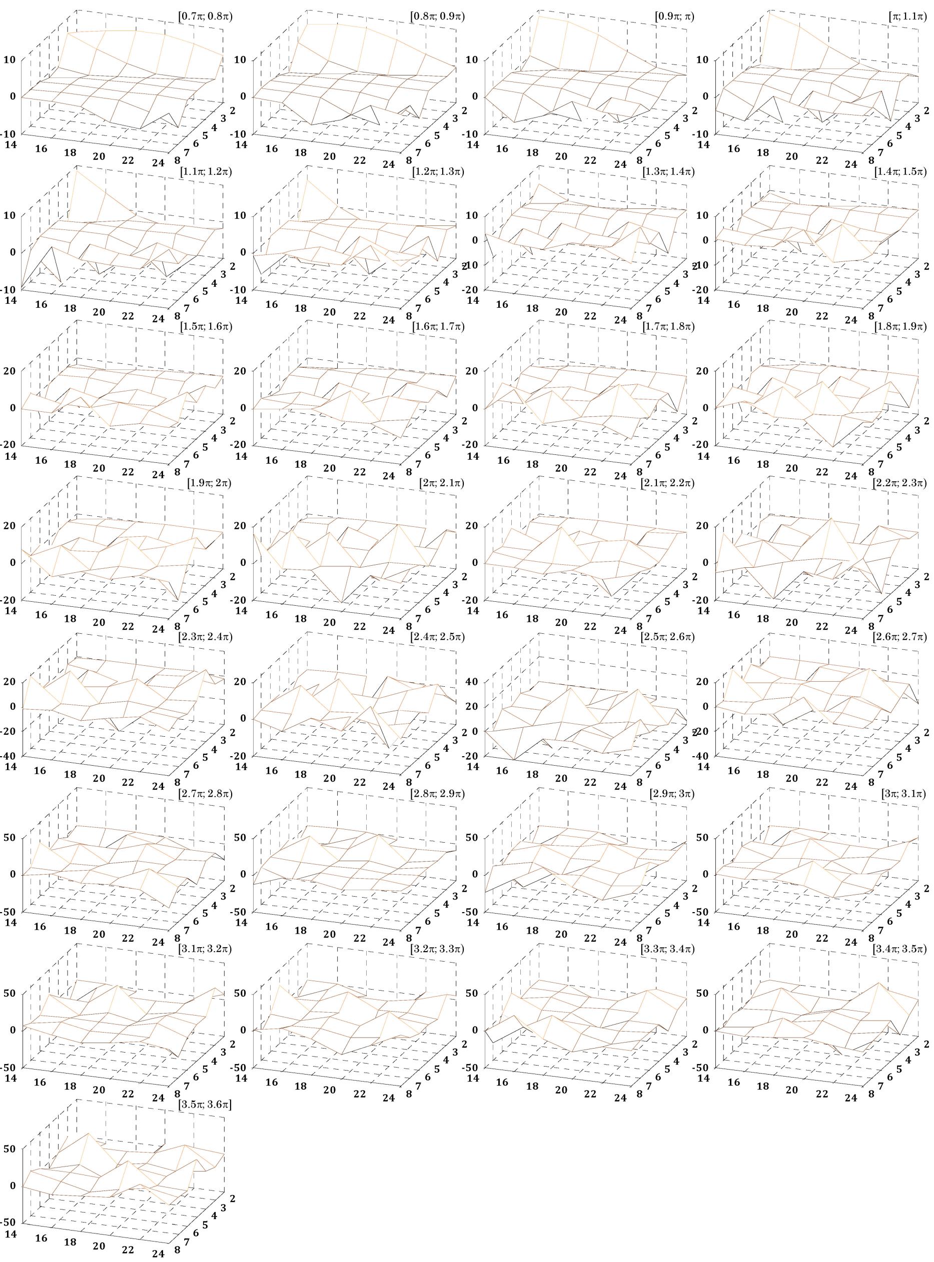

Figure 5: Second player's payoffs in matrix (133) as a meshed surface on the 29 subintervals of set (136)

as well (although some meshes on neighboring subintervals bear some resemblance). A distinctive feature here is that the payoff value scale of the second player is much wider than that of the first player. Whereas the first player's payoff varies between approximately -1.1144 and 1.3652, the second player's payoff varies between approximately -31.3741 and 30.4348, that means a potentially significant imbalance when the criterion of the payoff sum maximum is applied to select the best equilibrium.

The 7×6 bimatrix games (129) with (130) — (135) are solved in pure and mixed strategies, and there are multiple equilibrium situations on some time units. So, the best equilibrium situation on such time units is selected by the criterion of maximizing the players' payoffs sum. The stack of the 29 first player's equilibrium strategies in each of those 29 7×6 bimatrix games is shown in Figure 6, where the solid line corresponds to a pure strategy equilibrium and the dotted lines correspond to nonzero-probability pure strategies in a mixed strategy equilibrium. Similarly, the stack of the 29 second player's equilibrium strategies is shown in Figure 7. Thus, the solution to the "wider" game is the equilibrium situation formed subinterval-wise from the stacks in Figure 6 and Figure 7. Owing to Theorem 3, the equilibrium solutions for time intervals (120) — (122) are directly taken from the "wider" game equilibrium stack. In the solution for time interval (120), pure strategies $a^{(2)} = 4$ and $a^{(4)} = 6$ are not used by the first player, whereas pure strategy $b^{(1)} = 16$ is not used by the second player (Figure 8). In the solution for time interval (121), every player uses all one's pure strategies, only in mixed strategies (Figure 9). In the solution for time interval (122), pure strategy $a^{(4)} = 6$ is not used by the first player, whereas the second player uses all one's pure strategies (Figure 10) — either in mixed strategies or in pure strategies during $[3\pi;\ 3.4\pi)$. It is worth noting that there are no completely mixed strategies in the 29 time-unit equilibrium situations.

In the "wider" game equilibrium situation formed subinterval-wise from the stacks in Figure 6 and Figure 7, the players' payoffs are

$$v^* = \sum_{i=1}^{29} v_i^* \approx 7.4123 \tag{137}$$

and

$$z^* = \sum_{i=1}^{29} z_i^* \approx 99.8691, \tag{138}$$

provided by the criterion of maximizing the players' payoffs sum. However, it is worth noting that the presented game solution strongly depends on the criterion of selecting a single equilibrium situation (on each time unit). Inasmuch as the payoff ranges of the players differ severely, the applied above criterion may be unacceptable for the first player whose contribution to the sum is rather (insignificantly, on some time units) small. Thus, the same criterion can be used but only with payoff normalizations

$$\tilde{v}_{ij}^* = \frac{v_{ij}^* - \min\limits_{j=\overline{1,\ J_i}} v_{ij}^*}{\max\limits_{j=\overline{1,\ J_i}} v_{ij}^* - \min\limits_{j=\overline{1,\ J_i}} v_{ij}^*} \tag{139}$$

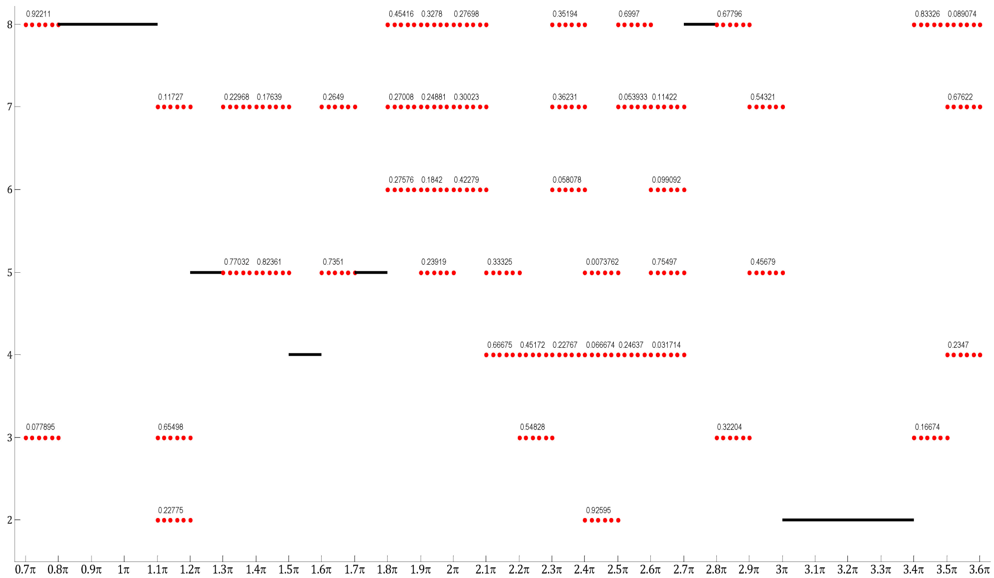

Figure 6: The stack of the 29 strategies as the first player's best equilibrium strategy in the best equilibrium situation in the "wider" game by (116) — (119) and (126) — (135)

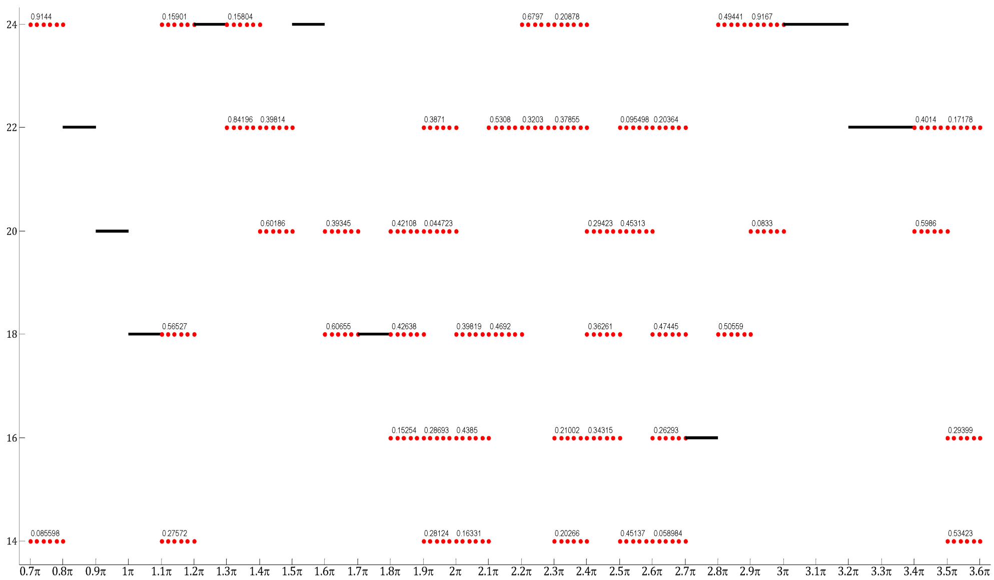

Figure 7: The stack of the 29 strategies as the second player's best equilibrium strategy in the best equilibrium situation in the "wider" game by (116) — (119) and (126) — (135)

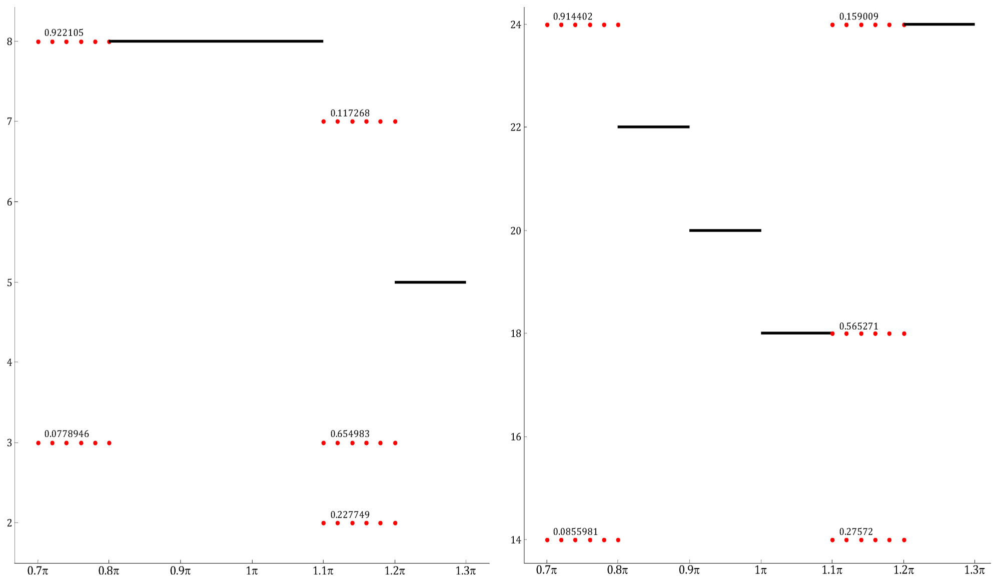

Figure 8: The stacks of the six best equilibrium strategies of the first (left) and second (right) players in the game played during time interval (120)

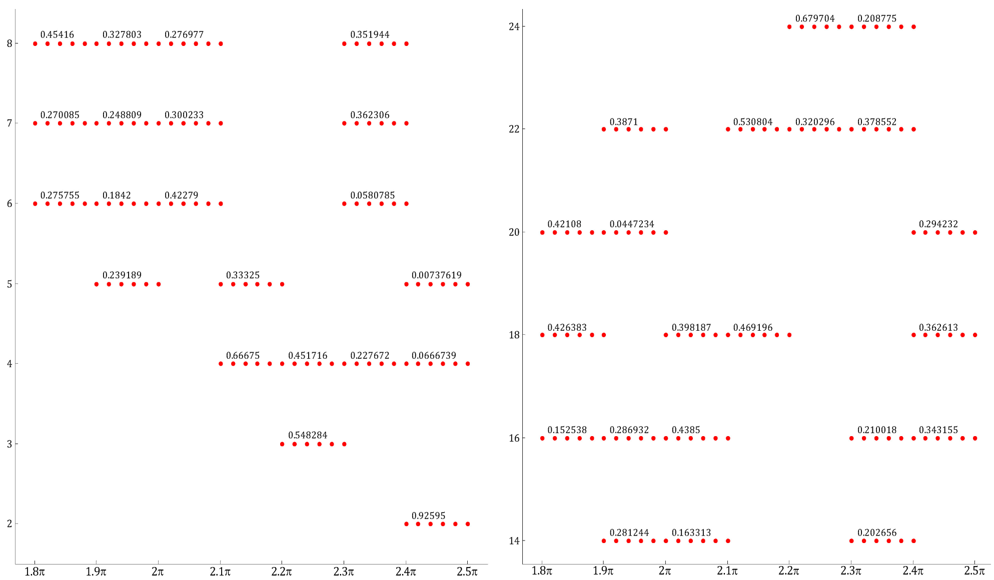

Figure 9: The stacks of the seven best equilibrium strategies of the first (left) and second (right) players in the game played during time interval (121)

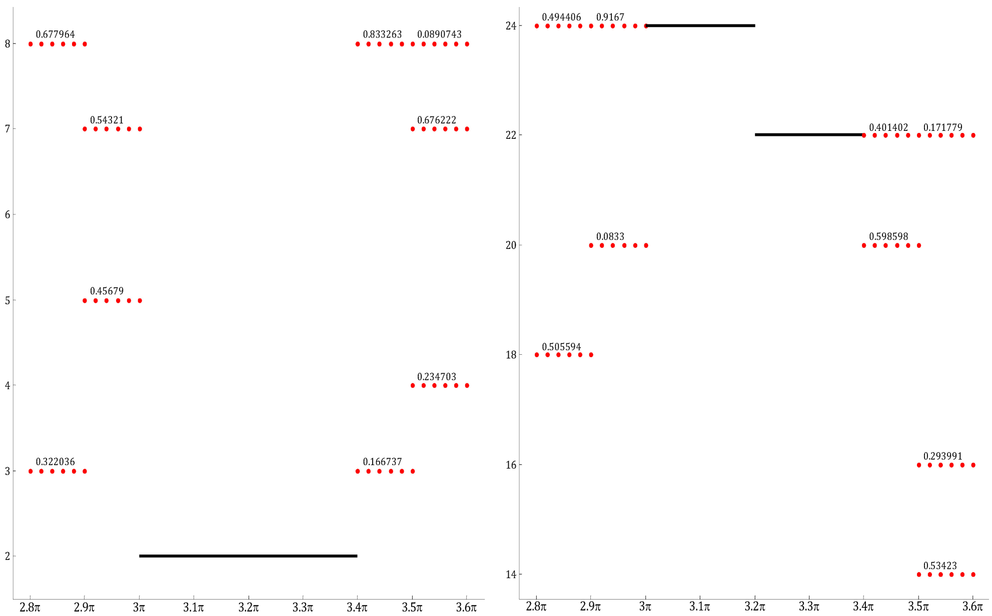

Figure 10: The stacks of the eight best equilibrium strategies of the first (left) and second (right) players in the game played during time interval (122)

and

$$\tilde{z}^*_{ij} = \frac{z^*_{ij} - \min\limits_{j=\overline{1,\ J_i}} z^*_{ij}}{\max\limits_{j=\overline{1,\ J_i}} z^*_{ij} - \min\limits_{j=\overline{1,\ J_i}} z^*_{ij}}, \tag{140}$$

where v^*_{ij} and z^*_{ij} are the first and second players payoffs in a j-th equilibrium situation on time unit i, on which there are J_i equilibria altogether. Then, similarly to (66), an equilibrium situation $\{\mathbf{P}^{**}_i,\ \mathbf{R}^{**}_i\}$ is selected such in which

$$\max_{j=\overline{1,\ J_i}} \left\{\tilde{v}^*_{ij} + \tilde{z}^*_{ij}\right\} \tag{141}$$

is reached. By using the criterion with (139) — (141), the equilibrium solution of the “wider” bimatrix staircase-function game changes (see the first player’s equilibrium stack in Figure 11 and the second player’s equilibrium stack in Figure 12, where the subintervals with the changes are segregated): the players’ equilibria on subintervals

$$[1.4\pi;\ 1.5\pi)\,, \tag{142}$$

$$[2.1\pi;\ 2.2\pi)\,, \tag{143}$$

$$[2.6\pi;\ 2.7\pi) \tag{144}$$

are different from the equilibria on subintervals (142) — (144) in both Figure 6 and Figure 7. The first player now mixes pure strategies $a^{(0)} = 2$ and $a^{(4)} = 6$ (instead

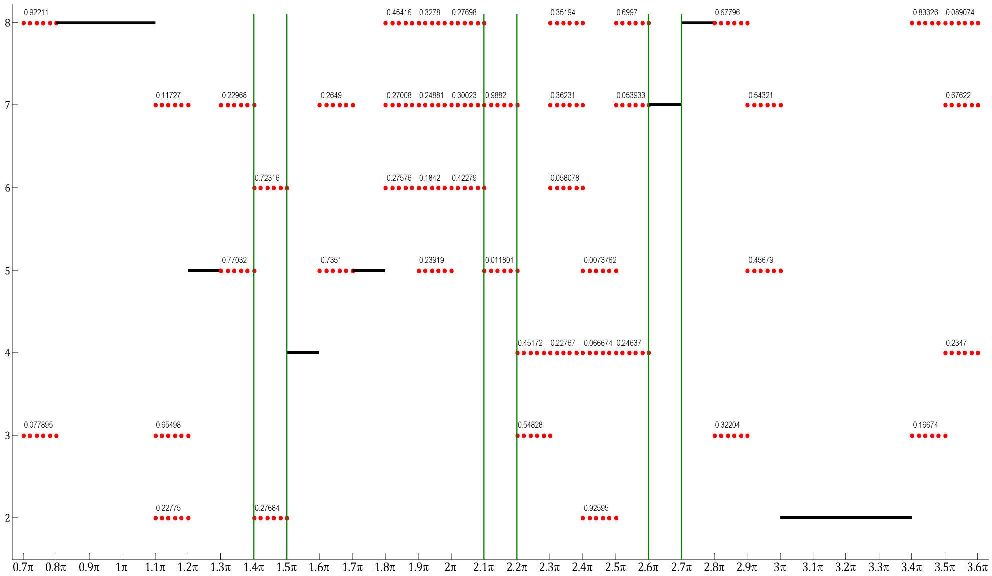

Figure 11: The best equilibrium situation of the first player in the "wider" game by (116) — (119) and (126) — (135) solved by selecting a single equilibrium situation on the time unit with (139) — (141)

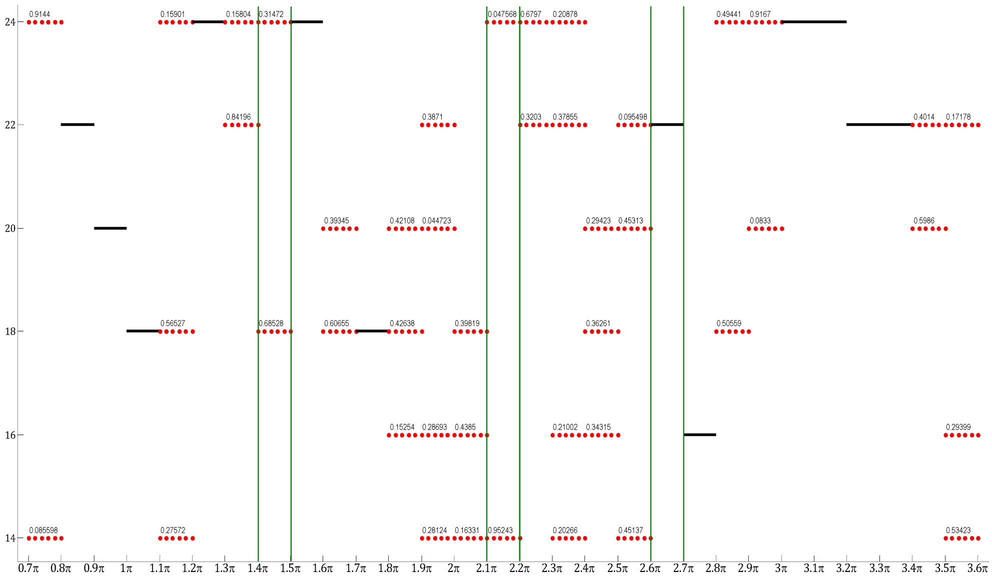

Figure 12: The best equilibrium situation of the second player in the "wider" game by (116) — (119) and (126) — (135) solved by selecting a single equilibrium situation on the time unit with (139) — (141)

of mixing $a^{(3)} = 5$ and $a^{(5)} = 7$ in Figure 6) on subinterval (142), and the second player now mixes pure strategies $b^{(2)} = 18$ and $b^{(5)} = 24$ (instead of mixing $b^{(3)} = 20$ and $b^{(4)} = 22$ in Figure 7) on subinterval (142). The equilibrium strategy support cardinality, which is

$$|\operatorname{supp} \mathbf{P}_8^*| = |\operatorname{supp} \mathbf{R}_8^*| = 2,$$

does not change. Nor does it change on subinterval (143):

$$|\operatorname{supp} \mathbf{P}_{15}^*| = |\operatorname{supp} \mathbf{R}_{15}^*| = 2,$$

where the first player now mixes pure strategy $a^{(3)} = 5$ with $a^{(5)} = 7$ (instead of mixing $a^{(3)} = 5$ with $a^{(2)} = 4$ in Figure 6), and the second player now mixes pure strategies $b^{(0)} = 14$ and $b^{(5)} = 24$ (instead of mixing $b^{(2)} = 18$ and $b^{(4)} = 22$ in Figure 7). The most radical change is on subinterval (144), whereon each of the players now does not mix one's four pure strategies ($a^{(2)} = 4$, $a^{(3)} = 5$, $a^{(4)} = 6$, $a^{(5)} = 7$ in Figure 6 and $b^{(0)} = 14$, $b^{(1)} = 16$, $b^{(2)} = 18$, $b^{(4)} = 22$ in Figure 7), but uses instead a single pure strategy: the first player just uses $a^{(5)} = 7$ and the second player uses $b^{(4)} = 22$. Now, in the "wider" game equilibrium situation formed subinterval-wise from the stacks in Figure 11 and Figure 12, the players' payoffs are

$$v^* = \sum_{i=1}^{29} v_i^* \approx 8.0337 \tag{145}$$

and

$$z^* = \sum_{i=1}^{29} z_i^* \approx 96.5492, \tag{146}$$

provided by the criterion of maximizing the players' payoffs sum as selecting a single equilibrium situation on the time unit with (139) — (141). The first player's payoff (145) is 8.3839 % greater than that (137), whereas the second player's payoff (146) is just 3.3242 % less than that (138). This is an example of that a proper selection of the single equilibrium criterion, e. g. using payoff normalizations like (139), (140), when the players' payoff ranges differ, can balance the player's eventual payoffs (making their distribution more fair). Obviously, equilibria on some time units may depend on the criterion (that is followed by the respective changes in the players' equilibrium stacks).

9. Discussion

In the sense of practical applicability, the presented method is a significant contribution to the 2-person game theory and operations research. It allows solving 2-person games played with staircase-function strategies in a far simpler manner just by considering a succession of time-unit subgames. In the case of a bimatrix staircase-function game, being "wider" one, its equilibrium situation is formed by solving and stacking equilibria of successive smaller-sized bimatrix games. Then, owing to Theorem 3, the respective equilibrium solution of any "narrower" subgame can be taken from the "wider" game equilibrium. The computational efficiency is only defined by and limited to the efficiency of finding equilibrium situations in an ordinary (time-unit) bimatrix game whose size is commonly not that large. Without considering the succession of time-unit bimatrix games, any straightforward approach to finding equilibrium situations in a bimatrix staircase-function game is intractable.

A special attention is paid to time variable t explicitly included into functions (8) and (11) to be integrated. The explicitness means that, as time goes by (and the players develop their actions), something is going on or changes within the process modeled by the staircase-function game. If, in a discrete-time staircase-function 2-person game, time t is not explicitly included into functions (8) and (11), then

$$K_i(\alpha_i,\ \beta_i) = \int\limits_{\left[\tau^{(i-1)};\ \tau^{(i)}\right)} f(\alpha_i,\ \beta_i)\, d\mu(t) = = f(\alpha_i,\ \beta_i) \cdot \left(\tau^{(i)} - \tau^{(i-1)}\right) \ \ \forall\, i = \overline{1,\ N-1} \tag{147}$$

and

$$K_N(\alpha_N,\ \beta_N) = \int\limits_{\left[\tau^{(N-1)};\ \tau^{(N)}\right]} f(\alpha_N,\ \beta_N)\, d\mu(t) = = f(\alpha_N,\ \beta_N) \cdot \left(\tau^{(N)} - \tau^{(N-1)}\right) \tag{148}$$

instead of (26) and (27), and

$$H_i(\alpha_i,\ \beta_i) = \int\limits_{\left[\tau^{(i-1)};\ \tau^{(i)}\right)} g(\alpha_i,\ \beta_i)\, d\mu(t) = = g(\alpha_i,\ \beta_i) \cdot \left(\tau^{(i)} - \tau^{(i-1)}\right) \ \ \forall\, i = \overline{1,\ N-1} \tag{149}$$

and

$$H_N(\alpha_N,\ \beta_N) = \int\limits_{\left[\tau^{(N-1)};\ \tau^{(N)}\right]} g(\alpha_N,\ \beta_N)\, d\mu(t) = = g(\alpha_N,\ \beta_N) \cdot \left(\tau^{(N)} - \tau^{(N-1)}\right) \tag{150}$$

instead of (28) and (29). Equalities (147) — (150) mean that the payoff value depends only on the length of the time unit. That is, the player's payoff then is equal to the subinterval length multiplied by the respective value of the function under the integral. If the length does not change in the case of bimatrix staircase-function game (49), then the time-unit bimatrix game does not change. If the length does not change in the case of discrete-time staircase-function 2-person game (33), the time-unit (ordinary) 2-person game defined on rectangle (32) does not change. Then the solution (of any type) to the initial (finite or uncountably infinite) discrete-time staircase-function 2-person game is determined just by the solution of a one time-unit game, and this solution will not change as the time units go by. Such a triviality of the equal-length-subinterval solution is explained by a standstill of the players' strategies. Consequently, the scientific significance of this trivial case is low — this is why it is not considered.

The scientific significance of the discrete-time staircase-function 2-person game and the methods of finding an equilibrium in it (provided by Theorems 2 and 3, and, under the supposition of that all the time-unit equilibria exist, by Theorems 4 and 5) is high. Owing to Theorems 2 and 3, such games, if finite, are very simple models to describe struggling for optimizing the distribution of some limited resources between two sides. Unlike ordinary bimatrix games, which model only static processes of the struggle, discrete-time staircase-function 2-person games allow considering discrete-time dynamics of the struggling processes. Such a simplification is similar to that when, e. g., the fuzzy logic facilitates the control of a complicated system without knowledge of its exact mathematical description.

10. Conclusion

Because of an intractably gigantic size, it is impracticable to solve 2-person games played in staircase-function finite spaces by directly rendering them to bimatrix games, where the solution is of the equilibrium type. Moreover, the time interval on which the discrete-time 2-person game is defined can vary by the number of time subintervals (time units), so a tractable and efficient method of finding an equilibrium in a 2-person game played in staircase-function finite spaces is to solve a succession of time-unit bimatrix games, whereupon their equilibria are stacked into pure-mixed-strategy equilibria. In the case of multiple equilibria on some time units, the criterion of the players' payoffs sum maximum is applied to select the best equilibrium. Owing to Theorems 2 and 3, the equilibrium of the initial finite game can be obtained by stacking the best equilibria of the smaller-sized bimatrix games, whichever the time interval is. If the game is uncountably infinite, i. e. a set of pure strategy possible values is uncountably infinite, and all time-unit equilibria exist, such a stack is possible as well owing to Theorems 4 and 5. So, the equilibrium of the initial discrete-time staircase-function 2-person game can be obtained by stacking the equilibria of the (ordinary) 2-person games defined on a rectangle, whichever the time interval is.

Solving games played in staircase-function finite spaces with possible time-unit shifting (when the initial time interval is narrowed by an integer number of time units) should be studied also for the case of three players. Then the presented assertions and conclusions are to be re-written for trimatrix games. A distinct peculiarity is that the equilibria multiplicity problem in trimatrix games is even trickier than that in bimatrix games. Moreover, the criterion of selecting a single equilibrium situation on each time unit in the case of a trimatrix staircase-function game becomes more disputable, especially when at least two players' payoff ranges differ significantly.

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