Journal of Mathematics and Applications

vol. 45 (2022)

Issued with the consent of the Rector

Editor–in–Chief
Publishing House of Ignacy Łukasiewicz Rzeszów
University of Technology, Poland
Lesław GNIEWEK

Open Access **Journal of Mathematics and Applications (JMA)** publishes
original research papers in the area of pure mathematics and its applications.
Two types of articles will be accepted for publication, namely research articles and review
articles. The authors are obligated to select the kind of their articles (research or review).
Manuscript, written in English and prepared using LaTeX, may be submitted
to the Editorial Office or one of the Editors or members of the Editorial Board.
Electronic submission of pdf file is required.
Detailed information for authors is given on the last page.

Text prepared to print in LATEX
by Szymon Dudek and Rafał Nalepa

The printed version of JMA is an original version.

ISSN 1733-6775

Publisher: Publishing House of Ignacy Łukasiewicz Rzeszów
University of Technology,
12 Powstańców Warszawy Ave., 35-959 Rzeszów (e-mail: oficyna@prz.edu.pl)
http://oficyna.prz.edu.pl/en/
Editorial Office: Ignacy Łukasiewicz Rzeszów University of Technology,
Faculty of Mathematics and Applied Physics, P.O. BOX 85
8 Powstańców Warszawy Ave., 35-959 Rzeszów (e-mail: jma@prz.edu.pl)
http://jma.prz.edu.pl/en/

Additional information and an imprint - p. 193

Journal of Mathematics and Applications

vol. 45 (2022)

Table of contents

Journal of
Mathematics
and Applications

JMA No 45, pp 5-20 (2022)

On the Hybrid Caputo-Proportional Fractional Differential Inclusions in Banach Spaces

Mohamed I. Abbas

Abstract: The current article concerns an existence criteria of solutions of nonlinear fractional differential inclusions in the sense of the hybrid Caputo-proportional fractional derivatives in Banach space. The investigation of the main result relies on the set-valued issue of Mönch fixed point theorem incorporated with the Kuratowski measure of noncompactness.

AMS Subject Classification: 34A08, 34A12, 34A08, 34A12, 34A37, 34G20.
Keywords and Phrases: Hybrid Caputo-proportional fractional derivatives; Measure of non-compactness; Mönch fixed point theorem.

1. Introduction

It is recently seen that there is a wide-spread of fractional differential systems because of their great relevance to reality and their dignified influence in describing several real-world problems in physics, mechanics and engineering. For intance, we refer the reader to the monographs of Baleanu et al.[7], Hilfer [21], Kilbas et al. [24], Mainardi [26], Miller and Ross [27], Podlubny [30], Samko et al. [32] and the papers [17, 33].

Due to the importance of fractional differential inclusions in mathematical modeling of problems in game theory, stability, optimal control, and so on. For this reason, many contributions have been investigated by some researchers [1, 4, 11, 12, 13, 18, 29].

On the other hand, the theory of measure of non-compactness is an essential tool in investigating the existence of solutions for nonlinear integral and differential equations, see, for example, the recent papers [5, 10, 15, 19, 31] and the references existing therein.

In [14], Benchohra et al. studied the existence of solutions for the fractional differential inclusions with boundary conditions

$$\begin{cases} {}^{C}\mathcal{D}^{r}y(t) \in G\left(t, y(t)\right), \quad \text{a.e. on } [0,T], \quad 1<r<2, \\ y(0)=y_0, \quad y(T)=y_T, \end{cases}$$

where ${}^{C}\mathcal{D}^{r}$ is the Caputo fractional derivative, $G:[0,T]\times\mathbf{E}\to\mathfrak{P}(\mathbf{E})$ is a multi-valued map, $y_0, y_T \in \mathbf{E}$ and $(\mathbf{E}, |\cdot|)$ is a Banach space.

Motivated by the above work, in this paper, we will extend the Caputo fractional derivative with a broader and more general one, which can be written as a Riemann-Liouville integral of a proportional derivative, or in some important special cases as a linear combination of a Riemann-Liouville integral and a Caputo derivative. To be more precise we will study the existence of solutions for the following nonlinear fractional differential inclusions with the hybrid Caputo-proportional fractional derivatives

$$\begin{cases} {}_{0}^{PC}\mathcal{D}_{t}^{\alpha}x(t) \in F\left(t, x(t)\right), \quad \text{a.e. on } \mathbf{J}:=[0,b], \quad 0<\alpha<1, \\ x(0)=x_0, \end{cases} \tag{1.1}$$

where ${}_{0}^{PC}\mathcal{D}_{t}^{\alpha}$ denotes the hybrid proportional-Caputo fractional derivative of order α, $(\mathbf{E}, |\cdot|)$ is a Banach space, $\mathfrak{P}(\mathbf{E})$ is the family of all nonempty subsets of $\mathbf{E}$, $x_0 \in \mathbf{E}$ and $F:\mathbf{J}\times\mathbf{E}\to\mathfrak{P}(\mathbf{E})$ is a given multi-valued map. We study the inclusion problem (1.1) in the case where the right hand side is convex-valued by means of the set-valued issue of Mönch fixed point theorem incorporated with the Kuratowski measure of non-compactness.

It is worth noting that the relevant results of fractional differential inclusions with the hybrid Caputo-proportional fractional derivatives are scarce. So the main goal of the present work is to contribute to the development of this area. Further, the topic of research has attracted lots of interests as a powerful tool for modeling scientific phenomena. Therefore, we refer the reader to some recent results which can be helpful for more related extensions or generalizations of the results in this paper in the future research works, see [22, 23, 28].

2. Preliminaries

First, we recall from [6] the following definition of the proportional (conformable) derivative of order α:

$${}_{0}^{P}D^{\alpha}g(t) = k_1(\alpha,t)g(t) + k_0(\alpha,t)g'(t),$$

where g is differentiable function and $k_0, k_1 : [0,1] \times \mathbb{R} \to [0,\infty)$ are continuous functions of the variable t and the parameter $\alpha \in [0,1]$ which satisfy the following conditions for all $t \in \mathbb{R}$:

$$\lim_{\alpha \to 0^+} k_0(\alpha, t) = 0, \quad \lim_{\alpha \to 1^-} k_0(\alpha, t) = 1, \quad k_0(\alpha, t) \neq 0, \ \alpha \in (0,1], \tag{2.1}$$

$$\lim_{\alpha \to 0^+} k_1(\alpha, t) = 1, \quad \lim_{\alpha \to 1^-} k_1(\alpha, t) = 0, \quad k_1(\alpha, t) \neq 0, \ \alpha \in [0,1). \tag{2.2}$$

Next, we explore the new definitions of the generalized hybrid proportional-Caputo fractional derivative.

Definition 2.1. [8] The hybrid Caputo-proportional fractional derivative of order $\alpha \in (0,1)$ of a differentiable function $g(t)$ is given by

$${}_{0}^{PC}\mathcal{D}_t^{\alpha} g(t) = \frac{1}{\Gamma(1-\alpha)} \int_0^t \big(k_1(\alpha,\tau) g(t) + k_0(\alpha,\tau) g'(t)\big)(t-\tau)^{-\alpha} \, d\tau, \tag{2.3}$$

where the function space domain is given by requiring that g is differentiable and both g and g' are locally L^1 functions on the positive reals.

Definition 2.2. [8] The inverse operator of the hybrid Caputo-proportional fractional derivative of order is given by

$${}_{0}^{PC}\mathcal{I}_t^{\alpha} g(t) = \int_0^t \exp\left(-\int_u^t \frac{k_1(\alpha,s)}{k_0(\alpha,s)} ds\right) \frac{{}_{0}^{RL}\mathcal{D}_u^{1-\alpha} g(u)}{k_0(\alpha,u)} \, du, \tag{2.4}$$

where ${}_{0}^{RL}\mathcal{D}_u^{1-\alpha}$ denotes the Riemann-Liouville fractional derivative of order $1-\alpha$ and is given by

$${}_{0}^{RL}\mathcal{D}_u^{1-\alpha} g(u) = \frac{1}{\Gamma(\alpha)} \frac{d}{du} \int_0^u (u-s)^{\alpha-1} g(s) \, ds. \tag{2.5}$$

For more details, we refer the reader to the book of Kilbas et al. [24].

Proposition 2.3. *[8] The following inversion relations:*

$${}_{0}^{PC}\mathcal{D}_t^{\alpha} \ {}_{0}^{PC}\mathcal{I}_t^{\alpha} g(t) = g(t) - \frac{t^{-\alpha}}{\Gamma(1-\alpha)} \lim_{t \to 0} \ {}_{0}^{RL}\mathcal{I}_t^{\alpha} g(t), \tag{2.6}$$

$${}_{0}^{PC}\mathcal{I}_t^{\alpha} \ {}_{0}^{PC}\mathcal{D}_t^{\alpha} g(t) = g(t) - \exp\left(-\int_0^t \frac{k_1(\alpha,s)}{k_0(\alpha,s)} ds\right) g(0) \tag{2.7}$$

are satisfied.

Proposition 2.4. *[8] The hybrid Caputo-proportional fractional derivative operator* ${}_{0}^{PC}\mathcal{D}_t^{\alpha}$ *is non-local and singular.*

Remark 2.5. [8] In the limiting cases $\alpha \to 0$ and $\alpha \to 1$, we recover the following special cases:

$$\begin{aligned}\lim_{\alpha\to 0} \quad {}^{PC}_{\;0}\mathcal{D}^{\alpha}_{t} g(t) &= \int_0^t g(\tau)\, d\tau,\\ \lim_{\alpha\to 1} \quad {}^{PC}_{\;0}\mathcal{D}^{\alpha}_{t} g(t) &= g(t).\end{aligned}$$

Denote by $C(\mathbf{J},\mathbf{E})$ the Banach space of all continuous functions from $\mathbf{J}$ to $\mathbf{E}$ with the norm $\|x\| = \sup_{t\in\mathbf{J}} |x(t)|$. By $L^1(\mathbf{J},\mathbf{E})$, we indicate the space of Bochner integrable functions from $\mathbf{J}$ to $\mathbf{E}$ with the norm $\|x\|_1 = \int_0^b |x(t)|\, dt$.

2.1. Multi-valued maps analysis

Let the Banach space be $(\mathbf{E}, |\cdot|)$. The expressions we have used are $\mathfrak{P}(\mathbf{E}) = \{Z \in \mathfrak{P}(\mathbf{E}) : Z \neq \emptyset\}$, $\mathfrak{P}_{\mathbf{cl}}(\mathbf{E}) = \{Z \in \mathfrak{P}(\mathbf{E}) : Z \textit{ is closed}\}$, $\mathfrak{P}_{\mathbf{bd}}(\mathbf{E}) = \{Z \in \mathfrak{P}(\mathbf{E}) : Z \textit{ is bounded}\}$, $\mathfrak{P}_{\mathbf{cp}}(\mathbf{E}) = \{Z \in \mathfrak{P}(\mathbf{E}) : Z \textit{ is compact}\}$, $\mathfrak{P}_{\mathbf{cvx}}(\mathbf{E}) = \{Z \in \mathfrak{P}(\mathbf{E}) : Z \textit{ is convex}\}$.

- A multi-valued map $\mathfrak{U} : \mathbf{E} \to \mathfrak{P}(\mathbf{E})$ is convex (closed) valued, if $\mathfrak{U}(x)$ is convex (closed) for all $x \in \mathbf{E}$.
- $\mathfrak{U}$ is bounded on bounded sets if $\mathfrak{U}(B) = \cup_{x\in B}\mathfrak{U}(x)$ is bounded in $\mathbf{E}$ for any $B \in \mathfrak{P}_{\mathbf{bd}}(\mathbf{E})$, i.e. $\sup_{x\in B}\{\sup\{\|y\| : y \in \mathfrak{U}(x)\}\} < \infty$.
- $\mathfrak{U}$ is called upper semi-continuous on $\mathbf{E}$ if for each $x^* \in \mathbf{E}$, the set $\mathfrak{U}(x^*)$ is nonempty, closed subset of $\mathbf{E}$, and if for each open set N of $\mathbf{E}$ containing $\mathfrak{U}(x^*)$, there exists an open neighborhood N^* of x^* such that $\mathfrak{U}(N^*) \subset N$.
- $\mathfrak{U}$ is completely continuous if $\mathfrak{U}(B)$ is relatively compact for each $B \in \mathfrak{P}_{\mathbf{bd}}(\mathbf{E})$.
- If $\mathfrak{U}$ is a multi-valued map that is completely continuous with nonempty compact values, then $\mathfrak{U}$ is u.s.c. if and only if $\mathfrak{U}$ has a closed graph (that is, if $x_n \to x_0, y_n \to y_0$, and $y_n \in \mathfrak{U}(x_n)$, then $y_0 \in \mathfrak{U}(x_0)$).

For more details about multi-valued maps, we refer to the book of Deimling [16].

Definition 2.6. A multi-valued map $F : \mathbf{J} \times \mathbf{E} \to \mathfrak{P}(\mathbf{E})$ is said to be Carathéodory if

(i) $t \mapsto F(t,x)$ is measurable for each $u \in \mathbf{E}$;

(ii) $x \mapsto F(t,x)$ is upper semi-continuous for almost all $t \in \mathbf{J}$.

We define the set of the selections of a multi-valued map F by

$$\mathcal{S}_{F,x} := \{f \in L^1(\mathbf{J},\mathbf{E}) : f(t) \in F(t,x(t)) \text{ for a.e. } t \in \mathbf{J}\}.$$

Lemma 2.7. *[25] Let $\mathbf{J}$ be a compact real interval and $\mathbf{E}$ be a Banach space. Let F be a multi-valued map satisfying the Caratheodory conditions with the set of L^1-selections $\mathcal{S}_{F,u}$ nonempty, and let $\Theta : L^1(\mathbf{J},\mathbf{E}) \to C(\mathbf{J},\mathbf{E})$ be a linear continuous mapping. Then the operator*

$$\Theta \circ \mathcal{S}_{F,x} : C(\mathbf{J},\mathbf{E}) \to \mathfrak{P}_{\mathbf{bd,cl,cvx}}(C(\mathbf{J},\mathbf{E})), \quad x \mapsto (\Theta \circ \mathcal{S}_{F,x})(x) := \Theta(\mathcal{S}_{F,x})$$

is a closed graph operator in $C(\mathbf{J},\mathbf{E}) \times C(\mathbf{J},\mathbf{E})$.

2.2. Measure of non-compactness

We specify this part of the paper to explore some important details of the Kuratowski measure of non-compactness.

Definition 2.8. [9] Let $\Lambda_{\mathbf{E}}$ be the family of bounded subsets of a Banach space $\mathbf{E}$. We define the Kuratowski measure of non-compactness $\kappa : \Lambda_{\mathbf{E}} \to [0,\infty]$ of $\mathbf{B} \in \Lambda_{\mathbf{E}}$ as

$$\kappa(\mathbf{B}) = \inf\{\epsilon > 0 : \mathbf{B} \subset \bigcup_{j=1}^{m} \mathbf{B}_j \text{ and } \operatorname{diam}(\mathbf{B}_j) \leq \epsilon\}.$$

Lemma 2.9. *[9] Let $\mathbf{C}, \mathbf{D} \subset \mathbf{E}$ be bounded, the Kuratowski measure of non-compactness possesses the next characteristics:*

i. $\kappa(\mathbf{C}) = 0 \Leftrightarrow \mathbf{C}$ *is relatively compact;*

ii. $\mathbf{C} \subset \mathbf{D} \Rightarrow \kappa(\mathbf{C}) \leq \kappa(\mathbf{D})$*;*

iii. $\kappa(\mathbf{C}) = \kappa(\overline{\mathbf{C}})$*, where $\overline{\mathbf{C}}$ is the closure of $\mathbf{C}$;*

iv. $\kappa(\mathbf{C}) = \kappa(conv(\mathbf{C}))$*, where $conv(\mathbf{C})$ is the convex hull of $\mathbf{C}$;*

v. $\kappa(\mathbf{C}+\mathbf{D}) \leq \kappa(\mathbf{C}) + \kappa(\mathbf{D})$*, where $\mathbf{C}+\mathbf{D} = \{u+v : u \in \mathbf{C},\ v \in \mathbf{D}\}$;*

vi. $\kappa(\nu\mathbf{C}) = |\nu|\kappa(\mathbf{C})$*, for any $\nu \in \mathbb{R}$.*

Theorem 2.10. *(Mönch's fixed point theorem) Let Ω be a closed and convex subset of a Banach space $\mathbf{E}$; $\mathcal{U}$ a relatively open subset of Ω, and $\mathcal{N} : \overline{\mathcal{U}} \to \mathfrak{P}(\Omega)$. Assume that graph $\mathcal{N}$ is closed, $\mathcal{N}$ maps compact sets into relatively compact sets and for some $x_0 \in \mathcal{U}$, the following two conditions are satisfied:*

(i) *$G \subset \overline{\mathcal{U}}$, $G \subset conv(x_0 \cup \mathcal{N}(G))$, $\overline{G} = \overline{C}$ implies $\overline{G}$ is compact, where C is a countable subset of G;*

(ii) $x \notin (1-\mu)x_0 + \mu\mathcal{N}(x) \quad \forall u \in \overline{\mathcal{U}} \backslash \mathcal{U}, \quad \mu \in (0,1)$.

Then there exists $x \in \overline{\mathcal{U}}$ with $x \in \mathcal{N}(x)$.

Theorem 2.11. *[20] Let $\mathbf{E}$ be a Banach space and $C \subset L^1(\mathbf{J},\mathbf{E})$ countable with $|u(t)| \leq h(t)$ for a.e. $t \in \mathbf{J}$, and every $u \in C$; where $h \in L^1(\mathbf{J},\mathbb{R}_+)$. Then the function $z(t) = \kappa(C(t))$ belongs to $L^1(\mathbf{J},\mathbb{R}_+)$ and satisfies*

$$\kappa\Big(\Big\{\int_0^b u(\tau)\,d\tau : u \in C\Big\}\Big) \leq 2\int_0^b \kappa(C(\tau))\,d\tau.$$

3. Main results

We start this section with the definition of a solution of the inclusion problem (1.1).

Definition 3.1. A function $x \in C(\mathbf{J}, \mathbf{E})$ is said to be a solution of the inclusion problem (1.1) if there exist a function $f \in L^1(\mathbf{J}, \mathbf{E})$ with $f(t) \in F(t, x(t))$ for a.e. $t \in \mathbf{J}$, such that ${}_0^{PC}D_t^{\alpha} x(t) = f(t)$ on $\mathbf{J}$, and the condition $x(0) = x_0$ is satisfied.

Lemma 3.2. *For $0 < \alpha \leq 1$ and $h \in C(\mathbf{J}, \mathbb{R})$ the solution x of the linear hybrid Caputo-proportional fractional differential equation*

$$\begin{cases} {}_0^{PC}D_t^{\alpha} x(t) = h(t), \quad t \in \mathbf{J}, \\ x(0) = x_0, \end{cases} \tag{3.1}$$

is given by the following integral equation

$$\begin{aligned} x(t) &= \exp\left(-\int_0^t \frac{k_1(\alpha, s)}{k_0(\alpha, s)}\, ds\right) x_0 \\ &\quad + \frac{1}{\Gamma(\alpha - 1)} \int_0^t \int_0^u \exp\left(-\int_u^t \frac{k_1(\alpha, s)}{k_0(\alpha, s)}\, ds\right) \frac{(u-\tau)^{\alpha-2}}{k_0(\alpha, u)} h(\tau)\, d\tau\, du, \quad t \in \mathbf{J}. \end{aligned} \tag{3.2}$$

Proof. Applying the operator ${}_0^{PC}I_t^{\alpha}(\cdot)$ on both sides of (3.1), we get

$${}_0^{PC}I_t^{\alpha}\, {}_0^{PC}D_t^{\alpha} x(t) = {}_0^{PC}I_t^{\alpha} h(t).$$

Using (2.4) and (2.5) together with Proposition 2.3, we get

$$\begin{aligned} x(t) - \exp\left(-\int_0^t \frac{k_1(\alpha, s)}{k_0(\alpha, s)} ds\right) x(0) &= \int_0^t \exp\left(-\int_u^t \frac{k_1(\alpha, s)}{k_0(\alpha, s)} ds\right) \frac{{}_0^{RL}D_u^{1-\alpha} h(u)}{k_0(\alpha, u)}\, du \\ = \frac{1}{\Gamma(\alpha)} \int_0^t \exp\left(-\int_u^t \frac{k_1(\alpha, s)}{k_0(\alpha, s)} ds\right) \frac{1}{k_0(\alpha, u)} \frac{d}{du} \int_0^u (u-\tau)^{\alpha-1} h(\tau)\, d\tau\, du \end{aligned} \tag{3.3}$$

Using the following Leibniz's rule:

$$\frac{d}{du} \int_{a_1(u)}^{a_2(u)} w(u, \tau)\ d\tau = \int_{a_1(u)}^{a_2(u)} \frac{\partial}{\partial u} w(u, \tau)\ d\tau + w(u, a_2(u)) a_2'(u) - w(u, a_1(u)) a_1'(u),$$

where $w(u, \tau) = (u - \tau)^{\alpha-1} h(\tau)$, $a_1(u) = 0$, and $a_2(u) = u$, we obtain that

$$\frac{d}{du} \int_0^u (u-\tau)^{\alpha-1} h(\tau)\ d\tau = (\alpha - 1) \int_0^u (u-\tau)^{\alpha-2} h(\tau)\ d\tau. \tag{3.4}$$

Therefore, the substitution from (3.4) in (3.3), we get

$$\begin{aligned} x(t) &= \exp\left(-\int_0^t \frac{k_1(\alpha, s)}{k_0(\alpha, s)}\, ds\right) x_0 \\ &\quad + \frac{1}{\Gamma(\alpha - 1)} \int_0^t \int_0^u \exp\left(-\int_u^t \frac{k_1(\alpha, s)}{k_0(\alpha, s)}\, ds\right) \frac{(u-\tau)^{\alpha-2}}{k_0(\alpha, u)} h(\tau)\, d\tau\, du. \end{aligned}$$

This completes the proof. $\square$

Remark 3.3. The result of Lemma 3.2 is true not only for real valued functions $x \in C(\mathbf{J}, \mathbb{R})$ but also for a Banach space functions $x \in C(\mathbf{J}, \mathbf{E})$.

Lemma 3.4. *Assume that $F : \mathbf{J} \times \mathbf{E} \to \mathfrak{P}(\mathbf{E})$ satisfies Carathèodory conditions, i.e., $t \mapsto F(t, x)$ is measurable for every $x \in \mathbf{E}$ and $x \mapsto F(t, x)$ is continuous for every $t \in \mathbf{J}$. A function $x \in C(\mathbf{J}, \mathbf{E})$ is a solution of the inclusion problem (1.1) if and only if it satisfies the integral equation*

$$\begin{aligned} x(t) = &\exp\left(-\int_0^t \frac{k_1(\alpha, s)}{k_0(\alpha, s)}\, ds\right) x_0 \\ &+ \frac{1}{\Gamma(\alpha-1)} \int_0^t \int_0^u \exp\left(-\int_u^t \frac{k_1(\alpha, s)}{k_0(\alpha, s)}\, ds\right) \frac{(u-\tau)^{\alpha-2}}{k_0(\alpha, u)} f(\tau)\, d\tau\, du, \end{aligned} \tag{3.5}$$

where $f \in L^1(\mathbf{J}, \mathbf{E})$ with $f(t) \in F(t, x(t))$ for a.e. $t \in \mathbf{J}$.

Now, we are ready to present the main result of the current paper.

Theorem 3.5. *Let $\varrho > 0$, $\mathcal{K} = \{x \in \mathbf{E} : \|x\| \le \varrho\}$, $\mathcal{U} = \{x \in C(\mathbf{J}, \mathbf{E}) : \|x\| < \varrho\}$, and suppose that:*

(H1) *The multi-valued map $F : \mathbf{J} \times \mathbf{E} \to \mathfrak{P}_{\mathbf{cp,cvx}}(\mathbf{E})$ is Carathèodory,*

(H2) *For each $\varrho > 0$, there exists a function $\varphi \in L^1(\mathbf{J}, \mathbb{R}_+)$ such that*

$$\|F(t, x)\|_{\mathfrak{P}} = \{|f| : f(t) \in F(t, x)\} \le \varphi(t),$$

for a.e. $t \in \mathbf{J}$ and $x \in \mathbf{E}$ with $|x| \le \varrho$, and

$$\lim_{\varrho \to \infty} \inf \frac{\int_0^b \varphi(t) dt}{\varrho} = \ell < \infty.$$

(H3) *There is a Carathèodory function $\vartheta : \mathbf{J} \times [0, 2\varrho] \to \mathbb{R}_+$ such that*

$$\kappa\left(F(t, G)\right) \le \vartheta(t, \kappa(G)),$$

a.e. $t \in \mathbf{J}$ and each $G \subset \mathcal{K}$, and the unique solution $\theta \in C(\mathbf{J}, [0, 2\varrho])$ of the inequality

$$\theta(t) \le 2\left\{\frac{1}{\Gamma(\alpha-1)} \int_0^t \int_0^u \exp\left(-\int_u^t \frac{k_1(\alpha, s)}{k_0(\alpha, s)} ds\right) \frac{(u-\tau)^{\alpha-2}}{k_0(\alpha, u)} \vartheta\left(\tau, \kappa\left(G(\tau)\right)\right) d\tau\, du\right\},$$

$t \in \mathbf{J}$,

is $\theta \equiv 0$.

Then the inclusion problem (1.1) possesses at least one solution, provided that

$$\ell < \frac{\Gamma(\alpha) M_{k_0}}{b}, \tag{3.6}$$

where $M_{k_0} := \inf_{t \in \mathbf{J}} |k_0(\alpha, t)| \neq 0$.

Proof. Define the multi-valued map $\mathcal{N} : C(\mathbf{J},\mathbf{E}) \to \mathfrak{P}(C(\mathbf{J},\mathbf{E}))$ by

$$(\mathcal{N}x)(t) = \begin{cases} f \in C(\mathbf{J},\mathbf{E}) : \\ f(t) = \exp\left(-\int_0^t \frac{k_1(\alpha,s)}{k_0(\alpha,s)}\,ds\right) x_0 \\ \qquad + \frac{1}{\Gamma(\alpha-1)}\int_0^t\int_0^u \exp\left(-\int_u^t \frac{k_1(\alpha,s)}{k_0(\alpha,s)}\,ds\right) \frac{(u-\tau)^{\alpha-2}}{k_0(\alpha,u)} w(\tau)\,d\tau\,du, \;\; w \in \mathcal{S}_{F,x}. \end{cases} \tag{3.7}$$

In accordance with Lemma 3.4, the fixed points of $\mathcal{N}$ are solutions to the inclusion problem (1.1). We shall show in five steps that the multi-valued operator $\mathcal{N}$ satisfies all assumptions of Mönch's fixed point theorem (Theorem 2.10) with $\overline{\mathcal{U}} = C(\mathbf{J},\mathcal{K})$.

Step 1. $\mathcal{N}(x)$ is convex, for any $x \in C(\mathbf{J},\mathcal{K})$.

For $f_1, f_2 \in \mathcal{N}(x)$, there exist $w_1, w_2 \in \mathcal{S}_{F,x}$ such that for each $t \in \mathbf{J}$, we have

$$f_i(t) = \exp\left(-\int_0^t \frac{k_1(\alpha,s)}{k_0(\alpha,s)}\,ds\right) x_0 + \frac{1}{\Gamma(\alpha-1)}\int_0^t\int_0^u \exp\left(-\int_u^t \frac{k_1(\alpha,s)}{k_0(\alpha,s)}\,ds\right) \frac{(u-\tau)^{\alpha-2}}{k_0(\alpha,u)} w_i(\tau)\,d\tau\,du, \;\; i = 1,2.$$

Let $0 \le \mu \le 1$. Then, for $t \in \mathbf{J}$,

$$(\mu f_1 + (1-\mu) f_2)(t) = \exp\left(-\int_0^t \frac{k_1(\alpha,s)}{k_0(\alpha,s)}\,ds\right) x_0 + \frac{1}{\Gamma(\alpha-1)}\int_0^t\int_0^u \exp\left(-\int_u^t \frac{k_1(\alpha,s)}{k_0(\alpha,s)}\,ds\right) \frac{(u-\tau)^{\alpha-2}}{k_0(\alpha,u)} (\mu w_1 + (1-\mu) w_2)(\tau)\,d\tau\,du.$$

Since $\mathcal{S}_{F,x}$ is convex (because F has convex values), then $\mu f_1 + (1-\mu) f_2 \in \mathcal{N}(x)$.

Step 2. $\mathcal{N}(G)$ is relatively compact for each compact $G \in \overline{\mathcal{U}}$.

Let $G \in \overline{\mathcal{U}}$ be a compact set and let $\{f_n\}$ be any sequence of elements of $\mathcal{N}(G)$. We show that $\{f_n\}$ has a convergent subsequence by using the Arzelà-Ascoli criterion of non-compactness in $C(\mathbf{J},\mathcal{K})$. Since $f_n \in \mathcal{N}(G)$, there exist $x_n \in G$ and $w_n \in \mathcal{S}_{F,x_n}$, such that

$$f_n(t) = \exp\left(-\int_0^t \frac{k_1(\alpha,s)}{k_0(\alpha,s)}\,ds\right) x_0 + \frac{1}{\Gamma(\alpha-1)}\int_0^t\int_0^u \exp\left(-\int_u^t \frac{k_1(\alpha,s)}{k_0(\alpha,s)}\,ds\right) \frac{(u-\tau)^{\alpha-2}}{k_0(\alpha,u)} w_n(\tau)\,d\tau\,du,$$

for $n \ge 1$. In view of Theorem 2.11 and the properties of the Kuratowski measure of non-compactness, we have

$$\kappa(\{f_n(t)\}) \le 2\left\{\frac{1}{\Gamma(\alpha-1)}\int_0^t\int_0^u \kappa\left(\left\{\exp\left(-\int_u^t \frac{k_1(\alpha,s)}{k_0(\alpha,s)}ds\right)\frac{(u-\tau)^{\alpha-2}}{k_0(\alpha,u)} w_n(\tau) : n \ge 1\right\}\right) d\tau du\right\}. \tag{3.8}$$

On the other hand, since G is compact, the set $\{w_n(\tau) : n \geq 1\}$ is compact. Consequently, $\kappa(\{w_n(\tau) : n \geq 1\}) = 0$ for a.e. $\tau \in \mathbf{J}$. Therefore, $\kappa(\{f_n(t)\}) = 0$ which implies that $\{f_n(t) : n \geq 1\}$ is relatively compact in $\mathcal{K}$ for each $t \in \mathbf{J}$. Furthermore, For each $t_1, t_2 \in \mathbf{J},\ t_1 < t_2$, one obtain that:

$$
\begin{aligned}
&|f_n(t_2) - f_n(t_1)| \\
\leq& \left| \exp\left(-\int_0^{t_2} \frac{k_1(\alpha,s)}{k_0(\alpha,s)}\, ds \right) x_0 - \exp\left(-\int_0^{t_1} \frac{k_1(\alpha,s)}{k_0(\alpha,s)}\, ds \right) x_0 \right| \\
&+ \frac{1}{\Gamma(\alpha-1)} \left| \int_0^{t_2}\int_0^{u} \left[\exp\left(-\int_0^{t_2} \frac{k_1(\alpha,s)}{k_0(\alpha,s)} ds \right) - \exp\left(-\int_0^{t_1} \frac{k_1(\alpha,s)}{k_0(\alpha,s)} ds \right) \right] \frac{(u-\tau)^{\alpha-2}}{k_0(\alpha,u)} w_n(\tau) d\tau du \right| \\
&+ \frac{1}{\Gamma(\alpha-1)} \left| \int_{t_1}^{t_2}\int_0^{u} \exp\left(-\int_0^{t_1} \frac{k_1(\alpha,s)}{k_0(\alpha,s)}\, ds \right) \frac{(u-\tau)^{\alpha-2}}{k_0(\alpha,u)} w_n(\tau)\, d\tau\, du \right|.
\end{aligned}
$$

By applying the mean value theorem to the function $\exp\left(-\int_0^{t} \frac{k_1(\alpha,s)}{k_0(\alpha,s)}\, ds \right)$ on (t_1, t_2), we obtain that

$$
\begin{aligned}
\left| \exp\left(-\int_0^{t_2} \frac{k_1(\alpha,s)}{k_0(\alpha,s)} ds \right) - \exp\left(-\int_0^{t_1} \frac{k_1(\alpha,s)}{k_0(\alpha,s)} ds \right) \right| &= \left| \frac{k_1(\alpha,\xi)}{k_0(\alpha,\xi)} \exp\left(-\int_0^{\xi} \frac{k_1(\alpha,s)}{k_0(\alpha,s)} ds \right) (t_2-t_1) \right| \\
&\leq \left| \frac{k_1(\alpha,\xi)}{k_0(\alpha,\xi)} \right| (t_2 - t_1), \quad \forall \xi \in (t_1, t_2).
\end{aligned}
$$

Therefore, we get

$$
\begin{aligned}
|f_n(t_2) - f_n(t_1)| \leq& \left| \frac{k_1(\alpha,\xi)}{k_0(\alpha,\xi)} \right| |x_0| (t_2 - t_1) \\
&+ \frac{1}{\Gamma(\alpha-1) M_{k_0}} \left| \frac{k_1(\alpha,\xi)}{k_0(\alpha,\xi)} \right| (t_2 - t_1) \int_0^{t_2}\int_0^{u} (u-\tau)^{\alpha-2} |w_n(\tau)|\, d\tau\, du \\
&+ \frac{1}{\Gamma(\alpha-1) M_{k_0}} \int_{t_1}^{t_2}\int_0^{u} (u-\tau)^{\alpha-2} |w_n(\tau)|\, d\tau\, du \\
\leq& \left| \frac{k_1(\alpha,\xi)}{k_0(\alpha,\xi)} \right| |x_0| (t_2 - t_1) \\
&+ \frac{1}{\Gamma(\alpha-1) M_{k_0}} \left| \frac{k_1(\alpha,\xi)}{k_0(\alpha,\xi)} \right| (t_2 - t_1) \int_0^{t_2}\int_0^{u} (u-\tau)^{\alpha-2} \varphi(\tau)\, d\tau\, du \\
&+ \frac{1}{\Gamma(\alpha-1) M_{k_0}} \int_{t_1}^{t_2}\int_0^{u} (u-\tau)^{\alpha-2} \varphi(\tau)\, d\tau\, du.
\end{aligned}
$$

As $t_1 \to t_2$, the right hand side of the above inequality tends to zero. Thus, $\{w_n(\tau) : n \geq 1\}$ is equicontinuous. Hence, $\{w_n(\tau) : n \geq 1\}$ is relatively compact in $C(\mathbf{J}, \mathcal{K})$.

Step 3. The graph of $\mathcal{N}$ is closed.

Let $x_n \to x_*, f_n \in \mathcal{N}(x_n)$, and $f_n \to f_*$. It must be to show that $f_* \in \mathcal{N}(x_*)$. Now, $f_n \in \mathcal{N}(x_n)$ means that there exists $w_n \in \mathcal{S}_{F,x_n}$ such that, for each $t \in \mathbf{J}$,

$$f_n(t) = \exp\left(-\int_0^t \frac{k_1(\alpha, s)}{k_0(\alpha, s)}\, ds\right) x_0 + \frac{1}{\Gamma(\alpha-1)} \int_0^t \int_0^u \exp\left(-\int_u^t \frac{k_1(\alpha, s)}{k_0(\alpha, s)}\, ds\right) \frac{(u-\tau)^{\alpha-2}}{k_0(\alpha, u)} w_n(\tau)\, d\tau\, du.$$

Consider the continuous linear operator $\Theta : L^1(\mathbf{J}, \mathbf{E}) \to C(\mathbf{J}, \mathbf{E})$,

$$\Theta(w)(t) \mapsto f_n(t) = \exp\left(-\int_0^t \frac{k_1(\alpha, s)}{k_0(\alpha, s)}\, ds\right) x_0 + \frac{1}{\Gamma(\alpha-1)} \int_0^t \int_0^u \exp\left(-\int_u^t \frac{k_1(\alpha, s)}{k_0(\alpha, s)}\, ds\right) \frac{(u-\tau)^{\alpha-2}}{k_0(\alpha, u)} w_n(\tau)\, d\tau\, du.$$

It is obvious that $\|f_n - f_*\| \to 0$ as $n \to \infty$. Therefore, in the light of Lemma 2.7, we infer that $\Theta \circ \mathcal{S}_F$ is a closed graph operator. Additionally, $f_n(t) \in \Theta(\mathcal{S}_{F,x_n})$. Since, $x_n \to x_*$, Lemma 2.7 gives

$$f_*(t) = \exp\left(-\int_0^t \frac{k_1(\alpha, s)}{k_0(\alpha, s)}\, ds\right) x_0 + \frac{1}{\Gamma(\alpha-1)} \int_0^t \int_0^u \exp\left(-\int_u^t \frac{k_1(\alpha, s)}{k_0(\alpha, s)}\, ds\right) \frac{(u-\tau)^{\alpha-2}}{k_0(\alpha, u)} w(\tau)\, d\tau\, du,$$

for some $w \in \mathcal{S}_{F,x}$.

Step 4. G is relatively compact in $C(\mathbf{J}, \mathcal{K})$.

Assume that $G \subset \overline{\mathcal{U}}$, $G \subset conv\,(\{0\} \cup \mathcal{N}(G))$, and $\overline{G} = \overline{C}$ for some countable set $C \subset G$. Using a similar approach as in Step 2, one can obtain that $\mathcal{N}(G)$ is equicontinuous. In accordance to $G \subset conv\,(\{0\} \cup \mathcal{N}(G))$, it follows that G is equicontinuous. In addition, since $C \subset G \subset conv\,(\{0\} \cup \mathcal{N}(G))$ and C is countable, then we can find a countable set $\mathbf{P} = \{f_n : n \geq 1\} \subset \mathcal{N}(G)$ with $C \subset conv\,(\{0\} \cup \mathbf{P})$. Thus, there exist $x_n \in G$ and $w_n \in \mathcal{S}_{F,x_n}$ such that

$$f_n(t) = \exp\left(-\int_0^t \frac{k_1(\alpha, s)}{k_0(\alpha, s)}\, ds\right) x_0 + \frac{1}{\Gamma(\alpha-1)} \int_0^t \int_0^u \exp\left(-\int_u^t \frac{k_1(\alpha, s)}{k_0(\alpha, s)}\, ds\right) \frac{(u-\tau)^{\alpha-2}}{k_0(\alpha, u)} w_n(\tau)\, d\tau\, du.$$

In the light of Theorem 2.11 and the fact that $G \subset \overline{C} \subset \overline{conv}\left(\{0\} \cup \mathbf{P}\right)$, we get

$$\kappa\left(G(t)\right) \leq \kappa\left(\overline{C}(t)\right) \leq \kappa\left(\mathbf{P}(t)\right) = \kappa\left(\{f_n(t) : n \geq 1\}\right).$$

By virtue of (3.8) and the fact that $w_n(\tau) \in G(\tau)$, we get

$$\begin{aligned}
&\kappa\left(G(t)\right)\\
&\leq 2\left\{\frac{1}{\Gamma(\alpha-1)}\int_0^t\int_0^u \kappa\left(\left\{\exp\left(-\int_u^t \frac{k_1(\alpha,s)}{k_0(\alpha,s)}ds\right)\frac{(u-\tau)^{\alpha-2}}{k_0(\alpha,u)}w_n(\tau) : n \geq 1\right\}\right) d\tau\, du\right\}\\
&\leq 2\left\{\frac{1}{\Gamma(\alpha-1)}\int_0^t\int_0^u \exp\left(-\int_u^t \frac{k_1(\alpha,s)}{k_0(\alpha,s)}\,ds\right)\frac{(u-\tau)^{\alpha-2}}{k_0(\alpha,u)}\kappa\left(G(\tau)\right)\, d\tau\, du\right\}\\
&\leq 2\left\{\frac{1}{\Gamma(\alpha-1)}\int_0^t\int_0^u \exp\left(-\int_u^t \frac{k_1(\alpha,s)}{k_0(\alpha,s)}\,ds\right)\frac{(u-\tau)^{\alpha-2}}{k_0(\alpha,u)}\vartheta\left(\tau,\kappa\left(G(\tau)\right)\right)\, d\tau\, du\right\}.
\end{aligned}$$

Also, the function θ given by $\theta(t) = \kappa\left(G(t)\right)$ belongs to $C(\mathbf{J}, [0, 2\varrho])$. Consequently by **(H3)**, $\theta \equiv 0$, that is $\kappa\left(G(t)\right) = 0$ for all $t \in \mathbf{J}$.

Now, by the Arzelà-Ascoli theorem, G is relatively compact in $C(\mathbf{J}, \mathcal{K})$.

Step 5. Let $f \in \mathcal{N}(x)$ with $x \in \overline{\mathcal{U}}$. Since $x(\tau) \leq \varrho$ and **(H2)**, we have $\mathcal{N}(\overline{\mathcal{U}}) \subset \overline{\mathcal{U}}$, because if it is not true, there exists a function $x \in \overline{\mathcal{U}}$ but $\|\mathcal{N}(x)\| > \varrho$ and

$$\begin{aligned}
f(t) &= \exp\left(-\int_0^t \frac{k_1(\alpha,s)}{k_0(\alpha,s)}\,ds\right) x_0\\
&\quad + \frac{1}{\Gamma(\alpha-1)}\int_0^t\int_0^u \exp\left(-\int_u^t \frac{k_1(\alpha,s)}{k_0(\alpha,s)}\,ds\right)\frac{(u-\tau)^{\alpha-2}}{k_0(\alpha,u)}w(\tau)\, d\tau\, du,
\end{aligned}$$

for some $w \in \mathcal{S}_{F,x}$. On the other hand we have

$$\begin{aligned}
\varrho < \|\mathcal{N}(x)\| &\leq \left|\exp\left(-\int_0^t \frac{k_1(\alpha,s)}{k_0(\alpha,s)}\,ds\right) x_0\right|\\
&\quad + \frac{1}{\Gamma(\alpha-1)}\int_0^t\int_0^u \left|\exp\left(-\int_u^t \frac{k_1(\alpha,s)}{k_0(\alpha,s)}\,ds\right)\right| \frac{(u-\tau)^{\alpha-2}}{|k_0(\alpha,u)|}\,|w(\tau)|\, d\tau\, du\\
&\leq |x_0| + \frac{1}{\Gamma(\alpha-1)M_{k_0}}\int_0^t\int_0^u (u-\tau)^{\alpha-2}\,|w(\tau)|\, d\tau\, du\\
&= |x_0| + \frac{1}{\Gamma(\alpha-1)M_{k_0}}\int_0^t\int_\tau^t (u-\tau)^{\alpha-2}\,|w(\tau)|\, du\, d\tau\\
&= |x_0| + \frac{1}{\Gamma(\alpha)M_{k_0}}\int_0^t (t-\tau)^{\alpha-1}\,|w(\tau)|\, d\tau
\end{aligned}$$

$$
\begin{aligned}
&\leq |x_0| + \frac{t}{\Gamma(\alpha) M_{k_0}} \int_0^t \varphi(\tau)\, d\tau \\
&\leq |x_0| + \frac{b}{\Gamma(\alpha) M_{k_0}} \int_0^b \varphi(\tau)\, d\tau.
\end{aligned}
$$

Dividing both sides by ϱ and taking the lower limit as $\varrho \to \infty$, we infer that $\frac{b}{\Gamma(\alpha) M_{k_0}} \ell \geq 1$ which contradicts (3.6). Hence $\mathcal{N}(\overline{\mathcal{U}}) \subset \overline{\mathcal{U}}$.

As a consequence of Steps 1-5 together with Theorem 2.10, we infer that $\mathcal{N}$ possesses a fixed point $x \in C(\mathbf{J}, \mathcal{K})$ which is a solution of the inclusion problem (1.1). $\square$

4. Example

Consider the fractional differential inclusion

$$
\begin{cases}
{}^{PC}_{\ 0}\mathcal{D}_t^{\frac{1}{2}} x(t) \in F\left(t, x(t)\right), & \text{a.e. on } [0,1], \\
x(0) = 0,
\end{cases}
\tag{4.1}
$$

where $\alpha = \frac{1}{2}, b = 1, x_0 = 0$, and $F : [0,1] \times \mathbb{R} \to \mathfrak{P}(\mathbb{R})$ is a multi-valued map given by

$$
x \mapsto F\left(t, x\right) = \left(e^{-|x|} + \sin t, 3 + \frac{|x|}{1+x^2} + 5t^3 \right).
$$

For $f \in F$, one has

$$
|f| = \max \left(e^{-|x|} + \sin t, 3 + \frac{|x|}{1+x^2} + 5t^3 \right) \leq 9, \quad x \in \mathbb{R}.
$$

Thus

$$
\begin{aligned}
\|F(t,x)\|_{\mathfrak{P}} &= \{|f| : f \in F(t,x)\} \\
&= \max \left(e^{-|x|} + \sin t, 3 + \frac{|x|}{1+x^2} + 5t^3 \right) \leq 9 = \varphi(t),
\end{aligned}
$$

for $t \in [0,1]$, $x \in \mathbb{R}$. Obviously, F is compact and convex valued, and it is upper semi-continuous.

Furthermore, for $(t,x) \in [0,1] \times \in \mathbb{R}$ with $|x| \leq \varrho$, one has

$$
\lim_{\varrho \to \infty} \inf \frac{\int_0^1 \varphi(t) dt}{\varrho} = 0 = \ell.
$$

Therefore, for a suitable M_{k_0}, the condition (3.6) implies that

$$
\frac{\Gamma(1/2) M_{k_0}}{b} = M_{k_0} \sqrt{\pi} > 0.
$$

Finally, we assume that there exists a Carathèodory function $\vartheta : [0,1] \times [0, 2\varrho] \to \mathbb{R}_+$ such that

$$\kappa\left(F(t,G)\right) \leq \vartheta(t, \kappa(G)),$$

a.e. $t \in [0,1]$ and each $G \subset \mathcal{K} = \{x \in \mathbb{R} : |x| \leq \varrho\}$, and the unique solution $\theta \in C([0,1],[0,2\varrho])$ of the inequality

$$\theta(t) \leq 2\left\{\frac{1}{\Gamma(\alpha-1)}\int_0^t\int_0^u \exp\left(-\int_u^t \frac{k_1(\alpha,s)}{k_0(\alpha,s)}ds\right)\frac{(u-\tau)^{\alpha-2}}{k_0(\alpha,u)}\vartheta\left(\tau,\kappa\left(G(\tau)\right)\right)d\tau du\right\}, \quad t \in \mathbf{J},$$

is $\theta \equiv 0$.

Hence all the assumptions of Theorem 3.5 hold true and we infer that the inclusion problem (4.1) possesses at least one solution on $[0,1]$.

5. Conclusions

In this paper, we extend the investigation of fractional differential inclusions to the case of hybrid Caputo-proportional fractional derivatives in Banach space. Based on the set-valued version of Mönch fixed point theorem together with the Kuratowski measure of non-compactness, the existence theorem of the solutions for the proposed inclusion problem is founded. An clarified example is suggested to understand the theoretical finding. Furthermore, the obtained results in this paper can be employed in future work in the sense of the generalized fractional derivative (GFD) definition which was recently proposed in [2, 3]. This new definition overcomes some issues associated with some conformable derivative and some other fractional derivatives.

References

[1] M.I. Abbas and S. Hristova, *Existence results of nonlinear generalized proportional fractional differential inclusions via the diagonalization technique*, AIMS Mathematics 6 (11) (2021) 12832–12844.

[2] M. Abu-Shady, Mohammed K. A. Kaabar, *A Generalized definition of the fractional derivative with applications*, Mathematical Problems in Engineering 2021, Article ID 9444803.

[3] M. Abu-Shady, M.K.A. Kaabar, *A Novel computational tool for the fractional-order special functions arising from modeling scientific pPhenomena via Abu-Shady-Kaabar fractional derivative*, Computational and Mathematical Methods in Medicine 2022, Article ID 2138775.

[4] B. Ahmad, A. Alsaedi, S.K. Ntouyas, H.H. Al-Sulami, *On neutral functional differential inclusions involving Hadamard fractional derivatives*, Mathematics 7 (11) (2019) 1–13, Article ID:1084.

[5] B. Alqahtani, S. Abbas, M. Benchohra, S.S. Alzaid, *Fractional q-difference inclusions in Banach spaces*, Mathematics 8 (91) (2020) 91, 1–12, DOI: 10.3390/math8010091.

[6] D.R. Anderson, D.J. Ulness, *Newly defined conformable derivatives*, Advances in Dynamical Systems and Applications 10 (2) (2015) 109–137.

[7] D. Baleanu, K. Diethelm, E. Scalas, J.J. Trujillo, *Fractional Calculus Models and Numerical Methods*, World Scientific: Singapore, 2012.

[8] D. Baleanu, A. Fernandez, A. Akgül, *On a fractional operator combining proportional and classical differintegrals*, Mathematics 8 (3) (2020) 360, 1–13.

[9] J. Banaś, K. Goebel, *Measures of Noncompactness in Banach Spaces*, Marcel Dekker, New York, 1980.

[10] J. Banaś, T. Zając, *On a measure of noncompactness in the space of regulated functions and its applications*, Adv. Nonlinear Anal. 8 (2019) 1099–1110.

[11] M. Benchohra, J.R. Graef, N. Guerraiche, S. Hamani, *Nonlinear boundary value problems for fractional differential inclusions with Caputo-Hadamard derivatives on the half line*, AIMS Mathematics 6 (6) (2021) 6278–6292.

[12] M. Benchohra, S. Hamani, *Nonlinear boundary value problems for differential inclusions with Caputo fractional derivative*, Topol. Meth. Nonlinear Anal. 32 (1) (2008) 115–130.

[13] M. Benchohra, N. Hamidi, J.J. Nieto, *Existence of solutions to differential inclusions with fractional order and impulses*, Electron. J. Qual. Theory Differ. Equ. 80 (2010) 1–18.

[14] M. Benchohra, J. Henderson, D. Seba, *Boundary value problems for fractional differential inclusions in Banach space*, Fract. Differ. Calc. 2 (1) (2012) 99–108.

[15] A. Das, B. Hazarika, V. Parvaneh, M. Mursaleen, *Solvability of generalized fractional order integral equations via measures of noncompactness*, Math. Sci. (2021) DOI:10.1007/s40096-020-00359-0.

[16] K. Deimling, *Multivalued Differential Equations*, Walter De Gruyter, Berlin-New York, 1992.

[17] K. Diethelm, A.D. Freed, *On the solution of nonlinear fractional-order differential equations used in the modeling of viscoelasticity*, in Scientific Computing in Chemical Engineering II-Computational Fluid Dynamics, Reaction Engineering and Molecular Properties, F. Keil, W. Mackens, H. Voss, and J. Werther, Eds., Springer-Verlag, Heidelberg, 1999, 217–224.

[18] J.R. Graef, N. Guerraiche, S. Hamani, *Boundary value problems for fractional differential inclusions with Hadamard type derivatives in Banach spaces*, Stud. Univ. Babeş-Bolyai Math. 62 (4) (2017) 427–438.

[19] B. Hazarika, R. Arab, M. Mursaleen, *Applications of measure of moncompactness and operator type contraction for existence of solution of functional integral equations*, Complex Anal. Oper. Theo. (2019), doi:10.1007/s11785-019-00933-y.

[20] H. P. Heinz, *On the behaviour of measures of noncompactness with respect to differentiation and integration of vector-valued functions*, Nonlinear Anal. 7 (1983) 1351–1371.

[21] R. Hilfer, *Applications of Fractional Calculus in Physics*, World Scientific, Singapore, 2000.

[22] M. Houas, F. Martinez, M.E. Samei, M.K.A. Kaabar, *Uniqueness and Ulam-Hyers-Rassias stability results for sequential fractional pantograph q- differential equations*, Journal of Inequalities and Applications 2022 (1) (2022) 1–24.

[23] M.K.A. Kaabar, A. Refice, M.S. Souid, F. Martinez, S. Etemad, Z. Siri, S. Rezapour, *Existence and UHR stability of solutions to the implicit nonlinear FBVP in the variable order settings*, Mathematics 9 (14) (2022) 1693, 1–17.

[24] A.A. Kilbas, H.M. Srivastava, J.J. Trujillo, *Theory and Applications of Fractional Differential Equations*, Elsevier B. V., Amsterdam, 2006.

[25] A. Lasota, Z. Opial, *An application of the Kakutani-Ky Fan theorem in the theory of ordinary differential equations*, Bull. Acad. Pol. Sci. Ser. Sci. Math. Astronom. Phys. 13 (1965) 781–786.

[26] F. Mainardi, *Fractional Calculus and Waves in Linear Viscoelasticity: An Introduction to Mathematical Models*, World Scientific Publishing Company: Singapore; Hackensack, NJ, USA, London, UK; Hong Kong, China, 2010.

[27] K.S. Miller, B. Ross, *An Introduction to the Fractional Calculus and Fractional Differential Equations*, Wiley-Interscience, John-Wiley and Sons: New York, NY, USA, 1993.

[28] A. Nouara, A. Amara, E. Kaslik, S. Etemad, S. Rezapour, F. Martinez, M.K.A. Kaabar, *A study on multiterm hybrid multi-order fractional boundary value problem coupled with its stability analysis of Ulam-Hyers type*, Advances in Difference Equations 2021 (1) (2021) 1–28.

[29] N. Nyamoradi, D. Baleanu, R. Agarwal, *On a multipoint boundary value problem for a fractional order differential inclusion on an infinite interval*, Adv. Math. Phys. 2013 (2013), Article ID:823961.

[30] I. Podlubny, *Fractional Differential Equations*, Academic Press, San Diego, 1999.

[31] A. Salem, H.M. Alshehri, L. Almaghamsi, *Measure of noncompactness for an infinite system of fractional Langevin equation in a sequence space*, Adv. Difference Equa. 132 (2021), DOI:10.1186/s13662-021-03302-2.

[32] S. Samko, A. Kilbas, O. Marichev, *Fractional Integrals and Drivatives*, Gordon and Breach Science Publishers, Longhorne, PA, 1993.

[33] H.M. Srivastava, K.M. Saad, *Some new models of the time-fractional gas dynamics equation*, Adv. Math. Models Appl. 3 (1) (2018) 5–17.

DOI: 10.7862/rf.2022.1

Mohamed I. Abbas
email: `miabbas@alexu.edu.eg`
ORCID: 0000-0002-3803-8114
Department of Mathematics and Computer Science, Faculty of Science,
Alexandria University
Alexandria 21511
EGYPT

Received 15.02.2022 *Accepted 13.09.2022*

Journal of
Mathematics
and Applications
JMA No 45, pp 21-58 (2022)

A Contribution on Real and Complex Convexity in Several Complex Variables

Jamel Abidi

ABSTRACT: Let $f, g : \mathbb{C}^n \to \mathbb{C}$ be holomorphic functions. Define $u(z,w) = |w - f(z)|^4 + |w - g(z)|^4$, $v(z,w) = |w - f(z)|^2 + |w - g(z)|^2$, for $(z,w) \in \mathbb{C}^n \times \mathbb{C}$. A comparison between the convexity of u and v is obtained under suitable conditions.
Now consider four holomorphic functions $\varphi_1, \varphi_2 : \mathbb{C}^m \to \mathbb{C}$ and $g_1, g_2 : \mathbb{C}^n \to \mathbb{C}$. We prove that $F = |\varphi_1 - g_1|^2 + |\varphi_2 - g_2|^2$ is strictly convex on $\mathbb{C}^n \times \mathbb{C}^m$ if and only if $n = m = 1$ and φ_1, φ_2, g_1, g_2 are affine functions with $(\varphi_1' g_2' - \varphi_2' g_1') \neq 0$.
Finally, it is shown that the product of four absolute values of pluriharmonic functions is plurisubharmonic if and only if the functions satisfy special conditions as well.

AMS Subject Classification: 32A10, 32A60, 32F32, 32U05, 32W50.
Keywords and Phrases: Holomorphic, convex, plurisubharmonic functions; Inequalities; Srictly; Maximum principle.

1. Introduction

Convex functions recently are studied in complex analysis because they appear in the theory of holomorphic functions, plurisubharmonic (psh) functions, currents, Lelong numbers, extension problems, holomorphic representation theory (see [2], [5], [6], [7], [8], [10], [11], [13], [14], [15], [16], [17] and [19]).
It is worth mentioning that an interesting relation between convex and plurisubharmonic functions has been obtained in [2].
Several papers appeared recently to this topic, let us mention [2], [3], [5], [6], [15], [19] and the monographs [11], [14], [19] and more recently [5].

Let $n \geq 1$. We can construct a C^∞ strictly psh function F defined on $\mathbb{C}^n \times \mathbb{C}$, such that F is not convex (and not concave) on each Euclidean not empty open ball subset in $\mathbb{C}^n \times \mathbb{C}$. For instance,

$$F(z,w) = |w - e^{\overline{z_1}}|^2 + ... + |w - e^{\overline{z_n}}|^2, \text{ for } z = (z_1, ..., z_n) \in \mathbb{C}^n, \ w \in \mathbb{C}.$$

Moreover, for the case of one complex variable, let $\lambda(z) = 2x^2 - y^2$, $z = (x + iy) \in \mathbb{C}$, $x = \mathrm{Re}(z)$. Then λ is a C^∞ strictly sh function on $\mathbb{C}$, while λ is not convex (respectively not concave) at each point of $\mathbb{C}$.
This proves that the new class of functions, consisting of convex and strictly psh functions, is well defined because we can not compare the two families (convex functions) and (convex and strictly psh functions).
Now thanks to [2], we know the holomorphic representation of each holomorphic function $f : \mathbb{C}^n \to \mathbb{C}$ under the suitable condition of the convexity of its modulus.

Let $\delta \in [1, +\infty[$. We have the following observation.
Put $K(z,w) = |w - f(z)|^\delta$ and $H(z,w) = |w - f(z)|$, for $(z,w) \in \mathbb{C}^2$, where $f : \mathbb{C} \to \mathbb{C}$ is holomorphic. Assume that K is convex on $\mathbb{C}^2$ and $\delta > 1$. Then H is convex on $\mathbb{C}^2$ and we have H^s is convex on $\mathbb{C}^2$, for each $s \in [1, +\infty[$ independently of δ and conversely.
Now let $f_1, f_2 : \mathbb{C} \to \mathbb{C}$ be two holomorphic functions and $s \in \mathbb{N}\backslash\{0\}$. Define $K_{2s}(z,w) = |w - f_1(z)|^{2s} + |w - f_2(z)|^{2s}$, for $(z,w) \in \mathbb{C}^2$. By theorem 10, we have that K_4 is convex on $\mathbb{C}^2$ implies that K_2 is convex on $\mathbb{C}^2$. But the converse is not true. For instance, let $f_1(z) = z^4$, $f_2(z) = -z^4$, $z \in \mathbb{C}$. Then K_2 is convex on $\mathbb{C}^2$. But K_4 is not convex on $\mathbb{C}^2$. This remark leads to the following problem.

Let $N \in \mathbb{N}\backslash\{0,1\}$ and $F_1, ..., F_N : \mathbb{C}^n \to \mathbb{C}$ be holomorphic functions. Define

$$\psi_\delta(z,w) = |w - F_1(z)|^\delta + ... + |w - F_N(z)|^\delta, \text{ for } (z,w) \in \mathbb{C}^n \times \mathbb{C}.$$

Suppose that ψ_δ is convex on $\mathbb{C}^n \times \mathbb{C}$.

Firstly, for the study of the convexity of ψ_δ, we observe that we study separately the following two cases.
Case 1. $\delta \in [1, +\infty[\backslash\{2\}$.
Case 2. $\delta = 2$.
Is it true that $\delta \in [1, +\infty[\backslash\{2\}$, implies that $F_1, ..., F_N$ are affine functions?
Recall that for $\delta = 2$, there exists several cases where ψ_2 is convex on $\mathbb{C}^n \times \mathbb{C}$, but $F_1, ..., F_N$ are not affine functions.
Moreover, for $N = 2$, by a limiting argument and a specific holomorphic differential equation, we prove that ψ_1 is convex on $\mathbb{C}^n \times \mathbb{C}$ if and only if F_1 and F_2 are affine functions. Indeed, ψ_{2k} is convex on $\mathbb{C}^n \times \mathbb{C}$ if and only if F_1 and F_2 are affine functions, for $k \in \mathbb{N}\backslash\{0,1\}$.

The paper is organized as follows. In section 2, we shall use an elementary holomorphic differential equation in the proofs of the following two technical questions.
Let $A_1, A_2 \in \mathbb{C}$ and $n, m \in \mathbb{N}\backslash\{0\}$. Characterize exactly all the 3 holomorphic functions $\varphi : \mathbb{C}^m \to \mathbb{C}$ and $g_1, g_2 : \mathbb{C}^n \to \mathbb{C}$ such that u is convex (respectively convex and strictly plurisubharmonic) on $\mathbb{C}^n \times \mathbb{C}^m$, where

$$u(z,w) = |A_1\varphi(w) - g_1(z)|^2 + |A_2\varphi(w) - g_2(z)|^2, \text{ for } (z,w) \in \mathbb{C}^n \times \mathbb{C}^m.$$

In this case find the expressions of φ, g_1 and g_2.
Moreover, find all the three holomorphic functions $\varphi : \mathbb{C}^m \to \mathbb{C}$ and $f_1, f_2 : \mathbb{C}^n \to \mathbb{C}$ such that v is convex and strictly psh on $\mathbb{C}^n \times \mathbb{C}^m$, where

$$v(z,w) = |A_1\varphi(w) - \overline{f_1}(z)|^2 + |A_2\varphi(w) - \overline{f_2}(z)|^2,\ (z,w) \in \mathbb{C}^n \times \mathbb{C}^m.$$

We prove that we have a great differences between the 2 classes of functions defined similar as u and v.

Now let $k_1, k_2 : G \to \mathbb{C}^t$ be two holomorphic functions. Then the functions $\parallel k_1 + \overline{k_2} \parallel^2$ and $(\parallel k_1 + \lambda \parallel^2 + \parallel k_2 + \delta \parallel^2)$ have the same hermitian Levi form on G, where G is a domain of $\mathbb{C}^s$, $\lambda, \delta \in \mathbb{C}^t$ and $s, t \in \mathbb{N}\backslash\{0\}$.
For the applications, we can see the proof of theorem 4, corollary 1, theorem 5 and others.

In section 3, we consider the following problems.
Problem 1. Let $n, m \geq 1$. Find all the 4 holomorphic functions $\varphi_1, \varphi_2 : \mathbb{C}^m \to \mathbb{C}$ and $g_1, g_2 : \mathbb{C}^n \to \mathbb{C}$ such that $\psi = |\varphi_1 - g_1|^2 + |\varphi_2 - g_2|^2$ is strictly convex on $\mathbb{C}^n \times \mathbb{C}^m$.
Problem 2. Characterize all the holomorphic functions $\varphi_1, \varphi_2 : \mathbb{C}^m \to \mathbb{C}$ and $g_1, g_2 : \mathbb{C}^n \to \mathbb{C}$ such that $\psi = |\varphi_1 - g_1|^2 + |\varphi_2 - g_2|^2$ is convex and strictly psh (respectively convex) on $\mathbb{C}^n \times \mathbb{C}^m$.

Before stating it, we can study the analysis question. Find all the holomorphic functions $\varphi_1, \varphi_2, \psi_1, \psi_2 : \mathbb{C}^m \to \mathbb{C}$ and $f_1, f_2, g_1, g_2 : \mathbb{C}^n \to \mathbb{C}$, such that u_1 and u_2 are convex and $u = (u_1 + u_2)$ is strictly psh on $\mathbb{C}^n \times \mathbb{C}^m$. Where $u_1(z,w) = |\varphi_1(w) - f_1(z)|^2 + |\varphi_2(w) - f_2(z)|^2$, $u_2(z,w) = |\psi_1(w) - g_1(z)|^2 + |\psi_2(w) - g_2(z)|^2$, for $(z,w) \in \mathbb{C}^n \times \mathbb{C}^m$.

In section 4, we use an algebraic method to mainly focus on properties of the new structure (convex and strictly psh) and their relations with the holomorphic representation theory.

In section 5 we study the product of several absolute values of pluriharmonic (prh) functions and some auxiliary results are proved.
Let U be a domain of $\mathbb{R}^d$, $(d \geq 2)$. Put sh(U) the set of all subharmonic functions on U.
For $f : U \to \mathbb{C}$ be a function, $|f|$ is the modulus of f. For $N \geq 1$ and $h = (h_1, ..., h_N)$, where $h_1, ..., h_N : U \to \mathbb{C}$, $\parallel h \parallel = (|h_1|^2 + ... + |h_N|^2)^{\frac{1}{2}}$.
Let $g : D \to \mathbb{C}$ be an analytic function, D is a domain of $\mathbb{C}$. We denote $\frac{\partial^m g}{\partial z^m}$ the holomorphic derivative of g of order m, for all $m \in \mathbb{N}\backslash\{0\}$.
If $\xi = (\xi_1, ..., \xi_n) \in \mathbb{C}^n$, and $z = (z_1, \ldots, z_n \in \mathbb{C}^n$ we write $< z/\xi >= z_1\overline{\xi_1} + ... + z_n\overline{\xi_n}$ and $B(\xi, r) = \{\zeta \in \mathbb{C}^n / \parallel \zeta - \xi \parallel < r\}$ for $r > 0$, where $\sqrt{< \xi/\xi >} = \parallel \xi \parallel$ is the Euclidean norm of ξ. The Lebesgue measure on $\mathbb{C}^n$ is denoted by m_{2n} and $C^k(U) = \{\varphi : U \to \mathbb{C} \ / \ \varphi \text{ is a function of class } C^k \text{ on } U\}$, $k \in \mathbb{N} \cup \{\infty\}\backslash\{0\}$.
Let D be a domain of $\mathbb{C}^n$, $(n \geq 1)$. An usual psh(D) and prh(D) are respectively the classes of plurisubharmonic and pluriharmonic functions on D. For all $a \in \mathbb{C}, |a|$ is the modulus of a, Re(a) is the real part of a and $D(a, r) = \{z \in \mathbb{C} \ / \ |z - a| < r\}$ for $r > 0$.

For the study of properties and extension problems of analytic and plurisubharmonic functions we cite the references [1], [6], [7], [8], [9], [10], [12], [13], [15], [16] and [17]. For the study of convex functions in complex convex domains, we cite [5], [11],

[14], [2] and [19].
For the theory of $n-$ subharmonic functions we cite [18].

2. A family of analytic functions and the holomorphic representation theory

We have

Lemma 1. *Let $g = (g_1, ..., g_N), f = (f_1, ..., f_N) : D \to \mathbb{C}^N$ be two holomorphic functions, $N \geq 1$, D is a domain of $\mathbb{C}^n$, $n \geq 1$ and $a, b \in \mathbb{C}^N$. Then*
$\parallel f + \overline{g} \parallel^2$ and $(\parallel f + a \parallel^2 + \parallel g + b \parallel^2)$ have the same hermitian Levi form on D.
On the other hand, let $u : D \to \mathbb{R}$ be a function of class C^2. Define $u_1 = (u+ \parallel f + \overline{g} \parallel^2)$, $u_2 = (u+ \parallel f + a \parallel^2 + \parallel g + b \parallel^2)$.
Then u_1 and u_2 are functions of class C^2 on D and we have the assertion.
The function u_1 is strictly psh on D if and only if u_2 is strictly psh on D.
(Observe that if $N < n$, then $\parallel g \parallel^2$ is not strictly psh at each point of D).

Proof. We have $\parallel f + \overline{g} \parallel^2 = |f_1 + \overline{g_1}|^2 + ... + |f_N + \overline{g_N}|^2 = |f_1|^2 + |g_1|^2 + ... + |f_N|^2 + |g_N|^2 + \sum_{j=1}^{N}(g_j f_j + \overline{g_j}\overline{f_j}) = \parallel g \parallel^2 + \parallel f \parallel^2 + \sum_{j=1}^{N}(g_j f_j + \overline{g_j}\overline{f_j})$.

Since $(g_j f_j + \overline{g_j}\overline{f_j})$ is prh on D, then $\sum_{j=1}^{N}(g_j f_j + \overline{g_j}\overline{f_j})$ is prh on D.

Consequently, $\parallel f + \overline{g} \parallel^2$ and $(\parallel f + a \parallel^2 + \parallel g + b \parallel^2)$ have the same hermitian Levi form on D.

By [4], we have

Theorem 1. *Let $\varphi : \mathbb{C}^m \to \mathbb{C}$ be a holomorphic nonconstant function, $m \geq 1$. Given $A_1, A_2 \in \mathbb{C} \backslash \{0\}$ and $n \geq 1$.*
The following conditions are equivalent
(I) There exists 2 holomorphic functions $g_1, g_2 : \mathbb{C}^n \to \mathbb{C}$ such that u is convex on $\mathbb{C}^n \times \mathbb{C}^m$, $u(z, w) = |A_1\varphi(w) - g_1(z)|^2 + |A_2\varphi(w) - g_2(z)|^2$, $(z, w) \in \mathbb{C}^n \times \mathbb{C}^m$;
(II) There exists $c \in \mathbb{C}$ such that $|\varphi + c|^2$ is convex on $\mathbb{C}^m$.

Now in all of this section, $(A_1, A_2) \in \mathbb{C}^2$. Let $\varphi : \mathbb{C}^m \to \mathbb{C}$ be a holomorphic nonconstant function, $m \geq 1$. Let $g_1, g_2 : \mathbb{C}^n \to \mathbb{C}$ be 2 holomorphic functions, $n \geq 1$. Define $u(z, w) = |A_1\varphi(w) - g_1(z)|^2 + |A_2\varphi(w) - g_2(z)|^2$, $u_1(z, w) = |A_1\varphi(w) - \overline{g_1}(z)|^2 + |A_2\varphi(w) - \overline{g_2}(z)|^2$, $u_2 = u + u_1$, for $(z, w) \in \mathbb{C}^n \times \mathbb{C}^m$. $v(z, w) = |A_1\overline{\varphi}(w) - g_1(z)|^2 + |A_2\overline{\varphi}(w) - g_2(z)|^2$, $v_1(z, w) = |A_1\overline{\varphi}(w) - \overline{g_1}(z)|^2 + |A_2\overline{\varphi}(w) - \overline{g_2}(z)|^2$ and $v_2 = v + v_1$, $(z, w) \in \mathbb{C}^n \times \mathbb{C}^m$. We have

Theorem 2. *Assume that $(A_1, A_2) \in \mathbb{C}^2 \backslash \{0\}$. The following conditions are equivalent*
(I) u is convex on $\mathbb{C}^n \times \mathbb{C}^m$;
(II) φ is an affine function on $\mathbb{C}^m$, or φ is not affine and there exists $c \in \mathbb{C}$ such that $|\varphi + c|^2$ is convex on $\mathbb{C}^m$ and we have the following cases.
Case 1. The function φ is affine on $\mathbb{C}^m$.

Then we have the representation

$$\begin{cases} g_1(z) = A_1(< z/\lambda_1 > +\mu_1) + \overline{A_2}\varphi_1(z) \\ g_2(z) = A_2(< z/\lambda_1 > +\mu_1) - \overline{A_1}\varphi_1(z) \end{cases}$$

for each $z \in \mathbb{C}^n$, *where* $\lambda_1 \in \mathbb{C}^n$, $\mu_1 \in \mathbb{C}$, $\varphi_1 : \mathbb{C}^n \to \mathbb{C}$ *is analytic,* $|\varphi_1|^2$ *is convex on* $\mathbb{C}^n$.
Case 2. φ *is not affine on* $\mathbb{C}^n$.
In this case there exists $c \in \mathbb{C}$ *such that* $|\varphi + c|^2$ *is convex on* $\mathbb{C}^m$. *Then we have the representation*

$$\begin{cases} g_1(z) = A_1c + \overline{A_2}\varphi_1(z) \\ g_2(z) = A_2c - \overline{A_1}\varphi_1(z) \end{cases}$$

for every $z \in \mathbb{C}^n$, *where* $\varphi_1 : \mathbb{C}^n \to \mathbb{C}$ *is analytic,* $|\varphi_1|^2$ *is convex on* $\mathbb{C}^n$.

We can discuss the cases $(A_1, A_2 \in \mathbb{C}\backslash\{0\})$, or $(A_1 \in \mathbb{C}\backslash\{0\}, A_2 = 0)$, or $(A_1 = 0, A_2 \in \mathbb{C}\backslash\{0\})$.
This theorem motivates the following questions. Find all the holomorphic representation of the analytic functions $f_1, f_2, f_3 : \mathbb{C}^n \to \mathbb{C}$, such that ψ is convex on $\mathbb{C}^n \times \mathbb{C}$. $\psi(z, w) = |B_1w - f_1(z)|^2 + |B_2w - f_2(z)|^2 + |B_3w - f_3(z)|^2$, for $(z, w) \in \mathbb{C}^n \times \mathbb{C}$, where $(B_1, B_2, B_3) \in \mathbb{C}^3\backslash\{0\}$.
Indeed, for instance, in harmonic analysis and convex analysis, actually the following question appeared naturally.
Find all the representation of the harmonic functions $F_1, F_2, F_3 : \mathbb{C} \to \mathbb{C}$, such that ψ_1 is convex and strictly $2-$sh on $\mathbb{C}^2$. Where $\psi_1(z, w) = |w - F_1(z)|^2 + |w - F_2(z)|^2 + |w - F_3(z)|^2$, $(z, w) \in \mathbb{C}^2$. (We study here functions on harmonic representation theory). Define $\psi_0(z, w) = |w - F_1(z)|^2 + |w - F_2(z)|^2$, for $(z, w) \in \mathbb{C}^2$. If we choose F_3 is affine on $\mathbb{C}$ and ψ_0 is convex and strictly $2-$ sh on $\mathbb{C}^2$, then we have a family of harmonic functions which satisfy the above condition.
The proof of this theorem is obvious and analogous to the proof of the following.

Theorem 3. *The following conditions are equivalent*
(I) u *is convex and strictly psh on* $\mathbb{C}^n \times \mathbb{C}^m$;
(II) $(A_1, A_2) \in \mathbb{C}^2\backslash\{0\}$, $n = m = 1$, *there exists* $c \in \mathbb{C}$ *such that* $|\varphi + c|^2$ *is convex on* $\mathbb{C}$ *and we have the following cases.*
Case 1. $A_1A_2 \neq 0$. *Then*

$$\begin{cases} g_1(z) + A_1c = A_1(az + b) + \overline{A_2}\psi(z) \\ g_2(z) + A_2c = A_2(az + b) - \overline{A_1}\psi(z) \end{cases}$$

for each $z \in \mathbb{C}$, *where* $a, b \in \mathbb{C}$, $\psi : \mathbb{C} \to \mathbb{C}$ *is holomorphic,* $|\psi|$ *is convex with* $|\psi'| > 0$ *and* $|\varphi'| > 0$ *on* $\mathbb{C}$.
Case 2. $A_1 \neq 0$ *and* $A_2 = 0$.
If φ *is affine and nonconstant on* $\mathbb{C}$. *Then we have the representation*

$$\begin{cases} g_1(z) = A_1(\lambda z + \mu) \\ g_2(z) = -\overline{A_1}\varphi_2(z) \end{cases}$$

for each $z \in \mathbb{C}$, *where* $\lambda \in \mathbb{C}\backslash\{0\}$, $\mu \in \mathbb{C}$, $\varphi_2 : \mathbb{C} \to \mathbb{C}$ *is analytic,* $|\varphi_2|^2$ *is convex and strictly subharmonic (sh) on* $\mathbb{C}$.
If φ *is not affine on* $\mathbb{C}$. *Then we have the representation*

$$\begin{cases} g_1(z) = -A_1 c \\ g_2(z) = -\overline{A_1}\varphi_3(z) \end{cases}$$

for every $z \in \mathbb{C}$, *where* $\varphi_3 : \mathbb{C} \to \mathbb{C}$ *is analytic,* $|\varphi_3|^2$ *is convex and strictly subharmonic on* $\mathbb{C}$. *In this situation we have* $\varphi(w) = e^{(aw+b)} - c$, *for each* $w \in \mathbb{C}$, *with* $a \in \mathbb{C}\backslash\{0\}$ *and* $b \in \mathbb{C}$.
Case 3. $A_1 = 0$ *and* $A_2 \neq 0$. *(Obviously analogous to case 2).*

Proof. (I) implies (II). We choose the following proof which have technical applications in the case when we study the convexity of the function F, $F(z,w) = |w - \psi_1(z)|^{2N} + |w - \psi_2(z)|^{2N}$, $N \in \mathbb{N}$, $N \geq 2$, $(z,w) \in \mathbb{C}^n \times \mathbb{C}$, $\psi_1, \psi_2 : \mathbb{C}^n \to \mathbb{C}$ be two holomorphic functions. In this situation we prove that ψ_1 and ψ_2 have analytic representations using the holomorphic differential equation $k''(k+\lambda) = \gamma(k')^2$, where $k : \mathbb{C} \to \mathbb{C}$ is a holomorphic function and $\lambda, \gamma \in \mathbb{C}$.
If $(A_1, A_2) = (0,0)$, then u is independent of w. Thus u is not strictly psh on $\mathbb{C}^n \times \mathbb{C}^m$. A contradiction.
The case where $A_1 \neq 0$ and $A_2 = 0$.
Since $u(0,.)$ is strictly psh on $\mathbb{C}^m$. Then the function $|A_1\varphi - g_1(0)|^2$ is strictly psh on $\mathbb{C}^m$. Thus by lemma 1, $m = 1$. Since $u(.,0)$ is convex on $\mathbb{C}$, then $|\varphi - \frac{g_1(0)}{A_1}|^2$ is convex and strictly sh on $\mathbb{C}$. Put $c = -\frac{g_1(0)}{A_1}$. Now $|\varphi + c|^2$ is convex and strictly sh on $\mathbb{C}$, therefore, by Abidi [2], we have
$\varphi(w) = aw + b$, for all $w \in \mathbb{C}$, where $a \in \mathbb{C}\backslash\{0\}$ and $b \in \mathbb{C}$, or
$\varphi(w) = e^{(a_1 w + b_1)} - c$, for all $w \in \mathbb{C}$, with $a_1 \in \mathbb{C}\backslash\{0\}$ and $b_1 \in \mathbb{C}$.
If $\varphi(w) = aw + b$, $\forall w \in \mathbb{C}$.
Then for each fixed $w_0 \in \mathbb{C}$, the function $u(., w_0)$ is convex on $\mathbb{C}^n$.
Therefore,

$$|-\sum_{j,k=1}^{n} \frac{\partial^2 g_1}{\partial z_j \partial z_k}(z)[\overline{A_1}(\overline{aw_0 + b}) - \overline{g_1}(z)]\alpha_j\alpha_k + \sum_{j,k=1}^{n} \frac{\partial^2 g_2}{\partial z_j \partial z_k}(z)\overline{g_2}(z)\alpha_j\alpha_k|$$
$$\leq |\sum_{j=1}^{n} \frac{\partial g_1}{\partial z_j}(z)\alpha_j|^2 + |\sum_{j=1}^{n} \frac{\partial g_2}{\partial z_j}(z)\alpha_j|^2,$$

for each $z \in \mathbb{C}^n$, $w_0 \in \mathbb{C}$ and $\alpha = (\alpha_1, ..., \alpha_n) \in \mathbb{C}^n$.
Since the right hand side of the above inequality is independent of $w_0 \in \mathbb{C}$, it follows that for every fixed $z \in \mathbb{C}^n$,

$$\sum_{j,k=1}^{n} \frac{\partial^2 g_1}{\partial z_j \partial z_k}(z)\alpha_j\alpha_k = 0, \text{ for all } \alpha = (\alpha_1, ..., \alpha_n) \in \mathbb{C}^n.$$

Therefore g_1 is affine on $\mathbb{C}^n$.
Put $g_1(z) = A_1(< z/\gamma > +\delta)$, for $z \in \mathbb{C}^n$, where $\gamma \in \mathbb{C}^n$ and $\delta \in \mathbb{C}$.

Let $T:\mathbb{C}^n\times\mathbb{C}\to\mathbb{C}^n\times\mathbb{C}$, $T(z,w)=(z,w+\frac{g_1(z)}{A_1a}-\frac{\delta}{a})$, for $(z,w)\in\mathbb{C}^n\times\mathbb{C}$.
Note that T is a $\mathbb{C}$ linear bijective transformation on $\mathbb{C}^n\times\mathbb{C}$.
Since u is convex and strictly psh on $\mathbb{C}^n\times\mathbb{C}$, then ψ is convex and strictly psh on $\mathbb{C}^n\times\mathbb{C}$, where $\psi(z,w)=u\,\mathrm{o}\,T(z,w)=|A_1(aw+b-\delta)|^2+|g_2(z)|^2$, for every $(z,w)\in\mathbb{C}^n\times\mathbb{C}$.
But ψ is convex and strictly psh on $\mathbb{C}^n\times\mathbb{C}$, then $|g_2|^2$ is convex and strictly psh on $\mathbb{C}^n$. Thus $n=1$.
Put $g_2(z)=-\overline{A_1}\varphi_2(z)$, for $z\in\mathbb{C}$ (φ_2 is analytic on $\mathbb{C}$). Thus $|\varphi_2|^2$ is convex and strictly sh on $\mathbb{C}$.
(II) implies (I). Obvious.

Question. Let $B_1,B_2\in\mathbb{C}\backslash\{0\}$. For $f_1,f_2:\mathbb{C}^n\to\mathbb{C}$, define $\psi(z,w)=|B_1w-f_1(z)|^2+|B_2w-f_2(z)|^2$, $(z,w)\in\mathbb{C}^n\times\mathbb{C}$. Find all the pluriharmonic (respectively $n-$ harmonic) functions $f_1,f_2:\mathbb{C}^n\to\mathbb{C}$, such that ψ is convex (respectively convex and strictly $n-$ subharmonic) on $\mathbb{C}^n\times\mathbb{C}$.

Theorem 4. *The following conditions are equivalent*
(I) u_1 is convex and strictly psh on $\mathbb{C}^n\times\mathbb{C}^m$;
(II) $m=1$, $n\in\{1,2\}$, $(A_1,A_2)\in\mathbb{C}^2\backslash\{0\}$, there exists $c\in\mathbb{C}$ such that $|\varphi+c|^2$ is convex on $\mathbb{C}$ and we have the following cases.
Case 1. For all $w\in\mathbb{C}$, $\varphi(w)=aw+b$, where $a\in\mathbb{C}\backslash\{0\}$ and $b\in\mathbb{C}$.
We have the representation

$$\begin{cases} g_1(z)=\overline{A_1}(<z/\lambda_1>+\mu_1)+A_2\varphi_1(z)\\ g_2(z)=\overline{A_2}(<z/\lambda_1>+\mu_1)-A_1\varphi_1(z)\end{cases}$$

for each $z\in\mathbb{C}^n$, where $\lambda_1\in\mathbb{C}^n$, $\mu_1\in\mathbb{C}$, $\varphi_1:\mathbb{C}^n\to\mathbb{C}$ is analytic, $|\varphi_1|^2$ is convex on $\mathbb{C}^n$, such that
$(n=1,\lambda_1\neq 0)$, *or*
$(n=1,\lambda_1=0,\frac{\partial\varphi_1}{\partial z}(z)\neq 0,$ for each $z\in\mathbb{C})$, *or*
$n=2,(\lambda_1,(\overline{\frac{\partial\varphi_1}{\partial z_1}}(z),\overline{\frac{\partial\varphi_1}{\partial z_2}}(z)))$ is a basis of the complex vector space $\mathbb{C}^2$,
for each $z=(z_1,z_2)\in\mathbb{C}^2$.
Case 2. For every $w\in\mathbb{C}$, $\varphi(w)=e^{(aw+b)}-c$, where $a\in\mathbb{C}\backslash\{0\}$ and $b\in\mathbb{C}$.
Then $n=1$ and we have the representation

$$\begin{cases} g_1(z)=-\overline{A_1}\overline{c}+A_2\psi_1(z)\\ g_2(z)=-\overline{A_2}\overline{c}-A_1\psi_1(z)\end{cases}$$

for each $z\in\mathbb{C}$, where $\psi_1:\mathbb{C}\to\mathbb{C}$ is analytic, $|\psi_1|^2$ is convex and strictly sh on $\mathbb{C}$.

The proof follows from the above 3 theorems and lemma 1.
We have

Corollary 1. *The following conditions are equivalent*
(I) u is convex on $\mathbb{C}^n\times\mathbb{C}^m$ and u_2 is strictly psh on $\mathbb{C}^n\times\mathbb{C}^m$;
(II) u is convex on $\mathbb{C}^n\times\mathbb{C}^m$ and u_1 is strictly psh on $\mathbb{C}^n\times\mathbb{C}^m$;
(III) $(A_1,A_2)\in\mathbb{C}^2\backslash\{0\}$, $m=1$, $n\in\{1,2\}$, there exists $c\in\mathbb{C}$ such that $|\varphi+c|^2$ is convex and strictly sh on $\mathbb{C}$ and we have the following 2 cases.

Case 1. For all $w \in \mathbb{C}$ $\varphi(w) = aw + b$, $(a \in \mathbb{C}\backslash\{0\}, b \in \mathbb{C})$.
Then we have the holomorphic representation

$$\begin{cases} g_1(z) = A_1(< z/\lambda_1 > +\mu_1) + \overline{A_2}\varphi_1(z) \\ g_2(z) = A_2(< z/\lambda_1 > +\mu_1) - \overline{A_1}\varphi_1(z) \end{cases}$$

for each $z \in \mathbb{C}^n$, *where* $\lambda_1 \in \mathbb{C}^n$, $\mu_1 \in \mathbb{C}$, $\varphi_1 : \mathbb{C}^n \to \mathbb{C}$ *is analytic,* $|\varphi_1|^2$ *is convex on* $\mathbb{C}^n$, *such that*
$(n = 1, \lambda_1 \neq 0)$, *or* $(n = 1, \lambda_1 = 0, \frac{\partial \varphi_1}{\partial z}(z) \neq 0, \text{ for each } z \in \mathbb{C})$, *or*
$(n = 2 \text{ and } (\lambda_1, (\overline{\frac{\partial \varphi_1}{\partial z_1}}(z), \overline{\frac{\partial \varphi_1}{\partial z_2}}(z))) \text{ is a basis of the complex vector space } \mathbb{C}^2,$
$\text{for every } z = (z_1, z_2) \in \mathbb{C}^2)$.
Case 2. For all $w \in \mathbb{C}$, $\varphi(w) = e^{(aw+b)} - c$, *where* $a \in \mathbb{C}\backslash\{0\}$ *and* $b \in \mathbb{C}$.
Then $n = 1$ *and we have the holomorphic representation*

$$\begin{cases} g_1(z) = -A_1 c + \overline{A_2}\psi_1(z) \\ g_2(z) = -A_2 c - \overline{A_1}\psi_1(z) \end{cases}$$

for every $z \in \mathbb{C}$, *where* $\psi_1 : \mathbb{C} \to \mathbb{C}$ *is analytic,* $|\psi_1|^2$ *is convex and strictly sh on* $\mathbb{C}$.

Proof. (I) implies (III). Note that u, u_1 and u_2 are functions of class C^∞ on $\mathbb{C}^n \times \mathbb{C}^m$. We have
u_2 is strictly psh on $\mathbb{C}^n \times \mathbb{C}^m$ if and only if u_1 is strictly psh on $\mathbb{C}^n \times \mathbb{C}^m$.
Assume that $(A_1, A_2) = (0, 0)$. Then u_1 is independent of $w \in \mathbb{C}^m$ and u_1 is strictly psh on $\mathbb{C}^n \times \mathbb{C}^m$. A contradiction.
Consequently, $(A_1, A_2) \in \mathbb{C}^2\backslash\{0\}$.
Define $u_3(z, w) = (|A_1|^2 + |A_2|^2)|\varphi(w)|^2 + |g_1(z)|^2 + |g_2(z)|^2$, $(z, w) \in \mathbb{C}^n \times \mathbb{C}^m$.
Then u_3 is a function of class C^∞ on $\mathbb{C}^n \times \mathbb{C}^m$. But u_1 is strictly psh on $\mathbb{C}^n \times \mathbb{C}^m$ if and only if u_3 is strictly psh on $\mathbb{C}^n \times \mathbb{C}^m$.
By lemma 1, we have $m = 1$ and $n \leq 2$.

Now $u(0, .)$ is convex on $\mathbb{C}$ and $u_3(0, .)$ is strictly sh on $\mathbb{C}$. In fact $(|A_1\varphi - g_1(0)|^2 + |A_2\varphi - g_2(0)|^2)$ is convex on $\mathbb{C}$ and $((|A_1|^2 + |A_2|^2)|\varphi|^2 + |g_1(0)|^2 + |g_2(0)|^2)$ is strictly sh on $\mathbb{C}$. Then there exists $c \in \mathbb{C}$ such that $|\varphi + c|^2$ is convex on $\mathbb{C}$ and $|\varphi|^2$ is strictly sh on $\mathbb{C}$. Which yields $|\varphi + c|^2$ is convex and strictly sh on $\mathbb{C}$.
By Abidi [2], using the holomorphic differential equation $k''(k + c) = \gamma(k')^2$ ($k : \mathbb{C} \to \mathbb{C}$ be a holomorphic function , $\gamma, c \in \mathbb{C}$), we have
$\varphi(w) = aw + b$, for all $w \in \mathbb{C}$, where $a \in \mathbb{C}\backslash\{0\}$ and $b \in \mathbb{C}$, or
$\varphi(w) = e^{(a_1 w + b_1)} - c$, for all $w \in \mathbb{C}$, with $a_1 \in \mathbb{C}\backslash\{0\}$ and $b_1 \in \mathbb{C}$.
The rest of the proof is now obvious.

Theorem 5. *The following conditions are equivalent*
(I) v *is convex and strictly psh on* $\mathbb{C}^n \times \mathbb{C}^m$;
(II) $m = 1$, $n \in \{1, 2\}$, $(A_1, A_2) \in \mathbb{C}^2\backslash\{0\}$, *there exists* $c \in \mathbb{C}$ *such that* $|\varphi + c|^2$ *is convex and strictly sh on* $\mathbb{C}$ *and we have the following 2 cases.*
Case 1. For all $w \in \mathbb{C}$, $\varphi(w) = aw + b$, $(a \in \mathbb{C}\backslash\{0\}, b \in \mathbb{C})$.
Then we have the representation

$$\begin{cases} g_1(z) = \overline{A_1}(< z/\lambda > +\mu) + A_2\varphi_1(z) \\ g_2(z) = \overline{A_2}(< z/\lambda > +\mu) - A_1\varphi_1(z) \end{cases}$$

for each $z \in \mathbb{C}^n$, *where* $\lambda \in \mathbb{C}^n$, $\mu \in \mathbb{C}$, $\varphi_1 : \mathbb{C}^n \to \mathbb{C}$ *is analytic,* $|\varphi_1|^2$ *is convex on* $\mathbb{C}^n$, *such that*
$(n = 1, \lambda \neq 0)$, *or* $(n = 1, \lambda = 0, \frac{\partial \varphi_1}{\partial z}(z) \neq 0,$ for every $z \in \mathbb{C})$, *or*
$(n = 2,$ and $(\lambda, (\frac{\partial \varphi_1}{\partial z_1}(z), \frac{\partial \varphi_1}{\partial z_2}(z)))$ is a basis of the complex vector space $\mathbb{C}^2$,
for any $z = (z_1, z_2) \in \mathbb{C}^2)$.
Case 2. For each $w \in \mathbb{C}$, $\varphi(w) = e^{(aw+b)} - c$, $(a \in \mathbb{C} \backslash \{0\}$ and $b \in \mathbb{C})$.
Then $n = 1$ *and we have the representation*

$$\begin{cases} g_1(z) = -\overline{A_1}\overline{c} + A_2\psi_1(z) \\ g_2(z) = -\overline{A_2}\overline{c} - A_1\psi_1(z) \end{cases}$$

for every $z \in \mathbb{C}$, *where* $\psi_1 : \mathbb{C} \to \mathbb{C}$ *is analytic,* $|\psi_1|^2$ *is convex and strictly sh on* $\mathbb{C}$.

Moreover, we can consider the function v_2 for a study. According to lemma 1, we obtain several holomorphic representations of g_1 and g_2 from the assumptions v and v_1 are convex on $\mathbb{C}^n \times \mathbb{C}^m$ and $v_2 = (v + v_1)$ is strictly psh on $\mathbb{C}^n \times \mathbb{C}^m$.

3. Some study in the theory of convex and strictly psh functions

3.1. The analysis of strictly convex functions

Put $u(z, w) = |\varphi_1(w) - g_1(z)|^2 + |\varphi_2(w) - g_2(z)|^2$, $\varphi_1, \varphi_2 : \mathbb{C}^m \to \mathbb{C}$ and $g_1, g_2 : \mathbb{C}^n \to \mathbb{C}$ be four holomorphic functions, $(z, w) \in \mathbb{C}^n \times \mathbb{C}^m$.
Recall that, for two holomorphic functions $\varphi : \mathbb{C}^m \to \mathbb{C}$ and $g : \mathbb{C}^n \to \mathbb{C}$, if we denote $\psi(z, w) = |\varphi(w) - g(z)|^2$, for $(z, w) \in \mathbb{C}^n \times \mathbb{C}^m$. ψ is not strictly convex at each point of $\mathbb{C}^n \times \mathbb{C}^m$ (this is the case of one absolute value of a holomorphic function). But, if we consider the sum of two absolute values of holomorphic functions, there exists several cases where ψ_1 is strictly convex on $\mathbb{C}^2$. For example

$$\psi_1(z, w) = |f_1(w) - k_1(z)|^2 + |f_2(w) - k_2(z)|^2$$

for $(z, w) \in \mathbb{C}^2$ and $f_1(w) = w$, $f_2(w) = 2w + 1$, $k_1(z) = 2z$, $k_2(z) = 0$.
Before the two above technical remarks, we pose the following question.
Question. Characterize all the holomorphic functions φ_1, φ_2, g_1, g_2 such that u is strictly convex on $\mathbb{C}^n \times \mathbb{C}^m$ (we prove that $n = m = 1$).

Remark 1. Let $F_1(z) = z^2$, $F_2(z) = -z^2$, $F_3(z) = z$, $K_1(w) = K_2(w) = K_3(w) = w$, $(z, w) \in \mathbb{C}^2$. $F_1, F_2, F_3, K_1, K_2, K_3$ are holomorphic functions on $\mathbb{C}$. Put $u(z, w) = |K_1(w) - F_1(z)|^2 + |K_2(w) - F_2(z)|^2 + |K_3(w) - F_3(z)|^2$. Observe that u is strictly convex on $\mathbb{C}^2$, but F_1 and F_2 are not affine functions.

We begin by

Lemma 2. *Let* $f_1, f_2 : \mathbb{C}^N \to \mathbb{C}$ *be two holomorphic functions,* $N \geq 1$. *Put* $v = |f_1|^2 + |f_2|^2$. *We have*
If v *is strictly psh on* $\mathbb{C}^N$, *then* $N \leq 2$.

Using the holomorphic differential equation $k''(k+c) = \gamma(k')^2$, for $k : \mathbb{C} \to \mathbb{C}$ be a holomorphic function and $(\gamma, c) \in \mathbb{C}^2$, we have

Lemma 3. *Let $g_1, g_2 : \mathbb{C}^n \to \mathbb{C}$ and $\varphi_2 : \mathbb{C}^m \to \mathbb{C}$ be three holomorphic functions and $a \in \mathbb{C}$.*
Put $u(z,w) = |g_1(z) - a|^2 + |\varphi_2(w) - g_2(z)|^2$, for $(z,w) \in \mathbb{C}^n \times \mathbb{C}^m$.
Then u is strictly convex on $\mathbb{C}^n \times \mathbb{C}^m$ if and only if $n = m = 1$, g_1 is affine nonconstant, g_2 is affine and φ_2 is affine nonconstant on $\mathbb{C}$.

Proof. Assume that u is strictly convex on $\mathbb{C}^n \times \mathbb{C}^m$. By lemma 2, it follows that $n = m = 1$. We have

$$|\varphi_2''(w)(\overline{\varphi_2}(w) - \overline{g_2}(z))| < |\varphi_2'(w)|^2$$

for each $w \in \mathbb{C}$ and for every fixed $z \in \mathbb{C}$.
Put $\psi_2(w) = |\varphi_2(w) - g_2(z)|^2$, for $w \in \mathbb{C}$. By Abidi [2], for each fixed $z \in \mathbb{C}$, the function ψ_2 is strictly convex in $\mathbb{C}$. Then φ_2 is affine nonconstant on $\mathbb{C}$, (see [2], [3]).
Now we have the inequality

$$|g_2''(z)(\overline{g_2}(z) - \overline{\varphi_2}(w)) + g_1''(z)\overline{g_1}(z)| < |g_1'(z)|^2 + |g_2'(z)|^2$$

for each $(z,w) \in \mathbb{C}^2$. Therefore the function $F(w) = \overline{g_2''}(z)\varphi_2(w)$ is holomorphic and bounded on $\mathbb{C}$, for every fixed $z \in \mathbb{C}$. Therefore F is constant on $\mathbb{C}$, for each fixed $z \in \mathbb{C}$.
Since φ_2 is affine nonconstant, it follows that $g_2'' = 0$ on $\mathbb{C}$. Then g_2 is affine on $\mathbb{C}$.
Now write $\varphi_2(w) = A_2 w + B_2$, $g_2(z) = a_2 z + b_2$, $A_2 \in \mathbb{C} \backslash \{0\}$, $B_2, a_2, b_2 \in \mathbb{C}$. Let $T(z,w) = (z, w + \frac{a_2}{A_2} z + \frac{b_2}{A_2})$.
Thus T is an affine holomorphic transformation and bijective on $\mathbb{C}^2$. Then $u_1 = u \,\mathrm{o}\, T$ is strictly convex on $\mathbb{C}^2$ and $u \,\mathrm{o}\, T(z,w) = |g_1(z) - a|^2 + |\varphi_2(w)|^2 = u_1(z,w)$.
Consequently, g_1 is affine nonconstant on $\mathbb{C}$.
The converse is obvious and the proof is complete.

Now let $\psi_1, \psi_2, f_1, f_2, k : \mathbb{C} \to \mathbb{C}$ be holomorphic functions and $\gamma, c \in \mathbb{C}$. Using the holomorphic differential equation $k''(k+c) = \gamma(\underline{k'})^2$ and the two partial differential equations $\psi_1''(w)\overline{f_1'}(z) + \psi_2''(w)\overline{f_2'}(z) = 0$, $f_1''(z)\overline{\psi_1'}(w) + f_2''(z)\overline{\psi_2'}(w) = 0$ on $\mathbb{C}^2$, we prove

Theorem 6. *Let $\varphi_1, \varphi_2 : \mathbb{C}^m \to \mathbb{C}$ and $g_1, g_2 : \mathbb{C}^n \to \mathbb{C}$ be four holomorphic functions. Put $u(z,w) = |\varphi_1(w) - g_1(z)|^2 + |\varphi_2(w) - g_2(z)|^2$, $(z,w) \in \mathbb{C}^n \times \mathbb{C}^m$.*
The following assertions are equivalent
(I) u is strictly convex on $\mathbb{C}^n \times \mathbb{C}^m$;
(II) $n = m = 1$, g_1, g_2, φ_1, φ_2 are affine functions on $\mathbb{C}$ and satisfying the condition $(g_1'\varphi_2' - g_2'\varphi_1') \neq 0$.

Proof. We have $n = m = 1$, because u is strictly psh on $\mathbb{C}^n \times \mathbb{C}^m$. Since u is strictly convex on $\mathbb{C}^n \times \mathbb{C}^m$, then the function $u(z,.)$ is strictly convex on $\mathbb{C}$, for each $z \in \mathbb{C}$. Therefore,

$$|\varphi_1''(w)(\overline{\varphi_1}(w) - \overline{g_1}(z)) + \varphi_2''(w)(\overline{\varphi_2}(w) - \overline{g_2}(z))| < |\varphi_1'(w)|^2 + |\varphi_2'(w)|^2$$

for each $w \in \mathbb{C}^m$ and for every fixed $z \in \mathbb{C}^n$. Thus, for every fixed $w \in \mathbb{C}$, the holomorphic function on the variable z, defined by $F(z) = (g_1(z)\overline{\varphi_1''}(w) + g_2(z)\overline{\varphi_2''}(w))$ is bounded on $\mathbb{C}$.

By Liouville theorem, F is constant on $\mathbb{C}$. Thus $(g_1'(z)\overline{\varphi_1''}(w) + g_2'(z)\overline{\varphi_2''}(w)) = 0$, for every $z, w \in \mathbb{C}$.
We discuss the cases $\varphi_1'' \neq 0$ or $\varphi_2'' \neq 0$ on $\mathbb{C}$. (Also we have $(\varphi_1'(w)\overline{g_1''}(z) + \varphi_2'(w)\overline{g_2''}(z)) = 0$ on $\mathbb{C}^2$).
Assume that $\varphi_1'' \neq 0$ and $\varphi_2'' \neq 0$. Therefore

$$\frac{\varphi_1''(w)}{\varphi_2''(w)} = -\frac{\overline{g_2'}(z)}{\overline{g_1'}(z)} = R, \; R \in \mathbb{C}.$$

Thus, $\varphi_1''(w) = R\varphi_2''(w)$ and $g_2'(z) = -\overline{R}g_1'(z)$, for each $z, w \in \mathbb{C}$. It follows that $\varphi_1(w) = R\varphi_2(w) + aw + b$ and $g_2(z) = -\overline{R}g_1(z) + \lambda$, $a, b, \lambda \in \mathbb{C}$.
The function F_1 is strictly convex on $\mathbb{C}^2$, where

$$F_1(z, w) = |R\varphi_2(w) + aw + b - g_1(z)|^2 + |\varphi_2(w) + \overline{R}g_1(z) - \lambda|^2.$$

This proves $|g_1 + \xi_1|^2$ is strictly convex on $\mathbb{C}$, where $\xi_1 \in \mathbb{C}$.
By the holomorphic differential equation $k''(k + c) = \gamma(k')^2$, ($k : \mathbb{C} \to \mathbb{C}$ be a holomorphic function and $c, \gamma \in \mathbb{C}$), we have g_1 is affine nonconstant on $\mathbb{C}$. Therefore, $|g_1 - \varphi_1|^2 + |\overline{R}g_1 - (\lambda - \varphi_2)|^2$ is strictly convex on $\mathbb{C}^2$.
By theorem 2, φ_1 and φ_2 are affine functions. A contradiction.
Consequently, $\varphi_1'' = 0$, or $\varphi_2'' = 0$ on $\mathbb{C}$.
Assume that $\varphi_1'' \neq 0$ and $\varphi_2'' = 0$ on $\mathbb{C}$. Therefore $\varphi_1''\overline{g_1'} = 0$ on $\mathbb{C}$. Thus $g_1' = 0$ on $\mathbb{C}$ and then g_1 is constant on $\mathbb{C}$. We have $|\varphi_1 - g_1(0)|^2 + |\varphi_2 - g_2|^2$ is strictly convex on $\mathbb{C}^2$. By lemma 3, we have φ_1 and g_2 are affine nonconstant, φ_2 is affine on $\mathbb{C}$. Therefore φ_1 is affine nonconstant on $\mathbb{C}$. A contradiction.
Consequently, φ_1 and φ_2 are affine functions on $\mathbb{C}$.
Now since the function $u(., w)$ is strictly convex on $\mathbb{C}$ (for each fixed $w \in \mathbb{C}$), then g_1, g_2, φ_1 and φ_2 satisfy the partial differential equation $g_1''\overline{\varphi_1'} + g_2''\overline{\varphi_2'} = 0$ in $\mathbb{C}^2$.
Using the last above partial differential equation, we prove that g_1 and g_2 are affine functions on $\mathbb{C}$. Note that if φ_1 and g_1 are constant functions, then $|g_2 - \varphi_2|^2$ is strictly convex on $\mathbb{C}^2$. This is impossible.
Therefore, we have
(φ_1 or g_1 is non constant) and (φ_2 or g_2 is non constant).
Analogously,
(g_1 or g_2 is non constant) and (φ_1 or φ_2 is non constant).
Since now u is strictly convex on $\mathbb{C}^2$, then

$$|\varphi_1'(w)\beta - g_1'(z)\alpha|^2 + |\varphi_2'(w)\beta - g_2'(z)\alpha|^2 > 0$$

for each $(z, w) \in \mathbb{C}^2$ and $(\alpha, \beta) \in \mathbb{C}^2 \backslash \{0\}$. Therefore, $(g_1'\varphi_2' - g_2'\varphi_1') \neq 0$.

3.2. The analysis of convex and strictly psh functions

Let $\psi_1, \psi_2, f_1, f_2, k : \mathbb{C} \to \mathbb{C}$ be holomorphic functions and $\gamma, c \in \mathbb{C}$. In the sequel, using the holomorphic differential equation $k''(k+c) = \gamma(k')^2$ and the two partial differential equations $\psi_1''(w)\overline{f_1'}(z) + \psi_2''(w)\overline{f_2'}(z) = 0$, $f_1''(z)\overline{\psi_1'}(w) + f_2''(z)\overline{\psi_2'}(w) = 0$ on $\mathbb{C}^2$, we have

Theorem 7. *Let $g_1, g_2 : \mathbb{C}^n \to \mathbb{C}$ and $\varphi_1, \varphi_2 : \mathbb{C}^m \to \mathbb{C}$ be four holomorphic functions. Put $u(z,w) = |\varphi_1(w) - g_1(z)|^2 + |\varphi_2(w) - g_2(z)|^2$, for $(z,w) \in \mathbb{C}^n \times \mathbb{C}^m$. The following conditions are equivalent*
(I) u is convex and strictly psh on $\mathbb{C}^n \times \mathbb{C}^m$;
(II) $n = m = 1$, $\varphi_1''\overline{g_1'} + \varphi_2''\overline{g_2'} = 0$ and $g_1''\overline{\varphi_1'} + g_2''\overline{\varphi_2'} = 0$ on $\mathbb{C}^2$,

(φ_1 or φ_2 is nonconstant) and (g_1 or g_2 is nonconstant) and we have the following cases.
Case 1. The functions φ_1 and φ_2 satisfies $\varphi_1'' \neq 0$ and $\varphi_2'' \neq 0$.
Assume that $g_1' \neq 0$.
If $g_1'' = 0$, then $g_2'' = 0$ on $\mathbb{C}$ (therefore g_1 and g_2 are affine functions with g_1 or g_2 is non constant. In this case, by theorem 2 or theorem 3, we can find φ_1 and φ_2 by their holomorphic expressions).
If $g_1'' \neq 0$. Thus $g_2'' \neq 0$. Since $u(z,.)$ is convex on $\mathbb{C}$ (for z fixed), then $\varphi_2 = c\varphi_1 + \xi_0$, $c, \xi_0 \in \mathbb{C}$.
$u = |\varphi_1 - g_1|^2 + |c\varphi_1 + \xi_0 - g_2|^2$, on $\mathbb{C}^2$.
Assume that $g_2' \neq 0$.
We have an analogous situation to the above case.
Case 2. The function φ_1 is not affine and the function φ_2 is affine on $\mathbb{C}$.
Then g_1 is constant on $\mathbb{C}$, $|\varphi_1 - g_1(0)|^2$ and $|g_2 - \varphi_2(0)|^2$ are convex functions and $|\varphi_1' g_2'| > 0$ on $\mathbb{C}^2$, or
g_2 is affine nonconstant and $|\varphi_1' g_2'| > 0$ on $\mathbb{C}^2$.
We can study also the case $\varphi_1'' = 0$ and $\varphi_2'' \neq 0$.
Case 3. The functions φ_1 and φ_2 are affine on $\mathbb{C}$.
The discussion is similar to cases 1, 2 and theorem 3.

Proof. (I) implies (II). By lemma 2, we have $2 \leq n+m \leq 2$. Then $n = m = 1$. Since u is convex and of class C^2 on $\mathbb{C}^2$, we have the inequality

$$|\frac{\partial^2 u}{\partial w^2}\beta^2 + \frac{\partial^2 u}{\partial z^2}\alpha^2 + \frac{\partial^2 u}{\partial z \partial w}\alpha\beta| \leq \frac{\partial^2 u}{\partial w \partial \overline{w}}|\beta|^2 + \frac{\partial^2 u}{\partial z \partial \overline{z}}|\alpha|^2 + 2\mathrm{Re}(\frac{\partial^2 u}{\partial \overline{z} \partial w}\overline{\alpha}\beta)$$

on $\mathbb{C}^2$. It follows that

$$|[\varphi_1''(\overline{\varphi_1} - \overline{g_1}) + \varphi_2''(\overline{\varphi_2} - \overline{g_2})]\beta^2 + [g_1''(\overline{g_1} - \overline{\varphi_1}) + g_2''(\overline{g_2} - \overline{\varphi_2})]\alpha^2| \leq |\varphi_1'\beta - g_1'\alpha|^2 + |\varphi_2'\beta - g_2'\alpha|^2$$

for each $(\alpha, \beta) \in \mathbb{C}^2$. If $\alpha = 0$ and $\beta \neq 0$, then

$$|\varphi_1''(\overline{\varphi_1} - \overline{g_1}) + \varphi_2''(\overline{\varphi_2} - \overline{g_2})| \leq |\varphi_1'|^2 + |\varphi_2'|^2$$

on $\mathbb{C}^2$. Now let $\psi(z) = g_1(z)\overline{\varphi_1''}(w) + g_2(z)\overline{\varphi_2''}(w) - \varphi_1(w)\overline{\varphi_1''}(w) - \varphi_2(w)\overline{\varphi_2''}(w)$, for $z \in \mathbb{C}$, (w is fixed on $\mathbb{C}$). ψ is holomorphic on $\mathbb{C}$ and $\psi(z)| \leq |\varphi_1'(w)|^2 + |\varphi_2''(w)|^2$, for every $z \in \mathbb{C}$, (w fixed). Thus ψ is constant on $\mathbb{C}$. Consequently, $\psi'(z) = 0$, for each $z \in \mathbb{C}$. Therefore

$$g_1'(z)\overline{\varphi_1''}(w) + g_2'(z)\overline{\varphi_2''}(w) = 0$$

for each $z, w \in \mathbb{C}$.

Now if $\alpha \neq 0$ and $\beta = 0$. We obtain $\varphi_1'(w)\overline{g_1''}(z) + \varphi_2'(w)\overline{g_2''}(z) = 0$, for every $(z,w) \in \mathbb{C}^2$.

For the rest of the proof we use theorem 1, theorem2, theorem 3 and the proof of theorem 7.

Remark 2. Using the above technical methods, the following three partial differential equations

$$k''(k+c) = \gamma(k')^2,$$
$$\psi_1''(w)\overline{f_1'}(z) + \psi_2''(w)\overline{f_2'}(z) = 0 \text{ on } \mathbb{C}^2,$$
$$f_1''(z)\overline{\psi_1'}(w) + f_2''(z)\overline{\psi_2'}(w) = 0 \text{ on } \mathbb{C}^2,$$

where ($\psi_1, \psi_2, f_1, f_2, k : \mathbb{C} \to \mathbb{C}$ are holomorphic functions and $\gamma, c \in \mathbb{C}$), we can solve the analogous problem when u is convex on $\mathbb{C}^n \times \mathbb{C}^m$ and $u = |\varphi_1 - g_1|^2 + |\varphi_2 - g_2|^2$; $\varphi_1, \varphi_2 : \mathbb{C}^m \to \mathbb{C}$ and $g_1, g_2 : \mathbb{C}^n \to \mathbb{C}$ are four holomorphic functions with the conditions (φ_1 or φ_2 is nonconstant) and (g_1 or g_2 is nonconstant).

3.3. Essential properties in function theory

In the sequel, we give technical tools for the study of the following families of functions consisting of: convex and not strictly psh functions on any not empty Euclidean open ball subset of $\mathbb{C}^n \times \mathbb{C}$; convex and strictly sh functions but not strictly psh on each Euclidean open ball; convex and $n-$ strictly sh functions but not strictly psh on every open ball,... . We have

Theorem 8. *Let $u : D \to \mathbb{R}$ be a function of class C^2, D is a domain of $\mathbb{C}^n, n \geq 1$. The following conditions are equivalent*
(I) u is not strictly psh on each not empty Euclidean open ball subset of D;
(II) u is not strictly psh at each point of D.

Example. Let $v(z,w) = |w^N - g_1(z)|^2 + |w^N - g_2(z)|^2$, $n, N \in \mathbb{N}$, $n, N \geq 2$, $g_1, g_2 : \mathbb{C}^n \to \mathbb{C}$ be two holomorphic functions. v is convex and not strictly psh at each point of $\mathbb{C}^n \times \mathbb{C}$, if for example $g_2(z) = -g_1(z)$, for each $z \in \mathbb{C}^n$ and $|g_1|^2$ is convex on $\mathbb{C}^n$.

Remark 3. (R1). Let $u_1(z,w) = |w - z|^2$, $u_2(z,w) = |w - 2z|^2$, $(z,w) \in \mathbb{C}^2$.
u_1 and u_2 are C^∞ and not strictly psh functions at each point of $\mathbb{C}^2$. But $u = (u_1 + u_2)$ is strictly psh on $\mathbb{C}^2$.
(R2). Put $v(z) = \| z \|^4$, $z = (z_1, ..., z_n) \in \mathbb{C}^n$. v is psh on $\mathbb{C}^n$ and strictly psh on $\mathbb{C}^n \backslash \{0\}$. Therefore v is strictly psh almost everywhere on $\mathbb{C}^n$. But v is not strictly psh on $\mathbb{C}^n$.

Example. Let $u = (u_1 + u_2)$, $v = (v_1 + v_2)$, where

$$u_1(z,w) = |w - f_1(z)|^2 + |w - f_2(z)|^2,$$

$$u_2(z,w) = |w - g_1(z)|^2 + |w - g_2(z)|^2,$$
$$v_1(z,w) = |w - \overline{f_1}(z)|^2 + |w - \overline{f_2}(z)|^2,$$
$$v_2(z,w) = |w - \overline{g_1}(z)|^2 + |w - \overline{g_2}(z)|^2,$$
$$f_1(z) = -f_2(z) = (z - z^2),\ g_1(z) = -g_2(z) = (z + z^2),\ \text{for } (z,w) \in \mathbb{C}^2.$$

f_1, f_2, g_1, g_2 are holomorphic functions on $\mathbb{C}$. We have u and v are strictly convex functions on $\mathbb{C}^2$. But u_1, u_2, v_1, v_2 are not convex functions on $\mathbb{C}^2$.

Example. Let $N \in \mathbb{N}$, $N \geq 2$ and $A \in \mathbb{R}_+$, $A \geq 2$ such that ψ is convex on $\mathbb{C}$, $\psi(z) = A|z|^2 + |z^N - 1|^2$, for $z \in \mathbb{C}$. Put $u = (u_1 + u_2)$, where $u_1(z,w) = |w - g_1(z)|^2 + |w - g_2(z)|^2$, $u_2(z,w) = |w - \overline{g_1}(z)|^2 + |w - \overline{g_2}(z)|^2$,
$g_1(z) = Az + (z^N - 1)$, $g_2(z) = Az - (z^N - 1)$, for $(z,w) \in \mathbb{C}^2$.
Note that g_1 and g_2 are holomorphic functions on $\mathbb{C}$. We have u_1 is not strictly psh and not convex on $\mathbb{C}^2$. u_2 is strictly psh and not convex on $\mathbb{C}^2$. But u is convex and strictly psh on $\mathbb{C}^2$.
We have

Proposition 1. *Let $g_1, g_2 : \mathbb{C} \to \mathbb{C}$ be two holomorphic functions. Put $u(z,w) = |w - g_1(z)|^4 + |w - g_2(z)|^4$, $v(z,w) = |w - \overline{g_1}(z)|^4 + |w - \overline{g_2}(z)|^4$, for $(z,w) \in \mathbb{C}^2$. We have u is not strictly psh on $\mathbb{C}^2$, for each tuple of holomorphic functions g_1 and g_2. But there exists several cases where v is strictly psh on $\mathbb{C}^2$.*

Proof. u and v are functions of class C^∞ on $\mathbb{C}^2$. The hermitian Levi form of u is $L(u)(z,w)(\alpha,\beta) = 4|w - g_1(z)|^2|\beta - g_1'(z)\alpha|^2 + 4|w - g_2(z)|^2|\beta - g_2'(z)\alpha|^2$, for $(z,w) \in \mathbb{C}^2$, $(\alpha,\beta) \in \mathbb{C}^2$.
Let $z_0 \in \mathbb{C}$. Put $w_0 = g_1(z_0)$. Let $\beta = g_2'(z_0)\alpha$, for $\alpha \in \mathbb{C}\backslash\{0\}$.
Then $L(u)(z_0,w_0)(\alpha, g_2'(z_0)\alpha) = 0$ and $\alpha \neq 0$.
The hermitian Levi form of v is
$L(v)(z,w)(\alpha,\beta) = (2|g_1'(z)|^2|w - \overline{g_1}(z)|^2 + 2|g_2'(z)|^2|w - \overline{g_2}(z)|^2)|\alpha|^2 + (2|w - \overline{g_1}(z)|^2 + 2|w - \overline{g_2}(z)|^2)|\beta|^2 + 2|g_1'(z)(w - \overline{g_1}(z))\alpha - (\overline{w} - g_1(z))\beta|^2 + 2|g_2'(z)(w - \overline{g_2}(z))\alpha - (\overline{w} - g_2(z))\beta|^2$, for $(z,w), (\alpha,\beta) \in \mathbb{C}^2$. Now choose $|g_1'| > 0$, $|g_2'| > 0$ and $|g_1 - g_2| > 0$ on $\mathbb{C}$. Let $(z,w) \in \mathbb{C}^2$. We discuss the following three cases $(\alpha \neq 0, \beta = 0)$, $(\alpha = 0, \beta \neq 0))$ and $(\alpha \neq 0, \beta \neq 0)$, we obtain $L(v)(z,w)(\alpha,\beta) > 0$ if $(\alpha,\beta) \in \mathbb{C}^2\backslash\{0\}$.
Then v is strictly psh on $\mathbb{C}^2$.
Let $\psi(z,w) = |w - \psi_1(z)|^2 + |w - \psi_2(z)|^2 + |w - \psi_3(z)|^2$,
$\varphi(z,w) = |w - \overline{\psi_1}(z)|^2 + |w - \overline{\psi_2}(z)|^2 + |w - \overline{\psi_3}(z)|^2$, for $(z,w) \in \mathbb{C}^2$, where $\psi_1, \psi_2, \psi_3 : \mathbb{C} \to \mathbb{C}$ are three holomorphic functions. Recall that if ψ is strictly psh on $\mathbb{C}^2$, then φ is strictly psh on $\mathbb{C}^2$. But we have

Proposition 2. *There exists three holomorphic functions $g_1, g_2, g_3 : \mathbb{C} \to \mathbb{C}$ such that if we define $u(z,w) = |w - g_1(z)|^4 + |w - g_2(z)|^4 + |w - g_3(z)|^4$ and $v(z,w) = |w - \overline{g_1}(z)|^4 + |w - \overline{g_2}(z)|^4 + |w - \overline{g_3}(z)|^4$, for $(z,w) \in \mathbb{C}^2$. We have u is convex on $\mathbb{C}^2$ and strictly psh on a neighborhood of $(0,i)$. But v is not strictly psh at $(0,i)$, while v is convex on $\mathbb{C}^2$.*

Example. Let $g_1(z) = z - i$, $g_2(z) = 2z - i$, $g_3(z) = 3z - i$, $z \in \mathbb{C}$. g_1, g_2 and g_3 are holomorphic functions on $\mathbb{C}$.

$z_0 = 0, w_0 = i$. Put $u(z,w) = |w - g_1(z)|^4 + |w - g_2(z)|^4 + |w - g_3(z)|^4$,
$v(z,w) = |w - \overline{g_1}(z)|^4 + |w - \overline{g_2}(z)|^4 + |w - \overline{g_3}(z)|^4$, for $(z,w) \in \mathbb{C}^2$.
Then u and v are functions of class C^∞ and convex on $\mathbb{C}^2$.
Let $\psi(z,w) = |w - g_1(z)|^4$, $(z,w) \in \mathbb{C}^2$. ψ is a C^∞ function on $\mathbb{C}^2$ and the hermitian Levi form of ψ is

$$L(\psi)(z,w)(\alpha,\beta) = 4|w - g_1(z)|^2|\beta - g_1'(z)\alpha|^2, \quad (\alpha,\beta) \in \mathbb{C}^2.$$

Denote by $L(u)(z,w)(\alpha,\beta)$ the hermitian Levi form of u at (z,w) and (α,β). Then $L(u)(z_0,w_0)(\alpha,\beta) = 16|\beta-\alpha|^2+16|\beta-2\alpha|^2+16|\beta-3\alpha|^2 = 0$ implies that $\alpha = \beta = 0$. Thus $L(u)(z_0,w_0)(\alpha,\beta) > 0$, for each $(\alpha,\beta) \in \mathbb{C}^2\backslash\{0\}$.
Let $S = \{(\alpha,\beta) \in \mathbb{C}^2 \ / \ |\alpha|^2 + |\beta|^2 = 1\}$. Thus $\{(z_0,w_0)\} \times S = K$ is a compact on $\mathbb{C}^2 \times \mathbb{C}^2$.
The function F, defined by

$$F(z,w)(\alpha,\beta) = \frac{\partial^2 u}{\partial z \partial \overline{z}}(z,w)|\alpha|^2 + \frac{\partial^2 u}{\partial w \partial \overline{w}}(z,w)|\beta|^2 + 2\mathrm{Re}(\frac{\partial^2 u}{\partial \overline{z} \partial w}(z,w)\overline{\alpha}\beta),$$

is continuous on $\mathbb{C}^2 \times \mathbb{C}^2$.
Since $F > 0$ on K, then $F > 0$ on $B((z_0,w_0),r) \times S$, where $r > 0$. Therefore u is strictly psh on a neighborhood of $(0,i)$ and convex on $\mathbb{C}^2$.
The hermitian Levi form of the C^∞ function θ on $\mathbb{C}^2$ is

$$\begin{aligned} L(\theta)(z,w)(\alpha,\beta) &= 2|g_1'(z)(w - \overline{g_1}(z))\alpha - (\overline{w} - g_1(z))\beta|^2 + 2|g_1'(z)(\overline{w} - g_1(z))\alpha|^2 \\ &+ 2|w - \overline{g_1}(z)|^2|\beta|^2, \end{aligned}$$

for $(z,w), (\alpha,\beta) \in \mathbb{C}^2$, where $\theta(z,w) = |w - \overline{g_1}(z)|^4$.
Observe that we have $w_0 - \overline{g_1}(z_0) = w_0 - \overline{g_2}(z_0) = w_0 - \overline{g_3}(z_0) = 0$. Therefore $L(v)(z_0,w_0)(\alpha,\beta) = 0$, for each $(\alpha,\beta) \in \mathbb{C}^2$.

We have the following technical remark.

Remark 4. Let $f_1, ..., f_N : \mathbb{C}^n \to \mathbb{C}$ be holomorphic functions, $n, N, k \in \mathbb{N}\backslash\{0\}$, $k \geq 2$. Put

$$u(z,w) = |w - f_1(z)|^{2k} + ... + |w - f_N(z)|^{2k},$$
$$v(z,w) = |w - \overline{f_1}(z)|^{2k} + ... + |w - \overline{f_N}(z)|^{2k},$$
$$u_1(z,w) = |w - f_1(z)|^2 + ... + |w - f_N(z)|^2,$$
$$v_1(z,w) = |w - \overline{f_1}(z)|^2 + ... + |w - \overline{f_N}(z)|^2,$$
$$\varphi = (u+v) \text{ and } \varphi_1 = (u_1 + v_1).$$

If u is strictly psh on $\mathbb{C}^n \times \mathbb{C}$, we can not deduce that v is strictly psh on $\mathbb{C}^n \times \mathbb{C}$.
If φ is strictly psh on $\mathbb{C}^n \times \mathbb{C}$, we can not conclude that u (or v) is strictly psh on $\mathbb{C}^n \times \mathbb{C}$.
But we have the technical properties.
(I) If u is strictly psh on $\mathbb{C}^n \times \mathbb{C}$, then u_1 is strictly psh on $\mathbb{C}^n \times \mathbb{C}$.
(II) v is strictly psh on $\mathbb{C}^n \times \mathbb{C}$ implies that v_1 is strictly psh on $\mathbb{C}^n \times \mathbb{C}$.

(III) If φ_1 is strictly psh on $\mathbb{C}^n \times \mathbb{C}$, then v_1 is strictly psh on $\mathbb{C}^n \times \mathbb{C}$.
(IV) If φ is strictly psh on $\mathbb{C}^n \times \mathbb{C}$, then φ_1 is strictly psh on $\mathbb{C}^n \times \mathbb{C}$.
(V) $(u+u_1)$ is strictly psh on $\mathbb{C}^n \times \mathbb{C}$, implies that u_1 is strictly psh on $\mathbb{C}^n \times \mathbb{C}$.
For example for the proof of the above property (I), since
$u(z,w) = |(w - f_1(z))^k|^2 + ... + |(w - f_N(z))^k|^2$, then u is a function of class C^∞ on $\mathbb{C}^n \times \mathbb{C}$. Therefore the hermitian Levi form of u is

$$\begin{aligned} L(u)(z,w)(\alpha,\beta) &= |w - f_1(z)|^{2k-2}|\beta - \sum_{j=1}^{n} \frac{\partial f_1}{\partial z_j}(z)\alpha_j|^2 + ... \\ &+ |w - f_N(z)|^{2k-2}|\beta - \sum_{j=1}^{n} \frac{\partial f_N}{\partial z_j}(z)\alpha_j|^2 \end{aligned}$$

for $z = (z_1, ..., z_n) \in \mathbb{C}^n$, $w \in \mathbb{C}$, $\alpha = (\alpha_1, ..., \alpha_n) \in \mathbb{C}^n$, $\beta \in \mathbb{C}$.
Now u_1 is a function of class C^∞ on $\mathbb{C}^n \times \mathbb{C}$. The hermitian Levi form of u_1 is

$$L(u_1)(z,w)(\alpha,\beta) = |\beta - \sum_{j=1}^{n} \frac{\partial f_1}{\partial z_j}(z)\alpha_j|^2 + ... + |\beta - \sum_{j=1}^{n} \frac{\partial f_N}{\partial z_j}(z)\alpha_j|^2.$$

Let $(z,w), (\alpha,\beta) \in \mathbb{C}^n \times \mathbb{C}$. Observe that $L(u)(z,w)(\alpha,\beta) > 0$ implies that $L(u_1)(z,w)(\alpha,\beta) > 0$, because the absolute value $|\beta - \sum_{j=1}^{n} \frac{\partial f_s}{\partial z_j}(z)\alpha_j|^2 \geq 0$, for each $s \in \{1, ..., N\}$.
The technical properties (II), (III), (IV) and (V) can be be proved similarly.
Observe that for $\psi : \mathbb{C}^n \to \mathbb{R}_+$, if ψ^2 is convex on $\mathbb{C}^n$, then ψ^4 is convex on $\mathbb{C}^n$. The converse, for instance, is in general not true. But in the sequel, using the holomorphic differential equation, $k''(k+c) = \gamma(k')^2$ ($k : \mathbb{C} \to \mathbb{C}$ be holomorphic and $c, \gamma \in \mathbb{C}$), we have

Theorem 9. *Let $g_1, g_2 : \mathbb{C}^n \to \mathbb{C}$ be two holomorphic functions. Put $u(z,w) = |w - g_1(z)|^2 + |w - g_2(z)|^2$, $v(z,w) = |w - g_1(z)|^4 + |w - g_2(z)|^4$, for $(z,w) \in \mathbb{C}^n \times \mathbb{C}$. We have*
(I) Assume that v is convex on $\mathbb{C}^n \times \mathbb{C}$, then u is convex on $\mathbb{C}^n \times \mathbb{C}$.
(II) Suppose that u is convex on $\mathbb{C}^n \times \mathbb{C}$, we can not conclude that v is convex on $\mathbb{C}^n \times \mathbb{C}$.

Proof. (I). Note that u and v are functions of class C^∞ on $\mathbb{C}^n \times \mathbb{C}$.
Assume that $n = 1$. We have

$$\left|\frac{\partial^2 v}{\partial z^2}(z,w)\alpha^2 + \frac{\partial^2 v}{\partial w^2}\beta^2 + 2\frac{\partial^2 v}{\partial z \partial w}\alpha\beta\right| \leq L(v)(z,w)(\alpha,\beta)$$

for each $(z,w), (\alpha,\beta) \in \mathbb{C}^2$, where

$$L(v)(z,w)(\alpha,\beta) = \frac{\partial^2 v}{\partial z \partial \overline{z}}(z,w)|\alpha|^2 + \frac{\partial^2 v}{\partial w \partial \overline{w}}|\beta|^2 + 2\mathrm{Re}(\frac{\partial^2 v}{\partial z \partial \overline{w}}\alpha\overline{\beta}).$$

We obtain the inequality
(E): $|[-2g_1''(z)w + 2g_1(z)g_1''(z) + 2(g_1'(z))^2](\overline{w}^2 - 2\overline{g_1}(z)\overline{w} + \overline{g_1}^2(z))\alpha^2 +$
$[-2g_2''(z)w + 2g_2(z)g_2''(z) + 2(g_2'(z))^2](\overline{w}^2 - 2\overline{g_2}(z)\overline{w} + \overline{g_2}^2(z))\alpha^2 + 2(\overline{w} - \overline{g_1}(z))^2\beta^2 +$
$2(\overline{w} - \overline{g_2}(z))^2\beta^2 - 2g_1'(z)(\overline{w} - \overline{g_1}(z))^2\alpha\beta - 2g_2'(z)(\overline{w} - \overline{g_2}(z))^2\alpha\beta| \leq$
$|2w\beta - 2\beta g_1(z) - 2wg_1'(z)\alpha + 2g_1'(z)g_1(z)\alpha|^2 + |2w\beta - 2\beta g_2(z) - 2wg_2'(z)\alpha + 2g_2'(z)g_2(z)\alpha|^2$,
for each (z, w), $(\alpha, \beta) \in \mathbb{C}^2$.
If $\beta = 0$ and $w \in \mathbb{R}$, the coefficient of w^3 is equal to 0. Therefore $(g_1''(z) + g_2''(z)) = 0$, for every $z \in \mathbb{C}$.
Now we divide the left hand side of the inequality (E) by $|\overline{w}|^2 > 0$ (for $w \in \mathbb{C}\backslash\{0\}$) and the right hand side of (E) by $|w|^2$ (observe that $|\overline{w}|^2 = |w|^2$), and letting $|w|$ go to $(+\infty)$, we obtain

$$|(4g_1''(z)\overline{g_1}(z) + 4g_2''(z)\overline{g_2}(z) + 2(g_1'(z))^2 + 2(g_2'(z))^2)\alpha^2 + 4\beta^2 - 4(g_1'(z) + g_2'(z))\alpha\beta| \leq |2\beta - 2g_1'(z)\alpha|^2 + |2\beta - 2g_2'(z)\alpha|^2.$$

Put $\beta = g_1'(z)\alpha$. Then

$$|4g_1''(z)\overline{g_1}(z) + 4g_2''(z)\overline{g_2}(z) + 2(g_1'(z) - g_2'(z))^2| \leq 4|g_1'(z) - g_2'(z)|^2.$$

Thus

$$|g_1''(z)(\overline{g_1}(z) - \overline{g_2}(z))|^2 \leq 6|g_1'(z) - g_2'(z)|^2$$

for each $z \in \mathbb{C}$. Now also we prove that

$$|g_2''(z)(\overline{g_1}(z) - \overline{g_2}(z))| \leq 6|g_1'(z) - g_2'(z)|^2$$

for every $z \in \mathbb{C}$. Using the triangle inequality, we have then

$$|g_1''(z)(g_1(z) - g_2(z)) - g_2''(z)(g_1(z) - g_2(z))| \leq 12|g_1'(z) - g_2'(z)|^2$$

for each $z \in \mathbb{C}$.
Therefore the function $(g_1 - g_2)$ satisfies

$$|(g_1''(z) - g_2''(z))(g_1(z) - g_2(z))| \leq 12|g_1'(z) - g_2'(z)|^2$$

for every $z \in \mathbb{C}$. Therefore the function $|g_1 - g_2|^2$ is convex on $\mathbb{C}$, by Abidi [2], (we can see [3]).
Since $(g_1 + g_2)$ is affine on $\mathbb{C}$, thus $g_1(z) = (az + b) + \varphi(z)$, $g_2(z) = (az + b) - \varphi(z)$, for each $z \in \mathbb{C}$, where $\varphi : \mathbb{C} \to \mathbb{C}$ is a holomorphic function such that $|\varphi|$ is convex on $\mathbb{C}$. Therefore u is convex on $\mathbb{C}^2$.
In the sequel, we can prove that g_1 and g_2 are affine functions on $\mathbb{C}$ (see proposition 3).
Assume that $n \geq 2$. Actually by the above case, it is easy to prove that g_1 and g_2 are affine functions on every complex line $L \subset \mathbb{C}^n$. Therefore, g_1 and g_2 are affine functions on $\mathbb{C}^n$.
(II). Assume that $n = 1$. Put $g_1(z) = z^2$, $g_2(z) = -z^2$, for $z \in \mathbb{C}$. Then

$$u(z, w) = |w - g_1(z)|^2 + |w - g_2(z)|^2 = 2|w|^2 + 2|z|^2, \ (z, w) \in \mathbb{C}^2.$$

Thus u is convex on $\mathbb{C}^2$. But v is not convex on $\mathbb{C}^2$, because $v(z,1) = |1 - z^2|^4 + +|1 + z^2|^4 = 2\psi(z)$, for each $z \in \mathbb{C}$. Observe that ψ is not convex in a neighborhood of $\frac{1}{2}$.

Proposition 3. *Let* $u(z,w) = |w + < z/a > +b + \varphi(z)|^4 + |w + < z/a > +b - \varphi(z)|^4$, $a \in \mathbb{C}^n$, $b \in \mathbb{C}$, $\varphi : \mathbb{C}^n \to \mathbb{C}$ *be holomorphic not affine,* $|\varphi|$ *is convex on* $\mathbb{C}^n$.
Then the function u *is not convex on* $\mathbb{C}^n \times \mathbb{C}$.

Proof. Define $v(z,w) = |w + \varphi(z)|^4 + |w - \varphi(z)|^4$, $(z,w) \in \mathbb{C}^n \times \mathbb{C}$. Observe that v is convex on $\mathbb{C}^n \times \mathbb{C}$ if and only if u is convex on $\mathbb{C}^n \times \mathbb{C}$.
Suppose that $n = 1$. Since φ is not affine and $|\varphi|$ is convex on $\mathbb{C}$, then by Abidi [3], we have the holomorphic representations
$\varphi(z) = (a_1 z + b_1)^k$, for each $z \in \mathbb{C}$, where $a_1 \in \mathbb{C}\backslash\{0\}$, $b_1 \in \mathbb{C}$, $k \in \mathbb{N}$, $k \geq 2$, or
$\varphi(z) = e^{(a_2 z + b_2)}$, for every $z \in \mathbb{C}$, with $a_2 \in \mathbb{C}\backslash\{0\}$ and $b_2 \in \mathbb{C}$.
Now for the study of the convexity of the function v, by an affine change of variable, we can assume that $\varphi(z) = z^k$, for any $z \in \mathbb{C}$, or $\varphi(z) = e^z$, for each $z \in \mathbb{C}$.
(I) Assume that $\varphi(z) = z^k$, $k \in \mathbb{N}$, $k \geq 2$.
If $k = 2$. We can see the above proof and we have the function $F = v(.,1)$ is not convex on $\mathbb{C}$.
Now suppose that $k \geq 3$.
Define $\psi(z) = v(z,1)$, for $z \in \mathbb{C}$. Then ψ is a function of class C^∞ on $\mathbb{C}$.
If ψ is convex on $\mathbb{C}$, then
$$\left|\frac{\partial^2 \psi}{\partial z^2}(z)\right| \leq \frac{\partial^2 \psi}{\partial z \partial \overline{z}}(z)$$
for each $z \in \mathbb{C}$.
$$\begin{aligned}
\frac{\partial^2 \psi}{\partial z^2}(z) &= [2k^2 z^{2k-2} + 2k(k-1)z^{k-2}(1 + z^k)](1 + \overline{z}^k)^2 \\
&+ [2k^2 z^{2k-2} + 2k(k-1)z^{k-2}(z^k - 1)](\overline{z}^k - 1)^2. \\
\frac{\partial^2 \psi}{\partial z \partial \overline{z}}(z) &= 4k^2 |z^{2k-2}| 1 + z^k|^2 + 4k^2 |z^k - 1|^2 |z|^{2k-2}.
\end{aligned}$$
For $z_0 = 1$, $\frac{\partial^2 \psi}{\partial z^2}(1) = 4(6k^2 - 4k) \geq 0$ and $\frac{\partial^2 \psi}{\partial z \partial \overline{z}}(1) = 16k^2$. Then $\frac{\partial^2 \psi}{\partial z^2}(1) = |\frac{\partial^2 \psi}{\partial z^2}(1)| \leq \frac{\partial^2 \psi}{\partial z \partial \overline{z}}(1)$. Therefore $6k^2 - 4k \leq 4k^2$ and $k \geq 3$. This is a contradiction.
(II) Assume that $\varphi(z) = e^z$, for $z \in \mathbb{C}$.
Let $\psi(z) = v(z,2)$, $z \in \mathbb{C}$. ψ is a function of class C^∞ on $\mathbb{C}$.
$$\begin{aligned}
\frac{\partial^2 \psi}{\partial z^2}(z) &= 2(2e^z + 2e^{2z})(2 + e^{\overline{z}})^2 + 2(2e^{2z} - 2e^z)(e^{\overline{z}} - 2)^2. \\
\frac{\partial^2 \psi}{\partial z \partial \overline{z}}(z) &= 4e^{(z+\overline{z})}(e^z + 2)(e^{\overline{z}} + 2) + 4e^{(z+\overline{z})}(e^z - 2)(e^{\overline{z}} - 2).
\end{aligned}$$
$\frac{\partial^2 \psi}{\partial z^2}(0) = 72$ and $\frac{\partial^2 \psi}{\partial z \partial \overline{z}}(0) = 40$. Therefore $|\frac{\partial^2 \psi}{\partial z^2}(0)| > \frac{\partial^2 \psi}{\partial z \partial \overline{z}}(0)$. Then ψ is not convex on $\mathbb{C}$. Consequently, v is not convex on $\mathbb{C}^2$.
Comparing the preceding theorem and proposition 3, we observe that the exponent 2 is

special in our considerations. For instance, let $u_k(z,w) = |w-f_1(z)|^{2k}+|w-f_2(z)|^{2k}$, $k \in \mathbb{N}\backslash\{0\}$, $f_1, f_2 : \mathbb{C}^n \to \mathbb{C}$ be two holomorphic functions and $(z,w) \in \mathbb{C}^n \times \mathbb{C}$. We can prove that u_k is convex on $\mathbb{C}^n \times \mathbb{C}$ implies that u_1 is convex on $\mathbb{C}^n \times \mathbb{C}$ if $(k \geq 2)$, but the converse is not true.
Let $v_\delta(z,w) = |A_1 w - f_1(z)|^\delta + |A_2 w - f_2(z)|^\delta$, $\delta \in [1, +\infty[$ and $(A_1, A_2) \in \mathbb{C}^2\backslash\{0\}$. Observe that the study of the convexity of the function v_δ is based on two additional cases.
Moreover, observe that by the above technical proof, we have

Theorem 10. *Let $f_1, f_2 : \mathbb{C}^n \to \mathbb{C}$ be two holomorphic functions. Define $u(z,w) = |w - f_1(z)|^4 + |w - f_2(z)|^4$, for $(z,w) \in \mathbb{C}^n \times \mathbb{C}$. We have u is convex on $\mathbb{C}^n \times \mathbb{C}$ if and only if f_1 and f_2 are affine functions on $\mathbb{C}^n$.*
Proof. We can see the proof of theorem 9 and proposition 3. □

Remark 5. Let $f_1(z) = z^N$, $f_2(z) = -z^N$, $f_3(z) = iz^N$ and $f_4(z) = -iz^N$, $N \in \mathbb{N}\backslash\{0,1\}$, for $z \in \mathbb{C}$.
Put $u(z,w) = |w - f_1(z)|^4 + |w - f_2(z)|^4 + |w - f_3(z)|^4 + |w - f_4(z)|^4$, $(z,w) \in \mathbb{C}^2$. u is convex on $\mathbb{C}^2$, because $u(z,w) = c(|w|^2 + |z^N|^2)^2$, where $c \in \mathbb{R}$, $c > 0$. But f_1, f_2, f_3 and f_4 are not affine functions.

We have the following.

Question 1. Let $F_1, F_2, F_3 : \mathbb{C}^n \to \mathbb{C}$ be holomorphic functions. Put $\psi_1(z) = (|F_1(z)|^4 + |F_2(z)|^4)$, $\psi_2(z) = (|F_1(z)|^4 + |F_2(z)|^4 + |F_3(z)|^4)$, $z \in \mathbb{C}^n$.
(I) Is it true that ψ_1 is convex on $\mathbb{C}^n$ implies that F_1 and F_2 are affine functions on $\mathbb{C}^n$?
(II) Assume that ψ_2 is convex on $\mathbb{C}^n$. Is it true that F_1, F_2 and F_3 are affine functions on $\mathbb{C}^n$?
The number of holomorphic functions is it fundamental in the above two situations?

We have

Proposition 4. *Let $k \in \mathbb{N}\backslash\{0,1\}$ and $\varphi : \mathbb{C}^n \to \mathbb{C}$ be holomorphic. Define $v(z,w) = |w+ < z/a > +b+\varphi(z)|^{2k}+|w+ < z/a > +b-\varphi(z)|^{2k}$, $a \in \mathbb{C}^n, b \in \mathbb{C}$, $(z,w) \in \mathbb{C}^n\times\mathbb{C}$. Assume that φ is not affine and $|\varphi|$ is convex on $\mathbb{C}^n$. Then v is not convex on $\mathbb{C}^n \times \mathbb{C}$.*
Proof. Obviously follows from the proof of proposition 3. Observe that, using the holomorphic differential equation cited above, we have the additional result. □

Theorem 11. *Let $g_1, g_2 : \mathbb{C}^n \to \mathbb{C}$ be two holomorphic functions and $k \in \mathbb{N}\backslash\{0,1\}$. Put $u(z,w) = |w - g_1(z)|^{2k} + |w - g_2(z)|^{2k}$ and $v(z,w) = |w - g_1(z)|^2 + |w - g_2(z)|^2$, $(z,w) \in \mathbb{C}^n \times \mathbb{C}$.*
(I) Assume that u is convex on $\mathbb{C}^n \times \mathbb{C}$. Then v is convex on $\mathbb{C}^n \times \mathbb{C}$.
(II) Suppose that v is convex on $\mathbb{C}^n \times \mathbb{C}$. We can not conclude that u is convex on $\mathbb{C}^n \times \mathbb{C}$. But we have
(III) u is convex on $\mathbb{C}^n \times \mathbb{C}$ if and only if g_1 and g_2 are affine functions.

Extension of the results. Let $\psi_\delta = |w - f_1(z)|^\delta + |w - f_2(z)|^\delta$, $\delta \in [1, +\infty[$, $f_1, f_2 : \mathbb{C}^n \to \mathbb{C}$ be two holomorphic functions and $(z,w) \in \mathbb{C}^n \times \mathbb{C}$. We observe without any assumption on $\delta \in [1, +\infty[$, for instance , for the study of the convexity

of the function ψ_δ, the proof is organized in two separately cases.
Case 1. $\delta = 2$. (In this case, we obtain several solutions not affine functions).
Case 2. $\delta \in [1, +\infty[\backslash\{2\}$.
In general we have the following two remarks (R1) and (R2).
(R1). Let $f : \mathbb{C} \to \mathbb{C}$ be a function. Put $\varphi_\delta(z,w) = |w - f(z)|^\delta$, $\delta \in [1, +\infty[$ and $(z,w) \in \mathbb{C}^2$. We have φ_δ is convex on $\mathbb{C}^2$ if and only if f is affine (and in particular f is a function of class C^∞ on $\mathbb{C}$).
(Let $N \in \mathbb{N}\backslash\{0\}$, $2N \geq \delta$ and put $G(z,w) = |w - f(z)|^{2N}$, $(z,w) \in \mathbb{C}^2$. Suppose that φ_δ is convex on $\mathbb{C}^2$. Consequently, G is psh on $\mathbb{C}^2$. By Abidi [1], it follows that f is harmonic on $\mathbb{C}$. Now let $T : \mathbb{C} \to \mathbb{C}$ be an $\mathbb{R}-$ linear bijective transformation. Consider $M(z,w) = (T(z), w)$, for $(z,w) \in \mathbb{C}^2$. Note that M is $\mathbb{R}-$ linear and a bijective transformation on $\mathbb{C}^2$. Therefore $G\,\mathrm{o}\,M$ is convex on $\mathbb{C}^2$ and consequently, $G\,\mathrm{o}\,M$ is psh on $\mathbb{C}^2$. Since $G\,\mathrm{o}\,M(z,w) = |w - f\,\mathrm{o}\,T(z)|$, for $(z,w) \in \mathbb{C}^2$. Then $f\,\mathrm{o}\,T$ is harmonic on $\mathbb{C}$, for any $\mathbb{R}-$ linear transformation T. Then f is affine on $\mathbb{C}$).
But if we define $F_\delta(z,w) = |w - f_1(z)|^\delta + |w - g_1(z)|^\delta$, where

$$f_1(z) = \begin{cases} 1 \text{ if } \mathrm{Re}(z) \geq 0 \\ -1 \text{ if } \mathrm{Re}(z) < 0 \end{cases}$$

and

$$g_1(z) = \begin{cases} -1 \text{ if } \mathrm{Re}(z) \geq 0 \\ 1 \text{ if } \mathrm{Re}(z) < 0 \end{cases}$$

for $(z,w) \in \mathbb{C}^2$. Then we have
$F_\delta(z,w) = |w-1|^\delta + |w+1|^\delta$ and consequently, the function F_δ is convex on $\mathbb{C}^2$, for each $\delta \geq 1$. But f_1 and g_1 are noncontinuous functions at any point of $\mathbb{C}$. Moreover, we have
(R2). There exists two continuous functions $f, g : \mathbb{C} \to \mathbb{C}$, with $K_\delta(z,w) = |w - f(z)|^\delta + |w - g(z)|^\delta$, $(z,w) \in \mathbb{C}^2$, K_δ is convex on $\mathbb{C}^2$ (for each $\delta \geq 1$), but f and g are not functions of class C^∞ on $\mathbb{C}$.
Example. Let $f(z) = |x|$, $g(z) = -|x|$, $z = (x + iy) \in \mathbb{C}$, $x = \mathrm{Re}(z)$.

Question 2. Let $\psi_1, ..., \psi_N : \mathbb{C}^n \to \mathbb{C}$ be analytic functions, $N, k \in \mathbb{N}$, $k \geq 2$. Define

$$\psi(z,w) = |w - \psi_1(z)|^{2k} + ... + |w - \psi_N(z)|^{2k},\ (z,w) \in \mathbb{C}^n \times \mathbb{C}.$$

Assume that $N \leq 2k-1$ and ψ is convex on $\mathbb{C}^n \times \mathbb{C}$. Characterize $\psi_1, ..., \psi_N$ by their analytic expressions.

Question 3. Let $\varphi_1, \varphi_2, \varphi_3, \varphi_4 : \mathbb{C}^m \to \mathbb{C}$ and $g_1, g_2, g_3, g_4 : \mathbb{C}^n \to \mathbb{C}$ be 8 holomorphic functions. Put $u = (u_1 + u_2)$, where $u_1(z,w) = |\varphi_1(w) - g_1(z)|^4 + |\varphi_2(w) - g_2(z)|^4$, $u_2(z,w) = |\varphi_3(w) - g_3(z)|^4 + |\varphi_4(w) - g_4(z)|^4$, $(z,w) \in \mathbb{C}^n \times \mathbb{C}$.
Characterize $\varphi_1, \varphi_2, \varphi_3, \varphi_4$, g_1, g_2, g_3, g_4 by their expressions such that u_1 and u_2 are convex functions on $\mathbb{C}^n \times \mathbb{C}$ and u is strictly psh on $\mathbb{C}^n \times \mathbb{C}$.
In the sequel, for instance, observe that there exists a great differences between the exponent 2 and the exponent 4 (or $2k, k \in \mathbb{N}\backslash\{0,1\}$) in real convexity.

We have

Lemma 4. *(I) There exists $\psi_1, \psi_2 : \mathbb{C}^n \to \mathbb{C}$ two holomorphic functions such that $|\psi_1|^2$ and $|\psi_2|^2$ are not convex functions, while $u = (|\psi_1|^2 + |\psi_2|^2)$ is convex on $\mathbb{C}^n$, but $v = (|\psi_1|^4 + |\psi_2|^4)$ is not convex on $\mathbb{C}^n$ (respectively $(|\psi_1|^{2k} + |\psi_2|^{2k})$ is not convex on $\mathbb{C}^n$ for each $k \in \mathbb{N}\backslash\{0,1\}$).*
(II) There exists $\varphi_1, \varphi_2 : \mathbb{C}^n \to \mathbb{C}$ holomorphic functions, with $|\varphi_1|^2$ is convex and $|\varphi_2|^2$ is not convex on $\mathbb{C}^n$, $(|\varphi_1|^2 + |\varphi_2|^2)$ is convex on $\mathbb{C}^n$, but $(|\varphi_1|^{2k} + |\varphi_2|^{2k})$ is not convex on $\mathbb{C}^n$, for each $k \in \mathbb{N}\backslash\{0,1\}$.
(Example. $\varphi_1(z) = 2z$, $\varphi_2(z) = z^2 - 1$, $z \in \mathbb{C}$).

We introduce this lemma because it yields the following questions.

Question 4. Let $f_1, f_2 : \mathbb{C}^n \to \mathbb{C}$ be analytic functions and $\delta \in [1, +\infty[$. Put $u = (|f_1|^\delta + |f_2|^\delta)$. Suppose that u is convex on $\mathbb{C}^n$ and $\delta \neq 2$. Is it true that $|f_1|$ and $|f_2|$ are convex functions on $\mathbb{C}^n$?

Question 5. Let $n, m \in \mathbb{N}\backslash\{0\}$. Find all the holomorphic functions $f_1, f_2 : \mathbb{C}^n \to \mathbb{C}$, $\varphi_1, \varphi_2 : \mathbb{C}^m \to \mathbb{C}$, such that ψ is convex on $\mathbb{C}^n \times \mathbb{C}^m$, where $\psi(z,w) = |\varphi_1(w) - f_1(z)|^\delta + |\varphi_2(w) - f_2(z)|^\delta$, for $(z,w) \in \mathbb{C}^n \times \mathbb{C}^m$.

4. Some study of a particular case and algebraic method

Theorem 12. *Let $A_1, A_2, A_3, A_4, A_5 \in \mathbb{C}\backslash\{0\}$. Consider $g_1, g_2, g_3, g_4, g_5 : \mathbb{C}^n \to \mathbb{C}$ be five holomorphic functions. Define $u_1(z,w) = |A_1 w - g_1(z)|^2 +$
$|A_2 w - g_2(z)|^2$, $v_1(z,w) = |A_3 w - g_3(z)|^2 + |A_4 w - g_4(z)|^2$, $u(z,w) = u_1(z,w) + v_1(z,w) + |A_5 w - g_5(z)|^2$, $(z,w) \in \mathbb{C}^n \times \mathbb{C}$.*
The following conditions are equivalent
(I) u_1 and v_1 are convex functions on $\mathbb{C}^n \times \mathbb{C}$ and u is (convex and strictly psh) on $\mathbb{C}^n \times \mathbb{C}$;
(II) $n \in \{1,2,3,4\}$ and we have

$$\begin{cases} g_1(z) = A_1(< z/a > +b) + \overline{A_2}\varphi(z) \\ g_2(z) = A_2(< z/a > +b) - \overline{A_1}\varphi(z) \end{cases}$$

$$\begin{cases} g_3(z) = A_3(< z/c > +d) + \overline{A_4}\psi(z) \\ g_4(z) = A_4(< z/c > +d) - \overline{A_3}\psi(z) \end{cases}$$

and $g_5(z) = (< z/\lambda > +\mu)$, (for all $z \in \mathbb{C}^n$, where $a, c, \lambda \in \mathbb{C}^n$, $b, d, \mu \in \mathbb{C}$, $\varphi, \psi : \mathbb{C}^n \to \mathbb{C}$ are 2 holomorphic functions, $|\varphi|$ and $|\psi|$ are convex functions on $\mathbb{C}^n$) with the following 4 cases.
*(1) $n = 4$. We have $(a - c, a - \lambda, (\overline{\frac{\partial\varphi}{\partial z_1}}(z), \overline{\frac{\partial\varphi}{\partial z_2}}(z), \overline{\frac{\partial\varphi}{\partial z_3}}(z), \overline{\frac{\partial\varphi}{\partial z_4}}(z)),$
$(\overline{\frac{\partial\psi}{\partial z_1}}(z), \overline{\frac{\partial\psi}{\partial z_2}}(z), \overline{\frac{\partial\psi}{\partial z_3}}(z), \overline{\frac{\partial\psi}{\partial z_4}}(z)))$ is a basis of the complex vector space $\mathbb{C}^4$, for all $z \in \mathbb{C}^4$.*
*(2) $n = 3$. Then we have for all $z \in \mathbb{C}^3$, $z = (z_1, z_2, z_3)$,
$(a - c, a - \lambda, (\overline{\frac{\partial\varphi}{\partial z_1}}(z), \overline{\frac{\partial\varphi}{\partial z_2}}(z), \overline{\frac{\partial\varphi}{\partial z_3}}(z)))$, or $(a - c, a - \lambda, (\overline{\frac{\partial\psi}{\partial z_1}}(z), \overline{\frac{\partial\psi}{\partial z_2}}(z), \overline{\frac{\partial\psi}{\partial z_3}}(z)))$, or*

$(a-c,(\overline{\frac{\partial\varphi}{\partial z_1}(z)},\overline{\frac{\partial\varphi}{\partial z_2}(z)},\overline{\frac{\partial\varphi}{\partial z_3}(z)}),(\overline{\frac{\partial\psi}{\partial z_1}(z)},\overline{\frac{\partial\psi}{\partial z_2}(z)},\overline{\frac{\partial\psi}{\partial z_3}(z)}))$, or $(a-\lambda,(\overline{\frac{\partial\varphi}{\partial z_1}(z)},\overline{\frac{\partial\varphi}{\partial z_2}(z)},\overline{\frac{\partial\varphi}{\partial z_3}(z)}),(\overline{\frac{\partial\psi}{\partial z_1}(z)},\overline{\frac{\partial\psi}{\partial z_2}(z)},\overline{\frac{\partial\psi}{\partial z_3}(z)}))$ is a basis of the complex vector space $\mathbb{C}^3$.

(3) $n=2$. Then for each $z=(z_1,z_2)\in\mathbb{C}^2$, the quantity $(a-c,a-\lambda)$, or $(a-c,(\overline{\frac{\partial\varphi}{\partial z_1}(z)},\overline{\frac{\partial\varphi}{\partial z_2}(z)}))$, or $(a-c,(\overline{\frac{\partial\psi}{\partial z_1}(z)},\overline{\frac{\partial\psi}{\partial z_2}(z)}))$, or $(a-\lambda,(\overline{\frac{\partial\varphi}{\partial z_1}(z)},\overline{\frac{\partial\varphi}{\partial z_2}(z)}))$, or $(a-\lambda,(\overline{\frac{\partial\psi}{\partial z_1}(z)},\overline{\frac{\partial\psi}{\partial z_2}(z)}))$, or $((\overline{\frac{\partial\varphi}{\partial z_1}(z)},\overline{\frac{\partial\varphi}{\partial z_2}(z)}),(\overline{\frac{\partial\psi}{\partial z_1}(z)},\overline{\frac{\partial\psi}{\partial z_2}(z)}))$ is a basis of the complex vector space $\mathbb{C}^2$.

(4) $n=1$. Then we have for all $z\in\mathbb{C}$, $(a-c)\neq 0$, or $(a-\lambda)\neq 0$, or $(\frac{\partial\varphi}{\partial z}(z)\neq 0)$, or $(\frac{\partial\psi}{\partial z}(z)\neq 0)$.

Proof. (I) implies (II). Since u_1 is convex on $\mathbb{C}^n\times\mathbb{C}$, then

$$\begin{cases} g_1(z)=A_1(<z/a>+b)+\overline{A_2}\varphi(z)\\ g_2(z)=A_2(<z/a>+b)-\overline{A_1}\varphi(z)\end{cases}$$

(for each $z\in\mathbb{C}^n$, where $a\in\mathbb{C}^n$, $b\in\mathbb{C}$, $\varphi:\mathbb{C}^n\to\mathbb{C}$ is analytic, $|\varphi|$ is convex on $\mathbb{C}^n$). u_2 is convex on $\mathbb{C}^n\times\mathbb{C}$, then

$$\begin{cases} g_3(z)=A_3(<z/c>+d)+\overline{A_4}\psi(z)\\ g_4(z)=A_4(<z/c>+d)-\overline{A_3}\psi(z)\end{cases}$$

(for every $z\in\mathbb{C}^n$, with $c\in\mathbb{C}^n$, $d\in\mathbb{C}$, $\psi:\mathbb{C}^n\to\mathbb{C}$ is analytic and $|\psi|$ is convex on $\mathbb{C}^n$).

Note that u is a function of class C^∞ on $\mathbb{C}^n\times\mathbb{C}$. Now since u is convex on $\mathbb{C}^n\times\mathbb{C}$, then if we put $z=(z_1,...,z_n)\in\mathbb{C}^n$, $w=z_{n+1}$, $\alpha=(\alpha_1,...,\alpha_n)\in\mathbb{C}^n$, $\beta=\alpha_{n+1}\in\mathbb{C}$, we have

$$\left|\sum_{j,k=1}^{n+1}\frac{\partial^2 u}{\partial z_j\partial z_k}(z)\alpha_j\alpha_k\right|\leq\sum_{j,k=1}^{n+1}\frac{\partial^2 u}{\partial z_j\partial\overline{z_k}}(z)\alpha_j\overline{\alpha}_k,\ \forall(z,w)\in\mathbb{C}^n\times\mathbb{C},\ \forall(\alpha,\beta)\in\mathbb{C}^n\times\mathbb{C}.$$

It follows that g_5 is an affine function on $\mathbb{C}^n$. Therefore $g_5(z)=(<z/\lambda>+\mu)$, for each $z\in\mathbb{C}^n$, where $\lambda\in\mathbb{C}^n$ and $\mu\in\mathbb{C}$. Then

$$\begin{aligned} u(z,w) &= (|A_1|^2+|A_2|^2)(|w-<z/a>-b|^2+|\varphi(z)|^2)\\ &+ (|A_3|^2+|A_4|^2)(|w-<z/c>-d|^2+|\psi(z)|^2)\\ &+ |A_5w-<z/\lambda>-\mu|^2,\ (z,w)\in\mathbb{C}^n\times\mathbb{C}.\end{aligned}$$

Define

$$\begin{aligned} v(z,w) &= |w-<z/a>-b|^2+|\varphi(z)|^2+|w-<z/c>-d|^2\\ &+ |\psi(z)|^2+|A_5w-<z/\lambda>-\mu|^2,\ (z,w)\in\mathbb{C}^n\times\mathbb{C}.\end{aligned}$$

Then v is a function of class C^∞ on $\mathbb{C}^n\times\mathbb{C}$ and we have (u is strictly psh on $\mathbb{C}^n\times\mathbb{C}$ if and only if v is strictly psh on $\mathbb{C}^n\times\mathbb{C}$). Therefore by lemma 1, $n+1\leq 5$. Consequently, $n\in\{1,2,3,4\}$.

Now let $T(z,w)=(z,w+<z/a>)$, $(z,w)\in\mathbb{C}^n\times\mathbb{C}$. T is a $\mathbb{C}-$ linear bijective transformation on $\mathbb{C}^n\times\mathbb{C}$.
Let $v_2(z,w)=v\,\mathrm{o}\,T(z,w)=|w-b|^2+|\varphi(z)|^2+|w+<z/a-c>-d|^2+|\psi(z)|^2+|A_5w+<z/a-\lambda>-\mu|^2$, $(z,w)\in\mathbb{C}^n\times\mathbb{C}$.
Therefore v_2 is a function of class C^∞ on $\mathbb{C}^n\times\mathbb{C}$. We have v is strictly psh on $\mathbb{C}^n\times\mathbb{C}$ if and only if v_2 is strictly psh on $\mathbb{C}^n\times\mathbb{C}$. The Levi hermitian form of v_2 is
$L(v_2)(z,w)(\alpha,\beta)=|\beta|^2+|\sum_{j=1}^{n}\frac{\partial\varphi}{\partial z_j}(z)\alpha_j|^2+|\beta+<\alpha/a-c>|^2+$
$|\sum_{j=1}^{n}\frac{\partial\psi}{\partial z_j}(z)\alpha_j|^2+|A_5\beta+<\alpha/a-\lambda>|^2$, $(z,w)=((z_1,...,z_n),w)\in\mathbb{C}^n\times\mathbb{C}$, $(\alpha,\beta)=((\alpha_1,...,\alpha_n),\beta)\in\mathbb{C}^n\times\mathbb{C}$.
Now $L(v_2)(z,w)(\alpha,\beta)=0$ if and only if $\beta=0$ and

$$|\sum_{j=1}^{n}\frac{\partial\varphi}{\partial z_j}(z)\alpha_j|^2+|\beta+<\alpha/a-c>|^2+|\sum_{j=1}^{n}\frac{\partial\psi}{\partial z_j}(z)\alpha_j|^2+|A_5\beta+<\alpha/a-\lambda>|^2=0,$$

$(z,w),(\alpha,\beta)\in\mathbb{C}^n\times\mathbb{C}$. It follows that, if we define $u_2(z)=|\varphi(z)|^2+|<z/a-c>-d|^2+|\psi(z)|^2+|<z/a-\lambda>-\mu|^2$, for $z\in\mathbb{C}^n$, then u_2 is a function of class C^∞ on $\mathbb{C}^n$.
Now Observe that v_2 is strictly psh on $\mathbb{C}^n\times\mathbb{C}$ if and only if u_2 is strictly psh on $\mathbb{C}^n$.
Case 1. $n=4$. In this case observe that u is strictly psh on $\mathbb{C}^4\times\mathbb{C}$ if and only if the quantity

$$(a-c,a-\lambda,(\overline{\frac{\partial\varphi}{\partial z_1}}(z),\overline{\frac{\partial\varphi}{\partial z_2}}(z),\overline{\frac{\partial\varphi}{\partial z_3}}(z),\overline{\frac{\partial\varphi}{\partial z_4}}(z)),(\overline{\frac{\partial\psi}{\partial z_1}}(z),\overline{\frac{\partial\psi}{\partial z_2}}(z),\overline{\frac{\partial\psi}{\partial z_3}}(z),\overline{\frac{\partial\psi}{\partial z_4}}(z)))$$

is a basis of the complex vector space $\mathbb{C}^4$, for all $z=(z_1,z_2,z_3,z_4)\in\mathbb{C}^4$.
Case 2. $n=3$. The Levi hermitian form of u_2 is
$L(u_2)(z)(\alpha)=|\sum_{j=1}^{3}\frac{\partial\varphi}{\partial z_j}(z)\alpha_j|^2+|<\alpha/a-c>|^2+|\sum_{j=1}^{3}\frac{\partial\psi}{\partial z_j}(z)\alpha_j|^2+$
$|<\alpha/a-\lambda>|^2$, for $z=(z_1,z_2,z_3)\in\mathbb{C}^3$, $\alpha=(\alpha_1,\alpha_2,\alpha_3)\in\mathbb{C}^3$.
$L(u_2)(z)(\alpha)=0$ if and only if

$$\begin{cases}<\alpha/a-c>=0,\\ <\alpha/a-\lambda>=0,\\ \sum_{j=1}^{3}\frac{\partial\varphi}{\partial z_j}(z)\alpha_j=0,\text{ and}\\ \sum_{j=1}^{3}\frac{\partial\psi}{\partial z_j}(z)\alpha_j=0.\end{cases}$$

Therefore u_2 is strictly psh on $\mathbb{C}^3$ if and only if for all $z=(z_1,z_2,z_3)\in\mathbb{C}^3$, we can choose a basis (of the complex vector space $\mathbb{C}^3$) consisting of 3 vectors from the set of vectors

$\{a-c, a-\lambda, (\overline{\frac{\partial\varphi}{\partial z_1}}(z), \overline{\frac{\partial\varphi}{\partial z_2}}(z), \overline{\frac{\partial\varphi}{\partial z_3}}(z)), (\overline{\frac{\partial\psi}{\partial z_1}}(z), \overline{\frac{\partial\psi}{\partial z_2}}(z), \overline{\frac{\partial\psi}{\partial z_3}}(z))\}$.
Case 3. $n = 2$. In this case u_2 is strictly psh on $\mathbb{C}^2$ if and only if for all $z = (z_1, z_2) \in \mathbb{C}^2$, we can choose a basis (consisting by 2 vectors basis of the complex vector space $\mathbb{C}^2$) from the set $\{a-c, a-\lambda, (\overline{\frac{\partial\varphi}{\partial z_1}}(z), \overline{\frac{\partial\varphi}{\partial z_2}}(z)), (\overline{\frac{\partial\psi}{\partial z_1}}(z), \overline{\frac{\partial\psi}{\partial z_2}}(z))\}$.
Case 4. $n = 1$. u_2 is strictly sh on $\mathbb{C}$ if and only if for all $z \in \mathbb{C}$, we have $(a-c) \neq 0$, or $(a-\lambda) \neq 0$, or $(\frac{\partial\varphi}{\partial z}(z) \neq 0)$, or $(\frac{\partial\psi}{\partial z}(z) \neq 0)$.
The proof is now finished.

Moreover, we have

Question 6. Let $n, m, N \in \mathbb{N}\backslash\{0\}$ and $(A_1, B_1), ..., (A_N, B_N) \in \mathbb{C}^2\backslash\{(0,0)\}$. Find all the holomorphic functions $g_1, f_1, ..., g_N, f_N : \mathbb{C}^n \to \mathbb{C}$ and all the holomorphic (respectively prh) nonconstant functions $k_1, ..., k_N : \mathbb{C}^m \to \mathbb{C}$ such that $u_1, ..., u_N$ are convex and $u = (u_1 + ... + u_N)$ is strictly psh on $\mathbb{C}^n \times \mathbb{C}^m$, where $u_j(z,w) = |A_j k_j(w) - f_j(z)|^2 + |B_j k_j(w) - g_j(z)|^2$, for $(z,w) \in \mathbb{C}^n \times \mathbb{C}^m$, $1 \leq j \leq N$.
In general we prove that this question have applications in the theory of (partial differential equations and (convex and strictly psh functions) in several variables), and therefore for the resolution of certain holomorphic partial differential equations in complex analysis. Because, in the sequel, we have a relation between partial differential equations and the subject (convex and strictly psh functions) in complex analysis and geometry.

Example. Find all the holomorphic functions $f, g : \mathbb{C} \to \mathbb{C}$, such that
(a) $|f^2 + f|$ and $|g^2 - g|$ are convex functions on $\mathbb{C}$, and
(b) ψ is strictly psh on $\mathbb{C}^2$, where $\psi(z_1, z_2) = |f^2(z_1) + f(z_1)|^2 + |g^2(z_2) - g(z_2)|^2$, for $(z_1, z_2) \in \mathbb{C}^2$.
In this case we solve the holomorphic differential equation
$(f^2 + f)''(f^2 + f) = \gamma(2ff' + f')^2$, where $\gamma \in \{\frac{s-1}{s}, 1 \,/\, s \in \mathbb{N}\backslash\{0\}\}, ...$

Example. Let $N \geq 2$. Find all the holomorphic functions $f_1, ..., f_N : \mathbb{C}^n \to \mathbb{C}$, such that v is convex and strictly psh on $\mathbb{C}^n \times \mathbb{C}$. We can see the problem v is convex and v_1 is strictly psh on $\mathbb{C}^n \times \mathbb{C}$ (in this case we apply lemma 1). Where

$$v(z,w) = |w - f_1(z)|^2 + ... + |w - f_N(z)|^2,$$

$$v_1(z,w) = |w - \overline{f_1}(z)|^2 + ... + |w - \overline{f_N}(z)|^2,$$

for $(z,w) \in \mathbb{C}^n \times \mathbb{C}$. In this situation, we solve several holomorphic partial differential equations which characterize the complex structure strictly psh. Finally, we choose the solution which gives the convexity of v (or conversely).

Question 7. Let $n, m, k \in \mathbb{N}\backslash\{0\}$. Find all the holomorphic functions $g_1, g_2, g_3, g_4 : \mathbb{C}^n \to \mathbb{C}$ and all the holomorphic functions $\varphi_1, \varphi_2, \varphi_3, \varphi_4 : \mathbb{C}^m \to \mathbb{C}$ such that v_1 and v_2 are convex and $v = (v_1 + v_2)$ is strictly convex on $\mathbb{C}^n \times \mathbb{C}^m$, where

$$v_1(z,w) = |\varphi_1(w) - g_1(z)|^{2k} + |\varphi_2(w) - g_2(z)|^{2k},$$

$$v_2(z,w) = |\varphi_3(w) - g_3(z)|^{2k} + |\varphi_4(w) - g_4(z)|^{2k},$$

for $(z,w) \in \mathbb{C}^n \times \mathbb{C}^m$.

5. The product of several psh functions and applications

The main objective of this section is to study the behaviour of the product of several absolute values of prh functions. Note that it is well known that the product of many psh functions is not in general psh.

Example. Let $v_1(z,w) = |w - \overline{z}||w - 2\overline{z}|$, for $(z,w) \in \mathbb{C}^2$. Then v_1 is not psh on $\mathbb{C}^2$. In the sequel, let D be a domain of $\mathbb{C}^n$, $n \in \mathbb{N}\backslash\{0\}$, $N \in \mathbb{N}\backslash\{0,1\}$ and $\varphi_1, ..., \varphi_N : D \to \mathbb{C}$ be holomorphic functions. Define $u(z,w) = \prod_{1\le j\le N} |w - \overline{\varphi_j}(z)|$, for $(z,w) \in D \times \mathbb{C}$. Find conditions should satisfy N, $\varphi_1, ..., \varphi_N$ so that u is psh on $D \times \mathbb{C}$.
Now let $f_0, ..., f_{N-1} : D \to \mathbb{C}$ be holomorphic functions. Put $v(z,w) = |w^N + \overline{f_{N-1}}(z)w^{N-1} + ... + \overline{f_1}(z)w + \overline{f_0}(z)|$, $(z,w) \in D \times \mathbb{C}$. Characterize $N, f_{N-1}, ..., f_1, f_0$ such that v is psh on $D \times \mathbb{C}$.

Proposition 5. Let $f, g : D \to \mathbb{C}$ be two functions, D is a domain of $\mathbb{C}^n$, $n \ge 1$. Put $u(z,w) = |w^2 + f(z)w + g(z)|$, $(z,w) \in D \times \mathbb{C}$. Assume that f is continuous and g of class C^2 on D. Then u is psh on $D \times \mathbb{C}$ if and only if we have one assertion of the following conditions.
(I) f is holomorphic on D and g is prh on D.
(II) f is prh and not holomorphic and $g = \frac{f^2}{4}$ on D.

Proof. Put $v = u^2$. Assume that u is psh on $D \times \mathbb{C}$. Then v is psh on $D \times \mathbb{C}$. By Abidi [2], f is pluriharmonic (prh) on D. Thus v is a function of class C^2 on $D \times \mathbb{C}$. Without loss of generality we assume that $n = 1$. Let $(z,w) \in D \times \mathbb{C}$.

$$\frac{\partial^2 v}{\partial w \partial \overline{w}}(z,w) = |2w + f(z)|^2.$$

$$\frac{\partial^2 v}{\partial \overline{z} \partial w}(z,w) = \frac{\partial f}{\partial \overline{z}}(z)((\overline{w})^2 + \overline{w}\overline{f}(z) + \overline{g}(z)) + (2w + f(z))(\overline{w}\frac{\partial \overline{f}}{\partial \overline{z}}(z) + \frac{\partial \overline{g}}{\partial \overline{z}}(z)).$$

We have

$$|\frac{\partial^2 v}{\partial \overline{z} \partial w}(z,w)|^2 \le \frac{\partial^2 v}{\partial w \partial \overline{w}}(z,w)\frac{\partial^2 v}{\partial z \partial \overline{z}}(z,w)$$

for each $(z,w) \in \mathbb{C}^2$. Now observe that if $w = -\frac{f(z)}{2}$, then $\frac{\partial^2 v}{\partial \overline{w} \partial w}(z, -\frac{f(z)}{2}) = 0$.
It follows that $\frac{\partial^2 v}{\partial \overline{z} \partial w}(z, -\frac{f(z)}{2}) = 0 = \frac{\partial f}{\partial \overline{z}}(z)(\overline{g}(z) - \frac{\overline{f^2}(z)}{4})$, for each $z \in D$. Now since f is real analytic on D, then $\frac{\partial f}{\partial \overline{z}}(z) = 0$, for every $z \in D$, or there exists $z_0 \in D$, such that $\frac{\partial f}{\partial \overline{z}}(z_0) \ne 0$.
Case 1. For each $z \in D$, $\frac{\partial f}{\partial \overline{z}}(z) = 0$.
Then f is holomorphic on D. Since $u(z,w) = |(w + \frac{f(z)}{2})^2 - \frac{f^2(z)}{4} + g(z)|$, for $(z,w) \in D \times \mathbb{C}$, we consider $T(z,w) = (z, w - \frac{f(z)}{2})$, for $(z,w) \in D \times \mathbb{C}$. T is a biholomorphism on $D \times \mathbb{C}$. Therefore $u \circ T$ is psh on $D \times \mathbb{C}$. $u \circ T(z,w) = |w^2 - \frac{f^2(z)}{4} + g(z)|$, $(z,w) \in D \times \mathbb{C}$. By Abidi [1], the function $(\frac{f^2}{4} - g)$ is harmonic on D. Consequently, g is harmonic on D.

Case 2. There exists $z_0 \in D$ such that $\frac{\partial f}{\partial \overline{z}}(z_0) \neq 0$.
We consider $E = \{\xi \in D \ / \ \frac{\partial f}{\partial \overline{\xi}}(\xi) = 0\}$. Since $\frac{\partial f}{\partial \overline{\xi}}$ is antianalytic on D, then E is an analytic closed subset on D. Therefore, $D \backslash E$ is a domain dense on D. Now since the function $(\frac{f^2}{4} - g)$ is continuous on D and $(\frac{f^2}{4} - g) = 0$ on $D \backslash E$, then $(\frac{f^2}{4} - g) = 0$ on D.
Let us mention that, if $n \geq 2$ and $f = (f_1 + \overline{f_2})$ is not holomorphic on an open polydisc $P = P_1 \times ... \times P_n \subset D$, where $f_1, f_2 : P \to \mathbb{C}$ are holomorphic functions, $P_1, ..., P_n$ are discs on $\mathbb{C}$. Since f_2 is nonconstant on P, we assume that $|\frac{\partial f_2}{\partial z_1}| > 0$ on P.
Thus $|\frac{\partial^2 v}{\partial \overline{z_1} \partial w}|^2 \leq \frac{\partial^2 v}{\partial \overline{z_1} \partial z_1} \frac{\partial^2 v}{\partial \overline{w} \partial w}$ on P. Since $\frac{\partial^2 v}{\partial \overline{w} \partial w}(z, -\frac{\overline{f(z)}}{2}) = 0$, then $\frac{\partial^2 v}{\partial \overline{z_1} \partial w}(z, -\frac{\overline{f(z)}}{2}) = 0$, for each $z \in P$. We obtain $\frac{\partial f_2}{\partial z_1}[\frac{f^2}{4} - g] = 0$ on P. Consequently, $g = \frac{f^2}{4}$ on P.
Now since f is not holomorphic on each not empty open polydisc subset of D, it follows that $g = \frac{f^2}{4}$ on D. The proof in now complete. $\square$

Now we have

Theorem 13. *Let $f, g, k : \mathbb{C}^n \to \mathbb{C}$, $n \geq 1$.*
Define $u(z, w) = |w^3 + w^2 f(z) + w g(z) + k(z)|$, for $(z, w) \in \mathbb{C}^n \times \mathbb{C}$.
Assume that f is continuous on $\mathbb{C}^n$ and g and k are functions of class C^2 on $\mathbb{C}^n$.
Then u is psh on $\mathbb{C}^n \times \mathbb{C}$ if and only if we have the following two cases.
Case 1. f and g are holomorphic functions and k is prh on $\mathbb{C}^n$.
Case 2. f is prh and not holomorphic on $\mathbb{C}^n$.
Put $q(w) = 3w^2 + 2w f(z) + g(z)$, for each $w \in \mathbb{C}$ and every fixed z on $\mathbb{C}^n$.
q have an only one zero on $\mathbb{C}$, for each z fixed on $\mathbb{C}^n$, (therefore $g(z) = \frac{f^2(z)}{3}$ and $k(z) = \frac{f^3(z)}{27}$).

Proof. Put $v = u^2$. Assume that u is psh on $\mathbb{C}^n \times \mathbb{C}$. Then v is psh on $\mathbb{C}^n \times \mathbb{C}$. We can prove that f is prh on $\mathbb{C}^n$, using Abidi [2]. Therefore v is a function of class C^2 on $\mathbb{C}^n \times \mathbb{C}$.
Case 1. The function f is holomorphic on $\mathbb{C}^n$.
$w^3 + w^2 f(z) + w g(z) + k(z) = (w + \frac{f(z)}{3})^3 + w(g(z) - \frac{f^2(z)}{3}) - \frac{f^3(z)}{27} + k(z)$, for $(z, w) \in \mathbb{C}^n \times \mathbb{C}$.
Since psh functions are invariant by any change by holomorphic functions, we can replace $(w + \frac{f(z)}{3})$ by w, we obtain
$w^3 + (w - \frac{f(z)}{3})(g(z) - \frac{f^2(z)}{3}) - \frac{f^3(z)}{27} + k(z) = w^3 + w(g(z) - \frac{f^2(z)}{3}) + k_1(z)$, k_1 is a function of class C^2 on $\mathbb{C}^n$.
Now using the proof described in [2], we can prove that g is prh on $\mathbb{C}^n$. Suppose that g is holomorphic on $\mathbb{C}^n$. We can prove that k is prh on $\mathbb{C}^n$. Therefore $u = |h|$, where $h : \mathbb{C}^n \times \mathbb{C} \to \mathbb{C}$ is prh. Then u is psh on $\mathbb{C}^n \times \mathbb{C}$. Suppose that g is not holomorphic on $\mathbb{C}^n$. Assume that $n = 1$. We have

$$
\begin{aligned}
\frac{\partial^2 v}{\partial w\partial\overline{w}}(z,w) &= |3w^2+2wf(z)+g(z)|^2.\\
\frac{\partial^2 v}{\partial w\partial\overline{z}}(z,w) &= (2w\frac{\partial f}{\partial\overline{z}}(z)+\frac{\partial g}{\partial\overline{z}}(z))(w^3+f(z)w^2+g(z)w+k(z))\\
&+ (3w^2+2wf(z)+g(z))((\overline{w})^2\frac{\partial\overline{f}}{\partial\overline{z}}(z)+\overline{w}\frac{\partial\overline{g}}{\partial\overline{z}}(z)+\frac{\partial\overline{k}}{\partial\overline{z}}(z)).
\end{aligned}
$$

Since v is psh then we have the inequality

$$
(E):\ |\frac{\partial^2 v}{\partial w\partial\overline{z}}(z,w)|^2 \leq \frac{\partial^2 v}{\partial w\partial\overline{w}}(z,w)\frac{\partial^2 v}{\partial z\partial\overline{z}}(z,w)
$$

for each $(z,w)\in\mathbb{C}^2$.
Since g is not holomorphic on $\mathbb{C}$, then there exists $z_0\in\mathbb{C}$, such that $|\frac{\partial g}{\partial\overline{z}}|>0$ on a neighborhood of z_0.
Let $q_1(w)=w^3+w^2f(z)+wg(z)+k(z)$ and $q_2(w)=3w^2+2wf(z)+g(z)$, for $(z,w)\in\mathbb{C}^2$.
Note that q_1 and q_2 are holomorphic polynomials in the variable $w\in\mathbb{C}$, for each fixed $z\in\mathbb{C}$. Also $q_1'=q_2$. The holomorphic polynomial q_2 has two zeros denoted w_1 and $w_2\in\mathbb{C}$.
Assume that $w_1\neq w_2$. Then w_1 and w_2 are distinct zeros of the polynomial q_1 by the inequality (E). Since $q_1'=q_2$ then w_1 and w_2 are two distinct zeros of order 2 of q_1. A contradiction because $\deg(q_1)=3$. Therefore $w_1=w_2$ is a zero of q_2 of order 2. Thus w_1 is a zero of q_1 of order 3. Then we have $q_1(w)=(w-w_1)^3$, for every $w\in\mathbb{C}$. Consequently, $f=-3w_1$ and then $q_1(w)=(w+\frac{f(z)}{3})^3$, for each z in a neighborhood of z_0. Then $g(z)=\frac{f^2(z)}{3}$ and therefore g is holomorphic in a neighborhood of z_0. A contradiction. This step is impossible.
Case 2. The function f is not holomorphic on $\mathbb{C}^n$.
Assume that $n=1$. Therefore $\frac{\partial f}{\partial\overline{z}}\neq 0$. Put $q_1(w)=w^3+w^2f(z)+wg(z)+k(z)$, $q_2(w)=3w^2+2wf(z)+g(z)$, $q_3(w)=2w\frac{\partial f}{\partial\overline{z}}(z)+\frac{\partial g}{\partial\overline{z}}(z)$, for $(z,w)\in\mathbb{C}^2$. Note that q_1, q_2 and q_3 are holomorphic polynomials in the variable $w\in\mathbb{C}$, for every fixed $z\in\mathbb{C}$. We have $q_1'=q_2$. Let $z_0\in\mathbb{C}$, such that $\frac{\partial f}{\partial\overline{z}}(z)\neq 0$, for every $z\in V_0$, where V_0 is an Euclidean open disc in $\mathbb{C}$, $z_0\in V_0$. Now q_2 have two zeros $w_0(z)$ and $w_1(z)\in\mathbb{C}$. Suppose that $w_0(z)=w_1(z)$. From the inequality (E), w_0 is a zero of q_1. Since $q_2'=q_1$, then w_0 is a zero of q_1 of order 3. Therefore $q_1(w)=(w-w_0)^3$. If for every $z\in V_0$, $w_0(z)=w_0=w_1(z)=w_1$, then $q_1(w)=(w-w_0(z))^3=(w+\frac{f(z)}{3})^3$, in $V_0\times\mathbb{C}$. Then $g=\frac{f^2}{3}$ and $k=\frac{f^3}{27}$. If there exists $z_1\in V_0$ such that $w_0(z_1)=a\neq w_1(z_1)=b$. The condition a and b are zeros of q_1 is impossible because $\deg(q_1)=3$. By the inequality (E), for example we have b is a zero of q_1 of order 2 and a is a zero of q_3.
Let w_2 the second zero of q_1 of order 1. Then we have the following relations between the zeros and the coefficients of the polynomial q_3, $a+b=-2\frac{f}{3}$, $ab=3g$ and $2b+w_2=-f$. Thus we have the equalities $3a+3b=4b+2w_2$, $3a=b+2w_2$ and $b^2+2bw_2=g=\frac{1}{3}ab$.

If $b \neq 0$ on a neighborhood of z_1., then $b + 2w_2 = \frac{1}{3}a = 3a$. Consequently, $a = 0$. Therefore $g = 0$ and $k \neq 0$. We have then $b = -2\frac{f}{3}$, $w_2 = \frac{f}{3}$. Thus u defined by, $u(z,w) = |w + 2\frac{f}{3}(z)|^2|w - \frac{f}{3}(z)|$, is psh on $\mathbb{C}^2$. Put $u_1(z,w) = |w+2f(z)|^2|w-f(z)|$, for $(z,w) \in \mathbb{C}^2$. Then u_1 is psh on $\mathbb{C}^2$. But it is obvious (by theorem 14 below), that f is holomorphic on $\mathbb{C}$. A contradiction. Consequently, $b = 0$ on a neighborhood of z_1. Thus $w_2 = -f$ and $a = 0$ (because $g = 0$ on a neighborhood of z_1). Therefore $q_2(w) = 3w^2$ and observe that $f = 0$. A contradiction. Therefore, the assumption $a \neq b$ is impossible. It follows that $w_0(z) = w_1(z)$, for each $z \in V_0$. Therefore $q_1(w) = (w - w_0)^3$, for each $w \in \mathbb{C}$. We obtain $g = \frac{f^2}{3}$ and $k = \frac{f^3}{27}$.
Assume now that $n \geq 2$. Obviously we consider in this situation an analogous proof of the above theorem as well. The proof is now complete.

Recall that for each $f : D \to \mathbb{C}$, ψ is psh on $D \times \mathbb{C}$ if and only if f is pluriharmonic (prh) on D, where $\psi(z,w) = |w - f(z)|^N$, $N \in \mathbb{N}\backslash\{0\}$, D is a domain of $\mathbb{C}^n$ and $(z,w) \in D \times \mathbb{C}$. Now we prove that there exists a similar characterization of holomorphic functions. We have

Theorem 14. *Let $f : \mathbb{C}^n \to \mathbb{C}$ be continuous. Put $u(z,w) = |w + 2f(z)|^2|w - f(z)|$, $(z,w) \in \mathbb{C}^n \times \mathbb{C}$. Then u is psh on $\mathbb{C}^n \times \mathbb{C}$ if and only if f is holomorphic on $\mathbb{C}^n$.*

Proof. Assume that u is psh on $\mathbb{C}^n \times \mathbb{C}$. Since $u(z,w) = |w^3 + 3f(z)w^2 - 4f^3(z)|$, for each $(z,w) \in \mathbb{C}^n \times \mathbb{C}$. Then f is prh on $\mathbb{C}^n$, (see [2], page 336). In particular, f is a function of class C^∞ on $\mathbb{C}^n$. If f is holomorphic on $\mathbb{C}^n$, then u is psh on $\mathbb{C}^n \times \mathbb{C}$. Assume that f is not holomorphic on $\mathbb{C}^n$. Then f is nonconstant. Without loss of generality we suppose that $n = 1$ in all of the rest of the proof.
Case 1. The function $g = \overline{f}$ is holomorphic on $\mathbb{C}$.
Put $v = u^2$. Then $v(z,w) = |w^3 + 3\overline{g}(z)w^2 - 4\overline{g^3}(z)|^2$, $(z,w) \in \mathbb{C}^2$. Note that v is a function of class C^∞ on $\mathbb{C}^2$. We have

$$\begin{aligned}
\frac{\partial^2 v}{\partial \overline{z}\partial w}(z,w) &= 6\frac{\partial \overline{g}}{\partial \overline{z}}(z)w[\overline{w^3 + 3\overline{g}(z)w^2 - 4\overline{g^3}(z)}],\\
\frac{\partial^2 v}{\partial \overline{w}\partial w}(z,w) &= |3w^2 + 6\overline{g}(z)w|^2,\\
\frac{\partial^2 v}{\partial \overline{z}\partial z}(z,w) &= |3\frac{\partial \overline{g}}{\partial \overline{z}}(z)w^2 - 12(\overline{g})^2(z)\frac{\partial \overline{g}}{\partial \overline{z}}(z)|^2.
\end{aligned}$$

Suppose that $\frac{\partial g}{\partial z} = 0$ on $\mathbb{C}$. Then g is constant on $\mathbb{C}$. It follows that f is constant on $\mathbb{C}$. A contradiction. Consequently, $\frac{\partial g}{\partial z} \neq 0$. Since $\frac{\partial^2 v}{\partial \overline{z}\partial z}(z, 2\overline{g}(z)) = 0$ and $|\frac{\partial^2 v}{\partial \overline{z}\partial w}(z,w)|^2 \leq \frac{\partial^2 v}{\partial \overline{z}\partial z}(z,w)\frac{\partial^2 v}{\partial \overline{w}\partial w}(z,w)$, for each $(z,w) \in \mathbb{C}^2$, then $\frac{\partial^2 v}{\partial \overline{z}\partial w}(z, 2\overline{g}(z)) = 0$, for any $z \in \mathbb{C}$. Thus $\overline{g}(z)[16(\overline{g}^3)(z)] = 0$, for all $z \in \mathbb{C}$. It follows that $g = 0$ on $\mathbb{C}$. A contradiction. Therefore this case is impossible.
Case 2. The function $g = \overline{f}$ is not holomorphic on $\mathbb{C}$.
Let $v = u^2$. Then v is a function of class C^∞ and psh on $\mathbb{C}^2$. Let $g_1, g_2 : \mathbb{C} \to \mathbb{C}$ be two harmonic functions and $(z,w) \in \mathbb{C}^2$. Define $F(z,w) = (w - g_1(z))^2(\overline{w} - \overline{g_1}(z))^2(w - g_2(z))(\overline{w} - \overline{g_2}(z))$. Note that F is a C^∞ function on $\mathbb{C}^2$. Assume that F is psh on $\mathbb{C}^2$. We have

$\frac{\partial^2 F}{\partial \overline{w} \partial w}(z,w) = |2(w-g_2(z)) + (w-g_1(z))|^2 |w-g_1(z)|^2$.

$\frac{\partial^2 F}{\partial \overline{z} \partial w}(z,w) = -2\frac{\partial g_1}{\partial \overline{z}}(z)(\overline{w}-\overline{g_1}(z))^2(w-g_2(z))(\overline{w}-\overline{g_2}(z)) - 4\frac{\partial \overline{g_1}}{\partial \overline{z}}(z)(w-g_1(z))(\overline{w}-\overline{g_1}(z))(w-g_2(z))(\overline{w}-\overline{g_2}(z)) - 2\frac{\partial g_2}{\partial \overline{z}}(z)(w-g_1(z))(\overline{w}-\overline{g_1}(z))^2(\overline{w}-\overline{g_2}(z)) - 2\frac{\partial \overline{g_2}}{\partial \overline{z}}(z)(w-g_1(z))(\overline{w}-\overline{g_1}(z))^2(w-g_2(z)) - \frac{\partial \overline{g_2}}{\partial \overline{z}}(z)(w-g_1(z))^2(\overline{w}-\overline{g_1}(z))^2 - 2\frac{\partial g_1}{\partial \overline{z}}(z)(w-g_1(z))(\overline{w}-\overline{g_2}(z))(\overline{w}-\overline{g_1}(z))^2 - 2\frac{\partial \overline{g_1}}{\partial \overline{z}}(z)(\overline{w}-\overline{g_1}(z))(\overline{w}-\overline{g_2}(z))(w-g_1(z))^2$.

$\frac{\partial^2 F}{\partial \overline{z} \partial z}(z,w) = 2\frac{\partial g_1}{\partial \overline{z}}(z)\frac{\partial g_1}{\partial z}(z)(\overline{w}-\overline{g_1}(z))^2(w-g_2(z))(\overline{w}-\overline{g_2}(z)) + 4\frac{\partial g_1}{\partial \overline{z}}(z)\frac{\partial \overline{g_1}}{\partial z}(z)(\overline{w}-\overline{g_1}(z))(w-g_1(z))(w-g_2(z))(\overline{w}-\overline{g_2}(z)) + 2\frac{\partial g_1}{\partial \overline{z}}(z)\frac{\partial g_2}{\partial z}(w-g_1(z))(z)(\overline{w}-\overline{g_1}(z))^2(\overline{w}-\overline{g_2}(z)) + 2\frac{\partial g_1}{\partial \overline{z}}(z)\frac{\partial \overline{g_2}}{\partial z}(z)(w-g_1(z))(\overline{w}-\overline{g_1}(z))^2(w-g_2(z)) + 2\frac{\partial \overline{g_1}}{\partial \overline{z}}(z)\frac{\partial \overline{g_1}}{\partial z}(z)(w-g_1(z))^2(w-g_2(z))(\overline{w}-\overline{g_2}(z)) + 4\frac{\partial \overline{g_1}}{\partial \overline{z}}(z)\frac{\partial g_1}{\partial z}(z)(w-g_1(z))(\overline{w}-\overline{g_1}(z))(w-g_2(z))(\overline{w}-\overline{g_2}(z)) + 2\frac{\partial \overline{g_1}}{\partial \overline{z}}(z)\frac{\partial g_2}{\partial z}(z)(\overline{w}-\overline{g_1}(z))(w-g_1(z))^2(\overline{w}-\overline{g_2}(z)) +$
$2\frac{\partial \overline{g_1}}{\partial \overline{z}}(z)\frac{\partial \overline{g_2}}{\partial z}(z)(\overline{w}-\overline{g_1}(z))(w-g_1(z))^2(w-g_2(z)) + 2\frac{\partial g_2}{\partial \overline{z}}(z)\frac{\partial g_1}{\partial z}(z)(w-g_1(z))(\overline{w}-\overline{g_1}(z))^2(\overline{w}-\overline{g_2}(z)) + 2\frac{\partial g_2}{\partial \overline{z}}(z)\frac{\partial g_1}{\partial z}(z)(w-g_1(z))(\overline{w}-\overline{g_1}(z))^2(\overline{w}-\overline{g_2}(z)) +$
$2\frac{\partial g_2}{\partial \overline{z}}(z)\frac{\partial \overline{g_1}}{\partial z}(z)(\overline{w}-\overline{g_1}(z))(\overline{w}-\overline{g_2}(z))(w-g_1(z))^2 + \frac{\partial g_2}{\partial \overline{z}}(z)\frac{\partial \overline{g_2}}{\partial z}(z)(w-g_1(z))^2(\overline{w}-\overline{g_1}(z))^2 + 2\frac{\partial \overline{g_2}}{\partial \overline{z}}(z)\frac{\partial g_1}{\partial z}(z)(w-g_1(z))(w-g_2(z))(\overline{w}-\overline{g_1}(z))^2 + 2\frac{\partial \overline{g_2}}{\partial \overline{z}}(z)\frac{\partial \overline{g_1}}{\partial z}(z)(\overline{w}-\overline{g_1}(z))(w-g_1(z))^2(w-g_2(z))$.

Let $\eta > 0$. Observe that if we replace g_1 and g_2 respectively by ηg_1 and ηg_2, the new function F_1, defined by $F_1(z,w) = |w-\eta g_1(z)|^4 |w-\eta g_2(z)|^2$ for $(z,w) \in \mathbb{C}^2$, is also of class C^∞ and psh on $\mathbb{C}^2$.

Therefore if we divide by η^2 and letting η go to 0, then

$$\lim_{\eta \to 0^+} \frac{1}{\eta^2} |\frac{\partial^2 F_1}{\partial \overline{z} \partial w}(z,w)|^2 \leq \lim_{\eta \to 0^+} [\frac{1}{\eta^2} \frac{\partial^2 F_1}{\partial \overline{w} \partial w}(z,w) \frac{\partial^2 F_1}{\partial \overline{z} \partial z}(z,w)].$$

Let $N \in \mathbb{N} \backslash \{0\}$. Write $f = f_1 + \overline{f_2}$, where f_1 and f_2 are holomorphic functions on $\mathbb{C}$. Consider $T(z,w) = (z, w + Nf_1(z))$, $(z,w) \in \mathbb{C}^2$. T is a biholomorphism of $\mathbb{C}^2$. Therefore $u \,\mathrm{o}\, T$ is a function of class C^∞ and psh on $\mathbb{C}^2$.

$$u \,\mathrm{o}\, T(z,w) = |w + (N+2)f_1(z) 2\overline{f_2}(z)|^2 |w + (N-1)f_1(z) - \overline{f_2}(z)|.$$

Define $g_1 = -(N+2)f_1 + 2\overline{f_2}$ and $g_2 = -(N-1)f_1 + \overline{f_2}$ on $\mathbb{C}$.
g_1 and g_2 are harmonic functions on $\mathbb{C}$.
Thus for $w_0 = 1$ and using the above inequality and letting η go to 0
(we replace g_1 and g_2 respectively by ηg_1 and ηg_2).
We obtain
$|2\frac{\partial g_1}{\partial \overline{z}}(z) + 4\frac{\partial \overline{g_1}}{\partial \overline{z}}(z) + 2\frac{\partial g_2}{\partial \overline{z}}(z) + 2\frac{\partial \overline{g_2}}{\partial \overline{z}}(z) + \frac{\partial \overline{g_2}}{\partial \overline{z}}(z) + 2\frac{\partial g_1}{\partial \overline{z}}(z) + 2\frac{\partial \overline{g_1}}{\partial \overline{z}}(z)|^2 \leq$
$9[2\frac{\partial g_1}{\partial \overline{z}}(z)\frac{\partial g_1}{\partial z}(z) + 4\frac{\partial g_1}{\partial \overline{z}}(z)\frac{\partial \overline{g_1}}{\partial z}(z) + 2\frac{\partial g_1}{\partial \overline{z}}(z)\frac{\partial g_2}{\partial z}(z) + 2\frac{\partial g_1}{\partial \overline{z}}(z)\frac{\partial \overline{g_2}}{\partial z}(z) + 2\frac{\partial \overline{g_1}}{\partial \overline{z}}(z)\frac{\partial \overline{g_1}}{\partial z}(z) + 4\frac{\partial \overline{g_1}}{\partial \overline{z}}(z)\frac{\partial g_1}{\partial z}(z) + 2\frac{\partial \overline{g_1}}{\partial \overline{z}}(z)\frac{\partial g_2}{\partial z}(z) + 2\frac{\partial \overline{g_1}}{\partial \overline{z}}(z)\frac{\partial \overline{g_2}}{\partial z}(z) + 2\frac{\partial g_2}{\partial \overline{z}}(z)\frac{\partial g_1}{\partial z}(z) + 2\frac{\partial g_2}{\partial \overline{z}}(z)\frac{\partial \overline{g_1}}{\partial z}(z) + \frac{\partial g_2}{\partial \overline{z}}(z)\frac{\partial \overline{g_2}}{\partial z}(z) + 2\frac{\partial \overline{g_2}}{\partial \overline{z}}(z)\frac{\partial g_1}{\partial z}(z) + \frac{\partial \overline{g_2}}{\partial \overline{z}}(z)\frac{\partial g_2}{\partial z}(z) + 2\frac{\partial \overline{g_2}}{\partial \overline{z}}(z)\frac{\partial \overline{g_1}}{\partial z}(z)]$.

Then $|4\frac{\partial \overline{f_2}}{\partial \overline{z}}(z) + 4(N+2)\frac{\partial \overline{f_1}}{\partial \overline{z}}(z) + 2\frac{\partial \overline{f_2}}{\partial \overline{z}}(z) + 3(N-1)\frac{\partial \overline{f_1}}{\partial \overline{z}}(z) + 4\frac{\partial \overline{f_2}}{\partial \overline{z}}(z) + 2(N+2)\frac{\partial \overline{f_1}}{\partial \overline{z}}(z)|^2 \leq$
$9[4(N+2)^2|\frac{\partial f_1}{\partial z}(z)|^2 + 2(N+2)(N-1)|\frac{\partial f_1}{\partial z}(z)|^2 + 2(N-1)(N+2)|\frac{\partial f_1}{\partial z}(z)|^2 + A(N,z)]$,

where $A(N,z)$ is a function defined on $\mathbb{N}\times\mathbb{C}$ and satisfy $\lim\limits_{N\to+\infty}\dfrac{1}{N^2}A(N,z)=0$, for each z fixed on $\mathbb{C}$.
We divide the last above inequality by N^2 and letting N go to $+\infty$. We obtain $9\times 9|\frac{\partial f_1}{\partial z}(z)|^2\le 9\times 8|\frac{\partial f_1}{\partial z}(z)|^2$, for all $z\in\mathbb{C}$. Thus $\frac{\partial f_1}{\partial z}(z)=0$, for each $z\in\mathbb{C}$.
Consequently, f_1 is constant on $\mathbb{C}$. Write $f_1=c$, $c\in\mathbb{C}$. Therefore $f=c+\overline{f_2}$ on $\mathbb{C}$. It follows that $g=\overline{f}$ is holomorphic on $\mathbb{C}$. A contradiction.
Consequently, this case is impossible. Therefore the above hypothesis is false and f is holomorphic on $\mathbb{C}$. The converse is obvious.

Remark 6. To compare the above theorem and some results of [2], observe that we can not write $(w+2f)(w-f)$ on the form $p(w-f)$, where p is a holomorphic polynomial on $\mathbb{C}$ and $f:\mathbb{C}\to\mathbb{C}$, $f\neq 0$. But if q is the following holomorphic polynomial on $\mathbb{C}^2$, defined by $q(\xi,w)=(w+2\xi)(w-\xi)$, for $(\xi,w)\in\mathbb{C}^2$, we can write $(w+2f)(w-f)=q(f,w)$. Denote $\psi(\xi,w)=|q(\overline{\xi},w)|$. Then ψ is not psh on $\mathbb{C}^2$. (We say in this case that $|q|$ characterize holomorphic functions).

Proposition 6. *Let $f_1,g_1,f_2,g_2,f_3,g_3:\mathbb{C}^n\to\mathbb{C}$ be holomorphic functions. Define $u(z,w)=|w-f_1(z)-\overline{g_1}(z)||w-f_2(z)-\overline{g_2}(z)||w-f_3(z)-\overline{g_3}(z)|$, for $(z,w)\in\mathbb{C}^n\times\mathbb{C}$. The following conditions are equivalent*
(I) u is psh on $\mathbb{C}^n\times\mathbb{C}$;
(II) We have only case 1, or case 2.
Case 1. $(g_1+g_2+g_3)$ is constant,
$(f_1+\overline{g_1})(f_2+\overline{g_2})+(f_1+\overline{g_1})(f_3+\overline{g_3})+(f_2+\overline{g_2})(f_3+\overline{g_3})$ is holomorphic on $\mathbb{C}^n$ and $(f_1+\overline{g_1})(f_2+\overline{g_2})(f_3+\overline{g_3})$ is prh on $\mathbb{C}^n$.
Case 2. $(g_1+g_2+g_3)$ is non constant and
$(f_1+\overline{g_1})(f_2+\overline{g_2})+(f_1+\overline{g_1})(f_3+\overline{g_3})+(f_2+\overline{g_2})(f_3+\overline{g_3})=\frac{1}{3}(f_1+f_2+f_3+\overline{g_1}+\overline{g_2}+\overline{g_3})^2$ on $\mathbb{C}^n$.

Proof. Obvious by the preceding theorem. In general, we have the following problems.

Problem 1. Let $n,N\in\mathbb{N}$, $N\ge 2$, D is a domain of $\mathbb{C}^n$. Find all the analytic functions $g_1,...,g_N:D\to\mathbb{C}$, such that u is psh on $D\times\mathbb{C}$. Here $u(z,w)=|w-\overline{g_1}(z)|...|w-\overline{g_N}(z)|$, for $(z,w)\in D\times\mathbb{C}$.

Problem 2. Let $v(z,w)=|f_1(z)-\overline{g_1}(w)|...|f_N(z)-\overline{g_N}(w)|$, $f_1,...,f_N:\mathbb{C}^n\to\mathbb{C}$ and $g_1,...,g_N:\mathbb{C}^m\to\mathbb{C}$ be $2N$ holomorphic functions, $N\ge 2$ and $(z,w)\in\mathbb{C}^n\times\mathbb{C}^m$. Find all the conditions described by $f_1,...,f_N,g_1,...,g_N$ such that v is convex on $\mathbb{C}^n\times\mathbb{C}^m$.

Problem 3. Put $v=|g_1-\varphi_1|...|g_N-\varphi_N|$, where $g_1,...,g_N:\mathbb{C}^m\to\mathbb{C}$ and $\varphi_1,...,\varphi_N:\mathbb{C}^n\to\mathbb{C}$ be $2N$ prh functions. Establish all the conditions satisfying by $g_1,...,g_N$, $\varphi_1,...,\varphi_N$ such that v is psh on $\mathbb{C}^n\times\mathbb{C}^m$.

Problem 4. Let $a_1,...,a_N\in\mathbb{C}^m$, $\varphi_1,...,\varphi_N:\mathbb{C}^n\to\mathbb{C}$ be holomorphic functions, $N\ge 2$. Put $v(z,w)=|<w/a_1>-\overline{\varphi_1}(z)|...|<w/a_N>-\overline{\varphi_N}(z)|$, $(z,w)\in\mathbb{C}^n\times\mathbb{C}^m$. Characterize $a_1,...,a_N$, $\varphi_1,...,\varphi_N$, such that v is psh on $\mathbb{C}^n\times\mathbb{C}^m$.

Remark 7. Let $v_N(z,w)=|w-\overline{\varphi_1}(z)|...|w-\overline{\varphi_N}(z)|$, $\varphi_1,...,\varphi_N:\mathbb{C}^n\to\mathbb{C}$ be holomorphic functions, $N\ge 2$, $(z,w)\in\mathbb{C}^n\times\mathbb{C}$.

Consider the problem $(E_N) : v_N$ is psh on $\mathbb{C}^n \times \mathbb{C}$.
A technical key for the study of the problem (E_N) is a consequence of the classical cases (E_2), (E_3) and (E_4) which are proved. Note that if $|w-\overline{\varphi_1}||w-\overline{\varphi_2}|$ is psh then for each holomorphic function φ_3, the new function $|w-\overline{\varphi_1}||w-\overline{\varphi_2}||w-\overline{\varphi_3}|$ is not psh on $\mathbb{C}^n \times \mathbb{C}$ if φ_1 and φ_2 are nonconstant functions and $\varphi_3 \neq \varphi_1$, or $\varphi_3 \neq \varphi_2$.
The converse. Let $u(z,w) = |w-\overline{\varphi_1}(z)||w-\overline{\varphi_2}(z)||w-\overline{\varphi_3}(z)|$. Suppose that u is psh on $\mathbb{C}^n \times \mathbb{C}$ and φ_j is non constant, $1 \leq j \leq 3$. Then $|w-\overline{\varphi_1}||w-\overline{\varphi_2}|$, $|w-\overline{\varphi_1}||w-\overline{\varphi_3}|$ and $|w-\overline{\varphi_2}||w-\overline{\varphi_3}|$ are not psh if $(\varphi_1+\varphi_2+\varphi_3)$ is constant, $\varphi_1 \neq \varphi_2$, $\varphi_1 \neq \varphi_3$ and $\varphi_2 \neq \varphi_3$.
Recall that u is psh on $\mathbb{C}^n \times \mathbb{C}$ if and only if $(\varphi_1+\varphi_2+\varphi_3)$ is constant and $(\varphi_1\varphi_2+\varphi_1\varphi_3+\varphi_2\varphi_3)$ is constant, or $(\varphi_1+\varphi_2+\varphi_3)$ is nonconstant and $\varphi_1=\varphi_2=\varphi_3$ on $\mathbb{C}^n$.

Remark 8. Consider the functions $g_1(z) = z$, $g_2(z) = -z$, $g_3(z) = iz$, $g_4(z) = -iz$, for $z \in \mathbb{C}$. g_1, g_2, g_3 and g_4 are holomorphic functions on $\mathbb{C}$. Let $(z,w) \in \mathbb{C}^2$.
$v(z,w) = |w-\overline{g_1}(z)||w-\overline{g_2}(z)||w-\overline{g_3}(z)||w-\overline{g_4}(z)| = |w^2-(\overline{z})^2||w^2+(\overline{z})^2| = |w^4-(\overline{z})^4|$.
$v = |h|$, where $h : \mathbb{C}^2 \to \mathbb{C}$ is prh. Then v is psh on $\mathbb{C}^2$. But v_1, v_2, v_3 and v_4 are not psh functions on $\mathbb{C}^2$, where

$$v_1(z,w) = |w-\overline{g_1}(z)||w-\overline{g_2}(z)||w-\overline{g_3}(z)|,$$

$$v_2(z,w) = |w-\overline{g_1}(z)||w-\overline{g_2}(z)||w-\overline{g_4}(z)|,$$

$$v_3(z,w) = |w-\overline{g_1}(z)||w-\overline{g_3}(z)||w-\overline{g_4}(z)|,$$

$$v_4(z,w) = |w-\overline{g_2}(z)||w-\overline{g_3}(z)||w-\overline{g_4}(z)|.$$

Note that a precise study of the plurisubharmonicity of the two functions ψ_1 and ψ_2 extends some interesting and sharp results in the framework of a slightly different direction. We can study the complex nature of the function $\psi_3(z,w) = |w-\overline{g_1}(z)|...|w-\overline{g_N}(z)|$, where ($N = 2^k$, or $N = 3\times 2^k$, $k \in \mathbb{N}$, $k \geq 2$), $g_1,...,g_N : \mathbb{C}^n \to \mathbb{C}$ are holomorphic functions, $\psi_1(z,w) = \prod_{1\leq j\leq 4} |w-\overline{\varphi_j}(z)|$, $\psi_2(z,w) = \prod_{1\leq j\leq 8} |w-\overline{\varphi_j}(z)|$ and $\varphi_j : \mathbb{C}^n \to \mathbb{C}$ is a holomorphic function, $1 \leq j \leq 8$.
In the sequel, the next result gives the exact characterization according to algebraic methods in the theory of holomorphic polynomials and related topics. We have

Theorem 15. *Let $\varphi_1,\varphi_2,\varphi_3,\varphi_4 : D \to \mathbb{C}$ be four holomorphic functions, D is a domain of $\mathbb{C}$. Put $u(z,w) = |w-\overline{\varphi_1}(z)||w-\overline{\varphi_2}(z)||w-\overline{\varphi_3}(z)||w-\overline{\varphi_4}(z)|$, $(z,w) \in D\times\mathbb{C}$. Let $v = u^2$. The following conditions are equivalent*
(I) u is psh on $D \times \mathbb{C}$;
(II) We have the following cases.
Case 1. $\frac{\partial^2 v}{\partial\overline{z}\partial w} = 0$ *on $D \times \mathbb{C}$.*
Case 2. $\frac{\partial^2 v}{\partial\overline{z}\partial w} \neq 0$ *on $D \times \mathbb{C}$ and we have the following two conditions.*
Step 1. $(\sum_{j=1}^{4}\varphi_j)$ *is nonconstant and $\varphi_1=\varphi_2=\varphi_3=\varphi_4$ on D.*

Step 2. $(\sum_{j=1}^{4}\varphi_j)$ *is constant on* D *and we have the following assertion.*
There exists j_1, j_2, j_3, j_4, *satisfying* $j_1 < j_2$, $j_3 < j_4$, $\{j_1, j_2, j_3, j_4\} = \{1, 2, 3, 4\}$, $\varphi_{j_1} = \varphi_{j_2}$, $\varphi_{j_3} = \varphi_{j_4}$ *and the function* $\varphi_{j_1}\varphi_{j_3}$ *is nonconstant on* D.

Proof. (I) implies (II). Let $(z, w) \in D \times \mathbb{C}$. We have
$v(z, w) = |w^4 - \overline{s_1}(z)w^3 + \overline{s_2}(z)w^2 - \overline{s_3}(z)w + \overline{s_4}(z)|^2$.
$s_1 = \sum_{j=1}^{4}\varphi_j$, $s_2 = \sum_{1\leq j<k\leq 4}\varphi_j\varphi_k$, $s_3 = \sum_{1\leq j<k<s\leq 4}\varphi_j\varphi_k\varphi_s$, $s_4 = \varphi_1\varphi_2\varphi_3\varphi_4$.
s_1, s_2, s_3 and s_4 are holomorphic functions on D.
v is a function of class C^∞ and psh on $D \times \mathbb{C}$.
$\frac{\partial^2 v}{\partial\overline{w}\partial w}(z, w) = |4w^3 - 3\overline{s_1}(z)w^2 + 2\overline{s_2}(z)w - \overline{s_3}(z)|^2 \geq 0$.
$\frac{\partial^2 v}{\partial\overline{z}\partial z}(z, w) = |-\overline{s_1'}(z)w^3 + \overline{s_2'}(z)w^2 - \overline{s_3'}(z)w + \overline{s_4'}(z)|^2 \geq 0$ and $\frac{\partial^2 v}{\partial\overline{z}\partial w}(z, w) = (-3\overline{s_1'}(z)w^2 + 2\overline{s_2'}(z)w - \overline{s_3'}(z))\overline{[w^4 - \overline{s_1}(z)w^3 + \overline{s_2}(z)w^2 - \overline{s_3}(z)w + \overline{s_4}(z)]}$.
Since v is psh on $D \times \mathbb{C}$, then we have the inequality

$$(E):\ |\frac{\partial^2 v}{\partial\overline{z}\partial w}(z, w)|^2 \leq \frac{\partial^2 v}{\partial\overline{z}\partial z}(z, w)\frac{\partial^2 v}{\partial\overline{w}\partial w}(z, w)$$

for each $(z, w) \in D \times \mathbb{C}$.
Put

$$q_1(w) = (w - \overline{\varphi_1}(z))(w - \overline{\varphi_2}(z))(w - \overline{\varphi_3}(z))(w - \overline{\varphi_4}(z)),$$
$$q_2(w) = -3\overline{s_1'}(z)w^2 + 2\overline{s_2'}(z)w - \overline{s_3'}(z),$$
$$q_3(w) = 4w^3 - 3\overline{s_1}(z)w^2 + 2\overline{s_2}(z)w - \overline{s_3}(z),$$
$$q_4(w) = -\overline{s_1'}(z)w^3 + \overline{s_2'}(z)w^2 - \overline{s_3'}(z)w + \overline{s_4'}(z).$$

q_1, q_2, q_3 and q_4 are holomorphic polynomials on $\mathbb{C}$, for each fixed z on D.
We have $q_1' = q_3$ and $q_4' = q_2$. By the inequality (E) we have then $|q_1 q_2| \leq |q_3 q_4|$ on $\mathbb{C}$.
Case 1. $q_2(w) = 0$, for every $w \in \mathbb{C}$ and for any $z \in D$.
Then s_1, s_2 and s_3 are constant functions on D. Therefore $\frac{\partial^2 v}{\partial\overline{z}\partial w} = 0$ on $D \times \mathbb{C}$. Thus we have

$$u(z, w) = |w^4 + c_1 w^3 + c_2 w^2 + c_3 w + \overline{\varphi_1}(z)\overline{\varphi_2}(z)\overline{\varphi_3}(z)\overline{\varphi_4}(z)|$$

where $c_1, c_2, c_3 \in \mathbb{C}$. Therefore $u = |h|$, when h is a pluriharmonic (prh) function on $D \times \mathbb{C}$. Consequently, u is psh on $D \times \mathbb{C}$.
Case 2. $q_2 \neq 0$ on $\mathbb{C}$.
Now fix $z \in D$, such that $[-3\overline{s_1'}(z)w^2 + 2\overline{s_2'}(z)w - \overline{s_3'}(z)] \neq 0$. Since $q_1 q_2 \neq 0$ and the inequality (E), there exists $c \in \mathbb{C}\backslash\{0\}$ such that $q_3 q_4 = c q_1 q_2$ on $\mathbb{C}$.
Step 1. $s_1' \neq 0$ on $\mathbb{C}$.
Then $c = \frac{4}{3}$. We have $A = \{\overline{\varphi_1}(z), \overline{\varphi_2}(z), \overline{\varphi_3}(z), \overline{\varphi_4}(z)\}$ is the set of all zeros of the analytic polynomial q_1. Assume that the cardinality of A is equal to 4. Observe that because of the property of the order of multiplicity of zeros of a polynomial and the

relation $q_3q_4 = \frac{4}{3}q_1q_2$, we have $\overline{\varphi_1}(z)$, $\overline{\varphi_2}(z)$, $\overline{\varphi_3}(z)$, $\overline{\varphi_4}(z)$ are distinct zeros of q_4. Therefore $\deg(q_4) \geq 4$. A contradiction. Consequently, the cardinal of the subset A is less than or equal 3.

Without loss of generality, we assume that $\varphi_1 = \varphi_2$. Assume that $\varphi_1 \neq \varphi_3$. We have $\overline{\varphi_1}(z)$ is a zero of q_3 and q_4. Note that $\{\overline{\varphi_1}(z), \overline{\varphi_3}(z), \overline{\varphi_4}(z)\}$ is exactly the set of zeros of the holomorphic polynomial q_4. Let $\overline{w_1}$ and $\overline{w_2}$ the two zeros of the polynomial q_2. Indeed, for instance, using possible relations between all the coefficients of a holomorphic polynomial and its zeros, we have then

$$\overline{w_1} + \overline{w_2} = \frac{2}{3}\frac{\overline{s'_2}(z)}{\overline{s'_1}(z)} = \frac{2}{3}(\overline{\varphi_1}(z) + \overline{\varphi_3}(z) + \overline{\varphi_4}(z)).$$

$$\overline{w_1 w_2} = \frac{\overline{s'_3}(z)}{3\overline{s'_1}(z)} = \frac{1}{3}(\overline{\varphi_1}(z)\overline{\varphi_3}(z) + \overline{\varphi_1}(z)\overline{\varphi_4}(z) + \overline{\varphi_3}(z)\overline{\varphi_4}(z)).$$

Assume that $\overline{\varphi_1}(z)$, $\overline{\varphi_3}(z)$ and $\overline{\varphi_4}(z)$ are zeros of q_4 of order 1. Then $\overline{w_1}$ and $\overline{w_2}$ are not zeros of q_4.

$\overline{w_1}$ and $\overline{w_2}$ are zeros of q_3.

Therefore $\overline{\varphi_1}(z) + \overline{w_1} + \overline{w_2} = \frac{3}{4}\overline{s_1}(z) = \frac{3}{4}(2\overline{\varphi_1}(z) + \overline{\varphi_3}(z) + \overline{\varphi_4}(z))$. Then $w_1 + w_2 = \frac{2}{3}(\varphi_1(z) + \varphi_3(z) + \varphi_4(z))$.

$w_1 + w_2 = \frac{1}{2}\varphi_1(z) + \frac{3}{4}(\varphi_3(z) + \varphi_4(z))$. Thus $\varphi_3(z) + \varphi_4(z) = 2\varphi_1(z)$ and then

$$\begin{aligned} w_1 + w_2 &= 2\varphi_1(z). \\ \overline{w_1 w_2} &= \frac{\overline{s'_3}(z)}{3\overline{s'_1}(z)} = \frac{1}{3}(\overline{\varphi_1}(z)\overline{\varphi_3}(z) + \overline{\varphi_1}(z)\overline{\varphi_4}(z) + \overline{\varphi_3}(z)\overline{\varphi_4}(z)) \\ &= \frac{1}{3}[2(\overline{\varphi_1}(z))^2 + \overline{\varphi_3}(z)\overline{\varphi_4}(z)]. \end{aligned}$$

We have also

$$w_1\varphi_1(z) + w_2\varphi_1(z) + w_1w_2 = \frac{2s_2(z)}{4} = \frac{s_2(z)}{2} =$$

$$\frac{1}{2}(\varphi_1^2(z) + 2\varphi_1(z)\varphi_3(z) + 2\varphi_1(z)\varphi_4(z) + \varphi_3(z)\varphi_4(z)) = \frac{1}{2}(5\varphi_1^2(z) + \varphi_3(z)\varphi_4(z))$$

$$= (w_1 + w_2)\varphi_1 + w_1w_2.$$

Therefore,

$$2\varphi_1^2(z) + w_1w_2 = \frac{1}{2}(5\varphi_1^2(z) + \varphi_3(z)\varphi_4(z)).$$

$$w_1w_2 = \frac{1}{2}\varphi_1^2(z) + \frac{1}{2}\varphi_3(z)\varphi_4(z) = \frac{1}{3}(2\varphi_1^2(z) + \varphi_3(z)\varphi_4(z)).$$

Thus $3\varphi_1^2(z) + 3\varphi_3(z)\varphi_4(z) = 4\varphi_1^2(z) + 2\varphi_3(z)\varphi_4(z)$. Then $\varphi_3(z)\varphi_4(z) = \varphi_1^2(z)$. Since $\varphi_3 + \varphi_4(z) = 2\varphi_1(z)$. Thus $\varphi_3(z) = \varphi_4(z) = \varphi_1(z)$. A contradiction.

Assume now that $\overline{\varphi_1}(z)$, $\overline{\varphi_3}(z)$ and $\overline{\varphi_4}(z)$ are not zeros of q_4 of order 1. Recall that $\{\overline{\varphi_1}(z), \overline{\varphi_3}(z), \overline{\varphi_4}(z)\}$ is exactly the set of zeros of q_4. Since $\varphi_1(z) \neq \varphi_3(z)$, then $\varphi_1(z) = \varphi_4(z)$.

Because if $\varphi_1(z) \neq \varphi_4(z)$, then $\varphi_3(z) = \varphi_4(z)$. Since $\varphi_1(z) = \varphi_2(z) \neq \varphi_3(z)$, then $u(\xi, w) = |w - \overline{\varphi_1}(\xi)|^2 |w - \overline{\varphi_3}(\xi)|^2$, for each $(\xi, w) \in G = D(z, R) \times \mathbb{C}$, where $R > 0$ satisfying $D(z, R) \subset D$. Since u is a function of class C^∞ and psh on the domain G, we can prove that we have the condition $\varphi_1 = \varphi_3$ on $D(z, R)$, or $(\varphi_1 + \varphi_3) = \frac{s_1}{2}$ is constant on $D(z, R)$.
Now since s_1 is holomorphic nonconstant on D, then s_1 is nonconstant on the open Euclidean disc $D(z, R)$. It follows that $\varphi_1 = \varphi_3$ on $D(z, R)$. A contradiction, because $\varphi_1(z) \neq \varphi_3(z)$. Consequently, $\varphi_1(z) = \varphi_4(z)$. Therefore $\overline{\varphi_1}(z)$ is a zero of q_2 of order 1. Assume that $\varphi_1(z) = w_1$. We have $\varphi_1(z) = \varphi_2(z) = \varphi_4(z)$, $\varphi_1(z) \neq \varphi_3(z)$. It follows that $\overline{\varphi_1}(z)$ is a zero of q_3 of order 2.
$\overline{\varphi_3}(z)$ is not a zero of q_3.
Now we use the classical relations between all the coefficients of a polynomial and its zeros, we have $\varphi_1(z) + w_2 = \frac{2s_2'(z)}{3s_1'(z)} = \frac{2}{3}(2\varphi_1(z) + \varphi_3(z))$. Also $2\varphi_1(z) + w_2 = \frac{3}{4}(3\varphi_1(z) + \varphi_3(z))$ and $\varphi_1(z) + w_2 = \frac{2}{3}(2\varphi_1(z) + \varphi_3(z))$. Then $\varphi_1(z) + \frac{2}{3}(2\varphi_1(z) + \varphi_3(z)) = \frac{3}{4}(3\varphi_1(z) + \varphi_3(z))$. Thus, $12\varphi_1(z) + 8(2\varphi_1(z) + \varphi_3(z)) = 9(3\varphi_1(z) + \varphi_3(z))$. Consequently, $\varphi_1(z) = \varphi_3(z)$. A contradiction. It follows that the assumption $\varphi_1(z) \neq \varphi_3(z)$ is impossible. Consequently, $\varphi_1(z) = \varphi_2(z) = \varphi_3(z)$.
Now assume that $\varphi_4(z) \neq \varphi_1(z)$. Let $\overline{w_0}$ the zero of q_2, $w_0 \neq \varphi_1(z)$. Note that $\overline{\varphi_1}(z)$ is a zero of the polynomial q_2 because $\overline{\varphi_1}(z)$ is a zero of q_4 of order 2.
$\overline{\varphi_1}(z)$ is a zero of q_1 of order 3.
Therefore $\overline{\varphi_1}(z)$ is a zero of q_3 of order 2. Consequently, $\overline{w_0}$ is a zero of q_3 of order 1. We have $w_0 + \varphi_1(z) = \frac{2s_2'(z)}{3s_1'(z)} = \frac{2}{3}(2\varphi_1(z) + \varphi_4(z))$. Also $2\varphi_1(z) + w_0 = \frac{3}{4}s_1(z) = \frac{3}{4}(3\varphi_1(z) + \varphi_4(z))$. Therefore, we have $\varphi_1(z) + \frac{2}{3}(2\varphi_1(z) + \varphi_4(z)) = \frac{3}{4}(3\varphi_1(z) + \varphi_4(z))$. Thus $\varphi_1(z) = \varphi_4(z)$. A contradiction. Consequently, the assumption $\varphi_1(z) \neq \varphi_4(z)$ is impossible. We conclude that $\varphi_1 = \varphi_2 = \varphi_3 = \varphi_4$ on D.
Step 2. s_1 is constant on D.
Let $(z, w) \in D \times \mathbb{C}$, such that $\frac{\partial^2 v}{\partial \bar{z} \partial w}(z, w) \neq 0$. Assume that $s_2'(z) \neq 0$. We have $q_1 q_2 = c q_3 q_4$, where $c = \frac{1}{2}$. Let $\overline{w_0} = \frac{\overline{s_3'(z)}}{2\overline{s_2'(z)}}$ be the only zero of q_2. Note that $\{\overline{\varphi_1}(z), \overline{\varphi_2}(z), \overline{\varphi_3}(z), \overline{\varphi_4}(z)\}$ is the set of zeros of the holomorphic polynomial q_1 on $\mathbb{C}$. If for example $\overline{\varphi_1}(z)$ is a zero of q_1 of order 1. Then $\overline{\varphi_1}(z)$ is not a zero of $q_3 = q_1'$. Since now $q_1 q_2 = \frac{1}{2} q_3 q_4$, then $\overline{\varphi_1}(z)$ is a zero of q_4.
Now if $\overline{\varphi_1}(z)$ is a zero of q_1 of order 2. Then $\overline{\varphi_1}(z)$ is a zero of $q_3 = q_1'$ of order 1. By the fundamental relation $q_1 q_2 = \frac{1}{2} q_3 q_4$, we obtain $\overline{\varphi_1}(z)$ is a zero of q_4. We conclude that the set of zeros of q_4 is $\{\overline{\varphi_1}(z), \overline{\varphi_2}(z), \overline{\varphi_3}(z), \overline{\varphi_4}(z)\}$.
Since now $\deg(q_4) = 2$ (because $s_2'(z) \neq 0$), then there exists j_1, j_2, j_3, j_4, $\{j_1, j_2, j_3, j_4\} = \{1, 2, 3, 4\}$, such that $\varphi_{j_1} = \varphi_{j_2} = \varphi_{j_3} \neq \varphi_{j_4}$ on D, or $\varphi_{j_1} = \varphi_{j_2}$ and $\varphi_{j_3} = \varphi_{j_4}$ on D.
Suppose that we have $\varphi_1 = \varphi_2 = \varphi_3 \neq \varphi_4$. Then $\overline{\varphi_1}(z)$ is a zero of q_1 of order 3. $\overline{\varphi_1}(z)$ is a zero of q_3 of order 2.
$\overline{w_0}$ is a zero of q_3 of order 1.
$\overline{\varphi_4}(z)$ is not a zero of q_3.
We have

$$2\varphi_1(z) + w_0 = \frac{3}{4}s_1(z) = \frac{3}{4}(3\varphi_1(z) + \varphi_4(z)).$$

$$w_0 = \frac{s_3'(z)}{2s_2'(z)} = \frac{1}{2}(\varphi_1(z) + \varphi_4(z)).$$

Thus $2\varphi_1(z) + \frac{1}{2}(\varphi_1(z) + \varphi_4(z)) = \frac{3}{4}(3\varphi_1(z) + \ \varphi_4(z))$. Therefore $\frac{5}{2}\varphi_1(z) + \frac{1}{2}\varphi_4(z) = \frac{9}{4}\varphi_1(z) + \frac{3}{4}\varphi_4(z)$. Then $\varphi_1(z) = \varphi_4(z)$. A contradiction. Consequently, the above assumption is impossible. It follows that $\varphi_{j_1} = \varphi_{j_2}$ and $\varphi_{j_3} = \varphi_{j_4}$ on D, (for example). We suppose without loss of generality that $j_1 < j_2$ and $j_3 < j_4$. Then $u(z, w) = |w - \overline{\varphi_{j_1}}(z)|^2 |w - \overline{\varphi_{j_3}}(z)|^2$, for $(z, w) \in D \times \mathbb{C}$.
Actually, we observe that $\varphi_{j_1} = \varphi_{j_3}$ on D, or $(\varphi_{j_1} + \varphi_{j_3})$ is constant on D. Suppose that $\varphi_{j_1} = \varphi_{j_3}$ on D. Then $\varphi_1 = \varphi_2 = \varphi_3 = \varphi_4$ on D. Since $s_1' = 0$ on D, then φ_1 is constant on D. Thus $\frac{\partial^2 v}{\partial \bar{z} \partial w} = 0$ on $D \times \mathbb{C}$. A contradiction. Consequently, $(\varphi_{j_1} + \varphi_{j_3})$ is constant on D and observe that the product $\varphi_{j_1}\varphi_{j_3}$ is nonconstant on D.
Assume that $s_2' = 0$ on D. Then $s_3' \neq 0$ on D, because $\frac{\partial^2 v}{\partial \bar{z} \partial w} \neq 0$ on $D \times \mathbb{C}$. The set of zeros of q_4 is $\{\overline{\varphi_1}(z), \overline{\varphi_2}(z), \overline{\varphi_3}(z), \overline{\varphi_4}(z)\}$. Since $\deg(q_4) = 1$, then $\varphi_1 = \varphi_2 = \varphi_3 = \varphi_4$ on D.
$s_1' = 0$ on D implies that φ_1 is constant on D. Therefore, $\frac{\partial^2 v}{\partial \bar{z} \partial w} = 0$ on $D \times \mathbb{C}$. It follows that this case is impossible.
(II) implies (I). Obvious.

Remark 9 . Let $F : \mathbb{C}^2 \to \mathbb{C}$ be holomorphic, $a_1, a_2, a_3, a_4 \in \mathbb{C}^4$ and $b_1, b_2, b_3, b_4 \in \mathbb{C}$. Define

$$\begin{aligned} v(z,w) \quad &= \quad |[w - \overline{< F(z)/a_1 >} - b_1][w - \overline{< F(z)/a_2 >} - b_2] \cdot \\ &\quad \cdot [w - \overline{< F(z)/a_3 >} - b_3][w - \overline{< F(z)/a_4 >} - b_4]|, \end{aligned}$$

for $(z, w) \in \mathbb{C}^2 \times \mathbb{C}$. We can characterize all the conditions on $a_1, a_2,\ a_3,\ a_4,\ b_1, b_2, b_3, b_4$, which ensure technical hypothesis for the plurisubharmonicity of v. Indeed, we have the following of various behaviour.

Theorem 16. *Let $\varphi_1, \varphi_2, \varphi_3, \varphi_4 : D \to \mathbb{C}$ be holomorphic functions, D is a domain of $\mathbb{C}^n$, $n \geq 1$.*
Put $u(z, w) = |w - \overline{\varphi_1}(z)||w - \overline{\varphi_2}(z)||w - \overline{\varphi_3}(z)||w - \overline{\varphi_4}(z)|$, $(z, w) \in D \times \mathbb{C}$.
Let $v = u^2$, $s_1 = \sum_{j=1}^{4} \varphi_j$, $s_2 = \sum_{1 \leq j < k \leq 4} \varphi_j \varphi_k$, $s_3 = \sum_{1 \leq j < k < s \leq 4} \varphi_j \varphi_k \varphi_s$, $s_4 = \varphi_1\varphi_2\varphi_3\varphi_4$, ($s_1, s_2, s_3, s_4$ are holomorphic functions on D).
The following assertions are equivalent
(I) u (respectively v) is psh on $D \times \mathbb{C}$;
(II) We have the following three cases.
Case 1. s_1, s_2 and s_3 are constant on D.
Case 2. s_1 is nonconstant on D and $\varphi_1 = \varphi_2 = \varphi_3 = \varphi_4$ on D.
Case 3. s_1 is constant on D, s_2 is nonconstant on D and there exits j_1, j_2, j_3, j_4, $\{j_1, j_2, j_3, j_4\} = \{1, 2, 3, 4\}$, $j_1 < j_2$, $j_3 < j_4$, with $\varphi_{j_1} = \varphi_{j_2}$ and $\varphi_{j_3} = \varphi_{j_4}$ on D.

Proof. Obvious by the above theorem.

Example. Let $a_1, a_2, a_3, a_4 \in \mathbb{C}^2$, $A_1, A_2, A_3, A_4 \in \mathbb{C}^m$ and $F : \mathbb{C}^n \to \mathbb{C}^2$ be a holomorphic function, $n, m \geq 1$. Define

$$\begin{aligned}\psi(z,w) &= |<w/A_1> - \overline{<F(z)/a_1>}||<w/A_2> - \overline{<F(z)/a_2>}| \cdot \\ &\cdot |<w/A_3> - \overline{<F(z)/a_3>}||<w/A_4> - \overline{<F(z)/a_4>}|,\end{aligned}$$

for $(z,w) \in \mathbb{C}^n \times \mathbb{C}^m$.
In a slightly different direction, we can show all the conditions formulated by the constants a_1, a_2, a_3, a_4, A_1, A_2, A_3, A_4, which characterize the plurisubharmonicity of ψ.

Example. Let $N \geq 2$ and $p(\xi_1, ..., \xi_N, w) = (w-\xi_1)...(w-\xi_N)$, for $(\xi_1, ..., \xi_N) \in \mathbb{C}^N$, $w \in \mathbb{C}$. Define $F(\xi_1, ..., \xi_N, w) = |p(\overline{\xi_1}, ..., \overline{\xi_N}, w)|$. Then for each Euclidean open ball $B(a,R) \subset \mathbb{C}^N$, $(a \in \mathbb{C}^N, R > 0)$, the function F is not psh on $B(a,R) \times \mathbb{C}$.

Remark 10. (I) Let $g_1(z) = z^2$, $g_2(z) = -z^2$, $g_3(z) = iz^2$, $g_4(z) = -iz^2$, $z \in \mathbb{C}$. g_1, g_2, g_3 and g_4 are holomorphic functions on $\mathbb{C}$. Let

$$u_1(z,w) = |w - \overline{g_1}(z)||w - \overline{g_2}(z)||w - \overline{g_3}(z)|,$$

$$u_2(z,w) = |w - \overline{g_1}(z)||w - \overline{g_2}(z)||w - \overline{g_4}(z)|,$$

$$u_3(z,w) = |w - \overline{g_1}(z)||w - \overline{g_3}(z)||w - \overline{g_4}(z)|,$$

$$u_4(z,w) = |w - \overline{g_2}(z)||w - \overline{g_3}(z)||w - \overline{g_4}(z)|,$$

$$u(z,w) = |w - \overline{g_1}(z)||w - \overline{g_2}(z)||w - \overline{g_3}(z)||w - \overline{g_4}(z)|,\ (z,w) \in \mathbb{C}^2.$$

We have u_1, u_2, u_3 and u_4 are not psh functions on $\mathbb{C}^2$. But u is psh on $\mathbb{C}^2$.
(II) $g_1(z) = g_2(z) = z + 1$, $g_3(z) = g_4(z) = -z + 1$, $z \in \mathbb{C}$.
g_1, g_2, g_3 and g_4 are holomorphic functions on $\mathbb{C}$.
$(g_1 + g_2 + g_3 + g_4)$ is constant on $\mathbb{C}$.
$(g_1g_2 + g_1g_3 + g_1g_4 + g_2g_3 + g_2g_4 + g_3g_4)$ is non constant on $\mathbb{C}$.
Let $(z,w) \in \mathbb{C}^2$. Put $u(z,w) = |w-\overline{g_1}(z)||w-\overline{g_2}(z)||w-\overline{g_3}(z)||w-\overline{g_4}(z)| = |w^4-4w^3+[6-2(\overline{z})^2]w^2-4w[1-(\overline{z})^2]+[1-(\overline{z})^2]^2|$. Observe that $(g_1g_2g_3+g_1g_2g_4+g_1g_3g_4+g_2g_3g_4)$ is nonconstant on $\mathbb{C}$. But u is psh on $\mathbb{C}^2$, because

$$u(z,w) = |w - 1 - \overline{z}|^2|w - 1 + \overline{z}|^2 = |(w-1)^2 - (\overline{z})^2|^2 = |h|^2,$$

where h is a prh function on $\mathbb{C}^2$.

Question 8. Let $N_1, ..., N_k, s_1, m_1, ..., s_t, m_t \in \mathbb{N}\backslash\{0\}$, $k, t \geq 1$ and $g_1, ..., g_k$, $\theta_1, ..., \theta_t : \mathbb{C}^n \to \mathbb{C}$ be prh functions. Put

$$u(z,w) = |w - g_1(z)|^{N_1}...|w - g_k(z)|^{N_k}|w^{s_1} - \theta_1^{m_1}(z)|...|w^{s_t} - \theta_t^{m_t}(z)|,$$

for $(z,w) \in \mathbb{C}^n \times \mathbb{C}$. Find conditions $g_1, ..., g_k, \theta_1, ..., \theta_t$ should satisfy so that u is psh on $\mathbb{C}^n \times \mathbb{C}$.

Question 9. Let $N \in \mathbb{N}\backslash\{0,1\}$ and $A_0, ..., A_{N-1} \in \mathbb{C}$. Define $v(z,w) = |w^N + A_{N-1}w^{N-1}\overline{z} + ... + A_1 w\overline{z^{N-1}} + A_0\overline{z^N}|$. Find all conditions on N, $A_0, ..., A_{N-1}$ such that v is psh on $\mathbb{C}^2$.

Conclude that we can characterize all the holomorphic polynomials q on $\mathbb{C}^2$, such that F is psh on $\mathbb{C}^2$, where $F(z,w) = |q(\overline{z},w)|$ for $(z,w) \in \mathbb{C}^2$.
Let p be a holomorphic polynomial on $\mathbb{C}^2$. Put $F_1(z,w) = |p(\overline{z},w)|$ and $F_2(z,w) = |p(z,\overline{w})|$, for $(z,w) \in \mathbb{C}^2$. Moreover, thanks to the above characterization, we can prove that F_1 is psh on $\mathbb{C}^2$ if and only if F_2 is psh on $\mathbb{C}^2$.
In the following question, we recall some properties and sharp results in the framework of complex analysis of the appeared function θ, defined by $\theta(z,w) = (w+\overline{z})^N$, for $N \in \mathbb{N}\backslash\{0,1\}$ and $(z,w) \in \mathbb{C}^2$.

Question 10. Let $N \in \mathbb{N}\backslash\{0,1\}$, $A \in \mathbb{C}\backslash\{0\}$, $(B_1,...,B_n) \in \mathbb{C}^n\backslash\{0\}$ and $s \in [1,+\infty[$. Let $g, f_0, ..., f_{N-2} : D \to \mathbb{C}$ be continuous functions, where D is a domain on $\mathbb{C}^n$. Define $\psi(z,w) = |(Aw + B_1\overline{z_1} + ... + B_n\overline{z_n})^N + g(z)|^s$ and $\varphi(z,w) = |(Aw + B_1\overline{z_1} + ... + B_n\overline{z_n})^N + f_{N-2}(z)w^{N-2} + ... + f_0(z)|$, for $(z,w) = (z_1,...,z_n,w) \in D\times\mathbb{C}$. Assume that ψ is psh on $D\times\mathbb{C}$. Prove that $g = 0$ on D. Suppose that φ is psh on $D\times\mathbb{C}$. Prove that $f_{N-2} = ... = f_0 = 0$.

References

[1] J. Abidi, *Sur quelques problèmes concernant les fonctions holomorphes et plurisousharmoniques*, Rend. Circ. Mat. Palermo 51 (2002) 411–424.

[2] J. Abidi, *Contribution à l'étude des fonctions plurisousharmoniques convexes et analytiques*, Serdica Math. J. 40 (2014) 329–388.

[3] J. Abidi, *The real and complex convexity*, J. Math. Appl. 41 (2018) 123–156.

[4] J. Abidi, *On convex and analytic functions*, An. Univ. Oradea Fasc. Mat. XXVI (2019) 107–128.

[5] M. Andersson, M. Passare, R. Sigurdsson, *Complex Convexity and Analytic Functionals*, Progress in Mathematics, Vol. 225, Birkhäuser, Berlin, 2004.

[6] H.J. Bremermann, *Complex convexity*, Trans. Amer. Math. Soc. 82 (1956) 17–51.

[7] U. Cegrell, *Removable singularities for plurisubharmonic functions and related problems*, Proc. Lond. Math. Soc. 36 (1978) 310–336.

[8] H. El Mir, *Sur le prolongement des courants positifs fermés*, Acta Math. 153 (1984) 1–45.

[9] G.M. Henkin, J. Leiterer, *Theory of Functions on Complex Manifolds*, Birkhäuser, Boston, Mass., 1984.

[10] L. Hörmander, *An Introduction to Complex Analysis in Several Variables*, Third Edition (revised), Mathematical Library, Vol. 7, North Holland, Amsterdam-New York-Oxford-Tokyo, 1990.

[11] L. Hörmander, L., *Notions of Convexity*, Birkhäuser, Basel-Boston-Berlin, 1994.

[12] M. Jarnicki, P. Pflug, *Extension of Holomorphic Functions*, de Gruyter, Berlin, 2000.

[13] M. Klimek, *Pluripotential Theory*, Clarendon Press, Oxford, 1991.

[14] S.G. Krantz, *Function Theory of Several Complex Variables*, Wiley, New York, 1982.

[15] P. Lelong, *Définition des fonctions plurisousharmoniques*, C. R. Acad. Sci. Paris 215 (1942) 398–400.

[16] P. Lelong, *Sur les suites des fonctions plurisousharmoniques*, C. R. Acad. Sci. Paris 215 (1942) 454–456.

[17] P. Lelong, *Les fonctions plurisousharmoniques*, Ann. Sci. Ecole Norm. Sup. 62 (1945) 301–338.

[18] W. Rudin, *Function Theory in Polydiscs*, Benjamin, New York, 1969.

[19] V.S. Vladimirov, *Les Fonctions de Plusieurs Variables Complexes (et Leur Application a la Théorie Quantique des Champs)*, Paris: Dunod, 1967.

DOI: 10.7862/rf.2022.2

Jamel Abidi
email: `abidijamel1@gmail.com`
ORCID: 0000-0002-2552-6848
Département de Mathématiques
Faculté des Sciences de Tunis
1060 Tunis
TUNISIA

Received 25.11.2021 *Accepted 14.05.2022*

Journal of
Mathematics
and Applications

JMA No 45, pp 59-86 (2022)

On the Basis Property of Root Vectors Related to a Non-Self-Adjoint Analytic Operator and Applications

Hanen Ellouz, Ines Feki and Aref Jeribi

Abstract: In the present paper, based on a separation condition on the spectrum of a self-adjoint operator T_0 on a separable Hilbert space $\mathcal{H}$, we prove that the system of root vectors of the perturbed operator $T(\varepsilon)$ given by

$$T(\varepsilon) := T_0 + \varepsilon T_1 + \varepsilon^2 T_2 + \ldots + \varepsilon^k T_k + \ldots$$

is complete and forms a basis with parentheses in $\mathcal{H}$, for small enough $|\varepsilon|$. Here $\varepsilon \in \mathbb{C}$ and $T_1, T_2, \ldots$ are linear operators on $\mathcal{H}$ having the same domain $\mathcal{D} \supset \mathcal{D}(T_0)$ and satisfying a specific growing inequality. The obtained results are of importance for applications to a non-self-adjoint Gribov operator in Bargmann space and to a non-self-adjoint problem deduced from a perturbation method for sound radiation.

AMS Subject Classification: 15A42, 47A55, 47B25, 46A35, 46B15, 65F15, 65H17.
Keywords and Phrases: Basis with parentheses; Elastic membrane; Gribov operator; Not condense; Spectrum.

1. Introduction

For non-self-adjoint perturbations of a self-adjoint operator, the crucial problem is the study of the spectral properties. For instance, the existence of a basis (possibly with parentheses) of root vectors is an important property. In order to prove the existence of such basis, several authors studied the comportment of the eigenvalues and established different conditions in terms of the spectrum (see [3]-[5], [8]-[13], [16]-[20], [22], [24] and [25]). Indeed, many non-self-adjoint ordinary differential operators can be considered as a perturbation $T+B$ of a leading self-adjoint component T by its subordinate B. In [22], A. S. Markus claimed that $G = T+B$ admits an unconditional basis with parentheses of root vectors if B is p-subordinate to T and the eigenvalue-counting function of T satisfy a certain asymptotic growth condition. One might ask whether we can construct a basis if the p-subordinate condition is relaxed. A positive answer is given by A. A. Shkalikov [24]. He assumed that T is positive, self-adjoint with discrete spectrum and its eigenvalues $(\mu_n)_{n\in\mathbb{N}^*}$ are not condense, i.e.,

$$\mu_{n+q} - \mu_n \geq 1, \quad \text{for some} \quad q \in \mathbb{N}^*. \tag{1.1}$$

Further, he required that B verify

$$\|B\psi_n\| \leq b, \tag{1.2}$$

where $(\psi_n)_{n\in\mathbb{N}^*}$ is an orthonormal system of eigenvectors associated to the eigenvalues $(\mu_n)_{n\in\mathbb{N}^*}$ of T. Under these assumptions, he established an asymptotic relation between the eigenvalue-counting functions of G and T and he claimed that the system of root vectors of G forms a basis with parentheses in $\mathcal{H}$. More precisely, he proved the existence of a spectral condition

$$n(r,G) - n(r,T) = O(1),$$

under which he guarantees the existence of a basis with parentheses of root vectors (see [24, $Theorem$ 2]).
Here $n(r,T)$ (respectively, $n(r,G)$) denotes the sum of multiplicities of all eigenvalues of T (respectively, G) contained in the disk $\{\lambda \in \mathbb{C} \text{ such that } |\lambda| \leq r\}$.

Notice that in classical perturbation theorems for bases or Riesz bases, the authors always required that the eigenvalues of T are with multiplicity one (for instance, see [6], [7, $Theorem\ XIX.2.7$] and [21, $Theorem\ V.4.15a$]). Although, by assuming that the eigenvalues are with finite multiplicity, several authors such as A. Jeribi [18, 19], A. S. Markus [22], A. A. Shkalikov [24] and C. Wyss [25] proved the existence of bases with parentheses or unconditional bases with parentheses.
It is interesting to note here that the concept of bases (or unconditional bases) with parentheses is a natural generalization of the one of the bases (or Riesz bases).
Furthermore, [24, $Theorem$ 2] ameliorates the result stated in [22]. Indeed, A. A. Shkalikov obtained a basis with parentheses under Eqs (1.1) and (1.2) which are much weaker.

Besides, in many situations, this result presents an important tool in the determining of the existence of bases. Among this direction we had the idea to exploit this outcome to study the Gribov operator (see [1], [2], [12] and [15]) originated from Reggeon field theory and constructed as a polynomial in the standard annihilation operator A and the standard creation operator A^*:

$$(A^*A)^3 + \varepsilon A^*(A + A^*)A + \varepsilon^2(A^*A)^{3u_2} + ... + \varepsilon^k(A^*A)^{3u_k} + ...,$$

where $\varepsilon \in \mathbb{C}$ and $(u_k)_{k\in\mathbb{N}}$ is a strictly decreasing sequence with strictly positive terms such that $u_0 = 1$ and $u_1 = \frac{1}{2}$; while the expressions of the operators A and A^* are given by:

$$\begin{cases} A : \mathcal{D}(A) \subset \mathcal{B} \longrightarrow \mathcal{B} \\ \qquad\qquad \varphi \longrightarrow A\varphi(z) = \dfrac{d\varphi}{dz}(z) \\ \mathcal{D}(A) = \{\varphi \in \mathcal{B} \text{ such that } A\varphi \in \mathcal{B}\} \end{cases}$$

and

$$\begin{cases} A^* : \mathcal{D}(A^*) \subset \mathcal{B} \longrightarrow \mathcal{B} \\ \qquad\qquad \varphi \longrightarrow A^*\varphi(z) = z\varphi(z) \\ \mathcal{D}(A^*) = \{\varphi \in \mathcal{B} \text{ such that } A^*\varphi \in \mathcal{B}\}, \end{cases}$$

$$\mathcal{B} = \left\{ \varphi : \mathbb{C} \longrightarrow \mathbb{C} \text{ entire such that } \int_{\mathbb{C}} e^{-|z|^2} |\varphi(z)|^2 dz d\bar{z} < \infty \right\}.$$

Since $\{\varphi_n := \frac{z^n}{\sqrt{n!}}\}_{n\geq 1}$ is an orthonormal basis of eigenvectors of $(A^*A)^3$ associated to the eigenvalues $\{n^3\}_{n\geq 1}$, then we have

$$\left\| \left(\varepsilon A^*(A + A^*)A + \sum_{k=2}^{\infty} \varepsilon^k (A^*A)^{3u_k} \right) \varphi_n \right\| \leq \frac{|\varepsilon|}{1 - |\varepsilon|}(1 + 2\sqrt{2})(1 + n^3), \quad \text{for } |\varepsilon| < 1. \tag{1.3}$$

It is clear here that Eq. (1.3) does not verify Eq. (1.2). Consequently, [24, *Theorem* 2] can not be applied.

Further, if we consider the following integro-differential operator initially motivated by P. J. T. Filippi et al. [14] and deduced from a perturbation method for sound radiation (see also [8], [11] and [13]):

$$(I + \varepsilon K)^{-1} \frac{d^4\varphi}{dx^4} + \varepsilon (I + \varepsilon K)^{-1} K \left(\frac{d^4}{dx^4} - \left(\frac{d^4}{dx^4} \right)^{\frac{1}{2}} \right) \varphi = \lambda(\varepsilon)\varphi,$$

where K is the integral operator with kernel the Hankel function of the first kind and order 0 and ε is a complex number such that $|\varepsilon| < \frac{1}{\|K\|}$; we obtain

$$\left\| \sum_{k=1}^{\infty} (-1)^k \varepsilon^k K^k \frac{d^2\varphi_n}{dx^2} \right\| \leq \frac{|\varepsilon|}{1 - |\varepsilon|\|K\|} \|K\| \; \kappa n^4, \quad \text{for} \quad |\varepsilon| < \frac{1}{\|K\|}.$$

Here $(\varphi_n)_{n\geq 1}$ denotes the system of eigenvectors of the operator

$$\begin{cases} \frac{d^4}{dx^4} : \mathcal{D}(\frac{d^4}{dx^4}) \subset L^2\big(]-L, L[\big) \longrightarrow L^2\big(]-L, L[\big) \\ \qquad\qquad \varphi \longrightarrow \dfrac{d^4\varphi}{dx^4} \\ \mathcal{D}(\frac{d^4}{dx^4}) = H_0^2\big(]-L, L[\big) \cap H^4\big(]-L, L[\big) \end{cases}$$

associated to the eigenvalues $(\lambda_n = \kappa n^4)_{n\geq 1}$ $(\kappa > 0)$. It is easy to check that $(\varphi_n)_{n\geq 1}$ forms an orthonormal basis of $L^2\big(]-L, L[\big)$.

Hence, Eq. (1.2) is not fulfilled and consequently [24, *Theorem* 2] can not be applied.

Among this direction and in order to overcome these bumps, we had the idea to extend [24, *Theorem* 2] to an abstract setting. More precisely, we continue the analysis started in [9] and we focus on the property of bases with parentheses of the analytic operator

$$T(\varepsilon) := T_0 + \varepsilon T_1 + \varepsilon^2 T_2 + \ldots + \varepsilon^k T_k + \ldots, \tag{1.4}$$

where $\varepsilon \in \mathbb{C}$, T_0 is a closed linear densely defined operator on a separable Hilbert space $\mathcal{H}$ with domain $\mathcal{D}(T_0)$ while $T_1, T_2, \ldots$ are linear operators on $\mathcal{H}$ having the same domain $\mathcal{D} \supset \mathcal{D}(T_0)$ and satisfying

$$\|T_k\varphi\| \leq q^{k-1}(a\|\varphi\| + b\|T_0\varphi\|^{\beta}\|\varphi\|^{1-\beta})$$

for all $\varphi \in \mathcal{D}(T_0)$ and for all $k \geq 1$, where $\beta \in]0, \frac{1}{2}[$ and a, b and $q > 0$.
We would like to mention here that the perturbed operator (1.4) was introduced by B. Sz. Nagy in [23] and considered later in some valuable papers such as [3], [5] and [8]-[13].
Furthermore, it is interesting to note here that in [9] we derived a precise description to the localization of the spectrum of the perturbed operator (1.4) and we proved an asymptotic relation between the eigenvalue-counting functions of T_0 and $T(\varepsilon)$. In other words, we claimed that the difference between the eigenvalue-counting functions of T_0 and $T(\varepsilon)$ is bounded by a constant. This generalization is of great importance. In fact, it allows us to control the jump of the eigenvalue-counting function of some analytic operators where the criteria of A. A. Shkalikov [24] can not be applied.

Now, based on the asymptotic relation between the eigenvalue-counting functions of T_0 and $T(\varepsilon)$ developed in [9], can we construct a basis with parentheses of root vectors of the perturbed operator $T(\varepsilon)$? Indeed, in view of [9, *Proposition* 3.1] the spectrum of $T(\varepsilon)$ is discrete for $|\varepsilon| < \frac{1}{q+\beta b}$. So, we consider $E_n = \cup_{m \geq 1} N(T(\varepsilon) - \lambda_n(\varepsilon))^m$ the root linear finite dimensional subspace whose dimension is called algebraic multiplicity of the eigenvalue $\lambda_n(\varepsilon)$. These subspaces are linearly independent and vectors in E_n are called root vectors of $T(\varepsilon)$. Following some ideas due to A. A. Shkalikov [24], we prove first that the system of root vectors of the perturbed operator $T(\varepsilon)$ is complete. Notice that our result improves Theorem 4.3 stated in [12]. In fact, not only the assumptions used in [12] are relaxed but also the values that takes $|\varepsilon|$ are greater than the one considered in [12, *Theorem* 4.3]. Furthermore, it can be considered as an extension of [24, *Lemma* 7] to an analytic operator.

Having obtained this aforementioned result, one might seek if it forms a basis with parentheses. Actually, using the spectral condition developed in [9], we prove that for $|\varepsilon|$ enough small, the system of root vectors of $T(\varepsilon)$ forms a basis with parentheses in $\mathcal{H}$.
We point out here that our result ameliorates [13, *Theorem* 3.4] since they established the existence of Riesz basis using a spectral analysis method based on the fact that the eigenvalues of T_0 are with multiplicity one; while we investigate the existence of basis with parentheses by supposing that the eigenvalues are with finite multiplicity. Further, our result might be regarded as an extension of [24, *Theorem* 2]. In fact, we guarantee the existence of basis with parentheses for some analytic operators where Eq. (1.2) considered by A. A. Shkalikov in [24] is not verified.

The present paper consists of four sections: In section 2, we introduce some basic definitions and auxiliary results connected to the main body of the paper. Section 3 is devoted to prove the completeness of the system of root vectors of $T(\varepsilon)$ and the existence of basis with parentheses of root vectors. In the last section, we apply the obtained results to a Gribov operator in Bargmann space and to a problem of radiation of a vibrating structure in a light fluid.

2. Preliminaries

In order to state our main results, let us begin with some definitions and preliminary results that we will need in the sequel. For this, let us consider a Hilbert space $\mathcal{H}$.

Definition 2.1. [22, *p.* 16] Let A be a linear operator such that its resolvent set, $\rho(A)$, is not empty. An operator B is said to be A-compact if its domain $\mathcal{D}(B)$ contains $\mathcal{D}(A)$ and if the operator $BR_\lambda(A)$ is compact, where $\lambda \in \rho(A)$. $\diamondsuit$

Definition 2.2. Let K be a compact operator on $\mathcal{H}$. K is said to belong to the Carleman-class $\mathcal{C}_p$ $(p > 0)$, if the series $\sum_{n=1}^{\infty}[s_n(\sqrt{K})]^p$ converges, where $s_n(\sqrt{K}), n = 1, 2, \ldots$, are the eigenvalues of the operator $\sqrt{K^*K}$. $\diamondsuit$

Definition 2.3. [22, *p.* 18] An operator K is said to be of finite order if it belongs to the Carleman-class $\mathcal{C}_p$ $(p > 0)$. $\diamondsuit$

Markus's theorem is formulated as:

Theorem 2.1. [*22*, *Theorem* 4.3] *Let A be a normal operator whose resolvent belongs to the Carleman-class $\mathcal{C}_p$ $(p > 0)$, and whose spectrum lies on a finite number of rays $\arg\lambda = \alpha_k (k = 1, \ldots, n)$. If B is A-compact, then the operator $G = A + B$ has a compact resolvent and the system of its root vectors is complete in $\mathcal{H}$.* $\diamondsuit$

Lemma 2.1. [*24*, *Lemma* 8] *Let $F(\lambda)$ be a scalar meromorphic function with finite order in an angle $\Lambda_\alpha = \{\lambda : |\arg\lambda| < \alpha\}$ and the poles of $F(\lambda)$ in this angle lie inside the strip $|Im\lambda| \le h$, $h > 0$. Suppose that $|F(\lambda)| \le M$ on the half-lines $Im\lambda = \pm(h + \delta)$, $\delta > 0$, inside the angle Λ_α. Then the following estimate holds inside the strip $|Im\lambda| \le h + \delta$ as $Re\lambda \to \infty$ outside an exceptional set of disks $\mathcal{D}$:*

$$\ln|F(\lambda)| \le C\left(M + \sup_{|t-r|\le r^\eta}(n(t+1,F) - n(t,F))\right), \qquad r = |\lambda|$$

where $n(t, F)$ is the pole-counting function for F and the number η can be taken arbitrarily small. For any $d > 0$, the exceptional set of disks $\mathcal{D}$ can be chosen in such a way that the total sum of the radii of the disks from $\mathcal{D}$ inside the rectangle $|Im\lambda| \le h$, $t \le Re\lambda \le t+1$ does not exceed d for any sufficiently large t. The constant C depends on δ, η, and d (the dependence on d is proportional to $\ln d$) but does not depend on r and F. $\diamondsuit$

In the remaining part of this section, we introduce the concept of basis (possibly with parentheses).

Definition 2.4. [22, *p.* 25] A sequence $\{V_n\}_{n=1}^{\infty}$ of subspaces of a Hilbert space $\mathcal{H}$ is called a basis (of subspaces), if any vector belonging to $\mathcal{H}$ can be uniquely represented as a series

$$\varphi = \sum_{n=1}^{\infty} \varphi_n \quad \text{such that} \quad \varphi_n \in V_n. \qquad \diamondsuit$$

Definition 2.5. [22, *p.* 27] A linearly independent sequence $\{\varphi_n\}_{n=1}^{\infty}$ is called a basis with parentheses for a Hilbert space $\mathcal{H}$, if there exists a sequence of positive integers $(n_k)_k$ such that $n_0 = 1$ and the subspaces spanned by the vectors $\{\varphi_n\}_{n_{k-1}}^{n_k - 1}$ form a basis for $\mathcal{H}$. $\diamondsuit$

Theorem 2.2. [*22*, *Lemma* 6.1] *Let $\{P_k\}_{k=1}^{\infty}$ be a sequence of disjoint projections (i.e.,*
$P_j P_k = \delta_{jk} P_k$). If the sequence of subspaces $R_k = Im P_k \quad (k \in \mathbb{N}^*)$ is complete in $\mathcal{H}$, then it is a basis for $\mathcal{H}$ if and only if

$$\sup_n \left\| \sum_{k=1}^{n} P_k \right\| < \infty. \qquad \diamondsuit$$

3. Main results

Let $\mathcal{H}$ be a separable Hilbert space and T_0 be a linear operator on $\mathcal{H}$ verifying the following hypotheses:
$(H1)$ T_0 is self-adjoint, positive and with domain $\mathcal{D}(T_0)$ in $\mathcal{H}$.
$(H2)$ The resolvent of T_0 is compact and its eigenvalues $(\lambda_n)_n$ verify

$$\lambda_{n+p} - \lambda_n \geq 1 \quad \text{for some} \quad p \in \mathbb{N}^*.$$

Let $T_1, T_2, T_3, \ldots$ be linear operators on $\mathcal{H}$ having the same domain $\mathcal{D}$ and satisfying the hypothesis:
$(H3)$ $\mathcal{D} \supset \mathcal{D}(T_0)$ and there exist $a, b, q > 0$ and $\beta \in]0, \frac{1}{2}[$ such that for all $k \geq 1$

$$\|T_k \varphi\| \leq q^{k-1}(a\|\varphi\| + b\|T_0\varphi\|^{\beta}\|\varphi\|^{1-\beta}) \quad \text{for all} \quad \varphi \in \mathcal{D}(T_0).$$

Let ε be a non zero complex number and consider the eigenvalue problem

$$\begin{cases} T_0\varphi + \varepsilon T_1 \varphi + \varepsilon^2 T_2 \varphi + \cdots + \varepsilon^k T_k \varphi + \cdots = \lambda \varphi \\ \varphi \in \mathcal{D}(T_0). \end{cases}$$

Before stating our main results, we shall recall the following theorem.

Theorem 3.1. [*12*, *Theorem* 2.1] *Suppose that hypotheses $(H1)$ and $(H3)$ hold. Then for $|\varepsilon| < q^{-1}$, the series $\sum_{i \geq 0} \varepsilon^i T_i \varphi$ converges for all $\varphi \in \mathcal{D}(T_0)$. If $T(\varepsilon)\varphi$ denotes its limit, then $T(\varepsilon)$ is a linear operator with domain $\mathcal{D}(T_0)$ and for $|\varepsilon| < (q + \beta b)^{-1}$, the operator $T(\varepsilon)$ is closed.* $\diamondsuit$

3.1. Completeness of the system of root vectors of $T(\varepsilon)$

The aim of this part is to establish the completeness of the system of root vectors of the perturbed operator $T(\varepsilon)$ in $\mathcal{H}$.
To this end, we need first to recall the following proposition developed in [9].
Taking into account Theorem 3.1, we denote by $B(\varepsilon) := \sum_{k=1}^{\infty} \varepsilon^k T_k$.

Proposition 3.1. [*9, Proposition* 3.1] *Assume that hypotheses* $(H1)$-$(H3)$ *hold. Then, for* $|\varepsilon| < \frac{1}{q+\beta b}$*, the operator* $B(\varepsilon)$ *is* T_0*-compact. Moreover, the operator* $T(\varepsilon)$ *is with compact resolvent.* ◇

Now, we are ready to state our result.

Theorem 3.2. *Assume that hypotheses* $(H1)$-$(H3)$ *are verified. Then, for* $|\varepsilon| < \frac{1}{q+\beta b}$*, the system of root vectors of the operator* $T(\varepsilon)$ *is complete in* $\mathcal{H}$*.* ◇

Remark 3.1.

(*i*) Theorem 3.2 extends [24, *Lemma* 7] to an analytic operator instead of the sum of two operators. Besides, we have proved that the system of root vectors of $T(\varepsilon)$ is complete even if the criteria of A. A. Shkalikov (Eq. (1.2)) is not satisfied.

(*ii*) Theorem 3.2 ameliorates Theorem 4.3 stated in [12]. Indeed, in order to guarantee that the operator $B(\varepsilon)$ is T_0-compact, the authors in [12] assumed that T_k is T_0-compact for all $k \geq 1$; whereas Proposition 3.1 ensure this result without this assumption. On the other hand, the values of $|\varepsilon|$ for which the system of root vectors of the operator $T(\varepsilon)$ is complete in $\mathcal{H}$, are greater than the one considered in [12, *Theorem* 4.3]. ◇

Proof of Theoerm 3.2.
In view of hypotheses $(H1)$ and $(H2)$, we have T_0 is self-adjoint with compact resolvent. Further, it follows from hypothesis $(H2)$ that

$$\begin{aligned} \lambda_{n+1} - \lambda_1 = {} & \underbrace{\lambda_{n+1} - \lambda_{(n+1)-p}}_{\geq 1} + \underbrace{\lambda_{(n+1)-p} - \lambda_{(n+1)-2p}}_{\geq 1} + \dots \\ & + \underbrace{\lambda_{1+p} - \lambda_{(n+1)-\frac{n}{p}p(=1)}}_{\geq 1} \geq \frac{n}{p}. \end{aligned} \tag{3.1}$$

Thus, Eq. (3.1) yields $\lambda_n \geq \frac{n-1}{p} + \lambda_1$. So, there exists $P > 1$ such that the series $\sum_{n\geq 1} (\frac{1}{\lambda_n})^P$ is convergent. Consequently, the resolvent of T_0 belongs to the Carleman-class $\mathcal{C}_P$. Moreover, in virtue of Proposition 3.1, the operator $B(\varepsilon)$ is T_0-compact for $|\varepsilon| < \frac{1}{q+\beta b}$. Consequently, Theorem 2.1 implies that for $|\varepsilon| < \frac{1}{q+\beta b}$, the system of root vectors of the operator $T(\varepsilon)$ is complete in $\mathcal{H}$. □

Corollary 3.1. *Suppose that hypotheses $(H1)$ and $(H3)$ are verified. Moreover, assume that*

$$\lambda_{n+p}^{1-\alpha} - \lambda_n^{1-\alpha} \geq 1, \quad \text{where} \ \ 0 \leq \alpha < 1. \tag{3.2}$$

Hence, for $\beta \in]0, 1 + \frac{\alpha-1}{2}[$ and $|\varepsilon| < \frac{1}{q+\beta b}$ the system of root vectors of the operator $T(\varepsilon)$ is complete in $\mathcal{H}$. ♢

Proof. It follows from Eq. (3.2) that $\lambda_n^{1-\alpha} \geq \frac{n-1}{p} + \lambda_1^{1-\alpha}$. Hence, there exists $P > 1 - \alpha$ such that the series $\sum_{n \geq 1} (\frac{1}{\lambda_n})^P$ is convergent. As T_0 is self-adjoint with compact resolvent, then the resolvent of T_0 belongs to the Carleman-class $\mathcal{C}_P$. Further, due to [9, *Corollary* 3.1] the operator $B(\varepsilon)$ is T_0-compact for $|\varepsilon| < \frac{1}{q+\beta b}$. Hence, according to Theorem 2.1, we deduce that the system of root vectors of the operator $T(\varepsilon)$ is complete in $\mathcal{H}$ for $|\varepsilon| < \frac{1}{q+\beta b}$. □

3.2. Basis with parentheses of root vectors of $T(\varepsilon)$

In Theorem 3.2, we have proved that the system of root vectors of the operator $T(\varepsilon)$ is complete. The question that occurs is whether this system forms a basis in $\mathcal{H}$. In other words, if

$$P_{n,\varepsilon} = \int_{\partial \Delta_n} (\lambda - T(\varepsilon))^{-1} d\lambda$$

denotes the spectral projection corresponding to the spectrum of $T(\varepsilon)$ inside Δ_n where Δ_n is a bounded closed isolated part of the spectrum of $T(\varepsilon)$, then the series $\sum_n P_{n,\varepsilon} f$ is convergent and its sum is it f.
To answer to this question, we shall prove some preliminary results.

Lemma 3.1. *Let τ be an arbitrary positive number. If $|Im\lambda| \geq \tau$, then for $|\varepsilon| < \frac{1}{q}$ there exists a positive number $N(\varepsilon, a, p, q, \tau)$ such that*

$$\sum_{n=1}^{\infty} \frac{\|B(\varepsilon)\varphi_n\|^2}{|\lambda - \lambda_n|^2} < N(\varepsilon, a, p, q, \tau). \tag{3.3}$$

If $Re\lambda \leq -\tau$, then for $|\varepsilon| < \frac{1}{q}$ there exists also a positive number $N_1(\varepsilon, a, p, q, \tau)$ such that

$$\sum_{n=1}^{\infty} \frac{\|B(\varepsilon)\varphi_n\|^2}{|\lambda - \lambda_n|^2} < N_1(\varepsilon, a, p, q, \tau). \tag{3.4}$$

♢

Proof. Let $n \in \mathbb{N}^*$ and λ_n be the eigenvalue number n of T_0. We have

$$\begin{aligned} \|B(\varepsilon)\varphi_n\| &= \|(\varepsilon T_1 + \varepsilon^2 T_2 + \ldots)\varphi_n\| \\ &\leq \sum_{i=1}^{\infty} \|\varepsilon^i T_i \varphi_n\|. \end{aligned}$$

Then, in view of hypothesis $(H3)$ we obtain

$$\begin{aligned}\|B(\varepsilon)\varphi_n\| &\leq \sum_{i=1}^{\infty}|\varepsilon|^i q^{i-1}(a\|\varphi_n\|+b\|T_0\varphi_n\|^{\beta}\|\varphi_n\|^{1-\beta}) \\ &\leq \sum_{i=1}^{\infty}|\varepsilon|^i q^{i-1}(a+b\lambda_n^{\beta}). \end{aligned} \tag{3.5}$$

Hence, for $|\varepsilon|<\frac{1}{q}$ it follows from Eq. (3.5) that

$$\frac{\|B(\varepsilon)\varphi_n\|^2}{|\lambda-\lambda_n|^2}\leq\frac{|\varepsilon|^2}{(1-|\varepsilon|q)^2}\left(\frac{a^2}{|\lambda-\lambda_n|^2}+\frac{2ab\lambda_n^{\beta}}{|\lambda-\lambda_n|^2}+\frac{b^2\lambda_n^{2\beta}}{|\lambda-\lambda_n|^2}\right). \tag{3.6}$$

Now, let $\sigma=Re\lambda$ and $\lambda=\sigma\pm i\tau$, where $\tau>0$. So, there exists $k\in\mathbb{N}^*$ such that $\lambda_{k-1}\leq\sigma$ and $\lambda_k>\sigma$. Thus, we have

$$\lambda_k-\sigma>\lambda_k^{\beta}\left(\lambda_k^{1-\beta}-\sigma^{1-\beta}\right)\geq C_1\lambda_k^{\beta}\ \ (C_1>0) \tag{3.7}$$

and

$$|\lambda-\lambda_{k-1}|\geq||\lambda|-\lambda_{k-1}|>|\lambda|^{\beta}\left(|\lambda|^{1-\beta}-\lambda_{k-1}^{1-\beta}\right)\geq C_2\lambda_{k-1}^{\beta}\ \ (C_2>0). \tag{3.8}$$

Then, Eqs (3.7), (3.8) imply that

$$\begin{aligned}\sum_{n=1}^{\infty}\frac{\lambda_n^{2\beta}}{|\lambda_n-\sigma|^2} &< \frac{1}{C_1^2}+\frac{1}{C_2^2}+\sum_{n<k-1}\frac{\lambda_n^{2\beta}}{|\lambda_n-\sigma|^2}+\sum_{n>k}\frac{\lambda_n^{2\beta}}{|\lambda_n-\sigma|^2} \\ &\leq \frac{2}{C^2}+\sum_{n<k-1}\frac{\lambda_n^{2\beta}}{|\lambda_n-\sigma|^2}+\sum_{n>k}\frac{\lambda_n^{2\beta}}{|\lambda_n-\sigma|^2}, \end{aligned} \tag{3.9}$$

where $C:=\min\{C_1,C_2\}$. Further, since $\beta\in]0,\frac{1}{2}[$, hence for $n<k-1$ we obtain

$$\begin{aligned}\sigma-\lambda_n &> \lambda_{k-1}-\lambda_n \\ &> (1-\beta)\lambda_n^{\beta}\left(\lambda_{k-1}-\lambda_n\right)^{1-\beta}\left(\lambda_{k-1}-\lambda_n\right)^{\beta}\lambda_n^{-\beta} \\ &\geq \gamma_1(1-\beta)\lambda_n^{\beta}\left(\lambda_{k-1}-\lambda_n\right)^{1-\beta},\quad 0<\gamma_1<1 \end{aligned} \tag{3.10}$$

and for $n>k$ we have

$$\begin{aligned}\lambda_n-\sigma &> \lambda_n-\lambda_k \\ &> (1-\beta)\lambda_n^{\beta}\left(\lambda_n-\lambda_k\right)^{1-\beta}\left(\lambda_n-\lambda_k\right)^{\beta}\lambda_n^{-\beta} \\ &\geq \gamma_2(1-\beta)\lambda_n^{\beta}\left(\lambda_n-\lambda_k\right)^{1-\beta},\quad 0<\gamma_2<1. \end{aligned} \tag{3.11}$$

So, Eqs (3.9), (3.10) and (3.11) yield

$$\sum_{n=1}^{\infty}\frac{\lambda_n^{2\beta}}{|\lambda_n-\sigma|^2}<\frac{2}{C^2}+\sum_{n<k-1}\frac{1}{\gamma_1^2(1-\beta)^2\left(\lambda_{k-1}-\lambda_n\right)^{2(1-\beta)}}$$
$$+\sum_{n>k}\frac{1}{\gamma_2^2(1-\beta)^2\left(\lambda_n-\lambda_k\right)^{2(1-\beta)}}.$$

Consequently, if we put $\gamma:=\min\{\gamma_1,\gamma_2\}$, we obtain

$$\sum_{n=1}^{\infty}\frac{\lambda_n^{2\beta}}{|\lambda_n-\sigma|^2} < \frac{2}{C^2}+\sum_{n<k-1}\frac{1}{\gamma^2(1-\beta)^2\left(\lambda_{k-1}-\lambda_n\right)^{2(1-\beta)}}+ \sum_{n>k}\frac{1}{\gamma^2(1-\beta)^2\left(\lambda_n-\lambda_k\right)^{2(1-\beta)}}. \tag{3.12}$$

As $\lambda_n\geq\frac{n-1}{p}+\lambda_1$, Eq. (3.12) yields

$$\sum_{n=1}^{\infty}\frac{\lambda_n^{2\beta}}{|\lambda-\lambda_n|^2}<\sum_{n=1}^{\infty}\frac{\lambda_n^{2\beta}}{|\lambda_n-\sigma|^2}$$
$$<\frac{2}{C^2}+\sum_{n<k-1}\frac{p^{2(1-\beta)}}{\gamma^2(1-\beta)^2\left(k-1-n\right)^{2(1-\beta)}}+\sum_{n>k}\frac{p^{2(1-\beta)}}{\gamma^2(1-\beta)^2\left(n-k\right)^{2(1-\beta)}}$$
$$\leq\frac{2}{C^2}+\sum_{n<k-1}\frac{p^{2(1-\beta)}}{\gamma^2(1-\beta)^2\left(k-1-n\right)^{2(1-\beta)}}+\sum_{m=1}^{\infty}\frac{p^{2(1-\beta)}}{\gamma^2(1-\beta)^2m^{2(1-\beta)}}$$
$$<\frac{2}{C^2}+\frac{2p^{2(1-\beta)}}{\gamma^2(1-\beta)^2}\sum_{m=1}^{\infty}\frac{1}{m^{2(1-\beta)}}=:\xi_1<\infty.$$

Moreover, if we use the same argument as above with $\frac{\beta}{2}$ we get

$$\sum_{n=1}^{\infty}\frac{\lambda_n^{\beta}}{|\lambda-\lambda_n|^2}<\frac{2}{C^2}+\frac{8p^{2-\beta}}{\gamma^2(2-\beta)^2}\sum_{m=1}^{\infty}\frac{1}{m^{2-\beta}}=:\xi_2<\infty.$$

Consequently, the series $\sum_n\frac{2ab\lambda_n^{\beta}}{|\lambda-\lambda_n|^2}$ and $\sum_n\frac{b^2\lambda_n^{2\beta}}{|\lambda-\lambda_n|^2}$ are convergent. So, there exists $\xi>0$ verifying

$$\sum_{n=1}^{\infty}\frac{2ab\lambda_n^{\beta}}{|\lambda-\lambda_n|^2}+\sum_{n=1}^{\infty}\frac{b^2\lambda_n^{2\beta}}{|\lambda-\lambda_n|^2}<b^2\xi_1+2ab\xi_2=:\xi. \tag{3.13}$$

To complete the proof of our result, we follow some ideas due to [24].

- Let us consider $|Im\lambda|\geq\tau$. For $|\varepsilon|<\frac{1}{q}$, it follows from Eqs (3.6) and (3.13) that

$$\sum_{n=1}^{\infty}\frac{\|B(\varepsilon)\varphi_n\|^2}{|\lambda-\lambda_n|^2} < \frac{|\varepsilon|^2}{(1-|\varepsilon|q)^2}\left(\xi+a^2\sum_{n=1}^{\infty}\frac{1}{(\lambda_n-\sigma)^2+\tau^2}\right).$$

On the other hand, hypothesis $(H2)$ implies that

$\lambda_{k+j+sp} - \sigma \geq s$ and $\sigma - \lambda_{k-j-sp-1} \geq s$, where $j = 0, \ldots, p-1$ and $s = 0, 1, \ldots$.

Hence, for $|\varepsilon| < \frac{1}{q}$ we obtain

$$\sum_{n=1}^{\infty} \frac{\|B(\varepsilon)\varphi_n\|^2}{|\lambda - \lambda_n|^2} < \frac{|\varepsilon|^2}{(1-|\varepsilon|q)^2} \left(\xi + a^2 p \left(\sum_{s=0}^{\frac{k}{p}-1} \frac{1}{s^2+\tau^2} + \sum_{s=0}^{\infty} \frac{1}{s^2+\tau^2} \right) \right).$$

So, for $|\varepsilon| < \frac{1}{q}$ we have

$$\begin{aligned} \sum_{n=1}^{\infty} \frac{\|B(\varepsilon)\varphi_n\|^2}{|\lambda - \lambda_n|^2} &< \frac{|\varepsilon|^2}{(1-|\varepsilon|q)^2} \left(\xi + 2pa^2 \left(\sum_{s=1}^{\infty} \frac{1}{s^2+\tau^2} + \frac{1}{\tau^2} \right) \right) \\ &< \frac{|\varepsilon|^2}{(1-|\varepsilon|q)^2} \left(\xi + 2pa^2 \left(\frac{1}{\tau^2} + \int_0^{\infty} \frac{dx}{x^2+\tau^2} \right) \right) \\ &\leq N(\varepsilon, a, p, q, \tau), \end{aligned}$$

where

$$N(\varepsilon, a, p, q, \tau) := \frac{|\varepsilon|^2}{(1-|\varepsilon|q)^2} \left(\xi + a^2 \frac{p}{\tau} \left(\pi + \frac{2}{\tau} \right) \right).$$

- Now, if $Re\lambda \leq -\tau$. It follows from hypothesis $(H2)$ that

$$\lambda_{1+j+sp} - \sigma \geq s - \sigma \geq s + \tau, \text{ where } j = 0, \ldots, p-1 \text{ and } s = 0, 1, \ldots, \tag{3.14}$$

since $\lambda_{1+j} > 0$ and $\lambda_{1+j+sp} - \sigma \geq s + \lambda_{1+j} - \sigma$. So, Eqs (3.6), (3.13) and (3.14) imply that for $|\varepsilon| < \frac{1}{q}$ we have

$$\begin{aligned} \sum_{n=1}^{\infty} \frac{\|B(\varepsilon)\varphi_n\|^2}{|\lambda - \lambda_n|^2} &< \frac{|\varepsilon|^2}{(1-|\varepsilon|q)^2} \left(\xi + a^2 \sum_{n=1}^{\infty} \frac{1}{|\sigma - \lambda_n|^2} \right) \\ &\leq \frac{|\varepsilon|^2}{(1-|\varepsilon|q)^2} \left(\xi + pa^2 \left(\sum_{s=0}^{\infty} \frac{1}{(\tau+s)^2} \right) \right) \\ &< \frac{|\varepsilon|^2}{(1-|\varepsilon|q)^2} \left(\xi + pa^2 \left(\int_0^{\infty} \frac{dx}{(\tau+x)^2} + \frac{1}{\tau^2} \right) \right) \\ &\leq N_1(\varepsilon, a, p, q, \tau), \end{aligned}$$

where

$$N_1(\varepsilon, a, p, q, \tau) := \frac{|\varepsilon|^2}{(1-|\varepsilon|q)^2} \left(\xi + a^2 \frac{p}{\tau} \left(1 + \frac{1}{\tau} \right) \right). \qquad \square$$

The following proposition holds (see [24]).

Proposition 3.2. *We have*

$$\|B(\varepsilon)(\lambda - T_0)^{-1}\|^2 \leq \sum_{n=1}^{\infty} \frac{\|B(\varepsilon)\varphi_n\|^2}{|\lambda - \lambda_n|^2}.$$

$\diamondsuit$

We denote by $S_h := \{\lambda$ such that $|Im\lambda| < h$ and $Re\lambda > -h\}$, with $h > 0$ (see Figure 1).

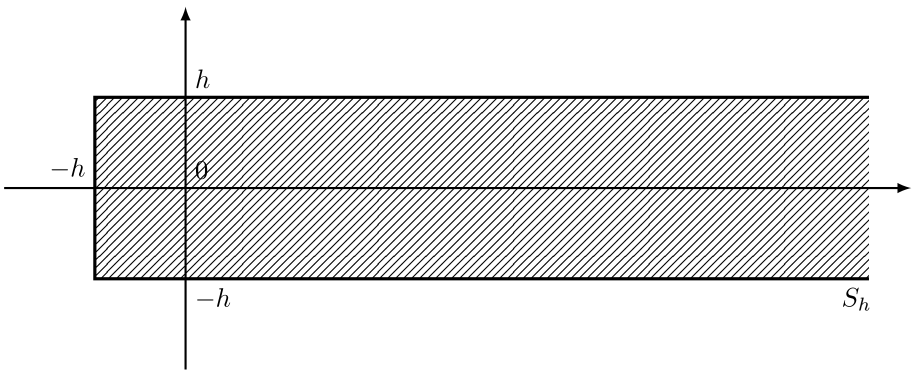

Figure 1

Proposition 3.3. *For small enough $|\varepsilon|$, the spectrum of the operator $T(\varepsilon)$ lies in the half-strip S_h.* $\diamondsuit$

Proof. Let $\lambda \in \mathbb{C}$ such that $|Im\lambda| \geq h$ or $Re\lambda \leq -h$. Since T_0 is self-adjoint and positive, then we have

$$\lambda - T(\varepsilon) = [I - B(\varepsilon)(\lambda - T_0)^{-1}](\lambda - T_0). \tag{3.15}$$

Further, combining Eq. (3.3) together with Proposition 3.2, we obtain for $|Im\lambda| \geq h$ and $|\varepsilon| < \frac{1}{q}$

$$\|B(\varepsilon)(\lambda - T_0)^{-1}\|^2 < N(\varepsilon, a, p, q, h). \tag{3.16}$$

So, for $|Im\lambda| \geq h$ and $|\varepsilon| < \frac{1}{q+\sqrt{\xi+a^2\frac{p}{h}(\pi+\frac{2}{h})}}$ we get

$$\|B(\varepsilon)(\lambda - T_0)^{-1}\| < 1.$$

On the other hand, for $Re\lambda \leq -h$ and $|\varepsilon| < \frac{1}{q}$, Eq. (3.4) and Proposition 3.2 yield

$$\|B(\varepsilon)(\lambda - T_0)^{-1}\|^2 < N_1(\varepsilon, a, p, q, h). \tag{3.17}$$

Hence, for $Re\lambda \leq -h$ and $|\varepsilon| < \frac{1}{q+\sqrt{\xi+a^2\frac{p}{h}(1+\frac{1}{h})}}$ we obtain

$$\|B(\varepsilon)(\lambda - T_0)^{-1}\| < 1.$$

Consequently, for

$$|\varepsilon| \quad < \min\left\{\frac{1}{q+\sqrt{\xi+a^2\frac{p}{h}(\pi+\frac{2}{h})}}, \frac{1}{q+\sqrt{\xi+a^2\frac{p}{h}(1+\frac{1}{h})}}\right\}$$

$$= \frac{1}{q+\sqrt{\xi+a^2\frac{p}{h}(\pi+\frac{2}{h})}},$$

we have

$$\|B(\varepsilon)(\lambda-T_0)^{-1}\| < 1 \quad \text{for } |Im\lambda| \geq h \text{ or } Re\lambda \leq -h.$$

Hence, $I - B(\varepsilon)(\lambda - T_0)^{-1}$ is invertible with bounded inverse outside S_h. Then, Eq. (3.15) implies that $\lambda - T(\varepsilon)$ is invertible with bounded inverse and we obtain

$$(\lambda - T(\varepsilon))^{-1} = (\lambda - T_0)^{-1}[I - B(\varepsilon)(\lambda - T_0)^{-1}]^{-1}. \tag{3.18}$$

Consequently $\lambda \in \rho(T(\varepsilon))$. So, the spectrum of the operator $T(\varepsilon)$ lies in the half-strip S_h. □

These results are of importance to prove the aim of this subsection.

Theorem 3.3. *Assume that hypotheses* $(H1)$*-*$(H3)$ *hold. Then, for small enough* $|\varepsilon|$*, the system of root vectors of the operator* $T(\varepsilon)$ *forms a basis with parentheses in* $\mathcal{H}$*.*$\diamondsuit$

Remark 3.2. (i) Theorem 3.3 guarantees basicity with parentheses not only for the sum of two operators such as in [24, *Theorem* 2] but for an analytic operator. Further, we prove that even if Eq. (1.2) considered in [24] is not verified, we can get a similar result.
(ii) Theorem 3.3 improves [13, *Theorem* 3.4] since we prove the existence of a basis with parentheses of root vectors of $T(\varepsilon)$ where the eigenvalues of T_0 are with finite multiplicity instead of multiplicity one. Indeed, in order to prove the existence of a Riesz basis related to the eigenvectors of $T(\varepsilon)$, the authors in [13] used a spectral analysis method based on the fact that the eigenvalues of T_0 are with multiplicity one. However, this spectral analysis can not be applied when the eigenvalues of T_0 are with finite multiplicity. $\diamondsuit$

Before going further, we recall the following result stated in [9].

Theorem 3.4. [*9*, *Theorem* 4.3.2] *Suppose that hypotheses* $(H1)$*-*$(H3)$ *are satisfied. Then, for small enough* $|\varepsilon|$*, the spectrum of the operator* $T(\varepsilon)$ *is constituted by isolated eigenvalues satisfying*

$$n(r, T(\varepsilon)) = n(r, T_0) + O(1) \quad i.e., \quad |n(r, T(\varepsilon)) - n(r, T_0)| < C_3 \quad as \quad r \to \infty,$$

where $n(r, T_0)$ *(respectively,* $n(r, T(\varepsilon))$*) denotes the sum of multiplicities of all eigenvalues of* T_0 *(respectively,* $T(\varepsilon)$*) contained in the disk* $\{\lambda \in \mathbb{C}$ *such that* $|\lambda| < r\}$ *and* C_3 *is a constant.* $\diamondsuit$

Proof of Theorem 3.3.
Let $\lambda \in \mathbb{C}$. In view of Proposition 3.3, the spectrum of $T(\varepsilon)$ lies in the half-strip $S_h := \{\lambda \text{ such that } |Im\lambda| < h \text{ and } Re\lambda > -h\}$, for $|\varepsilon| < \frac{1}{q+\sqrt{\xi+a^2\frac{p}{h}(\pi+\frac{2}{h})}}$. So, let $(\Delta_k)_{k\geq 1}$ be the rectangles bounded by the straight lines $Im\lambda = \pm h$, $Re\lambda = r_k$ and $Re\lambda = r_{k-1}$, where $r_0 = -h$ and $r_k \to \infty$ (see Figure 2).
We note here that the numbers r_k are chosen in such away that the boundary $\partial\Delta_k$ of any rectangle Δ_k does not pass through the eigenvalues of the operator $T(\varepsilon)$.

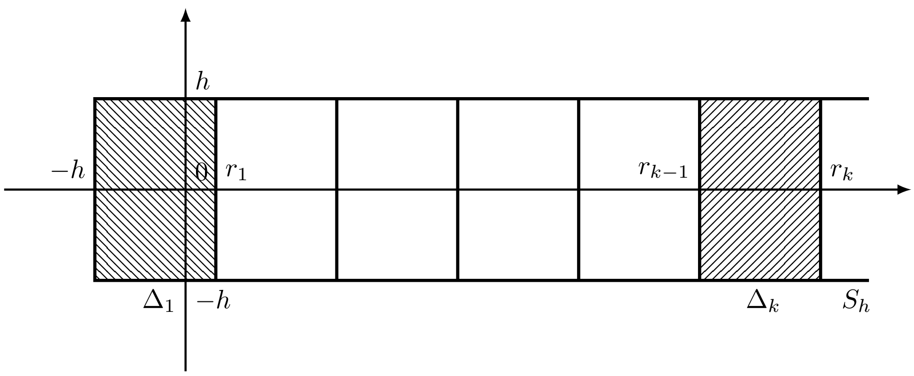

Figure 2

Then, for $|\varepsilon| < \frac{1}{q+\sqrt{\xi+a^2\frac{p}{h}(\pi+\frac{2}{h})}}$ we have

$$\sum_{k=1}^{n} \frac{-1}{2\pi i} \int_{\partial\Delta_k} (\lambda - T(\varepsilon))^{-1} d\lambda = \sum_{k=1}^{n} P_k(\varepsilon),$$

where $P_k(\varepsilon)$ designates the spectral projection corresponding to the spectrum of $T(\varepsilon)$ inside Δ_k.

To prove our result, it suffices to show that

$$\sup_n \left| \sum_{k=1}^{n} \frac{-1}{2\pi i} \int_{\partial\Delta_k} (\lambda - T(\varepsilon))^{-1} d\lambda \right| < \infty. \tag{3.19}$$

In order to do, so we are going first to estimate $\|(\lambda - T(\varepsilon))^{-1}\|$ for:

(i) $|Im\lambda| = \tau \geq h$ and $|\varepsilon| < \dfrac{1}{q+\sqrt{\xi+a^2\frac{p}{h}(\pi+\frac{2}{h})}}$.

(ii) $Re\lambda = -\tau$ and $|\varepsilon| < \dfrac{1}{q+\sqrt{\xi+a^2\frac{p}{h}(1+\frac{1}{h})}}$.

For this purpose, let us consider $\lambda = \sigma + i\tau$.

(i) In view of Eq. (3.16), we have

$$\|B(\varepsilon)(\lambda - T_0)^{-1}\| < \sqrt{N(\varepsilon, a, p, q, \tau)} < 1, \tag{3.20}$$

where

$$N(\varepsilon, a, p, q, \tau) := \frac{|\varepsilon|^2}{(1-|\varepsilon|q)^2}\left(\xi + a^2\frac{p}{\tau}\left(\pi + \frac{2}{\tau}\right)\right).$$

Further,

$$\|(\lambda - T_0)\|^{-1} \le \frac{1}{|Im\lambda|} = \frac{1}{\tau}. \tag{3.21}$$

Then, Eqs (3.20) and (3.21) yield

$$\begin{aligned}\left\|(\lambda - T(\varepsilon))^{-1}\right\| &= \left\|(\lambda - T_0)^{-1}\left[I - B(\varepsilon)(\lambda - T_0)^{-1}\right]^{-1}\right\| \\ &\le \left\|(\lambda - T_0)^{-1}\right\|\left\|\left[I - B(\varepsilon)(\lambda - T_0)^{-1}\right]^{-1}\right\| \\ &\le \frac{1}{\tau}\left(1 - \sqrt{N(\varepsilon, a, p, q, \tau)}\right)^{-1}.\end{aligned}$$

(ii) Eq. (3.17) implies that

$$\|B(\varepsilon)(\lambda - T_0)^{-1}\| < \sqrt{N_1(\varepsilon, a, p, q, \tau)} < 1, \tag{3.22}$$

where

$$N_1(\varepsilon, a, p, q, \tau) := \frac{|\varepsilon|^2}{(1-|\varepsilon|q)^2}\left(\xi + a^2\frac{p}{\tau}\left(1 + \frac{1}{\tau}\right)\right).$$

Furthermore, since

$$\begin{aligned}\|(\lambda - T_0)^{-1}\| &\le \frac{1}{d(\lambda, \sigma(T_0))} \\ &\le \frac{1}{|Re\lambda - \lambda_n|}, \qquad \lambda_n \in \sigma(T_0),\end{aligned}$$

then we get

$$\begin{aligned}\|(\lambda - T_0)^{-1}\| &\le \frac{1}{|-\tau - \lambda_n|}, \qquad Re\lambda = -\tau \\ &< \frac{1}{\tau}.\end{aligned} \tag{3.23}$$

Consequently, due to Eqs (3.22) and (3.23) we obtain

$$\begin{aligned}\|(\lambda - T(\varepsilon))^{-1}\| &= \left\|(\lambda - T_0)^{-1}\left[I - B(\varepsilon)(\lambda - T_0)^{-1}\right]^{-1}\right\| \\ &\le \frac{1}{\tau}\left(1 - \sqrt{N_1(\varepsilon, a, p, q, \tau)}\right)^{-1}.\end{aligned}$$

Now, to prove Eq. (3.19) it remains to show the existence of vertical segments in the half-strip S_h that tend to infinity and on which $(\lambda - T(\varepsilon))^{-1}$ is uniformly bounded (see [24, p. 292]).

Let us begin with the boundedness of $\|(\lambda - T(\varepsilon))^{-1}\|$. Let $f, g \in \mathcal{H}$ and consider the scalar function $F_\varepsilon(\lambda)$ defined as

$$F_\varepsilon(\lambda) = \langle (\lambda - T(\varepsilon))^{-1} f, g \rangle.$$

It is easy to see that
$*$ $F_\varepsilon(\lambda)$ is meromorphic and belongs to the Carleman-class $\mathcal{C}_P, P > 1$. In fact, due to [22, *p.* 13] the set of the Carleman-class $\mathcal{C}_P$ is a two-sided ideal of the algebra of bounded operators $L(H)$. Further, the resolvent of T_0 belongs to the Carleman-class $\mathcal{C}_P$ (see the proof of Theorem 3.2). Then, in view of Eq. (3.18) the resolvent of $T(\varepsilon)$ belongs to the Carleman-class $\mathcal{C}_P$.
$*$ The poles of $F_\varepsilon(\lambda)$ lie in the strip $|Im\lambda| < h$. Indeed, in view of [21, *p.* 38], the poles of $(\lambda - T(\varepsilon))^{-1}$ are exactly the eigenvalues of $T(\varepsilon)$ which lies in the half-strip S_h.
$*$ $|F_\varepsilon(\lambda)| \leq \frac{1}{\tau}\left(1 - \sqrt{N(\varepsilon, a, p, q, \tau)}\right)^{-1}$, for $|Im\lambda| = \tau = h + \delta$, $\delta > 0$ and $|\varepsilon| < \dfrac{1}{q + \sqrt{\xi + a^2 \frac{p}{h}(\pi + \frac{2}{h})}}$.

Then, in view of Lemma 2.1 we have

$$\ln |F_\varepsilon(\lambda)| \quad \leq \quad C' \left(\frac{1}{\tau}\left(1 - \sqrt{N(\varepsilon, a, p, q, \tau)}\right)^{-1} + \sup_{|t-r| \leq r^\eta} (n(t+1, F_\varepsilon) - n(t, F_\varepsilon)) \right),$$

for $|Im\lambda| \leq \tau$ and $Re\lambda = r_n \to \infty$ outside an exceptional set of disks $\mathcal{D}$, with $r = |\lambda|$. On the other hand, in virtue of Theorem 3.4 there exists a positive constant W such that for $|\varepsilon| < W$ we have

$$n(r, T(\varepsilon)) = n(r, T_0) + O(1). \tag{3.24}$$

Hence, hypothesis $(H2)$ and Eq. (3.24) imply that for $|\varepsilon| < W$

$$\begin{aligned} n(t+1, F_\varepsilon) - n(t, F_\varepsilon) \quad &= n(t+1, T(\varepsilon)) - n(t, T(\varepsilon)) \\ &= [n(t+1, T_0) + O(1)] - [n(t, T_0) + O(1)] \\ &= O(1) + [n(t+1, T_0) - n(t, T_0)] \\ &\leq O(1) + p = p'. \end{aligned}$$

Consequently, for $|\varepsilon| < V := \min \left\{ W, \frac{1}{q+\sqrt{\xi + a^2 \frac{p}{h}(\pi + \frac{2}{h})}} \right\}$ we have

$$|F_\varepsilon(\lambda)| \leq C_\varepsilon,$$

where C_ε is a constant independent of f, g.
Therefore, for $|\varepsilon| < V$ we obtain

$$\|(\lambda - T(\varepsilon))^{-1}\| \leq C_\varepsilon,$$

where $|Im\lambda| \le \tau$ and $Re\lambda = r_n \to \infty$ outside an exceptional set of disks $\mathcal{D}$.
Then, for $|\varepsilon| < V$ we have

$$\left|\sum_{k=1}^{n} \frac{-1}{2\pi i} \int_{\partial\Delta_k} (\lambda - T(\varepsilon))^{-1}\, d\lambda\right| < \infty.$$

Hence,

$$\sup_n \left|\sum_{k=1}^{n} \frac{-1}{2\pi i} \int_{\partial\Delta_k} (\lambda - T(\varepsilon))^{-1}\, d\lambda\right| < \infty.$$

Thus,

$$\sup_n \left\|\sum_{k=1}^{n} P_k(\varepsilon)\right\| < \infty.$$

As a consequence, due to Theorem 2.2, we claim that the family $(R(P_k(\varepsilon)))_{k\ge 1}$ forms a basis in $\mathcal{H}$ which means that the family of root vectors of $T(\varepsilon)$ forms a basis with parentheses in $\mathcal{H}$.
To complete the proof of our result, we show by a similar way as [24] the existence of vertical segments that do not pass through the eigenvalues of the operator $T(\varepsilon)$. Indeed, in each rectangle bounded by the straight lines $Re\lambda = n$, $Re\lambda = n+1$ and $Im\lambda = \pm h$, there are at most p' points of the eigenvalues $\lambda_k(\varepsilon)$ for $|\varepsilon| < W$. Hence, the projection of the disks from $\mathcal{D}$ onto the real axis does not fill the interval $[n, n+1]$. In fact, it suffices to choose $d < \frac{1}{2p'}$ (where d is the total radii of the disks from $\mathcal{D}$ inside each rectangle). So, there exists a vertical segment in this rectangle that does not intersect $\mathcal{D}$ (see Figure 3). Moreover, the vertical segments can be chosen in such a way that only points $\lambda_k(\varepsilon)$ with $|Re(\lambda_k(\varepsilon)) - Re(\lambda_j(\varepsilon))| < d$ fall between the neighboring segments.

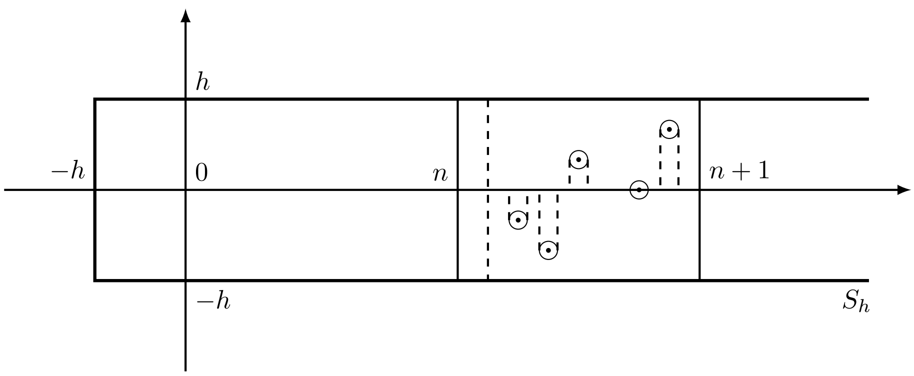

Figure 3

□

Corollary 3.2. *Assume that hypotheses* $(H1)$ *and* $(H3)$ *and Eq.* (3.2) *hold. Then, for* $\beta \in]0, 1+\frac{\alpha-1}{2}[$ *and small enough* $|\varepsilon|$*, the system of root vectors of the operator* $T(\varepsilon)$ *forms a basis with parentheses in* $\mathcal{H}$*.* $\diamondsuit$

Proof. Using $(H3)$ and making the same reasoning as the one developed in the proof of Lemma 3.1, we get for $|\varepsilon| < \frac{1}{q}$

$$\frac{\|B(\varepsilon)\varphi_n\|^2}{|\lambda-\lambda_n|^2} \leq \frac{|\varepsilon|^2}{(1-|\varepsilon|q)^2}\left(\frac{a^2}{|\lambda-\lambda_n|^2}+\frac{2ab\lambda_n^{\beta}}{|\lambda-\lambda_n|^2}+\frac{b^2\lambda_n^{2\beta}}{|\lambda-\lambda_n|^2}\right).$$

Now, let $\sigma = Re\lambda$. Then there exists $k \in \mathbb{N}^*$ such that $\lambda_{k-1} \leq \sigma$ and $\lambda_k > \sigma$. Since $\lambda_n^{1-\alpha} \geq \frac{n-1}{p} + \lambda_1^{1-\alpha}$, then $\lambda_n \geq \left(\frac{n-1}{p}+\lambda_1^{1-\alpha}\right)^{\frac{1}{1-\alpha}}$. Hence, for $n < k-1$ we obtain

$$\begin{aligned} \sigma - \lambda_n &> \lambda_{k-1} - \lambda_n \\ &\geq \lambda_n^{\beta}(\lambda_{k-1}^{1-\beta} - \lambda_n^{1-\beta}) \\ &\geq \lambda_n^{\beta}\left(\left(\frac{k-2}{p}+\lambda_1^{1-\alpha}\right)^{\frac{1-\beta}{1-\alpha}} - \left(\frac{n-1}{p}+\lambda_1^{1-\alpha}\right)^{\frac{1-\beta}{1-\alpha}}\right). \end{aligned} \tag{3.25}$$

Equivalently to Eq. (3.25), for $n > k$ we have

$$\lambda_n - \sigma > \lambda_n^{\beta}\left(\left(\frac{n-1}{p}+\lambda_1^{1-\alpha}\right)^{\frac{1-\beta}{1-\alpha}} - \left(\frac{k-1}{p}+\lambda_1^{1-\alpha}\right)^{\frac{1-\beta}{1-\alpha}}\right). \tag{3.26}$$

Two cases are presented: If $\beta \in]0, \alpha]$, then we have $\frac{1-\beta}{1-\alpha} \geq 1$. Hence, Eq. (3.25) implies that

$$\sigma - \lambda_n > \lambda_n^{\beta}\frac{(k-1-n)^{\frac{1-\beta}{1-\alpha}}}{p^{\frac{1-\beta}{1-\alpha}}} \tag{3.27}$$

and Eq. (3.26) yields

$$\lambda_n - \sigma > \lambda_n^{\beta}\frac{(n-k)^{\frac{1-\beta}{1-\alpha}}}{p^{\frac{1-\beta}{1-\alpha}}}. \tag{3.28}$$

Consequently, it follows from Eqs (3.9), (3.27) and (3.28) that

$$\begin{aligned} \sum_{n=1}^{\infty}\frac{\lambda_n^{2\beta}}{|\lambda_n-\sigma|^2} &< \frac{2}{C^2} + p^{\frac{2(1-\beta)}{1-\alpha}}\left(\sum_{n<k-1}\frac{1}{(k-1-n)^{\frac{2(1-\beta)}{1-\alpha}}} + \sum_{n>k}^{\infty}\frac{1}{(n-k)^{\frac{2(1-\beta)}{1-\alpha}}}\right) \\ &\leq \frac{2}{C^2} + p^{\frac{2(1-\beta)}{1-\alpha}}\left(\sum_{n<k-1}\frac{1}{(k-1-n)^{\frac{2(1-\beta)}{1-\alpha}}} + \sum_{m=1}^{\infty}\frac{1}{m^{\frac{2(1-\beta)}{1-\alpha}}}\right) \\ &< \frac{2}{C^2} + 2p^{\frac{2(1-\beta)}{1-\alpha}}\sum_{m=1}^{\infty}\frac{1}{m^{\frac{2(1-\beta)}{1-\alpha}}} =: \xi_1' < \infty. \end{aligned} \tag{3.29}$$

Now, if $\beta \in]\alpha, 1+\frac{\alpha-1}{2}[$. Then, we have $0 < \frac{-\alpha+\beta}{1-\alpha} < \frac{1}{2}$ and $\frac{1}{2} < 1 - \frac{-\alpha+\beta}{1-\alpha} = \frac{1-\beta}{1-\alpha} < 1$. So, in view of [22, *p.* 33] and Eq. (3.25) we get

$$\sigma - \lambda_n \quad > \quad \gamma^{'} \lambda_n^{\beta} \frac{\left(\frac{1-\beta}{1-\alpha}\right)(k-1-n)^{\frac{1-\beta}{1-\alpha}}}{p^{\frac{1-\beta}{1-\alpha}}}, \quad 0 < \gamma^{'} < 1. \tag{3.30}$$

Further, based on [22, *p.* 33] and Eq. (3.26) we obtain

$$\lambda_n - \sigma > \gamma^{''} \lambda_n^{\beta} \frac{\left(\frac{1-\beta}{1-\alpha}\right)(n-k)^{\frac{1-\beta}{1-\alpha}}}{p^{\frac{1-\beta}{1-\alpha}}}, \quad 0 < \gamma^{''} < 1. \tag{3.31}$$

Hence, similarly to Eq. (3.29), Eqs (3.9), (3.30) and (3.31) imply that

$$\sum_{n=1}^{\infty} \frac{\lambda_n^{2\beta}}{|\lambda_n - \sigma|^2} \quad < \quad \frac{2}{C^2} + \frac{2p^{\frac{2(1-\beta)}{1-\alpha}}}{\gamma_1^{'2}\left(\frac{1-\beta}{1-\alpha}\right)^2} \sum_{m=1}^{\infty} \frac{1}{m^{\frac{2(1-\beta)}{1-\alpha}}} =: \xi_2' < \infty,$$

where $\gamma_1^{'} := \min\{\gamma^{'}, \gamma^{''}\}$. Consequently, for $\beta \in]0, 1+\frac{\alpha-1}{2}[$ we get

$$\sum_{n=1}^{\infty} \frac{\lambda_n^{2\beta}}{|\lambda - \lambda_n|^2} < \sum_{n=1}^{\infty} \frac{\lambda_n^{2\beta}}{|\sigma - \lambda_n|^2} < \max\{\xi_1', \xi_2'\} =: \xi_2'.$$

On the other hand, if we replace β by $\frac{\beta}{2}$ we get

$$\sum_{n=1}^{\infty} \frac{\lambda_n^{\beta}}{|\lambda - \lambda_n|^2} < \frac{2}{C^2} + \frac{2p^{\frac{2-\beta}{1-\alpha}}}{\gamma_1^{'2}\left(\frac{1-\frac{\beta}{2}}{1-\alpha}\right)^2} \sum_{m=1}^{\infty} \frac{1}{m^{\frac{2-\beta}{1-\alpha}}} =: \xi_3' < \infty.$$

Hence, the series $\sum_n \frac{2ab\lambda_n^{\beta}}{|\sigma-\lambda_n|^2}$ and $\sum_n \frac{b^2\lambda_n^{2\beta}}{|\lambda-\lambda_n|^2}$ are convergent. So, let ξ' be a positive constant satisfying

$$\sum_{n=1}^{\infty} \frac{2ab\lambda_n^{\beta}}{|\lambda - \lambda_n|^2} + \sum_{n=1}^{\infty} \frac{b^2\lambda_n^{2\beta}}{|\lambda - \lambda_n|^2} < b^2\xi_2' + 2ab\xi_3' =: \xi'.$$

Furthermore, it follows from [9, Corollary 3.2] that for small enough $|\varepsilon|$ and $\beta \in]0, 1+\frac{\alpha-1}{2}[$ we have

$$n(r, T(\varepsilon)) = n(r, T_0) + O(1).$$

To get the desired result, we advise that the rest of the proof is similar to that of Theorem 3.3. $\square$

4. Applications

4.1. Application to a Gribov operator in Bargmann space

We are interested in a family of non self-adjoint operators, said of Gribov, studied by the specialists of physics of height energy. A representant of this family is a combination between the creation operator A^* and the annihilation operator A ([1], [2] and [15]) given by:

$$(A^*A)^3 + \varepsilon A^*(A + A^*)A + \varepsilon^2(A^*A)^{3u_2} + ... + \varepsilon^k(A^*A)^{3u_k} + ...,$$

where $\varepsilon \in \mathbb{C}$ and $(u_k)_{k\in\mathbb{N}}$ is a strictly decreasing sequence with strictly positive terms such that $u_0 = 1$ and $u_1 = \frac{1}{2}$.
We define the Bargmann space $\mathcal{B}$ by:

$$\mathcal{B} = \left\{ \varphi : \mathbb{C} \longrightarrow \mathbb{C} \text{ entire such that } \int_{\mathbb{C}} e^{-|z|^2} |\varphi(z)|^2 dz d\bar{z} < \infty \right\}.$$

This space is equipped with the following scalar product:

$$\begin{cases} \langle ., . \rangle : \mathcal{B} \times \mathcal{B} \longrightarrow \mathbb{C} \\ \qquad (\varphi, \psi) \longrightarrow \langle \varphi, \psi \rangle = \displaystyle\int_{\mathbb{C}} e^{-|z|^2} \varphi(z) \bar{\psi}(z) dz d\bar{z} \end{cases}$$

and its associated norm is denoted by $\|.\|$.

The expressions of the operators A and A^* are given by:

$$\begin{cases} A : \mathcal{D}(A) \subset \mathcal{B} \longrightarrow \mathcal{B} \\ \qquad \varphi \longrightarrow A\varphi(z) = \dfrac{d\varphi}{dz}(z) \\ \mathcal{D}(A) = \{\varphi \in \mathcal{B} \text{ such that } A\varphi \in \mathcal{B}\} \end{cases}$$

and

$$\begin{cases} A^* : \mathcal{D}(A^*) \subset \mathcal{B} \longrightarrow \mathcal{B} \\ \qquad \varphi \longrightarrow A^*\varphi(z) = z\varphi(z) \\ \mathcal{D}(A^*) = \{\varphi \in \mathcal{B} \text{ such that } A^*\varphi \in \mathcal{B}\}. \end{cases}$$

We consider the problem on $E = \{\varphi \in \mathcal{B} \text{ such that } \varphi(0) = 0\}$ and we denote by T_0 and H_1 the following operators:

$$\begin{cases} T_0 : \mathcal{D}(T_0) \subset E \longrightarrow E \\ \qquad \varphi \longrightarrow T_0\varphi(z) = (A^*A)^3\varphi(z) \\ \mathcal{D}(T_0) = \{\varphi \in E \text{ such that } T_0\varphi \in E\}, \end{cases}$$

and

$$\begin{cases} H_1 : \mathcal{D}(H_1) \subset E \longrightarrow E \\ \qquad \varphi \longrightarrow H_1\varphi(z) = A^*(A + A^*)A\varphi(z) \\ \mathcal{D}(H_1) = \{\varphi \in E \text{ such that } H_1\varphi \in E\}. \end{cases}$$

Now, we recall a straightforward, but useful result from [12].

Proposition 4.1. [*12*, *Proposition* 6.2] *We have the following assertions:*

(i) T_0 *is a self-adjoint operator.*
(ii) The resolvent set of T_0 *is compact.*
(iii) $\{e_n(z) = \frac{z^n}{\sqrt{n!}}\}_1^\infty$ *is a system of eigenvectors associated to the eigenvalues* $\{n^3\}_{n\geq 1}$ *of* T_0. ♢

Proposition 4.2. *The resolvent of the operator* T_0 *belongs to the Carleman-class* $\mathcal{C}_P$ *for any* $P > \frac{1}{3}$. ♢

Due to Proposition 4.1, T_0 is a self-adjoint operator with compact resolvent in E. Then, let

$$T_0 = \sum_{n=1}^{\infty} n^3 \langle ., e_n\rangle e_n$$

be its spectral decomposition. So, for a strictly decreasing sequence $(u_k)_{k\in\mathbb{N}}$ with strictly positive terms such that $u_0 = 1$ and $u_1 = \frac{1}{2}$, the operators $(T_0^{u_k})_{k\geq 0}$ are defined by:

$$\begin{cases} T_0^{u_k} : \mathcal{D}(T_0^{u_k}) \subset E \longrightarrow E \\ \qquad\qquad \varphi \longrightarrow T_0^{u_k}\varphi = \displaystyle\sum_{n=1}^{\infty} n^{3u_k}\langle \varphi, e_n\rangle e_n \\ \mathcal{D}(T_0^{u_k}) = \{\varphi \in E \text{ such that } \displaystyle\sum_{n=1}^{\infty} n^{6u_k} |\langle \varphi, e_n\rangle|^2 < \infty\}. \end{cases}$$

It is easy to check that for all $k \geq 0$, $\mathcal{D}(T_0^{u_k}) \subset \mathcal{D}(T_0^{u_{k+1}})$. Then, $\bigcap_{k\geq 2} \mathcal{D}(T_0^{u_k}) = \mathcal{D}(T_0^{u_2})$.
Let $\mathcal{D} = \mathcal{D}(T_0^{u_2}) \cap \mathcal{D}(H_1)$, T_1, $(T_k)_{k\geq 2}$ be the restrictions of H_1 and $T_0^{u_k}$ to $\mathcal{D}$, respectively. So, the operators $(T_k)_{k\geq 1}$ have the same domain $\mathcal{D}$ and we have $\mathcal{D}(T_0) \subset \mathcal{D}$.

Proposition 4.3. [*12*, *Proposition* 6.3] *There exist positive constants* $a, b, q > 0$ *and* $\beta \in \left[\frac{1}{2}, 1\right]$ *such that for all* $\varphi \in \mathcal{D}(T_0)$ *and for all* $k \geq 1$ *we have*

$$\|T_k\varphi\| \leq q^{k-1}(a\|\varphi\| + b\|T_0\varphi\|^\beta \|\varphi\|^{1-\beta}).$$ ♢

Remark 4.1. In Proposition 4.3, we take $q = 1$ and $a = b = 1 + 2\sqrt{2}$. ♢

Proposition 4.4. *For* $|\varepsilon| < 1$, *the series* $\sum_{k\geq 0} \varepsilon^k T_k \varphi$ *converges for all* $\varphi \in \mathcal{D}(T_0)$. *If we denote its sum by* $T(\varepsilon)\varphi$, *then we define a linear operator* $T(\varepsilon)$ *with domain* $\mathcal{D}(T_0)$. *Also, for* $|\varepsilon| < \frac{1}{1+\beta a}$, *the operator* $T(\varepsilon)$ *is closed.* ♢

The main results of this part are formulated as follows:

Proposition 4.5. *For* $|\varepsilon| < \frac{1}{1+\beta a}$ *and* $\beta \in [\frac{1}{2}, \frac{5}{6}[$, *the system of root vectors of the operator* $T(\varepsilon)$ *is complete in* E. ♢

Proof. Let λ_n be the eigenvalue number n of $(A^*A)^3$. It is easy to see that

$$\lambda_{n+p}^{\frac{1}{3}} - \lambda_n^{\frac{1}{3}} = (n+p) - n = p \geq 1, \quad (\text{where } \alpha = \tfrac{2}{3}). \tag{4.1}$$

Consequently, Corollary 3.1, Propositions 4.1, 4.2 and 4.4 and Eq. (4.1) imply that the system of root vectors of the operator $T(\varepsilon)$ is complete in E for $|\varepsilon| < \frac{1}{1+\beta a}$. $\square$

We have proved that the system of root vectors of the operator $T(\varepsilon)$ is complete in E. Now, it remains to show that it forms a basis with parentheses in E.

Theorem 4.1. *For small enough $|\varepsilon|$ and $\beta \in [\frac{1}{2}, \frac{5}{6}[$, the system of root vectors of the Gribov operator forms a basis with parentheses in E.* $\diamondsuit$

Proof. It suffices to apply Corollary 3.2, Propositions 4.1 and 4.4 and Eq. (4.1). $\square$

Remark 4.2. Theorem 4.1 ameliorates Theorem 4.1 stated in [4]. In fact, we have proved that for $\beta \in [\frac{1}{2}, \frac{5}{6}[$ the system of root vectors of the Gribov operator forms a basis with parentheses in E; while in [4], the authors showed the existence of a Riesz basis of finite-dimensional invariant subspaces for $\beta = \frac{2}{3}$. $\diamondsuit$

4.2. Application to a problem of radiation of a vibrating structure in a light fluid

An elastic membrane is stimulated by a harmonic force $F(x)e^{-i\omega t}$. It occupies the domain $-L < x < L$ of the plane $z = 0$. The two half-spaces $z < 0$ and $z > 0$ are filled with gas. The mechanical parameters of the membrane are E the Young modulus, ν the Poisson ratio, m the surface density, h the thickness of the membrane and $D := \frac{Eh^3}{12(1-\nu^2)}$ the rigidity. The fluid is characterized by ρ_0 the density, c the sound speed and $k := \frac{\omega}{c}$ the wave number.
Now, let us consider the following boundary value problem:

$$\left(\frac{d^4}{dx^4} - \frac{m\omega^2}{D}\right) u(x)$$
$$-i\rho_0 \int_{-L}^{L} H_0(k|x-x'|) \left(\frac{\omega^2}{D} - \frac{1}{m}\left(\frac{d^4}{dx^4} - \left(\frac{d^4}{dx^4}\right)^{\frac{1}{2}}\right)\right) u(x')dx' = \frac{F(x)}{D}, \qquad (4.2)$$

for all $x \in]-L, L[$ where u denotes the displacement of the membrane such that $u(x) = \dfrac{\partial u(x)}{\partial x} = 0$ for $x = -L$ and $x = L$ and H_0 is the Hankel function of the first kind and order 0 (see [20, *p.* 11]).

The problem (4.2) satisfy the following system:

$$\left(\frac{d^4}{dx^4} - \frac{m\omega^2}{D}\right) u(x) = \frac{1}{D}(F(x) - P(x)) \quad \text{for all} \quad x \in]-L, L[,$$

where

$$u(x) = \frac{\partial u(x)}{\partial x} = 0 \ \text{ for } \ x = -L \ \text{ and } \ x = L,$$

$$P(x) = \lim_{\eta \to 0^+} (p(x,\eta) - p(x,-\eta))$$

and

$$p(x,z)$$
$$= -sgn\ zi\frac{\rho_0}{2}\int_{-L}^{L} H_0(k\sqrt{(x-x')^2+z^2})\left(\omega^2 - \frac{D}{m}\left(\frac{d^4}{dx^4} - \left(\frac{d^4}{dx^4}\right)^{\frac{1}{2}}\right)\right)u(x')dx',$$

for $z < 0$ or $z > 0$ such that p designates the acoustic pressure in the fluid.

In order to study this problem, we consider the following operators:

$$\begin{cases} T_0 : \mathcal{D}(T_0) \subset L^2\big(]-L,L[\big) \longrightarrow L^2\big(]-L,L[\big) \\ \qquad\qquad\qquad\qquad \varphi \longrightarrow T_0\varphi(x) = \dfrac{d^4\varphi}{dx^4} \\ \mathcal{D}(T_0) = H_0^2\big(]-L,L[\big) \cap H^4\big(]-L,L[\big) \end{cases}$$

and

$$\begin{cases} K : L^2\big(]-L,L[\big) \longrightarrow L^2\big(]-L,L[\big) \\ \qquad\qquad\qquad \varphi \longrightarrow K\varphi(x) = \dfrac{i}{2}\displaystyle\int_{-L}^{L} H_0(k|x-x'|)\varphi(x')dx'. \end{cases}$$

Now, we recall the following result from [20].

Lemma 4.1. [*20*, *Lemmas* 3.1 *and* 3.2 *and Theorem* 3.1] *The following assertions hold:*
(i) *T_0 is a self-adjoint operator.*
(ii) *The injection from $\mathcal{D}(T_0)$ into $L^2\big(]-L,L[\big)$ is compact.*
(iii) *The spectrum of T_0 is constituted only of point spectrums which are positive, denumerable and of which the multiplicity is one and which have no finite limit points and satisfies*

$$0 < \lambda_1 \leq \lambda_2 \leq \ldots \leq \lambda_n \to +\infty.$$

Further,

$$\left(\frac{(2n+1)\pi}{4L}\right)^4 \leq \lambda_n \leq \left(\frac{(2n+3)\pi}{4L}\right)^4, \quad i.e., \ \lambda_n \sim_{+\infty} \left(\frac{n\pi}{2L}\right)^4.$$

(iv) *The resolvent of the operator T_0 belongs to the Carleman-class $\mathcal{C}_P$ for any $P > \frac{1}{4}$.*
$\diamondsuit$

Due to Lemma 4.1, T_0 is a self-adjoint operator and has a compact resolvent. Then, let

$$T_0\varphi = \sum_{n=1}^{\infty} \lambda_n \langle \varphi, \varphi_n \rangle \varphi_n$$

be its spectral decomposition, where $\lambda_n = \kappa n^4$ is the n^{th} eigenvalue of T_0 associated to the eigenvector $\varphi_n(x) = \mu e^{\sqrt[4]{\lambda_n}x} + \eta e^{-\sqrt[4]{\lambda_n}x} + \theta e^{i\sqrt[4]{\lambda_n}x} + \delta e^{-i\sqrt[4]{\lambda_n}x}$ (see [20, *p.* 7]). Hence, we define the operator B by:

$$\begin{cases} B = T_0^{\frac{1}{2}} : \mathcal{D}(B) \subset L^2\big(]-L,L[\big) \longrightarrow L^2\big(]-L,L[\big) \\ \qquad\qquad\qquad\qquad \varphi \longrightarrow B\varphi(x) = \left(\dfrac{d^4\varphi}{dx^4}\right)^{\frac{1}{2}} \\ \mathcal{D}(B) = \left\{ \varphi \in L^2(]-L,L[) \text{ such that } \displaystyle\sum_{n=1}^{\infty} \lambda_n |\langle \varphi, \varphi_n \rangle|^2 < \infty \right\} \end{cases}$$

and we consider the following eigenvalue problem:
Find the values $\lambda(\varepsilon) \in \mathbb{C}$ for which there is a solution $\varphi \in H_0^2\big(]-L,L[\big) \cap H^4\big(]-L,L[\big)$, $\varphi \neq 0$ for the equation

$$T_0\varphi + \varepsilon K(T_0 - B)\varphi = \lambda(\varepsilon)(I + \varepsilon K)\varphi \tag{4.3}$$

where $\lambda = \frac{m\omega^2}{D}$ and $\varepsilon = \frac{2\rho_0}{m}$.
Note that both λ and φ depend on the value of ε. So, we denote this by $\lambda := \lambda(\varepsilon)$ and $\varphi := \varphi(\varepsilon)$.

For $|\varepsilon| < \frac{1}{\|K\|}$, the operator $I + \varepsilon K$ is invertible. Then, the problem (4.3) becomes:
Find the values $\lambda(\varepsilon) \in \mathbb{C}$ for which there is a solution $\varphi \in H_0^2\big(]-L,L[\big) \cap H^4\big(]-L,L[\big)$, $\varphi \neq 0$ for the equation

$$(I + \varepsilon K)^{-1}T_0\varphi + \varepsilon(I + \varepsilon K)^{-1}K(T_0 - B)\varphi = \lambda(\varepsilon)\varphi. \tag{4.4}$$

The problem (4.4) is equivalent to:
Find the values $\lambda(\varepsilon) \in \mathbb{C}$ for which there is a solution $\varphi \in H_0^2\big(]-L,L[\big) \cap H^4\big(]-L,L[\big)$, $\varphi \neq 0$ for the equation

$$\big(T_0 + \varepsilon T_1 + \varepsilon^2 T_2 + \ldots + \varepsilon^n T_n + \ldots\big)\, \varphi = \lambda(\varepsilon)\varphi,$$

where $T_n := (-1)^n K^n \left(\dfrac{d^4}{dx^4}\right)^{\frac{1}{2}}$, for all $n \geq 1$.

Proposition 4.6. [*11*, *Proposition* 4.1] *The following properties hold:*
(i) There exist positive constants $a, b, q > 0$ *and* $\beta \in \left[\frac{1}{2}, 1\right]$ *such that for all* $\varphi \in \mathcal{D}(T_0)$ *and for all* $k \geq 1$ *we have*

$$\|T_k\varphi\| \leq q^{k-1}(a\|\varphi\| + b\|T_0\varphi\|^{\beta}\|\varphi\|^{1-\beta}).$$

Note that it suffices to take $a = b = q = \|K\|$.
(ii) For $|\varepsilon| < \frac{1}{\|K\|}$, *the series* $\sum_{k \geq 0} \varepsilon^k T_k\varphi$ *converges for all* $\varphi \in \mathcal{D}(T_0)$. *If we denote its sum by* $T(\varepsilon)\varphi$, *we define a linear operator* $T(\varepsilon)$ *with domain* $\mathcal{D}(T_0)$. *For* $|\varepsilon| < \frac{1}{\|K\|(1+\beta)}$, *the operator* $T(\varepsilon)$ *is closed.* ♢

Using the results described above, we can now prove the objective of this part.

Proposition 4.7. *For* $|\varepsilon| < \frac{1}{\|K\|(1+\beta)}$ *and* $\beta \in [\frac{1}{2}, \frac{7}{8}[$*, the system of root vectors of the operator* $T(\varepsilon)$ *is complete in* $L^2(]-L, L[)$*.* $\diamondsuit$

Proof. Let λ_n be the eigenvalue number n of T_0. We have

$$\lambda_{n+p}^{\frac{1}{4}} - \lambda_n^{\frac{1}{4}} = \kappa^{\frac{1}{4}}((n+p)-n) \geq 1, \quad \text{where } \alpha = \tfrac{3}{4} \text{ and } p \geq \tfrac{1}{\kappa^{\frac{1}{4}}}. \tag{4.5}$$

Then, in view of Corollary 3.1, Lemma 4.1 and Proposition 4.6 the system of root vectors of the operator $T(\varepsilon)$ is complete in $L^2(]-L, L[)$ for $|\varepsilon| < \frac{1}{\|K\|(1+\beta)}$. $\square$

Theorem 4.2. *For small enough* $|\varepsilon|$ *and* $\beta \in [\frac{1}{2}, \frac{7}{8}[$*, the system of root vectors of the operator* $T(\varepsilon)$ *forms a basis with parentheses in* $L^2(]-L, L[)$*.* $\diamondsuit$

Proof. The result follows immediately from Corollary 3.2, Lemma 4.1, Proposition 4.6 and Eq. (4.5). $\square$

Remark 4.3. Theorem 4.2 improves [11, *Theorem* 4.3]. Indeed, in [11] the authors proved that the system of root vectors of the operator $T(\varepsilon)$ forms an unconditional basis with parentheses in $L^2(]-L, L[)$ for $\beta \in [\frac{1}{2}, \frac{3}{4}]$, whereas in Theorem 4.2 we assure the existence of a basis with parentheses of root vectors for $\beta \in [\frac{1}{2}, \frac{7}{8}[$. $\diamondsuit$

References

[1] M. Aimar, A. IntissarA. Jeribi, *On an unconditional basis of generalized eigenvectors of the nonself-adjoint Gribov operator in Bargmann space.* J. Math. Anal. Appl. 231 (1999) 588–602.

[2] V. Bargmann, *On a Hilbert space of analytic functions and an associated integral transform,* Comm. Pure Appl. Math. 14 (1961) 187–214.

[3] N. Ben Ali, A. Jeribi, *On the Riesz basis of a family of analytic operators in the sense of Kato and application to the problem of radiation of a vibrating structure in a light fluid,* J. Math. Anal. Appl. 320 (2006) 78–94.

[4] S. Charfi, A. Damergi, A. Jeribi, *On a Riesz basis of finite-dimensional invariant subspaces and application to Gribov operator in Bargmann space,* Linear Multilinear Algebra 61 (2013) 1577–1591.

[5] S. Charfi, A. Jeribi, I. Walha, *Riesz basis property of families of nonharmonic exponentials and application to a problem of a radiation of a vibrating structure in a light fluid,* Numer. Funct. Anal. Optim. 32 (4) (2011) 370–382.

[6] C. Clark, *On relatively bounded perturbations of ordinary differential operators,* Pacific J. Math. 25 (1968) 59–70.

[7] N. Dunford, J.T. Schwartz, *Linear Operators, Part III.* New York, Wiley-Interscience, 1971.

[8] H. Ellouz, I. Feki, A. Jeribi, *On a Riesz basis of exponentials related to the eigenvalues of an analytic operator and application to a non-selfadjoint problem deduced from a perturbation method for sound radiation,* J. Math. Phys. 54 (2013) 112101 pp 15.

[9] H. Ellouz, I. Feki, A. Jeribi, *On the asymptotic behavior of the eigenvalues of an analytic operator in the sense of Kato and applications,* Serdica Math. J. 45 (2019) no. 1, 55–88.

[10] H. Ellouz, I. Feki, A. Jeribi, *On a Riesz basis of exponentials related to a family of analytic operators and application,* J. Pseudo-Differ. Oper. Appl. 10 (2019) no. 4, 999–1014.

[11] I. Feki, A. Jeribi, R. Sfaxi, *On an unconditional basis of generalized eigenvectors of an analytic operator and application to a problem of radiation of a vibrating structure in a light fluid,* J. Math. Anal. Appl. 375 (2011) 261–269.

[12] I. Feki, A. Jeribi, R. Sfaxi, *On a Schauder basis related to the eigenvectors of a family of non-selfadjoint analytic operators and applications,* Anal. Math. Phys. 3 (2013) 311–331.

[13] I. Feki, A. Jeribi, R. Sfaxi, *On a Riesz basis of eigenvectors of a nonself-adjoint analytic operator and applications,* Linear Multilinear Algebra 62 (2014) 1049–1068.

[14] P.J.T. Filippi, O. Lagarrigue, P.O. Mattei, *Perturbation method for sound radiation by a vibrating plate in a light fluid: comparison with the exact solution,* J. Sound Vib. 177 (1994) 259–275.

[15] A. Intissar, *Analyse Fonctionnelle et Théorie Spectrale Pour les Opérateurs Compacts Non-Autoadjoints,* Editions CEPADUES, 1997.

[16] A. Intissar, A. Jeribi, I. Walha, *Riesz basis of exponential family for a hyperbolic system,* J. Appl. Anal. 25 (1) (2019) 13–23.

[17] A. Jeribi, *Spectral Theory and Applications of Linear Operators and Block Operator Matrices,* New-York, Springer-Verlag, 2015.

[18] A. Jeribi, *Denseness, Bases and Frames in Banach Spaces and Applications,* De Gruyter, Berlin, 2018.

[19] A. Jeribi, *Perturbation Theory for Linear Operators: Denseness and Bases with Applications.* Springer-Verlag (ISBN 978-981-16-2527-5), Singapore, 2021.

[20] A. Jeribi, A. Intissar, *On an Riesz basis of generalized eigenvectors of the non-selfadjoint problem deduced from a perturbation method for sound radiation by a vibrating plate in a light fluid,* J. Math. Anal. Appl. 292 (2004) 1–16.

[21] T. Kato, *Perturbation Theory for Linear Operators*, Berlin, Springer, 1980.

[22] A.S. Markus, *Introduction to the spectral theory of polynomial operator pencils,* Translations of Mathematical Monographs, 71, American Mathematical Society, Providence, RI, 1988.

[23] B.Sz. Nagy, *Perturbations des transformations linéaires fermées,* Acta Sci. Math. Szeged 14 (1951) 125-137.

[24] A.A. Shkalikov, *On the basis property of root vectors of a perturbed selfadjoint operator,* Tr. Mat. Inst. Steklova, 269, 290-303 (2010) (Russian); translation in Proc. Steklov Inst. Math. 269 (2010) 284–298.

[25] C. Wyss, *Riesz bases for p-subordinate perturbations of normal operators,* J. Funct. Anal. 258 (2010) 208–240.

DOI: 10.7862/rf.2022.3

Hanen Ellouz
email: `ellouze.hanen@hotmail.fr`
ORCID: 0000-0003-3801-7898
Département de Mathématiques
Université de Sfax
Faculté des sciences de Sfax
Route de soukra Km 3.5
B.P. 1171, 3000, Sfax
TUNISIE

Ines Feki
email: `feki.ines@yahoo.fr`
ORCID: 0000-0003-0046-8298
Département de Mathématiques
Université de Sfax
Faculté des sciences de Sfax
Route de soukra Km 3.5
B.P. 1171, 3000, Sfax
TUNISIE

Aref Jeribi
email: `Aref.Jeribi@fss.rnu.tn`
ORCID: 0000-0001-6715-5996
Département de Mathématiques
Université de Sfax
Faculté des sciences de Sfax
Route de soukra Km 3.5

B.P. 1171, 3000, Sfax
TUNISIE

Received 11.06.2021 *Accepted 19.02.2022*

Journal of
Mathematics
and Applications

JMA No 45, pp 87-105 (2022)

Two Classes of Infinitely Many Solutions for Fractional Schrödinger-Maxwell System With Concave-Convex Power Nonlinearities

Mustapha Khiddi

Abstract: Employing critical theory and concentration estimates, we establish the existence of two classes of infinitely many weak solutions fractional Schrödinger-Poisson system involving critical Sobolev growth. The first classe of solutions with negative energy is found by using of notion genus while the second one contains infinitely many weak solutions with positive energy via Fountain theorem.

AMS Subject Classification: 35J60, 35Q55, 35R11.
Keywords and Phrases: Fractional system; Critical exponents; Fountain theorem; Genus; Concentration-compactness.

1. Introduction

In this paper we focus our attention on the following critical fractional system

$$\begin{cases} (-\Delta)^s u + u + \phi u = \lambda a(x)|u|^{r-2}u + b(x)|u|^{2_s^*-2}u & \text{in } \mathbb{R}^3, \\ (-\Delta)^t \phi = u^2 & \text{in } \mathbb{R}^3, \end{cases} \tag{1}$$

where $s \in (\frac{3}{4}, 1)$, $t \in (0,1)$ with $4s + 2t > 3$, $1 < r < 2 < 2_s^* := \frac{6}{3-2s}$, λ is a positive parameter, $a(x), b(x) \in \mathcal{C}(\mathbb{R}^3)$.
The system (1) is made up of a fractional Schrödinger equation coupled to a fractional poisson equation. It is well known that the system (1) has a strong physical

significance, because it appears in many quantum mechanics modules (see for example [5, 14]) and in semiconductor theory [3], and so on. In recent years, there has been an increasing attention to this type of system on the existence and the multiplicity of positive solutions, see the following references [2, 6, 8, 10, 11, 12, 15, 16, 21]. To our knowledge, there are few recent articles dealing with the result of the existence of two classes of solutions of infinite types and different signs of energies. By using the truncation tip at the level of the functional to make it bounded from below and satisfied the condition of $(P.S)_c$ for any $c < 0$. Following the Ljusternick-Schnirelmann theory, we obtain a negative class with infinitely solutions. Via the Fountain Theorem, we obtain the second class of infinitely positive solutions.

(A_1) Let $1 < r < 2 < 2^*_s$, $\sigma = \frac{2^*_s}{2^*_s - r}$ and $2^*_s = \frac{6}{3-2s}$, $a(x) \in \mathcal{C}(\mathbb{R}^3) \cap L^\sigma(\mathbb{R}^3)$, $b(x) \in \mathcal{C}(\mathbb{R}^3) \cap L^\infty(\mathbb{R}^3)$,

(A_2) $a(x) > 0$ in some open bounded subset Ω of $\mathbb{R}^3$ with strictly positive Lebesgue measure,

(G_1) Let G be a subgroup of O_3, $\#G = \infty$, $a(x), b(x)$ are $G-$invariant,

(G_2) $a(x) \in \mathcal{C}(\mathbb{R}^3, \mathbb{R}^+) \cap L^\sigma_G(\mathbb{R}^3)$, $b(x) \in \mathcal{C}(\mathbb{R}^3, \mathbb{R}^+)$, $b(x) = b(|x|)$ for any $x \in \mathbb{R}^3$ and $b(0) = b(\infty) = 0$.

Our first main result is the following:

Theorem 1.1.
If (A_1) and (A_2) are satisfied. Then there exists $\lambda_0 > 0$ such that, for each $\lambda \in (0, \lambda_0)$, the problem (1) has infinitely many solutions with negative energy.

Our next goal is the following:

Theorem 1.2.
If (G_1) and (G_2) are satisfied. Then for all $\lambda > 0$ the problem (1) has infinitely many solutions with positive energy.

The paper is organized as follows. In Section 2, we present some preliminaries results and we give the interval parameter λ for which the energy functional is compact. In Section 3, when λ is small enough, we prove the first Theorem 1.1 by application of genus. In Section 4, we give the proof of the second Theorem 1.2 without condition under the parameter $\lambda > 0$, we establish this result via Fountain theorem.

2. Functional framework and preliminary

For any $s \in (0, 1)$, we define the homogeneous fractional Sobolev space $\mathcal{D}^{s,2}(\mathbb{R}^3)$ as follows

$$\mathcal{D}^{s,2}(\mathbb{R}^3) = \left\{ u \in L^{2^*_s}(\mathbb{R}^3) : \ |\xi|^s \hat{u}(\xi) \in L^2(\mathbb{R}^3) \right\},$$

which is the completion of $C_0^\infty(\mathbb{R}^3)$ under the norm

$$\|u\|_{\mathcal{D}^{s,2}(\mathbb{R}^3)} = \left(\int_{\mathbb{R}^3} |\,(-\Delta)^{\frac{s}{2}}\, u|^2 dx\right)^{1/2} = \left(\int_{\mathbb{R}^3} |\xi|^{2s}|\hat{u}(\xi)|^2 d\xi\right)^{1/2}.$$

The fractional Sobolev space $H^s(\mathbb{R}^3)$ can be described by means of the Fourier transform, i.e.

$$H^s(\mathbb{R}^3) = \left\{u \in L^2(\mathbb{R}^3) : \int_{\mathbb{R}^3} |\xi|^{2s}|\hat{u}(\xi)|^2 + |\hat{u}(\xi)|^2 d\xi < +\infty\right\},$$

which is a Hilbert space under the norm. In this case, the inner product and the norm are defined as

$$\langle u, v\rangle = \int_{\mathbb{R}^3} |\xi|^{2s}\hat{u}(\xi)\overline{\hat{v}(\xi)} + \hat{u}(\xi)\overline{\hat{v}(\xi)} d\xi,$$

$$\|u\|_{H^s} = \left(\int_{\mathbb{R}^3} |\xi|^{2s}|\hat{u}(\xi)|^2 + |\hat{u}(\xi)|^2 d\xi\right)^{1/2}.$$

From Plancherel's theorem we have $\|u\|_{L^2(\mathbb{R}^3)} = \|\hat{u}\|_{L^2(\mathbb{R}^3)}$ and $\||\xi|^s|\hat{u}|\|_{L^2(\mathbb{R}^3)} = \|\,(-\Delta)^{\frac{s}{2}}\, u\|_{L^2(\mathbb{R}^3)}$. Hence

$$\|u\|_{H^s} = \left(\int_{\mathbb{R}^3} |\,(-\Delta)^{\frac{s}{2}}\, u(x)|^2 + |u(x)|^2 dx\right)^{1/2} \quad \forall u \in H^s(\mathbb{R}^3).$$

In our context, the Sobolev constant is given by

$$\mathbb{S} := \frac{\int_{\mathbb{R}^3} |(-\Delta)^{\frac{s}{2}} u(x)|^2 + |u(x)|^2 dx}{\left(\int_{\mathbb{R}^3} |u|^{2_s^*} dx\right)^{\frac{2}{2_s^*}}}. \tag{2}$$

From the embedding results, we know that $H^s(\mathbb{R}^3)$ is continuously and compactly embedded in $L^p(\mathbb{R}^3)$ when $1 \leq p < 2_s^*$, where $2_s^* = \frac{6}{3-2s}$ and the embedding is continuous but not compact if $p = 2_s^*$. For more general facts about the fractional Laplacian we refer the reader to the paper [7].
From [20], the author has proved that if $4s + 2t \geq 3$, for each $u \in H^s(\mathbb{R}^3)$, the Lax-Milgram theorem implies that there exists a unique $\phi_u^t \in \mathcal{D}^{1,2}(\mathbb{R}^3)$ such that

$$\int_{\mathbb{R}^3} (-\Delta)^{\frac{t}{2}} \phi_u^t (-\Delta)^{\frac{t}{2}} v dx = \int_{\mathbb{R}^3} u^2 v dx$$

$\forall v \in \mathcal{D}^{1,2}(\mathbb{R}^3)$, that is ϕ_u^t is a weak solution of

$$(-\Delta)^t \phi_u^t = u^2, \;\; x \in \mathbb{R}^3$$

and the representation formula holds

$$\phi_u^t(x) = c_t \int_{\mathbb{R}^3} \frac{u^2(y)}{|x-y|^{3-2t}} dy, \;\; x \in \mathbb{R}^3, \;\; c_t = \pi^{-\frac{3}{2}} 2^{-2t} \frac{\Gamma(\frac{3-2t}{2})}{\Gamma(t)},$$

which is called $t-$Riesz potential.

The properties of the function ϕ_u^t are given in the following lemma (see [[20], Lemma 2.3]).

Lemma 2.1. *If $4s + 2t \geq 3$, then for any $u \in H^s(\mathbb{R}^3)$, we have*

(i) $\phi_u^t \geq 0$;

(ii) $\phi_u^t : H^s(\mathbb{R}^3) \to \mathcal{D}^{s,2}(\mathbb{R}^3)$, *is continuous and maps bounded sets into bounded sets;*

(iii) $\int_{\mathbb{R}^3} \phi_u^t u^2 dx \leq S_s^2 \|u\|^2_{\frac{12}{3+2t}} \leq C\|u\|^4_{H^s(\mathbb{R}^3)}$;

(iv) *If $u_n \rightharpoonup u$ in $H^s(\mathbb{R}^3)$, then $\phi_{u_n}^t \to \phi_u^t$ in $\mathcal{D}^{s,2}(\mathbb{R}^3)$, and* $\int_{\mathbb{R}^3} \phi_{u_n}^t u_n^2 dx \to \int_{\mathbb{R}^3} \phi_u^t u^2 dx$.

Substituting ϕ_u^t in (1), it reduces as follows

$$(-\Delta)^s u + u + \phi_u^t u = \lambda a(x)|u|^{r-2}u + b(x)|u|^{2_s^*-2}u \text{ in } \mathbb{R}^3,$$

To find solutions of (1), we will use a variational approach. Hence, we will associate a suitable functional to our problem. More precisely, the Euler-Lagrange functional related to problem (1) is given by $I_\lambda : H^s(\mathbb{R}^3) \to \mathbb{R}$ defined as follows

$$I_\lambda(u) = \frac{1}{2}\|u\|^2_{H^s} + \frac{1}{4}\int_{\mathbb{R}^3} \phi_u^t u^2 dx - \frac{\lambda}{r}\int_{\mathbb{R}^3} a(x)|u|^r dx - \frac{1}{2_s^*}\int_{\mathbb{R}^3} b(x)|u|^{2_s^*} dx.$$

Obviously, $I_\lambda \in C^1(H^s(\mathbb{R}^3), \mathbb{R})$ and its critical points are weak solutions to (1). We call $u \in H^s(\mathbb{R}^3)$ is a weak solution of (1) if

$$\begin{aligned} \langle I'_\lambda(u), v\rangle = &\int_{\mathbb{R}^3} (-\Delta)^{\frac{s}{2}} u \, (-\Delta)^{\frac{s}{2}} v dx + \int_{\mathbb{R}^3} uv dx + \int_{\mathbb{R}^3} \phi_u^t uv dx \\ &- \lambda \int_{\mathbb{R}^3} a(x)|u|^{r-2}uv dx - \int_{\mathbb{R}^3} b(x)|u|^{2_s^*-2}uv dx = 0, \end{aligned}$$

for any $v \in H^s(\mathbb{R}^3)$.

Defined $N : H^s(\mathbb{R}^3) \to \mathbb{R}$ by $N(u) = \int_{\mathbb{R}^3} \phi_u^t u^2 dx$. The following lemma shows that the functional and possesses property which is similar to the well-known Brezis-Lieb lemma [4].

Lemma 2.2. *Assume that $4s + 2t > 3$. Let $u_n \rightharpoonup u$ in $H^s(\mathbb{R}^3)$ and $u_n \to u$ a.e. in $\mathbb{R}^3$. Then*

(i) $N(u_n - u) = N(u_n) - N(u) + o_n(1)$;

(ii) $N'(u_n - u) = N'(u_n) - N'(u) + o_n(1)$; *in $H^{-s}(\mathbb{R}^3)$.*

Proof. We can consult for example ([[20], Lemma 2.4]). □

Along the way one can easily the following lemma

Lemma 2.3. *Under the same conditions as the Lemma 2.2. Let $v_n = u_n - u \rightharpoonup 0$. Then*

$$\begin{cases} I_\lambda(v_n) \to c - I_\lambda(u), \\ I'_\lambda(v_n) \to 0. \end{cases} \tag{3}$$

We recall that

Definition 1. Let X be a Banach space

(i) For $c \in \mathbb{R}$, a sequence $\{u_n\} \subset H^s(\mathbb{R}^3)$ is a $(PS)_c$ for I_λ if $I_\lambda(u_n) = c + o(1)$ and $I_\lambda^{'}(u_n) = o(1)$ strongly in $H^{-s}(\mathbb{R}^3)$ as $n \to +\infty$;

(ii) I_λ satisfies the $(PS)_c$ condition in X if any $(PS)_c$ sequence for I_λ contains a convergent subsequence.

Let us show firstly the $(PS)_c$ sequence is bounded.

Lemma 2.4. *Let $c \in \mathbb{R}$. If $\{u_n\}$ is $(PS)_c$- sequence for I_λ, then $\{u_n\}$ is bounded in $H^s(\mathbb{R}^3)$.*

Proof. We have

$$I_\lambda(u_n) = c + o(1) \text{ and } I_\lambda^{'}(u_n) = o(1) \text{ in } H^{-s}(\mathbb{R}^3), \tag{4}$$

By contradiction, we assume that $\|u_n\|_{H^s} \to +\infty$.
Let $\widehat{u}_n = \frac{u_n}{\|u_n\|_{H^s}}$. Clearly, $\|\widehat{u}_n\|_{H^s} = 1$ is bounded in $H^s(\mathbb{R}^3)$. Up to a subsequence, we may assume that

$$\widehat{u}_n \rightharpoonup \widehat{u} \ \ in \ H^s(\mathbb{R}^3).$$

This implies

$$\widehat{u}_n \to \widehat{u} \quad in \ L^r(\mathbb{R}^3), \ 1 \leq r < 2_s^*.$$

By (4), we have

$$\begin{aligned} c + o(1) = &\frac{1}{2}\|u_n\|_{H^s}^2 \|\widehat{u}_n\|^2 + \frac{1}{4}\|u_n\|_{H^s}^2 \int_{\mathbb{R}^3} \phi_{\widehat{u}_n}^t \widehat{u}_n^2 dx - \frac{1}{2_s^*}\|u_n\|_{H^s}^{2_s^*} \int_{\mathbb{R}^3} b(x)|\widehat{u}_n|^{2_s^*} dx \\ &- \frac{\lambda}{r}\|u_n\|_{H^s}^r \int_{\mathbb{R}^3} a(x)|\widehat{u}_n|^r dx, \ as \ n \to +\infty, \end{aligned}$$

and

$$\begin{aligned} o(1) = &\|u_n\|_{H^s}^2 \|\widehat{u}_n\|^2 + \|u_n\|_{H^s}^2 \int_{\mathbb{R}^3} \phi_{\widehat{u}_n}^t \widehat{u}_n^2 dx - \|u_n\|_{H^s}^{2_s^*} \int_{\Omega} b(x)|\widehat{u}_n|^{2_s^*} dx \\ &- \lambda\|u_n\|_{H^s}^r \int_{\mathbb{R}^3} a(x)|\widehat{u}_n|^r dx, \ as \ n \to +\infty. \end{aligned}$$

By using the above two equalities, we have

$$\begin{aligned} o(1) = &(\frac{1}{r} - \frac{1}{2})\|\widehat{u}_n\|_{H^s}^2 + (\frac{1}{r} - \frac{1}{4}) \int_{\mathbb{R}^3} \phi_{\widehat{u}_n}^t \widehat{u}_n^2 dx \\ &+ \left(\frac{1}{2_s^*} - \frac{1}{r}\right) \|u_n\|_{H^s}^{2_s^*-2} \int_{\mathbb{R}^3} b(x)|\widehat{u}_n|^{2_s^*} dx, \end{aligned}$$

as $n \to +\infty$, that is

$$\begin{aligned}&(\frac{1}{r}-\frac{1}{2})\|\widehat{u}_n\|^2_{H^s}+(\frac{1}{r}-\frac{1}{4})\int_{\mathbb{R}^3}\phi^t_{\widehat{u}_n}\widehat{u}_n^2dx\\&=\left(\frac{1}{r}-\frac{1}{2^*_s}\right)\|u_n\|^{2^*_s-2}_{H^s}\int_{\mathbb{R}^3}b(x)|\widehat{u}_n|^{2^*_s}dx+o(1),\end{aligned}$$

as $n \to +\infty$.
This implies,

$$(\frac{1}{r}-\frac{1}{2})\|\widehat{u}_n\|^2_{H^s}+(\frac{1}{r}-\frac{1}{4})\int_{\mathbb{R}^3}\phi^t_{\widehat{u}_n}\widehat{u}_n^2dx\to+\infty,$$

as $n \to +\infty$. By Lemma 2.1 (iii), there $C > 0$ and $\|\widehat{u}_n\|_{H^s} = 1$ we have

$$\begin{aligned}+\infty\leftarrow&(\frac{1}{r}-\frac{1}{2})\|\widehat{u}_n\|^2_{H^s}+(\frac{1}{r}-\frac{1}{4})\int_{\mathbb{R}^3}\phi^t_{\widehat{u}_n}\widehat{u}_n^2dx\\&\le(\frac{1}{r}-\frac{1}{2})\|\widehat{u}_n\|^2_{H^s}+C\|\widehat{u}_n\|^4=(\frac{1}{r}-\frac{1}{2})+C,\ asn\to+\infty,\end{aligned}$$

which is a contradiction. Thus $\{u_n\}$ is bounded in $H^s(\mathbb{R}^3)$. □

Lemma 2.5. *There exists $\lambda_0 > 0$ such that for every $0 < \lambda < \lambda_0$ the functional I_λ satisfies $(PS)_c$ for all $c < 0$.*

Proof. Consider a $(PS)_c$ sequence $\{u_n\}$ for I_λ with $c < 0$. From Lemma 2.4 $\{u_n\}$ is bounded in $H^s(\mathbb{R}^3)$. Going if necessary to a subsequence, we can assume that

$$\begin{cases} u_n \rightharpoonup u, \text{ in } H^s(\mathbb{R}^3), \\ u_n \to u, \text{ in } L^r(\mathbb{R}^3),\ 1 \le r < 2^*_s. \end{cases} \tag{5}$$

By Lemma 2.3 we have

$$\langle I'_\lambda(u), \varphi\rangle = 0 \text{ for any } \varphi \in H^s(\mathbb{R}^3). \tag{6}$$

With (4) and $\sigma = \frac{2^*_s}{2^*_s - r}$ and the Hölder Inequality we get

$$\begin{aligned}I_\lambda(u_n)-&\frac{1}{2^*_s}\langle I'_\lambda(u_n),u_n\rangle=c+o(1)\|u_n\|_{H^s}\to c<0\\&\ge\left(\frac{1}{2}-\frac{1}{2^*_s}\right)\|u_n\|^2_{H^s}+\left(\frac{1}{2}-\frac{1}{2^*_s}\right)\int_{\mathbb{R}^3}\phi^t_{u_n}u_n^2dx+\lambda\left(\frac{1}{2^*_s}-\frac{1}{r}\right)\int_{\mathbb{R}^3}a(x)|u_n|^rdx\\&\ge\left(\frac{1}{2}-\frac{1}{2^s_*}\right)\mathbb{S}|u_n|^2_{2^*_s}-\lambda\left(\frac{1}{r}-\frac{1}{2^*_s}\right)|a|^\sigma_\sigma|u_n|^r_{2^*_s}.\end{aligned}$$

Then, there exists some constant $C > 0$ such that

$$|u_n|_{2^*_s} \le C\lambda^{\frac{1}{2-r}}, \tag{7}$$

and Brezis-Lieb Lemma [4] implies

$$|u|_{2_s^*} \leq C\lambda^{\frac{1}{2-r}}. \tag{8}$$

By (6), note that

$$\|u\|_{H^s}^2 + \int_{\mathbb{R}^3} \phi_u^t u^2 dx = \lambda \int_{\mathbb{R}^3} a(x)|u|^r dx + \int_{\mathbb{R}^3} b(x)|u|^{2_s^*} dx, \tag{9}$$

also, using Lemma 2.3, Lemma 2.1 (iv) and (4) we have

$$\|v_n\|_{H^s}^2 = \int_{\mathbb{R}^3} b(x)|v_n|^{2_s^*} dx + o(1). \tag{10}$$

Now, we suppose that

$$\lim_{n\to+\infty} \|v_n\|_{H^s}^2 = \lim_{n\to+\infty} \int_{\mathbb{R}^3} b(x)|v_n|^{2_s^*} dx = l \neq 0.$$

By Sobolev inequality, we have

$$\begin{aligned} \|v_n\|_{H^s}^2 &\geq \mathbb{S} \left(\int_{\mathbb{R}^3} |v_n|^{2_s^*} dx \right)^{\frac{2}{2_s^*}} \\ &\geq \mathbb{S} b_\infty^{-\frac{(3-2s)}{3}} \left(\int_{\mathbb{R}^3} b(x)|v_n|^{2_s^*} dx \right)^{\frac{2}{2_s^*}}, \end{aligned}$$

which implies that

$$l \geq \mathbb{S}^{\frac{3}{2s}} b_\infty^{\frac{2s-3}{2s}}. \tag{11}$$

Let $1 \leq r < 2 < 2_s^*$. By Lemmas 2.3, 2.1 (iv), (7),(9),(10), (11) and the Hölder inequality, we have

$$\begin{aligned} o(1) + c &= \frac{1}{2}\|u\|_{H^s}^2 + \frac{1}{4}\int_{\mathbb{R}^3} \phi_u^t u^2 dx - \frac{\lambda}{r}\int_{\mathbb{R}^3} a(x)|u|^r dx - \frac{1}{2_s^*}\int_{\mathbb{R}^3} b(x)|u|^{2_s^*} dx \\ &+ \frac{1}{2}\|v_n\|_{H^s}^2 - \frac{1}{2_s^*}\int_{\mathbb{R}^3} b(x)|v_n|^{2_s^*} dx + o(1) \\ &= \frac{1}{4}\|u\|_{H^s}^2 + \left(\frac{1}{2} - \frac{1}{2_s^*}\right)\|v_n\|_{H^s}^2 + \frac{1}{4}\left(\|u\|_{H^s}^2 + \int_{\mathbb{R}^3} \phi_u^t u^2 dx\right) \\ &- \frac{\lambda}{r}\int_{\mathbb{R}^3} a(x)|u|^r dx - \frac{1}{2_s^*}\int_{\mathbb{R}^3} b(x)|u|^{2_s^*} dx \\ &= \frac{1}{4}\|u\|_{H^s}^2 + \left(\frac{1}{2} - \frac{1}{2_s^*}\right)\|v_n\|_{H^s}^2 + \lambda\left(\frac{1}{4} - \frac{1}{r}\right)\int_{\mathbb{R}^3} a(x)|u|^r dx \\ &+ \left(\frac{1}{4} - \frac{1}{2_s^*}\right)\int_{\mathbb{R}^3} b(x)|u|^{2_s^*} dx \\ &\geq \frac{2s}{3}\mathbb{S}^{\frac{3}{2s}} b_\infty^{\frac{2s-3}{2s}} + \lambda\left(\frac{r-4}{4r}\right)|a|_\sigma^\sigma |u|_{2_s^*}^r. \\ &\geq \frac{2s}{3}\mathbb{S}^{\frac{3}{2s}} b_\infty^{\frac{2s-3}{2s}} + C\lambda^{\frac{2}{2-r}}\left(\frac{r-4}{4r}\right)|a|_\sigma^\sigma. \end{aligned}$$

Then there exists $K > 0$ such that

$$0 > c \geq \frac{2s}{3}\mathbb{S}^{\frac{3}{2s}} b_\infty^{\frac{2s-3}{2s}} - K\lambda^{\frac{2}{2-r}},$$

which is a contradiction for λ small enough. Then, $l = 0$, that is, $u_n \to u$ strongly in $H^s(\mathbb{R}^3)$. □

3. Proof of the first Theorem 1.1

First by the Sobolev inequality we obtain

$$I_\lambda(u) \geq h(\|u\|_{H^s}), \tag{12}$$

where

$$h(x) = \frac{1}{2}x^2 - \frac{b_\infty}{2_s^*\mathbb{S}^{\frac{2_s^*}{2}}}x^{2_s^*} - \frac{\lambda}{r}C_r x^r.$$

An easy computation shows that, there exists $\lambda_* > 0$ such that for all $0 < \lambda < \lambda_*$, the real valued function $x \mapsto h(x)$ has exactly two positive zeros denoted by R_0, R_1 and the point R is where h attains its nonnegative maximum, verifies $R_0 < R < R_1$. We now introduce the following truncation of the functional I_λ. Take the nonincreasing function $\tau : \mathbb{R}^+ \to [0, 1]$ and $C^\infty(\mathbb{R}^+)$ such that

$$\begin{cases} \tau(x) = 1 & \text{if } x < R_0, \\ \tau(x) = 0 & \text{if } x > R_1 \end{cases} \tag{13}$$

Let $\varphi(u) = \tau(\|u\|_{H^s})$. We consider the truncated functional

$$\tilde{I}_\lambda(u) = \frac{1}{2}\|u\|_{H^s}^2 + \frac{1}{4}\int_{\mathbb{R}^3}\phi_u^t u^2 dx - \frac{\lambda}{r}\int_{\mathbb{R}^3} a(x)|u|^r dx - \frac{1}{2_s^*}\int_{\mathbb{R}^3} b(x)|u|^{2_s^*}\varphi(u)dx. \tag{14}$$

Similar to 12, we have

$$\tilde{I}_\lambda(u) \geq \overline{h}(\|u\|_{H^s}) \tag{15}$$

where

$$\overline{h}(x) = \frac{1}{2}x^2 - \frac{b_\infty}{2_s^*\mathbb{S}^{\frac{2_s^*}{2}}}x^{2_s^*}\tau(x) - \frac{\lambda}{r}C_r x^r.$$

Clearly,

$$\overline{h}(x) \geq h(x) \tag{16}$$

for $x \geq 0$ and $\overline{h}(x) = h(x)$ if $0 \leq x \leq R_0$, $\overline{h}(x) \geq 0$, if $R_0 < x \leq R_1$ and if $x > R_1$, $\overline{h}(x) = x^r(\frac{1}{2}x^{2-r} - \frac{\lambda}{r}C_r)$ is strictly increasing and so $\overline{h}(x) > 0$, if $x > R_1$. Consequently

$$\overline{h}(x) \geq 0 \text{ for } x \geq R_0. \tag{17}$$

We have the following result.

Lemma 3.1. *This lemma can be expressed as three assertions:*

1. $\tilde{I}_\lambda \in \mathcal{C}^1(H^s(\mathbb{R}^3), \mathbb{R})$, *is even.*
2. *If* $\tilde{I}_\lambda(u_0) \leq 0$ *then* $\|u_0\|_{H^s} < R_0$. *Moreover,* $\tilde{I}_\lambda(u) = I_\lambda(u)$ *for all* u *in a small enough neighborhood of* u_0.
3. *There exists* $\lambda_0 > 0$, *such that if* $0 < \lambda < \lambda_0$, *then* $\tilde{I}_\lambda$ *verifies a local Palais-Smale condition for* $c < 0$.

Proof. Since $\varphi \in \mathcal{C}^\infty(H^s(\mathbb{R}^3), \mathbb{R})$ and $\varphi(u) = 1$ for u near 0, $\tilde{I}_\lambda \in \mathcal{C}^1(H^s(\mathbb{R}^3), \mathbb{R})$ and assertion **1** holds.
Note that $\tilde{I}_\lambda(u_0) \geq I_\lambda(u_0)$. By taking $\tilde{I}_\lambda(u_0) \leq 0$, we can deduce from 15 that

$$\overline{h}(\|u_0\|_{H^s}) \leq 0.$$

Then By (16) and (17) we have

$$\|u_0\|_{H^s} < R_0. \tag{18}$$

For the proof of (3), let $\{u_n\} \subset H^s(\mathbb{R}^3)$ is a $(PS)_c$ sequence $\tilde{I}_\lambda$, with $c < 0$. Then we may assume that $\tilde{I}_\lambda(u_n) < 0$, $\tilde{I}'_\lambda(u_n) \to 0$. By (2) and for $0 < \lambda < \lambda_0$, $\|u_n\|_{H^s} < R_0$, so $\tilde{I}_\lambda(u_n) = I_\lambda(u_n)$ and $\tilde{I}'_\lambda(u_n) = I'_\lambda(u_n)$. By Lemma 2.5, I_λ satisfies $(PS)_c$ condition for $c < 0$, so there is a subsequence $\{u_n\}$ such that $u_n \to u$ in $H^s(\mathbb{R}^3)$. Thus $\tilde{I}_\lambda$ satisfies $(PS)_c$ condition for $c < 0$. □

We first recall some concepts and results in minimax theory.
Let X be a Banach space, and $\sum$ denote all closed subsets of $X - \{0\}$ which are symmetric with respect to the origin. For $A \in \sum$, we define the genus $\gamma(A)$ by

$$\gamma(A) = \min\left\{k \in \mathbb{N} : \exists \phi \in C(A; \mathbb{R}^k - \{0\}), \phi(-x) = \phi(x)\right\},$$

if the minimum exists, and if such a minimum does not exist then we define $\gamma(A) = \infty$. The main properties of genus are contained in the following lemma (see[9] for the details).

Lemma 3.2. *Let* $A, B \in \sum$. *Then*

1. *If* $A \subset B$, *then* $\gamma(A) \leq \gamma(B)$.
2. *If there exists an odd homeomorphism between* A *and* B, *then* $\gamma(A) = \gamma(B)$.
3. *If* S^{N-1} *is the sphere in* $\mathbb{R}^N$, *then* $\gamma(S^{N-1}) = N$.
4. $\gamma(A \cup B) \leq \gamma(A) + \gamma(B)$.

5. *If $\gamma(A) < \infty$, then $\gamma(\overline{A - B}) \geq \gamma(A) - \gamma(B)$.*

6. *If A is compact, then $\gamma(A) < \infty$, and there exists $\delta > 0$ such that $\gamma(A) = \gamma(N_\delta(A))$ where $N_\delta(A) = \{x \in X : d(x, A) \leq \delta\}$.*

7. *If X_0 is a subspace of X with codimension k, and $\gamma(A) > k$, then $A \cap X_0 \neq \emptyset$.*

It is possible to prove the existence of level sets of $\tilde{I}_\lambda$ with arbitrarily large genus, more precisely:

Lemma 3.3. $\forall n \in \mathbb{N}$ $\exists \epsilon(n) > 0$ *such that*

$$\gamma(\{u \in H^s(\mathbb{R}^3) : \tilde{I}_\lambda(u) \leq -\epsilon(n)\}) \geq n.$$

Proof. Let Ω is an open bounded subset with strictly positive Lebesgue measure such that $a(x) > 0$ in Ω. Let $X_0^s(\Omega)$ be the function space defined as

$$X_0^s(\Omega) := \left\{u \in H^s(\mathbb{R}^3) : u = 0 \text{ a.e. in } \mathbb{R}^3 \backslash \Omega\right\}.$$

So, $X_0^s(\Omega) \subset H^s(\mathbb{R}^3)$. Observe that by [[7], Proposition 3.6] we have the following identity

$$\|u\|_{X_0^s(\Omega)} = \left(\int_\Omega |(-\Delta)^{s/2}u(x)|^2 + |u(x)|^2 dx\right)^{1/2} = \|u_n\|_{H^s}.$$

For $n \in \mathbb{N}$, we consider E_n be a $n-$dimensional subspace of $X_0^s(\Omega)$. Let $u_n \in E_n$ with norm $\|u_n\|_{H^s} = 1$. By (A_2) there exists a $c_n > 0$ such that

$$\int_\Omega a(x)|u_n|^r dx \geq c_n > 0.$$

For $0 < \rho < R_0$ and using Lemma 2.1 (iii), we get

$$\tilde{I}_\lambda(\rho u_n) \leq \frac{1}{2}\rho^2 + \frac{1}{4}C\rho^4 - \rho^{2_s^*} \int_{\mathbb{R}^3} b(x)|u_n|^{2_s^*} dx - \frac{\lambda}{r}\rho^r \int_\Omega a(x)|u_n|^r dx. \tag{19}$$

Since E_n is a finite-dimensional space, all the norms in E_n are equivalent. Thus we can define

$$\alpha_n := \inf\{\int_\Omega a(x)|u_n|^r dx : u_n \in E_n, \|u_n\|_{H^s} = 1\} \geq c_n > 0,$$

$$\beta_n := \inf\{\int_\Omega b(x)|u_n|^{2_s^*} dx : u_n \in E_n, \|u_n\|_{H^s} = 1\} > 0.$$

By using the definitions of α_n, β_n and inequality 19, we obtain

$$\tilde{I}_\lambda(\rho u_n) \leq \frac{1}{2}\rho^2 + \frac{1}{4}C\rho^4 - \rho^{2_s^*}\beta_n - \frac{\lambda}{r}\rho^r\alpha_n.$$

Then, there exists $\epsilon(n) > 0$ and $0 < \rho < R_0$ such that

$$\tilde{I}_\lambda(\rho u) \le -\epsilon(n)$$

for $u \in E_n$ and $\|u_n\|_{H^s} = 1$. Let $S_\eta = \{u \in H^s(\mathbb{R}^3)/\|u\|_{H^s} = \eta\}$, so

$$S_\eta \cap E_n \subset \{u \in H^s(\mathbb{R}^3)/\tilde{I}_\lambda(u) \le -\epsilon(n)\},$$

therefore, by Lemma 3.2 we see that

$$\gamma(\{u \in H^s(\mathbb{R}^3)/\tilde{I}_\lambda(u) \le -\epsilon\}) \ge \gamma(S_\eta \cap E_n) \ge n.$$

□

We are now in a position to prove the first result.

Proof of the Theorem 1.1.
For $n \in \mathbb{N}$, we define

$$\Gamma_n = \{A \subset H^s(\mathbb{R}^3) - \{0\}/A \text{ is close}, A = -A, \gamma(A) \ge n\}.$$

Let us set

$$c_n = \min_{A\in\Gamma_n} \max_{u\in A} \tilde{I}_\lambda(u),$$

and

$$K_c = \{u \in H^s(\mathbb{R}^3 : \tilde{I}'_\lambda(u) = 0, \tilde{I}_\lambda(u) = c\},$$

and suppose $0 < \lambda < \lambda_*$ where λ_* is the constant given by Lemma 3.1.
We claim if $n, r \in \mathbb{N}$ are such that $c = c_n = c_{n+1} = \cdots c_{n+r}$, then $\gamma(K_c) \ge r + 1$. For simplicity, we call

$$\tilde{I}_\lambda^{-\epsilon} = \{u \in H^s(\mathbb{R}^3)/\tilde{I}_\lambda(u) \le -\epsilon\}.$$

By lemma 3.3 there exists $\epsilon(n) > 0$ such that $\gamma(\tilde{I}_\lambda^{-\epsilon}) \ge n$, for all $n \in \mathbb{N}$. Because $\tilde{I}_\lambda(u)$ is continuous and even,$\tilde{I}_\lambda^{-\epsilon} \in \Gamma_n$, then $c_n \le -\epsilon(n) < 0$ for all n in $\mathbb{N}$. But $\tilde{I}_\lambda$ is bounded from below, hence $c_n > -\infty$ for all n in $\mathbb{N}$.
Let us assume that $c = c_n = c_{n+1} = ... = c_{n+r}$. Note that $c < 0$ therefore, $\tilde{I}_\lambda$ verifies the Plais-Smale condition in c, and it is easy to see that K_c is a compact set.
If $\gamma(K_c) \le r$, there exists a closed and symmetric set U verifying $K_c \subset U$, such that $\gamma(U) \le r$. By the deformation lemma (see [19]), we have an odd homeomorphism $\eta : H^s(\mathbb{R}^3) \to H^s(\mathbb{R}^3)$, such that $\eta(\tilde{I}_\lambda^{c+\delta} - U) \subset \tilde{I}_\lambda^{c-\delta}$, for some $\delta > 0$. By definition,

$$c = c_n = \inf_{A\in\Gamma_{n+r}} \sup_{u\in A} \tilde{I}_\lambda(u).$$

There exists then $A \in \Gamma_{n+r}$, such that $\sup_{u\in A} \tilde{I}_\lambda(u) < c + \delta$. i.e $A \subset \tilde{I}_\lambda^{c+\delta}$,

$$\eta(A - U) \subset \eta(\tilde{I}_\lambda^{c+\delta} - U) \subset \tilde{I}_\lambda^{c-\delta}.$$

By Lemma 3.2 (5) again $\gamma(\overline{A - U}) \ge \gamma(A) - \gamma(U) \ge n$, and $\gamma(\eta(\overline{A - U})) \ge \gamma(\overline{A - U})) \ge n$.

Then, $\eta(\overline{A-U}) \in \Gamma_n$. Impossible, in fact $\eta(\overline{A-U}) \in \Gamma_n$ implies $\sup_{u\in\eta(\overline{A-U})} \tilde{I}_\lambda(u) \geq c_n = c$.
So we have proved that $\gamma(K_c) \geq r+1$. We are now ready to show that I_λ has infinitely many critical point solutions. Note that c_n is non-decreasing and strictly negative. We distinguish two cases.
Case 1 Suppose that there are $1 < n_1 < \cdots n_i < \cdots$, satisfying

$$c_{n_1} < \cdots < c_{n_i} < \cdots.$$

In this case, we have infinitely many distinct critical points.
Case 2 We assume in this case, that for some positive integer n_0, there is a $r \geq 1$ such that $c = c_{n_0} = c_{n_0+1} = \cdots = c_{n_0+r}$, then $\gamma(K_{c_{n_0}}) \geq r+1$ which shows that $K_{c_{n_0}}$ contains infinitely many distinct elements. Since $\tilde{I}_\lambda(u) = I_\lambda(u)$ if $\tilde{I}_\lambda(u) < 0$, we see that there are infinitely many critical points of $I_\lambda(u)$. The theorem 1.1 is proved. □

4. Proof of the second Theorem 1.2

In this section, we show the existence of infinitely many solutions via the Fountain Theorem [22].
We consider

$$H^s_G(\mathbb{R}^3) := \left\{u \in H^s(\mathbb{R}^3) : u(\tau x) = u(x), \tau \in G\right\},$$

where G is a subgroup of the group of orthogonal linear transformations O_3. Let us consider the functional $I_{\lambda,G}: \ H^s_G(\mathbb{R}^3) \to \mathbb{R}$ as $I_{\lambda,G} = I_\lambda|_{H^s_G(\mathbb{R}^3)}$. By the principle of symmetric criticality of Krawcewicz-Marzantowicz [13], we know that u is a critical point of I_λ if and only if u is a critical point of $I_{\lambda,G} = I_\lambda|_{H^s_G(\mathbb{R}^3)}$.

Lemma 4.1. *For any $\lambda > 0$, $s \in (\frac{3}{4}, 1)$ and $t \in (0,1)$ such that $4s + 2t > 3$, the functional $I_{\lambda,G}$ satisfies $(PS)_c$ for all $c \in \mathbb{R}$.*

Proof. Let $\{u_n\}$ in $H^s_G(\mathbb{R}^3)$ such that $I_{\lambda,G}(u_n) \to c$ and $I'_{\lambda,G}(u_n) \to 0$ strongly in $H^{-s}_G(\mathbb{R}^3)$. Following the same arguments as in the proof of Lemma 2.4 we have $\{u_n\}$ is bounded. Therefore, up to a subsequence, we may assume that

$$\begin{cases} u_n \rightharpoonup u, \text{ in } H^s(\mathbb{R}^3); \\ u_n \to u, \text{ in } L^r(\mathbb{R}^3), \ 1 \leq r < 2^*_s; \\ u_n(x) \to u(x), \text{ a.e. in } \mathbb{R}^3. \end{cases} \tag{20}$$

From the concentration-compactness alternative for bounded sequences in the fractional space $H^s_G(\mathbb{R}^3)$, see [[18], Theorem 2.2]: There exists a subsequence, still denoted by $\{u_n\}$, at most countable set Λ, a set of points $\{x_j\}_{j\in\Lambda} \subset \mathbb{R}^3$ and real numbers $\mu_j, \nu_j \in [0, \infty)$ such that

$$|(-\Delta)^{s/2}u_n|^2 \rightharpoonup d\mu \geq |(-\Delta)^{s/2}u|^2 + \sum_{j\in\Lambda} \mu_j \delta_{x_j}, \ \mu_j = \mu(x_j), \tag{21}$$

$$|u_n|^{2_s^*} \rightharpoonup d\nu = |u|^{2_s^*} + \sum_{j\in\Lambda} \nu_j \delta_{x_j},\ \nu_j = \nu(x_j), \tag{22}$$

$$\mu_j \geq \mathbb{S}\nu_j^{\frac{2}{2_s^*}}. \tag{23}$$

We claim that the concentration of ν cannot occur at any $x \neq 0$. Now we suppose that there exists $x_j \neq 0$, where $j_0 \in \Lambda$ such that $\nu_{j_0} = \nu_{x_{j_0}} > 0$. The measure ν is $G-$invariant. For all $\tau \in G$, $\nu(x_{j_0}) = \nu(\tau x_{j_0}) > 0$. We know that $\#G = \infty$, thus

$$\nu\left(\{\tau x_{j_0} : \tau \in G\}\right) = \infty.$$

Note that the measure ν is finite, which is a contradiction. Then, for any $x_j \neq 0$ where $j \in \Lambda$, we get $\nu_j = \nu(x_j) = 0$. Now we suppose that $0 \notin \{x_j : j \in \Lambda\}$. In fact, assume $\varepsilon > 0$ small enough such that for any $0 \notin B_\varepsilon(0)$. Let $\varphi_\varepsilon \in \mathcal{C}_0^\infty(\mathbb{R}^3)$ be a cut-off function centered at 0 satisfying

$$0 \leq \varphi_\varepsilon \leq 1,\ \ \varphi_\varepsilon(x) = \begin{cases} 1 \text{ if } |x| \leq \frac{\varepsilon}{2}, \\ 0 \text{ if } |x| \geq \varepsilon. \end{cases}$$

Since $(\varphi_\varepsilon u_n)$ is bounded, $\langle I'_{\lambda,G}(u_n), \varphi_\varepsilon u_n\rangle \to 0$, that is

$$\begin{aligned}&\langle(-\Delta)^{\frac{s}{2}}(u_n), \varphi_\varepsilon(-\Delta)^{\frac{s}{2}}(u_n)\rangle + \langle(-\Delta)^{\frac{s}{2}}(u_n), u_n(-\Delta)^{\frac{s}{2}}(\varphi_\varepsilon)\rangle + \int_{\mathbb{R}^3} u_n^2\varphi_\varepsilon dx \\ &+ \int_{\mathbb{R}^3}\phi_{u_n}^t u_n^2\varphi_\varepsilon dx = \lambda\int_{\mathbb{R}^3} a(x)|u_n|^r\varphi_\varepsilon dx + \int_{\mathbb{R}^3} b(x)|u_n|^{2_s^*}\varphi_\varepsilon dx + o(1)\end{aligned}$$

$$\lim_{n\to+\infty}\langle(-\Delta)^{\frac{s}{2}}(u_n), \varphi_\varepsilon(-\Delta)^{\frac{s}{2}}(u_n)\rangle = \int_{\mathbb{R}^3}\varphi_\varepsilon d\mu \tag{24}$$

$$\lim_{n\to+\infty}\int_{\mathbb{R}^3} b(x)|u_n|^{2_s^*}\varphi_\varepsilon dx = \int_{\mathbb{R}^3} b(x)\varphi_\varepsilon d\nu = \int_{\mathbb{R}^3} b(x)|u|^{2_s^*}\varphi_\varepsilon dx + b(x_j)\nu_j \tag{25}$$

$$\begin{aligned}&\lim_{\varepsilon\to0}\lim_{n\to0}\left|\langle(-\Delta)^{\frac{s}{2}}(u_n), u_n(-\Delta)^{\frac{s}{2}}(\varphi_\varepsilon)\rangle\right| \\ &\leq \lim_{\varepsilon\to0}\lim_{n\to0}\left(\left(\int_{\mathbb{R}^3}|(-\Delta)^{\frac{s}{2}}u_n|^2dx\right)^{1/2}\times\left(\int_{\mathbb{R}^3}|u_n|^2|(-\Delta)^{\frac{s}{2}}\varphi_\varepsilon|^2dx\right)^{1/2}\right) \\ &\leq C\lim_{\varepsilon\to0}\left(\int_{\mathbb{R}^3}|u|^2|(-\Delta)^{\frac{s}{2}}\varphi_\varepsilon|^2dx\right)^{1/2} \\ &\leq C\lim_{\varepsilon\to0}\left(\int_{B_\varepsilon(0)}|u|^{2_s^*}|dx\right)^{1/2_s^*}\left(\int_{B_\varepsilon(0)}(-\Delta)^{\frac{s}{2}}\varphi_\varepsilon|^{\frac{3}{s}}dx\right)^{\frac{s}{3}} \\ &\leq C\lim_{\varepsilon\to0}\left(\int_{B_\varepsilon(0)}|u|^{2_s^*}|dx\right)^{1/2_s^*} = 0,\end{aligned} \tag{26}$$

and

$$\lim_{\varepsilon\to 0}\int_{\mathbb{R}^3} |\,(-\Delta)^{\frac{s}{2}}\, u|^2\varphi_\varepsilon dx = 0$$
$$\lim_{\varepsilon\to 0}\int_{\mathbb{R}^3} b(x)|u|^{2_s^*}\varphi_\varepsilon dx = 0, \tag{27}$$
$$\lim_{\varepsilon\to 0}\int_{\mathbb{R}^3} a(x)|u|^r\varphi_\varepsilon dx = 0,\;\; \lim_{\varepsilon\to 0}\int_{\mathbb{R}^3} |u|^2\varphi_\varepsilon dx = 0,$$
$$\lim_{\varepsilon\to 0}\int_{\mathbb{R}^3} \phi_u^t u^2\varphi_\varepsilon dx = 0.$$

Thus,

$$\mu(\{0\}) = b(0)\nu(\{0\}).$$

Note that $b(0) = 0$, then $\mu(\{0\}) = 0$. In the next step, we claim that the concentration of ν cannot occur at infinity.

$$\nu_\infty = \lim_{R\to+\infty}\limsup_{n\to+\infty}\int_{|x|>R} |u_n|^{2_s^*}dx,$$

$$\mu_\infty = \limsup_{n\to+\infty}\int_{x|>R} |\,(-\Delta)^{\frac{s}{2}}\, u_n|^2 dx.$$

Hence, by using the concept of the concentration-compactness in ([17],[18]) at infinity, ν_∞ and μ_∞ exist and satisfy :

$$\limsup_{n\to+\infty}\int_{\mathbb{R}^3} |u_n|^{2_s^*}dx = \int_{\mathbb{R}^3} d\nu + \nu_\infty$$

$$\limsup_{n\to+\infty}\int_{\mathbb{R}^3} |\,(-\Delta)^{\frac{s}{2}}\, u_n|^2 dx = \int_{\mathbb{R}^3} d\mu + \mu_\infty.$$

$$\mathbb{S}\nu_\infty^{2/2_s^*} \le \mu_\infty. \tag{28}$$

For any $R > 0$, take a radially symmetric function $\chi_R \in \mathcal{C}^\infty(\mathbb{R}^3)$ such that $0 \le \chi_R \le 1$, $\chi_R = 1$ in $\mathbb{R}^3\backslash B_{2R}$, $\chi_R = 0$ in B_R. It is easy to obtain that $\chi_R u_n$ is bounded on $H^s_G(\mathbb{R}^3)$. Then

$$\lim_{n\to+\infty}\langle I'_{\lambda,G}(u_n), \chi_R u_n\rangle = 0.$$

We have

$$\langle(-\Delta)^{\frac{s}{2}}\,(u_n), \chi_R\,(-\Delta)^{\frac{s}{2}}\,(u_n)\rangle + \langle(-\Delta)^{\frac{s}{2}}\,(u_n), u_n\,(-\Delta)^{\frac{s}{2}}\,(\chi_R)\rangle + \int_{\mathbb{R}^3} u_n^2\chi_R dx$$
$$+\int_{\mathbb{R}^3}\phi_{u_n}^t u_n^2\chi_R dx = \lambda\int_{\mathbb{R}^3} a(x)|u_n|^r\chi_R dx + \int_{\mathbb{R}^3} b(x)|u_n|^{2_s^*}\chi_R dx + o(1)$$

Similar to the proof of (26), we have

$$\lim_{R\to+\infty}\limsup_{n\to+\infty}\langle(-\Delta)^{\frac{s}{2}}(u_n), u_n(-\Delta)^{\frac{s}{2}}(\chi_R)\rangle \le C\lim_{R\to+\infty}\left(\int_{R<|x|<2R}|u|^{2_s^*}dx\right)^{1/2_s^*}=0.$$

Also,

$$\lim_{R\to+\infty}\limsup_{n\to+\infty}\int_{\mathbb{R}^3}a(x)|u_n|^r\chi_R dx=\lim_{R\to+\infty}\int_{\mathbb{R}^3}a(x)|u|^r\chi_R dx=0,$$

$$\lim_{R\to+\infty}\limsup_{n\to+\infty}\int_{\mathbb{R}^3}u_n^2\chi_R dx=\lim_{R\to+\infty}\int_{\mathbb{R}^3}u^2\chi_R dx=0,$$

$$\lim_{R\to+\infty}\limsup_{n\to+\infty}\int_{\mathbb{R}^3}\phi_{u_n}^t u_n^2\chi_R dx=\lim_{R\to+\infty}\int_{|x|>R}\phi_u^t u^2\chi_R dx=0.$$

Since $b(\infty)=0$,

$$\lim_{R\to+\infty}\limsup_{n\to+\infty}\int_{|x|>R}b(x)|u_n|^{2_s^*}dx=0.$$

Then,

$$\mu_\infty=\lim_{R\to+\infty}\limsup_{n\to+\infty}\int_{x|>R}|(-\Delta)^{\frac{s}{2}}u_n|^2dx\le\lim_{R\to+\infty}\limsup_{n\to+\infty}\int_{|x|>R}b(x)|u_n|^{2_s^*}dx=0.$$

Thus $\mu_\infty=0$. Then, from (28) we obtain $\nu_\infty=0$. Hence, up to a subsequence, we derive

$$\lim_{n\to+\infty}\int_{\mathbb{R}^3}|u_n|^{2_s^*}dx=\int_{\mathbb{R}^3}|u|^{2_s^*}dx.$$

By Brézis-Leib [4] $u_n\to u$ in $L_G^{2_s^*}(\mathbb{R}^3)$. Note that $b\in L_G^\infty(\mathbb{R}^3)$ we obtain

$$\lim_{n\to+\infty}\int_{\mathbb{R}^3}b(x)|u_n-u|^{2_s^*}dx=0.$$

Then $u_n\to u$ strongly in $H_G^s(\mathbb{R}^3)$. □

Since $H_G^s(\mathbb{R}^3)$ is separable (see [1]), there exist $\{e_n\}_{n\in\mathbb{N}}\subset H_G^s(\mathbb{R}^3)$ and $\{f_n\}_{n\in\mathbb{N}}\subset H_G^{-s}(\mathbb{R}^3)$ with

$$H_G^s(\mathbb{R}^3)=\overline{\operatorname{span}\{e_n\}_{n=1}^\infty},\ \ H_G^{-s}(\mathbb{R}^3)=\overline{\operatorname{span}\{f_n\}_{n=1}^\infty}$$

$$\langle f_i,e_j\rangle=\begin{cases}1 \text{ if } i=j,\\ 0 \text{ if } i\ne j,\end{cases}$$

where $\langle,\rangle$ is the duality pairing between $H_G^{-s}(\mathbb{R}^3)$ and $H_G^s(\mathbb{R}^3)$.

$$\textit{Let } X_j=\operatorname{span}\{e_j\},\ \ Y_n=\bigoplus_{j=0}^n X_j,\ \ Z_n=\overline{\bigoplus_{j=n}^\infty X_j}.$$

Let

Lemma 4.2. *([22] Fountain theorem)*
Consider an even functional $I_{\lambda,G} \in \mathcal{C}(H^s_G(\mathbb{R}^3), \mathbb{R})$. If, for every $k \in \mathbb{N}$, there exist $\rho_k > r_k > 0$ such that

1. $\alpha_k := \max\left\{I_{\lambda,G}(u) : u \in Y_k, \|u\|_{H^s_G} = \rho_k\right\} \leq 0.$
2. $\beta_k := \inf\left\{I_{\lambda,G}(u) : u \in Z_k, \|u\|_{H^s_G} = \rho_k\right\} \to \infty$ *as* $k \to +\infty$.
3. $I_{\lambda,G}$ *satisfying (PS) condition for every* $c > 0$.

Then $I_{\lambda,G}$ has an unbounded sequence of critical values.

Proof of Theorem 1.2.
The functional $I_{\lambda,G}$ is even, $I_{\lambda,G} \in \mathcal{C}(H^s_G(\mathbb{R}^3), \mathbb{R})$. By Lemma 4.1 $I_{\lambda,G}$ satisfying (PS) condition for any $c \in \mathbb{R}$. We only need to verify $I_{\lambda,G}$ satisfying (1) and (2) of Lemma 4.2. Since X_j is a finite-dimensional subspace of $H^s_G(\mathbb{R}^3)$ for each $j \in \mathbb{N}$ and $b(x) > 0$ a.e. in $\mathbb{R}^3$, this implies that there exists a constant $\varepsilon_j > 0$ such that for all $v \in X_j$ with $\|v\|_{H^s_G} = 1$ we have

$$\int_{\mathbb{R}^3} b(x)|v|^{2^*_s} dx \geq \varepsilon_j.$$

On the other hand,
for any $u \in X_j \backslash \{0\}$, with $\|u\|_{H^s_G} = 1$ and by using the Lemma Sobolev inequality we get

$$\begin{aligned} I_{\lambda,G}(tu) &\leq \frac{t^2}{2}\|u\|^2_{H^s_G} + C\frac{t^4}{2}\|u\|^4_{H^s_G} - \frac{\lambda t^r}{r}\int_{\mathbb{R}^3} a(x)|u|^r dx - \frac{t^{2^*_s}}{2^*_s}\int_{\mathbb{R}^3} b(x)|u|^{2^*_s} dx \\ &\leq \frac{t^2}{2} + C\frac{t^4}{2} - \frac{t^{2^*_s}}{2^*_s}\varepsilon_j. \end{aligned}$$

Since $4 < 2^*_s$, there exists $t_j > 1$ such that $e_j = t_j u$ satisfies $I_{\lambda,G}(e_j) \leq 0$. This proves (1) of Lemma 4.2.
Define

$$\beta_j = \sup_{u \in Z_j, \|u\|_{H^s_G} = 1} \left(\int_{\mathbb{R}^3} b(x)|u|^{2^*_s} dx\right)^{1/2^*_s}.$$

By the definition of Z_j, we get $u_j \rightharpoonup 0$ in $H^s_G(\mathbb{R}^3)$. Since $b(x)$ is continuous, $b(0) = 0$, $b(\infty) = 0$ and by the same argument using in Lemma 4.1 we see that a concentration of the measure ν can only occur at 0 and ∞. We deduce that

$$\int_{\mathbb{R}^3} b(x)|u_j|^{2^*_s} dx \to 0,$$

as $j \to \infty$, so

$$\beta_j \to 0.$$

For all $u \in Z_j$, we have

$$I_{\lambda,G}(u) \geq \frac{1}{2}\|u\|^2_{H^s_G} - \frac{\lambda C}{r}\|u\|^r_{H^s_G} - \frac{\beta_j^{2^*_s}}{2^*_s}\|u\|^{2^*_s}_{H^s_G}.$$

Let $u \in Z_j$, such that $\|u\|_{H^s_G} = A_j = \left(\frac{1}{\beta_j^{2^*_s}}\right)^{\frac{1}{2^*_s-2}}$ Since $\beta_j \to 0$ we have $A_j \to +\infty$ as $j \to +\infty$. Since $1 < r < 2$ we have

$$I_{\lambda,G}(u) \geq \left(\frac{1}{2} - \frac{1}{2^*_s}\right)A_j^2 - \frac{\lambda C}{r}A_j^r \to +\infty, \text{ as } j \to +\infty.$$

So, $I_{\lambda,G}$ satisfies (2). All the assumptions of Lemma 4.2 are satisfied. Therefore, this concludes the proof of Theorem 1.2. □

References

[1] R. Adams, *Sobolev Spaces*, Ac. Press, New York, 1975.

[2] A. Ambrosetti, D. Ruiz, *Multiple bound states for the Schrödinger-Poisson problem*, Communications in Contemporary Mathematics 10 (03) (2008) 391–404.

[3] V. Benci, D. F. Fortunato, *Solitary waves of the nonlinear Klein-Gordon equation coupled with the Maxwell equations*, Reviews in Mathematical Physics 14 (04) (2002) 409–420.

[4] H. Brézis, E. Lieb, *A relation between pointwise convergence of functions and convergence of functionals*, Proceedings of the American Mathematical Society 88 (3) (1983) 486–490.

[5] I. Catto, P.L. Lions, *A ecessary and sufficient condition for the stability of general molecular systems*, Communications in Partial Differential Equations 17 (7-8) (1992) 1051–1110.

[6] T. D'Aprile, D. Mugnai, *Non-existence results for the coupled Klein-Gordon-Maxwell equations*, Advanced Nonlinear Studies 4 (3) (2004) 307–322.

[7] E. Di Nezza, G. Palatucci, E. Valdinoci, *Hitchhiker's guide to the fractional Sobolev spaces*, Bulletin des Sciences Mathématiques 136 (5) (2012) 521–573.

[8] X. He, W. Zou, *Existence and concentration of ground states for Schrödinger-Poisson equations with critical growth*, Journal of Mathematical Physics 53 (2) (2012) 023702.

[9] O. Kavian, *Introduction à la théorie des points critiques et applications aux problèmes elliptiques*, Springer-Verlag, France, Paris, 1993.

[10] M. Khiddi, S.M. Sbai, *Infinitely many solutions for non-local elliptic non-degenerate p-Kirchhoff equations with critical exponent*, Complex Variables and Elliptic Equations (2019) pages 1–13.

[11] M. Khiddi, S.M. Sbai, *Infinitely many solutions for non-local elliptic non-degenerate p-Kirchhoff equations with critical exponent*, Complex Variables and Elliptic Equations 65 (3) (2020) 368–380.

[12] M. Khiddi, *Multiple solutions for the fractional Schrödinger–Poisson system with critical Sobolev exponent*, Rocky Mountain Journal of Mathematics 52 (2) (2022) 535–545.

[13] W. Krawcewicz, W. Marzantowicz, *Some remarks on the Lusternik-Schnirelman method for non-differentiable functionals invariant with respect to a finite group action*, The Rocky Mountain Journal of Mathematics 20 (4) (1990) 1041–1049.

[14] E.H. Lieb, B. Simon, *The Thomas-Fermi theory of atoms, molecules and solids*, Advances in Mathematics 23 (1) (1977) 22–116.

[15] Z. Liu, J. Zhang, *Multiplicity and concentration of positive solutions for the fractional Schrödinger-Poisson systems with critical growth*, ESAIM: Control, Optimisation and Calculus of Variations 23 (4) (2017) 1515–1542.

[16] E.G. Murcia, G. Siciliano, et al. *Positive semiclassical states for a fractional Schrödinger-Poisson system*, Differential and Integral Equations 30 (3/4) (2017) 231–258.

[17] G. Palatucci, A. Pisante, *Improved Sobolev embeddings, profile decomposition, and concentration-compactness for fractional Sobolev spaces*, Calculus of Variations and Partial Differential Equations 50 (3-4) (2014) 799–829.

[18] G. Palatucci, A. Pisante, Y. Sire, *Subcritical approximation of a Yamabe type non local equation: a gamma-convergence approach*, arXiv preprint arXiv:1306.6782, 2013.

[19] M. Struwe, *Variational Methods*, volume 31999, Springer, 1990.

[20] K. Teng, *Existence of ground state solutions for the nonlinear fractional Schrödinger–Poisson system with critical Sobolev exponent*, Journal of Differential Equations 261 (6) (2016) 3061–3106.

[21] J. Wang, L. Tian, J. Xu, F. Zhang, *Existence of multiple positive solutions for Schrödinger-Poisson systems with critical growth*, Zeitschrift für Angewandte Mathematik und Physik 66 (5) (2015) 2441–2471.

[22] M. Willem, *Minimax theorems: Progress in nonlinear differential equations and applications*, 24, Birkhäuser, Boston, Basel, Berlin, 1996.

DOI: 10.7862/rf.2022.4

Mustapha Khiddi
email: `mostapha-2@hotmail.com`
ORCID: 0000-0001-5749-5807
Département de Mathématique et Informatique
Laboratoire de Modélisation et Combinatoire
Université Cadi Ayyad
B.P. 4162 Safi
MOROCCO

Received 13.08.2020 *Accepted 18.10.2022*

Journal of
Mathematics
and Applications

JMA No 45, pp 107-128 (2022)

Karakostas Fixed Point Theorem and Semilinear Neutral Differential Equations with Impulses and Nonlocal Conditions

Hugo Leiva, Lenin Riera and Sebastián Lalvay

Abstract: This paper is concerned with the existence and uniqueness of solutions for a semilinear neutral differential equation with impulses and nonlocal conditions. First, we assume that the nonlinear terms are locally Lipschitz, and to achieve the existence of solutions, Karakostas Fixed Point Theorem is applied. After that, under some additional conditions, the uniqueness is proved as well. Next, assuming some bound on the nonlinear terms the global existence is proved by applying a generalization of Gronwall inequality for impulsive differential equations. Then, we suppose stronger hypotheses on the nonlinear functions, such as globally Lipschitz conditions, that allow us to appy Banach Fixed Point Theorem to prove the existence and uniqueness of solutions. Finally, we present an example as an application of our method.

AMS Subject Classification: 93B05, 93C10.
Keywords and Phrases: Semilinear neutral differential equations; Impulses; Delay; Nonlocal conditions; Karakostas fixed point theorem.

1. Introduction and Preliminaries

This work is devoted to study the existence of solutions for the following semilinear neutral differential equation with impulses and nonlocal conditions.

$$\begin{cases} \dfrac{d}{dt}[z(t) - f_{-1}(t, z_t)] = A_0(t)z(t) + f_1(t, z_t), \quad t \neq t_k, \quad t \in [0, \tau], \\ z(\theta) + h(z_{\tau_1}, z_{\tau_2}, \dots, z_{\tau_q})(\theta) = \eta(\theta), \quad \theta \in [-r, 0], \\ z(t_k^+) = z(t_k^-) + J_k(t_k, z(t_k)), \quad k = 1, 2, \dots, p, \end{cases} \tag{1.1}$$

where $A_0(t)$ is a $n \times n$ continuous matrix, the functions f_{-1}, f_1, and h are smooth enough and $0 < t_1 < t_2 < \cdots < t_p < \tau$, $0 < \tau_1 < \tau_2, \cdots < \tau_q < r < \tau$. Here, $z_t : [-r, 0] \longrightarrow \mathbb{R}^n$ is defined by $z_t(\theta) = z(t+\theta)$, and η belongs to the Banach space

$$\mathcal{PW}_r = \big\{\eta : [-r,0] \longrightarrow \mathbb{R}^n : \eta \text{ is continuous except at } s_{k\eta}, k = 1,2,\dots,p \text{ points} \\ \text{where the side limits exist } \eta(s_{k\eta}^+),\ \eta(s_{k\eta}^-) = \eta(s_{k\eta}), \quad \text{and are finite}\big\}$$

with the norm

$$\|\eta\|_r = \sup_{t\in[-r,0]} \|\eta(t)\|_{\mathbb{R}^n}.$$

There are many papers on the study of linear neutral differential equations, to mention [6, 12–14, 19, 20], particularly, the controllability of such equations has been studied in [12–14, 19, 20] where Kalman-type algebraic condition is proved (see [9]). In [6], the existence of solutions for an abstract neutral functional differential equations is discussed. To our knowledge, there are a few works on the existence of solutions for semilinear neutral equations with impulses and nonlocal conditions simultaneously. Karakostas Fixed Point Theorem will be applied to prove our main result on the existence of solutions of (1.1).

Theorem 1.1 (**Karakostas Fixed Point Theorem- see[7, 10, 11]**). *Let Z and Y be Banach spaces and D be a closed convex subset of Z, and let $\mathcal{B} : D \to Y$ be a continuous operator such that $\mathcal{B}(D)$ is a relatively compact subset of Y, and*

$$\mathcal{T} : D \times \overline{\mathcal{B}(D)} \to D$$

a continuous operator such that the family $\{\mathcal{T}(\cdot, y) : y \in \overline{\mathcal{B}(D)}\}$ is equicontractive. Then, the operator equation

$$\mathcal{T}(z, \mathcal{B}(z)) = z$$

admits a solution on D.

Now, we define natural Banach spaces where the solutions of problem (1.1) will take place and present some notations to be used through this work. We begin defining the Banach spaces

$$\mathcal{PW}_{t_1..t_p}([0,\tau];\mathbb{R}^n) = \{z : [0,\tau] \to \mathbb{R}^n : z \text{ is continuous except at } t_k, k = 1,\dots,p \\ \text{points where the side limits exist } z(t_k^+),\ z(t_k) = z(t_k^-), \\ \text{and are finite}\},$$

and

$$\mathcal{PW}_p = \big\{\eta : [-r,\tau] \longrightarrow \mathbb{R}^n : \eta\Big|_{[-r,0]} \in \mathcal{PW}_r \text{ and } \eta\Big|_{[0,\tau]} \in \mathcal{PW}_{t_1..t_p}\big\},$$

equipped with the supremum norm and

$$\|\eta\|_p = \sup_{t\in[-r,\tau]} \|\eta(t)\|_{\mathbb{R}^n},$$

respectively. We will also consider

$$\mathbb{R}^{qn} = \underbrace{\mathbb{R}^n \times \mathbb{R}^n \times \cdots \times \mathbb{R}^n}_{q-\text{times}} = \prod_{k=1}^{q} \mathbb{R}^n$$

equipped with the norm

$$\|y\|_q = \sum_{i=1}^{q} \|y_i\|_{\mathbb{R}^n}.$$

Analogously, we define the Banach space

$$\begin{aligned}\mathcal{PW}_{qp} = \big\{\eta : [-r,0] \longrightarrow \mathbb{R}^{qn} : \eta \text{ is continuous except at} s_{k\eta}, k = 1,2,\dots,p, \text{ points}\\ \text{where the side limits exist } \eta(s^+_{k\eta}),\ \eta(s^-_{k\eta}) = \eta(s_{k\eta}), \quad \text{and are finite}\big\}\end{aligned}$$

endowed with the norm

$$\|\eta\|_{qp} = \sup_{t\in[-r,0]} \|\eta(t)\|_q = \sup_{t\in[-r,0]} \left(\sum_{i=1}^{q} \|\eta_i(t)\|_{\mathbb{R}^n} \right).$$

The functions in system (1.1) are defined as follows:

$$f_{-1}, f_1 : [0,\tau] \times \mathcal{PW}_r \longrightarrow \mathbb{R}^n, \quad h : \mathcal{PW}_{qp} \longrightarrow \mathcal{PW}_r, \quad J_k : [0,\tau] \times \mathbb{R}^n \longrightarrow \mathbb{R}^n.$$

To conclude this section, we define the evolution operator $\mathcal{U}(t,\theta) = \Phi(t)\Phi^{-1}(\theta)$ where Φ is the fundamental matrix of the linear system of ordinary differential equations

$$y'(t) = A_0(t)y(t).$$

Also, we shall consider the following bound

$$M = \sup_{t,\theta\in[0,\tau]} \|\mathcal{U}(t,\theta)\|.$$

Remark 1.1. We will omit the subscript in the functions space norms defined above as long as this does not lead to confusion.

2. Formula for the solutions of system (1.1).

We devote this section to find a formula for solutions of the semilinear neutral differential equations with impulses and nonlocal conditions (1.1). Specifically, we transform problem (1.1) into an integral differential equation problem, which allows us to apply Karakostas Fixed Point Theorem to prove the existence of solutions for (1.1) in the next section.

Proposition 2.1. *The system* (1.1) *has solution* z *on* $[-r,\tau]$ *if, and only if,* z *is a solution of the following integral equation*

$$z(t)=\begin{cases}\mathcal{U}(t,0)\left[\eta(0)-h(z_{\tau_1},z_{\tau_2},\dots,z_{\tau_q})(0)-f_{-1}(0,\eta-h(z_{\tau_1},z_{\tau_2},\dots,z_{\tau_q}))\right]\\ +\displaystyle\int_0^t\mathcal{U}(t,\theta)\left[A_0(\theta)f_{-1}(\theta,z_\theta)+f_1(\theta,z_\theta)\right]d\theta+f_{-1}(t,z_t)\\ +\displaystyle\sum_{0<t_k<t}\mathcal{U}(t,t_k)J_k(t_k,z(t_k)),\quad t\in[0,\tau],\\ \eta(t)-h(z_{\tau_1},z_{\tau_2},\dots,z_{\tau_q})(t),\quad t\in[-r,0].\end{cases}\tag{2.1}$$

Proof. ($\Longrightarrow$) Suppose that z is a solution for system (1.1) on $[-r,\tau]$. Let

$$z_0=\eta(0)-h(z_{\tau_1},z_{\tau_2},\dots,z_{\tau_q})(0).$$

- On $[0,t_1)$, z is the solution of the following system

$$\begin{cases}\dfrac{d}{dt}[z(t)-f_{-1}(t,z_t)]=A_0(t)z(t)+f_1(t,z_t),\quad t\in[0,t_1),\\ z(t)+h(z_{\tau_1},\cdots,z_{\tau_q})(t)=\eta(t),\quad t\in[-r,0],\end{cases}$$

and by the variation of parameters formula

$$\begin{aligned}z(t)=&f_{-1}(t,z_t)+\mathcal{U}(t,0)[z_0-f_{-1}\left(0,\eta-h(z_{\tau_1},z_{\tau_2},\dots,z_{\tau_q})\right)]\\ &+\int_0^t\mathcal{U}(t,\theta)[A_0(\theta)f_{-1}(\theta,z_\theta)+f_1(\theta,z_\theta)]d\theta,\quad t\in[0,t_1).\end{aligned}$$

As $t\to t_1^-$,

$$\begin{aligned}z(t_1^-)=&f_{-1}(t_1,z_{t_1})+\mathcal{U}(t_1,0)[z_0-f_{-1}\left(0,\eta-h(z_{\tau_1},z_{\tau_2},\dots,z_{\tau_q})\right)]\\ &+\int_0^{t_1}\mathcal{U}(t_1,\theta)[A_0(\theta)f_{-1}(\theta,z_\theta)+f_1(\theta,z_\theta)]d\theta.\end{aligned}$$

- On $[t_1,t_2)$, z is the solution of the following system

$$\begin{cases}\dfrac{d}{dt}[z(t)-f_{-1}(t,z_t)]=A_0(t)z(t)+f_1(t,z_t),\quad t\in[t_1,t_2),\\ z(t_1^+)=z(t_1)+J_1(t_1,z(t_1))\end{cases}$$

and again the variation constant formula yields

$$\begin{aligned}z(t)=&f_{-1}(t,z_t)+\mathcal{U}(t,t_1)[z(t_1)+J_1(t_1,z(t_1))-f_{-1}\left(t_1,\eta-h(z_{\tau_1},z_{\tau_2},\dots,z_{\tau_q})\right)]\\ &+\int_{t_1}^t\mathcal{U}(t,\theta)[A_0(\theta)f_{-1}(\theta,z_\theta)+f_1(\theta,z_\theta)]d\theta,\quad t\in[t_1,t_2),\end{aligned}$$

therefore

$$
\begin{aligned}
z(t) =& f_{-1}(t, z_t) + \mathcal{U}(t, t_1)\big\{ f_{-1}(t_1, z_{t_1}) + \mathcal{U}(t_1, 0)[z_0 - f_{-1}\left(0, \eta - h(z_{\tau_1}, z_{\tau_2}, \ldots, z_{\tau_q})\right)] \\
&+ \int_0^{t_1} \mathcal{U}(t_1, \theta)[A_0(\theta) f_{-1}(\theta, z_\theta) + f_1(\theta, z_\theta)] d\theta + J_1(t_1, z(t_1)) \\
&- f_{-1}\left(t_1, \eta - h(z_{\tau_1}, z_{\tau_2}, \ldots, z_{\tau_q})\right) \big\} + \int_{t_1}^{t} \mathcal{U}(t, \theta)[A_0(\theta) f_{-1}(\theta, z_\theta) + f_1(\theta, z_\theta)] d\theta. \\
=& f_{-1}(t, z_t) + \mathcal{U}(t, t_1)\big\{ \mathcal{U}(t_1, 0)[z_0 - f_{-1}\left(0, \eta - h(z_{\tau_1}, z_{\tau_2}, \ldots, z_{\tau_q})\right)] \\
&+ \int_0^{t} \mathcal{U}(t_1, \theta)[A_0(\theta) f_{-1}(\theta, z_\theta) + f_1(\theta, z_\theta)] d\theta + J_1(t_1, z(t_1)) \big\} \\
&+ \int_{t_1}^{t} \mathcal{U}(t, \theta)[A_0(\theta) f_{-1}(\theta, z_\theta) + f_1(\theta, z_\theta)] d\theta.
\end{aligned}
$$

Using the cocycle property of $\mathcal{U}$,

$$
\begin{aligned}
z(t) =& f_{-1}(t, z_t) + \mathcal{U}(t, 0)[z_0 - f_{-1}\left(0, \eta - h(z_{\tau_1}, z_{\tau_2}, \ldots, z_{\tau_q})\right)] \\
&+ \int_0^{t_1} \mathcal{U}(t, \theta)[A_0(\theta) f_{-1}(\theta, z_\theta) + f_1(\theta, z_\theta)] d\theta + \mathcal{U}(t, t_1) J_1(t_1, z(t_1)) \\
&+ \int_{t_1}^{t} \mathcal{U}(t, \theta)[A_0(\theta) f_{-1}(\theta, z_\theta) + f_1(\theta, z_\theta)] d\theta \\
=& f_{-1}(t, z_t) + \mathcal{U}(t, 0)[z_0 - f_{-1}\left(0, \eta - h(z_{\tau_1}, z_{\tau_2}, \ldots, z_{\tau_q})\right)] \\
&+ \int_0^{t} \mathcal{U}(t, \theta)[A_0(\theta) f_{-1}(\theta, z_\theta) + f_1(\theta, z_\theta)] d\theta + \mathcal{U}(t, t_1) J_1(t_1, z(t_1)).
\end{aligned}
$$

Proceeding inductively as above, we have that for $t \in [t_p, t_{p+1})$,

$$
\begin{aligned}
& z(t) = f_{-1}(t, z_t) + \mathcal{U}(t, 0)[z_0 - f_{-1}\left(0, \eta - h(z_{\tau_1}, z_{\tau_2}, \ldots, z_{\tau_q})\right)] \\
&+ \int_0^{t} \mathcal{U}(t, \theta)[A_0(\theta) f_{-1}(\theta, z_\theta) + f_1(\theta, z_\theta)] d\theta + \sum_{k=1}^{p} \mathcal{U}(t, t_k) J_k(t_k, z(t_k)), \quad t \in [0, \tau] \\
&= f_{-1}(t, z_t) + \mathcal{U}(t, 0)[\eta(0) - h(z_{\tau_1}, z_{\tau_2}, \ldots, z_{\tau_q})(0) - f_{-1}\left(0, \eta - h(z_{\tau_1}, z_{\tau_2}, \ldots, z_{\tau_q})\right)] \\
&+ \int_0^{t} \mathcal{U}(t, \theta)[A_0(\theta) f_{-1}(\theta, z_\theta) + f_1(\theta, z_\theta)] d\theta + \sum_{0 < t_k < t} \mathcal{U}(t, t_k) J_k(t_k, z(t_k)), \quad t \in [0, \tau]
\end{aligned}
$$

($\Longleftarrow$) Assume that z is solution of the integral equation (2.1).

Then, at t_1,

$$
\begin{aligned}
z(t_1^-) =& f_{-1}(t_1, z_{t_1}) + \mathcal{U}(t_1,0)[z_0 - f_{-1}\left(0, \eta - h(z_{\tau_1}, z_{\tau_2}, \ldots, z_{\tau_q})\right)] \\
&+ \int_0^{t_1} \mathcal{U}(t_1,\theta)[A_0(\theta) f_{-1}(\theta, z_\theta) + f_1(\theta, z_\theta)]d\theta,
\end{aligned}
$$

$$
\begin{aligned}
z(t_1^+) =& f_{-1}(t_1, z_{t_1}) + \mathcal{U}(t_1,0)[z_0 - f_{-1}\left(0, \eta - h(z_{\tau_1}, z_{\tau_2}, \ldots, z_{\tau_q})\right)] \\
&+ \int_0^{t_1} \mathcal{U}(t_1,\theta)[A_0(\theta) f_{-1}(\theta, z_\theta) + f_1(\theta, z_\theta)]d\theta + \mathcal{U}(t_1,t_1)J_1(t_1, z(t_1)),
\end{aligned}
$$

which implies that

$$z(t_1^+) = z(t_1^-) + J_1(t_1, z(t_1)).$$

Near t_2,

$$
\begin{aligned}
z(t_2^-) =& f_{-1}(t_2, z_{t_2}) + \mathcal{U}(t_2,0)[z_0 - f_{-1}\left(0, \eta - h(z_{\tau_1}, z_{\tau_2}, \ldots, z_{\tau_q})\right)] \\
&+ \int_0^{t_2} \mathcal{U}(t_2,\theta)[A_0(\theta) f_{-1}(\theta, z_\theta) + f_1(\theta, z_\theta)]d\theta + \mathcal{U}(t_2,t_1)J_1(t_1, z(t_1)),
\end{aligned}
$$

$$
\begin{aligned}
z(t_2^+) =& f_{-1}(t_2, z_{t_2}) + \mathcal{U}(t_2,0)[z_0 - f_{-1}\left(0, \eta - h(z_{\tau_1}, z_{\tau_2}, \ldots, z_{\tau_q})\right)] \\
&+ \int_0^{t_2} \mathcal{U}(t_2,\theta)[A_0(\theta) f_{-1}(\theta, z_\theta) + f_1(\theta, z_\theta)]d\theta + \mathcal{U}(t_2,t_1)J_1(t_1, z(t_1)) \\
&+ \mathcal{U}(t_2,t_2)J_2(t_2, z(t_2)),
\end{aligned}
$$

which means that

$$z(t_2^+) = z(t_2^-) + J_2(t_2, z(t_2)).$$

Proceeding inductively as above, we get that for $k = 1, 2, \ldots, p$,

$$z(t_k^+) = z(t_k^-) + J_k(t_k, z(t_k)).$$

On the other hand, differentiating z with respect to t, for $t \in [0, \tau)$ and $t \neq t_k, k = 1, 2, \ldots, p$, we obtain that

$$
\begin{aligned}
\frac{d}{dt}(z(t)) =& \frac{d}{dt}\Bigg(f_{-1}(t, z_t) + \mathcal{U}(t,0)\left[z_0 - f_{-1}(0, \eta - h(z_{\tau_1}, z_{\tau_2}, \ldots, z_{\tau_q}))\right] \\
&+ \int_0^t \mathcal{U}(t,\theta)\left[A_0(\theta) f_{-1}(\theta, z_\theta) + f_1(\theta, z_\theta)\right]d\theta + \sum_{0<t_k<t} \mathcal{U}(t,t_k)J_k(t_k, z(t_k))\Bigg),
\end{aligned}
$$

$$
\begin{aligned}
\frac{d}{dt}(z(t)) =& \frac{d}{dt} f_{-1}(t, z_t) + A_0(t)\mathcal{U}(t,0)[z_0 - f_{-1}\left(0, \eta - h(z_{\tau_1}, z_{\tau_2}, \ldots, z_{\tau_q})\right)] \\
&+ A_0(t)\int_0^t \mathcal{U}(t,\theta)[A_0(\theta) f_{-1}(\theta, z_\theta) + f_1(\theta, z_\theta)]d\theta + A_0(t) f_{-1}(t, z_t) + f_1(t, z_t) \\
&+ A_0(t)\sum_{0<t_k<t} \mathcal{U}(t,t_k)J_k(t_k, z(t_k)).
\end{aligned}
$$

By rearranging terms it follows that

$$\begin{aligned}\frac{d}{dt}\left[z(t)-f_{-1}(t,z_t)\right]=&A_0(t)\Big\{f_{-1}(t,z_t)+\mathcal{U}(t,0)\left[z_0-f_{-1}(0,\eta-h(z_{\tau_1},z_{\tau_2},\dots,z_{\tau_q}))\right]\\&+\int_0^t\mathcal{U}(t,\theta)\left[A_0(\theta)f_{-1}(\theta,z_\theta)+f_1(\theta,z_\theta)\right]d\theta\\&+\sum_{0<t_k<t}\mathcal{U}(t,t_k)J_k(t_k,z(t_k))\Big\}+f_1(t,z_t)\\=&A_0(t)z(t)+f_1(t,z_t),\end{aligned}$$

that is to say, z is a solution of (1.1). □

3. Main Theorems

In this section we shall prove our main result about the existence of solutions for the semilinear neutral equation with impulses and nonlocal conditions (1.1) and their behavior.To achieve that, we consider the following hypotheses on the terms involving the system (1.1).

(H1) There exist constants $d_k, L_g, \gamma > 0, k = 1, 2, \dots, p$ such that $\forall y, z \in \mathbb{R}^n$, $t \in [0,\tau]$

i. $L_g qM < \gamma + M\sum_{k=1}^{p} d_k < \frac{1}{2}, \quad \|J_k(t,y)-J_k(t,z)\|_{\mathbb{R}^n} \leq d_k\|y-z\|_{\mathbb{R}^n}.$

ii. We have that $h(0)\equiv 0$ and

$$\|h(y)(t)-h(v)(t)\|_{\mathbb{R}^n}\leq L_g\sum_{i=1}^{q}\|y_i(t)-v_i(t)\|_{\mathbb{R}^n},\quad y,v\in\mathcal{PW}_{qp}.$$

(H2) The function f_{-1} satisfies

i.

$$\begin{aligned}&\|A_0(t)f_{-1}(t,\eta_1)-A_0(t)f_{-1}(t,\eta_2)\|_{\mathbb{R}^n}\leq\mathcal{K}\left(\|\eta_1\|_r,\|\eta_2\|_r\right)\|\eta_1-\eta_2\|_r,\ \eta_1,\eta_2\in\mathcal{PW}_r,\\&\|f_{-1}(t,\eta_1)-f_{-1}(t,\eta_2)\|_{\mathbb{R}^n}\leq\gamma\|\eta_1-\eta_2\|_r,\quad\eta_1,\eta_2\in\mathcal{PW}_r\\&\|A_0(t)f_{-1}(t,\eta)\|_{\mathbb{R}^n}\leq\Psi\left(\|\eta\|_r\right),\quad\eta\in\mathcal{PW}_r,\\&\|f_{-1}(t,\eta)\|_{\mathbb{R}^n}\leq\Psi\left(\|\eta\|_r\right),\quad\eta\in\mathcal{PW}_r.\end{aligned}$$

and f_1 satisfies

ii.

$$\begin{aligned}&\|f_1(t,\eta_1)-f_1(t,\eta_2)\|_{\mathbb{R}^n}\leq\mathcal{K}\left(\|\eta_1\|_r,\|\eta_2\|_r\right)\|\eta_1-\eta_2\|_r,\quad\eta_1,\eta_2\in\mathcal{PW}_r,\\&\|f_1(t,\eta)\|_{\mathbb{R}^n}\leq\Psi\left(\|\eta\|_r\right),\quad\eta\in\mathcal{PW}_r,\end{aligned}$$

where $\mathcal{K} : \mathbb{R}_+ \times \mathbb{R}_+ \longrightarrow \mathbb{R}_+, \Psi : \mathbb{R}_+ \longrightarrow \mathbb{R}_+$ are continuous and non decreasing functions.

(H3) There exists $\rho, \tau > 0$ such that

$$M\Psi\left(\|\eta\| + L_g q\left(\|\tilde{\eta}\| + \rho\right)\right) + \left(ML_g q + M\sum_{k=1}^{p} d_k\right)\left(\|\tilde{\eta}\| + \rho\right)$$
$$+ (2M\tau + 1)\Psi\left(\|\tilde{\eta}\| + \rho\right) < \rho$$

where the function $\tilde{\eta}$ is defined as follows

$$\tilde{\eta}(t) = \begin{cases} \mathcal{U}(t,0)\eta(0), & t \in [0,\tau], \\ \eta(t), & t \in [-r,0]. \end{cases}$$

(H4) Assume the following relation holds

$$M\left\{L_g q\,(1+\gamma) + 2\tau\mathcal{K}\left(\|\tilde{\eta}\| + \rho, \|\tilde{\eta}\| + \rho\right)\right\} < \frac{1}{2}.$$

Remark 3.1. The hypothesis **(H2)** is not a whim, it appears naturally when one studies the well-known Burgues equation and the Benjamin-Bona-Mahony equation; and since we will extend this work to infinite-dimensional Hilbert spaces, these hypotheses are considered here. For more details about it, one can see [10, 11].

Theorem 3.1. *Suppose that* **(H1)-(H3)** *hold. Then, the system* (1.1) *has at least one solution on* $[-r,\tau]$.

Proof. We shall transform the problem of proving the existence of solutions for system (1.1) into a fixed point problem. For this, we define the following operators

$$\mathcal{T} : \mathcal{PW}_p \times \mathcal{PW}_p \longrightarrow \mathcal{PW}_p,$$

and

$$\mathcal{B} : \mathcal{PW}_p \longrightarrow \mathcal{PW}_p$$

given by

$$\mathcal{T}(z,y)(t) = \begin{cases} y(t) + f_{-1}(t,z_t) + \displaystyle\sum_{0<t_k<t} \mathcal{U}(t,t_k)J_k(t_k, z(t_k)), & t \in [0,\tau], \\ \eta(t) - h(z_{\tau_1}, z_{\tau_2}, \ldots, z_{\tau_q})(t), & t \in [-r,0], \end{cases}$$

and

$$\mathcal{B}(y)(t) = \begin{cases} \mathcal{U}(t,0)\left[\eta(0) - h(y_{\tau_1}, y_{\tau_2}, \ldots, y_{\tau_q})(0) - f_{-1}(0, \eta - h(y_{\tau_1}, y_{\tau_2}, \ldots, y_{\tau_q}))\right] \\ + \displaystyle\int_0^t \mathcal{U}(t,\theta)\left[A_0(\theta)f_{-1}(\theta, y_\theta) + f_1(\theta, y_\theta)\right] d\theta, \quad t \in [0,\tau], \\ \eta(t), \quad t \in [-r,0], \end{cases}$$

respectively. We also consider the following closed and convex set

$$D = D(\rho, \tau, \eta) = \{y \in \mathcal{PW}_p : \|y - \tilde{\eta}\|_p \leq \rho\}.$$

With this setting, the problem of finding solutions for system (1.1) has been reduced to the problem of finding solutions of the following operator equation

$$\mathcal{T}(z, \mathcal{B}(z)) = z.$$

The rest of the proof will be given by statements as follows:

Statement 1. *$\mathcal{B}$ is a continuous mapping.*

For any $z, y \in \mathcal{PW}_p$ we have that

$$\begin{aligned}
\|\mathcal{B}(z)(t) - \mathcal{B}(y)(t)\| \; &\leq \|\mathcal{U}(t,0)\| \Big\{ \big\| h(y_{\tau_1}, y_{\tau_2}, \ldots, y_{\tau_q})(0) - h(z_{\tau_1}, z_{\tau_2}, \ldots, z_{\tau_q})(0) \big\| \\
&\quad + \big\| f_{-1}\big(0, \eta - h(y_{\tau_1}, y_{\tau_2}, ..., y_{\tau_q})\big) - f_{-1}\big(0, \eta - h(z_{\tau_1}, z_{\tau_2}, ..., z_{\tau_q})\big) \big\| \Big\} \\
&\quad + \int_0^t \|\mathcal{U}(t,\theta)\| \Big\{ \|A_0(\theta) f_{-1}(\theta, z_\theta) - A_0(\theta) f_{-1}(\theta, y_\theta)\| \\
&\quad + \|f_1(\theta, z_\theta) - f_1(\theta, y_\theta)\| \Big\} d\theta \\
&\leq M \left[L_g q \|z - y\| + \gamma \big\| g(z_{\tau_1}, z_{\tau_2}, \ldots, z_{\tau_q}) - h(y_{\tau_1}, y_{\tau_2}, \ldots, y_{\tau_q}) \big\| \right] \\
&\quad + M\tau \left[\mathcal{K}(\|z\|, \|y\|) \|z - y\| + \mathcal{K}(\|z\|, \|y\|) \|z - y\| \right], \\
&\leq M \left[L_g q \|z - y\| + \gamma L_g q \|z - y\| \right] \\
&\quad + 2M\tau \mathcal{K}(\|z\|, \|y\|) \|z - y\|,
\end{aligned}$$

where the last two inequality comes from **(H1-ii)** and **(H2)**. It follows that

$$\|\mathcal{B}(z) - \mathcal{B}(y)\| \; \leq M \left\{ L_g q \,(1 + \gamma) + 2\tau \mathcal{K}\left(\|z\|, \|y\|\right) \right\} \|z - y\|$$

by taking supremum over $t \in [-r, \tau]$. Hence $\mathcal{B}$ is locally Lipschitz, which implies the continuity of $\mathcal{B}$.

Statement 2. *$\mathcal{B}$ maps bounded sets of $\mathcal{PW}_p$ into bounded sets of $\mathcal{PW}_p$.*

In order to prove this statement, we will show that

$$\forall R > 0 \; \exists \lambda > 0 \; \forall y \in B_R : \|\mathcal{B}(y)\| \leq \lambda,$$

where $B_R = \{z \in \mathcal{PW}_p : \|z\| \leq R\}$. Let $R > 0$ and consider $\lambda = \max\{\vartheta, \|\eta\|\}$, ϑ to be determined later. Let $y \in B_R$. Then, on one hand, we have that

$$\|\mathcal{B}(y)(t)\| = \|\eta(t)\| \leq \|\eta\|,$$

if $t \in [-r, 0]$. While, on the other hand,

$$
\begin{aligned}
\|\mathcal{B}(y)(t)\| &\leq \|\mathcal{U}(t,0)\| \big\|\eta(0) - h(y_{\tau_1}, y_{\tau_2}, \ldots, y_{\tau_q})(0) - f_{-1}(0, \eta - h(y_{\tau_1}, y_{\tau_2}, \ldots, y_{\tau_q}))\big\| \\
&\quad + \int_0^t \|\mathcal{U}(t,\theta)\| \left[\|A_0(\theta) f_{-1}(\theta, y_\theta)\| + \|f_1(\theta, y_\theta)\|\right] d\theta \\
&\leq M\big\{\|\eta(0)\| + \big\|h(y_{\tau_1}, y_{\tau_2}, \ldots, y_{\tau_q})(0)\big\| + \big\|f_{-1}(0, \eta - h(y_{\tau_1}, y_{\tau_2}, \ldots, y_{\tau_q}))\big\|\big\} \\
&\quad + \tau M \left[\|A_0(\theta) f_{-1}(\theta, y_\theta)\| + \|f_1(\theta, y_\theta)\|\right] \\
&\leq M\big\{\|\eta(0)\| + L_g q\|y\| + \Psi\left(\big\|\eta - h(y_{\tau_1}, y_{\tau_2}, \ldots, y_{\tau_q})\big\|\right)\big\} + \tau M 2\Psi(\|y\|) \\
&\leq M\big\{\|\eta(0)\| + L_g q\|y\| + \Psi\left(\|\eta\| + \big\|h(y_{\tau_1}, y_{\tau_2}, \ldots, y_{\tau_q})\big\|\right)\big\} + \tau M 2\Psi(\|y\|) \\
&\leq M\big\{\|\eta(0)\| + L_g q\|y\| + \Psi\left(\|\eta\| + L_g q\|y\|\right)\big\} + \tau M 2\Psi(\|y\|) \\
&\leq M\big\{\|\eta(0)\| + L_g q R + \Psi\left(\|\eta\| + L_g q R\right) + \tau 2\Psi(R)\big\} = \vartheta,
\end{aligned}
$$

if $t \in [0, \tau]$. Here we have used **(H1-ii)** and **(H2)**. Now, taking supremum over $t \in [-r, \tau]$, we have that

$$\|\mathcal{B}(y)\| \leq \lambda.$$

Statement 3. *$\mathcal{B}$ maps bounded sets of $\mathcal{PW}_p$ into equicontinuous sets of $\mathcal{PW}_p$.*

Let us consider B_R as above and let us show that $\mathcal{B}(B_R)$ is equicontinuous on $[-r, \tau]$. On $[-r, 0]$, the continuity of η immediately implies the result. On $(0, \tau]$, we have that

$$
\begin{aligned}
\|\mathcal{B}(y)(t_2) - \mathcal{B}(y)(t_1)\| &\leq \|\mathcal{U}(t_2,0) - \mathcal{U}(t_1,0)\| \|\eta(0) - h(y_{\tau_1}, y_{\tau_2}, \ldots, y_{\tau_q})(0) \\
&\quad - f_{-1}\left(0, \eta - h(y_{\tau_1}, y_{\tau_2}, \ldots, y_{\tau_q})\right)\| \\
&\quad + \int_0^{t_1} \|\mathcal{U}(t_2,\theta) - \mathcal{U}(t_1,\theta)\| \|A_0(\theta) f_{-1}(\theta, y_\theta) + f_1(\theta, y_\theta)\| d\theta \\
&\quad + \int_{t_1}^{t_2} \|\mathcal{U}(t_2,\theta)\| \|A_0(\theta) f_{-1}(\theta, y_\theta) + f_1(\theta, y_\theta)\| d\theta \\
&\leq \|\mathcal{U}(t_2,0) - \mathcal{U}(t_1,0)\| \Big\{ \|\eta(0)\| + L_g q\|y\| \\
&\quad + \big\|f_{-1}\left(0, \eta - h(y_{\tau_1}, y_{\tau_2}, \ldots, y_{\tau_q})\right)\big\| \Big\} \\
&\quad + \int_0^{t_1} \|\mathcal{U}(t_2,\theta) - \mathcal{U}(t_1,\theta)\| \left[\|A_0(\theta) f_{-1}(\theta, y_\theta)\| + \|f_1(\theta, y_\theta)\|\right] d\theta \\
&\quad + \int_{t_1}^{t_2} \|\mathcal{U}(t_2,\theta)\| \left[\|A_0(\theta) f_{-1}(\theta, y_\theta)\| + \|f_1(\theta, y_\theta)\|\right] d\theta \\
&\leq \|\mathcal{U}(t_2,0) - \mathcal{U}(t_1,0)\| \Big\{ \|\eta(0)\| + L_g q\|y\| + \Psi\left(\|\eta\| + L_g q\|y\|\right) \Big\} \\
&\quad + 2\Psi\left(\|y\|\right) \int_0^{t_1} \|\mathcal{U}(t_2,\theta) - \mathcal{U}(t_1,\theta)\| d\theta + 2M\Psi\left(\|y\|\right)(t_2 - t_1)
\end{aligned}
$$

$$\leq \|\mathcal{U}(t_2,0)-\mathcal{U}(t_1,0)\|\Big\{\|\eta(0)\|+L_gqR+\Psi\left(\|\eta\|+L_gqR\right)\Big\}$$
$$+\,2\Psi\left(R\right)\int_0^{t_1}\|\mathcal{U}(t_2,\theta)-\mathcal{U}(t_1,\theta)\|d\theta+2M\Psi\left(R\right)(t_2-t_1)\to 0$$

as $t_2 \to t_1$ by the continuity of $\mathcal{U}$ and the fact that $\|\eta(0)\| + L_gqR + \Psi\left(\|\eta\| + L_gqR\right)$ is bounded. Here we have considered **(H1-ii)** and **(H2)**. This shows that $\mathcal{B}(B_R)$ is equicontinuous.

Statement 4. *The subset $\mathcal{B}(D)$ is relatively compact in $\mathcal{PW}_p$.*

Let us prove Statement 4. Let D be a bounded subset of $\mathcal{PW}_p$. By Statements 2 and 3, $\mathcal{B}(D)$ is bounded and equicontinuous in $\mathcal{PW}_p$. Let $\{y_n\}_{n\in\mathbb{N}} \subseteq \mathcal{B}(D)$, then

$$y_n\Big|_{[-r,0]} = \eta,\ \forall n \in \mathbb{N}.$$

Hence, $y_n\Big|_{[-r,0]}$ converges uniformly on $[-r,0]$.

Now, putting $\varphi_n = y_n\Big|_{[0,\tau]}$, we get that $\{\varphi_n\}_{n\in\mathbb{N}} \subseteq \mathcal{PW}_{t_1..t_p}$.
Let us put $t_0 = 0$ and $t_{p+1} = \tau$. Then, applying Arzela-Ascoli Theorem, the sequence $\{\varphi_n\}_{n\in\mathbb{N}}$ contains a subsequence $\{\varphi_n^1\}_{n\in\mathbb{N}}$ that converges in the interval $[t_0,t_1]$. Now, applying Arzela-Ascoli Theorem again, we get that the sequence $\{\varphi_n^1\}_{n\in\mathbb{N}}$ contains a subsequence $\{\varphi_n^2\}_{n\in\mathbb{N}}$ that converges in the interval $[t_1,t_2]$. Continuing with this process we find a subsequence $\{\varphi_n^{p+1}\}_{n\in\mathbb{N}}$ of $\{\varphi_n\}_{n\in\mathbb{N}}$ that converges in each interval $[t_k,t_{k+1}]$, with $k = 0,1,2,\ldots,p$. Therefore,

$$\varphi_n^{p+1} = y_n^{p+1}\Big|_{[0,\tau]} \text{ converges on } [0,\tau].$$

Consequently, $\{\varphi_n^{p+1}\}_{n\in\mathbb{N}} = \{y_n^{p+1}\}_{n\in\mathbb{N}}$ converges uniformly on $[-r,\tau]$. Thus, $\mathcal{B}(D)$ is relatively compact, and the proof of Statement 4 is completed.

Statement 5. *The family $\Big\{\mathcal{T}(\cdot,y) : y \in \overline{\mathcal{B}(D)}\Big\}$ is equicontractive.*

On the one hand, for any $u,v \in \mathcal{PW}_p$ and $t \in [-r,0]$, we get that

$$\begin{aligned}\|\mathcal{T}(u,\mathcal{B}(y))(t)-\mathcal{T}(v,\mathcal{B}(y))(t)\| &\leq \left\|h(u_{\tau_1},u_{\tau_2},\ldots,u_{\tau_q})(t)-h(v_{\tau_1},v_{\tau_2},\ldots,v_{\tau_q})(t)\right\|\\ &\leq L_gq\|u-v\|\\ &\leq ML_gq\|u-v\|.\end{aligned}$$

While on the other hand, by using **(H1-i)** and **(H2-i)**, for all $t \in (0,\tau]$ we obtain that

$$\begin{aligned}\|\mathcal{T}(u,\mathcal{B}(y))(t)-\mathcal{T}(v,\mathcal{B}(y))(t)\| &\leq \|f_{-1}(t,u_t)-f_{-1}(t,v_t)\| \\ &\quad + \sum_{0<t_k<t} \|\mathcal{U}(t,t_k)\left[J_k(t_k,u(t_k))-J_k(t_k,v(t_k))\right]\| \\ &\leq \gamma\|u-v\| + M\sum_{k=1}^{p}\|J_k(t_k,u(t_k))-J_k(t_k,v(t_k))\| \\ &\leq \gamma\|u-v\| + M\sum_{k=1}^{p} d_k\|u(t_k)-v(t_k)\| \\ &\leq \gamma\|u-v\| + M\|u-v\|\sum_{k=1}^{p} d_k \\ &\leq \left(\gamma + M\sum_{k=1}^{p} d_k\right)\|u-v\|.\end{aligned}$$

It follows that

$$\|\mathcal{T}(u,\mathcal{B}(y))-\mathcal{T}(v,\mathcal{B}(y))\| \leq \left(\gamma + M\sum_{k=1}^{p} d_k\right)\|u-v\| \leq \frac{1}{2}\|u-v\|$$

by taking supremum over $t\in[-r,\tau]$ and using **(H1-i)**. This shows that $\mathcal{T}(\cdot,\mathcal{B}(y))$ is a contraction which does not depend on $y\in\overline{\mathcal{B}(D)}$.

Statement 6. *The inclusion* $\mathcal{T}(\cdot,\mathcal{B}(\cdot))\,(D(\rho,\tau,\eta))\subset D(\rho,\tau,\eta)$ *holds.*

Let $z\in D(\rho,\tau,\eta)$ be arbitrary. Notice that

$$\mathcal{T}(z,\mathcal{B}(z))(t) = \begin{cases} \mathcal{U}(t,0)\big[\eta(0)-h(z_{\tau_1},z_{\tau_2},\ldots,z_{\tau_q})(0) - f_{-1}(0,\eta-h(z_{\tau_1},z_{\tau_2},\ldots,z_{\tau_q}))\big] \\ +\displaystyle\int_0^t \mathcal{U}(t,\theta)\left[A_0(\theta)f_{-1}(\theta,z_\theta)+f_1(\theta,z_\theta)\right]d\theta + f_{-1}(t,z_t) \\ +\displaystyle\sum_{0<t_k<t}\mathcal{U}(t,t_k)J_k(t_k,z(t_k)), & t\in[0,\tau], \\ \eta(t)-h(z_{\tau_1},z_{\tau_2},\ldots,z_{\tau_q})(t), & t\in[-r,0].\end{cases}$$

On the one hand, for $t\in[-r,0]$, we have that

$$\begin{aligned}\|\mathcal{T}(z,\mathcal{B}(z))(t)-\tilde{\eta}(t)\| &\leq \left\|g\left(z_{\tau_1},z_{\tau_2},\ldots,z_{\tau_q}\right)(t)\right\| \\ &\leq L_g q\|z\| \\ &\leq M L_g q\|z\| \\ &\leq M L_g q\left(\|\tilde{\eta}\|+\rho\right) \\ &< \rho.\end{aligned}$$

While on the other hand, for $t \in [0, \tau]$, we have that

$$
\begin{aligned}
\|\mathcal{T}(z, \mathcal{B}(z))(t) - \tilde{\eta}(t)\| \ &\leq M\left\|h(z_{\tau_1}, z_{\tau_2}, \ldots, z_{\tau_q})(0) - f_{-1}(0, \eta - h(z_{\tau_1}, z_{\tau_2}, \ldots, z_{\tau_q}))\right\| \\
&\quad + \int_0^t \|\mathcal{U}(t,\theta)\left[A_0(\theta) f_{-1}(\theta, z_\theta) + f_1(\theta, z_\theta)\right]\| d\theta + \|f_{-1}(t, z_t)\| \\
&\quad + \sum_{0<t_k<t} \|\mathcal{U}(t, t_k) J_k(t_k, z(t_k))\| \\
&\leq M\left\{L_g q\|z\| + \left\|f_{-1}(0, \eta - h(z_{\tau_1}, z_{\tau_2}, \ldots, z_{\tau_q}))\right\|\right\} \\
&\quad + 2M\tau\Psi(\|z\|) + \Psi(\|z\|) + M \sum_{0<t_k<t} \|J_k(t_k, z(t_k))\| \\
&\leq M\left\{L_g q\|z\| + \Psi\left(\|\eta\| + L_g q\|z\|\right)\right\} \\
&\quad + 2M\tau\Psi(\|z\|) + \Psi(\|z\|) + \left(M \sum_{k=1}^{p} d_k\right)\|z\| \\
&\leq M\left\{L_g q\left(\|\tilde{\eta}\| + \rho\right) + \Psi\left(\|\tilde{\eta}\| + L_g q\left(\|\tilde{\eta}\| + \rho\right)\right)\right\} \\
&\quad + 2M\tau\Psi\left(\|\tilde{\eta}\| + \rho\right) + \Psi\left(\|\tilde{\eta}\| + \rho\right) + \left(M \sum_{k=1}^{p} d_k\right)\left(\|\tilde{\eta}\| + \rho\right) \\
&\leq M\Psi\left(\|\tilde{\eta}\| + L_g q\left(\|\tilde{\eta}\| + \rho\right)\right) + \left(M L_g q + M \sum_{k=1}^{p} d_k\right)\left(\|\tilde{\eta}\| + \rho\right) \\
&\quad + (2M\tau + 1)\Psi\left(\|\tilde{\eta}\| + \rho\right) < \rho.
\end{aligned}
$$

Here we have used **(H3)**. Now, by taking supremum over $t \in [-r, \tau]$, we get that

$$\|\mathcal{T}(z, \mathcal{B}(z)) - \tilde{\eta}\| \leq \rho.$$

and by Karakostas Fixed Point Theorem the operator equation

$$\mathcal{T}(z, \mathcal{B}(z)) = z$$

admits a solution on D. This finishes the proof. □

Theorem 3.2. *System* (1.1) *has a unique solution if* ***(H4)*** *is additionally assumed.*

Proof. Suppose u and v are two solutions of system (1.1). Now, considering **(H1)** and **(H2)** we have that

$$
\begin{aligned}
\|u(t) - v(t)\| \leq \|\mathcal{U}(t,0)\| &\Big\{ \left\|h(u_{\tau_1}, u_{\tau_2}, \ldots, u_{\tau_q})(0) - h(v_{\tau_1}, v_{\tau_2}, \ldots, v_{\tau_q})(0)\right\| \\
&+ \left\|f_{-1}\left(0, \eta - h(u_{\tau_1}, u_{\tau_2}, \ldots, u_{\tau_q})\right) - f_{-1}\left(0, \eta - h(v_{\tau_1}, v_{\tau_2}, \ldots, v_{\tau_q})\right)\right\| \Big\} \\
&+ \int_0^t \|\mathcal{U}(t,\theta)\| \Big\{ \|A_0(\theta) f_{-1}(\theta, u_\theta) - A_0(\theta) f_{-1}(\theta, v_\theta)\| + \|f_1(\theta, u_\theta) - f_1(\theta, v_\theta)\| \Big\} d\theta
\end{aligned}
$$

$$+\|f_{-1}(t,u_t)-f_{-1}(t,v_t)\|+\sum_{0<t_k<t}\|\mathcal{U}(t,t_k)\|\|J_k(t_k,u(t_k))-J_k(t_k,v(t_k))\|$$

$$\leq M\left\{L_g q\,(1+\gamma)+2\tau\mathcal{K}\,(\|u\|,\|v\|)\right\}\|u-v\|+\left(\gamma+M\sum_{k=1}^{p}d_k\right)\|u-v\|$$

$$\leq M\left\{L_g q\,(1+\gamma)+2\tau\mathcal{K}\left(\|\tilde{\eta}\|+\rho,\|\tilde{\eta}\|+\rho\right)\right\}\|u-v\|+\frac{1}{2}\|u-v\|$$

Bearing in mind the hypothesis **(H4)**, and taking supremum over $t\in[-r,\tau]$, we have that

$$\|u-v\|\leq\omega\|u-v\|$$

with $0\leq\omega<1$. This implies $\|u-v\|=0$, and therefore $u=v$. □

Next, we consider the following subset $\tilde{D}$ of $\mathbb{R}^n$:

$$\tilde{D}=\{v\in\mathbb{R}^n:\|v\|_{\mathbb{R}^n}\leq\rho\}. \tag{3.1}$$

Therefore, for all $y\in D$ we have $y(t)-\tilde{\eta}(t)\in\tilde{D}$ for $t\in[-r,\tau]$.

Definition 3.1. We shall say that $[-r,\theta_1)$ is a maximal interval of existence for the solution z of problem (1.1) if there is not solution of (1.1) on $[-r,\theta_2)$ with $\theta_2>\theta_1$.

Theorem 3.3. *Suppose that the conditions of Theorem 3.1 hold. If z is a solution of problem (1.1) on $[-r,\theta_1)$ and θ_1 is maximal, then either $\theta_1=+\infty$ or there exists a sequence $\tau_n\to\theta_1$ as $n\to\infty$ such that $z(\tau_n)-\tilde{\eta}(\tau_n)\to\partial\tilde{D}$.*

Proof. Suppose $\theta_1<\infty$. For the purpose of contradiction assume the existence of a neighborhood N of $\partial\tilde{D}$ such that $\{z(t)-\tilde{\eta}(t)\}$ does not enter in it, for $0<\theta_2\leq t<\theta_1$. We can take $N=\tilde{D}\backslash B$, where B is a closed subset of $\tilde{D}$, then $z(t)-\tilde{\eta}(t)\in B$ for $0<t_p<\theta_2\leq t<\theta_1$. We need to prove that $\lim\limits_{t\to\theta_1^-}\{z(t)-\tilde{\eta}(t)\}=z_1-\tilde{\eta}(\theta_1)\in B$, which is enough to prove that $\lim\limits_{t\to\theta_1^-}z(t)=z_1$. Indeed, if we consider $0<t_p<\theta_2\leq\ell<t<\theta_1$, then:

$$\begin{aligned}\|z(t)-z(\ell)\|\leq&\|\mathcal{U}(t,0)-\mathcal{U}(\ell,0)\|\big(\|\eta(0)\|+\big\|h(z_{\tau_1},z_{\tau_2},\dots,z_{\tau_q})(0)\big\|\\&+\big\|f_{-1}(0,\eta-h(z_{\tau_1},z_{\tau_2},\dots,z_{\tau_q}))\big\|\big)\\&+\int_0^\ell\|\mathcal{U}(t,\theta)-\mathcal{U}(\ell,\theta)\|\|A(\theta)f_{-1}(\theta,z_\theta)+f_1(\theta,z_\theta)\|d\theta\\&+\int_\ell^t\|\mathcal{U}(t,\theta)\|\|A(\theta)f_{-1}(\theta,z_\theta)+f_1(\theta,z_\theta)\|d\theta+\|f_{-1}(t,z_t)-f_{-1}(\ell,z_\ell)\|\\&+\sum_{0<t_k<\ell}\|\mathcal{U}(t,t_k)-\mathcal{U}(\ell,t_k)\|\|J_k(t_k,z(t_k))\|\\&+\sum_{\ell<t_k<t}\|\mathcal{U}(t,t_k)\|\|J_k(t_k,z(t_k))\|\end{aligned}$$

$$
\begin{aligned}
&\leq \|\mathcal{U}(t,0)-\mathcal{U}(\ell,0)\|\big(\|\eta(0)\|+L_g q\|z\|+\Psi\left(\|\eta\|+L_g q\|z\|\right)\big)\\
&\quad+\Big(\int_0^\ell \|\mathcal{U}(t,\theta)-\mathcal{U}(\ell,\theta)\|d\theta+\int_\ell^t \|(\mathcal{U}(t,\theta)\|d\theta\Big)2\Psi(\|z\|)\\
&\quad+\|f_{-1}(t,z_t)-f_{-1}(\ell,z_\ell)\|+\|\mathcal{U}(t,\ell)-I\|\sum_{k=1}^{q}\|\mathcal{U}(\ell,t_k)\|\|J_k(z(t_k)\|\\
&\quad+\sum_{\ell<t_k<t}\|\mathcal{U}(t,t_k)\|\|J_k(t_k,z(t_k))\|\\
&\leq \|\mathcal{U}(t,0)-\mathcal{U}(\ell,0)\|\big(\|\eta(0)\|+L_g q\|z\|+\Psi\left(\|\eta\|+L_g q\|z\|\right)\big)\\
&\quad+\Big(\int_0^\ell \|\mathcal{U}(t,\theta)-\mathcal{U}(\ell,\theta)\|d\theta+\int_\ell^t \|(\mathcal{U}(t,\theta)\|d\theta\Big)2\Psi(\|z\|)\\
&\quad+\|f_{-1}(t,z_t)-f_{-1}(\ell,z_\ell)\|+\|\mathcal{U}(t,\ell)-I\|M\sum_{k=1}^{q}\|J_k(z(t_k)\|\\
&\quad+\sum_{\ell<t_k<t}\|\mathcal{U}(t,t_k)\|\|J_k(t_k,z(t_k))\|
\end{aligned}
$$

Since $\mathcal{U}$ is uniformly continuous for $t\geq 0$, then $\|z(t)-z(l)\|_{\mathbb{R}^n}$ goes to zero as $l\to\theta_1^-$. Therefore, $\lim_{t\to\theta_1^-} z(t)=z_1$ exists in $\mathbb{R}^n$ and, since B is closed, $z_1-\tilde{\eta}(\theta_1)$ belongs to B. This will contradict the maximality of θ_1. In fact, we have that $z_1\in B+\tilde{\eta}(\theta_1)$ is contained in the interior of the ball $\tilde{D}+\tilde{\eta}(\theta_1)$. Hence, $z(\cdot)$ can be extended to $[-r,\theta_1]$. In this regard, for ϵ small enough, the following initial value problem admit only one solutions on $[-r,\theta_1+\epsilon)$

$$
\begin{cases}
\dfrac{d}{dt}[u(t)-f_{-1}(t,u_t)]=A_0(t)u(t)+f_1(t,u_t), & t\in[\theta_1,\theta_1+\epsilon),\\
u(\theta)=z(\theta), \quad \theta\in[\theta_1-r,\theta_1].
\end{cases}
\tag{3.2}
$$

This is a contradiction with the maximality of θ_1. So, the proof is completed. □

Corollary 3.1. *In the conditions of Theorem 3.1, if the second part of hypothesis* **(*H*1)** *is changed to*

$$
\|f_1(t,\eta)\|\leq\mu(t)(1+\|\eta(0)\|_{\mathbb{R}^n}),\quad \eta\in\mathcal{PW}_r,\quad t\in[-r,\infty),
$$

where μ is a continuous function on $[-r,\infty)$, then a unique solution of problem (1.1) exists on $[-r,\infty)$.

Proof.

$$
\begin{aligned}
\|z(t)\|\leq&\|\mathcal{U}(t,0)\|\|\eta(0)-h(z_{\tau_1},z_{\tau_2},\ldots,z_{\tau_q})(0)-f_{-1}(0,\eta-h(z_{\tau_1},z_{\tau_2},\ldots,z_{\tau_q}))\|\\
&+M\int_0^t\|A_0(\theta)f_{-1}(\theta,z_\theta)\|+\|f_1(\theta,z_\theta)\|d\theta+\|f_{-1}(t,z_t)\|\\
&+\sum_{0<t_k<t}M\|J_k(t_k,z(t_k))\|
\end{aligned}
$$

$$
\begin{aligned}
&\leq \|\mathcal{U}(t,0)\| \big(\|\eta(0)\| + L_g q\|z\| + \Psi\left(\|\eta\| + L_g q \|z\|\right) \big) \\
&\quad + M\int_0^t \|A_0(\theta) f_{-1}(\theta, z_\theta)\| + \mu(\theta)(1 + \|z(\theta)\|)d\theta + \|f_{-1}(t, z_t)\| \\
&\quad + \sum_{0<t_k<t} M d_k \|z(t_k)\|.
\end{aligned}
$$

Then, applying Gronwall Inequality for impulsive differential equations(see [8, 15, 16, 18]), we obtain that

$$
\|z(t)\|_{\mathbb{R}^n} \leq M\left(\|z(0)\|_{\mathbb{R}^n} + \int_0^\tau \mu(\theta)d\theta \right) \prod_{t_0<t_k<t} (1 + Md_k) e^{\int_0^\tau M\mu(\theta)d\theta},
$$

This implies that $\|z(t)\|_{\mathbb{R}^n}$ remains bounded as $t \to \theta_1$ and applying Theorem 3.3 we get the result.

□

4. Global Lipschitz Conditions

This section will assume stronger hypotheses on the nonlinear terms that allow us to apply Banach Fixed Point Theorem. Specifically, we will suppose that the nonlinear functions that appear in our system are globally Lipschitz. Moreover, we shall consider the following simpler system

$$
\begin{cases}
\dfrac{d}{dt}[z(t) - f(t, z_t)] = A_0(t)z(t) + F(t, z_t), \quad t \in [0,\tau] \setminus \{t_1, t_2, \dots, t_p\} \\
z(s) = g(z)(s) + \phi(s), \quad s \in [-r, 0] \\
z(t_k^+) = z(t_k^-) + J_k(t_k, z(t_k)), \quad k = 1, 2, \dots, p,
\end{cases}
\tag{4.1}
$$

where the nonlocal condition $z(s) = g(z)(s) + \phi(s), \quad s \in [-r, 0]$ means

$$
z(s) = g\left(z\Big|_{[-r,0]} \right)(s) + \phi(s), \quad s \in [-r, 0].
$$

The functions $f, F : [0,\tau] \times \mathcal{PW}_r \longrightarrow \mathbb{R}^n$ are smooth enough satisfying certain conditions that will be specified later, and $J_k : [0,\infty) \times \mathbb{R}^n \longrightarrow \mathbb{R}^n$, $k = 1, 2, \dots, p$, are continuous and represents the impulsive effect in the system (4.1), the continuous function $g : \mathcal{PW}_r \longrightarrow \mathcal{PW}_r$ represent the nonlocal conditions, this function acts as a feedback operator which adjusts a part of the past when the initial function is present, or even, the whole past when the function ϕ is absent according to some precise future requirements (see [1]). The advantage of using nonlocal conditions is that measurements at more places can be incorporated to get better models. For more details and physical interpretations about nonlocal condition see [1–5, 21] and references therein.

Now, assuming a global Lipschitz condition, we will prove that system (4.1) admits a unique solution defined on $[0,\tau]$ by applying Banach Fixed Point Theorem. In this regards, we suppose the following global Lipschitz condition on the nonlinear terms:

(L1) There exist positive constants L_f and L_F such that for all $t \in [0, \tau]$, $\phi, \tilde{\phi} \in \mathcal{PW}_r$

$$\|f(t,\phi) - f(t,\tilde{\phi})\| \le L_f \|\phi - \tilde{\phi}\|_r,$$
$$\left\|F(t,\phi) - F(t,\tilde{\phi})\right\| \le L_F \|\phi - \tilde{\phi}\|_r.$$

(L2) There exist nonnegative constants d_k, $k = 1, 2, \dots, p$ such that for all $t \in [0, \infty)$, $z, \tilde{z} \in \mathbb{R}^n$

$$\|J_k(t,z) - J_k(t,\tilde{z})\|_{\mathbb{R}^n} \le d_k \|z - \tilde{z}\|_{\mathbb{R}^n}.$$

(L3) There exists a nonnegative constant L_g such that for all $\phi, \psi \in \mathcal{PW}_r$

$$\|g(\phi) - g(\psi)\|_r \le L_g \|\phi - \psi\|_r.$$

(L4)

$$L_f + M[L_g + L_f L_g + \|A_0\| L_f \tau + L_F \tau + \sum_{k=1}^{p} d_k] < 1,$$

where $\|A_0\| = \max\{\|A_0(t)\| : t \in [0, \tau]\}$.

Proposition 4.1. *Let $\phi \in \mathcal{PW}_r$. Then z is solution of system* (4.1) *if and only if z satisfies the integral equation*

$$z(t) = \begin{cases} g(z)(t) + \phi(t), \quad t \in [-r, 0], \\ f(t, z_t) + \mathcal{U}(t,0)\big[g(z)(0) + \phi(0) - f(0, g(z)(0) + \phi(0))\big] \\ + \displaystyle\int_0^t \mathcal{U}(t,s) A_0(s) f(s, z_s) ds \\ + \displaystyle\int_0^t \mathcal{U}(t,s) F(s, z_s) ds + \sum_{0 < t_k < t} \mathcal{U}(t, t_k) J_k(t_k, z(t_k)), \quad t \in [0, \tau], \end{cases} \tag{4.2}$$

Theorem 4.1. *Suppose that* (L1)-(L4) *hold. Then for $\phi \in \mathcal{PW}_r$ the system* (4.1) *has a unique solution defined on* $[0, \tau]$.

Proof. We shall apply Banach Contraction Mapping Theorem, in this regard, we will define the following operator $\mathcal{T} : \mathcal{PW}_p \longrightarrow \mathcal{PW}_p$ by

$$\mathcal{T}(t) = \begin{cases} g(z)(t) + \phi(t), \quad t \in [-r, 0], \\ f(t, z_t) + \mathcal{U}(t,0)\big[g(z)(0) + \phi(0) - f(0, g(z)(0) + \phi(0))\big] \\ + \displaystyle\int_0^t \mathcal{U}(t,s) A_0(s) f(s, z_s) ds \\ + \displaystyle\int_0^t \mathcal{U}(t,s) F(s, z_s) ds + \sum_{0 < t_k < t} \mathcal{U}(t, t_k) J_k(t_k, z(t_k)), \quad t \in [0, \tau]. \end{cases} \tag{4.3}$$

If $t \in [-r, 0]$, then

$$\begin{aligned} \|(\mathcal{T}z)(t) - (\mathcal{T}\tilde{z})(t)\| &= \|g(z)(t) - g(\tilde{z})(t)\| \le \|(g(z) - g(\tilde{z}))\Big|_{[-r,0]}\|_p \\ &\le L_g \|(z - \tilde{z})\Big|_{[-r,0]}\|_{\mathcal{PW}_r} \le L_g \|z - \tilde{z}\|_p. \end{aligned}$$

If $t \in [0, \tau]$, then

$$\begin{aligned}
\|(\mathcal{T}z)(t) - (\mathcal{T}\tilde{z})(t)\| &\leq \|f(t, z_t) - f(t, \tilde{z}_t)\| + \|\mathcal{U}(t, 0)\| \big[\|g(z)(0) - g(\tilde{z})(0)\| \\
&+ \|f(0, g(z)(0) + \phi(0)) - f(0, g(\tilde{z})(0) + \phi(0))\|\big] \\
&+ \int_0^t \|\mathcal{U}(t, s)\| \|A_0(s)\| \|f(s, z_s) - f(s, \tilde{z}_s)\| ds \\
&+ \int_0^t \|\mathcal{U}(t, s)\| \|F(s, z_s) - F(s, \tilde{z}_s)\| ds \\
&+ \sum_{0<t_k<t} \|\mathcal{U}(t, t_k)\| \|J_k(t_k, z(t_k)) - J_k(t_k, \tilde{z}(t_k))\| \\
&\leq L_f \|z_t - \tilde{z}_\tau(t)\| + M\big[L_g \|z - \tilde{z}\|_{\mathcal{PW}_p} + L_f \|g_\tau(z)(0) - g_\tau(\tilde{z}(0)\|\big] \\
&+ M\|A_0\| L_f \int_0^t \|z_s - \tilde{z}_s\| ds + M L_F \int_0^t \|z_s - \tilde{z}_s\| ds \\
&+ M \sum_{0<t_k<t} d_k \|z(t_k) - \tilde{z}(t_k)\| \\
&\leq \left(L_f + M\left[L_g + L_f L_g + \|A_0\| L_f \tau + L_F \tau + \sum_{k=1}^{p} d_k\right]\right) \|z - \tilde{z}\|_p.
\end{aligned}$$

Thus,

$$\|\mathcal{T}z - \mathcal{T}\tilde{z}\|_p \leq \left(L_f + M\left[L_g + L_f L_g + \|A_0\| L_f \tau + L_F \tau + \sum_{k=1}^{p} d_k\right]\right) \|z - \tilde{z}\|_p,$$

so, the operator $\mathcal{T}$ satisfies all the assumptions of the Banach Contraction Mapping Theorem, and therefore $\mathcal{T}$ has only one fixed point in the space $\mathcal{PW}_r$, which is the solution of problem (4.1). This completes the proof. □

5. Example

In this section, we consider an example of semilinear neutral differential equations with impulses, delay and nonlocal conditions such that Theorem 4.1 can be applied. Let us consider the following system

$$\begin{cases} \frac{d}{dt}\left[z(t) - \left(1 + \frac{\tan z(t-2)}{8(t+10)^2}\right)\right] = z(t) + e^{-\frac{z(t-2)}{10(t+5)^3}}, \quad t \in [0, \tau) \\ z(s) = \left(1 + \frac{\sin z}{30^2}\right)(s) + \phi(s), \quad s \in [-2, 0] \\ z(t_k^+) = z(t_k^-) + 1 + \frac{\cos\left(z(t_k^-)\right)}{4(t_k+8)^4}, \quad k = 1, 2. \end{cases} \tag{5.1}$$

Here $t_1 = \frac{5}{2}$, $t_2 = \frac{9}{2}$ and $\tau = 5$. In this example, the terms related to system (4.1) are given by: $f(t, z) = 1 + \frac{\tan(z)}{8(t+10)^2}$, $F(t, z) = e^{-\frac{z}{10(t+5)^3}}$, $g(z) = 1 + \frac{\sin(z)}{30^2}$, $J_k(t, z) = 1 + \frac{\cos(z)}{4(t+8)^4}$ and $A_0(t) = 1$. Then we have,

$$
\begin{array}{rcl}
|f(t,z)-f(t,\tilde{z})| & = & \frac{1}{8(t+10)^2}|\tan(z)-\tan(\tilde{z})| \leq \frac{1}{8\cdot 10^2}|z-\tilde{z}|_r, \\
|F(t,z)-F(t,\tilde{z})| & = & \left| e^{-\frac{z}{10(t+5)^3}} - e^{-\frac{\tilde{z}}{10(t+5)^3}} \right| \leq \frac{1}{10\cdot 5^3}|z-\tilde{z}|_r, \\
|J_k(t,z)-J_k(t,\tilde{z})| & = & \frac{1}{4(t+8)^4}|\cos(z)-\cos(\tilde{z})| \leq \frac{1}{4\cdot 8^4}|z-\tilde{z}|_r, \\
|g(z)-g(\tilde{z})|_r & = & \frac{1}{30^2}|\sin(z)-\sin(\tilde{z})|_{\mathcal{PW}_r} \leq \frac{1}{30^2}|z-\tilde{z}|_r,
\end{array}
$$

and

$$L_f + M[L_g + L_f L_g + |A_0|L_f\tau + L_F\tau + d_1 + d_2] \leq 0.63.$$

Hence, the conditions (L1)-(L4) are satisfied. Consequently, Theorem 4.1 ensures the existence of solutions for problem (5.1).

6. Final Remark

In this paper, first of all, we have proved the existence, uniqueness, and the globally defined solutions of a semilinear neutral differential equation with impulses and nonlocal conditions assuming that the nonlinear terms are locally Lipschitz. After that, we assume that the nonlinear functions that involve system (4.1) are globally Lipschitz, which allows us to prove the existence and uniqueness of solutions by applying Banach Fixed Point Theorem. Finally, we believe that this work can be extended to infinite dimension systems in Hilbert spaces, where the operator A_0 is no longer a matrix, instead, it will be the infinitesimal generator of a strongly continuous compact semigroup, and $-A_0$ a sectorial operator. In this way, the fractional powered spaces can be defined, allowing us to admit nonlinear terms involving spatial derivatives, like in the following neutral partial differential equations of Burges equation type:

$$
\begin{cases}
\dfrac{\partial}{\partial t}\left[z(t,x) + \displaystyle\int_0^t \int_0^\pi b(\theta - t, y, x) z(\theta, y)\,dy\,d\theta\right] = \nu z_{xx}(t,x) - z(t-r)z_x(t-r) \\
+f(t, z(t-r,x)), \quad t \neq t_k, \\
z(t,0) = z(t,1) = 0, \quad t \in [0,\tau] \\
z(\theta,x) + h(z(\tau_1+\theta,x),\ldots,z(\tau_q+\theta,x)) = \eta(\theta,x), \quad x \in [0,1], \\
z(t_k^+,x) = z(t_k^-,x) + J_k(z(t_k,x)), \quad x \in \Omega, \quad k = 1,2,3,\ldots,p,
\end{cases}
\tag{6.1}
$$

where $\eta \in \mathcal{PW}_{1/2}(-r,0;H_0^1) = \mathcal{PW}_{1/2}(-r,0;Z^{1/2})$, with $Z = L_2[0,1]$, $Z^{1/2} = D((-\Delta)^{1/2})$ and the functions f, J_k, h are locally Lipschitz.

Acknowledgment

The authors would like to express their thanks to the editor and anonymous referees for constructive comments and suggestions that improved the quality of this manuscript.

This work has been supported by Yachay Tech and ULA

References

[1] M. Burlica, M. Necula, D. Rosu, I. Vrabie, *Delay Differential Evolutions Subjected to Nonlocal Initial Conditions*, Monographs and Research Notes in Mathematics, Taylor and Francis Group, LLC, 2016.

[2] L. Byszewski, *Existence and uniqueness of solutions of nonlocal problems for hiperbolic equation $u_{xt} = F(x, t, u, u_x)$*, J. of App. Math. and Stoch. Anal. 3 (1990) 163–168.

[3] L. Byszewski, *Theorems about the existence and uniqueness of solutions of a semilinear evolution nonlocal Cauchy problem*, J. of Math. Anal. and Appl. 162 (1991) 494–505.

[4] L. Byszewski, V. Lakshmikantham, *Theorem about the existence and uniqueness of a solution of a nonlocal abstract Cauchy problem in a Banach space*, Appl. Anal. 40 (1990) 11–19.

[5] J. Chabrowski, *On nonlocal problems for parabolic equations*, Nagoya Math. J. 93 (1984) 109–131.

[6] E. Hernández, M. Pierri, *On abstract neutral differential equations with state-dependent delay*, Journal of Fixed Point Theory and Applications 20 (2018) 46–48.

[7] G.L. Karakostas, *An extension of Krasnosel'skii's fixed point theorem for contraction and compact mappings*, Topological Methods in Nonlinear Analysis, Journal of the Juliusz Schauder Center 22 (2003) 181–191.

[8] V. Lakshmikantham, D.D. Bainov and P.S. Simeonov, *Theory of Impulsive Differential Equations*, World Scientific, Singapore, 1989.

[9] E.B. Lee, L. Markus, *Foundations of Optimal Control Theory*, Wiley, New York, 1967.

[10] H. Leiva, P. Sundar, *Existence of solutions for a class of semilinear evolution equations whit impulses and delays*, Journal of Nonlinear Evolution Equations and Applications 2017 (2017) 95–108.

[11] H. Leiva, *Karakostas's fixed point theorem and the existence of solutions for impulsive semilinear evolution equations with delay and nonlocal conditions*, Communications in Mathematical Analysis 21 (2) (2018) 68–91.

[12] A. Manitius, *Necessary and sufficient conditions of approximate controllability for general linear retarded systems*, SIAM Journal on Control and Optimization 19 (1981) 516–532.

[13] R. Rabah, G.M. Sklyar. *Exact controllability of linear neutral type systems by the moment problem approach*, Proceedings of the IEEE Conference on Decision and Control, 2008, 2734–2739.

[14] H. Reinbacher, *New algebraic conditions for controllability of neutral differential equations*, Journal of Optimization Theory and Applications 54 (1987) 93–111.

[15] A.M. Samoilenko, N.A. Perestyuk, *Impulsive Differential Equations*, World Scientific Series on Nonlinear Science Series A, Vol. 14, 1995.

[16] S. Selvi, M. Mallika Arjunan, *Controllability results for impulsive differential systems with finite delay*, Journal Nonlinear Sciences and Applications 5 (2012) 206–219.

[17] A.S.C. Sinha, C.F. Yokomoto, *Null controllability of a nonlinear system with variable time delay*, IEEE Trans. Auto. Cont. AC-25 (1980) 1234–1236.

[18] R. Shikharchand Jain, M. Baburao Dhage, *On mild solutions of nonlocal semilinear impulsive functional integro-differential equations*, Applied Mathematics E-Notes 13 (2014) 109–119.

[19] D.A. O'Connor, T.J. Tarn. *On the function space controllability of linear neutral systems*, SIAM Journal on Control and Optimization 21 (1983) 306–328.

[20] G.R. Underwood, E.N. Chukwu, *Null controllability of nonlinear neutral differential equations*, Journal of Mathematical Analysis and Applications 129 (1982) 326–345.

[21] I.I. Vrabie, *Delay evolution equations with mixed nonlocal plus local initial conditions*, Comm. in Cont. Math. 17 (2) (2015) 1350035.

DOI: 10.7862/rf.2022.5

Hugo Leiva
email: `hleiva@yachaytech.edu.ec, hleiva@ula.ve`
ORCID: 0000-0002-3521-6253
Department of Mathematics
School of Mathematical and Computational Sciences
Yachay Tech
San Miguel de Urcuqui
Imbabura
ECUADOR

Lenin Riera
email: `lenin.riera@yachaytech.edu.ec`
ORCID: 0000-0002-4803-4918
Department of Mathematics
School of Mathematical and Computational Sciences
Yachay Tech
San Miguel de Urcuqui

Imbabura
ECUADOR

Sebastián Lalvay
email: sebastian.lalvay@yachaytech.edu.ec
ORCID: 0000-0002-0480-0828
Department of Mathematics
School of Mathematical and Computational Sciences
Yachay Tech
San Miguel de Urcuqui
Imbabura
ECUADOR

Received 14.02.2022 *Accepted 10.08.2022*

Journal of
Mathematics
and Applications

JMA No 45, pp 129-132 (2022)

On Some Inclusion in the Set Theory

Justyna Ochab and Marcin Ulaszek

ABSTRACT: This note contains the proof of an inclusion in the set theory. In that proof we use only basic laws appearing in the set theory. More precisely, using some basic laws of the set theory we provide the proof of an inclusion which is applied in the proof of certain theorem of the classical measure theory. The presented paper has an elementary character. Only the basic tools of the set theory are involved.

AMS Subject Classification: 03E20, 03E75.
Keywords and Phrases: Set theory; Inclusion; Distributivity over union; Distributivity over intersection.

1. Introduction

The purpose of this concise note is to present the proof of some inclusion of the set theory connected with a basic theorem concerning the property of the relation appearing in the measure theory which is called the equivalence with respect to a measure (cf. [5]; see also [6]).

In order to present the relation, assume that X is a nonempty set and S is a σ-field of some subsets of X, i. e., S is a family of some subsets of X which is σ-additive (i. e., if $A_i \in S$ for $i = 1, 2, ...$ then $\bigcup\limits_{i=1}^{\infty} A_i \in S$) and such that $A \setminus B \in S$ for arbitrary sets $A, B \in S$. Further, let m be a measure defined on S, i. e., $m : S \to \overline{\mathbb{R}}_+ = [0, +\infty]$ is σ-additive (that means $m\left(\bigcup\limits_{i=1}^{\infty} A_i\right) = \sum\limits_{i=1}^{\infty} m(A_i)$ for any sequence of sets belonging to S which are pairwise disjoint) and such that $m(\emptyset) = 0$.
We say that sets $A, B \in S$ *are equivalent with respect to the measure* m (we write $A \approx B$) if

$$m(A \setminus B) = m(B \setminus A) = 0.$$

It is easily seen that the relation of the equivalence with respect to the measure m is reflexive and symmetric. Thus, to prove that the relation $\approx$ is an equivalence relation it is only sufficient to prove that it is transitive, i. e., that the following implication holds

$$A \approx B \quad \text{and} \quad B \approx C \implies A \approx C \tag{1.1}$$

for arbitrary sets $A, B, C \in S$. The implication (1.1) will be proved if we show the following inclusion for arbitrary sets A, B, C :

$$A \setminus C \subset (A \setminus B) \cup (B \setminus C). \tag{1.2}$$

Observe that the proof of inclusion (1.2) can be performed in the standard way with the help of the transition of our problem to mathematical logic and then it is not difficult. However, it is interesting to conduct that proof by the use of basic laws of the theory of sets.
Since we have not found such a proof in popular mathematical literature (cf. [1–4,7,8]), we are going to present it in what follows.

2. Main result

The previously announced result is formulated in the form of the following theorem.

Theorem. *Let A, B, C be arbitrary sets. Then inclusion (1.2) holds.*

Proof. Denote by P the set appearing on the right-hand side of inclusion (1.2) i. e., $P = (A \setminus B) \cup (B \setminus C)$. Then, applying the well-known equality

$$X \setminus Y = X \cap Y'$$

we get

$$P = (A \setminus B) \cup (B \setminus C) = A \cap B' \cup B \cap C'.$$

Hence, in view of the distributivity of the union over the intersection, we obtain

$$\begin{aligned} P &= [(A \cap B') \cup B] \cap [(A \cap B') \cup C'] \\ &= [(A \cup B) \cap (B' \cup B)] \cap [(A \cup C') \cap (B' \cup C')] \\ &= (A \cup B) \cap (A \cup C') \cap (B' \cup C'). \end{aligned}$$

Now, applying the law of the associativity, we get

$$P = (B' \cup C') \cap [(A \cup C') \cap (A \cup B)]. \tag{2.1}$$

Further, using two times the law of the distributivity of the union over the intersection, we obtain consecutively the following equalities for the set L, where L denotes the left-hand side of inclusion (1.2):

$$\begin{aligned} L = A \setminus C = A \cap C' &= (A \cap C') \cup \emptyset = (A \cap C') \cup (B \cap B') \\ &= [(A \cap C') \cup B] \cap [(A \cap C') \cup B'] \\ &= [(A \cup B) \cap (C' \cup B)] \cap [(A \cup B') \cap (C' \cup B')]. \end{aligned}$$

Next, in virtue of the law of the associativity for the intersection, we get

$$L = [(A \cup B) \cap (A \cup B')] \cap [(C' \cup B) \cap (C' \cup B')].$$

Hence, taking into account the fact that the union is distributive over the intersection, we derive the equality

$$\begin{aligned} L &= [A \cup (B \cap B')] \cap [(C' \cup B) \cap (C' \cup B')] \\ &= A \cap [(C' \cup B) \cap (C' \cup B')]. \end{aligned}$$

Further, in view of the associativity of the intersection and the distributivity of the intersection over the union, we obtain

$$\begin{aligned} L &= (C' \cup B') \cap [A \cap (C' \cup B)] \\ &= (B' \cup C') \cap [(A \cap C') \cup (A \cap B)]. \end{aligned} \tag{2.2}$$

Now, comparing expressions (2.1) and (2.2) we see that in order to prove inclusion (1.2) it is sufficient to show that

$$(A \cap C') \cup (A \cap B) \subset (A \cup C') \cap (A \cup B). \tag{2.3}$$

To this end, similarly as before, let us denote

$$L = (A \cap C') \cup (A \cap B), \quad P = (A \cup C') \cap (A \cup B).$$

Then, keeping in mind the distributivity of the intersection over the union, we have

$$\begin{aligned} P = (A \cup C') \cap (A \cup B) &= [(A \cup B) \cap A] \cup [(A \cup B) \cap C'] \\ &= [(A \cap A) \cup (A \cap B)] \cup [(A \cap C') \cup (B \cap C')] \\ &= [A \cup (A \cap B)] \cup [(A \cap C') \cup (B \cap C')] \\ &= [(A \cap C') \cup (A \cap B)] \cup [A \cup (B \cap C')]. \end{aligned} \tag{2.4}$$

Finally, taking into account the form of the set L and (2.4) we conclude that

$$L = (A \cap C') \cup (A \cap B) \subset [(A \cap C') \cup (A \cap B)] \cup [A \cup (B \cap C')] = P.$$

Obviously, the above obtained inclusion completes the proof since we showed the desired inclusion (2.3).

□

References

[1] J. Banaś, S. Wędrychowicz, *Collection of Exercises in Mathematical Analysis*, Wydawnictwo Naukowe PWN, Warsaw, 2020 (in Polish).

[2] R. Engelking, *General Topology*, Wydawnictwo Naukowe PWN, Warsaw, 2007 (in Polish).

[3] K. Kuratowski, *Introduction to Set Theory and Topology*, Wydawnictwo Naukowe PWN, Warsaw, 2007 (in Polish).

[4] K. Kuratowski, A. Mostowski, *Set Theory*, PWN, Warsaw, 1966 (in Polish).

[5] S. Łojasiewicz, *Introduction to the Theory of Real Functions*, PWN, Warsaw, 1973 (in Polish).

[6] S. Łojasiewicz, *Introduction to the Theory of Real Functions*, J. Wiley & Sons, Chichester, 1988.

[7] W. Marek, J. Onyszkiewicz, *Elements of the Logic and Set Theory in Exercises*, Wydawnictwo Naukowe PWN, Warsaw, 2004 (in Polish).

[8] R. Sikorski, *Real Functions*, PWN, Warsaw, 1958 (in Polish).

DOI: 10.7862/rf.2022.6

Justyna Ochab
email: `j.ochab@prz.edu.pl`
ORCID: 0000-0001-5749-5807
Faculty of Mathematics and Applied Physics
Rzeszów University of Technology
al. Powstańców Warszawy 8,
35-959 Rzeszów
POLAND

Marcin Ulaszek
email: `marcinulaszek1998@gmail.com`
ORCID: 0000-0002-2624-4172
Faculty of Mathematics and Applied Physics
Rzeszów University of Technology
al. Powstańców Warszawy 8,
35-959 Rzeszów
POLAND

Received 22.03.2022 *Accepted 15.04.2022*

Journal of
Mathematics
and Applications

JMA No 45, pp 133-179 (2022)

Time-Unit Shifting in 2-Person Games Played in Finite and Uncountably Infinite Staircase-Function Spaces

Vadim Romanuke

Abstract: A computationally efficient and tractable method is presented to find the best equilibrium in a finite 2-person game played with staircase-function strategies. The method is based on stacking equilibria of smaller-sized bimatrix games, each defined on a time unit where the pure strategy value is constant. Every pure strategy is a staircase function defined on a time interval consisting of an integer number of time units (subintervals). If a time-unit shifting happens, where the initial time interval is narrowed by an integer number of time units, the respective equilibrium solution of any "narrower" subgame can be taken from the "wider" game equilibrium. If the game is uncountably infinite, i. e. a set of pure strategy possible values is uncountably infinite, and all time-unit equilibria exist, stacking equilibria of smaller-sized 2-person games defined on a rectangle works as well.

AMS Subject Classification: 91A05, 91A10, 91A50, 18F20.
Keywords and Phrases: Game theory; Payoff functional; Staircase-function strategy; Time unit (subinterval); Bimatrix game; Best equilibrium.

1. Staircase-function strategies

A noncooperative 2-person game is a model of process where two sides personified and referred to as persons or players interact in struggling for optimizing their own payoffs [24, 25, 7]. The players' payoffs are taken from some limited resources, so the distribution of the limited resources is optimized by the game model [25, 1, 13, 27]. The simplest 2-person game is a bimatrix game [7, 15, 25]. Whereas each of the players in a bimatrix game possesses a finite set (space) of pure strategies, the principles and theory of equilibrium, efficiency, profitability, and eventual optimality of bimatrix game solutions are thoroughly studied [7, 10, 14, 24]. However, the practice of bimatrix game solutions is not that simple. First, a problem may arise with multiplicity of the solutions. Second, a problem may arise with selecting a solution type (regarding equilibrium or profitability, which often are counteractive). Third, another problem does arise when the solution is in mixed strategies but the number of game iterations (moves, actions, plays, etc.) is limited and so a mixed strategy appears to be impracticable (for instance, it is impossible to practically realize a mixed strategy having probability of 7/19 if there are only 10 game iterations) [17, 18, 3, 7]. Furthermore, if at least two solutions are symmetric, they may be quite unstable due to cooperation between the players is excluded [7, 24, 25, 10, 23].

A far more complicated case is a 2-person game, in which the player's (pure) strategy is a function (usually, it is a function of time). In such a game, the player's payoff is a functional mapping every pair of functions (pure strategies of the players defined on a time interval) into a real value [20, 16]. In the case, when each of the players possesses a finite set of such function-strategies, the game might be rendered down to a bimatrix game [19, 13, 15]. The bimatrix game played with function-strategies, apart from those mentioned problems inherent in ordinary bimatrix games, is a far subtler model in the sense of its practicability.

The finiteness of a set of function-strategies is constituted by time interval discretization and discretization of possible values of the strategy. The time interval, on which the pure strategy is defined, is broken into a set of time subintervals (units), on which the strategy is (approximately considered) constant. This is so because there is no natural time continuity — every process is constant on some (usually, short) time period [2, 5, 8, 11, 12]. The continuity of possible values of the strategy on a subinterval is removed also by discretization (or sampling) [22, 18, 9] ruled by laws of the game-modeled system. Then the set of function-strategies becomes finite, where the strategy itself is a staircase function [22] but sometimes it can be conditionally interpreted as a point [30, 16, 18]. Compared to the most trivial strategy, which is a decision corresponding to a one-stage event whose duration through time is (usually, negligibly) short, a staircase-function strategy itself is a multi-stage process defined on a time interval [26, 30, 18, 4, 28, 29]. Nevertheless, the length of the time interval can be varied depending on properties of the process modeled by the game.

2. Multiplicity of equilibria and the time interval length

A 2-person game played in finite staircase-function spaces can be called the bimatrix staircase-function game. It is quite clear that the number of pure-strategy situations in a bimatrix staircase-function game grows immensely as the number of breakpoints ("stair" subintervals) increases, or the number of possible values of the player's pure strategy increases, or they both increase. For instance, if the number of time subintervals is just 5, and the number of possible values of the player's pure strategy is 6, then there is a finite set of

$$6^5 = 7776$$

possible pure strategies (i. e., 5-subinterval staircase functions of time) at this player. If the other player's pure strategy has, say, 8 possible values, then there are

$$8^5 = 32768$$

possible 5-subinterval staircase functions of time at this player, and the respective bimatrix staircase-function game has a size of either

$$7776 \times 32768$$

or

$$32768 \times 7776$$

and there are

$$6^5 \cdot 8^5 = 7776 \cdot 32768 = 254803968$$

pure-strategy situations. If an additional time subinterval is included, there are

$$6^6 \cdot 8^6 = 46656 \cdot 262144 = 12230590464$$

pure-strategy situations (more than 12.23 billion ones!). This is why a tractable method of solving 2-person games defined on a product of staircase-function spaces was presented in [21], where the spaces can be finite and continuous (uncountably infinite) as well. The method is based on stacking equilibria of "short" 2-person games, each defined on a subinterval where the pure strategy value is constant. It is proved in [21] that the bimatrix staircase-function game is solved as a stack of respective equilibria in the "short" (ordinary) bimatrix games (where the pure strategy is a very simple decision corresponding to a one-stage event). The stack is always possible, even when only time is discrete (and the set of pure strategy possible values is infinite or uncountably infinite). However, the problem of multiplicity of equilibria was not raised in [21]. The subinterval equilibrium multiplicity has a dramatic impact on the multiplicity of the equilibrium stack. For instance, if there are two equilibria on each of 5 subintervals, the game has altogether

$$2^5 = 32$$

equilibrium stacks. Then an open question is how to select a single equilibrium stack. Another open question is how to deal with a 2-person game in which the breakpoints of a function-strategy do not change but the time interval length can vary [2, 3, 7, 30, 18, 5].

3. Objective and six tasks to be fulfilled

Due to the above reasons, the objective is to expand and develop the tractable method of solving 2-person games played within players' finite sets of staircase functions [21] for the case when the length of the time interval on which the 2-person game is defined is varied. The case with an uncountably infinite set (space) of staircase functions is to be considered as well. To meet the objective, the following tasks are to be fulfilled:

1. To formalize a 2-person game, in which the players' strategies are staircase functions. In such a game, the set of the player's pure strategies is a continuum of staircase functions of time. Such function-strategies are presumed to be bounded and Lebesgue-integrable, and the time can be thought of as it is discrete.
2. To discretize the set of possible values of the player's pure strategy so that the game be defined on a product of staircase-function finite spaces.
3. To formalize the known method of solving 2-person games (the solution of the equilibrium type) played in staircase-function finite and uncountably infinite spaces by considering a possibility of narrowing the time interval on which the 2-person game is defined.
4. To give an example of how the suggested method is applied. A special attention must be paid to selecting a single equilibrium situation.
5. To discuss practical applicability and scientific significance of the method for the game theory and operations research.
6. To conclude on the study and make an outlook for furthering it.

4. 2-person game played with staircase-function strategies through discrete time

In a 2-person game, in which the player's pure strategy is a function of time, let each of the players use time-varying strategies defined almost everywhere on interval $[t_1;\ t_2]$ by $t_2 > t_1$. Denote a pure strategy of the first player by $x(t)$ and a pure strategy of the second player by $y(t)$. These functions are presumed to be bounded, i. e.

$$a_{\min} \leqslant x(t) \leqslant a_{\max} \ \text{ by } \ a_{\min} < a_{\max} \tag{1}$$

and

$$b_{\min} \leqslant y(t) \leqslant b_{\max} \ \text{ by } \ b_{\min} < b_{\max}, \tag{2}$$

defined almost everywhere on $[t_1;\ t_2]$. Besides, the square of the function-strategy is presumed to be Lebesgue-integrable. Thus, pure strategies of the player belong to a rectangular functional space of functions of time:

$$X = \{x(t),\ t \in [t_1;\ t_2],\ t_1 < t_2 :\ a_{\min} \leqslant x(t) \leqslant a_{\max} \text{ by } a_{\min} < a_{\max}\} \subset \\ \subset \mathbb{L}_2 [t_1;\ t_2] \tag{3}$$

and

$$Y = \{y(t),\ t \in [t_1;\ t_2],\ t_1 < t_2 :\ b_{\min} \leqslant y(t) \leqslant b_{\max} \text{ by } b_{\min} < b_{\max}\} \subset$$

$$\subset \mathbb{L}_2\left[t_1;\ t_2\right] \tag{4}$$

are the sets (sometimes referred to as action spaces) of the players' pure strategies.

The first player's payoff in situation (Figure 1)

$$\{x(t),\ y(t)\} \tag{5}$$

is

$$K\big(x(t),\ y(t)\big) \tag{6}$$

presumed to be an integral functional [21, 22]:

$$K\big(x(t),\ y(t)\big) = \int\limits_{[t_1;\ t_2]} f\big(x(t),\ y(t),\ t\big) d\mu(t), \tag{7}$$

where

$$f\big(x(t),\ y(t),\ t\big) \tag{8}$$

is a function of $x(t)$ and $y(t)$ explicitly including t. The second player's payoff in situation (5) is

$$H\big(x(t),\ y(t)\big) \tag{9}$$

presumed to be an integral functional also:

$$H\big(x(t),\ y(t)\big) = \int\limits_{[t_1;\ t_2]} g\big(x(t),\ y(t),\ t\big) d\mu(t), \tag{10}$$

where

$$g\big(x(t),\ y(t),\ t\big) \tag{11}$$

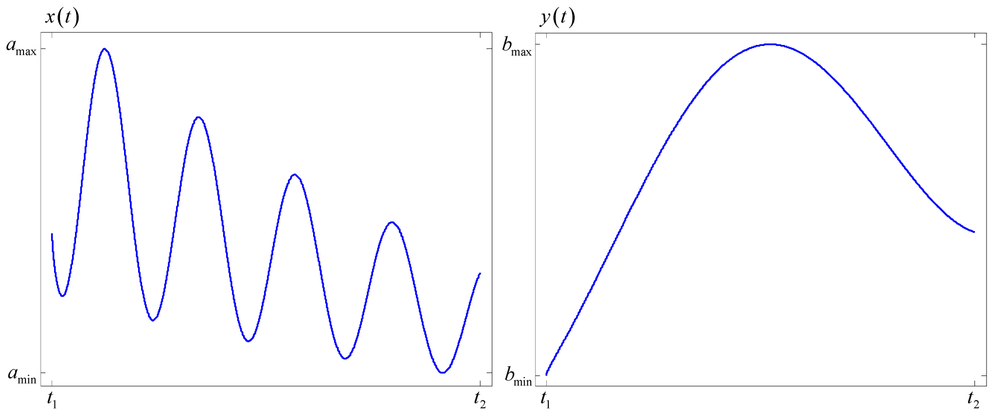

Figure 1: A situation (5) in 2-person game (12) played in uncountably infinite functional spaces (3) and (4)

is a function of $x(t)$ and $y(t)$ explicitly including t also. Therefore, a 2-person game

$$\left\langle \{X,\ Y\},\ \{K\bigl(x(t),\ y(t)\bigr),\ H\bigl(x(t),\ y(t)\bigr)\} \right\rangle \tag{12}$$

is uncountably infinite due to it is defined on product

$$X \times Y \subset \mathbb{L}_2\,[t_1;\ t_2] \times \mathbb{L}_2\,[t_1;\ t_2] \tag{13}$$

of uncountably infinite rectangular functional spaces (3) and (4) of players' pure strategies.

Each of sets (3) and (4) is a continuum of functions. It is worth noting that the game continuity is defined by the continuity of spaces (3) and (4), whereas payoff functionals (7) and (10) still can have discontinuities. In general, each of payoff functionals (6) and (9) may have a terminal component like

$$K\bigl(x(t),\ y(t)\bigr) = \\ = \int\limits_{[t_1;\ t_2]} f\bigl(x(t),\ y(t),\ t\bigr)d\mu(t) + T_f\bigl(x(t_2),\ y(t_2),\ t_2\bigr) \tag{14}$$

and

$$H\bigl(x(t),\ y(t)\bigr) = \\ = \int\limits_{[t_1;\ t_2]} g\bigl(x(t),\ y(t),\ t\bigr)d\mu(t) + T_g\bigl(x(t_2),\ y(t_2),\ t_2\bigr) \tag{15}$$

by some terminal functions

$$T_f\bigl(x(t_2),\ y(t_2),\ t_2\bigr) \tag{16}$$

and

$$T_g\bigl(x(t_2),\ y(t_2),\ t_2\bigr) \tag{17}$$

depending on only the final state of the player's strategy, but this case is not to be considered here.

Presume that the players' pure strategies $x(t)$ and $y(t)$ in game (12) can both change their values only for a finite number of times. Denote by N the number of subintervals at which the player's pure strategy is constant, where $N \in \mathbb{N} \backslash \{1\}$. In other words, when time is discrete, N is a number of time units. Then the player's pure strategy is a staircase function having at most N different values. Let

$$\Theta = \left\{ t_1 = \tau^{(0)} < \tau^{(1)} < \tau^{(2)} < \ldots < \tau^{(N-1)} < \tau^{(N)} = t_2 \right\}, \tag{18}$$

where $\left\{\tau^{(i)}\right\}_{i=1}^{N-1}$ are time points at which the staircase-function strategy can change its value. Time-interval breaking (18) is not necessarily to be equidistant. The staircase-function strategies are right-continuous [6, 21, 22]:

$$\lim_{\substack{\varepsilon > 0 \\ \varepsilon \to 0}} x\left(\tau^{(i)} + \varepsilon\right) = x\left(\tau^{(i)}\right) \tag{19}$$

and

$$\lim_{\substack{\varepsilon>0\\ \varepsilon\to 0}} y\left(\tau^{(i)}+\varepsilon\right)=y\left(\tau^{(i)}\right) \tag{20}$$

for $i=\overline{1,\ N-1}$, whereas (if the strategy value changes)

$$\lim_{\substack{\varepsilon>0\\ \varepsilon\to 0}} x\left(\tau^{(i)}-\varepsilon\right)\neq x\left(\tau^{(i)}\right) \tag{21}$$

and

$$\lim_{\substack{\varepsilon>0\\ \varepsilon\to 0}} y\left(\tau^{(i)}-\varepsilon\right)\neq y\left(\tau^{(i)}\right) \tag{22}$$

for $i=\overline{1,\ N-1}$. As an exception,

$$\lim_{\substack{\varepsilon>0\\ \varepsilon\to 0}} x\left(\tau^{(N)}-\varepsilon\right)=x\left(\tau^{(N)}\right) \tag{23}$$

and

$$\lim_{\substack{\varepsilon>0\\ \varepsilon\to 0}} y\left(\tau^{(N)}-\varepsilon\right)=y\left(\tau^{(N)}\right). \tag{24}$$

A 2-person game played with staircase-function strategies through discrete time can be defined by using (1) — (13), (18) — (24).

Definition 1. 2-person game (12) defined on product (13) of rectangular functional spaces (3) and (4) is called a discrete-time staircase-function 2-person game by time-interval breaking (18), if (19) — (24) hold and

$$\begin{gathered} x\,(t)=\alpha_i\in[a_{\min};\ a_{\max}] \text{ and } y\,(t)=\beta_i\in[b_{\min};\ b_{\max}] \\ \forall\, t\in\left[\tau^{(i-1)};\ \tau^{(i)}\right) \text{ for } i=\overline{1,\ N-1} \text{ and} \\ x\,(t)=\alpha_N\in[a_{\min};\ a_{\max}] \text{ and} \\ y\,(t)=\beta_N\in[b_{\min};\ b_{\max}]\ \ \forall\, t\in\left[\tau^{(N-1)};\ \tau^{(N)}\right], \end{gathered} \tag{25}$$

where the factual payoff of the first player in situation $\{\alpha_i,\ \beta_i\}$ is

$$K_i\,(\alpha_i,\ \beta_i)=\int\limits_{\left[\tau^{(i-1)};\ \tau^{(i)}\right)} f\,(\alpha_i,\ \beta_i,\ t)\,d\mu\,(t)\ \ \forall\, i=\overline{1,\ N-1} \tag{26}$$

and

$$K_N\,(\alpha_N,\ \beta_N)=\int\limits_{\left[\tau^{(N-1)};\ \tau^{(N)}\right]} f\,(\alpha_N,\ \beta_N,\ t)\,d\mu\,(t), \tag{27}$$

and the factual payoff of the second player in situation $\{\alpha_i,\ \beta_i\}$ is

$$H_i\,(\alpha_i,\ \beta_i)=\int\limits_{\left[\tau^{(i-1)};\ \tau^{(i)}\right)} g\,(\alpha_i,\ \beta_i,\ t)\,d\mu\,(t)\ \ \forall\, i=\overline{1,\ N-1} \tag{28}$$

and

$$H_N(\alpha_N,\ \beta_N) = \int\limits_{\left[\tau^{(N-1)};\ \tau^{(N)}\right]} g(\alpha_N,\ \beta_N,\ t)\,d\mu(t). \tag{29}$$

Situation (5) in the discrete-time staircase-function 2-person game is a stack of successive situations

$$\left\{\{\alpha_i,\ \beta_i\}\right\}_{i=1}^{N} \tag{30}$$

in a succession of N (ordinary) 2-person games

$$\left\langle \{[a_{\min};\ a_{\max}],\ [b_{\min};\ b_{\max}]\},\ \{K(\alpha_i,\ \beta_i),\ H(\alpha_i,\ \beta_i)\} \right\rangle \text{ for } i = \overline{1,\ N} \tag{31}$$

defined on product

$$[a_{\min};\ a_{\max}] \times [b_{\min};\ b_{\max}] \tag{32}$$

by (25) — (29).

Let a discrete-time staircase-function 2-person game by time-interval breaking (18) be denoted by

$$\left\langle \{X(\Theta),\ Y(\Theta)\},\ \left\{K_i\big(x(t),\ y(t)\big),\ H_i\big(x(t),\ y(t)\big)\right\} \right\rangle \tag{33}$$

with the players' pure strategy sets

$$X(\Theta) = \Big\{x(t) \in X([t_1;\ t_2]) : x(t) = \alpha_i \in [a_{\min};\ a_{\max}] \\ \forall\, t \in \left[\tau^{(i-1)};\ \tau^{(i)}\right) \text{ for } i = \overline{1,\ N-1} \text{ and} \\ x(t) = \alpha_N \in [a_{\min};\ a_{\max}]\ \forall\, t \in \left[\tau^{(N-1)};\ \tau^{(N)}\right]\Big\} \subset X([t_1;\ t_2]) \tag{34}$$

and

$$Y(\Theta) = \Big\{y(t) \in Y([t_1;\ t_2]) : y(t) = \beta_i \in [b_{\min};\ b_{\max}] \\ \forall\, t \in \left[\tau^{(i-1)};\ \tau^{(i)}\right) \text{ for } i = \overline{1,\ N-1} \text{ and} \\ y(t) = \beta_N \in [b_{\min};\ b_{\max}]\ \forall\, t \in \left[\tau^{(N-1)};\ \tau^{(N)}\right]\Big\} \subset Y([t_1;\ t_2]). \tag{35}$$

Obviously, discrete-time staircase-function 2-person game (33) is uncountably infinite as each of sets (34) and (35) contains a continuum of function-strategies. An example of situation (5) in a discrete-time staircase-function 2-person game played through seven time units (subintervals) is given in Figure 2. The exemplified pure-strategy situation of two staircase functions can be also represented as a stack of seven successive situations $\left\{\{\alpha_i,\ \beta_i\}\right\}_{i=1}^{7}$ of seven ordinary 2-person games (31), where each ordinary pure-strategy situation $\{\alpha_i,\ \beta_i\}$ for $i = \overline{1,\ 6}$ corresponds to a time unit (subinterval) $\left[\tau^{(i-1)};\ \tau^{(i)}\right)$ and ordinary pure-strategy situation $\{\alpha_7,\ \beta_7\}$ corresponds to a time unit (subinterval) $\left[\tau^{(6)};\ \tau^{(7)}\right] = \left[\tau^{(6)};\ t_2\right]$.

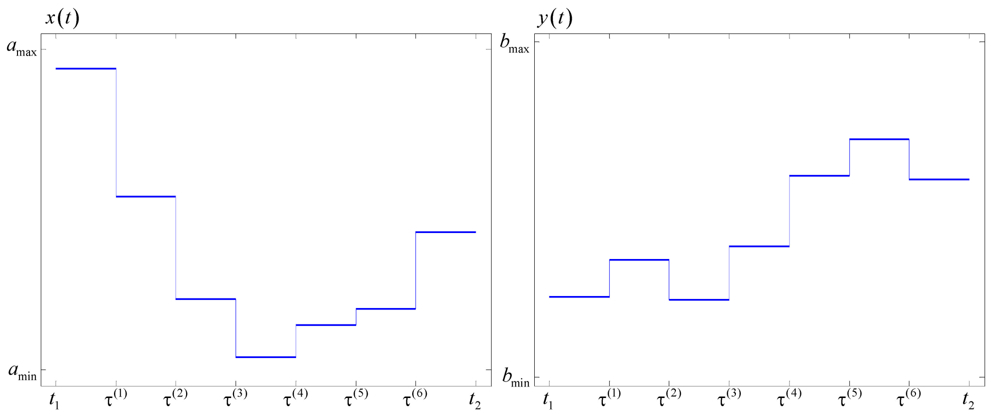

Figure 2: A situation (5) in discrete-time staircase-function 2-person game (33); the game is played in uncountably infinite functional spaces (34) and (35); the exemplified pure-strategy situation is a stack of seven successive situations $\left\{\{\alpha_i,\ \beta_i\}\right\}_{i=1}^{7}$

Time-interval breaking (18) allows considering payoffs (7) and (10) in situation (5) equivalent to the sum of respective payoffs (26)—(29). The proof can be found in [22].

Theorem 1. *In a pure-strategy situation (5) of discrete-time staircase-function 2-person game (33), payoff functionals (7) and (10) are re-written as subinterval-wise sums*

$$K\big(x(t),\ y(t)\big) = \sum_{i=1}^{N} K_i(\alpha_i,\ \beta_i) =$$

$$= \sum_{i=1}^{N-1} \int\limits_{\left[\tau^{(i-1)};\ \tau^{(i)}\right)} f(\alpha_i,\ \beta_i,\ t)\, d\mu(t) + \int\limits_{\left[\tau^{(N-1)};\ \tau^{(N)}\right]} f(\alpha_N,\ \beta_N,\ t)\, d\mu(t) \tag{36}$$

and

$$H\big(x(t),\ y(t)\big) = \sum_{i=1}^{N} H_i(\alpha_i,\ \beta_i) =$$

$$= \sum_{i=1}^{N-1} \int\limits_{\left[\tau^{(i-1)};\ \tau^{(i)}\right)} g(\alpha_i,\ \beta_i,\ t)\, d\mu(t) + \int\limits_{\left[\tau^{(N-1)};\ \tau^{(N)}\right]} g(\alpha_N,\ \beta_N,\ t)\, d\mu(t), \tag{37}$$

where situation (5) is a stack of successive situations (30) in a succession of N 2-person games (31).

Proof. Time interval $[t_1;\ t_2]$ can be re-written as

$$[t_1;\ t_2] = \left\{ \bigcup_{i=1}^{N-1} \left[\tau^{(i-1)};\ \tau^{(i)} \right) \right\} \cup \left[\tau^{(N-1)};\ \tau^{(N)} \right]. \tag{38}$$

Therefore, the property of countable additivity of the Lebesgue integral can be used:

$$K\big(x(t),\ y(t)\big) = \int\limits_{[t_1;\ t_2]} f\big(x(t),\ y(t),\ t\big) d\mu(t) =$$

$$= \int\limits_{\left\{ \bigcup_{i=1}^{N-1} \left[\tau^{(i-1)};\ \tau^{(i)} \right) \right\} \cup \left[\tau^{(N-1)};\ \tau^{(N)} \right]} f\big(x(t),\ y(t),\ t\big) d\mu(t) =$$

$$= \sum_{i=1}^{N-1} \int\limits_{\left[\tau^{(i-1)};\ \tau^{(i)} \right)} f\big(x(t),\ y(t),\ t\big) d\mu(t) + \int\limits_{\left[\tau^{(N-1)};\ \tau^{(N)} \right]} f\big(x(t),\ y(t),\ t\big) d\mu(t). \tag{39}$$

Owing to (25), $x(t) = \alpha_i$ and $y(t) = \beta_i$, so (39) is simplified as

$$\sum_{i=1}^{N-1} \int\limits_{\left[\tau^{(i-1)};\ \tau^{(i)} \right)} f\big(x(t),\ y(t),\ t\big) d\mu(t) + \int\limits_{\left[\tau^{(N-1)};\ \tau^{(N)} \right]} f\big(x(t),\ y(t),\ t\big) d\mu(t) =$$

$$= \sum_{i=1}^{N-1} \int\limits_{\left[\tau^{(i-1)};\ \tau^{(i)} \right)} f(\alpha_i,\ \beta_i,\ t)\, d\mu(t) + \int\limits_{\left[\tau^{(N-1)};\ \tau^{(N)} \right]} f(\alpha_N,\ \beta_N,\ t)\, d\mu(t) =$$

$$= \sum_{i=1}^{N} K_i(\alpha_i,\ \beta_i). \tag{40}$$

Consequently, in discrete-time staircase-function 2-person game (33), subinterval-wise sum (36) holds in any pure-strategy situation (5) consisting of staircase-function strategies $x(t) \in X(\Theta)$ and $y(t) \in Y(\Theta)$. Obviously, subinterval-wise sum (37) is proved similarly to (38) — (40). □

It is noteworthy that Theorem 1 can be proved also by considering function (8) on a time unit (subinterval) as a function of time t, due to $x(t) = \alpha_i$ and $y(t) = \beta_i$ on this subinterval. Denote this function by $\psi_i(t)$. Then this function appears to be zero on any other time unit. Subsequently, function (8) is presented as the sum of those subinterval functions:

$$f\big(x(t),\ y(t),\ t\big) = \sum_{i=1}^{N} \psi_i(t),$$

whereupon (40) is deduced.

Nevertheless, Theorem 1 does not provide a method of solving the discrete-time staircase-function 2-person game, but it hints about how the game might be solved in an easier way [21, 22]. Theorem 1 provides a fundamental decomposition of the staircase game based on the subinterval-wise summing in (36) and (37). This subinterval decomposition allows considering and solving each game (31) separately, whereupon the solutions are stitched (stacked) together, regardless of whether the player's action space is finite or not.

5. Finite discrete-time staircase-function 2-person game

In a discrete-time staircase-function 2-person game (33), let the set of possible values of the first player's pure strategy be discretized as

$$\mathrm{A} = \left\{ a_{\min} = a_i^{(0)} < a_i^{(1)} < a_i^{(2)} < \ldots < a_i^{(M-1)} < a_i^{(M)} = a_{\max} \right\} \tag{41}$$

and the set of possible values of the second player's pure strategy be discretized as

$$\mathrm{B} = \left\{ b_{\min} = b_i^{(0)} < b_i^{(1)} < b_i^{(2)} < \ldots < b_i^{(Q-1)} < b_i^{(Q)} = b_{\max} \right\} \tag{42}$$

by $M \in \mathbb{N}$ and $Q \in \mathbb{N}$, where

$$a_i^{(m-1)} = a^{(m-1)} \;\; \forall\, i = \overline{1,\; N} \;\; \text{for} \;\; m = \overline{1,\; M+1} \tag{43}$$

and

$$b_i^{(q-1)} = b^{(q-1)} \;\; \forall\, i = \overline{1,\; N} \;\; \text{for} \;\; q = \overline{1,\; Q+1}. \tag{44}$$

This means that along with the discrete time units (subintervals), the players are forced (somehow) to act within finite subsets of possible values of their pure strategies

$$A = \left\{ a^{(m-1)} \right\}_{m=1}^{M+1} \tag{45}$$

and

$$B = \left\{ b^{(q-1)} \right\}_{q=1}^{Q+1}. \tag{46}$$

Discretizations (41) — (44) allow defining a finite discrete-time staircase-function 2-person game.

Definition 2. Discrete-time staircase-function 2-person game (33) is called finite if it is played on a product of finite subsets

$$X(\Theta,\; A) = \left\{ x(t) \in X(\Theta) : \; x(t) \in \left\{ a^{(m-1)} \right\}_{m=1}^{M+1} \right\} \subset$$
$$\subset X(\Theta) \subset X([t_1;\; t_2]) \tag{47}$$

and

$$Y\left(\Theta,\ B\right) = \left\{ y\left(t\right) \in Y\left(\Theta\right):\ y\left(t\right) \in \left\{ b^{(q-1)} \right\}_{q=1}^{Q+1} \right\} \subset \subset Y\left(\Theta\right) \subset Y\left(\left[t_1;\ t_2\right]\right) \tag{48}$$

of sets (34) and (35).

So, let a finite discrete-time staircase-function 2-person game be denoted by

$$\left\langle \left\{ X\left(\Theta,\ A\right),\ Y\left(\Theta,\ B\right) \right\},\ \left\{ K\big(x\left(t\right),\ y\left(t\right)\big),\ H\big(x\left(t\right),\ y\left(t\right)\big) \right\} \right\rangle \tag{49}$$

with the players' pure strategy sets (47) and (48). In fact, this finite game is a bimatrix staircase-function game (see an example in Figure 3, where every pure strategy as a staircase function of time can be "imagined" as a conditional point pretended to be a simple decision to constitute an 81×256 bimatrix game) that is the succession of N bimatrix games

$$\left\langle \left\{ \left\{ a_i^{(m-1)} \right\}_{m=1}^{M+1},\ \left\{ b_i^{(q-1)} \right\}_{q=1}^{Q+1} \right\},\ \left\{ \mathbf{K}_i,\ \mathbf{H}_i \right\} \right\rangle \text{ for } i = \overline{1,\ N} \tag{50}$$

with the first player's payoff matrices

$$\mathbf{K}_i = \left[k_{imq}\right]_{(M+1)\times(Q+1)} \tag{51}$$

whose elements are

$$k_{imq} = \int\limits_{\left[\tau^{(i-1)};\ \tau^{(i)}\right)} f\left(a_i^{(m-1)},\ b_i^{(q-1)},\ t\right) d\mu\left(t\right) \text{ for } i = \overline{1,\ N-1} \tag{52}$$

and

$$k_{Nmq} = \int\limits_{\left[\tau^{(N-1)};\ \tau^{(N)}\right]} f\left(a_N^{(m-1)},\ b_N^{(q-1)},\ t\right) d\mu\left(t\right), \tag{53}$$

and with the second player's payoff matrices

$$\mathbf{H}_i = \left[h_{imq}\right]_{(M+1)\times(Q+1)} \tag{54}$$

whose elements are

$$h_{imq} = \int\limits_{\left[\tau^{(i-1)};\ \tau^{(i)}\right)} g\left(a_i^{(m-1)},\ b_i^{(q-1)},\ t\right) d\mu\left(t\right) \text{ for } i = \overline{1,\ N-1} \tag{55}$$

and

$$h_{Nmq} = \int\limits_{\left[\tau^{(N-1)};\ \tau^{(N)}\right]} g\left(a_N^{(m-1)},\ b_N^{(q-1)},\ t\right) d\mu\left(t\right). \tag{56}$$

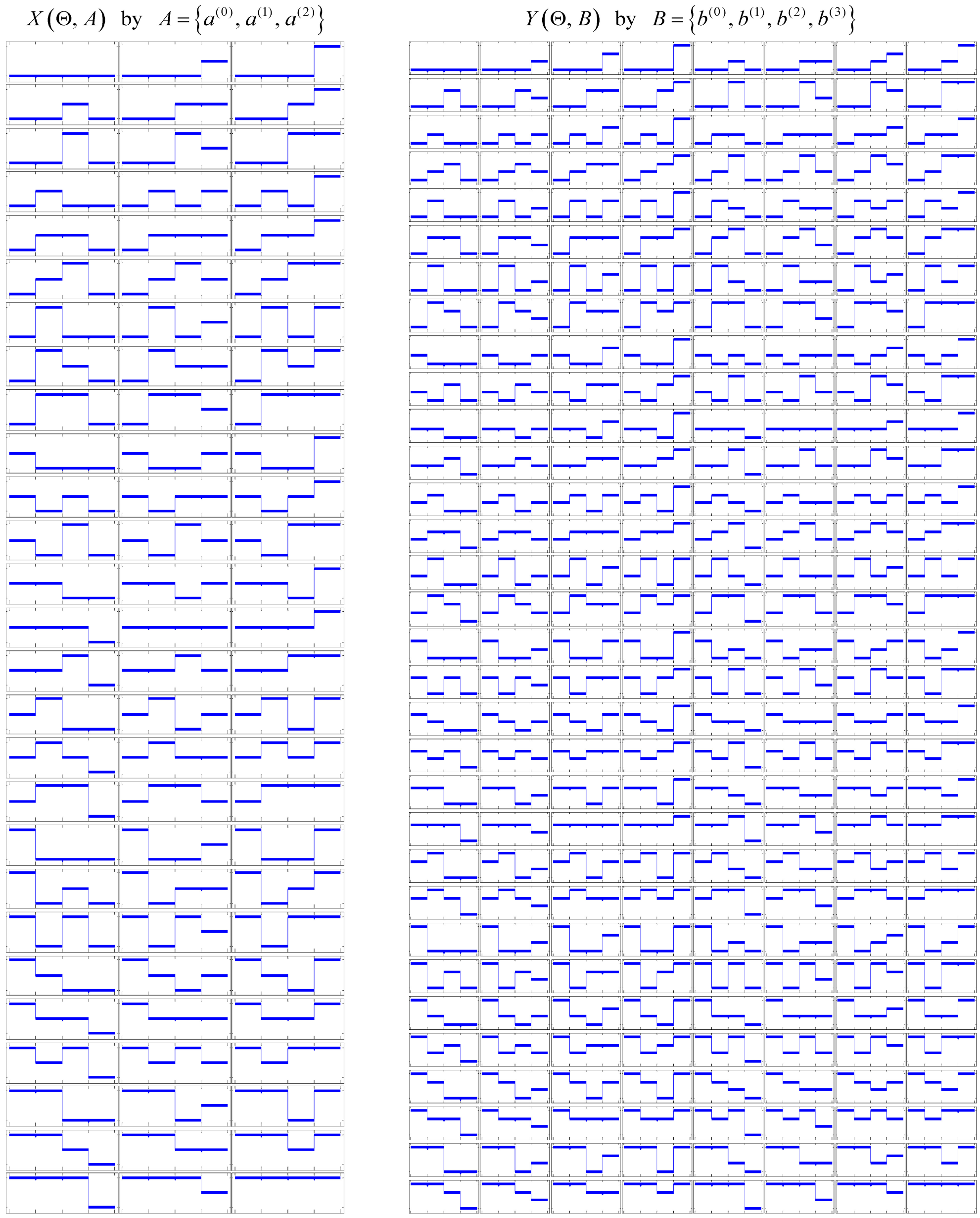

Figure 3: An example of finite pure strategy sets (47) and (48) in a bimatrix staircase-function game; the game is played with 4-subinterval staircase functions of time, where the first and second players have three and four possible values of their pure strategies, respectively

So, according with Definition 1, the first player's payoff in situation $\left\{a_i^{(m-1)},\ b_i^{(q-1)}\right\}$ is (52), (53), for $i = \overline{1,\ N}$, and the second player's payoff in situation $\left\{a_i^{(m-1)},\ b_i^{(q-1)}\right\}$ is (55), (56), for $i = \overline{1,\ N}$. In addition, situation (5) in the bimatrix staircase-function game is a stack of successive situations

$$\left\{\left\{a_i^{(m-1)},\ b_i^{(q-1)}\right\}\right\}_{i=1}^{N} \tag{57}$$

in a succession of N bimatrix games (50). Bimatrix staircase-function game (49) might be rendered down to the ordinary bimatrix game, wherein a pure strategy is a conditional point being in reality a staircase function. This rendering, however, is useless because the much more efficient method exists [21, 22] to consider game (49) as the succession of N bimatrix games (50) by (51) — (56) and find the solution of game (49) by stacking solutions of smaller-sized bimatrix games (50).

6. Time-unit shifting in bimatrix staircase-function games

An equilibrium situation in the bimatrix game always exists, either in pure or mixed strategies. Denote by

$$\mathbf{P}_i = \left[p_i^{(m)}\right]_{1\times(M+1)} \tag{58}$$

and

$$\mathbf{R}_i = \left[r_i^{(q)}\right]_{1\times(Q+1)} \tag{59}$$

the mixed strategies of the first and second players, respectively, in bimatrix game (50). The respective sets of mixed strategies of the first and second players are

$$\mathcal{P} = \left\{\mathbf{P}_i \in \mathbb{R}^{M+1} : p_i^{(m)} \geqslant 0,\ \sum_{m=1}^{M+1} p_i^{(m)} = 1\right\} \tag{60}$$

and

$$\mathcal{R} = \left\{\mathbf{R}_i \in \mathbb{R}^{Q+1} : r_i^{(q)} \geqslant 0,\ \sum_{q=1}^{Q+1} r_i^{(q)} = 1\right\}, \tag{61}$$

so $\mathbf{P}_i \in \mathcal{P}$, $\mathbf{R}_i \in \mathcal{R}$, and $\{\mathbf{P}_i,\ \mathbf{R}_i\}$ is a situation in this game.

Definition 3. A stack

$$\left\{\{\mathbf{P}_i,\ \mathbf{R}_i\}\right\}_{i=1}^{N} \tag{62}$$

of successive situations in bimatrix games (50) is called a (mixed-strategy) situation in bimatrix staircase-function game (49). Stacks $\{\mathbf{P}_i\}_{i=1}^{N}$ and $\{\mathbf{R}_i\}_{i=1}^{N}$ are the respective mixed strategies of the first and second players in this game.

It is clear that an equilibrium situation in a bimatrix staircase-function game is to be sought among stacks (62). The respective assertions can be found in [21, 22]. However, these papers do not directly show how to select the best equilibrium stack in the case of multiple equilibrium stacks.

Theorem 2. *If*

$$\{\mathbf{P}_i^*,\ \mathbf{R}_i^*\} = \left\{ \left[p_i^{(m)*} \right]_{1\times(M+1)},\ \left[r_i^{(q)*} \right]_{1\times(Q+1)} \right\} \tag{63}$$

is an equilibrium situation in bimatrix game (50) for $i = \overline{1,\ N}$, *then a stack*

$$\left\{ \{\mathbf{P}_i^*,\ \mathbf{R}_i^*\} \right\}_{i=1}^{N} = \left\{ \left\{ \left[p_i^{(m)*} \right]_{1\times(M+1)},\ \left[r_i^{(q)*} \right]_{1\times(Q+1)} \right\} \right\}_{i=1}^{N} \tag{64}$$

of such successive solutions is an equilibrium situation in bimatrix staircase-function game (49). If multiple equilibria exist (at one or more time units) and the maximum of the players' payoffs sum

$$\mathbf{P}_i^* \cdot \mathbf{K}_i \cdot (\mathbf{R}_i^*)^{\mathrm{T}} + \mathbf{P}_i^* \cdot \mathbf{H}_i \cdot (\mathbf{R}_i^*)^{\mathrm{T}} \tag{65}$$

is reached at $\mathbf{P}_i^* = \mathbf{P}_i^{**}$ *and* $\mathbf{R}_i^* = \mathbf{R}_i^{**}$, *i. e.*

$$\max_{\{\mathbf{P}_i^*,\ \mathbf{R}_i^*\}} \left\{ \mathbf{P}_i^* \cdot \mathbf{K}_i \cdot (\mathbf{R}_i^*)^{\mathrm{T}} + \mathbf{P}_i^* \cdot \mathbf{H}_i \cdot (\mathbf{R}_i^*)^{\mathrm{T}} \right\} = \\ = \mathbf{P}_i^{**} \cdot \mathbf{K}_i \cdot (\mathbf{R}_i^{**})^{\mathrm{T}} + \mathbf{P}_i^{**} \cdot \mathbf{H}_i \cdot (\mathbf{R}_i^{**})^{\mathrm{T}}, \tag{66}$$

then the maximum of the players' payoffs sum in an equilibrium stack of bimatrix staircase-function game (49) is reached at stack

$$\left\{ \{\mathbf{P}_i^{**},\ \mathbf{R}_i^{**}\} \right\}_{i=1}^{N} \tag{67}$$

and this maximum is

$$s^{**} = \sum_{i=1}^{N} \left(\mathbf{P}_i^{**} \cdot \mathbf{K}_i \cdot (\mathbf{R}_i^{**})^{\mathrm{T}} + \mathbf{P}_i^{**} \cdot \mathbf{H}_i \cdot (\mathbf{R}_i^{**})^{\mathrm{T}} \right). \tag{68}$$

Proof. As (63) is an equilibrium situation, then inequalities

$$\mathbf{P}_i \cdot \mathbf{K}_i \cdot (\mathbf{R}_i^*)^{\mathrm{T}} = \\ = \sum_{m=1}^{M+1} \sum_{q=1}^{Q+1} k_{imq} p_i^{(m)} r_i^{(q)*} = \\ = \sum_{m=1}^{M+1} \sum_{q=1}^{Q+1} p_i^{(m)} r_i^{(q)*} \int\limits_{\left[\tau^{(i-1)};\ \tau^{(i)}\right)} f\left(a_i^{(m-1)},\ b_i^{(q-1)},\ t \right) d\mu(t) \leqslant$$

$$\leqslant \sum_{m=1}^{M+1} \sum_{q=1}^{Q+1} p_i^{(m)*} r_i^{(q)*} \int\limits_{\left[\tau^{(i-1)};\, \tau^{(i)}\right)} f\left(a_i^{(m-1)},\, b_i^{(q-1)},\, t\right) d\mu\left(t\right) =$$

$$= \sum_{m=1}^{M+1} \sum_{j=1}^{Q+1} k_{imq} p_i^{(m)*} r_i^{(q)*} =$$

$$= \mathbf{P}_i^* \cdot \mathbf{K}_i \cdot \left(\mathbf{R}_i^*\right)^{\mathrm{T}} = v_i^* \;\; \forall\, \mathbf{P}_i \in \mathcal{P} \;\text{ for }\; i = \overline{1,\; N-1}, \tag{69}$$

$$\mathbf{P}_N \cdot \mathbf{K}_N \cdot \left(\mathbf{R}_N^*\right)^{\mathrm{T}} =$$

$$= \sum_{m=1}^{M+1} \sum_{q=1}^{Q+1} k_{Nmq} p_N^{(m)} r_N^{(q)*} =$$

$$= \sum_{m=1}^{M+1} \sum_{q=1}^{Q+1} p_N^{(m)} r_N^{(q)*} \int\limits_{\left[\tau^{(N-1)};\, \tau^{(N)}\right]} f\left(a_N^{(m-1)},\, b_N^{(q-1)},\, t\right) d\mu\left(t\right) \leqslant$$

$$\leqslant \sum_{m=1}^{M+1} \sum_{q=1}^{Q+1} p_N^{(m)*} r_N^{(q)*} \int\limits_{\left[\tau^{(N-1)};\, \tau^{(N)}\right]} f\left(a_N^{(m-1)},\, b_N^{(q-1)},\, t\right) d\mu\left(t\right) =$$

$$= \sum_{m=1}^{M+1} \sum_{q=1}^{Q+1} k_{Nmq} p_N^{(m)*} r_N^{(q)*} =$$

$$= \mathbf{P}_N^* \cdot \mathbf{K}_N \cdot \left(\mathbf{R}_N^*\right)^{\mathrm{T}} = v_N^* \;\; \forall\, \mathbf{P}_N \in \mathcal{P} \tag{70}$$

and inequalities

$$\mathbf{P}_i^* \cdot \mathbf{H}_i \cdot \mathbf{R}_i^{\mathrm{T}} =$$

$$= \sum_{m=1}^{M+1} \sum_{q=1}^{Q+1} h_{imq} p_i^{(m)*} r_i^{(q)} =$$

$$= \sum_{m=1}^{M+1} \sum_{q=1}^{Q+1} p_i^{(m)*} r_i^{(q)} \int\limits_{\left[\tau^{(i-1)};\, \tau^{(i)}\right)} g\left(a_i^{(m-1)},\, b_i^{(q-1)},\, t\right) d\mu\left(t\right) \leqslant$$

$$\leqslant \sum_{m=1}^{M+1} \sum_{q=1}^{Q+1} p_i^{(m)*} r_i^{(q)*} \int\limits_{\left[\tau^{(i-1)};\, \tau^{(i)}\right)} g\left(a_i^{(m-1)},\, b_i^{(q-1)},\, t\right) d\mu\left(t\right) =$$

$$= \sum_{m=1}^{M+1} \sum_{q=1}^{Q+1} h_{imq} p_i^{(m)*} r_i^{(q)*} =$$

$$= \mathbf{P}_i^* \cdot \mathbf{H}_i \cdot \left(\mathbf{R}_i^*\right)^{\mathrm{T}} = z_i^* \;\; \forall\, \mathbf{R}_i \in \mathcal{R} \;\text{ for }\; i = \overline{1,\; N-1}, \tag{71}$$

$$\mathbf{P}_N^* \cdot \mathbf{H}_N \cdot \mathbf{R}_N^{\mathrm{T}} =$$

$$= \sum_{m=1}^{M+1} \sum_{q=1}^{Q+1} h_{Nmq} p_N^{(m)*} r_N^{(q)} =$$

$$= \sum_{m=1}^{M+1} \sum_{q=1}^{Q+1} p_N^{(m)*} r_N^{(q)} \int\limits_{\left[\tau^{(N-1)};\, \tau^{(N)}\right]} g\left(a_N^{(m-1)},\ b_N^{(q-1)},\ t\right) d\mu\left(t\right) \leqslant$$

$$\leqslant \sum_{m=1}^{M+1} \sum_{q=1}^{Q+1} p_N^{(m)*} r_N^{(q)*} \int\limits_{\left[\tau^{(N-1)};\, \tau^{(N)}\right]} g\left(a_N^{(m-1)},\ b_N^{(q-1)},\ t\right) d\mu\left(t\right) =$$

$$= \sum_{m=1}^{M+1} \sum_{q=1}^{Q+1} h_{Nmq} p_N^{(m)*} r_N^{(q)*} =$$

$$= \mathbf{P}_N^* \cdot \mathbf{H}_N \cdot \left(\mathbf{R}_N^*\right)^{\mathrm{T}} = z_N^* \ \ \forall\, \mathbf{R}_N \in \mathcal{R} \tag{72}$$

hold. So, inequalities

$$\sum_{i=1}^{N-1} \mathbf{P}_i \cdot \mathbf{K}_i \cdot \left(\mathbf{R}_i^*\right)^{\mathrm{T}} + \mathbf{P}_N \cdot \mathbf{K}_N \cdot \left(\mathbf{R}_N^*\right)^{\mathrm{T}} =$$

$$= \sum_{i=1}^{N-1} \sum_{m=1}^{M+1} \sum_{q=1}^{Q+1} k_{imq} p_i^{(m)} r_i^{(q)*} + \sum_{m=1}^{M+1} \sum_{q=1}^{Q+1} k_{Nmq} p_N^{(m)} r_N^{(q)*} =$$

$$= \sum_{i=1}^{N-1} \left(\sum_{m=1}^{M+1} \sum_{q=1}^{Q+1} p_i^{(m)} r_i^{(q)*} \int\limits_{\left[\tau^{(i-1)};\, \tau^{(i)}\right)} f\left(a_i^{(m-1)},\ b_i^{(q-1)},\ t\right) d\mu\left(t\right) \right) +$$

$$+ \sum_{m=1}^{M+1} \sum_{q=1}^{Q+1} p_N^{(m)} r_N^{(q)*} \int\limits_{\left[\tau^{(N-1)};\, \tau^{(N)}\right]} f\left(a_N^{(m-1)},\ b_N^{(q-1)},\ t\right) d\mu\left(t\right) \leqslant$$

$$\leqslant \sum_{i=1}^{N-1} \left(\sum_{m=1}^{M+1} \sum_{q=1}^{Q+1} p_i^{(m)*} r_i^{(q)*} \int\limits_{\left[\tau^{(i-1)};\, \tau^{(i)}\right)} f\left(a_i^{(m-1)},\ b_i^{(q-1)},\ t\right) d\mu\left(t\right) \right) +$$

$$+ \sum_{m=1}^{M+1} \sum_{q=1}^{Q+1} p_N^{(m)*} r_N^{(q)*} \int\limits_{\left[\tau^{(N-1)};\, \tau^{(N)}\right]} f\left(a_N^{(m-1)},\ b_N^{(q-1)},\ t\right) d\mu\left(t\right) =$$

$$= \sum_{i=1}^{N-1} \sum_{m=1}^{M+1} \sum_{q=1}^{Q+1} k_{imq} p_i^{(m)*} r_i^{(q)*} + \sum_{m=1}^{M+1} \sum_{q=1}^{Q+1} k_{Nmq} p_N^{(m)*} r_N^{(q)*} =$$

$$= \sum_{i=1}^{N-1} \mathbf{P}_i^* \cdot \mathbf{K}_i \cdot \left(\mathbf{R}_i^*\right)^{\mathrm{T}} + \mathbf{P}_N^* \cdot \mathbf{K}_N \cdot \left(\mathbf{R}_N^*\right)^{\mathrm{T}} =$$

$$= \sum_{i=1}^{N} v_i^* = v^* \;\; \forall \, \mathbf{P}_i \in \mathcal{P} \;\; \text{for} \;\; i = \overline{1, \, N} \tag{73}$$

and

$$\sum_{i=1}^{N-1} \mathbf{P}_i^* \cdot \mathbf{H}_i \cdot \mathbf{R}_i^{\mathrm{T}} + \mathbf{P}_N^* \cdot \mathbf{H}_N \cdot \mathbf{R}_N^{\mathrm{T}} =$$
$$= \sum_{i=1}^{N-1} \sum_{m=1}^{M+1} \sum_{q=1}^{Q+1} h_{imq} p_i^{(m)*} r_i^{(q)} + \sum_{m=1}^{M+1} \sum_{q=1}^{Q+1} h_{Nmq} p_N^{(m)*} r_N^{(q)} =$$
$$= \sum_{i=1}^{N-1} \left(\sum_{m=1}^{M+1} \sum_{q=1}^{Q+1} p_i^{(m)*} r_i^{(q)} \int\limits_{\left[\tau^{(i-1)};\, \tau^{(i)}\right)} g\left(a_i^{(m-1)}, \, b_i^{(q-1)}, \, t\right) d\mu\left(t\right) \right) +$$
$$+ \sum_{m=1}^{M+1} \sum_{q=1}^{Q+1} p_N^{(m)*} r_N^{(q)} \int\limits_{\left[\tau^{(N-1)};\, \tau^{(N)}\right]} g\left(a_N^{(m-1)}, \, b_N^{(q-1)}, \, t\right) d\mu\left(t\right) \leqslant$$
$$\leqslant \sum_{i=1}^{N-1} \left(\sum_{m=1}^{M+1} \sum_{q=1}^{Q+1} p_i^{(m)*} r_i^{(q)*} \int\limits_{\left[\tau^{(i-1)};\, \tau^{(i)}\right)} g\left(a_i^{(m-1)}, \, b_i^{(q-1)}, \, t\right) d\mu\left(t\right) \right) +$$
$$+ \sum_{m=1}^{M+1} \sum_{q=1}^{Q+1} p_N^{(m)*} r_N^{(q)*} \int\limits_{\left[\tau^{(N-1)};\, \tau^{(N)}\right]} g\left(a_N^{(m-1)}, \, b_N^{(q-1)}, \, t\right) d\mu\left(t\right) =$$
$$= \sum_{i=1}^{N-1} \sum_{m=1}^{M+1} \sum_{q=1}^{Q+1} h_{imq} p_i^{(m)*} r_i^{(q)*} + \sum_{m=1}^{M+1} \sum_{q=1}^{Q+1} h_{Nmq} p_N^{(m)*} r_N^{(q)*} =$$
$$= \sum_{i=1}^{N-1} \mathbf{P}_i^* \cdot \mathbf{H}_i \cdot \left(\mathbf{R}_i^*\right)^{\mathrm{T}} + \mathbf{P}_N^* \cdot \mathbf{H}_N \cdot \left(\mathbf{R}_N^*\right)^{\mathrm{T}} =$$
$$= \sum_{i=1}^{N} z_i^* = z^* \;\; \forall \, \mathbf{R}_i \in \mathcal{R} \;\; \text{for} \;\; i = \overline{1, \, N} \tag{74}$$

hold as well. The assertion of Theorem 1 for bimatrix staircase-function game (49) can be re-written as

$$K\big(x\left(t\right), \, y\left(t\right)\big) = \sum_{i=1}^{N} k_{imq} =$$
$$= \sum_{i=1}^{N-1} \int\limits_{\left[\tau^{(i-1)};\, \tau^{(i)}\right)} f\left(a_i^{(m-1)}, \, b_i^{(q-1)}, \, t\right) d\mu\left(t\right) +$$

$$+ \int\limits_{\left[\tau^{(N-1)};\ \tau^{(N)}\right]} f\left(a_N^{(m-1)},\ b_N^{(q-1)},\ t\right) d\mu\,(t) \tag{75}$$

and

$$H\big(x\,(t)\,,\ y\,(t)\big) = \sum_{i=1}^{N} h_{imq} =$$
$$= \sum_{i=1}^{N-1} \int\limits_{\left[\tau^{(i-1)};\ \tau^{(i)}\right)} g\left(a_i^{(m-1)},\ b_i^{(q-1)},\ t\right) d\mu\,(t) +$$
$$+ \int\limits_{\left[\tau^{(N-1)};\ \tau^{(N)}\right]} g\left(a_N^{(m-1)},\ b_N^{(q-1)},\ t\right) d\mu\,(t). \tag{76}$$

Therefore, inequalities (73) and (74) along with using the payoff decomposition by (75) and (76) allow to conclude that the stack of successive equilibria (64) is an equilibrium situation in game (49).

As (66) holds, then

$$\sum_{i=1}^{N} \max_{\left\{\mathbf{P}_i^*,\ \mathbf{R}_i^*\right\}} \left\{\mathbf{P}_i^* \cdot \mathbf{K}_i \cdot \left(\mathbf{R}_i^*\right)^{\mathrm{T}} + \mathbf{P}_i^* \cdot \mathbf{H}_i \cdot \left(\mathbf{R}_i^*\right)^{\mathrm{T}}\right\} =$$
$$= \sum_{i=1}^{N} \left(\mathbf{P}_i^{**} \cdot \mathbf{K}_i \cdot \left(\mathbf{R}_i^{**}\right)^{\mathrm{T}} + \mathbf{P}_i^{**} \cdot \mathbf{H}_i \cdot \left(\mathbf{R}_i^{**}\right)^{\mathrm{T}}\right) =$$
$$= \sum_{i=1}^{N} \mathbf{P}_i^{**} \cdot \mathbf{K}_i \cdot \left(\mathbf{R}_i^{**}\right)^{\mathrm{T}} + \sum_{i=1}^{N} \mathbf{P}_i^{**} \cdot \mathbf{H}_i \cdot \left(\mathbf{R}_i^{**}\right)^{\mathrm{T}} =$$
$$= \sum_{i=1}^{N} v_i^{**} + \sum_{i=1}^{N} z_i^{**} = v^{**} + z^{**} = s^{**}, \tag{77}$$

i. e. the maximum of the players' payoffs sum is (68) reached at stack (67). □

Consider now the case when the bimatrix staircase-function game is played through a lesser number of time units. Thus, instead of time-interval breaking (18), the game is played by a narrower time-interval breaking

$$\Theta_* = \left\{t_1 \leqslant \tau_1 = \tau^{(n)} < \tau^{(n+1)} < \tau^{(n+2)} < \ldots < \tau^{(U-1)} < \tau^{(U)} = \tau_2 \leqslant t_2\right\}, \tag{78}$$

where

$$n \in \left\{\overline{0,\ N-1}\right\},\ \ U \in \left\{\overline{1,\ N}\right\},\ \ n < U, \tag{79}$$

and $\left\{\tau^{(i)}\right\}_{i=n+1}^{U-1}$ are time points at which the staircase-function strategy can change its value. So, $\Theta_* \subset \Theta$ in terms of the interval breaking.

Theorem 3. *In a bimatrix staircase-function game*

$$\Big\langle \{X(\Theta_*,\ A),\ Y(\Theta_*,\ B)\},\ \big\{K\big(x(t),\ y(t)\big),\ H\big(x(t),\ y(t)\big)\big\}\Big\rangle \tag{80}$$

by

$$X(\Theta_*,\ A) = \left\{x(t) \in X(\Theta_*):\ x(t) \in \left\{a^{(m-1)}\right\}_{m=1}^{M+1}\right\} \subset \\ \subset X(\Theta_*) \subset X([\tau_1;\ \tau_2]) \tag{81}$$

and

$$Y(\Theta_*,\ B) = \left\{y(t) \in Y(\Theta_*):\ y(t) \in \left\{b^{(q-1)}\right\}_{q=1}^{Q+1}\right\} \subset \\ \subset Y(\Theta_*) \subset Y([\tau_1;\ \tau_2]) \tag{82}$$

and a time-interval breaking (78) for (79), an equilibrium situation is a stack

$$\left\{\{\mathbf{P}_i^*,\ \mathbf{R}_i^*\}\right\}_{i=n+1}^{U} = \left\{\left\{\left[p_i^{(m)*}\right]_{1\times(M+1)},\ \left[r_i^{(q)*}\right]_{1\times(Q+1)}\right\}\right\}_{i=n+1}^{U} \tag{83}$$

of $U-n$ *successive equilibria (63) of bimatrix game (50) for* $i=\overline{n+1,\ U}$. *If multiple equilibria exist (at one or more time units) and the maximum of the players' payoffs sum (65) is reached at* $\mathbf{P}_i^* = \mathbf{P}_i^{**}$ *and* $\mathbf{R}_i^* = \mathbf{R}_i^{**}$, *i. e. (66) holds, then the maximum of the players' payoffs sum in an equilibrium stack of bimatrix staircase-function game (80) is reached at stack*

$$\left\{\{\mathbf{P}_i^{**},\ \mathbf{R}_i^{**}\}\right\}_{i=n+1}^{U} \tag{84}$$

and this maximum is

$$s^{**(\Theta_*)} = \sum_{i=n+1}^{U}\Big(\mathbf{P}_i^{**}\cdot\mathbf{K}_i\cdot(\mathbf{R}_i^{**})^{\mathrm{T}} + \mathbf{P}_i^{**}\cdot\mathbf{H}_i\cdot(\mathbf{R}_i^{**})^{\mathrm{T}}\Big). \tag{85}$$

Proof. As inequalities (69) — (72) hold $\forall\, i=\overline{1,\ N}$, they hold $\forall\, i=\overline{n+1,\ U}$. For time-interval breaking (78), time interval $[\tau_1;\ \tau_2]$ can be re-written as

$$[\tau_1;\ \tau_2] = \left\{\bigcup_{i=n+1}^{U-1}\left[\tau^{(i-1)};\ \tau^{(i)}\right)\right\} \cup \left[\tau^{(U-1)};\ \tau^{(U)}\right]. \tag{86}$$

So, owing to Theorem 1,

$$K\big(x(t),\ y(t)\big) = \sum_{i=n+1}^{U} k_{imq} = \\ = \sum_{i=n+1}^{U-1}\ \int\limits_{\left[\tau^{(i-1)};\ \tau^{(i)}\right)} f\left(a_i^{(m-1)},\ b_i^{(q-1)},\ t\right) d\mu(t) +$$

$$+ \int\limits_{\left[\tau^{(U-1)};\ \tau^{(U)}\right]} f\left(a_U^{(m-1)},\ b_U^{(q-1)},\ t\right) d\mu\left(t\right) \tag{87}$$

and

$$H\bigl(x\left(t\right),\ y\left(t\right)\bigr) = \sum_{i=n+1}^{U} h_{imq} =$$

$$= \sum_{i=n+1}^{U-1} \int\limits_{\left[\tau^{(i-1)};\ \tau^{(i)}\right)} g\left(a_i^{(m-1)},\ b_i^{(q-1)},\ t\right) d\mu\left(t\right) +$$

$$+ \int\limits_{\left[\tau^{(U-1)};\ \tau^{(U)}\right]} g\left(a_U^{(m-1)},\ b_U^{(q-1)},\ t\right) d\mu\left(t\right). \tag{88}$$

So, inequalities

$$\sum_{i=n+1}^{U-1} \mathbf{P}_i \cdot \mathbf{K}_i \cdot \left(\mathbf{R}_i^*\right)^{\mathrm{T}} + \mathbf{P}_U \cdot \mathbf{K}_U \cdot \left(\mathbf{R}_U^*\right)^{\mathrm{T}} =$$

$$= \sum_{i=n+1}^{U-1} \sum_{m=1}^{M+1} \sum_{q=1}^{Q+1} k_{imq} p_i^{(m)} r_i^{(q)*} + \sum_{m=1}^{M+1} \sum_{q=1}^{Q+1} k_{Umq} p_U^{(m)} r_U^{(q)*} =$$

$$= \sum_{i=n+1}^{U-1} \left(\sum_{m=1}^{M+1} \sum_{q=1}^{Q+1} p_i^{(m)} r_i^{(q)*} \int\limits_{\left[\tau^{(i-1)};\ \tau^{(i)}\right)} f\left(a_i^{(m-1)},\ b_i^{(q-1)},\ t\right) d\mu\left(t\right) \right) +$$

$$+ \sum_{m=1}^{M+1} \sum_{q=1}^{Q+1} p_U^{(m)} r_U^{(q)*} \int\limits_{\left[\tau^{(U-1)};\ \tau^{(U)}\right]} f\left(a_U^{(m-1)},\ b_U^{(q-1)},\ t\right) d\mu\left(t\right) \leqslant$$

$$\leqslant \sum_{i=n+1}^{U-1} \left(\sum_{m=1}^{M+1} \sum_{q=1}^{Q+1} p_i^{(m)*} r_i^{(q)*} \int\limits_{\left[\tau^{(i-1)};\ \tau^{(i)}\right)} f\left(a_i^{(m-1)},\ b_i^{(q-1)},\ t\right) d\mu\left(t\right) \right) +$$

$$+ \sum_{m=1}^{M+1} \sum_{q=1}^{Q+1} p_U^{(m)*} r_U^{(q)*} \int\limits_{\left[\tau^{(U-1)};\ \tau^{(U)}\right]} f\left(a_U^{(m-1)},\ b_U^{(q-1)},\ t\right) d\mu\left(t\right) =$$

$$= \sum_{i=n+1}^{U-1} \sum_{m=1}^{M+1} \sum_{q=1}^{Q+1} k_{imq} p_i^{(m)*} r_i^{(q)*} + \sum_{m=1}^{M+1} \sum_{q=1}^{Q+1} k_{Umq} p_U^{(m)*} r_U^{(q)*} =$$

$$= \sum_{i=n+1}^{U-1} \mathbf{P}_i^* \cdot \mathbf{K}_i \cdot \left(\mathbf{R}_i^*\right)^{\mathrm{T}} + \mathbf{P}_U^* \cdot \mathbf{K}_U \cdot \left(\mathbf{R}_U^*\right)^{\mathrm{T}} =$$

$$= \sum_{i=n+1}^{U} v_i^{*(\Theta_*)} = v^{*(\Theta_*)} \;\; \forall\, \mathbf{P}_i \in \mathcal{P} \;\text{ for }\; i = \overline{n+1,\, U} \tag{89}$$

and

$$\sum_{i=n+1}^{U-1} \mathbf{P}_i^* \cdot \mathbf{H}_i \cdot \mathbf{R}_i^{\mathrm{T}} + \mathbf{P}_U^* \cdot \mathbf{H}_U \cdot \mathbf{R}_U^{\mathrm{T}} =$$

$$= \sum_{i=n+1}^{U-1} \sum_{m=1}^{M+1} \sum_{q=1}^{Q+1} h_{imq} p_i^{(m)*} r_i^{(q)} + \sum_{m=1}^{M+1} \sum_{q=1}^{Q+1} h_{Umq} p_U^{(m)*} r_U^{(q)} =$$

$$= \sum_{i=n+1}^{U-1} \left(\sum_{m=1}^{M+1} \sum_{q=1}^{Q+1} p_i^{(m)*} r_i^{(q)} \int\limits_{\left[\tau^{(i-1)};\, \tau^{(i)}\right)} g\left(a_i^{(m-1)},\, b_i^{(q-1)},\, t\right) d\mu\left(t\right) \right) +$$

$$+ \sum_{m=1}^{M+1} \sum_{q=1}^{Q+1} p_U^{(m)*} r_U^{(q)} \int\limits_{\left[\tau^{(U-1)};\, \tau^{(U)}\right]} g\left(a_U^{(m-1)},\, b_U^{(q-1)},\, t\right) d\mu\left(t\right) \leqslant$$

$$\leqslant \sum_{i=n+1}^{U-1} \left(\sum_{m=1}^{M+1} \sum_{q=1}^{Q+1} p_i^{(m)*} r_i^{(q)*} \int\limits_{\left[\tau^{(i-1)};\, \tau^{(i)}\right)} g\left(a_i^{(m-1)},\, b_i^{(q-1)},\, t\right) d\mu\left(t\right) \right) +$$

$$+ \sum_{m=1}^{M+1} \sum_{q=1}^{Q+1} p_U^{(m)*} r_U^{(q)*} \int\limits_{\left[\tau^{(U-1)};\, \tau^{(U)}\right]} g\left(a_U^{(m-1)},\, b_U^{(q-1)},\, t\right) d\mu\left(t\right) =$$

$$= \sum_{i=n+1}^{U-1} \sum_{m=1}^{M+1} \sum_{q=1}^{Q+1} h_{imq} p_i^{(m)*} r_i^{(q)*} + \sum_{m=1}^{M+1} \sum_{q=1}^{Q+1} h_{Umq} p_U^{(m)*} r_U^{(q)*} =$$

$$= \sum_{i=n+1}^{U-1} \mathbf{P}_i^* \cdot \mathbf{H}_i \cdot \left(\mathbf{R}_i^*\right)^{\mathrm{T}} + \mathbf{P}_U^* \cdot \mathbf{H}_U \cdot \left(\mathbf{R}_U^*\right)^{\mathrm{T}} =$$

$$= \sum_{i=n+1}^{U} z_i^{*(\Theta_*)} = z^{*(\Theta_*)} \;\; \forall\, \mathbf{R}_i \in \mathcal{R} \;\text{ for }\; i = \overline{n+1,\, U} \tag{90}$$

hold. Therefore, inequalities (89) and (90) along with using the payoff decomposition by (87) and (88) allow to conclude that the stack of successive equilibria (83) is an equilibrium situation in bimatrix staircase-function game (80).

As (66) holds, then

$$\sum_{i=n+1}^{U} \max_{\left\{\mathbf{P}_i^*,\, \mathbf{R}_i^*\right\}} \left\{ \mathbf{P}_i^* \cdot \mathbf{K}_i \cdot \left(\mathbf{R}_i^*\right)^{\mathrm{T}} + \mathbf{P}_i^* \cdot \mathbf{H}_i \cdot \left(\mathbf{R}_i^*\right)^{\mathrm{T}} \right\} =$$

$$= \sum_{i=n+1}^{U} \left(\mathbf{P}_i^{**} \cdot \mathbf{K}_i \cdot \left(\mathbf{R}_i^{**}\right)^{\mathrm{T}} + \mathbf{P}_i^{**} \cdot \mathbf{H}_i \cdot \left(\mathbf{R}_i^{**}\right)^{\mathrm{T}} \right) =$$

$$= \sum_{i=n+1}^{U} \mathbf{P}_i^{**} \cdot \mathbf{K}_i \cdot \left(\mathbf{R}_i^{**}\right)^{\mathrm{T}} + \sum_{i=n+1}^{U} \mathbf{P}_i^{**} \cdot \mathbf{H}_i \cdot \left(\mathbf{R}_i^{**}\right)^{\mathrm{T}} =$$

$$= \sum_{i=n+1}^{U} v_i^{**(\Theta_*)} + \sum_{i=n+1}^{U} z_i^{**(\Theta_*)} = v^{**(\Theta_*)} + z^{**(\Theta_*)} = s^{**(\Theta_*)}, \tag{91}$$

i. e. the maximum of the players' payoffs sum is (85) reached at stack (84). □

It is quite obvious that

$$\left\{ \left\{ \mathbf{P}_i^*, \mathbf{R}_i^* \right\} \right\}_{i=n+1}^{U} \subset \left\{ \left\{ \mathbf{P}_i^*, \mathbf{R}_i^* \right\} \right\}_{i=1}^{N}. \tag{92}$$

So, Theorem 3 implies that the time-unit shifting does not change the structure and number of equilibria in a bimatrix staircase-function game, nor does it change the structure of the best equilibrium stack determined by the maximum of the players' payoffs sum. In fact, game (80) is a subgame of bimatrix staircase-function game (49). An equilibrium solution of the subgame can be easily taken from the respective equilibrium solution of ("wider") game (49).

7. Time-unit shifting in discrete-time staircase-function 2-person games

See whether the inference above is valid for discrete-time staircase-function 2-person game (33), which, generally speaking, is played within uncountably infinite sets of players' staircase-function strategies. Denote by

$$p_i\left(\alpha_i\right) \tag{93}$$

and

$$r_i\left(\beta_i\right) \tag{94}$$

the mixed strategies of the first and second players, respectively, in (subinterval) infinite 2-person game (31), where

$$P = \left\{ p_i\left(\alpha_i\right) \in \mathbb{L}_2\left[a_{\min};\ a_{\max}\right] : p_i\left(\alpha_i\right) \geqslant 0, \int\limits_{[a_{\min};\ a_{\max}]} p_i\left(\alpha_i\right) d\mu\left(\alpha_i\right) = 1 \right\} \tag{95}$$

and

$$R = \left\{ r_i\left(\beta_i\right) \in \mathbb{L}_2\left[b_{\min};\ b_{\max}\right] : r_i\left(\beta_i\right) \geqslant 0, \int\limits_{[b_{\min};\ b_{\max}]} r_i\left(\beta_i\right) d\mu\left(\beta_i\right) = 1 \right\} \tag{96}$$

are the respective sets of mixed strategies of the players. So, $p_i\left(\alpha_i\right) \in P$, $r_i\left(\beta_i\right) \in R$, and

$$\left\{ p_i\left(\alpha_i\right),\ r_i\left(\beta_i\right) \right\} \tag{97}$$

is a situation in this game.

Definition 4. A stack
$$\left\{\left\{p_i\left(\alpha_i\right),\ r_i\left(\beta_i\right)\right\}\right\}_{i=1}^{N} \tag{98}$$
of successive situations in (ordinary) 2-person games (31) is called a (mixed-strategy) situation in discrete-time staircase-function 2-person game (33). Stacks $\left\{p_i\left(\alpha_i\right)\right\}_{i=1}^{N}$ and $\left\{r_i\left(\beta_i\right)\right\}_{i=1}^{N}$ are the respective mixed strategies of the first and second players in this game.

Just like in the case of a finite discrete-time staircase-function 2-person game, it is clear that an equilibrium situation in a discrete-time staircase-function 2-person game is to be sought among stacks (98). The respective assertions in [21], however, concern only the case of equilibrium situations of pure strategies.

Theorem 4. *If $p_i^*\left(\alpha_i\right) \in P$, $r_i^*\left(\beta_i\right) \in R$, and*
$$\left\{p_i^*\left(\alpha_i\right),\ r_i^*\left(\beta_i\right)\right\} \tag{99}$$
is an equilibrium situation in 2-person game (31) for $i = \overline{1,\ N}$, then a stack
$$\left\{\left\{p_i^*\left(\alpha_i\right),\ r_i^*\left(\beta_i\right)\right\}\right\}_{i=1}^{N} \tag{100}$$
of such successive equilibria is an equilibrium situation in discrete-time staircase-function 2-person game (33).

Proof. As (99) is an equilibrium situation, and all these subinterval equilibria exist, then inequalities
$$\int\limits_{\left[b_{\min};\ b_{\max}\right]} r_i^*\left(\beta_i\right)\left(\int\limits_{\left[a_{\min};\ a_{\max}\right]} p_i\left(\alpha_i\right) K_i\left(\alpha_i,\ \beta_i\right) d\mu\left(\alpha_i\right)\right) d\mu\left(\beta_i\right) =$$
$$= \int\limits_{\left[b_{\min};\ b_{\max}\right]} r_i^*\left(\beta_i\right)\left(\int\limits_{\left[a_{\min};\ a_{\max}\right]} p_i\left(\alpha_i\right) \int\limits_{\left[\tau^{(i-1)};\ \tau^{(i)}\right)} f\left(\alpha_i,\ \beta_i,\ t\right) d\mu\left(t\right) d\mu\left(\alpha_i\right)\right) d\mu\left(\beta_i\right) \leqslant$$
$$\leqslant \int\limits_{\left[b_{\min};\ b_{\max}\right]} r_i^*\left(\beta_i\right)\left(\int\limits_{\left[a_{\min};\ a_{\max}\right]} p_i^*\left(\alpha_i\right) \int\limits_{\left[\tau^{(i-1)};\ \tau^{(i)}\right)} f\left(\alpha_i,\ \beta_i,\ t\right) d\mu\left(t\right) d\mu\left(\alpha_i\right)\right) d\mu\left(\beta_i\right) =$$
$$= \int\limits_{\left[b_{\min};\ b_{\max}\right]} r_i^*\left(\beta_i\right)\left(\int\limits_{\left[a_{\min};\ a_{\max}\right]} p_i^*\left(\alpha_i\right) K_i\left(\alpha_i,\ \beta_i\right) d\mu\left(\alpha_i\right)\right) d\mu\left(\beta_i\right)$$
$$\forall\, p_i\left(\alpha_i\right) \in P \ \text{ for }\ i = \overline{1,\ N-1}, \tag{101}$$

$$\int\limits_{[b_{\min};\, b_{\max}]} r_N^*(\beta_N) \left(\int\limits_{[a_{\min};\, a_{\max}]} p_N(\alpha_N)\, K_N(\alpha_N,\ \beta_N)\, d\mu(\alpha_N) \right) d\mu(\beta_N) =$$

$$= \int\limits_{[b_{\min};\, b_{\max}]} r_N^*(\beta_N)$$

$$\left(\int\limits_{[a_{\min};\, a_{\max}]} p_N(\alpha_N) \int\limits_{\left[\tau^{(N-1)};\, \tau^{(N)}\right]} f(\alpha_N,\ \beta_N,\ t)\, d\mu(t) d\mu(\alpha_N) \right) d\mu(\beta_N) \leqslant$$

$$\leqslant \int\limits_{[b_{\min};\, b_{\max}]} r_N^*(\beta_N)$$

$$\left(\int\limits_{[a_{\min};\, a_{\max}]} p_N^*(\alpha_N) \int\limits_{\left[\tau^{(N-1)};\, \tau^{(N)}\right]} f(\alpha_N,\ \beta_N,\ t)\, d\mu(t) d\mu(\alpha_N) \right) d\mu(\beta_N) =$$

$$= \int\limits_{[b_{\min};\, b_{\max}]} r_N^*(\beta_N) \left(\int\limits_{[a_{\min};\, a_{\max}]} p_N^*(\alpha_N)\, K_N(\alpha_N,\ \beta_N)\, d\mu(\alpha_N) \right) d\mu(\beta_N)$$

$$\forall\, p_N(\alpha_N) \in P \qquad (102)$$

and inequalities

$$\int\limits_{[b_{\min};\, b_{\max}]} r_i(\beta_i) \left(\int\limits_{[a_{\min};\, a_{\max}]} p_i^*(\alpha_i)\, H_i(\alpha_i,\ \beta_i)\, d\mu(\alpha_i) \right) d\mu(\beta_i) =$$

$$= \int\limits_{[b_{\min};\, b_{\max}]} r_i(\beta_i) \left(\int\limits_{[a_{\min};\, a_{\max}]} p_i^*(\alpha_i) \int\limits_{\left[\tau^{(i-1)};\, \tau^{(i)}\right)} g(\alpha_i,\ \beta_i,\ t)\, d\mu(t) d\mu(\alpha_i) \right) d\mu(\beta_i) \leqslant$$

$$\leqslant \int\limits_{[b_{\min};\, b_{\max}]} r_i^*(\beta_i) \left(\int\limits_{[a_{\min};\, a_{\max}]} p_i^*(\alpha_i) \int\limits_{\left[\tau^{(i-1)};\, \tau^{(i)}\right)} g(\alpha_i,\ \beta_i,\ t)\, d\mu(t) d\mu(\alpha_i) \right) d\mu(\beta_i) =$$

$$= \int\limits_{[b_{\min};\, b_{\max}]} r_i^*(\beta_i) \left(\int\limits_{[a_{\min};\, a_{\max}]} p_i^*(\alpha_i)\, H_i(\alpha_i,\ \beta_i)\, d\mu(\alpha_i) \right) d\mu(\beta_i)$$

$$\forall\, r_i(\beta_i) \in R \ \text{ for } \ i = \overline{1,\ N-1}, \qquad (103)$$

$$\int\limits_{[b_{\min};\, b_{\max}]} r_N(\beta_N) \left(\int\limits_{[a_{\min};\, a_{\max}]} p_N^*(\alpha_N)\, H_N(\alpha_N,\ \beta_N)\, d\mu(\alpha_N) \right) d\mu(\beta_N) =$$

$$= \int\limits_{[b_{\min};\, b_{\max}]} r_N(\beta_N)$$

$$\left(\int\limits_{[a_{\min};\, a_{\max}]} p_N^*(\alpha_N) \int\limits_{\left[\tau^{(N-1)};\, \tau^{(N)}\right]} g(\alpha_N,\ \beta_N,\ t)\, d\mu(t) d\mu(\alpha_N) \right) d\mu(\beta_N) \leqslant$$

$$\leqslant \int\limits_{[b_{\min};\, b_{\max}]} r_N^*(\beta_N)$$

$$\left(\int\limits_{[a_{\min};\, a_{\max}]} p_N^*(\alpha_N) \int\limits_{\left[\tau^{(N-1)};\, \tau^{(N)}\right]} g(\alpha_N,\ \beta_N,\ t)\, d\mu(t) d\mu(\alpha_N) \right) d\mu(\beta_N) =$$

$$= \int\limits_{[b_{\min};\, b_{\max}]} r_N^*(\beta_N) \left(\int\limits_{[a_{\min};\, a_{\max}]} p_N^*(\alpha_N)\, H_N(\alpha_N,\ \beta_N)\, d\mu(\alpha_N) \right) d\mu(\beta_N)$$

$$\forall\ r_N(\beta_N) \in R \quad (104)$$

hold. So, inequalities

$$\sum_{i=1}^{N} \int\limits_{[b_{\min};\, b_{\max}]} r_i^*(\beta_i) \left(\int\limits_{[a_{\min};\, a_{\max}]} p_i(\alpha_i)\, K_i(\alpha_i,\ \beta_i)\, d\mu(\alpha_i) \right) d\mu(\beta_i) =$$

$$= \sum_{i=1}^{N-1} \int\limits_{[b_{\min};\, b_{\max}]} r_i^*(\beta_i)$$

$$\left(\int\limits_{[a_{\min};\, a_{\max}]} p_i(\alpha_i) \int\limits_{\left[\tau^{(i-1)};\, \tau^{(i)}\right)} f(\alpha_i,\ \beta_i,\ t)\, d\mu(t) d\mu(\alpha_i) \right) d\mu(\beta_i) +$$

$$+ \int\limits_{[b_{\min};\, b_{\max}]} r_N^*(\beta_N)$$

$$\left(\int\limits_{[a_{\min};\, a_{\max}]} p_N(\alpha_N) \int\limits_{\left[\tau^{(N-1)};\, \tau^{(N)}\right]} f(\alpha_N,\ \beta_N,\ t)\, d\mu(t) d\mu(\alpha_N) \right) d\mu(\beta_N) \leqslant$$

$$\leqslant \sum_{i=1}^{N-1} \int\limits_{[b_{\min};\, b_{\max}]} r_i^*(\beta_i)$$

$$\left(\int\limits_{[a_{\min};\, a_{\max}]} p_i^*(\alpha_i) \int\limits_{\left[\tau^{(i-1)};\, \tau^{(i)}\right)} f(\alpha_i,\ \beta_i,\ t)\, d\mu(t) d\mu(\alpha_i) \right) d\mu(\beta_i) +$$

$$+ \int\limits_{[b_{\min};\, b_{\max}]} r_N^*(\beta_N)$$

$$\left(\int\limits_{[a_{\min};\, a_{\max}]} p_N^*(\alpha_N) \int\limits_{\left[\tau^{(N-1)};\, \tau^{(N)}\right]} f(\alpha_N,\ \beta_N,\ t)\, d\mu(t) d\mu(\alpha_N) \right) d\mu(\beta_N) =$$

$$= \sum_{i=1}^{N} \int\limits_{[b_{\min};\, b_{\max}]} r_i^*(\beta_i) \left(\int\limits_{[a_{\min};\, a_{\max}]} p_i^*(\alpha_i)\, K_i(\alpha_i,\ \beta_i)\, d\mu(\alpha_i) \right) d\mu(\beta_i) =$$

$$= \sum_{i=1}^{N} v_i^* = v^* \ \ \forall\, p_i(\alpha_i) \in P \ \text{ for } \ i = \overline{1,\ N} \quad (105)$$

and

$$\sum_{i=1}^{N} \int\limits_{[b_{\min};\, b_{\max}]} r_i(\beta_i) \left(\int\limits_{[a_{\min};\, a_{\max}]} p_i^*(\alpha_i)\, H_i(\alpha_i,\ \beta_i)\, d\mu(\alpha_i) \right) d\mu(\beta_i) =$$

$$= \sum_{i=1}^{N-1} \int\limits_{[b_{\min};\, b_{\max}]} r_i(\beta_i)$$

$$\left(\int\limits_{[a_{\min};\, a_{\max}]} p_i^*(\alpha_i) \int\limits_{\left[\tau^{(i-1)};\, \tau^{(i)}\right)} g(\alpha_i,\ \beta_i,\ t)\, d\mu(t) d\mu(\alpha_i) \right) d\mu(\beta_i) +$$

$$+ \int\limits_{[b_{\min};\, b_{\max}]} r_N(\beta_N)$$

$$\left(\int\limits_{[a_{\min};\, a_{\max}]} p_N^*(\alpha_N) \int\limits_{\left[\tau^{(N-1)};\, \tau^{(N)}\right]} g(\alpha_N,\ \beta_N,\ t)\, d\mu(t) d\mu(\alpha_N) \right) d\mu(\beta_N) \leqslant$$

$$\leqslant \sum_{i=1}^{N-1} \int\limits_{[b_{\min};\, b_{\max}]} r_i^*(\beta_i)$$

$$\left(\int\limits_{[a_{\min};\ a_{\max}]} p_i^*(\alpha_i) \int\limits_{\left[\tau^{(i-1)};\ \tau^{(i)}\right)} g(\alpha_i,\ \beta_i,\ t)\, d\mu(t) d\mu(\alpha_i) \right) d\mu(\beta_i) +$$

$$+ \int\limits_{[b_{\min};\ b_{\max}]} r_N^*(\beta_N)$$

$$\left(\int\limits_{[a_{\min};\ a_{\max}]} p_N^*(\alpha_N) \int\limits_{\left[\tau^{(N-1)};\ \tau^{(N)}\right]} g(\alpha_N,\ \beta_N,\ t)\, d\mu(t) d\mu(\alpha_N) \right) d\mu(\beta_N) =$$

$$= \sum_{i=1}^{N} \int\limits_{[b_{\min};\ b_{\max}]} r_i^*(\beta_i) \left(\int\limits_{[a_{\min};\ a_{\max}]} p_i^*(\alpha_i)\, H_i(\alpha_i,\ \beta_i)\, d\mu(\alpha_i) \right) d\mu(\beta_i) =$$

$$= \sum_{i=1}^{N} z_i^* = z^* \ \ \forall\, r_i(\beta_i) \in R \ \text{ for } \ i = \overline{1,\ N} \tag{106}$$

hold as well. Therefore, inequalities (105) and (106) along with using the payoff decomposition by (36) and (37) allow to conclude that the stack of successive equilibria (100) is an equilibrium situation in game (33). □

Theorem 5. *In a discrete-time staircase-function 2-person game*

$$\left\langle \{X(\Theta_*),\ Y(\Theta_*)\},\ \left\{K\big(x(t),\ y(t)\big),\ H\big(x(t),\ y(t)\big)\right\} \right\rangle \tag{107}$$

by

$$X(\Theta_*) = \Big\{ x(t) \in X([\tau_1;\ \tau_2]) : x(t) = \alpha_i \in [a_{\min};\ a_{\max}] \\ \forall\, t \in \left[\tau^{(i-1)};\ \tau^{(i)}\right) \ \ \textit{for}\ i = \overline{n+1,\ U-1} \ \ \textit{and} \\ x(t) = \alpha_U \in [a_{\min};\ a_{\max}] \ \forall\, t \in \left[\tau^{(U-1)};\ \tau^{(U)}\right] \Big\} \subset X([\tau_1;\ \tau_2]) \tag{108}$$

and

$$Y(\Theta_*) = \Big\{ y(t) \in Y([\tau_1;\ \tau_2]) : y(t) = \beta_i \in [b_{\min};\ b_{\max}] \\ \forall\, t \in \left[\tau^{(i-1)};\ \tau^{(i)}\right) \ \ \textit{for}\ i = \overline{n+1,\ U-1} \ \ \textit{and} \\ y(t) = \beta_U \in [b_{\min};\ b_{\max}] \ \forall\, t \in \left[\tau^{(U-1)};\ \tau^{(U)}\right] \Big\} \subset Y([\tau_1;\ \tau_2]) \tag{109}$$

and a time-interval breaking (78) for (79), an equilibrium situation is a stack

$$\left\{ \{p_i^*(\alpha_i),\ r_i^*(\beta_i)\} \right\}_{i=n+1}^{U} \tag{110}$$

of $U - n$ successive equilibria (99) in 2-person game (31) for $i = \overline{n+1,\ U}$.

Proof. As inequalities (101) — (104) hold $\forall\, i = \overline{1,\ N}$, they hold $\forall\, i = \overline{n+1,\ U}$. For time-interval breaking (78), time interval $[\tau_1;\ \tau_2]$ can be re-written as (86), so, owing to Theorem 1,

$$K\bigl(x\,(t)\,,\ y\,(t)\bigr) = \sum_{i=n+1}^{U} K_i\,(\alpha_i,\ \beta_i) =$$

$$= \sum_{i=n+1}^{U-1} \int\limits_{\left[\tau^{(i-1)};\ \tau^{(i)}\right)} f\,(\alpha_i,\ \beta_i,\ t)\,d\mu\,(t) + \int\limits_{\left[\tau^{(U-1)};\ \tau^{(U)}\right]} f\,(\alpha_U,\ \beta_U,\ t)\,d\mu\,(t) \tag{111}$$

and

$$H\bigl(x\,(t)\,,\ y\,(t)\bigr) = \sum_{i=n+1}^{U} H_i\,(\alpha_i,\ \beta_i) =$$

$$= \sum_{i=n+1}^{U-1} \int\limits_{\left[\tau^{(i-1)};\ \tau^{(i)}\right)} g\,(\alpha_i,\ \beta_i,\ t)\,d\mu\,(t) + \int\limits_{\left[\tau^{(U-1)};\ \tau^{(U)}\right]} g\,(\alpha_U,\ \beta_U,\ t)\,d\mu\,(t). \tag{112}$$

So, inequalities

$$\sum_{i=n+1}^{U} \int\limits_{[b_{\min};\ b_{\max}]} r_i^*\,(\beta_i) \left(\int\limits_{[a_{\min};\ a_{\max}]} p_i\,(\alpha_i)\,K_i\,(\alpha_i,\ \beta_i)\,d\mu\,(\alpha_i) \right) d\mu\,(\beta_i) =$$

$$= \sum_{i=n+1}^{U-1} \int\limits_{[b_{\min};\ b_{\max}]} r_i^*\,(\beta_i)$$

$$\left(\int\limits_{[a_{\min};\ a_{\max}]} p_i\,(\alpha_i) \int\limits_{\left[\tau^{(i-1)};\ \tau^{(i)}\right)} f\,(\alpha_i,\ \beta_i,\ t)\,d\mu\,(t) d\mu\,(\alpha_i) \right) d\mu\,(\beta_i) +$$

$$+ \int\limits_{[b_{\min};\ b_{\max}]} r_U^*\,(\beta_U)$$

$$\left(\int\limits_{[a_{\min};\ a_{\max}]} p_U\,(\alpha_U) \int\limits_{\left[\tau^{(U-1)};\ \tau^{(U)}\right]} f\,(\alpha_U,\ \beta_U,\ t)\,d\mu\,(t) d\mu\,(\alpha_U) \right) d\mu\,(\beta_U) \leqslant$$

$$\leqslant \sum_{i=n+1}^{U-1} \int\limits_{[b_{\min};\ b_{\max}]} r_i^*\,(\beta_i)$$

$$\left(\int\limits_{[a_{\min};\, a_{\max}]} p_i^*(\alpha_i) \int\limits_{\left[\tau^{(i-1)};\, \tau^{(i)}\right)} f(\alpha_i,\ \beta_i,\ t)\, d\mu(t) d\mu(\alpha_i) \right) d\mu(\beta_i) +$$

$$+ \int\limits_{[b_{\min};\, b_{\max}]} r_U^*(\beta_U)$$

$$\left(\int\limits_{[a_{\min};\, a_{\max}]} p_U^*(\alpha_U) \int\limits_{\left[\tau^{(U-1)};\, \tau^{(U)}\right]} f(\alpha_U,\ \beta_U,\ t)\, d\mu(t) d\mu(\alpha_U) \right) d\mu(\beta_U) =$$

$$= \sum_{i=n+1}^{U} \int\limits_{[b_{\min};\, b_{\max}]} r_i^*(\beta_i) \left(\int\limits_{[a_{\min};\, a_{\max}]} p_i^*(\alpha_i)\, K_i(\alpha_i,\ \beta_i)\, d\mu(\alpha_i) \right) d\mu(\beta_i) =$$

$$= \sum_{i=n+1}^{U} v_i^{*(\Theta_*)} = v^{*(\Theta_*)} \ \ \forall\, p_i(\alpha_i) \in P \ \text{ for } \ i = \overline{n+1,\ U} \tag{113}$$

and

$$\sum_{i=n+1}^{U} \int\limits_{[b_{\min};\, b_{\max}]} r_i(\beta_i) \left(\int\limits_{[a_{\min};\, a_{\max}]} p_i^*(\alpha_i)\, H_i(\alpha_i,\ \beta_i)\, d\mu(\alpha_i) \right) d\mu(\beta_i) =$$

$$= \sum_{i=n+1}^{U-1} \int\limits_{[b_{\min};\, b_{\max}]} r_i(\beta_i)$$

$$\left(\int\limits_{[a_{\min};\, a_{\max}]} p_i^*(\alpha_i) \int\limits_{\left[\tau^{(i-1)};\, \tau^{(i)}\right)} g(\alpha_i,\ \beta_i,\ t)\, d\mu(t) d\mu(\alpha_i) \right) d\mu(\beta_i) +$$

$$+ \int\limits_{[b_{\min};\, b_{\max}]} r_U(\beta_U)$$

$$\left(\int\limits_{[a_{\min};\, a_{\max}]} p_U^*(\alpha_U) \int\limits_{\left[\tau^{(U-1)};\, \tau^{(U)}\right]} g(\alpha_U,\ \beta_U,\ t)\, d\mu(t) d\mu(\alpha_U) \right) d\mu(\beta_U) \leqslant$$

$$\leqslant \sum_{i=n+1}^{U-1} \int\limits_{[b_{\min};\, b_{\max}]} r_i^*(\beta_i)$$

$$\left(\int\limits_{[a_{\min};\, a_{\max}]} p_i^*(\alpha_i) \int\limits_{\left[\tau^{(i-1)};\, \tau^{(i)}\right)} g(\alpha_i,\ \beta_i,\ t)\, d\mu(t) d\mu(\alpha_i) \right) d\mu(\beta_i) +$$

$$+ \int\limits_{[b_{\min};\, b_{\max}]} r_U^*(\beta_U)$$

$$\left(\int\limits_{[a_{\min};\, a_{\max}]} p_U^*(\alpha_U) \int\limits_{\left[\tau^{(U-1)};\, \tau^{(U)}\right]} g(\alpha_U,\ \beta_U,\ t)\, d\mu(t) d\mu(\alpha_U) \right) d\mu(\beta_U) =$$

$$= \sum_{i=n+1}^{U} \int\limits_{[b_{\min};\, b_{\max}]} r_i^*(\beta_i) \left(\int\limits_{[a_{\min};\, a_{\max}]} p_i^*(\alpha_i)\, H_i(\alpha_i,\ \beta_i)\, d\mu(\alpha_i) \right) d\mu(\beta_i) =$$

$$= \sum_{i=n+1}^{U} z_i^{*(\Theta_*)} = z^{*(\Theta_*)}\ \ \forall\, r_i(\beta_i) \in R \text{ for } i = \overline{n+1,\ U} \tag{114}$$

hold. Therefore, inequalities (113) and (114) along with using the payoff decomposition by (111) and (112) allow to conclude that the stack of successive equilibria (110) is an equilibrium situation in discrete-time staircase-function 2-person game (107). □

The assertion about the maximum of the players' payoffs sum in an equilibrium stack in game (33) could have been proved in a way similar to that in the proof of Theorem 2. However, this question has far less practical sense compared to that for bimatrix staircase-function games (which always have equilibrium solutions). This is so because discrete-time staircase-function 2-person games are played, generally speaking, within uncountably infinite sets of players' staircase-function strategies (93) and (94), and even the latter may have pretty tricky structure, let alone a subinterval game may have no equilibrium at all.

Similarly to game (80) being a subgame of bimatrix staircase-function game (49), and an inclusion by (92), it is quite obvious that game (107) is a subgame of discrete-time staircase-function 2-person game (33) and

$$\left\{ \{ p_i^*(\alpha_i),\ r_i^*(\beta_i) \} \right\}_{i=n+1}^{U} \subset \left\{ \{ p_i^*(\alpha_i),\ r_i^*(\beta_i) \} \right\}_{i=1}^{N}. \tag{115}$$

Theorem 5 being a generalization of Theorem 3 implies that the time-unit shifting does not change the structure of equilibria in a discrete-time staircase-function 2-person game. If an equilibrium solution of ("wider") game (33) exists, the respective equilibrium solution of the ("narrower") subgame can be taken from it.

8. An example of the bimatrix staircase-function game

Consider an example of the bimatrix staircase-function game, in which functions (8) and (11) in integral functionals (7) and (10) are

$$f\big(x(t),\ y(t),\ t\big) = \\ = \sin\left(0.05xt - 0.01yt^2 - \frac{\pi}{4}\right) + \cos(0.04xyt)\, e^{1.3\cos(0.01xyt)} \tag{116}$$

and

$$g\big(x(t),\ y(t),\ t\big) = t\sin\left(0.03xyt - \frac{\pi}{5}\right) e^{-2.44\cos\left(0.02xyt + \frac{\pi}{3}\right)}, \tag{117}$$

where the players are forced (somehow) to act within finite subsets of possible values of their pure strategies (45) and (46):

$$A = \left\{a^{(m-1)}\right\}_{m=1}^{M+1} = \left\{a_i^{(m-1)}\right\}_{m=1}^{7} = \{m+1\}_{m=1}^{7} \tag{118}$$

and

$$B = \left\{b^{(q-1)}\right\}_{q=1}^{Q+1} = \left\{b_i^{(q-1)}\right\}_{q=1}^{6} = \{12+2q\}_{q=1}^{6}. \tag{119}$$

The time unit (or the time subinterval length) is 0.1π, i.e. the players may (synchronously, simultaneously) change their pure strategies values only through this time step. The tasks are to solve such bimatrix staircase-function game (80) for time intervals

$$[\tau_1;\ \tau_2] = [0.7\pi;\ 1.3\pi], \tag{120}$$

$$[\tau_1;\ \tau_2] = [1.8\pi;\ 2.5\pi], \tag{121}$$

$$[\tau_1;\ \tau_2] = [2.8\pi;\ 3.6\pi], \tag{122}$$

where

$$X(\Theta_*,\ A) = \left\{x(t) \in X(\Theta_*) :\ x(t) \in \{m+1\}_{m=1}^{7}\right\} \subset \\ \subset X(\Theta_*) \subset X([\tau_1;\ \tau_2]) \tag{123}$$

and

$$Y(\Theta_*,\ B) = \left\{y(t) \in Y(\Theta_*) :\ y(t) \in \{12+2q\}_{q=1}^{6}\right\} \subset \\ \subset Y(\Theta_*) \subset Y([\tau_1;\ \tau_2]) \tag{124}$$

by

$$\tau^{(i)} - \tau^{(i-1)} = 0.1\pi \ \text{ for } \ i = \overline{n+1,\ U} \tag{125}$$

in time-interval breaking (78).

According with Theorem 3, it is sufficient to find an equilibrium stack of the bimatrix staircase-function game with (116) — (119) played during time interval

$$[t_1;\ t_2] = [0.7\pi;\ 3.6\pi] \tag{126}$$

where the respective payoff functionals

$$K\big(x(t),\ y(t)\big) = \int\limits_{[0.7\pi;\ 3.6\pi]} f\big(x(t),\ y(t),\ t\big)d\mu(t) \tag{127}$$

and

$$H\big(x(t),\ y(t)\big) = \int\limits_{[0.7\pi;\ 3.6\pi]} g\big(x(t),\ y(t),\ t\big)d\mu(t), \tag{128}$$

due to there are 29 time units in (126), are transformed into 29 payoff 7×6 matrices of the first player and 29 payoff 7×6 matrices of the second player. So, the "wider" bimatrix staircase-function game is the succession of 29 bimatrix games

$$\left\langle \left\{ \left\{ a_i^{(m-1)} \right\}_{m=1}^{7},\ \left\{ b_i^{(q-1)} \right\}_{q=1}^{6} \right\},\ \{\mathbf{K}_i,\ \mathbf{H}_i\} \right\rangle \text{ for } i = \overline{1,\ 29} \tag{129}$$

with the first player's payoff matrices

$$\mathbf{K}_i = [k_{imq}]_{7\times 6} \tag{130}$$

whose elements are

$$k_{imq} = \int\limits_{\left[\tau^{(i-1)};\ \tau^{(i)}\right)} f\left(a_i^{(m-1)},\ b_i^{(q-1)},\ t\right) d\mu(t) =$$
$$= \int\limits_{[0.6\pi+0.1\pi i;\ 0.7\pi+0.1\pi i)} f(m+1,\ 12+2q,\ t)\, d\mu(t) =$$
$$= \int\limits_{[0.6\pi+0.1\pi i;\ 0.7\pi+0.1\pi i)} \left[\sin\left(0.05\cdot(m+1)\,t - 0.01\cdot(12+2q)\,t^2 - \frac{\pi}{4}\right) + \right.$$
$$\left. + \cos\big(0.04\cdot(m+1)\cdot(12+2q)\,t\big)e^{1.3\cos\left(0.01\cdot(m+1)\cdot(12+2q)t\right)}\right] d\mu(t)$$
$$\text{for } i = \overline{1,\ 28} \tag{131}$$

and

$$k_{29mq} = \int\limits_{\left[\tau^{(28)};\ \tau^{(29)}\right]} f\left(a_{29}^{(m-1)},\ b_{29}^{(q-1)},\ t\right) d\mu(t) =$$
$$= \int\limits_{[3.5\pi;\ 3.6\pi]} f(m+1,\ 12+2q,\ t)\, d\mu(t) =$$

$$= \int\limits_{[3.5\pi;\ 3.6\pi]} \left[\sin\left(0.05 \cdot (m+1)\,t - 0.01 \cdot (12+2q)\,t^2 - \frac{\pi}{4}\right) + \right.$$

$$\left. + \cos\bigl(0.04 \cdot (m+1) \cdot (12+2q)\,t\bigr) e^{1.3 \cos\left(0.01 \cdot (m+1) \cdot (12+2q) t\right)} \right] d\mu\,(t)\,, \tag{132}$$

and with the second player's payoff matrices

$$\mathbf{H}_i = [h_{imq}]_{7 \times 6} \tag{133}$$

whose elements are

$$h_{imq} = \int\limits_{\left[\tau^{(i-1)};\ \tau^{(i)}\right)} g\left(a_i^{(m-1)},\ b_i^{(q-1)},\ t\right) d\mu\,(t) =$$

$$= \int\limits_{[0.6\pi+0.1\pi i;\ 0.7\pi+0.1\pi i)} g\,(m+1,\ 12+2q,\ t)\, d\mu\,(t) =$$

$$= \int\limits_{[0.6\pi+0.1\pi i;\ 0.7\pi+0.1\pi i)} t \sin\left(0.03 \cdot (m+1) \cdot (12+2q)\,t - \frac{\pi}{5}\right)$$

$$e^{-2.44 \cos\left(0.02 \cdot (m+1) \cdot (12+2q) t + \frac{\pi}{3}\right)} d\mu\,(t) \quad \text{for} \quad i = \overline{1,\ 28} \tag{134}$$

and

$$h_{29mq} = \int\limits_{\left[\tau^{(28)};\ \tau^{(29)}\right]} g\left(a_{29}^{(m-1)},\ b_{29}^{(q-1)},\ t\right) d\mu\,(t) =$$

$$= \int\limits_{[3.5\pi;\ 3.6\pi]} g\,(m+1,\ 12+2q,\ t)\, d\mu\,(t) =$$

$$= \int\limits_{[3.5\pi;\ 3.6\pi]} t \sin\left(0.03 \cdot (m+1) \cdot (12+2q)\,t - \frac{\pi}{5}\right)$$

$$e^{-2.44 \cos\left(0.02 \cdot (m+1) \cdot (12+2q) t + \frac{\pi}{3}\right)} d\mu\,(t)\,. \tag{135}$$

In the "wider" bimatrix staircase-function game, each of the players is allowed to change its pure strategy value only at time points

$$\left\{\tau^{(i)}\right\}_{i=1}^{28} = \{0.7\pi + 0.1\pi i\}_{i=1}^{28}\,.$$

Payoff matrix (130) on each subinterval of set

$$\left\{ \{[0.6\pi + 0.1\pi i;\ 0.7\pi + 0.1\pi i)\}_{i=1}^{28}\,,\ [3.5\pi;\ 3.6\pi] \right\} \tag{136}$$

is shown in Figure 4 as a meshed surface, where a close-to-chaotic payoff distribution can be seen. Payoff matrix (133) on each subinterval of set (136) is shown in Figure 5 as a meshed surface also, where a close-to-chaotic payoff distribution is seen

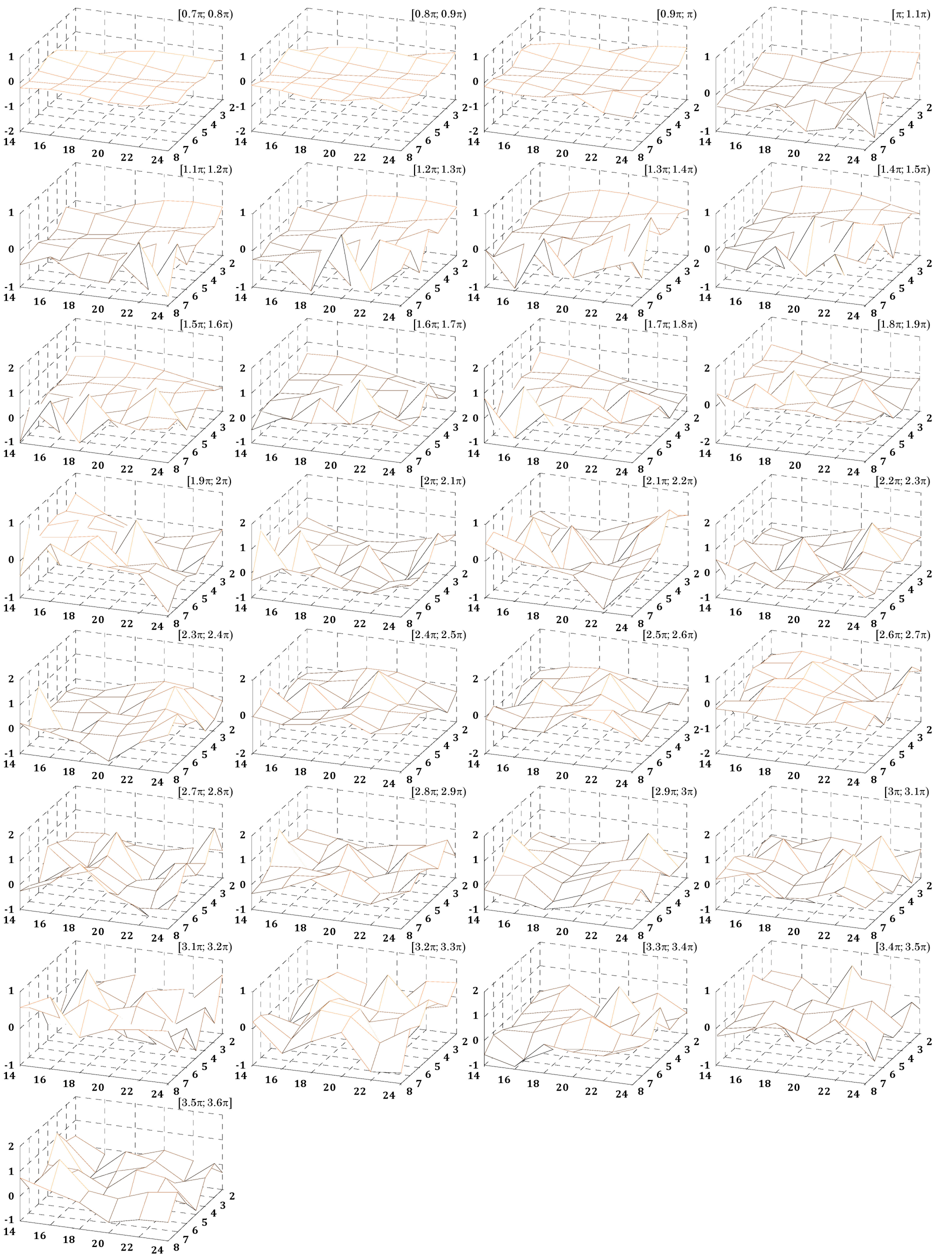

Figure 4: First player's payoffs in matrix (130) as a meshed surface on the 29 subintervals of set (136)

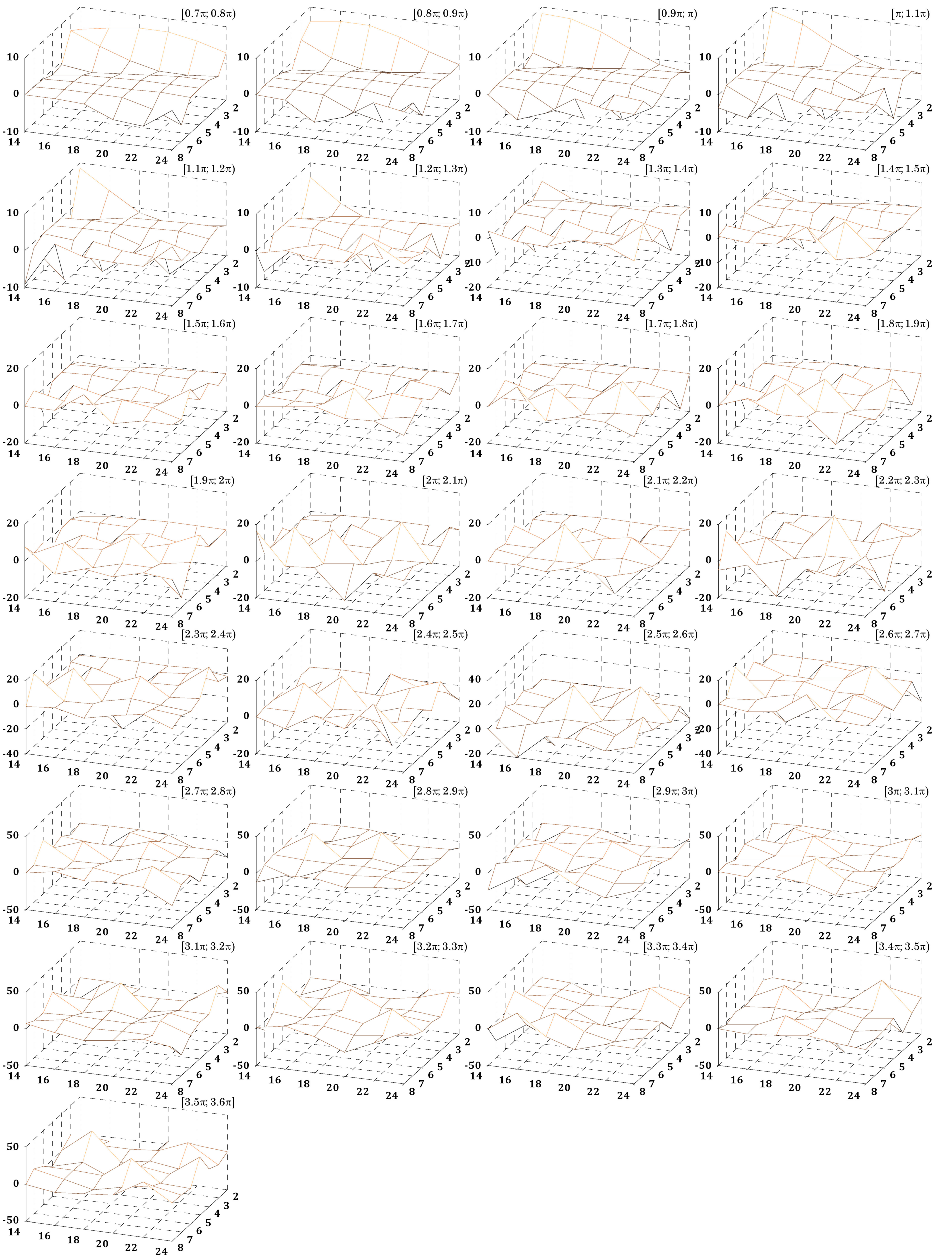

Figure 5: Second player's payoffs in matrix (133) as a meshed surface on the 29 subintervals of set (136)

as well (although some meshes on neighboring subintervals bear some resemblance). A distinctive feature here is that the payoff value scale of the second player is much wider than that of the first player. Whereas the first player's payoff varies between approximately -1.1144 and 1.3652, the second player's payoff varies between approximately -31.3741 and 30.4348, that means a potentially significant imbalance when the criterion of the payoff sum maximum is applied to select the best equilibrium.

The 7×6 bimatrix games (129) with (130) — (135) are solved in pure and mixed strategies, and there are multiple equilibrium situations on some time units. So, the best equilibrium situation on such time units is selected by the criterion of maximizing the players' payoffs sum. The stack of the 29 first player's equilibrium strategies in each of those 29 7×6 bimatrix games is shown in Figure 6, where the solid line corresponds to a pure strategy equilibrium and the dotted lines correspond to nonzero-probability pure strategies in a mixed strategy equilibrium. Similarly, the stack of the 29 second player's equilibrium strategies is shown in Figure 7. Thus, the solution to the "wider" game is the equilibrium situation formed subinterval-wise from the stacks in Figure 6 and Figure 7. Owing to Theorem 3, the equilibrium solutions for time intervals (120) — (122) are directly taken from the "wider" game equilibrium stack. In the solution for time interval (120), pure strategies $a^{(2)} = 4$ and $a^{(4)} = 6$ are not used by the first player, whereas pure strategy $b^{(1)} = 16$ is not used by the second player (Figure 8). In the solution for time interval (121), every player uses all one's pure strategies, only in mixed strategies (Figure 9). In the solution for time interval (122), pure strategy $a^{(4)} = 6$ is not used by the first player, whereas the second player uses all one's pure strategies (Figure 10) — either in mixed strategies or in pure strategies during $[3\pi;\ 3.4\pi)$. It is worth noting that there are no completely mixed strategies in the 29 time-unit equilibrium situations.

In the "wider" game equilibrium situation formed subinterval-wise from the stacks in Figure 6 and Figure 7, the players' payoffs are

$$v^* = \sum_{i=1}^{29} v_i^* \approx 7.4123 \tag{137}$$

and

$$z^* = \sum_{i=1}^{29} z_i^* \approx 99.8691, \tag{138}$$

provided by the criterion of maximizing the players' payoffs sum. However, it is worth noting that the presented game solution strongly depends on the criterion of selecting a single equilibrium situation (on each time unit). Inasmuch as the payoff ranges of the players differ severely, the applied above criterion may be unacceptable for the first player whose contribution to the sum is rather (insignificantly, on some time units) small. Thus, the same criterion can be used but only with payoff normalizations

$$\tilde{v}_{ij}^* = \frac{v_{ij}^* - \min\limits_{j=\overline{1,\ J_i}} v_{ij}^*}{\max\limits_{j=\overline{1,\ J_i}} v_{ij}^* - \min\limits_{j=\overline{1,\ J_i}} v_{ij}^*} \tag{139}$$

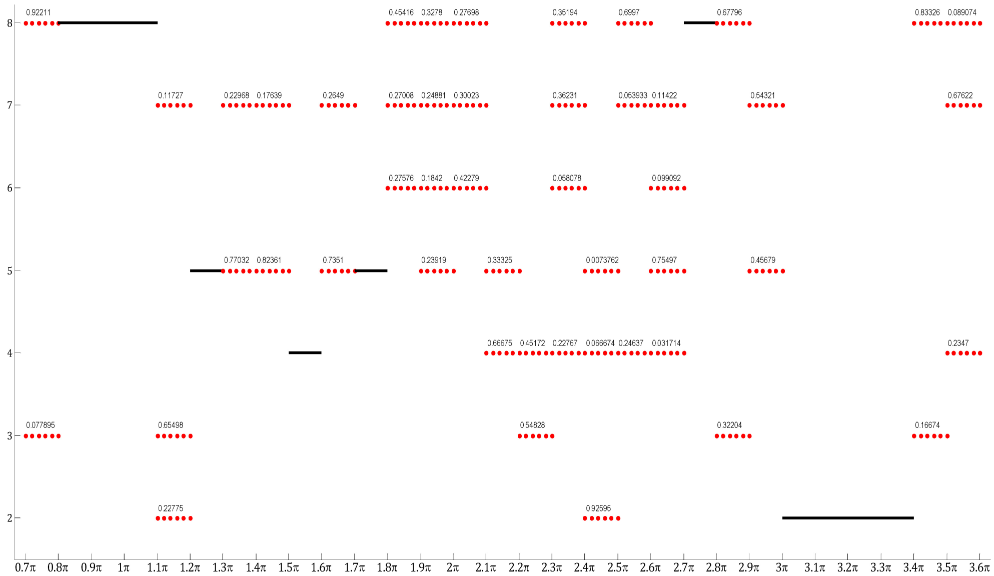

Figure 6: The stack of the 29 strategies as the first player's best equilibrium strategy in the best equilibrium situation in the "wider" game by (116) — (119) and (126) — (135)

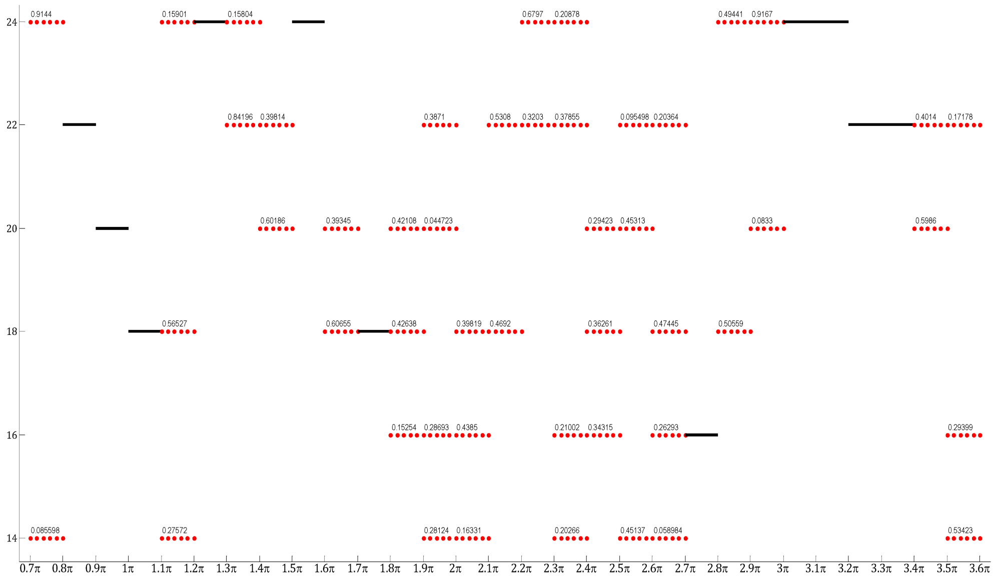

Figure 7: The stack of the 29 strategies as the second player's best equilibrium strategy in the best equilibrium situation in the "wider" game by (116) — (119) and (126) — (135)

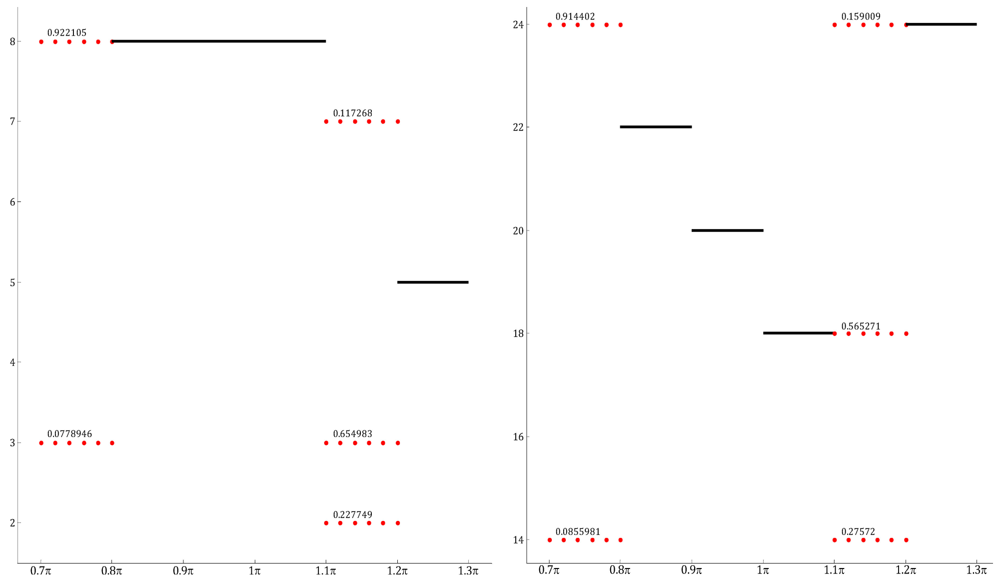

Figure 8: The stacks of the six best equilibrium strategies of the first (left) and second (right) players in the game played during time interval (120)

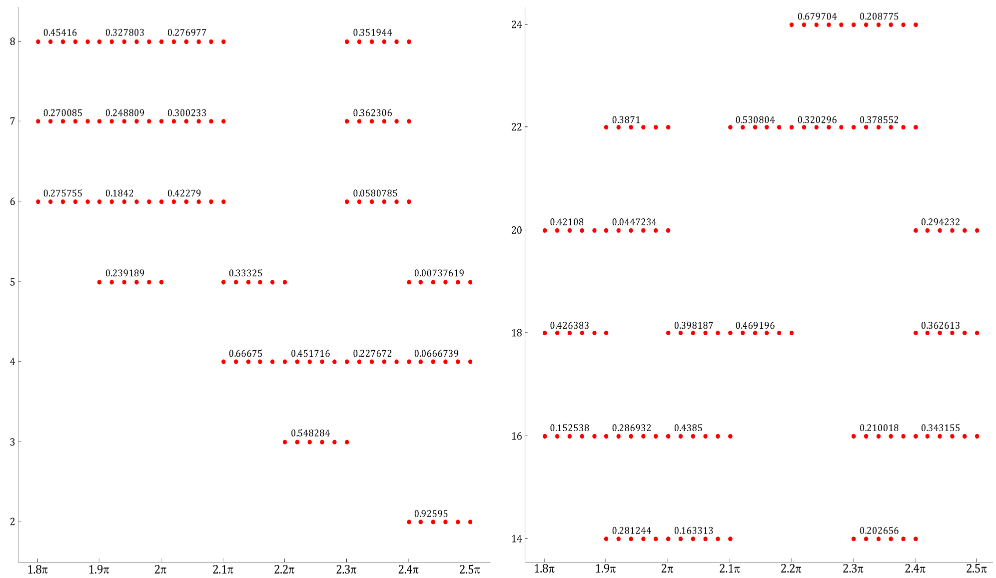

Figure 9: The stacks of the seven best equilibrium strategies of the first (left) and second (right) players in the game played during time interval (121)

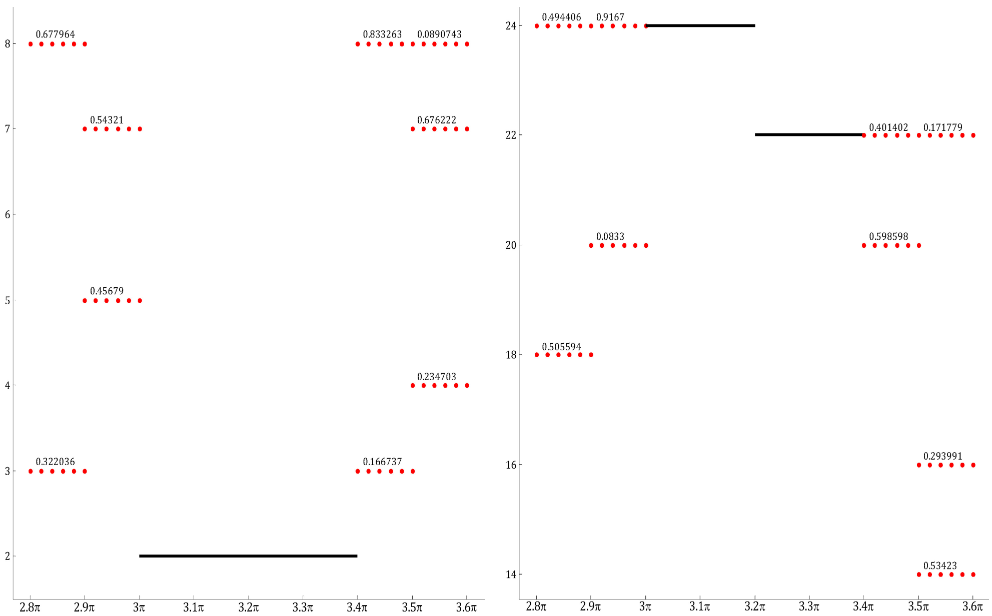

Figure 10: The stacks of the eight best equilibrium strategies of the first (left) and second (right) players in the game played during time interval (122)

and

$$\tilde{z}^*_{ij} = \frac{z^*_{ij} - \min\limits_{j=\overline{1,\ J_i}} z^*_{ij}}{\max\limits_{j=\overline{1,\ J_i}} z^*_{ij} - \min\limits_{j=\overline{1,\ J_i}} z^*_{ij}}, \tag{140}$$

where v^*_{ij} and z^*_{ij} are the first and second players payoffs in a j-th equilibrium situation on time unit i, on which there are J_i equilibria altogether. Then, similarly to (66), an equilibrium situation $\{\mathbf{P}^{**}_i,\ \mathbf{R}^{**}_i\}$ is selected such in which

$$\max_{j=\overline{1,\ J_i}} \left\{\tilde{v}^*_{ij} + \tilde{z}^*_{ij}\right\} \tag{141}$$

is reached. By using the criterion with (139) — (141), the equilibrium solution of the “wider” bimatrix staircase-function game changes (see the first player's equilibrium stack in Figure 11 and the second player's equilibrium stack in Figure 12, where the subintervals with the changes are segregated): the players' equilibria on subintervals

$$[1.4\pi;\ 1.5\pi)\,, \tag{142}$$

$$[2.1\pi;\ 2.2\pi)\,, \tag{143}$$

$$[2.6\pi;\ 2.7\pi) \tag{144}$$

are different from the equilibria on subintervals (142) — (144) in both Figure 6 and Figure 7. The first player now mixes pure strategies $a^{(0)} = 2$ and $a^{(4)} = 6$ (instead

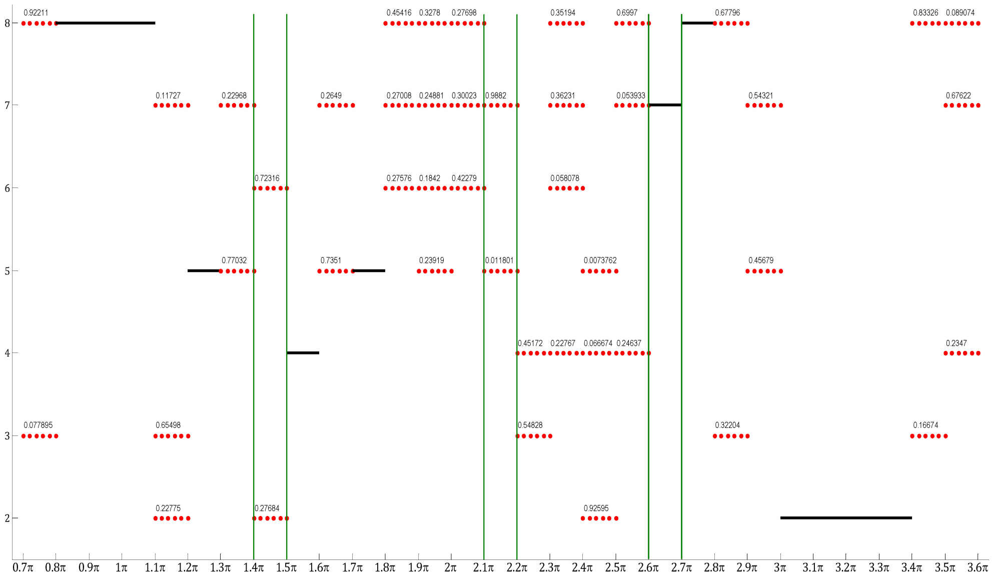

Figure 11: The best equilibrium situation of the first player in the “wider” game by (116) — (119) and (126) — (135) solved by selecting a single equilibrium situation on the time unit with (139) — (141)

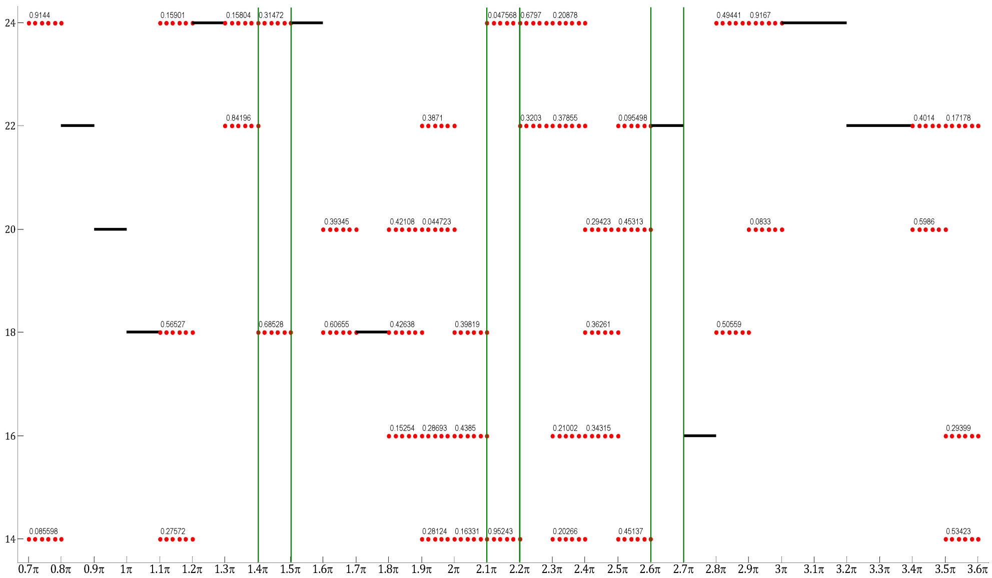

Figure 12: The best equilibrium situation of the second player in the “wider” game by (116) — (119) and (126) — (135) solved by selecting a single equilibrium situation on the time unit with (139) — (141)

of mixing $a^{(3)} = 5$ and $a^{(5)} = 7$ in Figure 6) on subinterval (142), and the second player now mixes pure strategies $b^{(2)} = 18$ and $b^{(5)} = 24$ (instead of mixing $b^{(3)} = 20$ and $b^{(4)} = 22$ in Figure 7) on subinterval (142). The equilibrium strategy support cardinality, which is

$$|\mathrm{supp}\, \mathbf{P}_8^*| = |\mathrm{supp}\, \mathbf{R}_8^*| = 2,$$

does not change. Nor does it change on subinterval (143):

$$|\mathrm{supp}\, \mathbf{P}_{15}^*| = |\mathrm{supp}\, \mathbf{R}_{15}^*| = 2,$$

where the first player now mixes pure strategy $a^{(3)} = 5$ with $a^{(5)} = 7$ (instead of mixing $a^{(3)} = 5$ with $a^{(2)} = 4$ in Figure 6), and the second player now mixes pure strategies $b^{(0)} = 14$ and $b^{(5)} = 24$ (instead of mixing $b^{(2)} = 18$ and $b^{(4)} = 22$ in Figure 7). The most radical change is on subinterval (144), whereon each of the players now does not mix one's four pure strategies ($a^{(2)} = 4$, $a^{(3)} = 5$, $a^{(4)} = 6$, $a^{(5)} = 7$ in Figure 6 and $b^{(0)} = 14$, $b^{(1)} = 16$, $b^{(2)} = 18$, $b^{(4)} = 22$ in Figure 7), but uses instead a single pure strategy: the first player just uses $a^{(5)} = 7$ and the second player uses $b^{(4)} = 22$. Now, in the "wider" game equilibrium situation formed subinterval-wise from the stacks in Figure 11 and Figure 12, the players' payoffs are

$$v^* = \sum_{i=1}^{29} v_i^* \approx 8.0337 \tag{145}$$

and

$$z^* = \sum_{i=1}^{29} z_i^* \approx 96.5492, \tag{146}$$

provided by the criterion of maximizing the players' payoffs sum as selecting a single equilibrium situation on the time unit with (139) — (141). The first player's payoff (145) is 8.3839 % greater than that (137), whereas the second player's payoff (146) is just 3.3242 % less than that (138). This is an example of that a proper selection of the single equilibrium criterion, e. g. using payoff normalizations like (139), (140), when the players' payoff ranges differ, can balance the player's eventual payoffs (making their distribution more fair). Obviously, equilibria on some time units may depend on the criterion (that is followed by the respective changes in the players' equilibrium stacks).

9. Discussion

In the sense of practical applicability, the presented method is a significant contribution to the 2-person game theory and operations research. It allows solving 2-person games played with staircase-function strategies in a far simpler manner just by considering a succession of time-unit subgames. In the case of a bimatrix staircase-function game, being "wider" one, its equilibrium situation is formed by solving and stacking equilibria of successive smaller-sized bimatrix games. Then, owing to Theorem 3, the respective equilibrium solution of any "narrower" subgame can be taken from the "wider" game equilibrium. The computational efficiency is only defined by and limited to the efficiency of finding equilibrium situations in an ordinary (time-unit) bimatrix game whose size is commonly not that large. Without considering the succession of time-unit bimatrix games, any straightforward approach to finding equilibrium situations in a bimatrix staircase-function game is intractable.

A special attention is paid to time variable t explicitly included into functions (8) and (11) to be integrated. The explicitness means that, as time goes by (and the players develop their actions), something is going on or changes within the process modeled by the staircase-function game. If, in a discrete-time staircase-function 2-person game, time t is not explicitly included into functions (8) and (11), then

$$K_i(\alpha_i,\ \beta_i) = \int\limits_{[\tau^{(i-1)};\ \tau^{(i)})} f(\alpha_i,\ \beta_i)\, d\mu(t) = \\ = f(\alpha_i,\ \beta_i) \cdot \left(\tau^{(i)} - \tau^{(i-1)}\right) \ \ \forall\, i = \overline{1,\ N-1} \tag{147}$$

and

$$K_N(\alpha_N,\ \beta_N) = \int\limits_{[\tau^{(N-1)};\ \tau^{(N)}]} f(\alpha_N,\ \beta_N)\, d\mu(t) = \\ = f(\alpha_N,\ \beta_N) \cdot \left(\tau^{(N)} - \tau^{(N-1)}\right) \tag{148}$$

instead of (26) and (27), and

$$H_i(\alpha_i,\ \beta_i) = \int\limits_{[\tau^{(i-1)};\ \tau^{(i)})} g(\alpha_i,\ \beta_i)\, d\mu(t) = \\ = g(\alpha_i,\ \beta_i) \cdot \left(\tau^{(i)} - \tau^{(i-1)}\right) \ \ \forall\, i = \overline{1,\ N-1} \tag{149}$$

and

$$H_N(\alpha_N,\ \beta_N) = \int\limits_{[\tau^{(N-1)};\ \tau^{(N)}]} g(\alpha_N,\ \beta_N)\, d\mu(t) = \\ = g(\alpha_N,\ \beta_N) \cdot \left(\tau^{(N)} - \tau^{(N-1)}\right) \tag{150}$$

instead of (28) and (29). Equalities (147) — (150) mean that the payoff value depends only on the length of the time unit. That is, the player's payoff then is equal to the subinterval length multiplied by the respective value of the function under the integral. If the length does not change in the case of bimatrix staircase-function game (49), then the time-unit bimatrix game does not change. If the length does not change in the case of discrete-time staircase-function 2-person game (33), the time-unit (ordinary) 2-person game defined on rectangle (32) does not change. Then the solution (of any type) to the initial (finite or uncountably infinite) discrete-time staircase-function 2-person game is determined just by the solution of a one time-unit game, and this solution will not change as the time units go by. Such a triviality of the equal-length-subinterval solution is explained by a standstill of the players' strategies. Consequently, the scientific significance of this trivial case is low — this is why it is not considered.

The scientific significance of the discrete-time staircase-function 2-person game and the methods of finding an equilibrium in it (provided by Theorems 2 and 3, and, under the supposition of that all the time-unit equilibria exist, by Theorems 4 and 5) is high. Owing to Theorems 2 and 3, such games, if finite, are very simple models to describe struggling for optimizing the distribution of some limited resources between two sides. Unlike ordinary bimatrix games, which model only static processes of the struggle, discrete-time staircase-function 2-person games allow considering discrete-time dynamics of the struggling processes. Such a simplification is similar to that when, e. g., the fuzzy logic facilitates the control of a complicated system without knowledge of its exact mathematical description.

10. Conclusion

Because of an intractably gigantic size, it is impracticable to solve 2-person games played in staircase-function finite spaces by directly rendering them to bimatrix games, where the solution is of the equilibrium type. Moreover, the time interval on which the discrete-time 2-person game is defined can vary by the number of time subintervals (time units), so a tractable and efficient method of finding an equilibrium in a 2-person game played in staircase-function finite spaces is to solve a succession of time-unit bimatrix games, whereupon their equilibria are stacked into pure-mixed-strategy equilibria. In the case of multiple equilibria on some time units, the criterion of the players' payoffs sum maximum is applied to select the best equilibrium. Owing to Theorems 2 and 3, the equilibrium of the initial finite game can be obtained by stacking the best equilibria of the smaller-sized bimatrix games, whichever the time interval is. If the game is uncountably infinite, i. e. a set of pure strategy possible values is uncountably infinite, and all time-unit equilibria exist, such a stack is possible as well owing to Theorems 4 and 5. So, the equilibrium of the initial discrete-time staircase-function 2-person game can be obtained by stacking the equilibria of the (ordinary) 2-person games defined on a rectangle, whichever the time interval is.

Solving games played in staircase-function finite spaces with possible time-unit shifting (when the initial time interval is narrowed by an integer number of time units) should be studied also for the case of three players. Then the presented assertions and conclusions are to be re-written for trimatrix games. A distinct peculiarity is that the equilibria multiplicity problem in trimatrix games is even trickier than that in bimatrix games. Moreover, the criterion of selecting a single equilibrium situation on each time unit in the case of a trimatrix staircase-function game becomes more disputable, especially when at least two players' payoff ranges differ significantly.

References

[1] S. Adlakha, R. Johari, G. Y. Weintraub, *Equilibria of dynamic games with many players: Existence, approximation, and market structure*, Journal of Economic Theory 156 (2015) 269–316.

[2] M. Arblaster, *Air Traffic Management. Economics, Regulation and Governance*, Elsevier, 2018.

[3] J. P. Benoit, V. Krishna, *Finitely repeated games*, Econometrica 53 (4) (1985) 905–922.

[4] J. Bergin, B. MacLeod, *Continuous time repeated games*, International Economic Review 34 (1993) 21–37.

[5] S. P. Coraluppi, S. I. Marcus, *Risk-sensitive and minimax control of discrete-time, finite-state Markov decision processes*, Automatica 35 (2) (1999) 301–309.

[6] R. E. Edwards, *Functional Analysis: Theory and Applications*, Holt, Rinehart and Winston, 1965.

[7] D. Fudenberg, J. Tirole, *Game Theory*, MIT Press, Cambridge, MA, 1991.

[8] M. C. Gelhausen, P. Berster, D. Wilken, *Airport Capacity Constraints and Strategies for Mitigation*, Academic Press, 2019.

[9] N. A. Gershenfeld, *The Nature of Mathematical Modeling*, Cambridge University Press, Cambridge, UK, 1999.

[10] J. C. Harsanyi, R. Selten, *A General Theory of Equilibrium Selection in Games*, The MIT Press, 1988.

[11] D. Hirshleifer, D. Jiang, Y. M. DiGiovanni, *Mood beta and seasonalities in stock returns*, Journal of Financial Economics 137 (1) (2020) 272–295.

[12] S. Kalaiselvam, R. Parameshwaran, *Chapter 7. Seasonal Thermal Energy Storage*, in: *Thermal Energy Storage Technologies for Sustainability. Systems Design, Assessment and Applications*, Academic Press, 2014, 145–162.

[13] C A. Kamhoua, C. D. Kiekintveld, F. Fang, Q. Zhu, *Game Theory and Machine Learning for Cyber Security*, Wiley-IEEE Press, 2021.

[14] C. E. Lemke, J. T. Howson, *Equilibrium points of bimatrix games*, SIAM Journal on Applied Mathematics 12 (2) (1964) 413–423.

[15] N. Nisan, T. Roughgarden, É. Tardos, V. V. Vazirani, *Algorithmic Game Theory*, Cambridge University Press, Cambridge, UK, 2007.

[16] V. V. Romanuke, *Theory of Antagonistic Games*, New World — 2000, Lviv, 2010.

[17] V. V. Romanuke, *Convergence and estimation of the process of computer implementation of the optimality principle in matrix games with apparent play horizon*, Journal of Automation and Information Sciences 45 (10) (2013) 49–56.

[18] V. V. Romanuke, *Theoretic-game methods of identification of models for multistage technical control and run-in under multivariate uncertainties* (a Dissertation for the Doctoral Degree of Technical Sciences in Speciality 01.05.02 Mathematical Modeling and Computational Methods), Vinnytsia National Technical University, Vinnytsia, Ukraine, 2014 (in Ukrainian).

[19] V. V. Romanuke, V. G. Kamburg, *Approximation of isomorphic infinite two-person noncooperative games via variously sampling the players' payoff functions and reshaping payoff matrices into bimatrix game*, Applied Computer Systems 20 (2016) 5–14.

[20] V. V. Romanuke, *Finite approximation of continuous noncooperative two-person games on a product of linear strategy functional spaces*, Journal of Mathematics and Applications 43 (2020) 123–138.

[21] V. V. Romanuke, *Two-person games on a product of staircase-function continuous and finite spaces*, Visnyk of the Lviv University. Series Appl. Math. and Informatics 29 (2021) 67–90.

[22] V. V. Romanuke, *Finite uniform approximation of two-person games defined on a product of staircase-function infinite spaces*, International Journal of Approximate Reasoning 145 (2022) 139–162.

[23] Y. Teraoka, *A two-person game of timing with random arrival time of the object*, Mathematica Japonica 24 (1979) 427–438.

[24] N. N. Vorob'yov, *Game Theory Fundamentals. Noncooperative Games*, Nauka, Moscow, 1984 (in Russian).

[25] N. N. Vorob'yov, *Game Theory for Economists-Cyberneticists*, Nauka, Moscow, 1985 (in Russian).

[26] J. Yang, Y.-S. Chen, Y. Sun, H.-X. Yang, Y. Liu, *Group formation in the spatial public goods game with continuous strategies*, Physica A: Statistical Mechanics and its Applications 505 (2018) 737–743.

[27] P. Young, S. Zamir, *Handbook of Game Theory. Volume 4*, Elsevier, North Holland, Netherlands, 2015.

[28] J. Zarei, M. Rasti-Barzoki, S. R. Hejazi, *A game theoretic approach for integrated pricing, lot-sizing and advertising decisions in a dual-channel supply chain*, International Journal of Operational Research 40 (3) (2021) 342–365.

[29] R. Zhao, G. Neighbour, J. Han, M. McGuire, P. Deutz, *Using game theory to describe strategy selection for environmental risk and carbon emissions reduction in the green supply chain*, Journal of Loss Prevention in the Process Industries 25 (6) (2012) 927–936.

[30] Z. Zhou, Z. Jin, *Optimal equilibrium barrier strategies for time-inconsistent dividend problems in discrete time*, Insurance: Mathematics and Economics 94 (2020) 100–108.

DOI: 10.7862/rf.2022.7

Vadim Romanuke
email: `v.romanuke@amw.gdynia.pl`
ORCID: 0000-0003-3543-3087
Faculty of Mechanical and Electrical Engineering
Polish Naval Academy
Gdynia
POLAND

Received 03.04.2022 *Accepted 09.08.2022*

Journal of
Mathematics
and Applications
JMA No 45, pp 181-190 (2022)

On the Analytic α-Lipschitz Vector-Valued Operators

Abbasali Shokri

ABSTRACT: Let (X, d) be a non-empty compact metric space in $\mathbb{C}$, $(B, \parallel . \parallel)$ be a commutative unital Banach algebra over the scalar field $\mathbb{F}(= \mathbb{R}\ or\ \mathbb{C})$ and $\alpha \in \mathbb{R}$ with $0 < \alpha \leq 1$. In this work, first we define the analytic α-Lipschitz B-valued operators on X and denote the Banach algebra of all these operators by $Lip_A^\alpha(X, B)$. When $B = \mathbb{F}$, we write $Lip_A^\alpha(X)$ instead of $Lip_A^\alpha(X, B)$. Then we study some interesting results about $Lip_A^\alpha(X, B)$, including the relationship between $Lip_A^\alpha(X, B)$ with $Lip_A^\alpha(X)$ and B, and also characterize the characters on $Lip_A^\alpha(X, B)$.

AMS Subject Classification: 46B28, 46E15, 46E40.
Keywords and Phrases: Analytic operator; Banach algebra; Vector-valued operator; Lipschitz operator.

1. Introduction

Throughout this paper, let (X, d) be a compact metric space in $\mathbb{C}$, $(B, \parallel . \parallel)$ be a commutative unital Banach algebra over the scalar field $\mathbb{F}(= \mathbb{R}\ or\ \mathbb{C})$ with unit $\mathbf{e}$, $C(X, B)$ be the set of all B-valued continuous operators on X, and also $\alpha \in \mathbb{R}$ with $0 < \alpha \leq 1$.

The *dual space* of B is the vector space B^* whose elements are the continuous linear functionals on B. The set of all multiplicative functionals on B is called *spectrum* of B; we denote it by $\sigma(B)$. Suppose that throughout this article, $\Lambda \in \sigma(B)$ is arbitrary and fixed. Since $\sigma(B)$ is a subset of the closed unit ball of B^*, $\parallel \Lambda \parallel$ is bounded, where

$$\parallel \Lambda \parallel = \sup\{\ \mid \Lambda x \mid\ :\ x \in B\ ,\ \parallel x \parallel \leq 1\ \}.$$

When $B = \mathbb{F}$, take Λ as the identity function $\Lambda x = x$.

Consider the set Y as follows

$$Y := \{(x, y) \ : \ x, y \in X \ , \ x \neq y\}.$$

For an operator $f : X \to B$ and any $(x, y) \in Y$ define

$$L_f^\alpha(x, y) := \frac{\Big|\big(\Lambda o f\big)(x) - \big(\Lambda o f\big)(y)\Big|}{d^\alpha(x, y)} \ ,$$

where $d^\alpha(x, y) = \big(d(x, y)\big)^\alpha$ and $0 < \alpha \leq 1$. Now define

$$p_\alpha(f) := \sup_{x \neq y} \ L_f^\alpha(x, y) \ , \ 0 < \alpha \leq 1,$$

which is called the *Lipschitz constant* of f. Also for $0 < \alpha \leq 1$ define

$$Lip^\alpha(X, B) := \{f \in C(X, B) \ : \ p_\alpha(f) < +\infty\},$$

and for $0 < \alpha < 1$ define

$$lip^\alpha(X, B) := \{f \in Lip_\alpha(X, B) \ : \ \lim_{d(x,y) \to 0} L_f^\alpha(x, y) = 0\}.$$

The elements of $Lip^\alpha(X, B)$ and $lip^\alpha(X, B)$ are called *big* and *little* α-Lipschitz B-valued operators, respectively.

Now, for each $\lambda \in \mathbb{F}$, $x \in X$ and $f, g \in C(X, B)$ define

$$\big(f + g\big)(x) := f(x) + g(x) \ , \ \big(\lambda f\big)(x) := \lambda f(x) \ ,$$

and the uniform norm $\| \ . \ \|_\infty$ on $C(X, B)$ by

$$\| \ f \ \|_\infty := \sup_{x \in X} \| \ f(x) \ \| \quad ; \quad f \in C(X, B).$$

Also for any $f \in Lip^\alpha(X, B)$ define

$$\| \ f \ \|_\alpha := p_\alpha(f) + \| \ f \ \|_\infty \ .$$

It is easy to see that $\big(C(X, B), \| \ . \ \|_\infty \ \big)$ becomes a Banach algebra over $\mathbb{F}$.

Cao, Zhang and Xu in [6] proved that $\big(Lip^\alpha(X, B), \| \ . \ \|_\alpha \ \big)$ is a Banach space over $\mathbb{F}$ and $\big(lip^\alpha(X, B), \| \ . \ \|_\alpha \ \big)$ is a closed linear subspace of $\big(Lip^\alpha(X, B), \| \ . \ \|_\alpha \ \big)$ when B is a Banach space. We also studied some of the properties of these algebras in [14-17] when B is a commutative unital Banach algebra.

Note that for $\alpha = 1$ and $B = \mathbb{F}$, the space $Lip^1(X, \mathbb{F})$ consisting of all Lipschitz functions from X into $\mathbb{F}(= \mathbb{R} \ or \ \mathbb{C})$ has a series of interesting and important properties, which has been studied by many mathematicians, including the first of them Sherbert [13]. In [7, 18] some properties of Lipschitz scalar-valued functions are mentioned.

Let D be an open subset of X. An operator f of D into B is said to be *analytic* on D if, for every continuous linear functional $\phi \in B^*$, the scalar-valued function $\phi o f$

is analytic on D in the usual sense. Note that we do not require D to be connected. For a full discussion of analytic complex-valued and vector-valued functions, see [2, 7]. The algebra of all continuous B-valued operators on X whose analytic in interior X is denoted by $A(X,B)$. We write $A(X)$ instead of $A(X,\mathbb{F})$ ($\mathbb{F} = \mathbb{R}$ *or* $\mathbb{C}$). Some of the properties of these algebras have been studied by certain mathematicians, see [1, 3-5, 8-11].

Finally, in this article, we introduce the analytic α-Lipschitz B-valued operator algebras $Lip_A^\alpha(X,B)$ and we characterize their characters, also we study the relationship between of $Lip_A^\alpha(X,B)$ and B. We prove the main results of the article in several theorems.

2. Lip-analytic Operators

In this section, we introduce the analytic α-Lipschitz vector-valued operator algebras $Lip_A^\alpha(X,B)$, and we study some of their properties.

We write $C(X)$ and $Lip^\alpha(X)$ instead of $C(X,\mathbb{F})$ and $Lip^\alpha(X,\mathbb{F})$ respectively. By the Stone-Weierstrass theorem, we have

Theorem 2.1. [7]. *$A(X)$ is uniformly dense in $C(X)$.*

It is obvious that $A(X,B)$ is a subalgebra of $C(X,B)$. We have

Theorem 2.2. *$A(X,B)$ is uniformly dense in $C(X,B)$.*

Proof. Let $\epsilon > 0$ and $f \in C(X,B)$ be arbitrary. We show that there exists $g \in A(X,B)$ such that $\| f - g \|_\infty < \epsilon$. Since $f \in C(X,B)$, $\Lambda o f \in C(X)$. Then by Theorem 2.1, there is $h \in A(X)$ such that $\| \Lambda o f - h \|_\infty < \epsilon$. So

$$\sup_{x\in X} \Big|(\Lambda o f)(x) - h(x)\Big| < \epsilon.$$

Since $\Lambda(\mathbf{e}) = 1$, $h(x) = \Lambda(h(x)\mathbf{e})$ for all $x \in X$. Then

$$\sup_{x\in X} \Big|\Lambda(f(x)) - \Lambda(h(x)\mathbf{e})\Big| < \epsilon.$$

Hence

$$\sup_{x\in X} \Big|\Lambda\big((f - h.\mathbf{e})(x)\big)\Big| < \epsilon.$$

Since $\Lambda \in \sigma(B)$ is arbitrary, $\sup_{x\in X} \| (f - h.\mathbf{e})(x) \| < \epsilon$. Thus $\| f - h.\mathbf{e} \|_\infty < \epsilon$. Now, take $g := h.\mathbf{e}$. Since $h \in A(X)$ and $\mathbf{e} \in B$, $g \in A(X,B)$. Therefore we conclude that $\| f - g \| < \epsilon$ where $g \in A(X,B)$. □

We have the similar Theorem 2.1 for the algebra of Lipschitz scalar-valued functions:

Theorem 2.3. [18]. *$Lip^\alpha(X)$ is uniformly dense in $C(X)$.*

Theorem 2.4. *$Lip^{\alpha}(X,B)$ is uniformly dense in $C(X,B)$.*

Proof. Let $\epsilon > 0$ and $f \in C(X,B)$ be arbitrary. We show that there exists $h \in Lip^{\alpha}(X,B)$ such that $\| h - f \|_{\infty} < \epsilon$. Since $f \in C(X,B)$, $\Lambda o f \in C(X)$. So by Theorem 2.3, there exists $g \in Lip^{\alpha}(X)$ such that $\| g - \Lambda o f \|_{\infty} < \epsilon$. Define

$$\eta : \mathbb{C} \to B$$

$$\eta(\lambda) := \lambda \mathbf{e}.$$

Since g is continuous, $\eta o g$ is continuous. Also

$$\begin{aligned} p_{\alpha}(\eta o g) &= \sup_{x \neq y} L^{\alpha}_{\eta o g}(x,y) \\ &= \sup_{x \neq y} \frac{\| (\eta o g)(x) - (\eta o g)(y) \|}{d^{\alpha}(x,y)} \\ &= \sup_{x \neq y} \frac{\| g(x)\mathbf{e} - g(y)\mathbf{e} \|}{d^{\alpha}(x,y)} \quad (\| \mathbf{e} \| = 1) \\ &\leq p_{\alpha}(g) < \infty. \end{aligned}$$

So $\eta o g \in Lip^{\alpha}(X,B)$. Set $h := \eta o g$. Now we show that $\| h - f \|_{\infty} < \epsilon$. Since $\Lambda(\mathbf{e}) = 1$, for all $x \in X$ we have

$$\begin{aligned} \big|\Lambda\big(g(x)\mathbf{e} - f(x)\big)\big| &= \big|g(x) - (\Lambda o f)(x)\big| \\ &\leq \| g - \Lambda o f \|_{\infty} \\ &< \epsilon. \end{aligned}$$

This implies that

$$\big|\Lambda\big(\eta o g - f\big)(x)\big| < \epsilon \ , \ x \in X.$$

Since $\Lambda \in \sigma(B)$ is arbitrary, $\| (\eta o g - f)(x) \| < \epsilon$ for all $x \in X$. Consequently, $\| \eta o g - f \|_{\infty} < \epsilon$ or $\| h - f \|_{\infty} < \epsilon$. This completes the proof. $\square$

Corollary 2.5. *By using Theorems 2.2 and 2.4, each element of $A(X,B)$ can be approximated by elements of $Lip^{\alpha}(X,B)$ with sup-norm. Also each element of $Lip^{\alpha}(X,B)$ can be approximated by elements of $A(X,B)$ with sup-norm.*

Definition 2.6. Let D be an open subset of X. An operator f of D into B is said to be Lip-analytic on D if $f \in Lip^{\alpha}(X,B) \cap A(X,B)$.

The algebra of all Lip-analytic B-valued operators on X whose analytic in interior X is denoted by $Lip^{\alpha}_{A}(X,B)$. When $B = \mathbb{F}$, we write $Lip^{\alpha}_{A}(X)$ instead of $Lip^{\alpha}_{A}(X,B)$.

By Theorems 2.2 and 2.4, we can prove that:

Theorem 2.7. *$Lip^{\alpha}_{A}(X,B)$ is uniformly dense in $C(X,B)$.*

Let E_1 and E_2 be linear spaces. From [12], a *tensor product* of E_1 and E_2 is a pair (T, τ), where T is a linear space and $\tau : E_1 \times E_2 \to T$ is a bilinear map with the following (universal) property: For each linear space F and for each bilinear map $V : E_1 \times E_2 \to F$, there is a unique linear map $U : T \to F$ such that $V = U o \tau$.
We shall also use the standard notation for tensor products, we write $E_1 \otimes E_2$ for T and $x_1 \otimes x_2 = \tau(x_1, x_2)$ for $x_1 \in E_1$ and $x_2 \in E_2$. If $Z \in E_1 \otimes E_2$, then there is $m \in \mathbb{N}$, and for each $j = 1, 2$ there are $x_j^{(1)}, ..., x_j^{(m)} \in E_j$ such that $Z = \sum_{k=1}^{m} x_1^{(k)} \otimes x_2^{(k)}$.

Let E_1 and E_2 be Banach spaces with dual spaces E_1^* and E_2^* . Then we define for $Z \in E_1 \otimes E_2$

$$\| Z \|_\epsilon = \sup \Big\{ \big| \langle Z, \phi_1 \otimes \phi_2 \rangle \big| \ : \ \phi_j \in N_1[0, E_j^*] \ , \ j = 1, 2 \Big\},$$

where

$$Z = \sum_{k=1}^{m} x_1^{(k)} \otimes x_2^{(k)} \ ; \ \big(m \in \mathbb{N}, \ x_j^{(k)} \in E_j, \ j = 1, 2, \ 1 \leq k \leq m \big),$$

and

$$\begin{aligned} \langle \ Z, \phi_1 \otimes \phi_2 \ \rangle &= \langle \ \sum_{k=1}^{m} x_1^{(k)} \otimes x_2^{(k)}, \phi_1 \otimes \phi_2 \ \rangle \\ &= (\phi_1 \otimes \phi_2) \Big(\sum_{k=1}^{m} x_1^{(k)} \otimes x_2^{(k)} \Big) \\ &= \sum_{k=1}^{m} (\phi_1 \otimes \phi_2) \big(x_1^{(k)} \otimes x_2^{(k)} \big) \\ &= \sum_{k=1}^{m} \phi_1 \big(x_1^{(k)} \big) \ \phi_2 \big(x_2^{(k)} \big), \end{aligned}$$

and $N_1[0, E_j^*]$ is closed ball in E_j^* with radius 1 centered at 0 . We call $\| \ . \ \|_\epsilon$ the *injective norm* on $E_1 \otimes E_2$.

Let $(E_1, \| \ . \ \|_1)$ and $(E_2, \| \ . \ \|_2)$ be Banach spaces. Then their *injective tensor product* $E_1 \check{\otimes} E_2$ is the completion of $E_1 \otimes E_2$ with respect to $\| \ . \ \|_\epsilon$. For every $Z \in E_1 \check{\otimes} E_2$, we have

$$\| Z \|_\epsilon = \sup \Big\{ \ \| \ \big(id \otimes \phi \big)(Z) \ \|_1 \ \ : \ \phi \in N_1[0, E_2^*] \Big\},$$

where

$$\big(id \otimes \phi \big)(a \otimes b) = a\phi(b) \ ; \ (a \in E_1 \ , \ b \in E_2).$$

Definition 2.8. Let E_1 and E_2 be Banach spaces. A norm $\| \ . \ \|$ on $E_1 \otimes E_2$ is called a cross norm if

$$\| x_1 \otimes x_2 \| = \| x_1 \| \| x_2 \| \quad (x_1 \in E_1 \ , \ x_2 \in E_2).$$

Proposition 2.9. [12]. *Let E_1 and E_2 be Banach spaces. Then $\| \ . \ \|_\epsilon$ is a cross norm on $E_1 \otimes E_2$.*

3. The Main Results

In this section, we present the main results of the article.

Theorem 3.1. *$Lip_A^\alpha(X,B)$ is isometrically isomorphic to $Lip_A^\alpha(X)\check{\otimes}B$.*

Proof. It is straightforward to prove that the mapping

$$Lip_A^\alpha(X)\times B\to Lip_A^\alpha(X,B),\quad (f,b)\longmapsto fb \tag{3.1}$$

is bilinear. So from the defining property of the algebraic tensor product $Lip_A^\alpha(X)\otimes B$, it follows that (1) extends to a linear map

$$S: Lip_A^\alpha(X)\otimes B\longrightarrow Lip_A^\alpha(X,B)$$

$$S(f\otimes b):=fb\ ,$$

where

$$(fb)(x):=f(x)b\ ;\ (x\in X).$$

Then

$$\begin{aligned}\| S(f\otimes b)\|_\alpha &= \| fb\|_\alpha = \| fb\|_\infty + p_\alpha(fb)\\ &= \| f\|_\infty \| b\| + p_\alpha(f)\| b\|\\ &= (\| f\|_\infty + p_\alpha(f))\| b\|\\ &= \| f\|_\alpha \| b\|\\ &= \| f\otimes b\|_\epsilon\ .\end{aligned}$$

Therefore S is an isometry and thus injective with closed range. It remains to be shows that it has dense range as well.

Let $f\in Lip_A^\alpha(X,B)$ and $\epsilon>0$. Being the continuous image of a compact space, $K:=f(X)\subset B$ is compact. We may thus find $b_1,...,b_n\in B$ such that $K\subset \cup_{i=1}^n N(b_i,\epsilon)$, where $N(b_i,\epsilon)$ is a neighborhood with radius ϵ centered at b_i. Let $U_j:=f^{-1}\big(N(b_j,\epsilon)\big)$ for $j=1,...,n$. Choose $f_1,...,f_n\in Lip_A^\alpha(X,B)$ such that $supp(f_j)\subset U_j$ for j = 1 , ..., n, and $\Lambda o(\sum_{i=1}^n f_i)=1$. Then for every $x\in X$ we have

$$\begin{aligned}\| \big(f-\sum_{i=1}^n S(\Lambda o f_i\otimes b_i)\big)(x)\| &= \| \big(f-\sum_{i=1}^n(\Lambda o f_i)b_i\big)(x)\|\\ &= \| f(x)-\sum_{i=1}^n(\Lambda o f_i)(x)b_i\|\\ &= \| f(x)\big(\Lambda o(\sum_{i=1}^n f_i)\big)(x)-\sum_{i=1}^n(\Lambda o f_i)(x)b_i\|\\ &= \| f(x)\sum_{i=1}^n(\Lambda o f_i)(x)-\sum_{i=1}^n(\Lambda o f_i)(x)b_i\|\end{aligned}$$

$$= \quad \| \sum_{i=1}^{n} (\Lambda o f_i)(x)\big(f(x) - b_i\big) \|$$
$$\leq \quad \sum_{i=1}^{n} | (\Lambda o f_i)(x) | \| f(x) - b_i \| .$$

It easy to see that the right hand side of the above relation is less than ϵ. So we conclude that $\overline{R_S} = Lip^{\alpha}_{A}(X, B)$. This completes the proof. □

With an argument similar to the proof of Theorem 3.1, we can prove that:

Theorem 3.2. *$A(X, B)$ is isometrically isomorphic to $A(X)\check{\otimes}B$.*

Define the canonical embedding

$$j : Lip^{\alpha}_{A}(X) \to Lip^{\alpha}_{A}(X, B)$$
$$j(h) := h \otimes \mathbf{e} \ ,$$

such that

$$(h \otimes \mathbf{e})(x) := h(x)\mathbf{e} \ ; \ x \in X.$$

By Theorem 3.1, the map j is well defined. Let χ be a arbitrary and fixed character on $Lip^{\alpha}_{A}(X, B)$. Then there is $z \in X$ such that $\chi o j$ is the evaluation at z, indeed $\chi o j = \delta_z$ where $\delta_z(f) = f(z)$.

Define $\varphi(\omega) := \omega - z, (\omega \in X)$. It is clear that $\varphi \in A(X)$, and we have

$$\begin{aligned} p_{\alpha}(\varphi) = \sup_{x \neq y} \frac{| \varphi(x) - \varphi(y) |}{| x - y |^{\alpha}} \quad &= \quad \sup_{x \neq y} \frac{| (x - z) - (y - z) |}{| x - y |^{\alpha}} \\ &= \quad \sup_{x \neq y} | x - y |^{1-\alpha} < \infty. \end{aligned}$$

So $\varphi \in Lip^{\alpha}(X)$, and consequently $\varphi \in Lip^{\alpha}_{A}(X)$.

Now consider

$$I := \{ f \in Lip^{\alpha}_{A}(X, B) \ : \ f(z) = 0 \}.$$

It is obvious that I is nonempty and an ideal in $Lip^{\alpha}_{A}(X, B)$.

Theorem 3.3. *I is contained in the kernel of χ.*

Proof. Let $f \in I$ be arbitrary. Then $f \in A(X, B)$. So f has a Taylor series expansion $f(\omega) = \Sigma_{n=1}^{\infty} \frac{f^{(n)}(z)}{n!} (\omega - z)^n$ around z. Define

$$g(\omega) := \begin{cases} \frac{f(\omega)}{\omega - z} \ ; & \omega \neq z \ , \\ f'(z) \ ; & \omega = z. \end{cases}$$

It is clear that Λog is analytic in the interior of X, so $g \in A(X, B)$. For $\omega = z$, it is obvious that $g \in Lip_A^\alpha(X, B)$, and for $\omega \neq z$ we have

$$f(\omega) = (\omega - z)g(\omega) = \varphi(\omega)g(\omega).$$

It can be easily proved that $g \in Lip_A^\alpha(X, B)$. Then for every $\omega \in X$ with $\omega \neq z$, we have

$$\begin{aligned} f(\omega) &= \varphi(\omega)g(\omega) = \varphi(\omega)\mathbf{e}g(\omega) \\ &= (\varphi \otimes \mathbf{e})(\omega)g(\omega) = \big((\varphi \otimes \mathbf{e})g\big)(\omega) \\ &= \big(j(\varphi)g\big)(\omega). \end{aligned}$$

So $f = j(\varphi)g$. Therefore

$$\begin{aligned} \chi(f) &= \chi\big(j(\varphi)g\big) = \chi\big(j(\varphi)\big)\chi(g) \\ &= \big(\chi oj\big)(\varphi)\chi(g) = \delta_z(\varphi)\chi(g) \\ &= \varphi(z)\chi(g) = 0 \times \chi(g) = 0. \end{aligned}$$

So $f \in ker\chi$, and that means $I \subset ker\chi$. This completes the proof. □

Theorem 3.4. *Every character χ on $Lip_A^\alpha(X, B)$ is of form $\chi = \psi o\delta_z$ for some character ψ on B and some $z \in X$, where $\delta_z(f) = f(z)$.*

Proof. Let χ be an arbitrary character on $Lip_A^\alpha(X, B)$. Then there is $z \in X$ such that χoj is the evaluation at z, indeed $\chi oj = \delta_z$ where $\delta_z(f) = f(z)$. Define

$$I := \{f \in Lip_A^\alpha(X, B) \;:\; f(z) = 0\}.$$

By Theorem 3.3, I is contained in the kernel of χ. It is clear that $ker\delta_z = I$. Therefore $ker\delta_z \subset ker\chi$. We obtain the desired factorization $\chi = \psi o\delta_z$ for some character ψ on B. □

References

[1] O. Blasco, G. Stylogiannis, *Lipschitz-type conditions on homogeneous Banach spaces of analytic functions*, J. Math. Anal. Appl. 445 (2017) 612–630.

[2] F.F. Bonsall, J. Duncan, *Complete Normed Algebras*, Springer-Verlag, 1973.

[3] B. Bouya, *Closed ideals in analytic weighted Lipschitz algebras*, Advances in Mathematics 219 (2008) 1446–1468.

[4] B. Bouya, M. Zarrabi, *On closed ideals in the big Lipschits algebras of analytic functions*, Bull. Sci. Math. 137 (2013) 575–583.

[5] C. Boyd, R. Ryan, N. Snigireva, *Radius of analyticity of analytic functions on Banach spaces*, J. Math. Anal. Appl. 463 (1) (2018) 40–49.

[6] H.X. Cao, J.H. Zhang, Z.B. Xu, *Characterizations and extensions of Lipschitz-α operators*, Acta Mathematica Sinica, English series 22 (3) (2006) 671–678.

[7] H.G. Dales, *Banach Algebras and Automatic Continuty*, Clarendon press, Oxford, 2000.

[8] J. Dziok, *A unified class of analytic functions with fixed argument of coefficients*, Acta Mathematica Scientia 31 (4) (2011) 1357–1366.

[9] C. Flavia, T. Maria. *Operator norms and essential norms of weighted composition operators between Banach spaces of analytic functions,* J. Math. Anal. Appl. 434 (2016) 93–124.

[10] Y. Fuxiang, K. Ker-I, *On parallel complexity of analytic functions*, Theoretical Computer Science 489-490 (2013) 48–57.

[11] M. Mastylo, P. Mleczko, *Composition operators on Banach spaces of analytic functions*, Anal. Acad. Scientiarum Fennicæ 44 (2019) 601–613.

[12] V. Runde, *Lectures on Amenability*, Springer, 2002.

[13] D.R. Sherbert, *The structure of ideals and point derivations in Banach algebras of Lipschitz functions*, Trans. Amer. Math. Soc. 111 (1964) 240–272.

[14] A.A. Shokri, A. Ebadian, A.R. Medghalchi. *Amenability and weak aminability of Lipschitz operator algebras*, Acta Universitatis Apulensis 18 (2009) 87–96.

[15] A. Shokri, *Second dual space of little α- Lipschitz vector-valued operator algebras*, Sahand Commun. Math. Anal. (SCMA) 8 (1) (2017) 33–41.

[16] A.A. Shokri, A. Shokri, *Homomorphisms of certain α-Lipschitz operator Algebras*, Acta Universitatis Apulensis 27 (2011) 9–13.

[17] A.A. Shokri, A. Shokri, *Maximal ideal space of certain α-Lipschitz operator algebras*, J. Math. and Appl. (JMA) 35 (2012) 141–147.

[18] N. Weaver, *Lipschitz Algebras*, World scientific publishing Co., Inc., River Edge, NJ, 1999.

DOI: 10.7862/rf.2022.8

Abbasali Shokri
email: `Ab.Shokri@iau.ac.ir, shokrimath@yahoo.com`
ORCID: 0000-0002-1759-1439
Department of Mathematics,
Ahar Branch, Islamic Azad University,

Ahar,
IRAN

Received 16.09.2021 *Accepted 20.10.2022*

JOURNAL OF MATHEMATICS AND APPLICATIONS – 2022

COOPERATING REVIEWERS

Radoš Bakić (Serbia)
Józef Banaś (Poland)
Lixin Cheng (China)
Dariusz Cichoń (Poland)
Sławomir Dinew (Poland)
Janusz Dronka (Poland)
Szymon Dudek (Poland)
Cosme Duque (Venezuela)
Małgorzata Gajewska (Poland)
Francisco Martínez González (Spain)
Jarosław Górnicki (Poland)
Mohammed K. A. Kaabar (China)
Bilel Krichen (Tunisia)
Adam Lecko (Poland)
Eberhard Malkowsky (Serbia)
Stefan Milius (Germany)
Leszek Olszowy (Poland)
Dariusz Partyka (Poland)
Dariusz Pączko (Poland)
Andriy Romanov (Ukraine)
Yevgeniy M. Rudnichenko (Ukraine)
Beata Rzepka (Poland)
Mohamed Aziz-Taoudi (Morocco)
Mariola Walczyk (Poland)
Ahmed Zeghal (Morocco)

The Journal of Mathematics and Applications publishes
a list of reviewers on the websites:

https://jma.prz.edu.pl/en/current-issue

https://oficyna.prz.edu.pl/en/scientific-research-papers/the-faculty-of-mathematics-and-a/jurnal-of-mathematics-and-applic

ADDITIONAL INFORMATION

The Journal of Mathematics and Applications publishes a list of reviewers on the websites:

https://jma.prz.edu.pl/en/current-issue

https://oficyna.prz.edu.pl/en/scientific-research-papers/the-faculty-of-mathematics-and-a/jurnal-of-mathematics-and-applic

The journal uses the procedure for reviewing as described on the websites:

https://jma.prz.edu.pl/en/reviewing-procedure

https://oficyna.prz.edu.pl/en/scientific-research-papers/the-faculty-of-mathematics-and-a/jurnal-of-mathematics-and-applic/jma-info-eng/reviewing-procedure

Information and instruction for authors available at:

https://jma.prz.edu.pl/en/information-for-authors

https://oficyna.prz.edu.pl/en/scientific-research-papers/the-faculty-of-mathematics-and-a/jurnal-of-mathematics-and-applic/jma-info-eng/information-for-authors

Review form available at:

https://jma.prz.edu.pl/en/reviewing-procedure

https://oficyna.prz.edu.pl/en/scientific-research-papers/the-faculty-of-mathematics-and-a/jurnal-of-mathematics-and-applic/jma-info-eng/reviewing-procedure

Contact details to Editorial Office available at:

https://jma.prz.edu.pl/en/contact-information

https://oficyna.prz.edu.pl/en/scientific-research-papers/the-faculty-of-mathematics-and-a/jurnal-of-mathematics-and-applic/jma-info-eng/contact-information

Electronic version of the published articles available at:

https://jma.prz.edu.pl/en/current-issue

https://oficyna.prz.edu.pl/en/scientific-research-papers/the-faculty-of-mathematics-and-a/jurnal-of-mathematics-and-applic

INFORMATION FOR AUTHORS

Journal of Mathematics and Applications (JMA) is an Open Access journal.
Journal of Mathematics and Applications publishes original research papers in the area of pure mathematics and its applications. Two types of articles will be accepted for publication, namely research articles and review articles. The authors are obligated to select the kind of their articles (research or review). Manuscript, written in English and prepared using LaTeX, may be submitted to the Editorial Office or one of the Editors or members of the Editorial Board. Electronic submission of pdf file is required.

Manuscripts should be written in English, and the first page should contain: title, name(s) of author(s), abstract not exceeding 200 words, primary and secondary 2010 Mathematics Subject Classification codes, list of key words and phrases.

Manuscripts should be produced using TeX (LaTeX) on one side of A4 (recommended format: 12-point type, including references, text width 12.5 cm, long 19 cm).

Authors' **affiliations** and full addresses (with e-mail addresses) should be given at the end of the article.

Figures, if not prepared using TeX, must be provided electronically in one of the following formats: EPS, CorelDraw, PDF, JPG, GIF.

References should be arranged in alphabetical order, and styled and punctuated according to the examples given below. Abbreviations of journal names should follow Mathematical Reviews.

Examples:

[6] D. Beck, *Introduction to Dynamical System*, Progr. Math. 54, Birkhäuser, Basel, 1978.
[7] R. Hill, A. James, *A new index formula*, J. Geometry 15 (1982) 19–3l.
[8] J. Kowalski, *Some remarks on J(X)*, in: Algebra and Analysis (Edmonton, 1973), E. Brook (ed.), Lecture Notes in Math. 867, Springer, Berlin, 1974, 115–124.

After the acceptance of the paper the authors will be asked to transmit the final source TeX file and pdf file corrected according to the reviewers' suggestions to jma@prz.edu.pl.

The **galley proofs** will be sent electronically to the corresponding author.

Publisher's sheet 8,58. Printer's sheet 12,00.
Manuscript completed in December 2022, Printed in December 2022.
Printing Publishing House, 12 Powstańców Warszawy Ave., 35-959 Rzeszów
Order no. 70/22