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A PROOF OF NON-EXISTENCE
OF SELF-IMAGING PHENOMENON
IN THE INCOHERENT CASE

The existing description of incoherent wave field propagation in terms of Fourier transformation has made possible to prove non-existence of the self-imaging phenomenon for incoherent images.

Keywords: optical data processing, image formation theory, optical transfer functions, optical morphological transformations, spatial filtering, Fourier optics, Fourier transforms

INTRODUCTION

Self-imaging is meant in this paper as the phenomenon that occurs during propagation of an image in homogeneous isotropic stationary dielectric medium and consists in reconstruction of the original wave field intensity distribution in the plane distance z away from the original image. The authors approach to the self-imaging problem in the analogous way as W.D. Montgomery did in coherent case [1], i.e. by using the propagation operator in the diagonal form. In incoherent case, the diagonalization by the Fourier transformation was presented in [2]. The essential results of [2] are collected in Section 2 of the present paper where monotonicity of the propagation operator has also been pointed out. These results has made possible to prove non-existence of self-imaging phenomenon for incoherent images, which is showed in Section 3.

1. THE DIAGONAL OPERATOR OF INCOHERENT
WAVE FIELD PROPAGATION

As well as in the coherent case [1, 3], the propagation of an incoherent wave field can be described by the linear transformation of the wave field intensity distribution [4]:

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$$I(x, y; z) = \hat{P}_z I(x, y; 0) = \int\limits_{-\infty}^{+\infty} \int g(x - x', y - y'; z) I(x', y'; 0) \, dx' dy' \qquad (1)$$

where $I(x,y;0)$ and $I(x,y;z)$ are wave field intensity distributions in the original plane and in the plane distance z away, respectively, and $\hat{P}_z$ is the incoherent propagation operator [5]. The integral operator kernel $g(x,y;z)$ in Eq. (1) is a well-known function of x, y, z [2].

There exists a diagonal Fourier representation of the incoherent propagation operator, converting the Fourier transform $J(\omega_x, \omega_y; 0)$ of wave field intensity distribution in starting plane to intensity transform $J(\omega_x, \omega_y; z)$ of wave field formed at the distance z [2]. This diagonal Fourier representation $G(\omega_x, \omega_y; z)$, defined by

$$J(\omega_x, \omega_y; z) = G(\omega_x, \omega_y; z) J(\omega_x, \omega_y; 0) \qquad (2)$$

is the Fourier transform of the kernel $g(x,y;z)$. It was obtained by using [6]:

$$G(\rho; z) = F[g(x, y; z)] = \frac{\kappa\, k^2}{8\pi^2} z\rho K_1(z\rho) + \frac{\kappa}{32\pi^2} \rho^2 K_2(z\rho) \qquad (3)$$

where ω_x, ω_y are spatial angular frequencies having the sense of wave-vector projection on the axes x and y, K_1 and K_2 are the modified Bessel functions of second kind (MacDonald functions) of first and second order, respectively, κ is a positive real constant, k is the wave number, and ρ is the radius in spatial angular frequency domain defined by

$$\rho = \sqrt{\omega_x^2 + \omega_y^2} \qquad (4)$$

Selection rule of the constant κ was showed in [2]. Equation (2) together with the transform $G(\rho;z)$ is a more convenient tool for calculations than Eq. (1) with kernel $g(x,y;z)$ thanks to the diagonalization of the incoherent propagation operator and to fast Fourier transformation efficiency.

When propagation distance z is fixed, the transform $G(\rho;z)$ is a decreasing function of angular frequency radius ρ. It can be proved by using the formula for modified Bessel function differentiation [7]:

$$K'_\nu(x) = -\tfrac{1}{2}\left(K_{\nu-1}(x) + K_{\nu+1}(x)\right) \qquad (5)$$

and the formula for replacing Bessel function of higher order with functions of lower orders [8]:

$$K_{\nu+1}(x) = K_{\nu-1}(x) + \frac{2\nu}{x} K_\nu(x) \tag{6}$$

which yields after substituting:

$$K'_\nu(x) = -K_{\nu-1}(x) - \frac{\nu}{x} K_\nu(x) \tag{7}$$

The function $G(\rho;z)$ derivative is equal:

$$\frac{\partial G(\rho;z)}{\partial \rho} = \frac{\kappa k^2}{8\pi^2} z K_1(z\rho) - \frac{\kappa k^2}{8\pi^2} z^2 \rho \left(K_0(z\rho) + \frac{1}{z\rho} K_1(z\rho) \right) +$$

$$+ \frac{\kappa}{16\pi^2} \rho K_2(z\rho) - \frac{\kappa}{32\pi^2} z\rho^2 \left(K_1(z\rho) + \frac{2}{z\rho} K_2(z\rho) \right) =$$

$$= -\frac{\kappa k^2}{8\pi^2} z^2 \rho K_0(z\rho) - \frac{\kappa}{32\pi^2} z\rho^2 K_1(z\rho) < 0 \tag{8}$$

There are the minus signs in front of both the derivative components, which together with the fact that modified Bessel functions are positive in real domain yields function $G(\rho;z)$ monotonicity.

2. THE PROBLEM OF SELF-IMAGING OF INCOHERENT WAVE FIELDS

If the self-imaging effect occurs in incoherent case than there is at least one wave field intensity distribution that maps to the identical distribution as a result of propagation at the distance z. Using the propagation Eq. (2) and allowing the two intensity distributions to differ by a multiplicative real constant, we can write this assumption as an eigenequation in the Fourier representation:

$$J(\omega_x, \omega_y; z) = G(\omega_x, \omega_y; z) J(\omega_x, \omega_y; 0) = C J(\omega_x, \omega_y; 0) \tag{9}$$

where C is a real constant. Like W.D. Montgomery [1] in coherent case, we can formulate a condition for the incoherent self-imaging wave field on the basis of the above equation: the Fourier transform $J(\omega_x, \omega_y; 0)$ of intensity distribution of such wave field, being an eigenfunction of Eq. (9), must take on non-zero values only in the angular spatial frequency region $\{(\omega_x, \omega_y)\}$ that satisfy the condition:

$$G(\omega_x, \omega_y; z) \equiv G(\rho; z) = C \tag{10}$$

As it has been shown in Section 2, the function $G(\rho;z)$ is monotonic in the whole frequency domain. Therefore the eigenequation (9) may only have such nontrivial eigenfunctions $J(\omega_x,\omega_y;0)$ that take on non-zero values only at one spatial frequency radius ρ, i.e. in one circle-shaped spatial frequency region with a radius $\rho_o \neq 0$. The transform $J(\omega_x,\omega_y;0)$ is meant here to be trivial if it takes on non-zero value at $\rho = 0$ only, i.e. the corresponding intensity distribution $I(x,y;0)$ is uniform. On the other hand, intensity distribution of every image is non-negative and has positive average, and hence its Fourier transform is positive at the zero spatial frequency ρ. Because there is only one ρ_o, it yields $\rho_o = 0$, i.e. the wave field is trivial. Therefore there is a contradiction in the demand that a non-trivial transform of physical wave field intensity distribution is an eigenfunction of eigenequation (9). In other words, the self-imaging effect does not exist in incoherent case.

3. CONCLUSIONS

The diagonalized operator of incoherent propagation has made possible to prove non-existence of incoherent self-imaging phenomenon for non-trivial images. Only infinite incoherent image with uniform intensity distribution does not change as a result of propagation at a certain distance.

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DOWÓD NIEISTNIENIA ZJAWISKA SAMOOBRAZOWANIA
W PRZYPADKU NIEKOHERENTNYM

Na podstawie istniejącego opisu optycznej propagacji niekoherentnych pól falowych z zastosowaniem transformacji Fouriera udowodniono teoretycznie nieistnienie zjawiska samoobrazowania dla niekoherentnych obrazów.

Słowa kluczowe: przetwarzanie danych optycznych, teoria powstawania obrazów, transformaty optyczne, filtracja częstości przestrzennych, optyka fourierowska, transformaty Fouriera