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Inequality for Polynomials with Prescribed Zeros

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Abstract: For a polynomial $p(z)$ of degree n with a zero at β, of order at least $k(\geq 1)$, it is known that

$$\int_0^{2\pi}\left|\frac{p(e^{i\theta})}{(e^{i\theta}-\beta)^k}\right|^2 d\theta \leq \left\{\prod_{j=1}^{k}\left(1+|\beta|^2-2|\beta|\cos\frac{\pi}{n+2-j}\right)\right\}^{-1}\int_0^{2\pi}|p(e^{i\theta})|^2 d\theta.$$

By considering polynomial $p(z)$ of degree n in the form

$p(z)=(z-\beta_1)(z-\beta_2)\dots(z-\beta_k)q(z),\ k\geq 1$ and $q(z),$ a polynomial of degree $n-k$, with

$$S=\{\ \gamma_{l_1}\gamma_{l_2}\dots\gamma_{l_k}:\gamma_{l_1}\gamma_{l_2}\dots\gamma_{l_k} \text{ is a permutation of } k \text{ objects } \beta_1,\beta_2,\dots,\beta_k \text{ taken all at a time}\},$$

we have obtained

$$\int_0^{2\pi}\left|\frac{p(e^{i\theta})}{(e^{i\theta}-\beta_1)(e^{i\theta}-\beta_2)\dots(e^{i\theta}-\beta_k)}\right|^2 d\theta$$
$$\leq\left[\min_{\gamma_{l_1}\gamma_{l_2}\cdots\gamma_{l_k}\in S}\left\{\prod_{j=1}^{k}\left(1+|\gamma_{l_j}|^2-2|\gamma_{l_j}|\cos\frac{\pi}{n+2-j}\right)\right\}^{-1}\right]\int_0^{2\pi}|p(e^{i\theta})|^2 d\theta,$$

a generalization of the known result.

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1. Introduction and statement of result

While thinking of polynomials vanishing at β, Donaldson and Rahman [1] had considered the problem:

How large can $\left(\frac{1}{2\pi}\int_0^{2\pi}|\frac{p(e^{i\theta})}{e^{i\theta}-\beta}|^2 d\theta\right)^{1/2}$ be, for a polynomial $p(z)$ of degree n with

$$\left(\frac{1}{2\pi}\int_0^{2\pi}|p(e^{i\theta})|^2 d\theta\right)^{1/2} = 1?$$

and they had obtained

Theorem A. *If $p(z)$ is a polynomial of degree n such that $p(\beta) = 0$, where β is an arbitrary non-negative number then*

$$\int_0^{2\pi}\left|\frac{p(e^{i\theta})}{e^{i\theta}-\beta}\right|^2 d\theta \leq \left(1+\beta^2-2\beta\cos\left(\frac{\pi}{n+1}\right)\right)^{-1}\int_0^{2\pi}|p(e^{i\theta})|^2 d\theta.$$

In [2] Jain had considered the zero of polynomial $p(z)$ at β to be of order at least $k(\geq 1)$, with β being an arbitrary complex number and had obtained the following generalization of Theorem A.

Theorem B. *If $p(z)$ is a polynomial of degree n such that $p(z)$ has a zero at β, of order at least $k(\geq 1)$, with β being an arbitrary complex number then*

$$\int_0^{2\pi}\left|\frac{p(e^{i\theta})}{(e^{i\theta}-\beta)^k}\right|^2 d\theta \leq \left\{\prod_{j=1}^{k}\left(1+|\beta|^2-2|\beta|\cos\frac{\pi}{n+2-j}\right)\right\}^{-1}\int_0^{2\pi}|p(e^{i\theta})|^2 d\theta.$$

In this paper we have obtained a generalization of Theorem B by considering polynomial $p(z)$ of degree n in the form

$$p(z) = (z-\beta_1)(z-\beta_2)\dots(z-\beta_k)q(z), k \geq 1.$$

More precisely we have proved

Theorem. *Let $p(z)$ be a polynomial of degree n such that*

$$p(z) = (z-\beta_1)(z-\beta_2)\dots(z-\beta_k)q(z), k \geq 1. \tag{1.1}$$

Further let

$$S = \{\gamma_{l_1}\gamma_{l_2}\dots\gamma_{l_k} : \gamma_{l_1}\gamma_{l_2}\dots\gamma_{l_k} \text{ is a permutation of } k \text{ objects } \beta_1,\beta_2,\dots,\beta_k \text{ taken all at a time}\}.$$

Then

$$\int_0^{2\pi}\left|\frac{p(e^{i\theta})}{(e^{i\theta}-\beta_1)(e^{i\theta}-\beta_2)\dots(e^{i\theta}-\beta_k)}\right|^2 d\theta$$

$$\leq \left[\min_{\gamma_{l_1}\gamma_{l_2}\dots\gamma_{l_k}\in S}\left\{\prod_{j=1}^{k}\left(1+|\gamma_{l_j}|^2-2|\gamma_{l_j}|\cos\frac{\pi}{n+2-j}\right)\right\}^{-1}\right]\int_0^{2\pi}|p(e^{i\theta})|^2 d\theta.$$

2. Lemma

For the proof of Theorem we require the following lemma.

Lemma 1. *If $p(z)$ is a polynomial of degree n such that*

$$p(\beta) = 0,$$

where β is an arbitray complex number then

$$\int_0^{2\pi} \left| \frac{p(e^{i\theta})}{e^{i\theta} - \beta} \right|^2 d\theta \le \left(1 + |\beta|^2 - 2|\beta| \cos \frac{\pi}{n+1}\right)^{-1} \int_0^{2\pi} |p(e^{i\theta})|^2 d\theta.$$

This lemma is due to Jain [2].

3. Proof of Theorem

Theorem is trivially true for $k = 1$, by Lemma 1. Accordingly we assume that $k > 1$. The polynomial

$$T_1(z) = (z - \beta_1) q(z) \tag{3.1}$$

is of degree $n - k + 1$ and therefore by Lemma 1 we have

$$\int_0^{2\pi} |q(e^{i\theta})|^2 d\theta = \int_0^{2\pi} \left| \frac{T_1(e^{i\theta})}{e^{i\theta} - \beta_1} \right|^2 d\theta \le \left(1 + |\beta_1|^2 - 2|\beta_1| \cos \frac{\pi}{n-k+2}\right)^{-1} \int_0^{2\pi} |T_1(e^{i\theta})|^2 d\theta. \tag{3.2}$$

Further the polynomial

$$T_2(z) = (z - \beta_2) T_1(z), = (z - \beta_1)(z - \beta_2) q(z), \quad (\text{by}(3.1)), \tag{3.3}$$

is of degree $n - k + 2$ and by Lemma 1 we have

$$\int_0^{2\pi} |T_1(e^{i\theta})|^2 d\theta = \int_0^{2\pi} \left| \frac{T_2(e^{i\theta})}{e^{i\theta} - \beta_2} \right|^2 d\theta \le \left(1 + |\beta_2|^2 - 2|\beta_2| \cos \frac{\pi}{n-k+3}\right)^{-1} \int_0^{2\pi} |T_2(e^{i\theta})|^2 d\theta. \tag{3.4}$$

On combining (3.2) and (3.4) we get

$$\int_0^{2\pi} |q(e^{i\theta})|^2 d\theta$$

$$\le \left\{ \left(1 + |\beta_1|^2 - 2|\beta_1| \cos \frac{\pi}{n-k+2}\right) \left(1 + |\beta_2|^2 - 2|\beta_2| \cos \frac{\pi}{n-k+3}\right) \right\}^{-1} \int_0^{2\pi} |T_2(e^{i\theta})|^2 d\theta.$$

We can now continue and obtain similarly

$$\int_0^{2\pi} |q(e^{i\theta})|^2 \leq \left\{ \left(1+|\beta_1|^2 - 2|\beta_1| \cos \frac{\pi}{n-k+2}\right)\left(1+|\beta_2|^2 - 2|\beta_2| \cos \frac{\pi}{n-k+3}\right) \times \left(1+|\beta_3|^2 - 2|\beta_3| \cos \frac{\pi}{n-k+4}\right)\right\}^{-1} \int_0^{2\pi} |T_3(e^{i\theta})|^2 d\theta,$$

(with

$$T_3(z) = (z-\beta_3)T_2(z), = (z-\beta_1)(z-\beta_2)(z-\beta_3)q(z), \quad \text{(by (3.3))}), \tag{3.5}$$

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$$\int_0^{2\pi} |q(e^{i\theta})|^2 d\theta \leq \left\{\left(1+|\beta_1|^2-2|\beta_1|\cos\frac{\pi}{n-k+2}\right)\left(1+|\beta_2|^2-2|\beta_2|\cos\frac{\pi}{n-k+3}\right)\ldots \ldots\left(1+|\beta_k|^2 - 2|\beta_k| \cos \frac{\pi}{n-k+k+1}\right)\right\}^{-1} \int_0^{2\pi} |T_k(e^{i\theta})|^2 d\theta, \tag{3.6}$$

(with

$$\begin{aligned} T_k(z) &= (z-\beta_k)T_{k-1}(z), \\ &= (z-\beta_1)(z-\beta_2)\ldots(z-\beta_k)q(z), \text{(similar to (3.3) and (3.5)))}. \end{aligned} \tag{3.7}$$

On using (1.1) and (3.7) in (3.6) we get

$$\int_0^{2\pi} \left| \frac{p(e^{i\theta})}{(e^{i\theta}-\beta_1)(e^{i\theta}-\beta_2)\ldots(e^{i\theta}-\beta_k)} \right|^2 d\theta \leq \left\{ \left(1+|\beta_1|^2 - 2|\beta_1| \cos \frac{\pi}{n-k+2}\right) \left(1+|\beta_2|^2 - 2|\beta_2| \cos \frac{\pi}{n-k+3}\right)\ldots \ldots\ldots\left(1+|\beta_k|^2 - 2|\beta_k| \cos \frac{\pi}{n+1}\right)\right\}^{-1} \times \int_0^{2\pi} |p(e^{i\theta})|^2 d\theta$$

and as the order of $\beta_1, \beta_2, \ldots, \beta_k$ is immaterial, Theorem follows.

References

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