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Influence of boundary conditions on 2D wave propagation in a rectangle

N. K. Ashirbayev, J. N. Ashirbayeva

Submitted by: *Jan Stankiewicz*

ABSTRACT: Work is devoted to generalization of a differential method of spatial characteristics to case of the flat task about distribution of waves in rectangular area of the final sizes with gaps in boundary conditions. On the basis of the developed numerical technique are received the settlement certainly - differential ratios of dynamic tasks in special points of front border of rectangular area, where boundary conditions on coordinate aren't continuous. They suffer a rupture of the first sort in points in which action P - figurative dynamic loading begins. Results of research are brought to the numerical decision.

AMS Subject Classification: *isotropic environment, dynamic load, plane deformation, special point, tension, speed, wave progress, numerical solution, algorithm*
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1. Introduction

Modern engineering and technology widely employ massive elements of constructions, containing cracks, holes, inclusions and other inhomogeneities of various nature and purpose. Performance of these elements under dynamic loads puts a number of questions concerning with dynamic problems of solid mechanics. In particular, evaluation of dynamic stresses near cuts, holes, pores, inclusions and singular points of a boundary is of great practical importance for mechanical and civil engineering, rock mechanics, seismology and fault detection. Solving arising problems and studying unsteady wave fields discloses significant physical features and provides data on the strength and reliability of a construction. Meanwhile, the problem of finding unsteady wave fields is quite difficult. In many practically important cases, the problem is additionally complicated by discontinuous behaviour of a solution. Such are cases when a finite elastic region contains discontinuities in boundary conditions, holes or inclusions with corner points and/or cuts with corners, which are sources of high stress concentration. It is impossible to solve such problems without developing efficient numerical methods. Accordingly, modern studies of unsteady waves in solids focus on the development and improvement of numerical techniques. For the dynamic problems, they include various modifications of finite differences, discrete steps, spatial characteristics, finite elements, Godunov's mesh-characteristic method, boundary integral equations, method of sources, etc. Among the methods, the finite difference methods, based on using characteristic surfaces and compatibility equations on them, have certain advantages. They provide utmost correspondence between the dependence regions of the starting differential equations and approximating difference equations, what notably increases the accuracy of results for smooth and discontinuous solutions; they also provide correct identification of boundaries and contacts. In 1960, an explicit scheme of second order was suggested for a system of partial differential equations of second order in three variables [1]. The scheme employed characteristics and it was used for studying plane waves [2]. Later on, the method of spatial characteristics has been developed for solving particular dynamic problems of solid mechanics [3], [4], [5], [6], [7], [8], [9] [10], [11], [12], [13].

2. Problem formulation.

Consider plane-strain deformation of an elastic rectangle $0_1 \leq \ell, -L_2 \leq L$ The conventional dynamic equations of plane-strain elasticity (see [14]) are used in the form suggested in the paper [2]:

$$v_1,_t - p,_1 - q,_1 - \tau,_2 = 0; \quad v_2,_t - p,_2 + q,_2 - \tau,_1 = 0; \tag{2.1}$$

$$\gamma^2(\gamma^2-1)^{-1}p,_t - v_1,_1 - v_2,_2 = 0; \quad \gamma^2 q,_t - v_1,_1 + v_2,_2 = 0; \gamma^2\tau,_t - v_1,_2 - v_2,_1 = 0,$$

Herein, the dimensionless time $\bar{t}$, spatial coordinates $\bar{x_i}$, stresses p, q, τ and velocities v_1, v_2 are defined via the corresponding physical time t, coordinates x_i, stresses $\sigma_{11}, \sigma_{22}, \sigma_{12}$ and displacements u_i in accordance with [2], as

$$\bar{t} = \frac{tc_1}{b}; \bar{x_i} = \frac{x_i}{b}; \quad v_i = \frac{1}{c_1}\frac{\partial u_i}{\partial t}, (i = 1, 2) \quad p = \frac{\sigma_{11} + \sigma_{22}}{2\rho c_1^2};$$

$$q = \frac{\sigma_{11} - \sigma_{22}}{2\rho c_1^2}; \quad \tau = \frac{\sigma_{12}}{\rho c_1^2}; \gamma = \frac{c_1}{c_2},$$

with b being a characteristic length. Further on, the overbar in the notation of the dimensionless time and coordinates is omitted.

We assume that before loading, the body does not move and it is stress-free. Therefore, the initial conditions are:

$$v_1(x_1, x_2, 0) = v_2(x_1, x_2; 0) = p(x_1, x_2, 0) = q(x_1, x_2, 0) = \tau(x_1, x_2, 0) = 0. \tag{2.2}$$

The boundary conditions (BC) for solving the system (2.1) are as follows. The boundary $x_1 = 0$ of the rectangle is loaded by the normal traction p + q, prescribed on the part $L^* \leq x_2 \leq L^{**}$ as a step function, changing in time t with the amplitude A and the angular frequency T. The shear traction τ is zero. Hence, at $L^* \leq x_2 \leq L^{**}$, the BC are:

$$p + q = f(x_2, t) = Asin(wt), \tau = 0 \text{ for } 0 \leq t \leq t^*. \tag{2.3}$$

The load acts from the moment $t = 0$ till $t = t^*$ and then ceases to zero, so that

$$p + q = 0, \tau = 0 \text{ for } t \geq t^* \tag{2.4}$$

The remaining part of the upper boundary and the entire lower boundary ($x_1 = l$) of the rectangle are traction-free:

$$p + q = 0, \tau = 0 \text{ for } t \geq 0. \tag{2.5}$$

The boundaries $x_2 = \pm L$ are clamped. Hence at any time, the velocity at their points is zero:

$$v_1(x_1, t) = v_2(x_1, t) = 0 \text{ for } t \geq 0. \tag{2.6}$$

We are interested in finding fields of stresses and velocities caused by the fronts of incidental and diffracted elastic waves for $t > 0$. The problem consists in solving the system of partial differential equations (2.1) under the initial condition (2.2) and the boundary conditions (2.3) - (2.6). The solution is obtained by the method of spatial characteristics, presented in detail in [2]. Note, however, that the method, as it is suggested in [2], is applicable only to regions with continuous change of the input parameters. Thus we have developed an algorithm, presented below for finding the solution near the singular points $x_2 = L^*$ and $x_2 = L^{**}$ of the boundary $x_1 = 0$, where the load suffers the discontinuity of the first kind.

We represent the sides of the rectangle by n_1 and n_2 segments, respectively. Thus the division steps are $h_1 = l/n_1$ and $h_2 = L/n_2$. The nodal points are $(x_1^i, x_2^j$ with $x_1^i = ih_1(i = 0, 1, 2, ..., n_1)$ and $x_2^j = jh_2(j = -n_2, -n_2 + 1, -n_2 + 2, ..., -1, 0, 1, 2, ..., n_2 - 1, n_2)$. These points coincide with those, which appear at lines of boundary nodes of a rectangular mesh covering the considered rectangle.

Consider for certainty the point $E_1(x_2 = L^{**})$ of the boundary $x_1 = 0$ (fig. 1). In its vicinity, two corner points I and II are distinguished. For the corners I and II, we derive and employ finite difference approximations, obtained by integration along bi-characteristics and the axis of the characteristic cone. Note that for the corner I the equations are similar to those for the upper right corner R of the considered region:

$$\delta v_1^I - \delta v_2^I + \alpha_8 \delta p^I = A_1, \quad \delta v_1^I + \delta v_2^I + \alpha_2 \delta q^I = A_2, \tag{2.7}$$

while for the corner II they are similar to those for the upper left corner M:

$$\delta v_1^{II} + \delta v_2^{II} + \alpha_8 \delta p^{II} = A_3, \quad \delta v_1^{II} - \delta v_2^{II} + \alpha_2 \delta q^{II} = A_4, \tag{2.8}$$

The right-hand sides in (2.8) and (2.9) are defined by equations:

$$
\begin{aligned}
A_1 =& k(v_{1,1} + p_{,1} + q_{,1} - \tau_{,1} + v_{2,2} - p_{,2} + q_{,2} + \tau_{,2})\\
&- \alpha_0(v_{1,2} + v_{2,1}) - \alpha_9(v_{1,12 - v_{2,12}} - \alpha_5 p_{,12} + \alpha_3 \tau_{,12};\\
A_2 =& k(v_{1,1} + p_{,1} + q_{,1} + \tau_{,1} - v_{2,2} + p_{,2} - q_{,2} + \tau_{,2})\\
&- \alpha_0(v_{1,2} - v_{2,1}) + \alpha_1(v_{1,2} - v_{2,1}) - \alpha_5 q_{,12} + \alpha_9(v_{1,12} + v_{2,12});\\
A_3 =& k(v_{1,1} + p_{,1} + q_{,1} + \tau_{,1} + v_{2,2} + p_{,2} - q_{,2} + \tau_{,2})\\
&+ \alpha_0(v_{1,2} + v_{2,1}) + \alpha_9(v_{1,12} + v_{2,12}) + \alpha_5 p_{,12} + \alpha_3 \tau_{,12};\\
A_4 =& k(v_{1,1} + p_{,1} + q_{,1} - \tau_{,1} - v_{2,2} - p_{,2} + q_{,2} + \tau_{,2})\\
&+ \alpha_0(v_{1,2} - v_{2,1}) - \alpha_1(v_{1,2} - v_{2,1}) + \alpha_5 q_{,12} - \alpha_9(v_{1,12} - v_{2,12}).
\end{aligned}
$$

In accordance with (2.3), to the left of the point $_1$ and at the point $_1$ itself, we have prescribed the normal traction $p + q$. For its increment $\delta p^I + \delta q^I$, we may write:

$$\delta p^I + \delta q^I = A[sin(wt) - sin(w(t-k))]. \tag{2.9}$$

where k is the number of a time step. Besides, we need to meet the continuity conditions for the normal velocities and the normal and shear tractions at adjacent points of corners:

$$\delta v_1^I = \delta v_1^{II}, \quad \delta v_2^I = \delta v_2^{II},$$

$$\delta p^I - \delta q^I = \delta p^{II} - \delta q^{II}, \quad \delta \tau^I = \delta \tau^I = \delta \tau^{II}. \tag{2.10}$$

The system (2.8) - (2.11) uniquely defines the increments of the velocities δv_1^I, δv_1^{II}, δv_2^I, δv_2^{II} and stresses δp^I, δq^I, $\delta \tau^I$, δp^{II}, δq^{II}, $\delta \tau^{II}$ at the point $_1$, where the BC is discontinuous:

$$\delta v_1 = \Delta_1/\Delta, \quad \delta v_2 = \Delta_2/\Delta, \quad \delta p^I = \Delta_3/\Delta, \tag{2.11}$$

$$\delta q^I = \Delta_4/\Delta, \quad \delta \tau = 0, \quad \delta p^{II} = \Delta_5/\Delta, \quad \delta q^{II} = \Delta_6/\Delta.$$

The determinants entering (2.10) are given by formulae:

$$
\begin{aligned}
\Delta_1 =& -[\alpha_2\alpha_8(3(A_1 + A_2) - 2(\alpha_2 + \alpha_8)f(x_2,t) - A_3\\
&- A_4) + \alpha_8^2(A_2 + A_4) + \alpha_2^2(A_1 + A_3)],\\
\Delta_2 =& \alpha_2\alpha_8(A_1 - A_2 - A_3 + A_4) - \alpha_8^2(A_2 - A_4) + \alpha_2^2(A_1 - A_3),\\
\Delta_3 =& 2\alpha_2(A_2 - A_3 - \alpha_2 f(x_2,t)) + 2\alpha_8(A_4 - A_1 - \alpha_2 f(x_2,t)),\\
\Delta_4 =& \alpha_2(A_3 - A_2) - 2\alpha_8(A_4 - A_1 + (\alpha_2 + \alpha_8)f(x_2,t),\\
\Delta_5 =& 2\alpha_8(A_2 - A_3) + 2\alpha_2(2A_2 - 2A_3 - A_4 + A_1) - 2\alpha_2(\alpha_2 + \alpha_8)f(x_2,t),\\
\Delta_6 =& 2\alpha_2(A_1 - A_4) + 2\alpha_8(2A_1 + A_2 - A_3 - 2A_4) - 2\alpha_8(\alpha_2 + \alpha_8)f(x_2,t),\\
\Delta =& -2(\alpha_2 + \alpha_8)^2.
\end{aligned}
$$

3. Main results

Equations (2.12) serve us for finding the solution at the right singular point E_1. Similar equations are used for the left singular point E_2 (fig . 1). They present the basis of an algorithm for solving unsteady problems of dynamic elasticity involving discontinuities of the first kind at points of the boundary. The algorithm is employed in a subroutine, which is included into a general program for calculations on a conventional laptop.

As an illustration, we present results for the rectangular region $0 \le x_1 \le 5$ and $|x_2| \le 5$. The part, at which the external load acts, is: $-4.85 \le x_2 \le 4.85$; hence only 3 percent of the upper boundary is free of the loads. The load, being symmetric with respect to the x_1-axis, we consider only the right half of the rectangle ($x_2 \ge 0$). The spatial steps are taken equal: $h_1 = h_2 = h = 0.05$. The time step k is chosen to meet the stability condition for an explicit finite-difference scheme [2]:

$$(\frac{k}{h})^2 \le min\{\frac{\gamma^2}{\gamma^2+1}, \frac{\gamma^2}{2(\gamma^2-1)}\}.$$

In calculations, we set $k = 0.025$. The amplitude A of the applied load entering (2.3) is taken unit, and the period is $= 100k$; consequently, the angular frequency is $\omega = \pi/(100k) = 0.4\pi$. The duration of the external pulse is $t^* = T = 2.5$. Fig. 2 presents the velocity v_1 at five 'observation' points on the boundary of $x_1 = 0 : x_2 = 0$ (point 1), $x_2 = 20h$ (2), $x_2 = 40h$ (3), $x_2 = 60h$ (4), and $x_2 = 80h$ (5). At each of the points, for the first hundred time steps, the form of the curve is defined by the form of the applied sinusoidal pulse. Firstly, distortion arises at the point (5) with the coordinate $x_2 = 80h$, which is closest to the singular point E_1 having the coordinate $x_2 = 97h$. It arises because of the influence of waves diffracted by the singular point and propagating with the normalized speed $c_1 = 1$ of longitudinal waves. Then, in certain intervals, the influence of this point appears at points 4, 3, 2, and 1, successively. The intensity of the influence is relatively small. Notably more strong effect is caused by the longitudinal wave diffracted by the corner point R of the rectangle ($x_1 = 0, x_2 = 100h$), which propagates with the speed c_1, as well. For the point 1 at the center of the rectangle, the waves, diffracted by the corners M and R, arrive at the moment $t = 400k = 10$ simultaneously with the wave reflected from the lower boundary ($x_1 = 100h$). Consequently, for the time exceeding 400 k, we may see the result of interference of various waves. Summarizing, we conclude that the suggested finite-difference equations provide a means for solving dynamic problems involving points of a boundary, where BC suffer discontinuity of the first kind. Numerical realization of the approach has shown its stability at a sufficiently long interval of time. The results correctly reproduce the general picture and specific features of wave processes. The approach may be used for studying dynamic stresses and strains in homogeneous and layered media.

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N. K. Ashirbayev - corresponding author
email: ank_56@mail.ru,

J. N. Ashirbayeva
email: saya_270681@mail.ru

South Kazakhstan State University,
M. Auezov Shymkent University,
160000 Shymkent. Kazakhstan

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