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On the zeros of an analytic function

V.K. Jain

ABSTRACT: Kuniyeda, Montel and Toya had shown that the polynomial $p(z) = \sum_{k=0}^{n} a_k z^k; a_0 \neq 0$, of degree n, does not vanish in

$$|z| \leq \{1 + (\sum_{j=1}^{n} |a_j/a_0|^p)^{q/p}\}^{-1/q},$$

where $p > 1$, $q > 1$, $(1/p) + (1/q) = 1$ and we had proved that $p(z)$ does not vanish in $|z| \leq \alpha^{1/q}$, where

$$\alpha = \text{unique root in } \ (0,1) \ \text{ of } \quad D_n x^3 - D_n S x^2 + (1 + D_n S)x - 1 = 0,$$

$$D_n = (\sum_{j=1}^{n} |a_j/a_0|^p)^{q/p},$$

$$S = (|a_1| + |a_2|)^q (|a_1|^p + |a_2|^p)^{-(q-1)},$$

a refinement of Kuniyeda et al.'s result under the assumption

$$D_n < (2 - S)/(S - 1).$$

Now we have obtained a generalization of our old result and proved that the function

$$f(z) = \sum_{k=0}^{\infty} a_k z^k, (\not\equiv \text{aconstant}); a_0 \neq 0,$$

analytic in $|z| \leq 1$, does not vanish in $|z| < \alpha_m^{1/q}$, where

$$\alpha_m = \text{unique root in } (0,1) \text{ of } \ D x^{m+1} - D M_m x^2 + (1 + D M_m)x - 1 = 0,$$

$$D = (\sum_{k=1}^{\infty} |a_k/a_0|^p)^{q/p},$$

$$M_m = (\sum_{k=1}^{m} |a_k|)^q (\sum_{k=1}^{m} |a_k|^p)^{-q/p},$$

$m =$ any positive integer with the characteristic that there exists a positive integer $k(\leq m)$ with $a_k \neq 0$.

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1 Introduction and statement of results

Let

$$P(z) = b_0 + b_1 z + \ldots + b_n z^n$$

be a polynomial of degree n. Then according to a classical result of Kuniyeda, Montel and Toya [3, p. 124] on the location of zeros of a polynomial we have

Theorem A. *All the zeros of the polynomial $P(z)$ lie in*

$$|z| < \{1 + (\sum_{j=0}^{n-1} |b_j/b_n|^p)^{q/p}\}^{1/q},$$

where

$$p > 1, \quad q > 1, \quad (1/p) + (1/q) = 1. \tag{1.1}$$

On applying Theorem A to the polynomial $z^n p(1/z)$, we have the following equivalent formulation of Theorem A.

Theorem B. *The polynomial*

$$p(z) = a_0 + a_1 z + a_2 z^2 + \ldots + a_n z^n; a_0 \neq 0, \tag{1.2}$$

of degree n does not vanish in

$$|z| \leq (1 + D_n)^{-1/q}, \tag{1.3}$$

where p, q are given in (1.1) and

$$D_n = (\sum_{j=1}^{n} |a_j/a_0|^p)^{q/p}. \tag{1.4}$$

We [2] had obtained

Theorem C. *All the zeros of $P(z)$ lie in*

$$|z| < \chi^{1/q},$$

where χ is the unique root of the equation

$$x^3 - (1 + LM)x^2 + LMx - L = 0,$$

in $(1, \infty)$,

$$\begin{aligned} L &= (\sum_{j=0}^{n-1} |b_j/b_n|^p)^{q/p}, \\ M &= (|b_{n-1}| + |b_{n-2}|)^q (|b_{n-1}|^p + |b_{n-2}|^p)^{-(q-1)}. \end{aligned}$$

Theorem C is a refinement of Theorem A, under the assumption

$$L < (2 - M)/(M - 1).$$

The equivalent formulation of Theorem C, (similar to the formulation of Theorem B from Theorem A) is

Theorem D. *The polynomial*

$$p(z) = a_0 + a_1 z + \ldots + a_n z^n; a_0 \neq 0,$$

of degree n *does not vanish in*

$$|z| \leq \alpha^{1/q},$$

where α *is the unique root of the equation*

$$D_n x^3 - D_n S x^2 + (1 + D_n S)x - 1 = 0,$$

in $(0, 1)$*,*

$$S = (|a_1| + |a_2|)^q (|a_1|^p + |a_2|^p)^{-(q-1)},$$

and D_n *is as in Theorem B.*

Theorem D is a refinememnt of Theorem B, under the assumption

$$D_n < (2 - S)/(S - 1).$$

In this paper we have obtained a generalization of Theorem D for the functions, analytic in $|z| \leq 1$. More precisely we have proved

Theorem 1. *Let*

$$f(z) = \sum_{k=0}^{\infty} a_k z^k, (\not\equiv \text{aconstant}); a_0 \neq 0, \tag{1.5}$$

be analytic in $|z| \leq 1$. *Then* $f(z)$ *does not vanish in*

$$|z| < \alpha_m^{1/q}, \tag{1.6}$$

where

$$\begin{aligned}
q &> 1, \quad p > 1, \quad (1/p) + (1/q) = 1, \\
m &= \text{any positive integer with the characteristic that} \qquad (1.7) \\
&\quad\ \text{there exists a positive integer } k(\leq m) \text{ with } a_k \neq 0, \\
\alpha_m &= \text{unique root in } (0,1), \text{ of} \\
&\quad\ \{g(x) \equiv\},\ Dx^{m+1} - DM_m x^2 + (1 + DM_m)x - 1 = 0, \qquad (1.8) \\
D &= (\sum_{k=1}^{\infty} |a_k/a_0|^p)^{q/p}, (> 0, \text{by}(1.5)), \qquad (1.9) \\
M_m &= (\sum_{k=1}^{m} |a_k|)^q (\sum_{k=1}^{m} |a_k|^p)^{-q/p}, (> 0, \text{by}(1.7)). \qquad (1.10)
\end{aligned}$$

From Theorem 1 we easily get

***Corollary* 1.** *Under the same hypothesis as in Theorem 1, $f(z)$ does not vanish in*

$$|z| < \sup_{m \geq M, q>1} {\alpha_m}^{1/q},$$

where

$$M = \textit{least positive integer } k \quad \textit{such that } a_k \neq 0.$$

2 Lemmas

For the proof of the theorem, we require the following lemmas.

***Lemma* 1.** *Let*

$$\begin{aligned} &\alpha_j > 0, \quad \beta_j > 0, \text{ for } j = 1, 2, \ldots, n, \\ &q > 1, \quad p > 1, \quad (1/p) + (1/q) = 1, \\ &1 \leq m < n. \end{aligned}$$

Then

$$\sum_{j=1}^{n} \alpha_j \beta_j \leq ((\sum_{j=1}^{n} \beta_j^p)^{1/p} (\sum_{j=1}^{m} \beta_j^p)^{-1/p}) \left\{ (\sum_{j=1}^{m} \alpha_j \beta_j)^q + ((\sum_{j=1}^{m} \beta_j^p)^{q-1})(\sum_{j=m+1}^{n} \alpha_j^q) \right\}^{1/q}. \tag{2.1}$$

This lemma is due to Beckenbach [1].
From Lemma 1 we easily obtain

***Lemma* 2.** *Inequality (2.1) is true even if*

$$\begin{aligned} \alpha_j &\geq 0, \qquad j = 1, 2, \ldots, n, \\ \beta_j &\geq 0, \qquad j = 1, 2, \ldots, n, \end{aligned}$$

with

$$\beta_j \neq 0, \mathrm{for at least one} j, 1 \leq j \leq m.$$

***Lemma* 3.** *The equation*

$$Dx^{m+1} - DM_m x^2 + (1 + DM_m)x - 1 = 0 \tag{2.2}$$

has a unique root α_m in $(0, 1)$ where m, D and M_m are as in Theorem 1.

Proof of Lemma 3. We firstly assume that

$$m > 1.$$

Now we consider the transformation

$$x = 1/t$$

in equation (2.2), thereby giving the transformed equation

$$t^{m+1} - (1 + DM_m)t^m + DM_m t^{m-1} - D = 0, \tag{2.3}$$

and then the transformation

$$t = 1 + y$$

in (2.3), thereby giving the transformed equation

$$(1 + y)^{m+1} - (1 + DM_m)(1 + y)^m + DM_m(1 + y)^{m-1} - D = 0, \tag{2.4}$$

i.e.

$$\begin{aligned} y^{m+1} &+ y^m((m/1) - DM_m) + ((m-1)/1!)((m/2) - DM_m)y^{m-1} \\ &+ ((m-1)(m-2)/2!)((m/3) - DM_m)y^{m-2} + \dots \\ &+ ((m-1)(m-2)\dots(m-j+1)/(j-1)!)((m/j) - DM_m)y^{m+1-j} + \dots \\ &+ ((m-1)(m-2)\dots(m-m+1)/(m-1)!)((m/m) - DM_m)y - D \\ =&0. \end{aligned} \tag{2.5}$$

By using Déscarte's rule of signs we can say that equation (2.5) (i.e. equation (2.4)) will have a unique positive root and accordingly the equation (2.3) will have a unique root in $(1, \infty)$. Hence the equation (2.2) will have a unique root α_m, (say), in $(0, 1)$, thereby proving Lemma 3 for the possibility under consideration.

For the possibility

$$m = 1,$$

the transformed equation, similar to equation (2.5), (i.e. equation (2.4)), is

$$y^2 + y(1 - DM_m) - D = 0.$$

Now Lemma 3 follows for this possibility, by using arguments similar to those used for proving Lemma 3 for the possibility

$$m > 1.$$

This completes the proof of Lemma 3.

3 Proof of Theorem 1

Let

$$f_n(z) = \sum_{k=0}^{n} a_k z^k, n = 1, 2, 3, \dots \quad .$$

Then for $|z|<1$ and $n>m$

$$\begin{aligned}
|f_n(z)| &\geq |a_0| - \sum_{k=1}^{n} |z|^k |a_k|,\\
&\geq |a_0| - \left\{ (\sum_{k=1}^{n} |a_k|^p)^{1/p} (\sum_{k=1}^{m} |a_k|^p)^{-1/p} \right\} \left[(\sum_{k=1}^{m} |z|^k |a_k|)^q \right.\\
&\quad \left. + \left\{ (\sum_{k=1}^{m} |a_k|^p)^{q-1} \right\} (\sum_{k=m+1}^{n} |z|^{kq}) \right]^{1/q}, \text{(by Lemma 2)},\\
&\geq |a_0| - (\sum_{k=1}^{n} |a_k|^p)^{1/p} \left[(\sum_{k=1}^{m} |a_k||z|^k)^q (\sum_{k=1}^{m} |a_k|^p)^{-q/p} \right.\\
&\quad \left. + (\sum_{k=m+1}^{n} |z|^{kq}) \right]^{1/q}, \text{(by 1.1)},\\
&\geq |a_0| - (\sum_{k=1}^{n} |a_k|^p)^{1/p} \left[M_m |z|^q + (\sum_{k=m+1}^{n} |z|^{kq}) \right]^{1/q}, \text{(by 1.10)},
\end{aligned}$$

which, by making

$$n \to \infty,$$

implies that

$$\begin{aligned}
|f(z)| &\geq |a_0| - (\sum_{k=1}^{\infty} |a_k|^p)^{1/p} \left[M_m |z|^q + (\sum_{k=m+1}^{\infty} |z|^{kq}) \right]^{1/q}, (\sum_{k=1}^{\infty} |a_k|^p \text{ will converge}\\
&\qquad \text{as } \sum_{k=1}^{\infty} |a_k| \text{ converges and } (\sum_{k=1}^{n} |a_k|^p)^{1/p} \leq \sum_{k=1}^{n} |a_k|, n = 1, 2, \ldots),\\
&= |a_0| \left[1 - \{ D(M_m |z|^q + (|z|^{(m+1)q}/(1-|z|^q))) \}^{1/q} \right], \text{(by 1.9)},\\
&> 0, \qquad (3.1)
\end{aligned}$$

if

$$D|z|^{(m+1)q} - DM_m |z|^{2q} + (1 + DM_m)|z|^q - 1 < 0. \qquad (3.2)$$

Now as

$$g(0) = -1, \text{(by(1.8))},$$

we can say by Lemma 3, (3.1) and (3.2) that

$$|f(z)| > 0,$$

if

$$|z|^q < \alpha_m,$$

thereby proving Theorem 1.

***Remark* 1.** *Theorem 1 gives better bound than that given by the result, that* $f(z)$ *does not vanish in*

$$|z| < \{1/(1+D)\}^{1/q},$$

obtained by using Hölder's inequality instead of Lemma 2 and following the method of proof of Theorem 1, provided

$$\begin{aligned} m = 1 \quad &\& \quad M_m < m, \\ m \geq 2 \quad &\& \quad M_m \leq 1, \\ m \geq 2, 1 < M_m < m \quad &\text{and} \quad D < D_0, \end{aligned} \tag{3.3}$$

where D_0 *is the unique positive root of the equation*

$$\begin{aligned} (M_m - 1)D^{m-1} &+ (m-1)(M_m - (m/(m-1)))D^{m-2} \\ &+ ((m-1)(m-2)/2)(M_m - (m/(m-2)))D^{m-3} \\ &+ \ldots + (m-1)(M_m - (m/2))D + (M_m - m) \\ &= 0, \quad (m \geq 2\&1 < M_m < m), \end{aligned}$$

as for $m = 1\&M_m < m$

$$g(1/(1+D)) < 0,$$

and for $m \geq 2$

$$g(1/(1+D)) < 0,$$

is equivalent to

$$\begin{aligned} (M_m - 1)D^{m-1} &+ (m-1)(M_m - (m/(m-1)))D^{m-2} \\ &+ ((m-1)(m-2)/2)(M_m - (m/(m-2)))D^{m-3} \\ &+ \ldots + (m-1)(M_m - (m/2))D + (M_m - m) \\ &< 0. \end{aligned}$$

The function

$$f(z) = 1 + z + (z/(2i))^3 + (z/(2i))^4 + (z/(2i))^5 + \ldots$$

satisfies (3.3) with

$$p = q = m = 2$$

and the corresponding $\alpha_m^{1/q}$ *is* .752.

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V.K. Jain
email: vkj@maths.iitkgp.ernet.in
Mathematics Department,
I.I.T. Kharagpur - 721302, INDIA
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