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Starlikeness and convexity of certain integral operators defined by convolution

Jyoti Aggarwal and Rachana Mathur

ABSTRACT: We define two new general integral operators for certain analytic functions in the unit disc \mathcal{U} and give some sufficient conditions for these integral operators on some subclasses of analytic functions.

AMS Subject Classification: *30C45*

Keywords and Phrases: *Multivalent functions, Starlike Functions, Convex Functions, Convolution*

1 Introduction

Let $\mathcal{A}_p(n)$ denote the class of all functions of the form

$$f(z) = z^p + \sum_{k=p+n}^{\infty} a_k z^k \quad (p, n \in N = \{1, 2, 3, \dots\}). \quad (1.1)$$

which is analytic in open unit disc $\mathcal{U} = \{z \in \mathbb{C} \mid |z| < 1\}$.

In particular, we set

$$\mathcal{A}_p(1) = \mathcal{A}_p, \mathcal{A}_1(1) = \mathcal{A}_1 := \mathcal{A}.$$

If $f \in \mathcal{A}_p(n)$ is given by (1.1) and $g \in \mathcal{A}_p(n)$ is given by

$$g(z) = z^p + \sum_{k=p+n}^{\infty} b_k z^k \quad (p, n \in N = \{1, 2, 3, \dots\}). \quad (1.2)$$

then the Hadamard product (or convolution) $f * g$ of f and g is given by

$$(f * g)(z) = z^p + \sum_{k=p+n}^{\infty} a_k b_k z^k = (g * f)(z). \quad (1.3)$$

We observe that several known operators are deducible from the convolutions. That is, for various choices of g in (1.3), we obtain some interesting operators. For example, for functions $f \in \mathcal{A}_p(n)$ and the function g is defined by

$$g(z) = z^p + \sum_{k=p+n}^{\infty} \psi_{k,m}(\alpha, \lambda, l, p) z^k \quad (m \in N_0 = N \cup \{0\}) \quad (1.4)$$

where

$$\psi_{k,m}(\alpha, \lambda, l, p) = \left[\frac{\Gamma(k+1)\Gamma(p-\alpha+1)}{\Gamma(p+1)\Gamma(k-\alpha+1)} \cdot \frac{p+\lambda(k-p)+l}{p+l} \right]^m.$$

The convolution (1.3) with the function g is defined by (1.4) gives an operator studied by Bulut ([1]).

$$(f * g)(z) = D_{\lambda, l, p}^{m, \alpha} f(z)$$

Using convolution we introduce the new classes $\mathcal{US}_g^p(\delta, \beta, b)$ and $\mathcal{UK}_g^p(\delta, \beta, b)$ as follows

Definition 1.1 A functions $f \in \mathcal{A}_p(n)$ is in the class $\mathcal{US}_g^p(\delta, \beta, b)$ if and only if f satisfies

$$\operatorname{Re} \left\{ p + \frac{1}{b} \left(\frac{z(f * g)'(z)}{(f * g)(z)} - p \right) \right\} > \delta \left| \frac{1}{b} \left(\frac{z(f * g)'(z)}{(f * g)(z)} - p \right) \right| + \beta, \quad (1.5)$$

where $z \in \mathcal{U}$, $b \in \mathbb{C} - \{0\}$, $\delta \geq 0$, $0 \leq \beta < p$.

Definition 1.2 A functions $f \in \mathcal{A}_p(n)$ is in the class $\mathcal{US}_g^p(\delta, \beta, b)$ if and only if f satisfies

$$\operatorname{Re} \left\{ p + \frac{1}{b} \left(1 + \frac{z(f * g)''(z)}{(f * g)'(z)} - p \right) \right\} > \delta \left| \frac{1}{b} \left(1 + \frac{z(f * g)''(z)}{(f * g)'(z)} - p \right) \right| + \beta, \quad (1.6)$$

where $z \in \mathcal{U}$, $b \in \mathbb{C} - \{0\}$, $\delta \geq 0$, $0 \leq \beta < p$.

Note that

$$f \in \mathcal{UK}_g^p(\delta, \beta, b) \iff \frac{zf'(z)}{p} \in \mathcal{US}_g^p(\delta, \beta, b).$$

Remark 1.1 (i) For $\delta = 0$, we have

$$\begin{aligned} \mathcal{UK}_g^p(0, \beta, b) &= \mathcal{K}_g^p(\beta, b) \\ \mathcal{US}_g^p(0, \beta, b) &= \mathcal{S}_g^p(\beta, b) \end{aligned}$$

(ii) For $\delta = 0$ and $\beta = 0$

$$\begin{aligned} \mathcal{UK}_g^p(0, 0, b) &= \mathcal{K}_g^p(b) \\ \mathcal{US}_g^p(0, 0, b) &= \mathcal{S}_g^p(b) \end{aligned}$$

(iii) For $\delta = 0$, $\beta = 0$ and $b = 1$

$$\begin{aligned} \mathcal{UK}_g^p(0, 0, 1) &= \mathcal{K}_g^p \\ \mathcal{US}_g^p(0, 0, 1) &= \mathcal{S}_g^p \end{aligned}$$

(iv) For $(f_j * g)(z) = D_{\lambda, l, p}^{m, \alpha} f_j(z)$, we have two classes $\mathcal{UK}_{\alpha, \lambda, l}^{m, j, p, n}(\delta_j, \beta_j, b)$ and $\mathcal{US}_{\alpha, \lambda, l}^{m, j, p, n}(\delta_j, \beta_j, b)$ which is introduced by Guney and Bulut [1].

Definition 1.3 Let $\eta \in N, m = (m_1, \dots, m_\eta) \in N_0^\eta$ and $k = (k_1, \dots, k_\eta) \in R_+^\eta$. One defines the following general integral operators:

$$\begin{aligned} \mathcal{I}_g^{p,\eta,m,k} &: \mathcal{A}_p(n)^\eta \rightarrow \mathcal{A}_p(n) \\ \mathcal{G}_g^{p,\eta,m,k} &: \mathcal{A}_p(n)^\eta \rightarrow \mathcal{A}_p(n) \end{aligned} \quad (1.7)$$

such that

$$\begin{aligned} \mathcal{I}_g^{p,\eta,m,k}(z) &= \int_0^z pt^{p-1} \prod_{j=1}^{\eta} \left(\frac{(f_j * g)(t)}{t^p} \right)^{k_j} dt, \\ \mathcal{G}_g^{p,\eta,m,k}(z) &= \int_0^z pt^{p-1} \prod_{j=1}^{\eta} \left(\frac{(f_j * g)'(t)}{pt^{p-1}} \right)^{k_j} dt, \end{aligned} \quad (1.8)$$

where $z \in \mathcal{U}, f_j, g \in \mathcal{A}_p(n), 1 \leq j \leq \eta$.

Remark 1.2 (i) For $\eta = 1, m_1 = m, k_1 = k$, and $f_1 = f$, we have the new two new integral operators

$$\begin{aligned} \mathcal{I}_g^{p,\eta,m,k}(z) &= \int_0^z pt^{p-1} \left(\frac{(f_j * g)(t)}{t^p} \right)^{k_j} dt, \\ \mathcal{G}_g^{p,\eta,m,k}(z) &= \int_0^z pt^{p-1} \left(\frac{(f_j * g)'(t)}{pt^{p-1}} \right)^{k_j} dt, \end{aligned} \quad (1.9)$$

(ii) For $(f_j * g)(z) = D_{\lambda,l,p}^{m,\alpha} f_j(z)$, we have

$$\begin{aligned} \mathcal{I}_g^{p,\eta,m,k}(z) &= \int_0^z pt^{p-1} \prod_{j=1}^{\eta} \left(\frac{D_{\lambda,l,p}^{m,\alpha} f_j(t)}{t^p} \right)^{k_j} dt, \\ \mathcal{G}_g^{p,\eta,m,k}(z) &= \int_0^z pt^{p-1} \prod_{j=1}^{\eta} \left(\frac{D_{\lambda,l,p}^{m,\alpha} f_j(t)'(t)}{pt^{p-1}} \right)^{k_j} dt, \end{aligned} \quad (1.10)$$

These operator were introduced by Bulut [1].

(iii) If we take $g(z) = z^p/(1-z)$, the we have

$$\begin{aligned} \mathcal{I}_g^{p,\eta,m,k}(z) &= \int_0^z pt^{p-1} \prod_{j=1}^{\eta} \left(\frac{(f_j)(t)}{t^p} \right)^{k_j} dt, \\ \mathcal{G}_g^{p,\eta,m,k}(z) &= \int_0^z pt^{p-1} \prod_{j=1}^{\eta} \left(\frac{(f_j)'(t)}{pt^{p-1}} \right)^{k_j} dt, \end{aligned} \quad (1.11)$$

These two operators were introduced by Frasin [3].

2 Sufficient Conditions for $\mathcal{I}_g^{p,\eta,m,k}(z)$

Theorem 2.1 Let $\eta \in N, m = (m_1, \dots, m_\eta) \in N_0^\eta$ and $k = (k_1, \dots, k_\eta) \in R_+^\eta$. Also let $b \in \mathbb{C} - \{0\}, \delta \geq 0, 0 \leq \beta < p$, and $f_j \in \mathcal{US}_g^p(\delta, \beta, b)$ for $1 \leq j \leq \eta$. If

$$0 \leq p + \sum_{j=1}^{\eta} k_j(\beta_j - p) < p, \quad (2.1)$$

then the integral operator $\mathcal{I}_g^{p,\eta,m,k}(z)$, defined by (1.8), is in the class $\mathcal{K}_g^p(\tau, b)$ where

$$\tau = p + \sum_{j=1}^{\eta} k_j(\beta_j - p).$$

Proof. From the definition (1.8), we observe that $\mathcal{I}_g^{p,\eta,m,k}(z) \in \mathcal{A}_p(n)$. We can easily see that

$$(\mathcal{I}^{p,\eta,m,k}(z))' = pz^{p-1} \prod_{j=1}^{\eta} \left(\frac{(f_j * g)(z)}{z^p} \right)^{k_j}. \quad (2.2)$$

Differentiating (2.2) logarithmically and multiplying by 'z', we obtain

$$\frac{z(\mathcal{I}^{p,\eta,m,k}(z))''}{(\mathcal{I}^{p,\eta,m,k}(z))'} = p - 1 + \sum_{j=1}^{\eta} k_j \left(\frac{z((f_j * g)(z))'}{(f_j * g)(z)} - p \right) \quad (2.3)$$

or equivalently

$$1 + \frac{z(\mathcal{I}^{p,\eta,m,k}(z))''}{(\mathcal{I}^{p,\eta,m,k}(z))'} - p = \sum_{j=1}^{\eta} k_j \left(\frac{z((f_j * g)(z))'}{(f_j * g)(z)} - p \right) \quad (2.4)$$

Then, by multiplying (2.4) with '1/b', we have

$$\frac{1}{b} \left(1 + \frac{z(\mathcal{I}^{p,\eta,m,k}(z))''}{(\mathcal{I}^{p,\eta,m,k}(z))'} - p \right) = \sum_{j=1}^{\eta} k_j \frac{1}{b} \left(\frac{z((f_j * g)(z))'}{(f_j * g)(z)} - p \right) \quad (2.5)$$

or

$$\begin{aligned} & p + \frac{1}{b} \left(1 + \frac{z(\mathcal{I}^{p,\eta,m,k}(z))''}{(\mathcal{I}^{p,\eta,m,k}(z))'} - p \right) \\ &= p + \sum_{j=1}^{\eta} k_j \frac{1}{b} \left(\frac{z((f_j * g)(z))'}{(f_j * g)(z)} - p + p - p \sum_{j=1}^{\eta} k_j \right) \end{aligned} \quad (2.6)$$

Since $f_j \in \mathcal{US}_g^p(\delta_j, \beta_j, b)$ ($1 \leq j \leq \eta$), we get

$$\begin{aligned} & \operatorname{Re} \left\{ p + \frac{1}{b} \left(1 + \frac{z(\mathcal{I}^{p,\eta,m,k}(z))''}{(\mathcal{I}^{p,\eta,m,k}(z))'} - p \right) \right\} \\ &= p + \sum_{j=1}^{\eta} k_j \operatorname{Re} \left\{ \frac{1}{b} \left(\frac{z((f_j * g)(z))'}{(f_j * g)(z)} - p \right) \right\} + p - \sum_{j=1}^{\eta} pk_j \\ &> \sum_{j=1}^{\eta} k_j \delta_j \left| \frac{1}{b} \left(\frac{z((f_j * g)(z))'}{(f_j * g)(z)} - p \right) \right| + p + \sum_{j=1}^{\eta} k_j(\beta_j - p). \end{aligned} \quad (2.7)$$

Since

$$\sum_{j=1}^{\eta} k_j \delta_j \left| \frac{1}{b} \left(\frac{z((f_j * g)(z))'}{(f_j * g)(z)} - p \right) \right| > 0$$

because the integral operator $\mathcal{I}_g^{p,\eta,m,k}(z)$, defined by (1.8), is in the class $\mathcal{K}_g^p(\tau, b)$ with

$$\tau = p + \sum_{j=1}^{\eta} k_j (\beta_j - p).$$

■

3 Sufficient Conditions for $\mathcal{G}_g^{p,\eta,m,k}(z)$

Theorem 3.1 Let $\eta \in \mathbb{N}$, $m = (m_1, \dots, m_\eta) \in \mathbb{N}_0^\eta$ and $k = (k_1, \dots, k_\eta) \in \mathbb{R}_+^\eta$. Also let $b \in \mathbb{C} - \{0\}$, $\delta \geq 0$, $0 \leq \beta < p$, and $f_j \in \mathcal{US}_g^p(\delta, \beta, b)$ for $1 \leq j \leq \eta$. If

$$0 \leq p + \sum_{j=1}^{\eta} k_j (\beta_j - p) < p, \quad (3.1)$$

then the integral operator $\mathcal{G}_g^{p,\eta,m,k}(z)$, defined by (1.8), is in the class $\mathcal{K}_g^p(\tau, b)$ where

$$\tau = p + \sum_{j=1}^{\eta} k_j (\beta_j - p).$$

Proof. From the definition (1.8), we observe that $\mathcal{I}_g^{p,\eta,m,k}(z) \in \mathcal{A}_p(n)$. We can easily see that

$$(\mathcal{G}^{p,\eta,m,k}(z))' = pz^{p-1} \prod_{j=1}^{\eta} \left(\frac{(f_j * g)'(z)}{pz^{p-1}} \right)^{k_j}. \quad (3.2)$$

Differentiating (3.2) logarithmically and multiplying by 'z', we obtain

$$\frac{z(\mathcal{G}^{p,\eta,m,k}(z))''}{(\mathcal{G}^{p,\eta,m,k}(z))'} = p - 1 + \sum_{j=1}^{\eta} k_j \left(\frac{z((f_j * g)(z))''}{(f_j * g)'(z)} + 1 - p \right) \quad (3.3)$$

or equivalently

$$1 + \frac{z(\mathcal{G}^{p,\eta,m,k}(z))''}{(\mathcal{G}^{p,\eta,m,k}(z))'} - p = \sum_{j=1}^{\eta} k_j \left(\frac{z((f_j * g)(z))''}{((f_j * g)(z))'} + 1 - p \right) \quad (3.4)$$

Then, by multiplying (3.4) with '1/b', we have

$$\frac{1}{b} \left(1 + \frac{z(\mathcal{G}^{p,\eta,m,k}(z))''}{(\mathcal{G}^{p,\eta,m,k}(z))'} - p \right) = \sum_{j=1}^{\eta} k_j \frac{1}{b} \left(\frac{z((f_j * g)(z))''}{(f_j * g)'(z)} + 1 - p \right) \quad (3.5)$$

or

$$p + \frac{1}{b} \left(\frac{z (\mathcal{G}^{p,\eta,m,k}(z))''}{(\mathcal{G}^{p,\eta,m,k}(z))'} + 1 - p \right) = p + \sum_{j=1}^{\eta} k_j \frac{1}{b} \left(\frac{z ((f_j * g)(z))''}{(f_j * g)'(z)} + 1 - p + p - p \sum_{j=1}^{\eta} k_j \right) \quad (3.6)$$

Since $f_j \in \mathcal{UK}_g^p(\delta_j, \beta_j, b)$ ($1 \leq j \leq \eta$), we get

$$\begin{aligned} & \operatorname{Re} \left\{ p + \frac{1}{b} \left(1 + \frac{z (\mathcal{G}^{p,\eta,m,k}(z))''}{(\mathcal{G}^{p,\eta,m,k}(z))'} - p \right) \right\} \quad (3.7) \\ &= p + \sum_{j=1}^{\eta} k_j \operatorname{Re} \left\{ \frac{1}{b} \left(\frac{z ((f_j * g)(z))''}{(f_j * g)'(z)} + 1 - p \right) \right\} + p - \sum_{j=1}^{\eta} p k_j + p + \sum_{j=1}^{\eta} k_j (\beta_j - p). \\ &> \sum_{j=1}^{\eta} k_j \delta_j \left| \frac{1}{b} \left(\frac{z ((f_j * g)(z))''}{(f_j * g)'(z)} + 1 - p \right) \right| + p + \sum_{j=1}^{\eta} k_j (\beta_j - p). \end{aligned}$$

Since

$$\sum_{j=1}^{\eta} k_j \delta_j \left| \frac{1}{b} \left(\frac{z ((f_j * g)(z))''}{(f_j * g)'(z)} + 1 - p \right) \right| > 0$$

because the integral operator $\mathcal{G}_g^{p,\eta,m,k}(z)$, defined by (1.8), is in the class $\mathcal{K}_g^p(\tau, b)$ with

$$\tau = p + \sum_{j=1}^{\eta} k_j (\beta_j - p).$$

■

4 Corollaries and Consequences

For $\eta = 1, m_1 = m, k_1 = k$, and $f_1 = f$, we have

Corollary 4.1 *Let $\eta \in N, m \in N_0^\eta$ and $k \in R_+^\eta$. Also let $b \in \mathbb{C} - \{0\}, \delta \geq 0, 0 \leq \beta < p$, and $f \in \mathcal{US}_g^p(\delta, \beta, b)$ for $1 \leq j \leq \eta$. If*

$$0 \leq p + k(\beta - p) < p, \quad (4.1)$$

then the integral operator $\mathcal{I}_g^{p,\eta,m,k}(z)$ is in the class $\mathcal{K}_g^p(\tau, b)$ where

$$\tau = p + k(\beta - p).$$

Corollary 4.2 *Let $\eta \in N, m \in N_0^\eta$ and $k \in R_+^\eta$. Also let $b \in \mathbb{C} - \{0\}, \delta \geq 0, 0 \leq \beta < p$, and $f \in \mathcal{US}_g^p(\delta, \beta, b)$ for $1 \leq j \leq \eta$. If*

$$0 \leq p + k(\beta - p) < p, \quad (4.2)$$

then the integral operator $\mathcal{G}_g^{p,\eta,m,k}(z)$ is in the class $\mathcal{K}_g^p(\tau, b)$ where

$$\tau = p + k(\beta - p).$$

For $(f_j * g)(z) = D_{\lambda, l, p}^{m, \alpha} f_j(z)$, we have

Corollary 4.3 Let $\eta \in N, m = (m_1, \dots, m_\eta) \in N_0^\eta$ and $k = (k_1, \dots, k_\eta) \in R_+^\eta$. Also let $b \in \mathbb{C} - \{0\}, \delta \geq 0, 0 \leq \beta < p$, and $f_j \in \mathcal{US}_{\alpha, \lambda, l}^{m, j, p, n}(\delta_j, \beta_j, b)$ for $1 \leq j \leq \eta$. If

$$0 \leq p + \sum_{j=1}^{\eta} k_j(\beta_j - p) < p, \quad (4.3)$$

then the integral operator $\mathcal{I}_{p, \eta, m, k}(z)$ is in the class $\mathcal{K}^{p, n}(\tau, b)$ where

$$\tau = p + \sum_{j=1}^{\eta} k_j(\beta_j - p).$$

Corollary 4.4 Let $\eta \in N, m = (m_1, \dots, m_\eta) \in N_0^\eta$ and $k = (k_1, \dots, k_\eta) \in R_+^\eta$. Also let $b \in \mathbb{C} - \{0\}, \delta \geq 0, 0 \leq \beta < p$, and $\mathcal{UK}_{\alpha, \lambda, l}^{m, j, p, n}(\delta_j, \beta_j, b)$ for $1 \leq j \leq \eta$. If

$$0 \leq p + \sum_{j=1}^{\eta} k_j(\beta_j - p) < p, \quad (4.4)$$

then the integral operator $\mathcal{G}_{p, \eta, m, k}(z)$ is in the class $\mathcal{K}^{p, n}(\tau, b)$ where

$$\tau = p + \sum_{j=1}^{\eta} k_j(\beta_j - p).$$

which are known results obtained by Guney and Bulut [2].

Further, if put $p = 1$, we have

Corollary 4.5 Let $\eta \in N, m = (m_1, \dots, m_\eta) \in N_0^\eta$ and $k = (k_1, \dots, k_\eta) \in R_+^\eta$. Also let $b \in \mathbb{C} - \{0\}, \delta \geq 0, 0 \leq \beta < 1$, and $f_j \in \mathcal{US}_g^p(\delta, \beta, b)$ for $1 \leq j \leq \eta$. If

$$0 \leq 1 + \sum_{j=1}^{\eta} k_j(\beta_j - 1) < 1, \quad (4.5)$$

then the integral operator $\mathcal{I}_g^{1, \eta, m, k}(z)$ is in the class $\mathcal{K}_g^1(\tau, b)$ where

$$\tau = 1 + \sum_{j=1}^{\eta} k_j(\beta_j - 1).$$

Corollary 4.6 Let $\eta \in N, m = (m_1, \dots, m_\eta) \in N_0^\eta$ and $k = (k_1, \dots, k_\eta) \in R_+^\eta$. Also let $b \in \mathbb{C} - \{0\}, \delta \geq 0, 0 \leq \beta < 1$, and $f_j \in \mathcal{US}_g^1(\delta, \beta, b)$ for $1 \leq j \leq \eta$. If

$$0 \leq 1 + \sum_{j=1}^{\eta} k_j(\beta_j - 1) < 1, \quad (4.6)$$

then the integral operator $\mathcal{G}_g^{1,\eta,m,k}(z)$ is in the class $\mathcal{K}_g^1(\tau, b)$ where

$$\tau = 1 + \sum_{j=1}^{\eta} k_j(\beta_j - 1).$$

Upon setting $g(z) = z^p/(1-z)$, we have

Corollary 4.7 *Let $\eta \in N, m = (m_1, \dots, m_\eta) \in N_0^\eta$ and $k = (k_1, \dots, k_\eta) \in R_+^\eta$. Also let $b \in \mathbb{C} - \{0\}, \delta \geq 0, 0 \leq \beta < p$, and $f_j \in \mathcal{US}^p(\delta, \beta, b)$ for $1 \leq j \leq \eta$. If*

$$0 \leq p + \sum_{j=1}^{\eta} k_j(\beta_j - p) < p, \quad (4.7)$$

then the integral operator $\mathcal{G}^{p,\eta,m,k}(z)$ is in the class $\mathcal{K}^p(\tau, b)$ where

$$\tau = p + \sum_{j=1}^{\eta} k_j(\beta_j - p).$$

Corollary 4.8 *Let $\eta \in N, m = (m_1, \dots, m_\eta) \in N_0^\eta$ and $k = (k_1, \dots, k_\eta) \in R_+^\eta$. Also let $b \in \mathbb{C} - \{0\}, \delta \geq 0, 0 \leq \beta < p$, and $f_j \in \mathcal{US}^p(\delta, \beta, b)$ for $1 \leq j \leq \eta$. If*

$$0 \leq p + \sum_{j=1}^{\eta} k_j(\beta_j - p) < p, \quad (4.8)$$

then the integral operator $\mathcal{G}^{p,\eta,m,k}(z)$ is in the class $\mathcal{K}^p(\tau, b)$ where

$$\tau = p + \sum_{j=1}^{\eta} k_j(\beta_j - p).$$

Upon setting $g(z) = z^p/(1-z)$ and $\delta = 0$, we have

Corollary 4.9 *Let $\eta \in N, m = (m_1, \dots, m_\eta) \in N_0^\eta$ and $k = (k_1, \dots, k_\eta) \in R_+^\eta$. Also let $b \in \mathbb{C} - \{0\}, 0 \leq \beta < p$, and $f_j \in \mathcal{US}^p(0, \beta, b)$ for $1 \leq j \leq \eta$. If*

$$0 \leq p + \sum_{j=1}^{\eta} k_j(\beta_j - p) < p, \quad (4.9)$$

then the integral operator $\mathcal{G}^{p,\eta,m,k}(z)$ is in the class $\mathcal{K}^p(\tau, b)$ where

$$\tau = p + \sum_{j=1}^{\eta} k_j(\beta_j - p).$$

Corollary 4.10 Let $\eta \in N$, $m = (m_1, \dots, m_\eta) \in N_0^\eta$ and $k = (k_1, \dots, k_\eta) \in R_+^\eta$. Also let $b \in \mathbb{C} - \{0\}$, $\delta \geq 0$, $0 \leq \beta < p$, and $f_j \in \mathcal{US}^p(0, \beta, b)$ for $1 \leq j \leq \eta$. If

$$0 \leq p + \sum_{j=1}^{\eta} k_j(\beta_j - p) < p, \quad (4.10)$$

then the integral operator $\mathcal{G}^{p,\eta,m,k}(z)$ is in the class $\mathcal{K}^p(\tau, b)$ where

$$\tau = p + \sum_{j=1}^{\eta} k_j(\beta_j - p).$$

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On e - \mathcal{I} -open sets, e - \mathcal{I} -continuous functions and decomposition of continuity

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ABSTRACT: In this paper, we introduce the notations of e - \mathcal{I} -open sets and strong \mathcal{B}_I^* -set to obtain a decomposition of continuity via idealization. Additionally, we investigate properties of e - \mathcal{I} -open sets and strong \mathcal{B}_I^* -set. Also we studied some more properties of e - \mathcal{I} -open sets and obtained several characterizations of e - \mathcal{I} -continuous functions and investigate their relationship with other types of functions.

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1 Introduction and Preliminaries

The subject of ideals in topological spaces has been studied by Kuratowski [12] and Vaidyanathaswamy [25]. Jankovic and Hamlett [11] investigated further properties of ideal space. The importance of continuity and generalized continuity is significant in various areas of mathematics and related sciences. One of them, which has been in recent years of interest to general topologists, is its decomposition. The decomposition of continuity has been studied by many authors. The class of e -open sets contains all δ -preopen [15] sets and δ -semiopen [14] sets. In this paper, we introduce the notation of e - \mathcal{I} -open sets which is a generalization of $semi^*$ - \mathcal{I} -open sets [8] and pre^* - \mathcal{I} -open [5] sets is introduced, and strong \mathcal{B}_I^* -set to obtain a decomposition of continuity via idealization. Additionally, we investigate properties of e - \mathcal{I} -open sets and strong \mathcal{B}_I^* -set. Also we studied some more properties of e - \mathcal{I} -open sets and obtained several characterizations of e - \mathcal{I} -continuous functions and investigate their relationship with other types of functions.

A subset A of a space (X, τ) is said to be regular open (resp. regular closed) [23] if $A = Int(Cl(A))$ (resp. $A = Cl(Int(A))$). A is called δ -open [26] if for each $x \in A$,

there exist a regular open set G such that $x \in G \subset A$. The complement of δ -open set is called δ -closed. A point $x \in X$ is called a δ -cluster point of A if $Int(Cl(U)) \cap A \neq \emptyset$ for each open set U containing x . The set of all δ -cluster points of A is called the δ -closure of A and is denoted by $Cl_\delta(A)$ [26]. The set δ -interior of A [26] is the union of all regular open sets of X contained in A and its denoted by $Int_\delta(A)$. A is δ -open if $Int_\delta(A) = A$. The collection of all δ -open sets of (X, τ) is denoted by $\delta O(X)$ and forms a topology τ^δ . The topology τ^δ is called the semi regularization of τ and is denoted by τ_s .

An ideal \mathcal{I} on a topological space (X, τ) is a nonempty collection of subsets of X which satisfies the following conditions:

$A \in \mathcal{I}$ and $B \subset A$ implies $B \in \mathcal{I}$; $A \in \mathcal{I}$ and $B \in \mathcal{I}$ implies $A \cup B \in \mathcal{I}$. Applications to various fields were further investigated by Jankovic and Hamlett [11] Dontchev et al. [3]; Mukherjee et al. [13]; Arenas et al. [2]; et al. Nasef and Mahmoud [18], etc. Given a topological space (X, τ) with an ideal \mathcal{I} on X and if $\wp(X)$ is the set of all subsets of X , a set operator $(\cdot)^* : \wp(X) \rightarrow \wp(X)$, called a local function [24, 11] of A with respect to τ and \mathcal{I} is defined as follows: for $A \subseteq X$,

$$A^*(\mathcal{I}, \tau) = \{x \in X \mid U \cap A \notin \mathcal{I} \text{ for every } U \in \tau(x)\}$$

where $\tau(x) = \{U \in \tau \mid x \in U\}$. A Kuratowski closure operator $Cl^*(x) = A \cup A^*(\mathcal{I}, \tau)$. When there is no chance for confusion, we will simply write A^* for $A^*(\mathcal{I}, \tau)$. X^* is often a proper subset of X .

A subset A of an ideal space (X, τ) is said to be R - \mathcal{I} -open (resp. R - \mathcal{I} -closed) [28] if $A = Int(Cl^*(A))$ (resp. $A = Cl^*(Int(A))$). A point $x \in X$ is called δ - \mathcal{I} -cluster point of A if $Int(Cl^*(U)) \cap A \neq \emptyset$ for each open set U containing x . The family of all δ - \mathcal{I} -cluster points of A is called the δ - \mathcal{I} -closure of A and is denoted by $\delta Cl_{\mathcal{I}}(A)$. The set δ - \mathcal{I} -interior of A is the union of all R - \mathcal{I} -open sets of X contained in A and its denoted by $\delta Int_{\mathcal{I}}(A)$. A is said to be δ - \mathcal{I} -closed if $\delta Cl_{\mathcal{I}}(A) = A$ [28].

Definition 1.1. A subset A of a topological space X is called

1. β -open [1] if $A \subset Cl(Int(Cl(A)))$.
2. α -open [19] if $A \subset Int(Cl(Int(A)))$.
3. t -set [22] if $Int(A) = Int(Cl(A))$.
4. e -open set [7] if $A \subset Int(\delta Cl(A)) \cup Cl(\delta Int(A))$.
5. strongly B -set [7] if $A = U \cap V$ where U is an open set and V is a t -set and $Int(Cl(A)) = Cl(Int(A))$.
6. δ -preopen [15] if $A \subset Int(\delta Cl(A))$.
7. δ -semiopen [14] if $A \subset Cl(\delta Int(A))$.
8. a -open [4] if $A \subset Int(Cl(\delta Int(A)))$.

The class of all δ -preopen (resp. δ -semiopen, α -open) sets of (X, τ) is denoted by $\delta PO(X)$ (resp. $\delta SO(X)$, $\alpha O(X)$).

Definition 1.2. A subset A of an ideal topological space (X, τ, \mathcal{I}) is called

1. $\delta\alpha\mathcal{I}$ -open [8] if $A \subset \text{Int}(Cl(\delta\text{Int}_I(A)))$.
2. semi* \mathcal{I} -open [8] if $A \subset Cl(\delta\text{Int}_I(A))$.
3. pre* \mathcal{I} -open [5] if $A \subseteq \text{Int}(\delta Cl_I(A))$.
4. Strongly $t\mathcal{I}$ -set [5] if $\text{Int}(A) = \text{Int}(\delta Cl_I(A))$.
5. Strongly $B\mathcal{I}$ -set [5] if $A = U \cap V$ where U is an open set and V is a Strongly $t\mathcal{I}$ -set.
6. $\delta\beta_I$ -open [8] if $A \subset \text{Int}(Cl(\delta\text{Int}_I(A)))$.
7. B_I -set [9] if $A = U \cap V$ where U is an open set and V is a $t\mathcal{I}$ -set.

The class of all semi* \mathcal{I} -open (resp. pre* \mathcal{I} -open, $\delta\beta_I$ -open, $\delta\alpha\mathcal{I}$ -open) sets of (X, τ, \mathcal{I}) is denoted by $S^*IO(X)$ (resp. $P^*IO(X)$, $\delta\beta IO(X)$, $\delta\alpha IO(X)$). [8, 5].

2 $e\mathcal{I}$ -open

Definition 2.1. A subset A of an ideal topological space (X, τ, \mathcal{I}) is said to be $e\mathcal{I}$ -open if $A \subset Cl(\delta\text{Int}_I(A)) \cup \text{Int}(\delta Cl_I(A))$.

The class of all $e\mathcal{I}$ -open sets in X will be denoted by $EIO(X, \tau)$.

Proposition 2.2. Let A be an $e\mathcal{I}$ -open such that $\delta\text{Int}_I(A) = \emptyset$, then A is pre* \mathcal{I} -open. For a subset of an ideal topological space the following hold:

1. Every semi* \mathcal{I} -open is $e\mathcal{I}$ -open,
2. Every pre* \mathcal{I} -open is $e\mathcal{I}$ -open,
3. Every $e\mathcal{I}$ -open is $\delta\beta_I$ -open.

Proof. (1) Obvious.

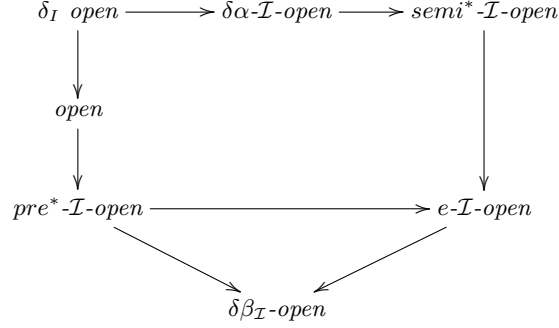
(2) Obvious.

(3) Let A be $e\mathcal{I}$ -open. Then we have

$$\begin{aligned}
 A &\subset Cl(\delta\text{Int}_I(A)) \cup \text{Int}(\delta Cl_I(A)) \\
 &\subset Cl(\text{Int}(\delta\text{Int}_I(A))) \cup \text{Int}(\text{Int}(\delta Cl_I(A))) \\
 &\subset Cl(\text{Int}(\delta\text{Int}_I(A)) \cup \text{Int}(\delta Cl_I(A))) \\
 &\subset Cl[\text{Int}(\delta\text{Int}_I(A)) \cup \delta Cl_I(A)] \\
 &\subset Cl[\text{Int}(\delta Cl_I(A \cup A))] \\
 &= Cl(\text{Int}(\delta Cl_I(A))).
 \end{aligned}$$

This show that A is an $\delta\beta_I$ -open set. □

Remark 2.3. From above the following implication and none of these implications is reversible as shown by examples given below



Example 2.4. Let $X = \{a, b, c, d\}$ with a topology $\tau = \{\emptyset, X, \{b\}, \{a, d\}, \{a, b, d\}\}$ and an ideal $\mathcal{I} = \{\emptyset, \{b\}\}$. Then the set $A = \{b, d\}$ is $e\text{-}\mathcal{I}\text{-open}$, but is not $\text{semi}^*\text{-}\mathcal{I}\text{-open}$. Because $Cl(\delta Int_{\mathcal{I}}(A)) \cup Int(\delta Cl_{\mathcal{I}}(A)) = Cl(\emptyset) \cup Int(X) = \emptyset \cup X = X \supset A$ and hence A is $e\text{-}\mathcal{I}\text{-open}$. Since $Cl(\delta Int_{\mathcal{I}}(A)) = Cl(\emptyset) = \emptyset \not\supseteq A$. So A is not $\text{semi}^*\text{-}\mathcal{I}\text{-open}$.

Example 2.5. Let $X = \{a, b, c\}$ with a topology $\tau = \{\emptyset, X, \{a\}, \{b\}, \{a, b\}\}$ and an ideal $\mathcal{I} = \{\emptyset, \{b\}\}$. Then the set $A = \{a, c\}$ is $e\text{-}\mathcal{I}\text{-open}$, but is not $\text{pre}^*\text{-}\mathcal{I}\text{-open}$. For $Cl(\delta Int_{\mathcal{I}}(A)) \cup Int(\delta Cl_{\mathcal{I}}(A)) = Cl(\{a, b\}) \cup Int(\{a, c\}) = \{a, b, c\} \cup \{a\} = X \supset A$ and hence A is $e\text{-}\mathcal{I}\text{-open}$. Since $Int(\delta Cl_{\mathcal{I}}(A)) = Int(\{a, c\}) = \{a\} \not\supseteq A$. Hence A is not $\text{Pre}^*\text{-}\mathcal{I}\text{-open}$.

Example 2.6. Let $X = \{a, b, c, d\}$ with a topology $\tau = \{\emptyset, X, \{b\}, \{a, d\}, \{a, b, d\}\}$ and an ideal $\mathcal{I} = \{\emptyset, \{b\}\}$. Then the set $A = \{a, c\}$ is $\delta\beta_{\mathcal{I}}\text{-open}$, but is not $e\text{-}\mathcal{I}\text{-open}$. Since $Cl(\delta Int_{\mathcal{I}}(A)) \cup Int(\delta Cl_{\mathcal{I}}(A)) = Cl(\emptyset) \cup Int(\{a, c, d\}) = \{a, d\} \not\supseteq A$ and hence A is not $e\text{-}\mathcal{I}\text{-open}$. For $Cl(Int(\delta Cl_{\mathcal{I}}(A))) = Cl(Int(\{a, c, d\})) = Cl(\{a, d\}) = \{a, c, d\} \supseteq A$. Hence A is $\delta\beta_{\mathcal{I}}\text{-open}$.

Proposition 2.7. Let (X, τ, \mathcal{I}) be an ideal topological space and let $A, U \subseteq X$. If A is $e\text{-}\mathcal{I}\text{-open}$ set and $U \in \tau$. Then $A \cap U$ is an $e\text{-}\mathcal{I}\text{-open}$.

Proof. By assumption $A \subset Cl(\delta Int_{\mathcal{I}}(A)) \cup Int(\delta Cl_{\mathcal{I}}(A))$ and $U \subseteq Int(U)$. Then

$$\begin{aligned}
 A \cap U &\subset (Cl(\delta Int_{\mathcal{I}}(A)) \cup Int(\delta Cl_{\mathcal{I}}(A))) \cap Int(U) \\
 &\subset (Cl(\delta Int_{\mathcal{I}}(A)) \cap Int(U)) \cup (Int(\delta Cl_{\mathcal{I}}(A)) \cap Int(U)) \\
 &\subset (Cl(\delta Int_{\mathcal{I}}(A)) \cap Cl(Int(U))) \cup (Int(\delta Cl_{\mathcal{I}}(A)) \cap Cl(Int(U))) \\
 &\subset (Cl(\delta Int_{\mathcal{I}}(A)) \cap Int(U)) \cup (Int(Cl(\delta Cl_{\mathcal{I}}(A)) \cap Cl(Cl(Int(U)))))) \\
 &\subset Cl(\delta Int_{\mathcal{I}}(A \cap U)) \cup (Int(Cl(\delta Cl_{\mathcal{I}}(A)) \cap Cl(Int(U)))) \\
 &\subset Cl(\delta Int_{\mathcal{I}}(A \cap U)) \cup (Int(Cl(\delta Cl_{\mathcal{I}}(A)) \cap Int(U))) \\
 &\subset Cl(\delta Int_{\mathcal{I}}(A \cap U)) \cup (Int(\delta Cl_{\mathcal{I}}(A \cap U))).
 \end{aligned}$$

Thus $A \cap U$ is $e\text{-}\mathcal{I}\text{-open}$. □

Definition 2.8. A subset A of an ideal topological space (X, τ, \mathcal{I}) is said to be $e\mathcal{I}$ -closed if its complement is $e\mathcal{I}$ -open.

Theorem 2.9. A subset A of an ideal topological space (X, τ, \mathcal{I}) is $e\mathcal{I}$ -closed, then $Cl(\delta Int_I(A)) \cap Int(\delta Cl_I(A)) \subset A$.

Proof. Since A is $e\mathcal{I}$ -closed, $X - A$ is $e\mathcal{I}$ -open, from the fact τ^* finer than τ , and the fact $\tau^\delta \subset \tau^{\delta\mathcal{I}}$ we have,

$$\begin{aligned} X - A &\subset Cl(\delta Int_I(X - A)) \cup Int(\delta Cl_I(X - A)) \\ &\subset Cl(\delta Int(X - A)) \cup Int(\delta Cl(X - A)) \\ &= [X - [Cl(\delta Int(A))]] \cup [X - [Int(\delta Cl(A))]] \\ &\subset [X - [Cl(\delta Int_I(A))]] \cup [X - [Int(\delta Cl_I(A))]] \\ &= X - [[Cl(\delta Int_I(A)) \cap Int(\delta Cl_I(A))]]. \end{aligned}$$

Therefore we obtain $[Cl(\delta Int_I(A)) \cap Int(\delta Cl_I(A))] \subset A$. □

Corollary 2.10. A subset A of an ideal topological space (X, τ, \mathcal{I}) such that $X - [Cl(\delta Int_I(A))] = Int(\delta Cl_I(X - A))$ and $X - [Int(\delta Cl_I(A))] = Cl(\delta Int_I(X - A))$. Then A is $e\mathcal{I}$ -closed if and only if $[Cl(\delta Int_I(A)) \cap Int(\delta Cl_I(A))] \subset A$.

Proof. Necessity: This is immediate consequence of Theorem 2.9

Sufficiency: Let $[Cl(\delta Int_I(A)) \cap Int(\delta Cl_I(A))] \subset A$. Then

$$\begin{aligned} X - A &\subset X - [Cl(\delta Int_I(A)) \cap Int(\delta Cl_I(A))] \\ &\subset [X - [Cl(\delta Int_I(A))]] \cup [X - [Int(\delta Cl_I(A))]] \\ &= Cl(\delta Int_I(X - A)) \cup Int(\delta Cl_I(X - A)) \end{aligned}$$

Thus $X - A$ is $e\mathcal{I}$ -open and hence A is $e\mathcal{I}$ -closed. □

If (X, τ, \mathcal{I}) is an ideal topological space and A is a subset of X , we denote by $\mathcal{I}|_A$. If (X, τ, \mathcal{I}) relative ideal on A and $\mathcal{I}|_A = \{A \cap I : I \in \mathcal{I}\}$ is obviously an ideal on A .

Lemma 2.11. [11] Let (X, τ, \mathcal{I}) be an ideal topological space and A, B subsets of X such that $B \subset A$. Then $B^*(\tau|_A, \mathcal{I}|_A) = B^*(\tau, \mathcal{I}) \cap A$.

Proposition 2.12. Let (X, τ, \mathcal{I}) be ideal topological space and let $A, U \subseteq X$. If A is an $e\mathcal{I}$ -open set and $U \in \tau$. Then $A \cap U \in EIO(U, \tau|_U, \mathcal{I}|_U)$.

Proof. Straight forward from Proposition 2.7 □

Theorem 2.13. If $A \in EIO(X, \tau, \mathcal{I})$ and $B \subset \tau$, then $A \cap B \in EIO(X, \tau, \mathcal{I})$.

Proof. Let $A \in EIO(X, \tau, \mathcal{I})$ and $B \subset \tau$ then $A \subset Cl(\delta Int_I(A)) \cup Int(\delta Cl_I(A))$ and

$$\begin{aligned} A \cap B &\subset [Cl(\delta Int_I(A)) \cup Int(\delta Cl_I(A))] \cap B \\ &\subset [Cl(\delta Int_I(A)) \cap B] \cup [Int(\delta Cl_I(A)) \cap B] \\ &\subset [Cl(\delta Int_I(A \cap B))] \cup [Int(\delta Cl_I(A \cap B))]. \end{aligned}$$

This proof come from the fact $\delta Int_I(A)$ is the union of all $R\mathcal{I}$ -open of X contained in A . Then

$$\begin{aligned} A = Int(Cl^*(A)) &\Rightarrow A \cap B = Int(Cl^*(A)) \cap B \\ &= Int(A^* \cup A) \cap B \\ &= Int[(A \cap B) \cup (A^* \cap B)] \\ &\subset Int[Cl^*(A \cap B)] = A \cap B \end{aligned}$$

Hence $Cl(\delta Int_I(A)) \cap B \subset Cl(\delta Int_I(A \cap B))$, and other part is obvious. \square

Proposition 2.14. *for any ideal topological space (X, τ, \mathcal{I}) and $A \subset X$ we have:*

1. *If $I = \emptyset$, then A is $e\mathcal{I}$ -open if and only if A is e -open.*
2. *If $I = \wp(X)$, then A is $e\mathcal{I}$ -open if and only if $A \in \tau$.*
3. *If $I = N$, then A is $e\mathcal{I}$ -open if and only if A is e -open.*

Proof. (1) Let $I = \emptyset$ and $A \subset X$. We have $\delta Cl_I(A) = \delta Cl(A)$, $\delta Int_I(A) = \delta Int(A)$ and $A^* = Cl(A)$. on other hand, $Cl^*(A) = A^* \cup A = Cl(A)$. Hence $A^* = Cl(A) = Cl^*(A)$. Since A is $e\mathcal{I}$ -open

$$A \subset Cl(\delta Int_I(A)) \cup Int(\delta Cl_I(A)) = Cl(\delta Int(A)) \cup Int(\delta Cl(A))$$

Thus, A is e -open.

Conversely, let A is e -open. Since $I = \emptyset$, then

$$A \subset Cl(\delta Int(A)) \cup Int(\delta Cl(A)) = Cl(\delta Int_I(A)) \cup Int(\delta Cl_I(A))$$

Thus A is $e\mathcal{I}$ -open.

(2) Let $I = P(X)$ and $A \subset X$. We have $A^* = \emptyset$. Since $\delta Int_I(A)$ is the union of all $R\mathcal{I}$ -open contained in A , since $A^* = \emptyset$, then $Int(A) = A$, and $\delta Cl_I(A)$ is the family of all $\delta\mathcal{I}$ -cluster points of A , since $A^* = \emptyset$, then $Int(A) \cap A \neq \emptyset$ On other hand

$$\begin{aligned} A &\subset Cl(\delta Int_I(A)) \cup Int(\delta Cl_I(A)) \\ &= Cl(Int(A)) \cup Int(Cl(A)) \\ &\subset Int(Cl(Int(A))) \cup Int(Cl(A)) \\ &= Int(Cl(Int(A)) \cup Int(Cl(A))) \\ &\subset Int(Cl(Int(A) \cup Cl(A))) \\ &\subset Int(Cl(Cl(A \cup A))) \\ &\subset Int(Cl(A \cup A)) = Int(Cl(A)). \end{aligned}$$

This show $A \in \tau$.

Conversely, It is shown in Remark 2.3 .

(3) Every $e\mathcal{I}$ -open is e -open.

Let A be $e\mathcal{I}$ -open then, $A \subset Cl(\delta Int_I(A)) \cup Int(\delta Cl_I(A))$. by using this fact $A^* = Cl(A) = Cl^*(A)$, we have $\delta Cl_I(A) = \delta Cl(A)$, $\delta Int_I(A) = \delta Int(A)$, since $\delta Cl_I(A)$ is the family of all $\delta\mathcal{I}$ -cluster point of A , and $\delta Int_I(A)$ the union of all $R\mathcal{I}$ -open set of X we have respectively,

$$\begin{aligned} \emptyset \neq Int(Cl^*(U)) \cap A &= Int(U^* \cup U) \cap A = Int(Cl(U) \cup U) \cap A \\ &= Int(Cl(U)) \cap A \neq \emptyset \end{aligned}$$

From this we get $\delta Cl_I(A) = \delta Cl(A)$, and

$$\begin{aligned} A &= Int(Cl^*(A)) = Int(A^* \cup A) = Int[Cl(A) \cup A] \\ &= Int(Cl(A)) = A \end{aligned}$$

From this we get $\delta Int_I(A) = \delta Int(A)$. This show that

$$A \subset Cl(\delta Int_I(A)) \cup Int(\delta Cl_I(A)) \subset Cl(\delta Int(A)) \cup Int(\delta Cl(A))$$

Hence (3) is proved

Let us consider $I = N$ and A is e -open

If $I = N$ then $A^* = Cl^*(Int(Cl^*A))$.

Since A is e -open then $A \subset Cl(\delta Int(A)) \cup Int(\delta Cl(A))$. Then

$$\begin{aligned} \emptyset \neq Int(Cl(U)) \cap A &= Int(U \cup U) \cap A = Int(Cl(Int(Cl(U)) \cup U) \cap A \\ &\subset Int(Cl^*(Int(Cl^*(U))) \cup U) \cap A = Int(U^* \cup U) \cap A = Int(Cl^*(U)) \cap A \neq \emptyset \end{aligned}$$

From this we get $\delta Cl(A) \subset \delta Cl_I(A)$, and

$$\begin{aligned} A &= Int(Cl(A)) = Int(A \cup A) = Int[Cl(Int(Cl(A))) \cup A] \\ &\subset Int[Cl^*(Int(Cl^*(A))) \cup A] = Int(A^* \cup A) = Int(Cl^*(A)) = A \end{aligned}$$

From this we get $\delta Int(A) \subset \delta Int_I(A)$.

A is $e\mathcal{I}$ -open. Hence the proof. □

Proposition 2.15. 1. The union of any family of $e\mathcal{I}$ -open sets is an $e\mathcal{I}$ -open set.

2. The intersection of even two $e\mathcal{I}$ -open sets need not to be $e\mathcal{I}$ -open as shown in the following example.

Proof. (1) Let $\{A_\alpha/\alpha \in \Delta\}$ be a family of $e\mathcal{I}$ -open set,
 $A_\alpha \subset Cl(\delta Int_I(A_\alpha)) \cup Int(\delta Cl_I(A_\alpha))$

Hence

$$\begin{aligned}
\cup_{\alpha} A_{\alpha} &\subset \cup_{\alpha} [Cl(\delta Int_I(A_{\alpha})) \cup Int(\delta Cl_I(A_{\alpha}))] \\
&\subset \cup_{\alpha} [Cl(\delta Int_I(A_{\alpha}))] \cup \cup_{\alpha} [Int(\delta Cl_I(A_{\alpha}))] \\
&\subset [Cl(\cup_{\alpha} (\delta Int_I(A_{\alpha}))) \cup [Int(\cup_{\alpha} (\delta Cl_I(A_{\alpha}))]] \\
&\subset [Cl(\cup_{\alpha} (\delta Int_I(A_{\alpha}))) \cup [Int(\cup_{\alpha} (\delta Cl_I(A_{\alpha}))]] \\
&\subset [Cl(\delta Int_I(\cup_{\alpha} A_{\alpha}))] \cup [Int(\delta Cl_I(\cup_{\alpha} A_{\alpha}))].
\end{aligned}$$

$\cup_{\alpha} A_{\alpha}$ is $e\mathcal{I}$ -open. □

Example 2.16. Let $X = \{a, b, c, d\}$ with a topology $\tau = \{\emptyset, X, \{a\}, \{b, d\}, \{a, b, d\}\}$ and $\mathcal{I} = \{\emptyset, \{c\}, \{d\}, \{c, d\}\}$. Then the set $A = \{a, c\}$ and $A = \{b, c\}$ are $e\mathcal{I}$ -open, but $A \cap B = \{c\}$ is not $e\mathcal{I}$ -open. Since $\{b, c\}$ and $\{b, c\} \subset Cl(\delta Int_I(A)) \cup Int(\delta Cl_I(A))$. For $Cl(\delta Int_I(A)) \cup Int(\delta Cl_I(A)) = Cl(\emptyset) \cup Int(\{c, d\}) = Cl(\emptyset) \cup \emptyset = \emptyset \not\supseteq \{c\}$. So $A \cap B \not\subseteq Cl(\delta Int_I(A \cap B)) \cup Int(\delta Cl_I(A \cap B))$.

Definition 2.17. Let A be a subset of X .

1. The intersection of all $e\mathcal{I}$ -closed containing A is called the $e\mathcal{I}$ -closure of A and its denoted by $Cl_e^*(A)$,
2. The $e\mathcal{I}$ -interior of A , denoted by $Int_e^*(A)$, is defined by the union of all $e\mathcal{I}$ -open sets contained in A .

Proposition 2.18. Let (X, τ, \mathcal{I}) be an ideal topological space. Then if $A \in EIO(X, \tau)$ and $B \in \tau^a$, then $A \cap B \in eO(X, \tau)$.

Proof. Let $A \in EIO(X, \tau)$, i.e., $A \subset Cl(\delta Int_I(A)) \cup Int(\delta Cl_I(A))$ and $B \in \tau^a$, i.e., $B \subset Int(Cl(\delta Int(B)))$. Then

$$\begin{aligned}
A \cap B &\subset Cl(\delta Int_I(A)) \cup Int(\delta Cl_I(A)) \cap Int(Cl(\delta Int(B))) \\
&= [Cl(\delta Int_I(A)) \cap Int(Cl(\delta Int(B)))] \cup [Int(\delta Cl_I(A)) \cap Int(Cl(\delta Int(B)))] \\
&\subset [Cl(Cl(\delta Int_I(A))) \cap Cl(Cl(\delta Int(B)))] \cup [Int(\delta Cl_I(A)) \cap Cl(\delta Int(B))] \\
&\subset [Cl(Cl(\delta Int_I(A)) \cap Cl(\delta Int(B)))] \cup [Int(Cl(\delta Cl_I(A)) \cap Cl(\delta Int(B)))] \\
&\subset [Cl(Cl(\delta Int_I(A) \cap \delta Int(B)))] \cup [Int(Cl(\delta Cl_I(A) \cap \delta Int(B)))] \\
&\subset [Cl(\delta Int_I(A \cap \delta Int(B)))] \cup [Int(\delta Cl_I(\delta Cl_I(A \cap B)))] \\
&\subset [Cl(\delta Int(A \cap B))] \cup [Int(\delta Cl(A \cap B))].
\end{aligned}$$

Then $A \cap B \in eO(X, \tau)$. □

Remark 2.19. 1. Let A be a subset of an ideal topological space (X, τ, \mathcal{I}) . Then A is $e\mathcal{I}$ -closed if and only if $Cl_e^*(A) = A$,

2. Let B be a subset of an ideal topological space (X, τ, \mathcal{I}) . Then B is $e\mathcal{I}$ -open if and only if $Int_e^*(B) = B$,

Proposition 2.20. *Let A, B be a subsets of an ideal topological space (X, τ, \mathcal{I}) such that A is $e\mathcal{I}$ -open and B is $e\mathcal{I}$ -closed in X . Then there exist $e\mathcal{I}$ -open set H and $e\mathcal{I}$ -closed set K such that $A \cap B \subset H$ and $K \subset A \cup B$.*

Proof. Let $K = Cl_e^*(A) \cap B$ and $H = A \cup Int_e^*(B)$. Then, K is $e\mathcal{I}$ -closed and H is $e\mathcal{I}$ -open. $A \subset Cl_e^*(A)$ implies $A \cap B \subset Cl_e^*(A) \cap B = K$ and $Int_e^*(B) \subset B$ implies $A \cup Int_e^*(B) = H \subset A \cup B$. \square

Definition 2.21. 1. *A subset S of an ideal topological space (X, τ, \mathcal{I}) is called e -dense if $Cl_e(S) = X$, where $Cl_e(S)$ [7] (Def 2.9) is the smallest e -closed sets containing S ,*

2. *A subset S of an ideal topological space (X, τ, \mathcal{I}) is called $e\mathcal{I}$ -dense if $Cl_e^*(S) = X$.*

3 strong \mathcal{B}_I^* -set

Definition 3.1. *Let (X, τ, \mathcal{I}) be an ideal topological space. A subset A of X is called strong \mathcal{B}_I^* -set if $A = U \cap V$, where $U \in \tau$ and V is a strongly $t\mathcal{I}$ -set and $Int(\delta Cl_I(V)) = Cl(\delta Int_I(V))$.*

Proposition 3.2. *Let (X, τ, \mathcal{I}) be an ideal topological space and A be a subset of X . The following hold:*

1. *If A is strong \mathcal{B}_I^* -set, then A is a B_I -set,*
2. *If A is strongly $t\mathcal{I}$ -set, then A is a $t\mathcal{I}$ -set.*

Proof. 1. It follows from the fact every strongly $t\mathcal{I}$ -set is $t\mathcal{I}$ -set, the proof is obvious.

2. It follows from ([5] Theorem 21 (3)).

\square

Remark 3.3. *The following diagram holds for a subset A of a space X :*

$$\begin{array}{ccccc}
 \text{open} & \longrightarrow & \text{strong } \mathcal{B}_I^*\text{-set} & \longrightarrow & B_I\text{-set} \\
 & & \uparrow & & \uparrow \\
 & & \text{strongly } t_I\text{-set} & \longrightarrow & t_I\text{-set}
 \end{array}$$

Remark 3.4. *The converses of proposition 3.2 (1), (2) need not to be true as the following examples show.*

Example 3.5. Let $X = \{a, b, c, d\}$ with a topology $\tau = \{\emptyset, X, \{a\}, \{a, c\}, \{a, b, c\}\}$ and an ideal $\mathcal{I} = \{\emptyset, \{c\}, \{a, c\}\}$. Then the set $A = \{a, c\}$ is B_I -set, but not a strong \mathcal{B}_I^* -set and hence A is a t_I -set but not strongly $t\mathcal{I}$ -set. For $Int(Cl^*(A)) = Int(\{a, c\}) = \{a\} = Int(A)$ and hence A is a t_I -set. It is obvious that A is a B_I -set. But $Int(\delta Cl_I(A)) = Int(\{X\}) = X$ and $Cl(\delta Int_I(A)) = Cl(\{a\}) = \{a, d\}$ i.e. $Int(\delta Cl_I(A)) \neq Cl(\delta Int_I(A))$. So A is not strong \mathcal{B}_I^* -set.

Example 3.6. Let $X = \{a, b, c, d\}$ with a topology $\tau = \{\emptyset, X, \{b\}, \{b, c\}, \{b, c, d\}\}$ and an ideal $\mathcal{I} = \{\emptyset, \{b\}, \{c\}, \{b, c\}\}$. Then the set $A = \{b, c\}$ is strong \mathcal{B}_I^* -set, but not a strongly $t\mathcal{I}$ -set. $Int(\delta Cl_I(A)) = Int(\{X\}) = X$ and $Cl(\delta Int_I(A)) = Cl(\{b, c\}) = \{X\}$ i.e. $Int(\delta Cl_I(A)) = Cl(\delta Int_I(A))$. So A is strong \mathcal{B}_I^* -set. But, $Int(\delta Cl_I(A)) = Int(\{X\}) = X \neq Int(A)$. Therefor A is not a strongly $t\mathcal{I}$ -set.

Proposition 3.7. Let A be subset of an ideal topological space (X, τ, \mathcal{I}) . Then the following condition are equivalent:

1. A is open.
2. A is $e\mathcal{I}$ -open and strong \mathcal{B}_I^* -set.

Proof. (1) \Rightarrow (2): By Remark 2.3 and Remark 3.3, every open set is $e\mathcal{I}$ -open. On other hand every open set is strongly \mathcal{B}_I^* -set.

(2) \Rightarrow (1): Let A is $e\mathcal{I}$ -open and strong \mathcal{B}_I^* -set. Then $A \subset Cl(\delta Int_I(A)) \cup Int(\delta Cl_I(A)) = Cl(\delta Int_I(U \cap V)) \cup Int(\delta Cl_I(U \cap V))$, where U is open and V is strongly $t\mathcal{I}$ -set and $Int(\delta Cl_I(V)) = Int(V)$, $Int(\delta Cl_I(V)) = Cl(\delta Int_I(V))$. Hence

$$\begin{aligned} A &\subset [Int(\delta Cl_I(U)) \cap Int(\delta Cl_I(V))] \cup [Cl(\delta Int_I(U)) \cap Cl(\delta Int_I(V))] \\ &= [U \cap Int(\delta Cl_I(V))] \cup [U \cap Cl(\delta Int_I(V))] \\ &\subset [U] \cap [Int(\delta Cl_I(V)) \cup Cl(\delta Int_I(V))] \\ &\subset [U] \cup [Int(\delta Cl_I(V)) \cap Int(\delta Int_I(V))] \\ &\subset [U] \cup [Int(\delta Cl_I(V))] \\ &\subset U \cup Int(V) = Int(A). \end{aligned}$$

On other hand, we have $U \cap Int(V) \subset U \cap V = A$. Thus, $A = U \cap Int(V)$ and A is open. \square

4 decomposition of continuity

Definition 4.1. [7] A function $f : (X, \tau) \longrightarrow (Y, \sigma)$ is said to be e -continuous if for each open set V of (Y, σ) , $f^{-1}(V)$ is e -open.

Definition 4.2. A function $f : (X, \tau, \mathcal{I}) \longrightarrow (Y, \sigma)$ is said to be $e\mathcal{I}$ -continuous (resp. $pre^*\mathcal{I}$ -continuous [5], strong \mathcal{B}_I^* -continuous) if for each open set V of (Y, σ) , $f^{-1}(V)$ is $e\mathcal{I}$ -open (resp. $pre^*\mathcal{I}$ -open, strong \mathcal{B}_I^* -set) in (X, τ, \mathcal{I}) .

Definition 4.3. A function $f : (X, \tau, \mathcal{I}) \longrightarrow (Y, \sigma)$ is said to be $semi^*\mathcal{I}$ -continuous if for each open set V of (Y, σ) , $f^{-1}(V)$ is $semi^*\mathcal{I}$ -open in (X, τ, \mathcal{I}) .

Proposition 4.4. *If a function $f : (X, \tau, \mathcal{I}) \longrightarrow (Y, \sigma)$ is semi* \mathcal{I} -continuous (pre* \mathcal{I} -continuous), then f is $e\mathcal{I}$ -continuous.*

Proof. This is immediate consequence of Proposition 2.2 (2) and (3). □

Proposition 4.5. *If a function $f : (X, \tau, \mathcal{I}) \longrightarrow (Y, \sigma)$ is strong \mathcal{B}_I^* -continuous, then f is B_I -continuous*

Proof. This is immediate consequence of Proposition 3.2 (1). □

Theorem 4.6. *For a function $f : (X, \tau, \mathcal{I}) \rightarrow (Y, \sigma)$. Then the following properties are equivalent,*

1. f is continuous.
2. f is $e\mathcal{I}$ -continuous and strong \mathcal{B}_I^* -continuous.

Proof. This is immediate consequence of Proposition 3.7. □

5 $e\mathcal{I}$ - continuous mappings

Definition 5.1. 1. *A function $f : (X, \tau) \longrightarrow (Y, \sigma)$ is called δ -almost-continuous if the inverse image of each open set in Y is δ -preopen set in X [15].*

2. *A function $f : (X, \tau, \mathcal{I}) \longrightarrow (Y, \sigma)$ is called δ -semicontinuous if the inverse image of each open set in Y is δ -semiopen set in X [6].*

3. *A function $f : (X, \tau, \mathcal{I}) \longrightarrow (Y, \sigma)$ is called α -continuous if for each open set V of (Y, σ) , $f^{-1}(V)$ is α -open [4].*

4. *A function $f : (X, \tau, \mathcal{I}) \longrightarrow (Y, \sigma)$ is called $\delta\alpha\mathcal{I}$ -continuous if for each δ_I -open set V of (Y, σ) , $f^{-1}(V)$ is $\delta\alpha\mathcal{I}$ -open [8].*

Definition 5.2. [16] *Let (X, τ) be topological space and $A \subseteq X$. Then the set $\cap\{U \in \tau : A \subset U\}$ is called the kernel of A and denoted by $Ker(A)$.*

Lemma 5.3. [10] *Let (X, τ) be topological space and $A \subseteq X$.*

1. $x \in Ker(A)$ if and only if $A \cap F \neq \emptyset$ for any closed subset of X with $x \in F$,
2. $A \subset Ker(A)$ and $A = Ker(A)$ if A is open in X ,
3. if $A \subset B$, then $Ker(A) \subset Ker(B)$.

Definition 5.4. *Let N be a subset of a space (X, τ, \mathcal{I}) , and let $x \in X$. Then N is called $e\mathcal{I}$ -neighborhood of x , if there exist $e\mathcal{I}$ -open set U containing x such that $U \subset N$.*

Theorem 5.5. *The following statement are equivalent for a function $f : (X, \tau, \mathcal{I}) \longrightarrow (Y, \sigma)$:*

1. f is $e\mathcal{I}$ -continuous,
2. for each $x \in X$ and each open set V in Y with $f(x) \in V$, there exist $e\mathcal{I}$ -open set U containing x such that $f(U) \subset V$,
3. for each $x \in X$ and each open set V in Y with $f(x) \in V$, $f^{-1}(V)$ is $e\mathcal{I}$ -neighborhood of x ,
4. for every subset A of X , $f(Int_e^*(A)) \subset Ker(f(A))$,
5. for every subset B of Y , $Int_e^*(f^{-1}(B)) \subset f^{-1}(Ker(B))$.

Proof. (1) \Rightarrow (2): Let $x \in X$ and let V be an open set in Y such that $f(x) \in V$. Since f is $e\mathcal{I}$ -continuous, $f^{-1}(V)$ is $e\mathcal{I}$ -open. By putting $U = f^{-1}(V)$ which is containing x , we have $f(U) \subset V$.

(2) \Rightarrow (3): Let V be an open set in Y such that $f(x) \in V$. Then by (2) there exists a $e\mathcal{I}$ -open set U containing x such that $f(U) \subset V$. So $x \in U \subset f^{-1}(V)$. Hence $f^{-1}(V)$ is $e\mathcal{I}$ -neighborhood of x .

(3) \Rightarrow (1): Let V be an open set in Y such that $f(x) \in V$. Then by (3), $f^{-1}(V)$ is $e\mathcal{I}$ -neighborhood of x . Thus for each $x \in f^{-1}(V)$, there exists a $e\mathcal{I}$ -open set U_x containing x such that $x \in U_x \subset f^{-1}(V)$. Hence $f^{-1}(V) \subset \bigcup_{x \in f^{-1}(V)} U_x$ and so $f^{-1}(V) \in EIO(X, \tau)$.

(1) \Rightarrow (5): Let A be any subset of X . Suppose that $y \notin Ker(A)$. Then, by Lemma 5.3, there exists a closed subset F of Y such that $y \in F$ and $f(A) \cap F = \emptyset$. Thus we have $A \cap f^{-1}(F) = \emptyset$ and $(Int_e^*(A)) \cap f^{-1}(F) = \emptyset$. Therefore, we obtain $f(Int_e^*(A)) \cap (F) = \emptyset$ and $y \notin f(Int_e^*(A))$. This implies that $f(Int_e^*(A)) \subset Ker(f(A))$.

(5) \Rightarrow (6): Let B be any subset of Y . By (5) and Lemma 5.3, we have $f(Int_e^*(f^{-1}(B))) \subset Ker(f(f^{-1}(B))) \subset Ker(B)$ and $Int_e^*(f^{-1}(B)) \subset f^{-1}(Ker(B))$.

(6) \Rightarrow (1): Let V be any subset of Y . By (6) and Lemma 5.3, we have $Int_e^*(f^{-1}(V)) \subset f^{-1}(Ker(V)) = f^{-1}(V)$ and $Int_e^*(f^{-1}(V)) = f^{-1}(V)$. This shows that $f^{-1}(V)$ is $e\mathcal{I}$ -open. \square

The following examples show that $e\mathcal{I}$ -continuous functions do not need to be semi* \mathcal{I} -continuous and pre* \mathcal{I} -continuous, and e -continuous function does not need to be $e\mathcal{I}$ -continuous.

Example 5.6. Let $X = Y = \{a, b, c, d\}$ be a topology space by setting $\tau = \sigma = \{\emptyset, X, \{a\}, \{d\}, \{a, d\}\}$ and $\mathcal{I} = \{\emptyset, \{c\}\}$ on X . Define a function $f : (X, \tau, \mathcal{I}) \rightarrow (Y, \sigma)$ as follows $f(a) = f(c) = d$ and $f(b) = f(d) = b$. Then f is $e\mathcal{I}$ -continuous but it is not pre* \mathcal{I} -continuous.

Example 5.7. Let $X = Y = \{a, b, c\}$ be a topology space by setting $\tau = \sigma = \{\emptyset, X, \{a, b\}\}$ and $\mathcal{I} = \{\emptyset, \{c\}\}$ on X . Define a function $f : (X, \tau, \mathcal{I}) \rightarrow (Y, \sigma)$ as follows $f(a) = a$, $f(b) = c$, $f(c) = b$. Then f is $e\mathcal{I}$ -continuous but it is not semi* \mathcal{I} -continuous.

Example 5.8. Let (X, τ) be the real line with the indiscrete topology and (Y, τ) the real line with the usual topology and $\mathcal{I} = \{\emptyset\}$. Then the identity function $f : (X, \tau, \mathcal{I}) \rightarrow (Y, \sigma)$ is e -continuous but not $e\mathcal{I}$ -continuous.

Proposition 5.9. *Let $f : (X, \tau, \mathcal{I}) \longrightarrow (Y, \sigma, \mathcal{J})$ and $g : (Y, \sigma, \mathcal{J}) \longrightarrow (Z, \rho)$ be two functions, where \mathcal{I} and \mathcal{J} are ideals on X and Y , respectively. Then $g \circ f$ is $e\mathcal{I}$ -continuous if f is $e\mathcal{I}$ -continuous and g is continuous.*

Proof. The proof is clear. □

Proposition 5.10. *Let $f : (X, \tau, \mathcal{I}) \longrightarrow (Y, \sigma)$ be $e\mathcal{I}$ -continuous and $U \in \tau$. Then the restriction $f|_U : (X, \tau|_U, \mathcal{I}|_U) \longrightarrow (Y, \sigma)$ is $e\mathcal{I}$ -continuous.*

Proof. Let V be any open set of (Y, σ) . Since f is $e\mathcal{I}$ -continuous, $f^{-1}(V) \in EIO(X, \tau)$ and by Lemma 2.11, $f|_U^{-1}(V) = f^{-1}(V) \cap U \in EIO(U, \mathcal{I}|_U)$. This shows that $f|_U : (X, \tau|_U, \mathcal{I}|_U) \longrightarrow (Y, \sigma)$ is $e\mathcal{I}$ -continuous. □

Theorem 5.11. *Let $f : (X, \tau, \mathcal{I}) \longrightarrow (Y, \sigma)$ be a function and let $\{U_\alpha : \alpha \in \Delta\}$ be an open cover of X . If the restriction function $f|_{U_\alpha}$ is $e\mathcal{I}$ -continuous for each $\alpha \in \Delta$, then f is $e\mathcal{I}$ -continuous.*

Proof. The proof is similar to that of Theorem 5.10 □

Lemma 5.12. [20] *For any function $f : (X, \tau, \mathcal{I}) \longrightarrow (Y, \sigma, \mathcal{J})$, $f(\mathcal{I})$ is an ideal on Y .*

Definition 5.13. [20, 21] *A subset A of an ideal topological space (X, τ, \mathcal{I}) is said to be I -compact if for every τ -open cover $\{\omega_\alpha : \alpha \in \Delta\}$ of A , there exists a finite subset Δ_o of Δ such that $(X - \cup\{\omega_\alpha : \alpha \in \Delta\}) \in \mathcal{I}$.*

Definition 5.14. *An ideal topological space (X, τ, \mathcal{I}) is said to be $e\mathcal{I}$ -compact if for every $e\mathcal{I}$ -open cover $\{\omega_\alpha : \alpha \in \Delta\}$ of X , there exists a finite subset Δ_o of Δ such that $(X - \cup\{\omega_\alpha : \alpha \in \Delta\}) \in \mathcal{I}$.*

Theorem 5.15. *The image of $e\mathcal{I}$ -compact space under $e\mathcal{I}$ -continuous surjective function is $f(\mathcal{I})$ -compact.*

Proof. Let $f : (X, \tau, \mathcal{I}) \longrightarrow (Y, \sigma)$ be a $e\mathcal{I}$ -continuous surjection and $\{V_\alpha : \alpha \in \Delta\}$ be an open cover of Y . Then $\{f^{-1}(V_\alpha) : \alpha \in \Delta\}$ is a $e\mathcal{I}$ -open cover of X due to our assumption on f . Since X is $e\mathcal{I}$ -compact, then there exists a finite subset Δ_o of Δ such that $(X - \cup\{f^{-1}(V_\alpha) : \alpha \in \Delta_o\}) \in \mathcal{I}$. Therefore $(Y - \cup\{V_\alpha : \alpha \in \Delta_o\}) \in f(\mathcal{I})$, which shows that $(Y, \sigma, f(\mathcal{I}))$ is $f(\mathcal{I})$ -compact. □

Theorem 5.16. *A $e\mathcal{I}$ -continuous image of an $e\mathcal{I}$ -connected space is connected.*

Proof. Let $f : (X, \tau, \mathcal{I}) \longrightarrow (Y, \sigma)$ is $e\mathcal{I}$ -continuous function of $e\mathcal{I}$ -connected space X onto a topological space Y . If possible, let Y be disconnected. Let A and B form a disconnected set of Y . Then A and B are clopen and $Y = A \cup B$, where $A \cap B = \emptyset$. Since f is $e\mathcal{I}$ -continuous, $X = f^{-1}(Y) = f^{-1}(A \cup B) = \emptyset$, where $f^{-1}(A)$ and $f^{-1}(B)$ are nonempty $e\mathcal{I}$ -open sets in X . Also $f^{-1}(A) \cap f^{-1}(B) = \emptyset$. Hence X is non $e\mathcal{I}$ -connected, which is contradiction. Therefore, Y is connected. □

Definition 5.17. A function $f : (X, \tau, \mathcal{I}) \longrightarrow (Y, \sigma, \mathcal{J})$ is called $e\text{-}\mathcal{J}$ -open (resp., $e\text{-}\mathcal{J}$ -closed) if for each $U \in \tau$ (resp., closed set M in X), $f(U)$ (resp., $f(M)$) is $e\text{-}\mathcal{J}$ -open (resp., $e\text{-}\mathcal{J}$ -closed)

Remark 5.18. Every $e\text{-}\mathcal{I}$ -open (resp., $e\text{-}\mathcal{I}$ -closed) function is e -open (resp., e -closed) and the converses are false in general.

Example 5.19. Let $X = \{a, b, c\}$ be a topology space by setting $\tau_1 = \{\emptyset, X, \{b, c\}\}$ and $\tau_2 = \{\emptyset, X, \{a, b\}, \{b\}, \{a\}\}$ and an ideal $\mathcal{I} = \{\emptyset, \{a\}\}$. Then the identity function $f : (X, \tau_1) \longrightarrow (X, \tau_2, \mathcal{I})$ is e -open but not $e\text{-}\mathcal{I}$ -open.

Example 5.20. Let $X = \{a, b, c\}$ be a topology space by setting $\tau_1 = \{\emptyset, X, \{a\}\}$ and $\tau_2 = \{\emptyset, X, \{b, c\}, \{b\}, \{c\}\}$ and an ideal $\mathcal{I} = \{\emptyset, \{c\}\}$. Defined function $f : (X, \tau_1) \longrightarrow (X, \tau_2, \mathcal{I})$ as follows: $f(a) = a$, $f(b) = f(c) = b$. Then f is e -closed but not $e\text{-}\mathcal{I}$ -closed.

Theorem 5.21. A function $f : (X, \tau, \mathcal{I}) \longrightarrow (Y, \sigma, \mathcal{J})$ is $e\text{-}\mathcal{J}$ -open if and only if for each $x \in X$ and each neighborhood U of x , there exists $V \in EJO(Y, \sigma)$ containing $f(x)$ such that $V \subset f(U)$.

Proof. Suppose that f is a $e\text{-}\mathcal{J}$ -open function. For each $x \in X$ and each neighborhood U of x , there exists $U_o \in \tau$ such that $x \in U_o \subset U$. Since f is $e\text{-}\mathcal{J}$ -open, $V = f(U_o) \in EJO(Y, \sigma)$ and $f(x) \in V \subset f(U)$. Conversely, let U be an open set of (X, τ) . For each $x \in U$, there exists $V_x \in EJO(Y, \sigma)$ such that $f(x) \in V_x \subset f(U)$. Therefore we obtain $f(U) = \bigcup \{V_x : x \in U\}$ and hence by Proposition 2.7, $f(U) \in EJO(Y, \sigma)$. This shows that f is $e\text{-}\mathcal{J}$ -open. \square

Theorem 5.22. A function $f : (X, \tau, \mathcal{I}) \longrightarrow (Y, \sigma, \mathcal{J})$ be $e\text{-}\mathcal{J}$ -open (resp., $e\text{-}\mathcal{J}$ -closed). If W is any subset of Y and F is a closed (resp., open) set of X containing $f^{-1}(W)$, then there exists $e\text{-}\mathcal{J}$ -closed (resp., $e\text{-}\mathcal{J}$ -open) subset H of Y containing W such that $f^{-1}(W) \subset F$.

Proof. Suppose that f is $e\text{-}\mathcal{J}$ -open function. Let W be any subset of Y and F a closed subset of X containing $f^{-1}(W)$. Then $X - F$ is open and since f is $e\text{-}\mathcal{J}$ -open, $f(X - F)$ $e\text{-}\mathcal{J}$ -open. Hence $H = Y - f(X - F)$ is $e\text{-}\mathcal{J}$ -closed. It follows from $f^{-1}(W) \subset F$ that $W \subset H$. Moreover, we obtain $f^{-1}(H) \subset F$. For $e\text{-}\mathcal{J}$ -closed function. \square

Theorem 5.23. For any objective function $f : (X, \tau) \longrightarrow (Y, \sigma, \mathcal{J})$, the following are equivalent:

1. $f^{-1} : (Y, \sigma, \mathcal{J}) \longrightarrow (X, \tau)$ is $e\text{-}\mathcal{J}$ -continuous,
2. f is $e\text{-}\mathcal{J}$ -open,
3. f is $e\text{-}\mathcal{J}$ -closed,

Proof. It is straightforward. \square

Definition 5.24. A space (X, τ) is called

1. e -space if every e -open set of X is open in X .
2. submaximal if every dense subset of X is open in X [17].
3. extremely disconnected if the closure of every open set of X is open in X [27].

Corollary 5.25. *If a function $f : (X, \tau, \mathcal{I}) \rightarrow (Y, \sigma)$ is continuous, then f is $e\mathcal{I}$ -continuous.*

Corollary 5.26. *If (X, τ) is extremely disconnected and submaximal, then for any ideal \mathcal{I} on X , $P^*IO(X, \tau) = S^*IO(X, \tau) = \delta SO(X, \tau) = \delta PO(X, \tau) = \delta\alpha IO(X, \tau) = aO(X, \tau) = \tau$.*

Corollary 5.27. *If (X, τ) is e -space, then for any ideal \mathcal{I} on X , $EIO(X, \tau) = eO(X, \tau) = P^*IO(X, \tau) = S^*IO(X, \tau) = \delta SO(X, \tau) = \delta PO(X, \tau) = \delta\alpha IO(X, \tau) = aO(X, \tau) = \tau$.*

Corollary 5.28. *Let $f : (X, \tau, \mathcal{I}) \rightarrow (Y, \sigma)$ be a function and let (X, τ) be e -space, then the following are equivalent:*

1. f is $e\mathcal{I}$ -continuous,
2. f is e -continuous,
3. f is $pre^*\mathcal{I}$ -continuous,
4. f is δ -almostcontinuous,
5. f is $semi^*\mathcal{I}$ -continuous,
6. f is δ -semicontinuous,
7. f is $\delta\alpha\mathcal{I}$ -continuous,
8. f is $\delta\alpha$ -continuous,
9. f is continuous,

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On some differential sandwich theorems using an extended generalized Sălăgean operator and extended Ruscheweyh operator

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ABSTRACT: In this work we define a new operator using the extended generalized Sălăgean operator and extended Ruscheweyh operator. Denote by $DR_{\lambda}^{m,n}$ the Hadamard product of the extended generalized Sălăgean operator D_{λ}^m and extended Ruscheweyh operator R^n , given by $DR_{\lambda}^{m,n} : \mathcal{A}_{\zeta}^* \rightarrow \mathcal{A}_{\zeta}^*$, $DR_{\lambda}^{m,n} f(z, \zeta) = (D_{\lambda}^m * R^n) f(z, \zeta)$ and $\mathcal{A}_{n\zeta}^* = \{f \in \mathcal{H}(U \times \overline{U}), f(z, \zeta) = z + a_{n+1}(\zeta) z^{n+1} + \dots, z \in U, \zeta \in \overline{U}\}$ is the class of normalized analytic functions with $\mathcal{A}_{1\zeta}^* = \mathcal{A}_{\zeta}^*$. The purpose of this paper is to introduce sufficient conditions for strong differential subordination and strong differential superordination involving the operator $DR_{\lambda}^{m,n}$ and also to obtain sandwich-type results.

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Keywords and Phrases: *analytic functions, differential operator, differential subordination, differential superordination.*

1 Introduction

Denote by U the unit disc of the complex plane $U = \{z \in \mathbb{C} : |z| < 1\}$, $\overline{U} = \{z \in \mathbb{C} : |z| \leq 1\}$ the closed unit disc of the complex plane and $\mathcal{H}(U \times \overline{U})$ the class of analytic functions in $U \times \overline{U}$.

Let

$$\mathcal{A}_{n\zeta}^* = \{f \in \mathcal{H}(U \times \overline{U}), f(z, \zeta) = z + a_{n+1}(\zeta) z^{n+1} + \dots, z \in U, \zeta \in \overline{U}\},$$

with $\mathcal{A}_{1\zeta}^* = \mathcal{A}_{\zeta}^*$, where $a_k(\zeta)$ are holomorphic functions in \overline{U} for $k \geq 2$, and $\mathcal{H}^*[a, n, \zeta] = \{f \in \mathcal{H}(U \times \overline{U}), f(z, \zeta) = a + a_n(\zeta) z^n + a_{n+1}(\zeta) z^{n+1} + \dots, z \in U, \zeta \in \overline{U}\}$, for $a \in \mathbb{C}$ and $n \in \mathbb{N}$, $a_k(\zeta)$ are holomorphic functions in \overline{U} for $k \geq n$.

Generalizing the notion of differential subordinations, J.A. Antonino and S. Romaguera have introduced in [17] the notion of strong differential subordinations, which was developed by G.I. Oros and Gh. Oros in [18].

Definition 1.1 [18] Let $f(z, \zeta), H(z, \zeta)$ analytic in $U \times \overline{U}$. The function $f(z, \zeta)$ is said to be strongly subordinate to $H(z, \zeta)$ if there exists a function w analytic in U , with $w(0) = 0$ and $|w(z)| < 1$ such that $f(z, \zeta) = H(w(z), \zeta)$ for all $\zeta \in \overline{U}$. In such a case we write $f(z, \zeta) \prec\prec H(z, \zeta)$, $z \in U, \zeta \in \overline{U}$.

Remark 1.1 [18] (i) Since $f(z, \zeta)$ is analytic in $U \times \overline{U}$, for all $\zeta \in \overline{U}$, and univalent in U , for all $\zeta \in \overline{U}$, Definition 1.1 is equivalent to $f(0, \zeta) = H(0, \zeta)$, for all $\zeta \in \overline{U}$, and $f(U \times \overline{U}) \subset H(U \times \overline{U})$.

(ii) If $H(z, \zeta) \equiv H(z)$ and $f(z, \zeta) \equiv f(z)$, the strong subordination becomes the usual notion of subordination.

As a dual notion of strong differential subordination G.I. Oros has introduced and developed the notion of strong differential superordinations in [19].

Definition 1.2 [19] Let $f(z, \zeta), H(z, \zeta)$ analytic in $U \times \overline{U}$. The function $f(z, \zeta)$ is said to be strongly superordinate to $H(z, \zeta)$ if there exists a function w analytic in U , with $w(0) = 0$ and $|w(z)| < 1$, such that $H(z, \zeta) = f(w(z), \zeta)$, for all $\zeta \in \overline{U}$. In such a case we write $H(z, \zeta) \prec\prec f(z, \zeta)$, $z \in U, \zeta \in \overline{U}$.

Remark 1.2 [19] (i) Since $f(z, \zeta)$ is analytic in $U \times \overline{U}$, for all $\zeta \in \overline{U}$, and univalent in U , for all $\zeta \in \overline{U}$, Definition 1.2 is equivalent to $H(0, \zeta) = f(0, \zeta)$, for all $\zeta \in \overline{U}$, and $H(U \times \overline{U}) \subset f(U \times \overline{U})$.

(ii) If $H(z, \zeta) \equiv H(z)$ and $f(z, \zeta) \equiv f(z)$, the strong superordination becomes the usual notion of superordination.

Definition 1.3 [1] We denote by Q^* the set of functions that are analytic and injective on $\overline{U} \times \overline{U} \setminus E(f, \zeta)$, where $E(f, \zeta) = \{y \in \partial U : \lim_{z \rightarrow y} f(z, \zeta) = \infty\}$, and are such that $f'_z(y, \zeta) \neq 0$ for $y \in \partial U \times \overline{U} \setminus E(f, \zeta)$. The subclass of Q^* for which $f(0, \zeta) = a$ is denoted by $Q^*(a)$.

For two functions $f(z, \zeta) = z + \sum_{j=2}^{\infty} a_j(\zeta) z^j$ and $g(z, \zeta) = z + \sum_{j=2}^{\infty} b_j(\zeta) z^j$ analytic in $U \times \overline{U}$, the Hadamard product (or convolution) of $f(z, \zeta)$ and $g(z, \zeta)$, written as $(f * g)(z, \zeta)$ is defined by

$$f(z, \zeta) * g(z, \zeta) = (f * g)(z, \zeta) = z + \sum_{j=2}^{\infty} a_j(\zeta) b_j(\zeta) z^j.$$

Definition 1.4 ([2]) For $f \in \mathcal{A}_{\zeta}^*$, $\lambda \geq 0$ and $m \in \mathbb{N}$, the extended generalized Sălăgean operator D_{λ}^m is defined by $D_{\lambda}^m : \mathcal{A}_{\zeta}^* \rightarrow \mathcal{A}_{\zeta}^*$,

$$\begin{aligned} D_{\lambda}^0 f(z, \zeta) &= f(z, \zeta) \\ D_{\lambda}^1 f(z, \zeta) &= (1 - \lambda) f(z, \zeta) + \lambda z f'_z(z, \zeta) = D_{\lambda} f(z, \zeta) \\ &\dots \\ D_{\lambda}^{m+1} f(z, \zeta) &= (1 - \lambda) D_{\lambda}^m f(z, \zeta) + \lambda z (D_{\lambda}^m f(z, \zeta))'_z = D_{\lambda} (D_{\lambda}^m f(z, \zeta)), \end{aligned}$$

for $z \in U, \zeta \in \overline{U}$.

Remark 1.3 If $f \in \mathcal{A}_\zeta^*$ and $f(z, \zeta) = z + \sum_{j=2}^{\infty} a_j(\zeta) z^j$, then $D_\lambda^m f(z, \zeta) = z + \sum_{j=2}^{\infty} [1 + (j-1)\lambda]^m a_j(\zeta) z^j$, for $z \in U, \zeta \in \overline{U}$.

Definition 1.5 ([3]) For $f \in \mathcal{A}_\zeta^*$, $m \in \mathbb{N}$, the extended Ruscheweyh derivative R^m is defined by $R^m : \mathcal{A}_\zeta^* \rightarrow \mathcal{A}_\zeta^*$,

$$\begin{aligned} R^0 f(z, \zeta) &= f(z, \zeta) \\ R^1 f(z, \zeta) &= z f'_z(z, \zeta) \\ &\dots \\ (m+1) R^{m+1} f(z, \zeta) &= z (R^m f(z, \zeta))'_z + m R^m f(z, \zeta), \end{aligned}$$

$z \in U, \zeta \in \overline{U}$.

Remark 1.4 If $f \in \mathcal{A}_\zeta^*$, $f(z, \zeta) = z + \sum_{j=2}^{\infty} a_j(\zeta) z^j$, then $R^m f(z, \zeta) = z + \sum_{j=2}^{\infty} \frac{(m+j-1)!}{m!(j-1)!} a_j(\zeta) z^j$, $z \in U, \zeta \in \overline{U}$.

In order to prove our strong subordination and strong superordination results, we make use of the following known results.

Lemma 1.1 Let the function q be univalent in $U \times \overline{U}$ and θ and ϕ be analytic in a domain D containing $q(U \times \overline{U})$ with $\phi(w) \neq 0$ when $w \in q(U \times \overline{U})$. Set $Q(z, \zeta) = z q'_z(z, \zeta) \phi(q(z, \zeta))$ and $h(z, \zeta) = \theta(q(z, \zeta)) + Q(z, \zeta)$. Suppose that

1. Q is starlike univalent in $U \times \overline{U}$ and
2. $\operatorname{Re} \left(\frac{z h'_z(z, \zeta)}{Q(z, \zeta)} \right) > 0$ for $z \in U, \zeta \in \overline{U}$.

If p is analytic with $p(0, \zeta) = q(0, \zeta)$, $p(U \times \overline{U}) \subseteq D$ and

$$\theta(p(z, \zeta)) + z p'_z(z, \zeta) \phi(p(z, \zeta)) \prec \prec \theta(q(z, \zeta)) + z q'_z(z, \zeta) \phi(q(z, \zeta)),$$

then $p(z, \zeta) \prec \prec q(z, \zeta)$ and q is the best dominant.

Lemma 1.2 Let the function q be convex univalent in $U \times \overline{U}$ and ν and ϕ be analytic in a domain D containing $q(U \times \overline{U})$. Suppose that

1. $\operatorname{Re} \left(\frac{\nu'_z(q(z, \zeta))}{\phi(q(z, \zeta))} \right) > 0$ for $z \in U, \zeta \in \overline{U}$ and
2. $\psi(z, \zeta) = z q'_z(z, \zeta) \phi(q(z, \zeta))$ is starlike univalent in $U \times \overline{U}$.

If $p(z, \zeta) \in \mathcal{H}^*[q(0, \zeta), 1, \zeta] \cap Q^*$, with $p(U \times \overline{U}) \subseteq D$ and $\nu(p(z, \zeta)) + z p'_z(z, \zeta) \phi(p(z, \zeta))$ is univalent in $U \times \overline{U}$ and

$$\nu(q(z, \zeta)) + z q'_z(z, \zeta) \phi(q(z, \zeta)) \prec \prec \nu(p(z, \zeta)) + z p'_z(z, \zeta) \phi(p(z, \zeta)),$$

then $q(z, \zeta) \prec \prec p(z, \zeta)$ and q is the best subordinant.

2 Main results

Extending the results from [11] to the class \mathcal{A}_ζ^* we obtain:

Definition 2.1 ([12]) Let $\lambda \geq 0$ and $n, m \in \mathbb{N}$. Denote by $DR_\lambda^{m,n} : \mathcal{A}_\zeta^* \rightarrow \mathcal{A}_\zeta^*$ the operator given by the Hadamard product of the extended generalized Sălăgean operator D_λ^m and the extended Ruscheweyh operator R^n ,

$$DR_\lambda^{m,n} f(z, \zeta) = (D_\lambda^m * R^n) f(z, \zeta),$$

for any $z \in U$, $\zeta \in \overline{U}$, and each nonnegative integers m, n .

Remark 2.1 If $f \in \mathcal{A}_\zeta^*$ and $f(z, \zeta) = z + \sum_{j=2}^{\infty} a_j(\zeta) z^j$, then

$$DR_\lambda^{m,n} f(z, \zeta) = z + \sum_{j=2}^{\infty} [1 + (j-1)\lambda]^m \frac{(n+j-1)!}{n!(j-1)!} a_j^2(\zeta) z^j, \text{ for } z \in U, \zeta \in \overline{U}.$$

Remark 2.2 For $m = n$ we obtain the operator DR_λ^m studied in [13], [14], [15], [16], [4], [5], [6].

For $\lambda = 1$, $m = n$, we obtain the Hadamard product SR^n [7] of the Sălăgean operator S^n and Ruscheweyh derivative R^n , which was studied in [8], [9], [10].

Using simple computation one obtains the next result.

Proposition 2.1 For $m, n \in \mathbb{N}$ and $\lambda \geq 0$ we have For $m, n \in \mathbb{N}$ and $\lambda \geq 0$ we have

$$DR_\lambda^{m+1,n} f(z, \zeta) = (1 - \lambda) DR_\lambda^{m,n} f(z, \zeta) + \lambda z (DR_\lambda^{m,n} f(z, \zeta))'_z \quad (2.1)$$

and

$$z (DR_\lambda^{m,n} f(z, \zeta))'_z = (n+1) DR_\lambda^{m,n+1} f(z, \zeta) - n DR_\lambda^{m,n} f(z, \zeta). \quad (2.2)$$

Proof. We have

$$\begin{aligned} DR_\lambda^{m+1,n} f(z, \zeta) &= z + \sum_{j=2}^{\infty} [1 + (j-1)\lambda]^{m+1} \frac{(n+j-1)!}{n!(j-1)!} a_j^2(\zeta) z^j \\ &= z + \sum_{j=2}^{\infty} [(1-\lambda) + \lambda j] [1 + (j-1)\lambda]^m \frac{(n+j-1)!}{n!(j-1)!} a_j^2(\zeta) z^j \\ &= z + (1-\lambda) \sum_{j=2}^{\infty} [1 + (j-1)\lambda]^m \frac{(n+j-1)!}{n!(j-1)!} a_j^2(\zeta) z^j \\ &\quad + \lambda \sum_{j=2}^{\infty} [1 + (j-1)\lambda]^m \frac{(n+j-1)!}{n!(j-1)!} j a_j^2(\zeta) z^j \\ &= (1-\lambda) DR_\lambda^{m,n} f(z, \zeta) + \lambda z (DR_\lambda^{m,n} f(z, \zeta))'_z, \end{aligned}$$

and

$$\begin{aligned}
 & (n+1)DR_{\lambda}^{m,n+1}f(z,\zeta) - nDR_{\lambda}^{m,n}f(z,\zeta) \\
 &= (n+1)z + (n+1)\sum_{j=2}^{\infty} [1+(j-1)\lambda]^m \frac{(n+j)!}{(n+1)!(j-1)!} a_j^2(\zeta) z^j \\
 &\quad -nz - n\sum_{j=2}^{\infty} [1+(j-1)\lambda]^m \frac{(n+j-1)!}{n!(j-1)!} a_j^2(\zeta) z^j \\
 &= z + (n+1)\sum_{j=2}^{\infty} [1+(j-1)\lambda]^m \frac{n+j}{n+1} \frac{(n+j-1)!}{n!(j-1)!} a_j^2(\zeta) z^j \\
 &\quad -n\sum_{j=2}^{\infty} [1+(j-1)\lambda]^m \frac{(n+j-1)!}{n!(j-1)!} a_j^2(\zeta) z^j \\
 &= z + \sum_{j=2}^{\infty} [1+(j-1)\lambda]^m \frac{(n+j-1)!}{n!(j-1)!} ja_j^2(z) z^j \\
 &= z(DR_{\lambda}^{m,n}f(z,\zeta))'_z.
 \end{aligned}$$

■

We begin with the following

Theorem 2.2 Let $\frac{DR_{\lambda}^{m+1,n}f(z,\zeta)}{DR_{\lambda}^{m,n}f(z,\zeta)} \in \mathcal{H}(U \times \bar{U})$, $z \in U$, $\zeta \in \bar{U}$, $f \in \mathcal{A}_{\zeta}^*$, $m, n \in \mathbb{N}$, $\lambda \geq 0$ and let the function $q(z, \zeta)$ be convex and univalent in $U \times \bar{U}$ such that $q(0, \zeta) = 1$. Assume that

$$\operatorname{Re} \left(1 + \frac{\alpha}{\mu} + \frac{2\beta}{\mu} q(z, \zeta) + \frac{zq''_z(z, \zeta)}{q'_z(z, \zeta)} \right) > 0, \quad z \in U, \zeta \in \bar{U}, \quad (2.3)$$

for $\alpha, \beta, \mu \in \mathbb{C}, \mu \neq 0$, $z \in U, \zeta \in \bar{U}$, and

$$\begin{aligned}
 \psi_{\lambda}^{m,n}(\alpha, \beta, \mu; z, \zeta) &:= \left(\frac{1 - \lambda(n+1)}{\lambda} \mu + \alpha \right) \frac{DR_{\lambda}^{m+1,n}f(z, \zeta)}{DR_{\lambda}^{m,n}f(z, \zeta)} \\
 &+ \mu(n+1) [1 - \lambda(n+2)] \frac{DR_{\lambda}^{m,n+1}f(z, \zeta)}{DR_{\lambda}^{m,n}f(z, \zeta)} \\
 &+ \lambda\mu(n+1)(n+2) \frac{DR_{\lambda}^{m,n+2}f(z, \zeta)}{DR_{\lambda}^{m,n}f(z, \zeta)} + \left(\beta - \frac{\mu}{\lambda} \right) \left(\frac{DR_{\lambda}^{m+1,n}f(z, \zeta)}{DR_{\lambda}^{m,n}f(z, \zeta)} \right)^2.
 \end{aligned} \quad (2.4)$$

If q satisfies the following strong differential subordination

$$\psi_{\lambda}^{m,n}(\alpha, \beta, \mu; z, \zeta) \prec \prec \alpha q(z, \zeta) + \beta (q(z, \zeta))^2 + \mu z q'_z(z, \zeta), \quad (2.5)$$

for $\alpha, \beta, \mu \in \mathbb{C}, \mu \neq 0$ then

$$\frac{DR_{\lambda}^{m+1,n}f(z, \zeta)}{DR_{\lambda}^{m,n}f(z, \zeta)} \prec \prec q(z, \zeta), \quad z \in U, \zeta \in \bar{U}, \quad (2.6)$$

and q is the best dominant.

Proof. Let the function p be defined by $p(z, \zeta) := \frac{DR_\lambda^{m+1, n} f(z, \zeta)}{DR_\lambda^{m, n} f(z, \zeta)}$, $z \in U$, $z \neq 0$, $\zeta \in \overline{U}$, $f \in \mathcal{A}_\zeta^*$. The function p is analytic in U and $p(0, \zeta) = 1$.

Differentiating with respect to z this function, we get

$$zp'_z(z, \zeta) = \frac{z(DR_\lambda^{m+1, n} f(z, \zeta))'_z}{DR_\lambda^{m, n} f(z, \zeta)} - \frac{DR_\lambda^{m+1, n} f(z, \zeta)}{DR_\lambda^{m, n} f(z, \zeta)} \frac{z(DR_\lambda^{m, n} f(z, \zeta))'_z}{DR_\lambda^{m, n} f(z, \zeta)}$$

By using the identity (2.1) and (2.2), we obtain

$$\begin{aligned} zp'_z(z, \zeta) &= \frac{1 - \lambda(n+1)}{\lambda} \frac{DR_\lambda^{m+1, n} f(z, \zeta)}{DR_\lambda^{m, n} f(z, \zeta)} \\ &+ (n+1) [1 - \lambda(n+2)] \frac{DR_\lambda^{m, n+1} f(z, \zeta)}{DR_\lambda^{m, n} f(z, \zeta)} \\ &+ \lambda(n+1)(n+2) \frac{DR_\lambda^{m, n+2} f(z, \zeta)}{DR_\lambda^{m, n} f(z, \zeta)} - \frac{1}{\lambda} \left(\frac{DR_\lambda^{m+1, n} f(z, \zeta)}{DR_\lambda^{m, n} f(z, \zeta)} \right)^2 \\ &+ \lambda(n+1)(n+2) \frac{DR_\lambda^{m, n+2} f(z, \zeta)}{DR_\lambda^{m, n} f(z, \zeta)} - \frac{1}{\lambda} \left(\frac{DR_\lambda^{m+1, n} f(z, \zeta)}{DR_\lambda^{m, n} f(z, \zeta)} \right)^2 \end{aligned} \quad (2.7)$$

By setting $\theta(w) := \alpha w + \beta w^2$ and $\phi(w) := \mu$, $\alpha, \beta, \mu \in \mathbb{C}$, $\mu \neq 0$ it can be easily verified that θ is analytic in \mathbb{C} , ϕ is analytic in $\mathbb{C} \setminus \{0\}$ and that $\phi(w) \neq 0$, $w \in \mathbb{C} \setminus \{0\}$.

Also, by letting $Q(z, \zeta) = zq'_z(z, \zeta) \phi(q(z, \zeta)) = \mu zq'_z(z, \zeta)$, we find that $Q(z, \zeta)$ is starlike univalent in $U \times \overline{U}$.

Let $h(z, \zeta) = \theta(q(z, \zeta)) + Q(z, \zeta) = \alpha q(z, \zeta) + \beta (q(z, \zeta))^2 + \mu zq'_z(z, \zeta)$, $z \in U$, $\zeta \in \overline{U}$.

If we derive the function Q , with respect to z , perform calculations, we have $\operatorname{Re} \left(\frac{zh'_z(z, \zeta)}{Q(z, \zeta)} \right) = \operatorname{Re} \left(1 + \frac{\alpha}{\mu} + \frac{2\beta}{\mu} q(z, \zeta) + \frac{zq''_z(z, \zeta)}{q'_z(z, \zeta)} \right) > 0$.

By using (2.7), we obtain $\alpha p(z, \zeta) + \beta (p(z, \zeta))^2 + \mu zp'_z(z, \zeta) = \left(\frac{1 - \lambda(n+1)}{\lambda} \mu + \alpha \right) \frac{DR_\lambda^{m+1, n} f(z, \zeta)}{DR_\lambda^{m, n} f(z, \zeta)} + \mu(n+1) [1 - \lambda(n+2)] \frac{DR_\lambda^{m, n+1} f(z, \zeta)}{DR_\lambda^{m, n} f(z, \zeta)} + \lambda \mu(n+1)(n+2) \frac{DR_\lambda^{m, n+2} f(z, \zeta)}{DR_\lambda^{m, n} f(z, \zeta)} + \left(\beta - \frac{\mu}{\lambda} \right) \left(\frac{DR_\lambda^{m+1, n} f(z, \zeta)}{DR_\lambda^{m, n} f(z, \zeta)} \right)^2$.

By using (2.5), we have $\alpha p(z, \zeta) + \beta (p(z, \zeta))^2 + \mu zp'_z(z, \zeta) \prec\prec \alpha q(z, \zeta) + \beta (q(z, \zeta))^2 + \mu zq'_z(z, \zeta)$.

Therefore, the conditions of Lemma 1.1 are met, so we have $p(z, \zeta) \prec\prec q(z, \zeta)$, $z \in U$, $\zeta \in \overline{U}$, i.e. $\frac{DR_\lambda^{m+1, n} f(z, \zeta)}{DR_\lambda^{m, n} f(z, \zeta)} \prec\prec q(z, \zeta)$, $z \in U$, $\zeta \in \overline{U}$, and q is the best dominant. ■

Corollary 2.3 Let $q(z, \zeta) = \frac{\zeta + Az}{\zeta + Bz}$, $-1 \leq B < A \leq 1$, $m, n \in \mathbb{N}$, $\lambda \geq 0$, $z \in U$, $\zeta \in \overline{U}$. Assume that (2.3) holds. If $f \in \mathcal{A}_\zeta^*$ and

$$\psi_\lambda^{m, n}(\alpha, \beta, \mu; z, \zeta) \prec\prec \alpha \frac{\zeta + Az}{\zeta + Bz} + \beta \left(\frac{\zeta + Az}{\zeta + Bz} \right)^2 + \mu \frac{\zeta(A - B)z}{(\zeta + Bz)^2},$$

for $\alpha, \beta, \mu \in \mathbb{C}$, $\mu \neq 0$, $-1 \leq B < A \leq 1$, where $\psi_\lambda^{m,n}$ is defined in (2.4), then

$$\frac{DR_\lambda^{m+1,n} f(z, \zeta)}{DR_\lambda^{m,n} f(z, \zeta)} \prec\prec \frac{\zeta + Az}{\zeta + Bz}$$

and $\frac{\zeta + Az}{\zeta + Bz}$ is the best dominant.

Proof. For $q(z, \zeta) = \frac{\zeta + Az}{\zeta + Bz}$, $-1 \leq B < A \leq 1$, in Theorem 2.2 we get the corollary. ■

Corollary 2.4 Let $q(z, \zeta) = \left(\frac{\zeta+z}{\zeta-z}\right)^\gamma$, $m, n \in \mathbb{N}$, $\lambda \geq 0$, $z \in U$. Assume that (2.3) holds. If $f \in \mathcal{A}_\zeta^*$ and

$$\psi_\lambda^{m,n}(\alpha, \beta, \mu; z, \zeta) \prec \alpha \left(\frac{\zeta+z}{\zeta-z}\right)^\gamma + \beta \left(\frac{\zeta+z}{\zeta-z}\right)^{2\gamma} + \mu \frac{2\zeta\gamma z}{(\zeta-z)^2} \left(\frac{\zeta+z}{\zeta-z}\right)^{\gamma-1}$$

for $\alpha, \mu, \beta \in \mathbb{C}$, $0 < \gamma \leq 1$, $\mu \neq 0$, where $\psi_\lambda^{m,n}$ is defined in (2.4), then

$$\frac{DR_\lambda^{m+1,n} f(z, \zeta)}{DR_\lambda^{m,n} f(z, \zeta)} \prec\prec \left(\frac{\zeta+z}{\zeta-z}\right)^\gamma,$$

and $\left(\frac{\zeta+z}{\zeta-z}\right)^\gamma$ is the best dominant.

Proof. Corollary follows by using Theorem 2.2 for $q(z, \zeta) = \left(\frac{\zeta+z}{\zeta-z}\right)^\gamma$, $0 < \gamma \leq 1$. ■

Theorem 2.5 Let q be convex and univalent in $U \times \overline{U}$, such that $q(0, \zeta) = 1$, $m, n \in \mathbb{N}$, $\lambda \geq 0$. Assume that

$$\operatorname{Re} \left(\frac{q'_z(z, \zeta)}{\mu} (\alpha + 2\beta q(z, \zeta)) \right) > 0, \text{ for } \alpha, \mu, \beta \in \mathbb{C}, \mu \neq 0, \quad (2.8)$$

$z \in U$, $\zeta \in \overline{U}$.

If $f \in \mathcal{A}_\zeta^*$, $\frac{DR_\lambda^{m+1,n} f(z, \zeta)}{DR_\lambda^{m,n} f(z, \zeta)} \in \mathcal{H}_\lambda^*[q(0, \zeta), 1, \zeta] \cap Q^*$ and $\psi_\lambda^{m,n}(\alpha, \beta, \mu; z, \zeta)$ is univalent in $U \times \overline{U}$, where $\psi_\lambda^{m,n}(\alpha, \beta, \mu; z, \zeta)$ is as defined in (2.4), then

$$\alpha q(z, \zeta) + \beta (q(z, \zeta))^2 + \mu z q'_z(z, \zeta) \prec\prec \psi_\lambda^{m,n}(\alpha, \beta, \mu; z, \zeta), \quad (2.9)$$

$z \in U$, $\zeta \in \overline{U}$, implies

$$q(z, \zeta) \prec\prec \frac{DR_\lambda^{m+1,n} f(z, \zeta)}{DR_\lambda^{m,n} f(z, \zeta)}, \quad z \in U, \zeta \in \overline{U}, \quad (2.10)$$

and q is the best subordinant.

Proof. Let the function p be defined by $p(z, \zeta) := \frac{DR_\lambda^{m+1, n} f(z, \zeta)}{DR_\lambda^{m, n} f(z, \zeta)}$, $z \in U$, $z \neq 0$, $\zeta \in \bar{U}$, $f \in \mathcal{A}_\zeta^*$.

By setting $\nu(w) := \alpha w + \beta w^2$ and $\phi(w) := \mu$ it can be easily verified that ν is analytic in \mathbb{C} , ϕ is analytic in $\mathbb{C} \setminus \{0\}$ and that $\phi(w) \neq 0$, $w \in \mathbb{C} \setminus \{0\}$.

Since $\frac{\nu'_z(q(z, \zeta))}{\phi(q(z, \zeta))} = \frac{q'_z(z, \zeta)}{\mu} (\alpha + 2\beta q(z, \zeta))$, it follows that

$$\operatorname{Re} \left(\frac{\nu'_z(q(z, \zeta))}{\phi(q(z, \zeta))} \right) = \operatorname{Re} \left(\frac{q'_z(z, \zeta)}{\mu} (\alpha + 2\beta q(z, \zeta)) \right) > 0,$$

for $\mu, \xi, \beta \in \mathbb{C}$, $\mu \neq 0$.

By using (2.9) we obtain

$$\begin{aligned} \alpha q(z, \zeta) + \beta (q(z, \zeta))^2 + \mu z q'_z(z, \zeta) &\prec\prec \\ \alpha q(z, \zeta) + \beta (q(z, \zeta))^2 + \mu z q'_z(z, \zeta). \end{aligned}$$

Using Lemma 1.2, we have

$$q(z, \zeta) \prec\prec p(z, \zeta) = \frac{DR_\lambda^{m+1, n} f(z, \zeta)}{DR_\lambda^{m, n} f(z, \zeta)}, \quad z \in U, \quad \zeta \in \bar{U},$$

and q is the best subordinator. ■

Corollary 2.6 Let $q(z, \zeta) = \frac{\zeta + Az}{\zeta + Bz}$, $-1 \leq B < A \leq 1$, $m, n \in \mathbb{N}$, $\lambda \geq 0$. Assume that (2.8) holds.

If $f \in \mathcal{A}_\zeta^*$, $\frac{DR_\lambda^{m+1, n} f(z, \zeta)}{DR_\lambda^{m, n} f(z, \zeta)} \in \mathcal{H}^*[q(0, \zeta), 1, \zeta] \cap Q^*$ and

$$\alpha \frac{\zeta + Az}{\zeta + Bz} + \beta \left(\frac{\zeta + Az}{\zeta + Bz} \right)^2 + \mu \frac{\zeta(A - B)z}{(\zeta + Bz)^2} \prec\prec \psi_\lambda^{m, n}(\alpha, \beta, \mu; z, \zeta),$$

for $\alpha, \mu, \beta \in \mathbb{C}$, $\mu \neq 0$, $-1 \leq B < A \leq 1$, where $\psi_\lambda^{m, n}$ is defined in (2.4), then

$$\frac{\zeta + Az}{\zeta + Bz} \prec\prec \frac{DR_\lambda^{m+1, n} f(z, \zeta)}{DR_\lambda^{m, n} f(z, \zeta)}$$

and $\frac{\zeta + Az}{\zeta + Bz}$ is the best subordinator.

Proof. For $q(z, \zeta) = \frac{\zeta + Az}{\zeta + Bz}$, $-1 \leq B < A \leq 1$ in Theorem 2.5 we get the corollary. ■

Corollary 2.7 Let $q(z, \zeta) = \left(\frac{\zeta + z}{\zeta - z} \right)^\gamma$, $m, n \in \mathbb{N}$, $\lambda \geq 0$. Assume that (2.8) holds.

If $f \in \mathcal{A}_\zeta^*$, $\frac{DR_\lambda^{m+1, n} f(z, \zeta)}{DR_\lambda^{m, n} f(z, \zeta)} \in \mathcal{H}^*[q(0, \zeta), 1, \zeta] \cap Q^*$ and

$$\begin{aligned} \alpha \left(\frac{\zeta + z}{\zeta - z} \right)^\gamma + \beta \left(\frac{\zeta + z}{\zeta - z} \right)^{2\gamma} + \mu \frac{2\zeta\gamma z}{(\zeta - z)^2} \left(\frac{\zeta + z}{\zeta - z} \right)^{\gamma-1} \\ \prec\prec \psi_\lambda^{m, n}(\alpha, \beta, \mu; z, \zeta), \end{aligned}$$

for $\alpha, \mu, \beta \in \mathbb{C}$, $0 < \gamma \leq 1$, $\mu \neq 0$, where $\psi_\lambda^{m,n}$ is defined in (2.4), then

$$\left(\frac{\zeta+z}{\zeta-z}\right)^\gamma \prec\prec \frac{DR_\lambda^{m+1,n}f(z,\zeta)}{DR_\lambda^{m,n}f(z,\zeta)}$$

and $\left(\frac{\zeta+z}{\zeta-z}\right)^\gamma$ is the best subordinant.

Proof. Corollary follows by using Theorem 2.5 for $q(z,\zeta) = \left(\frac{\zeta+z}{\zeta-z}\right)^\gamma$, $0 < \gamma \leq 1$.

Combining Theorem 2.2 and Theorem 2.5, we state the following sandwich theorem.

Theorem 2.8 Let q_1 and q_2 be analytic and univalent in $U \times \overline{U}$ such that $q_1(z,\zeta) \neq 0$ and $q_2(z,\zeta) \neq 0$, for all $z \in U$, $\zeta \in \overline{U}$, with $z(q_1)'_z(z,\zeta)$ and $z(q_2)'_z(z,\zeta)$ being starlike univalent. Suppose that q_1 satisfies (2.3) and q_2 satisfies (2.8). If $f \in \mathcal{A}_\zeta^*$, $\frac{DR_\lambda^{m+1,n}f(z,\zeta)}{DR_\lambda^{m,n}f(z,\zeta)} \in \mathcal{H}^*[q(0,\zeta), 1, \zeta] \cap Q^*$ and $\psi_\lambda^{m,n}(\alpha, \beta, \mu; z, \zeta)$ is as defined in (2.4) univalent in $U \times \overline{U}$, then

$$\begin{aligned} \alpha q_1(z,\zeta) + \beta (q_1(z,\zeta))^2 + \mu z (q_1)'_z(z,\zeta) &\prec\prec \psi_\lambda^{m,n}(\alpha, \beta, \mu; z, \zeta) \\ &\prec\prec \alpha q_2(z,\zeta) + \beta (q_2(z,\zeta))^2 + \mu z (q_2)'_z(z,\zeta), \end{aligned}$$

for $\alpha, \mu, \beta \in \mathbb{C}$, $\mu \neq 0$, implies

$$q_1(z,\zeta) \prec\prec \frac{DR_\lambda^{m+1,n}f(z,\zeta)}{DR_\lambda^{m,n}f(z,\zeta)} \prec\prec q_2(z,\zeta), \quad \delta \in \mathbb{C}, \delta \neq 0,$$

and q_1 and q_2 are respectively the best subordinant and the best dominant.

For $q_1(z,\zeta) = \frac{\zeta+A_1z}{\zeta+B_1z}$, $q_2(z,\zeta) = \frac{\zeta+A_2z}{\zeta+B_2z}$, where $-1 \leq B_2 < B_1 < A_1 < A_2 \leq 1$, we have the following corollary.

Corollary 2.9 Let $m, n \in \mathbb{N}$, $\lambda \geq 0$. Assume that (2.3) and (2.8) hold for $q_1(z,\zeta) = \frac{\zeta+A_1z}{\zeta+B_1z}$ and $q_2(z,\zeta) = \frac{\zeta+A_2z}{\zeta+B_2z}$, respectively. If $f \in \mathcal{A}_\zeta^*$, $\frac{DR_\lambda^{m+1,n}f(z,\zeta)}{DR_\lambda^{m,n}f(z,\zeta)} \in \mathcal{H}^*[q(0,\zeta), 1, \zeta] \cap Q^*$ and

$$\begin{aligned} \alpha \frac{\zeta+A_1z}{\zeta+B_1z} + \beta \left(\frac{\zeta+A_1z}{\zeta+B_1z}\right)^2 + \mu \frac{(A_1-B_1)\zeta z}{(\zeta+B_1z)^2} &\prec\prec \psi_\lambda^{m,n}(\alpha, \beta, \mu; z, \zeta) \\ &\prec\prec \alpha \frac{\zeta+A_2z}{\zeta+B_2z} + \beta \left(\frac{\zeta+A_2z}{\zeta+B_2z}\right)^2 + \mu \frac{(A_2-B_2)\zeta z}{(\zeta+B_2z)^2}, \end{aligned}$$

for $\alpha, \mu, \beta \in \mathbb{C}$, $\mu \neq 0$, $-1 \leq B_2 \leq B_1 < A_1 \leq A_2 \leq 1$, where $\psi_\lambda^{m,n}$ is defined in (2.4), then

$$\frac{\zeta+A_1z}{\zeta+B_1z} \prec\prec \frac{DR_\lambda^{m+1,n}f(z,\zeta)}{DR_\lambda^{m,n}f(z,\zeta)} \prec\prec \frac{\zeta+A_2z}{\zeta+B_2z},$$

hence $\frac{\zeta+A_1z}{\zeta+B_1z}$ and $\frac{\zeta+A_2z}{\zeta+B_2z}$ are the best subordinant and the best dominant, respectively.

Theorem 2.10 Let $\left(\frac{DR_\lambda^{m+1,n}f(z,\zeta)}{DR_\lambda^{m,n}f(z,\zeta)}\right)^\delta \in \mathcal{H}(U \times \overline{U})$, $f \in \mathcal{A}_\zeta^*$, $z \in U$, $\zeta \in \overline{U}$, $\delta \in \mathbb{C}$, $\delta \neq 0$, $m, n \in \mathbb{N}$, $\lambda \geq 0$ and let the function $q(z, \zeta)$ be convex and univalent in $U \times \overline{U}$ such that $q(0, \zeta) = 1$, $\zeta \in \overline{U}$. Assume that

$$\operatorname{Re} \left(\frac{\alpha + \beta}{\beta} + \frac{zq''_{z^2}(z, \zeta)}{q'_z(z, \zeta)} \right) > 0, \quad (2.11)$$

for $\alpha, \beta \in \mathbb{C}$, $\beta \neq 0$, $z \in U$, $\zeta \in \overline{U}$, and

$$\begin{aligned} \psi_\lambda^{m,n}(\alpha, \beta; z, \zeta) &:= \left(\frac{DR_\lambda^{m+1,n}f(z, \zeta)}{DR_\lambda^{m,n}f(z, \zeta)} \right)^\delta \\ &\cdot \left[\alpha + \delta\beta \frac{1 - \lambda(n+1)}{\lambda} + \delta\beta(n+1)[1 - \lambda(n+2)] \frac{DR_\lambda^{m,n+1}f(z, \zeta)}{DR_\lambda^{m+1,n}f(z, \zeta)} \right. \\ &\left. + \delta\beta\lambda(n+1)(n+2) \frac{DR_\lambda^{m,n+2}f(z, \zeta)}{DR_\lambda^{m+1,n}f(z, \zeta)} - \frac{\delta\beta}{\lambda} \frac{DR_\lambda^{m+1,n}f(z, \zeta)}{DR_\lambda^{m,n}f(z, \zeta)} \right] \end{aligned} \quad (2.12)$$

If q satisfies the following strong differential subordination

$$\psi_\lambda^{m,n}(\alpha, \beta; z, \zeta) \prec\prec \alpha q(z, \zeta) + \beta z q'_z(z, \zeta), \quad (2.13)$$

for $\alpha, \beta \in \mathbb{C}$, $\beta \neq 0$, $z \in U$, $\zeta \in \overline{U}$, then

$$\left(\frac{DR_\lambda^{m+1,n}f(z, \zeta)}{DR_\lambda^{m,n}f(z, \zeta)} \right)^\delta \prec\prec q(z, \zeta), \quad z \in U, \zeta \in \overline{U}, \delta \in \mathbb{C}, \delta \neq 0, \quad (2.14)$$

and q is the best dominant.

Proof. Let the function p be defined by $p(z, \zeta) := \left(\frac{DR_\lambda^{m+1,n}f(z,\zeta)}{DR_\lambda^{m,n}f(z,\zeta)}\right)^\delta$, $z \in U$, $z \neq 0$, $\zeta \in \overline{U}$, $f \in \mathcal{A}_\zeta^*$. The function p is analytic in $U \times \overline{U}$ and $p(0, \zeta) = 1$. We have

$$\begin{aligned} zp'_z(z, \zeta) &= \delta z \left(\frac{DR_\lambda^{m+1,n}f(z, \zeta)}{DR_\lambda^{m,n}f(z, \zeta)} \right)^\delta \frac{DR_\lambda^{m,n}f(z, \zeta)}{DR_\lambda^{m+1,n}f(z, \zeta)} \left(\frac{DR_\lambda^{m+1,n}f(z, \zeta)}{DR_\lambda^{m,n}f(z, \zeta)} \right)'_z \\ &= \delta \left(\frac{DR_\lambda^{m+1,n}f(z, \zeta)}{DR_\lambda^{m,n}f(z, \zeta)} \right)^\delta \frac{DR_\lambda^{m,n}f(z, \zeta)}{DR_\lambda^{m+1,n}f(z, \zeta)} \\ &\quad \cdot \left(\frac{z \left(DR_\lambda^{m+1,n}f(z, \zeta) \right)'_z}{DR_\lambda^{m,n}f(z, \zeta)} - \frac{DR_\lambda^{m+1,n}f(z, \zeta)}{DR_\lambda^{m,n}f(z, \zeta)} \frac{z \left(DR_\lambda^{m,n}f(z, \zeta) \right)'_z}{DR_\lambda^{m,n}f(z, \zeta)} \right). \end{aligned}$$

By using the identity (2.1) and (2.2), we obtain

$$\begin{aligned}
 zp'_z(z, \zeta) &= \delta \left(\frac{DR_\lambda^{m+1, n} f(z, \zeta)}{DR_\lambda^{m, n} f(z, \zeta)} \right)^\delta \frac{DR_\lambda^{m, n} f(z, \zeta)}{DR_\lambda^{m+1, n} f(z, \zeta)} \\
 &\cdot \left[\left(\frac{1 - \lambda(n+1)}{\lambda} \right) \frac{DR_\lambda^{m+1, n} f(z, \zeta)}{DR_\lambda^{m, n} f(z, \zeta)} + n + 1 \right] \\
 &\cdot [1 - \lambda(n+2)] \frac{DR_\lambda^{m, n+1} f(z, \zeta)}{DR_\lambda^{m, n} f(z, \zeta)} + \lambda(n+1)(n+2) \frac{DR_\lambda^{m, n+2} f(z, \zeta)}{DR_\lambda^{m, n} f(z, \zeta)} \\
 &- \frac{1}{\lambda} \left(\frac{DR_\lambda^{m+1, n} f(z, \zeta)}{DR_\lambda^{m, n} f(z, \zeta)} \right)^2 \Big] \tag{2.15}
 \end{aligned}$$

so, we obtain

$$\begin{aligned}
 zp'_z(z, \zeta) &= \delta \left(\frac{DR_\lambda^{m+1, n} f(z, \zeta)}{DR_\lambda^{m, n} f(z, \zeta)} \right)^\delta \left[\frac{1 - \lambda(n+1)}{\lambda} + \right. \\
 &\quad \left. (n+1) [1 - \lambda(n+2)] \frac{DR_\lambda^{m, n+1} f(z, \zeta)}{DR_\lambda^{m+1, n} f(z, \zeta)} + \right. \\
 &\quad \left. \lambda(n+1)(n+2) \frac{DR_\lambda^{m, n+2} f(z, \zeta)}{DR_\lambda^{m+1, n} f(z, \zeta)} - \frac{1}{\lambda} \frac{DR_\lambda^{m+1, n} f(z, \zeta)}{DR_\lambda^{m, n} f(z, \zeta)} \right] \tag{2.16}
 \end{aligned}$$

By setting $\theta(w) := \alpha w$ and $\phi(w) := \beta$, it can be easily verified that θ is analytic in \mathbb{C} , ϕ is analytic in $\mathbb{C} \setminus \{0\}$ and that $\phi(w) \neq 0, w \in \mathbb{C} \setminus \{0\}$.

Also, by letting $Q(z, \zeta) = zq'_z(z, \zeta)\phi(q(z, \zeta)) = \beta zq'_z(z, \zeta)$, we find that $Q(z, \zeta)$ is starlike univalent in $U \times \bar{U}$.

Let $h(z, \zeta) = \theta(q(z, \zeta)) + Q(z, \zeta) = \alpha q(z, \zeta) + \beta zq'_z(z, \zeta)$.

We have $\operatorname{Re} \left(\frac{zh'_z(z, \zeta)}{Q(z, \zeta)} \right) = \operatorname{Re} \left(\frac{\alpha + \beta}{\beta} + \frac{zq''_z(z, \zeta)}{q'_z(z, \zeta)} \right) > 0$.

By using (2.16), we obtain

$$\begin{aligned}
 \alpha p(z, \zeta) + \beta zp'_z(z, \zeta) &= \left(\frac{DR_\lambda^{m+1, n} f(z, \zeta)}{DR_\lambda^{m, n} f(z, \zeta)} \right)^\delta \\
 &\cdot \left[\alpha + \delta\beta \frac{1 - \lambda(n+1)}{\lambda} + \delta\beta(n+1) [1 - \lambda(n+2)] \frac{DR_\lambda^{m, n+1} f(z, \zeta)}{DR_\lambda^{m+1, n} f(z, \zeta)} \right. \\
 &\quad \left. + \delta\beta\lambda(n+1)(n+2) \frac{DR_\lambda^{m, n+2} f(z, \zeta)}{DR_\lambda^{m+1, n} f(z, \zeta)} - \frac{\delta\beta}{\lambda} \frac{DR_\lambda^{m+1, n} f(z, \zeta)}{DR_\lambda^{m, n} f(z, \zeta)} \right].
 \end{aligned}$$

By using (2.13), we have $\alpha p(z, \zeta) + \beta zp'_z(z, \zeta) \prec\prec \alpha q(z, \zeta) + \beta zq'_z(z, \zeta)$.

From Lemma 1.1, we have $p(z, \zeta) \prec\prec q(z, \zeta), z \in U, \zeta \in \bar{U}$, i.e. $\left(\frac{DR_\lambda^{m+1, n} f(z, \zeta)}{DR_\lambda^{m, n} f(z, \zeta)} \right)^\delta \prec\prec q(z, \zeta), z \in U, \zeta \in \bar{U}, \delta \in \mathbb{C}, \delta \neq 0$ and q is the best dominant. ■

Corollary 2.11 Let $q(z, \zeta) = \frac{\zeta + Az}{\zeta + Bz}$, $z \in U$, $\zeta \in \overline{U}$, $-1 \leq B < A \leq 1$, $m, n \in \mathbb{N}$, $\lambda \geq 0$. Assume that (2.11) holds. If $f \in \mathcal{A}_\zeta^*$ and

$$\psi_\lambda^{m,n}(\alpha, \beta; z, \zeta) \prec\prec \alpha \frac{\zeta + Az}{\zeta + Bz} + \beta \frac{(A-B)\zeta z}{(\zeta + Bz)^2},$$

for $\alpha, \beta \in \mathbb{C}$, $\beta \neq 0$, $-1 \leq B < A \leq 1$, where $\psi_\lambda^{m,n}$ is defined in (2.12), then

$$\left(\frac{DR_\lambda^{m+1,n} f(z, \zeta)}{DR_\lambda^{m,n} f(z, \zeta)} \right)^\delta \prec\prec \frac{\zeta + Az}{\zeta + Bz}, \quad \delta \in \mathbb{C}, \delta \neq 0,$$

and $\frac{\zeta + Az}{\zeta + Bz}$ is the best dominant.

Proof. For $q(z, \zeta) = \frac{\zeta + Az}{\zeta + Bz}$, $-1 \leq B < A \leq 1$, in Theorem 2.10 we get the corollary. ■

Corollary 2.12 Let $q(z, \zeta) = \left(\frac{\zeta + z}{\zeta - z} \right)^\gamma$, $m, n \in \mathbb{N}$, $\lambda \geq 0$. Assume that (2.11) holds. If $f \in \mathcal{A}_\zeta^*$ and

$$\psi_\lambda^{m,n}(\alpha, \beta, \mu; z, \zeta) \prec\prec \alpha \left(\frac{\zeta + z}{\zeta - z} \right)^\gamma + \beta \frac{2\gamma\zeta z}{(\zeta - z)^2} \left(\frac{\zeta + z}{\zeta - z} \right)^{\gamma-1},$$

for $\alpha, \beta \in \mathbb{C}$, $0 < \gamma \leq 1$, $\beta \neq 0$, where $\psi_\lambda^{m,n}$ is defined in (2.12), then

$$\left(\frac{DR_\lambda^{m+1,n} f(z, \zeta)}{DR_\lambda^{m,n} f(z, \zeta)} \right)^\delta \prec\prec \left(\frac{\zeta + z}{\zeta - z} \right)^\gamma, \quad \delta \in \mathbb{C}, \delta \neq 0,$$

and $\left(\frac{\zeta + z}{\zeta - z} \right)^\gamma$ is the best dominant.

Proof. Corollary follows by using Theorem 2.10 for $q(z, \zeta) = \left(\frac{\zeta + z}{\zeta - z} \right)^\gamma$, $0 < \gamma \leq 1$. ■

Theorem 2.13 Let q be convex and univalent in $U \times \overline{U}$ such that $q(0, \zeta) = 1$. Assume that

$$\operatorname{Re} \left(\frac{\alpha}{\beta} q'_z(z, \zeta) \right) > 0, \quad \text{for } \alpha, \beta \in \mathbb{C}, \beta \neq 0. \quad (2.17)$$

If $f \in \mathcal{A}_\zeta^*$, $\left(\frac{DR_\lambda^{m+1,n} f(z, \zeta)}{DR_\lambda^{m,n} f(z, \zeta)} \right)^\delta \in \mathcal{H}^*[q(0, \zeta), 1, \zeta] \cap Q^*$ and $\psi_\lambda^{m,n}(\alpha, \beta; z, \zeta)$ is univalent in $U \times \overline{U}$, where $\psi_\lambda^{m,n}(\alpha, \beta; z, \zeta)$ is as defined in (2.12), then

$$\alpha q(z, \zeta) + \beta z q'_z(z, \zeta) \prec\prec \psi_\lambda^{m,n}(\alpha, \beta; z, \zeta) \quad (2.18)$$

implies

$$q(z, \zeta) \prec\prec \left(\frac{DR_\lambda^{m+1,n} f(z, \zeta)}{DR_\lambda^{m,n} f(z, \zeta)} \right)^\delta, \quad \delta \in \mathbb{C}, \delta \neq 0, z \in U, \zeta \in \overline{U}, \quad (2.19)$$

and q is the best subordinant.

Proof. Let the function p be defined by $p(z, \zeta) := \left(\frac{DR_\lambda^{m+1, n} f(z, \zeta)}{DR_\lambda^{m, n} f(z, \zeta)} \right)^\delta$, $z \in U$, $z \neq 0$, $\zeta \in \overline{U}$, $\delta \in \mathbb{C}, \delta \neq 0$, $f \in \mathcal{A}_\zeta^*$. The function p is analytic in $U \times \overline{U}$ and $p(0, \zeta) = 1$.

By setting $\nu(w) := \alpha w$ and $\phi(w) := \beta$ it can be easily verified that ν is analytic in \mathbb{C} , ϕ is analytic in $\mathbb{C} \setminus \{0\}$ and that $\phi(w) \neq 0$, $w \in \mathbb{C} \setminus \{0\}$.

Since $\frac{\nu'_z(q(z, \zeta))}{\phi(q(z, \zeta))} = \frac{\alpha}{\beta} q'_z(z, \zeta)$, it follows that

$$\operatorname{Re} \left(\frac{\nu'_z(q(z, \zeta))}{\phi(q(z, \zeta))} \right) = \operatorname{Re} \left(\frac{\alpha}{\beta} q'_z(z, \zeta) \right) > 0, \text{ for } \alpha, \beta \in \mathbb{C}, \beta \neq 0.$$

Now, by using (2.18) we obtain

$$\alpha q(z, \zeta) + \beta z q'_z(z, \zeta) \prec\prec \alpha q(z, \zeta) + \beta z q'_z(z, \zeta), \quad z \in U, \zeta \in \overline{U}.$$

From Lemma 1.2, we have

$$q(z, \zeta) \prec\prec p(z, \zeta) = \left(\frac{DR_\lambda^{m+1, n} f(z, \zeta)}{DR_\lambda^{m, n} f(z, \zeta)} \right)^\delta,$$

$z \in U, \zeta \in \overline{U}, \delta \in \mathbb{C}, \delta \neq 0$, and q is the best subordinated. ■

Corollary 2.14 Let $q(z, \zeta) = \frac{\zeta + Az}{\zeta + Bz}$, $-1 \leq B < A \leq 1$, $z \in U, \zeta \in \overline{U}$, $m, n \in \mathbb{N}$, $\lambda \geq 0$. Assume that (2.17) holds. If $f \in \mathcal{A}_\zeta^*$, $\left(\frac{DR_\lambda^{m+1, n} f(z, \zeta)}{DR_\lambda^{m, n} f(z, \zeta)} \right)^\delta \in \mathcal{H}^*[q(0, \zeta), 1, \zeta] \cap Q^*$, $\delta \in \mathbb{C}, \delta \neq 0$ and

$$\alpha \frac{\zeta + Az}{\zeta + Bz} + \beta \frac{(A - B)\zeta z}{(\zeta + Bz)^2} \prec\prec \psi_\lambda^{m, n}(\alpha, \beta; z, \zeta),$$

for $\alpha, \beta \in \mathbb{C}, \beta \neq 0, -1 \leq B < A \leq 1$, where $\psi_\lambda^{m, n}$ is defined in (2.12), then

$$\frac{\zeta + Az}{\zeta + Bz} \prec\prec \left(\frac{DR_\lambda^{m+1, n} f(z, \zeta)}{DR_\lambda^{m, n} f(z, \zeta)} \right)^\delta, \quad \delta \in \mathbb{C}, \delta \neq 0,$$

and $\frac{\zeta + Az}{\zeta + Bz}$ is the best subordinated.

Proof. For $q(z, \zeta) = \frac{\zeta + Az}{\zeta + Bz}$, $-1 \leq B < A \leq 1$, in Theorem 2.13 we get the corollary. ■

Corollary 2.15 Let $q(z, \zeta) = \left(\frac{\zeta + z}{\zeta - z} \right)^\gamma$, $m, n \in \mathbb{N}, \lambda \geq 0$. Assume that (2.17) holds. If $f \in \mathcal{A}_\zeta^*$, $\left(\frac{DR_\lambda^{m+1, n} f(z, \zeta)}{DR_\lambda^{m, n} f(z, \zeta)} \right)^\delta \in \mathcal{H}^*[q(0, \zeta), 1, \zeta] \cap Q^*$ and

$$\alpha \left(\frac{\zeta + z}{\zeta - z} \right)^\gamma + \beta \frac{2\gamma\zeta z}{(\zeta - z)^2} \left(\frac{\zeta + z}{\zeta - z} \right)^{\gamma-1} \prec\prec \psi_\lambda^{m, n}(\alpha, \beta, \mu; z, \zeta),$$

for $\alpha, \beta \in \mathbb{C}$, $0 < \gamma \leq 1$, $\beta \neq 0$, where $\psi_\lambda^{m,n}$ is defined in (2.12), then

$$\left(\frac{\zeta+z}{\zeta-z}\right)^\gamma \prec\prec \left(\frac{DR_\lambda^{m+1,n}f(z,\zeta)}{DR_\lambda^{m,n}f(z,\zeta)}\right)^\delta, \quad \delta \in \mathbb{C}, \delta \neq 0,$$

and $\left(\frac{\zeta+z}{\zeta-z}\right)^\gamma$ is the best subdominant.

Proof. Corollary follows by using Theorem 2.13 for $q(z, \zeta) = \left(\frac{\zeta+z}{\zeta-z}\right)^\gamma$, $0 < \gamma \leq 1$.

Combining Theorem 2.10 and Theorem 2.13, we state the following sandwich theorem.

Theorem 2.16 Let q_1 and q_2 be convex and univalent in $U \times \overline{U}$ such that $q_1(z, \zeta) \neq 0$ and $q_2(z, \zeta) \neq 0$, for all $z \in U$, $\zeta \in \overline{U}$. Suppose that q_1 satisfies (2.11) and q_2 satisfies (2.17). If $f \in \mathcal{A}_\zeta^*$, $\left(\frac{DR_\lambda^{m+1,n}f(z,\zeta)}{DR_\lambda^{m,n}f(z,\zeta)}\right)^\delta \in \mathcal{H}^*[q(0, \zeta), 1, \zeta] \cap Q^*$, $\delta \in \mathbb{C}$, $\delta \neq 0$ and $\psi_\lambda^{m,n}(\alpha, \beta; z, \zeta)$ is as defined in (2.12) univalent in $U \times \overline{U}$, then

$$\begin{aligned} \alpha q_1(z, \zeta) + \beta z (q_1)'_z(z, \zeta) &\prec\prec \psi_\lambda^{m,n}(\alpha, \beta; z, \zeta) \\ &\prec\prec \alpha q_2(z, \zeta) + \beta z (q_2)'_z(z, \zeta), \end{aligned}$$

for $\alpha, \beta \in \mathbb{C}$, $\beta \neq 0$, implies

$$q_1(z, \zeta) \prec\prec \left(\frac{DR_\lambda^{m+1,n}f(z,\zeta)}{DR_\lambda^{m,n}f(z,\zeta)}\right)^\delta \prec\prec q_2(z, \zeta),$$

$z \in U$, $\zeta \in \overline{U}$, $\delta \in \mathbb{C}$, $\delta \neq 0$, and q_1 and q_2 are respectively the best subdominant and the best dominant.

For $q_1(z, \zeta) = \frac{\zeta+A_1z}{\zeta+B_1z}$, $q_2(z, \zeta) = \frac{\zeta+A_2z}{\zeta+B_2z}$, where $-1 \leq B_2 < B_1 < A_1 < A_2 \leq 1$, we have the following corollary.

Corollary 2.17 Let $m, n \in \mathbb{N}$, $\lambda \geq 0$. Assume that (2.11) and (2.17) hold for $q_1(z, \zeta) = \frac{\zeta+A_1z}{\zeta+B_1z}$ and $q_2(z, \zeta) = \frac{\zeta+A_2z}{\zeta+B_2z}$, respectively. If $f \in \mathcal{A}_\zeta^*$, $\left(\frac{DR_\lambda^{m+1,n}f(z,\zeta)}{DR_\lambda^{m,n}f(z,\zeta)}\right)^\delta \in \mathcal{H}^*[q(0, \zeta), 1, \zeta] \cap Q^*$ and

$$\begin{aligned} \alpha \frac{\zeta+A_1z}{\zeta+B_1z} + \beta \frac{(A_1-B_1)\zeta z}{(\zeta+B_1z)^2} &\prec\prec \psi_\lambda^{m,n}(\alpha, \beta, \mu; z, \zeta) \\ &\prec\prec \alpha \frac{\zeta+A_2z}{\zeta+B_2z} + \beta \frac{(A_2-B_2)\zeta z}{(\zeta+B_2z)^2}, \quad z \in U, \zeta \in \overline{U}, \end{aligned}$$

for $\alpha, \beta \in \mathbb{C}$, $\beta \neq 0$, $-1 \leq B_2 \leq B_1 < A_1 \leq A_2 \leq 1$, where $\psi_\lambda^{m,n}$ is defined in (2.4), then

$$\frac{\zeta+A_1z}{\zeta+B_1z} \prec\prec \left(\frac{DR_\lambda^{m+1,n}f(z,\zeta)}{DR_\lambda^{m,n}f(z,\zeta)}\right)^\delta \prec\prec \frac{\zeta+A_2z}{\zeta+B_2z},$$

$z \in U$, $\zeta \in \overline{U}$, $\delta \in \mathbb{C}$, $\delta \neq 0$, hence $\frac{\zeta+A_1z}{\zeta+B_1z}$ and $\frac{\zeta+A_2z}{\zeta+B_2z}$ are the best subdominant and the best dominant, respectively.

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Majorization problems for classes of analytic functions

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ABSTRACT: The main object of the present paper is to investigate problems of majorization for certain classes of analytic functions of complex order defined by an operator related to the modified Bessel functions of first kind. These results are obtained by investigating appropriate class of admissible functions. Various known or new special cases of our results are-

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1 Introduction

Let \mathcal{A} be the class of functions of the form

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n, \quad (1.1)$$

which are analytic in the open unit disk $\mathbb{U} = \{z \in \mathbb{C} : |z| < 1\}$.

For given $g(z) = z + \sum_{n=2}^{\infty} b_n z^n \in \mathcal{A}$ the Hadamard product of f and g is denoted by

$$(f * g)(z) = z + \sum_{n=2}^{\infty} a_n b_n z^n, \quad z \in \mathbb{U}. \quad (1.2)$$

Note that $f * g \in \mathcal{A}$ which are analytic in the open disc \mathbb{U} .

We say that $f \in \mathcal{A}$ is subordinate to $g \in \mathcal{A}$ denoted by $f \prec g$ if there exists a Schwarz function ω which is analytic in \mathbb{U} with $\omega(0) = 0$ and $|\omega(z)| < 1$ for all $z \in \mathbb{U}$, such that $f(z) = g(\omega(z))$ for $z \in \mathbb{U}$.

Note that, if the function g is univalent in \mathbb{U} , due to Miller and Mocanu [9], we have

$$f(z) \prec g(z) \iff [f(0) = g(0) \text{ and } f(\mathbb{U}) \subset g(\mathbb{U})].$$

If f and g are analytic functions in \mathbb{U} , following MacGregor [8], we say that f is majorized by g in \mathbb{U} that is $f(z) \ll g(z)$ if there exists a function ϕ , analytic in \mathbb{U} , such that

$$|\phi(z)| < 1 \text{ and } f(z) = \phi(z)g(z), \quad z \in \mathbb{U}.$$

It is of interest to note that the notation of majorization is closely related to the concept of quasi-subordination between analytic functions.

Let $\mathcal{C}^*(\gamma)$ denote the class of starlike functions of complex order γ ($\gamma \in \mathbb{C} \setminus \{0\}$), satisfying the following condition

$$\frac{f(z)}{z} \neq 0 \text{ and } \Re \left(1 + \frac{1}{\gamma} \left[\frac{zf'(z)}{f(z)} - 1 \right] \right) > 0, \quad z \in \mathbb{U}.$$

In particular, the class

$$\mathcal{S}^*(\alpha, \lambda) := \mathcal{C}^*((1 - \alpha)\cos\lambda e^{-i\lambda}), \quad |\lambda| < \frac{\pi}{2}; \quad 0 \leq \alpha \leq 1$$

denotes the class of λ -spiral function of order α investigated by Libera [6]. Moreover, the classes

$$\widehat{\mathcal{S}}^*(\lambda) := \mathcal{S}^*(0, \lambda), \quad \mathcal{S}^*(\alpha) := \mathcal{S}^*(\alpha, 0)$$

are the class of spiral functions introduced by Špaček [12] (see also [13]) and the class of starlike functions of order α , respectively. For $\alpha = 0$, we obtain the familiar class $\mathcal{S}^* := \mathcal{S}^*(0)$ of starlike functions.

We recall here a generalized Bessel function of first kind of order p denoted by $\omega_{p,b,c} =: \omega$ defined in [1] and given by

$$\omega(z) = \omega_{p,b,c}(z) = \sum_{n=0}^{\infty} \frac{(-1)^n c^n}{n! \Gamma(p+n+(b+1)/2)} \left(\frac{z}{2}\right)^{2n+p}, \quad z \in \mathbb{C} \quad (1.3)$$

which is the particular solution of the second order linear homogeneous differential equation

$$z^2\omega''(z) + bz\omega'^2 - [p^2 + (1-b)]\omega(z) = 0, \quad (1.4)$$

where $b, p, c \in \mathbb{C}$, which is natural generalization of Bessel's equation.

The differential equation (1.4) permits the study of Bessel function, modified Bessel function, spherical Bessel function and modified spherical Bessel functions all together. Solutions of (1.4) are referred to as the generalized Bessel function of order p . The particular solution given by (1.3) is called the generalized Bessel function of the first kind of order p . Although the series defined in (1.3) is convergent everywhere, the function $\omega_{p,b,c}$ is generally not univalent in \mathbb{U} .

It is of interest to note that when $b = c = 1$, we reobtain the Bessel function of the first kind $\omega_{p,1,1} = j_p$, and for $b = 1, c = -1$ the function $\omega_{p,1,-1}$ becomes the modified Bessel function I_p . Further note that $b = 2$ and $c = 1$ the function $w_{p,2,1}(z)$

reduces to $\sqrt{\frac{2}{\pi}} J_p(z)$ becomes the spherical Bessel function of the first kind of order p . Now, we consider the function $u_{p,b,c}(z)$ defined by the transformation

$$u_{p,b,c}(z) = 2^p \Gamma\left(p + \frac{b+1}{2}\right) z^{1-\frac{p}{2}} \omega_{p,b,c}(\sqrt{z}).$$

By using well known Pochhammer symbol (or the shifted factorial) defined, in terms of the familiar Gamma function, by

$$(a)_n := \frac{\Gamma(a+n)}{\Gamma(a)} = \begin{cases} 1 & (n=0), \\ a(a+1)(a+2)\cdots(a+n-1) & (n=1,2,\dots). \end{cases}$$

we can express $u_{p,b,c}(z)$ as

$$u_{p,b,c}(z) = z + \sum_{n=1}^{\infty} \frac{(-c/4)^n z^{n+1}}{(m)_n n!}, \quad (1.5)$$

where $m = p + \frac{b+1}{2} \notin \mathbb{Z}_0^-$. This function is analytic on \mathbb{C} and satisfies the second-order linear differential equation

$$4z^2 u''(z) + 2(2p+b+1)zu'(z) + czu(z) = 0.$$

Now, we consider the linear operator

$$\mathfrak{B}_m^c f : \mathcal{A} \rightarrow \mathcal{A}$$

defined by

$$\mathfrak{B}_m^c f(z) := u_{p,b,c}(z) * f(z) = z + \sum_{n=1}^{\infty} \frac{(-c/4)^n}{(m)_n (n)!} a_{n+1} z^{n+1}, \quad z \in \mathbb{U}, \quad (1.6)$$

where $m = p + \frac{b+1}{2} \notin \mathbb{Z}_0^-$. It is easy to verify from the definition (1.6) that

$$z(\mathfrak{B}_{m+1}^c f(z))' = m\mathfrak{B}_m^c f(z) - (m-1)\mathfrak{B}_{m+1}^c f(z). \quad (1.7)$$

We recall the special cases of \mathfrak{B}_m^c -operator due to Baricz et al [3].

- Setting $b = c = 1$ in (1.6) or (1.7), we obtain the operator $\mathcal{J}_p : \mathcal{A} \rightarrow \mathcal{A}$ related with Bessel function, given by

$$\mathcal{J}_p f(z) = zu_{p,1,1}(z) * f(z) = z + \sum_{n=1}^{\infty} \frac{(-1/4)^n}{(p+1)_n (n)!} a_{n+1} z^{n+1}, \quad z \in \mathbb{U} \quad (1.8)$$

and its recursive relation

$$z(\mathcal{J}_{p+1} f(z))' = (p+1)\mathcal{J}_p f(z) - p\mathcal{J}_{p+1} f(z), \quad z \in \mathbb{U}.$$

- Setting $b = 1$ and $c = -1$ in (1.6) or (1.7), we obtain the operator $\mathcal{I}_p : \mathcal{A} \rightarrow \mathcal{A}$ related with modified Bessel function, given by

$$\mathcal{I}_p f(z) = zu_{p,1,-1}(z) * f(z) = z + \sum_{n=1}^{\infty} \frac{(1/4)^n}{(p+1)_n (n)!} a_{n+1} z^{n+1}, \quad z \in \mathbb{U} \quad (1.9)$$

and its recursive relation

$$z(\mathcal{I}_{p+1} f(z))' = (p+1)\mathcal{I}_p f(z) - p\mathcal{I}_{p+1} f(z), \quad z \in \mathbb{U}.$$

- Setting $b = 2$ and $c = 1$ in (1.6) or (1.7), we obtain the operator $\mathcal{K}_p : \mathcal{A} \rightarrow \mathcal{A}$ related with spherical Bessel function, given by

$$\mathcal{K}_p f(z) = zu_{p,2,1}(z) * f(z) = z + \sum_{n=1}^{\infty} \frac{(-1/4)^n}{(p+\frac{3}{2})_n (n)!} a_{n+1} z^{n+1}, \quad z \in \mathbb{U} \quad (1.10)$$

and its recursive relation

$$z(\mathcal{K}_{p+1} f(z))' = (p+\frac{3}{2})\mathcal{K}_p f(z) - (p+\frac{1}{2})\mathcal{K}_{p+1} f(z), \quad z \in \mathbb{U}.$$

It is of interest to note that the function \mathcal{B}_m^c given by (1.6) is an elementary transformation of the generalized hypergeometric function, i.e it is easy to see that $\mathcal{B}_m^c f(z) = z {}_0F_1(m; \frac{-c}{4}z) * f(z)$ and also $u_{p,b,c}(\frac{-4}{c}z) * f(z) = z {}_0F_1(m; z)$.

The generalized Bessel function is a recent topic of study in Geometric Function Theory (e.g. see the work of [1, 2, 3]). Using the \mathcal{B}_m^c -linear operator due to Baricz et al [3] given by (1.6), we now define the following new subclass of \mathcal{A} .

Definition 1 A function $f(z) \in \mathcal{A}$ is said to be in the class $\mathcal{S}_m^c(A, B; \gamma)$, if

$$1 + \frac{1}{\gamma} \left[\frac{z(\mathfrak{B}_{m+1}^c f(z))'}{\mathfrak{B}_{m+1}^c f(z)} - 1 \right] \prec \frac{1 + Az}{1 + Bz}, \quad (1.11)$$

where $-1 \leq B < A \leq 1$; $\gamma, c, m \in \mathbb{C}$, $\gamma \neq 0$, $m \neq 0, -1, -2, \dots$

In particular, the class

$$\mathcal{S}_m^c(\gamma) := \mathcal{S}_m^c(1, -1; \gamma),$$

denote the class of functions $f \in \mathcal{A}$ satisfying the following condition:

$$\Re \left(1 + \frac{1}{\gamma} \left[\frac{z(\mathfrak{B}_{m+1}^c f(z))'}{\mathfrak{B}_{m+1}^c f(z)} - 1 \right] \right) > 0, \quad z \in \mathbb{U}. \quad (1.12)$$

Moreover, let us denote

$$\mathcal{S}_m^c(\alpha, \lambda) := \mathcal{S}_m^c((1-\alpha)\cos\lambda e^{-i\lambda}), \quad \mathcal{S}_m^c(\alpha) := \mathcal{S}_m^c(\alpha, 0), \quad |\lambda| < \frac{\pi}{2}; \quad 0 \leq \alpha \leq 1.$$

Majorization problems for the class \mathcal{S}^* had been studied by MacGregor [8]. Recently Altintas et al. [4] investigated a majorization problem for the class $\mathcal{C}^*(\gamma)$ and Goyal and Goswami [5] generalized these results for the class of analytic functions involving fractional operator. In this paper we investigated a majorization problem for the class $\mathcal{S}_m^c(A, B; \gamma)$ associated with Bessel functions and point out some special cases of our result.

2 The main results

First we show that the class $\mathcal{S}_m^c(A, B; \gamma)$ is not empty.

Theorem 1 *A function $f \in \mathcal{A}$ of the form (1.1) belongs to the class $\mathcal{S}_m^c(A, B; \gamma)$ if it satisfies the condition*

$$\sum_{n=2}^{\infty} d_n |a_n| \leq (B - A) |\gamma|, \quad (2.1)$$

where

$$d_n = \frac{(|c|/4)^{n-1} \{(B+1)(n-1) + (B-A)|\gamma|\}}{|(m)_{n-1}| (n-1)!}, \quad n = 2, 3, \dots$$

Proof. A function f of the form (1.1) belongs to the class $\mathcal{S}_m^c(A, B; \gamma)$ if and only if there exists a function ω , $|\omega(z)| \leq |z|$ ($z \in \mathbb{U}$), such that for $z \in \mathbb{U}$ we have

$$1 + \frac{1}{\gamma} \left[\frac{z(\mathfrak{B}_{m+1}^c f(z))'}{\mathfrak{B}_{m+1}^c f(z)} - 1 \right] = \frac{1 + A\omega(z)}{1 + B\omega(z)}$$

or equivalently

$$z(\mathfrak{B}_{m+1}^c f(z))' - \mathfrak{B}_{m+1}^c f(z) = \omega(z) \{Bz(\mathfrak{B}_{m+1}^c f(z))' + [(B-A)\gamma - B] \mathfrak{B}_{m+1}^c f(z)\}.$$

Thus, it is sufficient to prove that for $z \in \mathbb{U}$ we have

$$\left| z(\mathfrak{B}_{m+1}^c f(z))' - \mathfrak{B}_{m+1}^c f(z) \right| - \left| Bz(\mathfrak{B}_{m+1}^c f(z))' + [(B-A)\gamma - B] \mathfrak{B}_{m+1}^c f(z) \right| \leq 0.$$

Indeed, letting $|z| = r$ ($0 \leq r < 1$) and $\alpha_n = \frac{(-c/4)^{n-1}}{(m)_{n-1} (n-1)!}$ we have

$$\begin{aligned} & \left| z(\mathfrak{B}_{m+1}^c f(z))' - \mathfrak{B}_{m+1}^c f(z) \right| - \left| Bz(\mathfrak{B}_{m+1}^c f(z))' + [(B-A)\gamma - B] \mathfrak{B}_{m+1}^c f(z) \right| \\ &= \left| \sum_{n=2}^{\infty} (n-1) \alpha_n a_n z^n \right| - \left| (B-A)\gamma z - \sum_{n=2}^{\infty} (Bn + (B-A)\gamma - B) \alpha_n a_n z^n \right| \\ &\leq \sum_{n=2}^{\infty} (n-1) |\alpha_n| |a_n| r^{n-1} - (B-A)|\gamma| + \sum_{n=2}^{\infty} (Bn + (B-A)|\gamma| - B) |\alpha_n| |a_n| r^{n-1} \\ &\leq \sum_{n=2}^{\infty} d_n |a_n| r^{n-1} - (B-A)|\gamma| \leq 0, \end{aligned}$$

whence $f \in \mathcal{S}_m^c(A, B; \gamma)$. ■

Remark 1 *By Theorem 1 we see that a function f of the form (1.1) belongs to the class $\mathcal{S}_m^c(A, B; \gamma)$ if it has "sufficiently small" coefficients. In particular, the functions*

$$f(z) = z + az^n, \quad z \in \mathbb{U},$$

where

$$|a| \leq \frac{(|c|/4)^n \{(B+1)(n-1) + (B-A)|\gamma|\}}{|(m)_n| (n)! (B-A)|\gamma|}$$

belong to the class $\mathcal{S}_m^c(A, B; \gamma)$. The convex combinations of these functions belong to the class $\mathcal{S}_m^c(A, B; \gamma)$ too.

Theorem 2 Let $f \in \mathcal{A}$ and suppose that $g \in \mathcal{S}_m^c(A, B; \gamma)$ with $|m| \geq |\gamma(A-B) + mB|$. If $\mathfrak{B}_{m+1}^c f(z)$ is majorized by $\mathfrak{B}_{m+1}^c g(z)$, then

$$|\mathfrak{B}_m^c f(z)| \leq |\mathfrak{B}_m^c g(z)|, \quad |z| \leq r_1, \quad (2.2)$$

where r_1 is the smallest positive root of the equation

$$|\gamma(A-B) + mB|r^3 - (|m| + 2|B|)r^2 - (|\gamma(A-B) + mB| + 2)r + |m| = 0. \quad (2.3)$$

Proof. Since $g \in \mathcal{S}_m^c(A, B; \gamma)$, we find from (1.11) that

$$1 + \frac{1}{\gamma} \left(\frac{z(\mathfrak{B}_{m+1}^c g(z))'}{\mathfrak{B}_{m+1}^c g(z)} - 1 \right) = \frac{1 + Aw(z)}{1 + Bw(z)}, \quad (2.4)$$

where w is analytic in \mathbb{U} , with $w(0)$ and $|w(z)| \leq |z|$ for all $z \in \mathbb{U}$.

From (2.4), we get

$$\frac{z(\mathfrak{B}_{m+1}^c g(z))'}{\mathfrak{B}_{m+1}^c g(z)} = \frac{1 + [\gamma(A-B) + B]w(z)}{1 + Bw(z)}. \quad (2.5)$$

Now, by applying the relation (1.7) in (2.5), we get

$$\frac{m\mathfrak{B}_m^c g(z)}{\mathfrak{B}_{m+1}^c g(z)} = \frac{m + [\gamma(A-B) + mB]w(z)}{1 + Bw(z)} \quad (2.6)$$

which yields that,

$$|\mathfrak{B}_{m+1}^c g(z)| \leq \frac{|m| [1 + |B||z|]}{|m| - |\gamma(A-B) + mB||z|} |\mathfrak{B}_m^c g(z)|. \quad (2.7)$$

Since $\mathfrak{B}_{m+1}^c f(z)$ is majorized by $\mathfrak{B}_{m+1}^c g(z)$, then there exist a function ϕ analytic in \mathbb{U} , with $\phi(0)$ and $|\phi(z)| \leq |z|$ for all $z \in \mathbb{U}$, such that

$$\mathfrak{B}_{m+1}^c f(z) = \phi(z)\mathfrak{B}_{m+1}^c g(z).$$

By differentiating with respect to z we get

$$z(\mathfrak{B}_{m+1}^c f(z))' = z\phi'(z)\mathfrak{B}_{m+1}^c g(z) + z\phi(z)(\mathfrak{B}_{m+1}^c g(z))'. \quad (2.8)$$

Noting that the Schwarz function ϕ satisfies (cf. [10])

$$|\phi'(z)| \leq \frac{1 - |\phi(z)|^2}{1 - |z|^2} \quad (2.9)$$

and using (1.7), (2.7) and (2.9) in (2.8), we have

$$|\mathfrak{B}_m^c f(z)| \leq \left(|\phi(z)| + \frac{1 - |\phi(z)|^2}{1 - |z|^2} \frac{(1 + |B||z|)|z|}{|m| - |\gamma(A-B) + mB||z|} \right) |\mathfrak{B}_m^c g(z)|,$$

Setting $|z| = r$ and $|\phi(z)| = \rho$, $0 \leq \rho \leq 1$ the above inequality leads us to the inequality

$$|\mathfrak{B}_m^c f(z)| \leq F(\rho, r) |\mathfrak{B}_m^c g(z)|, \quad (2.10)$$

where

$$F(\rho, r) = \frac{\Phi(\rho)}{(1-r^2)(|m| - |\gamma(A-B) + mB|r)},$$

with

$$\Phi(\rho) = -\rho^2(1 + |B|r)r + \rho(1 - r^2)(|m| - |mB + \gamma(A-B)|r) + r(1 + |B|r).$$

It is clear that if

$$\frac{(1-r^2)(|m| - |mB + \gamma(A-B)|r)}{2(1 + |B|r)r} \geq 1,$$

then the function Φ takes its maximum value in the interval $\langle 0, 1 \rangle$ at $\rho = 1$. Since the above inequality holds for $0 \leq r \leq r_1 = r_1(\gamma, A, B)$, where r_1 is the smallest positive root of the equation (2.3), then there is $0 < F(\rho, r) \leq F(\rho, 1) = 1$ for $r \in \langle 0, r_1 \rangle$ and $\rho \in \langle 0, 1 \rangle$. This gives (2.2) and completes the proof. ■

Putting $A = 1$, $B = -1$ in Theorem 2, we have the following corollary:

Corollary 1 *Let $f \in \mathcal{A}$ and suppose that $g \in \mathcal{S}_m^c(\gamma)$ with $|m| \geq |2\gamma - m|$. If $\mathfrak{B}_{m+1}^c f(z)$ is majorized by $\mathfrak{B}_{m+1}^c g(z)$, then*

$$|\mathfrak{B}_m^c f(z)| \leq |\mathfrak{B}_m^c g(z)|, \quad |z| \leq r_2, \quad (2.11)$$

where r_2 is the smallest positive root of the equation

$$|2\gamma - m|r^3 - (|m| + 2)r^2 - (|2\gamma - m| + 2)r + |m| = 0, \quad (2.12)$$

given by

$$r_2 = \frac{\kappa - \sqrt{\kappa^2 - 4|m||2\gamma - m|}}{2|2\gamma - m|}, \quad \kappa = (|m| + 2) + |2\gamma - m|.$$

Putting $\gamma = (1 - \alpha)\cos\lambda e^{-i\lambda}$, $|\lambda| < \frac{\pi}{2}$; $0 \leq \alpha \leq 1$, in corollary 1, we have the following corollary.

Corollary 2 *Let $f \in \mathcal{A}$ and suppose that $g \in \mathcal{S}_m^c(\alpha, \lambda)$ with $|m| \geq |2(1 - \alpha)\cos\lambda e^{-i\lambda} - m|$. If $\mathfrak{B}_{m+1}^c f(z)$ is majorized by $\mathfrak{B}_{m+1}^c g(z)$, then*

$$|\mathfrak{B}_{m+1}^c f(z)| \leq |\mathfrak{B}_{m+1}^c g(z)|, \quad |z| \leq r_3, \quad (2.13)$$

where r_3 is the smallest positive root of the equation

$$|2(1 - \alpha)\cos\lambda e^{-i\lambda} - m|r^3 - (|m| + 2)r^2 - (|2(1 - \alpha)\cos\lambda e^{-i\lambda} - m| + 2)r + |m| = 0, \quad (2.14)$$

given by

$$r_3 = \frac{\delta - \sqrt{\delta^2 - 4|m||2(1 - \alpha)\cos\lambda e^{-i\lambda} - m|}}{2|2(1 - \alpha)\cos\lambda e^{-i\lambda} - m|} \quad (2.15)$$

and

$$\delta = (|m| + 2) + |2(1 - \alpha)\cos\lambda e^{-i\lambda} - m|.$$

Further, by taking $\lambda = 0$ we obtain the next corollary.

Corollary 3 *Let $f \in \mathcal{A}$ and suppose that $g \in \mathcal{S}_m^c(\alpha)$ with $\operatorname{Re} m \geq 1 - \alpha$. If $\mathfrak{B}_{m+1}^c f(z)$ is majorized by $\mathfrak{B}_{m+1}^c g(z)$, then*

$$|\mathfrak{B}_m^c f(z)| \leq |\mathfrak{B}_m^c g(z)|, \quad |z| \leq r_4, \quad (2.16)$$

where

$$r_4 = \frac{\delta - \sqrt{\delta^2 - 4|m| |2(1 - \alpha) - m|}}{2|2(1 - \alpha) - m|}$$

and

$$\delta = (|m| + 2) + |2(1 - \alpha) - m|.$$

For $\alpha = 0$ and $m = 1$ Corollary 3 reduces to the following result.

Corollary 4 [8] *Let $f \in \mathcal{A}$ and suppose that $g \in \mathcal{S}_1^c(0)$. If $\mathfrak{B}_2^c f(z)$ is majorized by $\mathfrak{B}_2^c g(z)$, then*

$$|\mathfrak{B}_1^c f(z)| \leq |\mathfrak{B}_1^c g(z)|, \quad |z| \leq r_5, \quad (2.17)$$

where $r_5 := 2 - \sqrt{3}$.

Concluding Remarks: Further specializing the parameters b, c one can define the various other interesting subclasses of $\mathcal{S}_m^c(A, B; \gamma)$, involving the types of Bessel functions as stated in equations (1.8) to (1.10), and one can easily derive the result as in Theorem 2 and the corresponding corollaries as mentioned above. The details involved may be left as an exercise for the interested reader.

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On a class of meromorphic functions defined by the convolution

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ABSTRACT: In the present paper we define some classes of meromorphic functions with fixed argument of coefficients. Next we obtain coefficient estimates, distortion theorems, integral means inequalities, the radii of convexity and starlikeness and convolution properties for the defined class of functions.

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Keywords and Phrases: *meromorphic functions, fixed argument, subordination, convolution*

Dedicated to Professor Leon Mikołajczyk

1 Introduction

Let $\widetilde{\mathcal{M}}$ denote the class of functions which are *analytic* in $\mathcal{D} = \mathcal{D}(1)$, where

$$\mathcal{D}(r) = \{z \in \mathbb{C} : 0 < |z| < r\} \quad (r \in (0, 1])$$

and let \mathcal{M}^k ($k \in \mathbb{N}_0 := \{0, 1, 2, \dots\}$) denote the class of functions $f \in \widetilde{\mathcal{M}}$ of the form

$$f(z) = \frac{1}{z} + \sum_{n=k}^{\infty} a_n z^n \quad (z \in \mathcal{D}). \quad (1)$$

Moreover, let $\mathcal{M} := \mathcal{M}^0$. Also, by \mathcal{T}_θ ($\theta \in \mathbb{R}$) we denote the class of functions $f \in \mathcal{M}$ of the form

$$f(z) = \frac{1}{z} + e^{i\theta} \sum_{n=0}^{\infty} |a_n| z^n \quad (z \in \mathcal{D}). \quad (2)$$

The class \mathcal{T}_θ is called the class of meromorphic functions with fixed argument of coefficients. For $\theta = \pi$ we obtain the class \mathcal{T}_π of meromorphic functions with negative

coefficients. Classes of functions with fixed argument of coefficients were considered in [1, 2, 3, 4].

A function $f \in \mathcal{M}$ is said to be *convex* in $\mathcal{D}(r)$ if

$$\Re \left(1 + \frac{zf''(z)}{f'(z)} \right) < 0 \quad (z \in \mathcal{D}(r)).$$

A function $f \in \mathcal{M}$ is said to be *starlike* in $\mathcal{D}(r)$ if

$$\Re \frac{zf'(z)}{f(z)} < 0 \quad (z \in \mathcal{D}(r)). \quad (3)$$

Let \mathcal{B} be a subclass of the class \mathcal{M} . We define *the radius of starlikeness of order α* and *the radius of convexity of order α* for the class \mathcal{B} by

$$R_\alpha^*(\mathcal{B}) = \inf_{f \in \mathcal{B}} \{ \sup \{ r \in (0, 1] : f \text{ is starlike in } \mathcal{D}(r) \} \},$$

$$R_\alpha^c(\mathcal{B}) = \inf_{f \in \mathcal{B}} \{ \sup \{ r \in (0, 1] : f \text{ is convex in } \mathcal{D}(r) \} \},$$

respectively.

Let functions f, F be analytic in $\mathcal{U} := \mathcal{D} \cup \{0\}$. We say that f is *subordinate* to F , and write $f(z) \prec F(z)$ (or simply $f \prec F$), if and only if there exists a function ω analytic in \mathcal{U} , $|\omega(z)| \leq |z|$ ($z \in \mathcal{U}$), such that

$$f(z) = F(\omega(z)) \quad (z \in \mathcal{U}).$$

In particular, if F is univalent in \mathcal{U} , we have the following equivalence:

$$f(z) \prec F(z) \iff f(0) = F(0) \text{ and } f(\mathcal{U}) \subset F(\mathcal{U}).$$

For functions $f, g \in \widetilde{\mathcal{M}}$ of the form

$$f(z) = \sum_{n=-1}^{\infty} a_n z^n \quad \text{and} \quad g(z) = \sum_{n=-1}^{\infty} b_n z^n,$$

by $f * g$ we denote *the Hadamard product* (or *convolution*) of f and g , defined by

$$(f * g)(z) = \sum_{n=-1}^{\infty} a_n b_n z^n \quad (z \in \mathcal{D}).$$

Let $\varphi \in \mathcal{M}^k$ be a given function of the form

$$\varphi(z) = \frac{1}{z} + \sum_{n=k}^{\infty} \alpha_n z^n \quad (z \in \mathcal{D}; \alpha_n > 0, n = k, k+1, \dots). \quad (4)$$

Assume that A, B are real parameters, $-1 \leq A < B \leq 1$, ($\cos \theta < 0$ or $B \neq 1$). By $\mathcal{M}^k(\varphi; A, B)$ we denote the class of functions $f \in \mathcal{M}^k$ such that

$$z(\varphi * f)(z) \prec \frac{1 + Az}{1 + Bz}. \quad (5)$$

Now, we define the classes of functions with fixed argument of coefficients related to the class $\mathcal{M}^k(\varphi; A, B)$. Let us denote

$$\mathcal{M}_\theta^k(\varphi; A, B) := \mathcal{T}_\theta \cap \mathcal{M}^k(\varphi; A, B), \quad \mathcal{M}(\varphi; A, B) := \mathcal{M}^0(\varphi; A, B).$$

In the present paper we obtain coefficient estimates, distortion theorems, integral means inequalities, and the radii of convexity and starlikeness for the class $\mathcal{M}_\theta^k(\varphi; A, B)$. We also derive convolution properties for the class of functions.

2 Coefficient estimates

Before stating and proving coefficient estimates in the class $\mathcal{M}(\varphi; A, B)$ we need the following lemma.

Lemma 1 [6] *Let f be a function of the form*

$$f(z) = \sum_{n=0}^{\infty} a_n z^n,$$

which is analytic in \mathcal{D} . If $f \prec g$ and g is convex univalent in \mathcal{U} , then

$$|a_n| \leq 1 \quad (n \in \mathbb{N}).$$

Theorem 1 *If a function f of the form (1) belongs to the class $\mathcal{M}(\varphi; A, B)$, then*

$$|a_n| \leq \frac{B-A}{\alpha_n} \quad (n = 0, 1, \dots), \quad (6)$$

The result is sharp.

Proof. Let a function f of the form (1) belong to the class $\mathcal{M}(\varphi; A, B)$ and let us put

$$g(z) = \frac{z(\varphi * f)(z) - 1}{A - B} \quad \text{and} \quad h(z) = \frac{z}{1 + Bz}.$$

Then, by (5), we have $g \prec h$. Since the function g is given by

$$g(z) = \sum_{n=0}^{\infty} \frac{\alpha_n}{A - B} a_n z^{n+1}$$

and the function h is convex univalent in \mathcal{U} , by Lemma 1 we obtain

$$\frac{\alpha_n}{B - A} |a_n| \leq 1 \quad (n \in \mathbb{N}_0). \quad (7)$$

Thus we have (6). The Equality in (7) holds for the functions g_n of the form

$$g_n(z) = h(z^{n+1}) = z^{n+1} + \sum_{j=n+2}^{\infty} b_j z^j \quad (n = 0, 1, \dots),$$

for some b_j ($j = n + 2, n + 3, \dots$). Consequently, the equality in (6) holds true for the functions f_n of the form

$$f_n(z) = \frac{1}{z} + \frac{A-B}{\alpha_n} z^n + \sum_{j=n+1}^{\infty} \frac{A-B}{\alpha_j} b_{j+1} z^j \quad (n = 0, 1, \dots).$$

■

Theorem 2 *If a function f of the form (2) belongs to the class $\mathcal{M}_\theta^k(\varphi; A, B)$, then*

$$\sum_{n=k}^{\infty} \alpha_n |a_n| \leq \delta(\theta; A, B), \quad (8)$$

where

$$\delta(\theta; A, B) := \frac{B-A}{\sqrt{1-B^2 \sin^2 \theta} - B \cos \theta}. \quad (9)$$

Proof. Let a function f belong to the class $\mathcal{M}_\theta^k(\varphi; A, B)$. Then, by (5) and the definition of subordination, we have

$$z(\varphi * f)(z) = \frac{1 + A\omega(z)}{1 + B\omega(z)},$$

where $\omega(0) = 0$ and $|\omega(z)| < 1$ for $z \in \mathcal{U}$. Thus we obtain

$$|z(\varphi * f)(z) - 1| < |Bz(\varphi * f)(z) - A| \quad (z \in \mathcal{D}).$$

Hence, by (2), we have

$$\left| \sum_{n=k}^{\infty} \alpha_n |a_n| z^{n+1} \right| < \left| B - A + B e^{i\theta} \sum_{n=k}^{\infty} \alpha_n |a_n| z^{n+1} \right| \quad (z \in \mathcal{D}). \quad (10)$$

Putting $z = r$ ($0 \leq r < 1$), we find that

$$|w| < |B - A + B w e^{i\theta}|, \quad (11)$$

where, for convenience,

$$w = \sum_{n=k}^{\infty} \alpha_n |a_n| r^{n+1}.$$

Since w is a real number, by (11) we have

$$(1 - B^2)w^2 - [2B(B - A) \cos \theta] w - (B - A)^2 < 0.$$

Solving this inequality with respect to w , we obtain

$$\sum_{n=k}^{\infty} \alpha_n |a_n| r^{n+1} < \delta(\theta; A, B),$$

which, upon letting $r \rightarrow 1^-$, readily yields the assertion (8) of Theorem 1. ■

Theorem 3 A function f of the form (2) belongs to the class $\mathcal{M}_\pi^k(\varphi; A, B)$ if and only if

$$\sum_{n=k}^{\infty} \alpha_n |a_n| \leq \frac{B-A}{1+B}. \quad (12)$$

Proof. By virtue of Theorem 1, we only need to show that the condition (12) is the sufficient condition. Let a function f of the form (2) satisfy the condition (12). Then, in view of (10), it is sufficient to prove that

$$\left| \sum_{n=k}^{\infty} \alpha_n |a_n| z^{n+1} \right| - \left| B - A - B \sum_{n=k}^{\infty} \alpha_n |a_n| z^{n+1} \right| < 0 \quad (z \in \mathcal{D}).$$

Indeed, letting $|z| = r$ ($0 < r < 1$), we have

$$\begin{aligned} & \left| \sum_{n=k}^{\infty} \alpha_n |a_n| z^{n+1} \right| - \left| B - A - B \sum_{n=k}^{\infty} \alpha_n |a_n| z^{n+1} \right| \\ & \leq \left(\sum_{n=k}^{\infty} \alpha_n |a_n| r^{n+1} \right) - \left(B - A - B \sum_{n=k}^{\infty} \alpha_n |a_n| r^{n+1} \right) \\ & < (1+B) \sum_{n=k}^{\infty} \alpha_n |a_n| - (B-A) \leq 0, \end{aligned}$$

which implies that $f \in \mathcal{M}_\pi^k(\varphi; A, B)$. ■

Theorem 2 readily yields

Corollary 1 If a function f of the form (2) belongs to the class $\mathcal{M}_\theta^k(\varphi; A, B)$, then

$$|a_n| \leq \frac{\delta(\theta; A, B)}{\alpha_n} \quad (n = k, k+1, \dots), \quad (13)$$

where $\delta(\theta; A, B)$ is defined by (9). The result is sharp for $\theta = \pi$. Then the functions f_n of the form

$$f_n(z) = \frac{1}{z} - \frac{B-A}{(1+B)\alpha_n} z^n \quad (z \in \mathcal{D}; n = k, k+1, \dots) \quad (14)$$

are the extremal functions.

3 Distortion theorems

From Theorem 2 we have the following lemma.

Lemma 2 Let a function f of the form (2) belong to the class $\mathcal{M}_\theta^k(\varphi; A, B)$. If the sequence $\{\alpha_n\}$ defined by (4) satisfies the inequality

$$\alpha_k \leq \alpha_n \quad (n = k, k+1, \dots), \quad (15)$$

then

$$\sum_{n=k}^{\infty} |a_n| \leq \frac{\delta(\theta; A, B)}{\alpha_k}.$$

Moreover, if

$$n\alpha_k \leq \alpha_n \quad (k \geq 1, n = k, k+1, \dots), \quad (16)$$

then

$$\sum_{n=k}^{\infty} n |a_n| \leq \frac{k\delta(\theta; A, B)}{\alpha_k}.$$

Theorem 4 Let a function f belong to the class $\mathcal{M}_\theta^k(\varphi; A, B)$. If the sequence $\{\alpha_n\}$ defined by (4) satisfies (15), then

$$\frac{1}{r} - \frac{\delta(\theta; A, B)}{\alpha_k} r^k \leq |f(z)| \leq \frac{1}{r} + \frac{\delta(\theta; A, B)}{\alpha_k} r^k \quad (|z| = r < 1). \quad (17)$$

Moreover, if (16) holds, then

$$\frac{1}{r^2} - \frac{k\delta(\theta; A, B)}{\alpha_k} r^{k-1} \leq |f'(z)| \leq \frac{1}{r^2} + \frac{k\delta(\theta; A, B)}{\alpha_k} r^{k-1} \quad (|z| = r < 1). \quad (18)$$

The result is sharp for $\theta = \pi$, with the extremal function f_k of the form (14).

Proof. Let a function f of the form (2) belong to the class $\mathcal{M}_\theta^k(\varphi; A, B)$, $|z| = r < 1$. Since

$$|f(z)| = \left| \frac{1}{z} + e^{i\theta} \sum_{n=k}^{\infty} a_n z^n \right| \leq \frac{1}{r} + \sum_{n=k}^{\infty} |a_n| r^n \leq \frac{1}{r} + \sum_{n=k}^{\infty} |a_n|$$

and

$$|f(z)| = \left| \frac{1}{z} + e^{i\theta} \sum_{n=k}^{\infty} a_n z^n \right| \geq \frac{1}{r} - \sum_{n=k}^{\infty} |a_n| r^n \geq \frac{1}{r} - \sum_{n=k}^{\infty} |a_n|,$$

then by Lemma 2 we have (17). Analogously we prove (18). ■

4 Integral means inequalities

Due to Littlewood [7] we obtain integral means inequalities for the functions from the class $\mathcal{M}_\theta^k(\varphi; A, B)$.

Lemma 3 [7]. Let function f, g be analytic in \mathcal{U} . If $f \prec g$, then

$$\int_0^{2\pi} |f(re^{it})|^\lambda dt \leq \int_0^{2\pi} |g(re^{it})|^\lambda dt \quad (0 < r < 1, \lambda > 0). \quad (19)$$

Silverman [8] found that the function

$$g(z) = z - \frac{z^2}{2} \quad (z \in \mathcal{D}),$$

is often extremal over the family of functions with negative coefficients. He applied this function to resolve integral means inequality, conjectured in [9] and settled in [10], that (19) holds true for all functions f with negative coefficients. In [10] he also proved his conjecture for some subclasses of \mathcal{T}_π .

Applying Lemma 3 and Theorem 2 we prove the following result.

Theorem 5 *Let the sequence $\{\alpha_n\}$ defined by (4) satisfy the inequality (15). If $f \in \mathcal{M}_\theta^0(\varphi; A, B)$, then*

$$\int_0^{2\pi} |f(re^{it})|^\lambda dt \leq \int_0^{2\pi} |g(re^{it})|^\lambda dt \quad (0 < r < 1, \lambda > 0), \quad (20)$$

where

$$g(z) = \frac{1}{z} + e^{i\theta} \frac{\delta(\theta; A, B)}{\alpha_0} \quad (z \in \mathcal{D}).$$

Proof. For function f of the form (2), the inequality (20) is equivalent to the following:

$$\int_0^{2\pi} \left| 1 + e^{i\theta} \sum_{n=0}^{\infty} |a_n| z^{n+1} \right|^\lambda dt \leq \int_0^{2\pi} \left| 1 + e^{i\theta} \frac{\delta(\theta; A, B)}{\alpha_0} z \right|^\lambda dt.$$

By Lemma 3, it suffices to show that

$$\sum_{n=0}^{\infty} |a_n| z^{n+1} \prec \frac{\delta(\theta; A, B)}{\alpha_0} z. \quad (21)$$

Setting

$$w(z) = \sum_{n=0}^{\infty} \frac{\alpha_0}{\delta(\theta; A, B)} a_n z^{n+1} \quad (z \in \mathcal{D})$$

and using (15) and Theorem 2 we obtain

$$|w(z)| = \left| \sum_{n=0}^{\infty} \frac{\alpha_0}{\delta(\theta; A, B)} a_n z^{n+1} \right| \leq |z| \sum_{n=0}^{\infty} \frac{\alpha_n}{\delta(\theta; A, B)} |a_n| \leq |z| \quad (z \in \mathcal{D}).$$

Since

$$\sum_{n=0}^{\infty} a_n z^{n+1} = \frac{\delta(\theta; A, B)}{\alpha_0} w(z) \quad (z \in \mathcal{D}),$$

by definition of subordination we have (21) and this completes the proof. ■

5 The radii of convexity and starlikeness

Theorem 6 *If a function f belongs to the class $\mathcal{M}_\theta^k(\varphi; A, B)$, $k \geq 1$, then f is starlike in the disk $\mathcal{D}(r^*)$, where*

$$r^* := \inf_{n \geq k} \left(\frac{\alpha_n}{n\delta(\theta, A, B)} \right)^{\frac{1}{n+1}} \quad (22)$$

and $\delta(\theta, A, B)$, $\{\alpha_n\}$ are defined by (9) and (4), respectively. For $\theta = \pi$, the result is sharp, that is

$$R^*(\mathcal{M}_\pi^k(\varphi; A, B)) = r^*.$$

Proof. A function $f \in \mathcal{M}^k$ of the form (2) is starlike in the disk $\mathcal{D}(r)$ if and only if it satisfies the condition (3) or if

$$\left| \frac{zf'(z) + f(z)}{zf'(z) - f(z)} \right| < 1 \quad (z \in \mathcal{D}(r)). \quad (23)$$

Since

$$\left| \frac{zf'(z) + f(z)}{zf'(z) - f(z)} \right| = \left| \frac{e^{i\theta} \sum_{n=k}^{\infty} (n+1) |a_n| z^n}{\frac{2}{z} - e^{i\theta} \sum_{n=k}^{\infty} (n-1) |a_n| z^n} \right| \leq \frac{\sum_{n=k}^{\infty} (n+1) |a_n| |z|^{n+1}}{2 - \sum_{n=k}^{\infty} (n-1) |a_n| |z|^{n+1}},$$

putting $|z| = r$ the condition (23) be true if

$$\sum_{n=k}^{\infty} n |a_n| r^{n+1} \leq 1. \quad (24)$$

By Theorem 2, we have

$$\sum_{n=k}^{\infty} \frac{\alpha_n}{\delta(\theta, A, B)} |a_n| \leq 1,$$

Thus, the condition (24) be true if

$$nr^{n+1} \leq \frac{\alpha_n}{\delta(\theta, A, B)} \quad (n = k, k+1, \dots),$$

that is, if

$$r \leq \left(\frac{\alpha_n}{n\delta(\theta, A, B)} \right)^{\frac{1}{n+1}} \quad (n = k, k+1, \dots).$$

It follows that each function $f \in \mathcal{M}_\theta^k(\varphi; A, B)$ is starlike in the disk $\mathcal{D}(r^*)$, where r^* is defined by (22). For $\theta = \pi$ the functions f_n of the form (14) are extremal functions. ■

Theorem 7 *If a function f belongs to the class $\mathcal{M}_\theta^k(\varphi; A, B)$, then f is convex in the disk $\mathcal{D}(r^c)$, where*

$$r^c := \inf_{n \geq k} \left(\frac{\alpha_n}{n^2 \delta(\theta, A, B)} \right)^{\frac{1}{n+1}}$$

and $\delta(\theta, A, B)$, $\{\alpha_n\}$ are defined by (9) and (4), respectively. For $\theta = \pi$, the result is sharp, that is,

$$R^c(\mathcal{M}_\pi^k(\varphi; A, B)) = r^c.$$

Proof. The proof is analogous to that of Theorem 4, and we omit the details. ■

6 Cnonvolution properties

Let

$$f(z) = \frac{1}{z} + e^{i\alpha} \sum_{n=k}^{\infty} |a_n| z^n, \quad g(z) = \frac{1}{z} + e^{i\beta} \sum_{n=k}^{\infty} |b_n| z^n \quad (z \in \mathcal{D}). \quad (25)$$

We define modified Hadamard product for the functions f, g as follows:

$$f \circledast g(z) = \frac{1}{z} - \sum_{n=k}^{\infty} |a_n| |b_n| z^n \quad (z \in \mathcal{D}).$$

Theorem 8 *Let $f \in \mathcal{M}_\alpha^k(\varphi; A, B)$ and $g \in \mathcal{M}_\beta^k(\psi; C, D)$. Then $f \circledast g \in \mathcal{M}_\pi^k(\varphi * \psi; E, F)$, whenever*

$$\delta(\pi, E, F) \geq \delta(\alpha, A, B) \delta(\beta, C, D). \quad (26)$$

Proof. Let

$$\psi(z) = \frac{1}{z} + \sum_{n=k}^{\infty} \beta_n z^n \quad (z \in \mathcal{D}; \beta_n > 0, n = k, k+1, \dots)$$

and let functions f, g of the form (25) belong to the classes $\mathcal{M}_\alpha^k(\varphi; A, B)$ and $\mathcal{M}_\beta^k(\psi; C, D)$, respectively. From Theorem 2 we have

$$\sum_{n=k}^{\infty} \frac{\alpha_n}{\delta(\alpha; A, B)} |a_n| \leq 1, \quad \sum_{n=k}^{\infty} \frac{\beta_n}{\delta(\beta; C, D)} |b_n| \leq 1.$$

Thus, by (26) we obtain

$$\begin{aligned} \sum_{n=k}^{\infty} \frac{\alpha_n \beta_n}{\delta(\pi, E, F)} |a_n b_n| &\leq \sum_{n=k}^{\infty} \frac{\alpha_n \beta_n}{\delta(\alpha; A, B) \delta(\beta; C, D)} |a_n| |b_n| \\ &\leq \sum_{n=k}^{\infty} \frac{\alpha_n}{\delta(\alpha; A, B)} |a_n| \sum_{n=k}^{\infty} \frac{\beta_n}{\delta(\beta; C, D)} |b_n| \leq 1. \end{aligned}$$

Applying Theorem 3 we get $f \circledast g \in \mathcal{M}_\pi^k(\varphi * \psi; E, F)$. ■

Theorem 9 Let the sequence $\{\alpha_n\}$ defined by (4) satisfy the inequalities (15). If $f, g \in \mathcal{M}_\theta^k(\varphi; A, B)$, then $f \otimes g \in \mathcal{M}_\pi^k(\varphi; C, D)$, whenever

$$(D - C)\alpha_0 \geq (1 + D)[\delta(\theta, A, B)]^2. \quad (27)$$

Proof. Let a functions f, g of the form (25) belong to the class $\mathcal{M}_\alpha^k(\varphi; A, B)$. Then by Theorem 2 we have

$$\sum_{n=k}^{\infty} \frac{\alpha_n}{\delta(\alpha; A, B)} |a_n| \leq 1, \quad \sum_{n=k}^{\infty} \frac{\alpha_n}{\delta(\alpha; A, B)} |b_n| \leq 1.$$

Thus, by the Cauchy-Schwarz inequality we obtain

$$\sum_{n=k}^{\infty} \frac{\alpha_n}{\delta(\theta, A, B)} \sqrt{|a_n b_n|} \leq 1. \quad (28)$$

We have to prove that

$$\sum_{k=2}^{\infty} \alpha_n \frac{1+D}{D-C} |a_n b_n| \leq 1.$$

Therefore, by (28) it is sufficient to show that

$$\frac{1+D}{D-C} |a_n b_n| \leq \frac{1}{\delta(\theta, A, B)} \sqrt{|a_n b_n|} \quad (n \geq 2)$$

or equivalently

$$\sqrt{|a_n b_n|} \leq \frac{D-C}{(1+D)\delta(\theta, A, B)} \quad (n \geq 2).$$

From (28) we have

$$\sqrt{|a_n b_n|} \leq \frac{\delta(\theta, A, B)}{\alpha_n} \quad (n \geq 2).$$

Consequently, we need only to prove that

$$\frac{D-C}{(1+D)\delta(\theta, A, B)} \geq \frac{\delta(\theta, A, B)}{\alpha_n} \quad (n \geq 2),$$

and this inequality follows from (27) and (15). ■

We note that for functions $f \in \mathcal{M}_\alpha^k(\varphi; A, B)$ and $g \in \mathcal{M}_{\pi-\alpha}^k(\psi; C, D)$ we have $f * g = f \otimes g$. Thus from Theorem 8 obtain following corollary.

Corollary 2 If $f \in \mathcal{M}_\alpha^k(\varphi; A, B)$ and $g \in \mathcal{M}_{\pi-\alpha}^k(\psi; C, D)$, then $f * g \in \mathcal{M}_\pi^k(\varphi * \psi; E, F)$, whenever

$$\delta(\pi, E, F) \geq \delta(\alpha, A, B)\delta(\pi - \alpha, C, D).$$

Putting $\theta = \pi$ in Theorem 9 we obtain following corollary.

Corollary 3 *Let the sequence $\{\alpha_n\}$ defined by (4) satisfy (15). If $f, g \in \mathcal{M}_\pi^k(\varphi; A, B)$, then $f \otimes g \in \mathcal{M}_\pi^k(\varphi; C, D)$, whenever*

$$(D - C)(1 + B)^2 \alpha_0 \geq (1 + D)(B - A)^2.$$

Putting $C = A$ and $D = B$ in Corollary 3 we obtain following corollary.

Corollary 4 *Let the sequence $\{\alpha_n\}$ defined by (4) satisfy (15). If $f, g \in \mathcal{M}_\pi^k(\varphi; A, B)$, then $f \otimes g \in \mathcal{M}_\pi^k(\varphi; A, B)$, whenever*

$$\alpha_0 \geq \frac{B - A}{1 + B}.$$

Since for $\alpha = \beta = \pi$, $E = A$ and $F = B$ the condition (26) is true, then from Theorem 8 we have following corollary.

Corollary 5 *If $f \in \mathcal{M}_\pi^k(\varphi; A, B)$ and $g \in \mathcal{M}_\pi^k(\psi; C, D)$, then*

$$f \otimes g \in \mathcal{M}_\pi^k(\varphi * \psi; A, B) \cap \mathcal{M}_\pi^k(\varphi * \psi; C, D).$$

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Problem with integral condition for evolution equation

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ABSTRACT: We propose a method of solving the problem with non-homogeneous integral condition for homogeneous evolution equation with abstract operator in a linear space H . For right-hand side of the integral condition which belongs to the special subspace $L \subseteq H$, in which the vectors are represented using Stieltjes integrals over a certain measure, the solution of the problem is represented in the form of Stieltjes integral over the same measure.

AMS Subject Classification: *35M10, 35M20*

Keywords and Phrases: *differential-symbol method, evolution equation, problem with integral condition*

1. Statement of the problem.

The significant place in the research on problems for evolution equations in Banach spaces is taken by the semigroup theory (see, e.g., [2, 1, 3, 4]).

In the recent years, problems with integral conditions have been intensively studied while investigating the process of diffusion of particles in a turbulent medium, processes of heat conduction, moisture transfer in capillary-porous media, problems of describing the dynamics of population abundance as well as problems of demography (see, e. g., works [8, 9, 5, 6, 7, 10]).

Let A be a given linear operator acting in the linear space H and, for this operator, arbitrary powers A^n , $n = 2, 3, \dots$, be also defined in H . We consider the problem

$$\left[\frac{d}{dt} - a(A) \right] U(t) = 0, \quad t \in (0; h), \quad (1)$$

$$\int_0^h U(t) dt = \varphi, \quad (2)$$

where $\varphi \in H$, $(0, h) \subset \mathbb{R}$, $h > 0$, $U: (0, h) \rightarrow H$ is an unknown vector-function, $a(A)$ is an abstract operator with analytical on $\Lambda \subseteq \mathbb{C}$ symbol $a(\lambda) \neq \text{const}$.

Let $\eta(\lambda)$ be the entire function

$$\eta(\lambda) = \frac{\exp[a(\lambda)h] - 1}{a(\lambda)} \quad (3)$$

and P be the set of zeros of function (3). If $a(\lambda_0) = 0$, then we assume that $\eta(\lambda_0) = h$. Hence, $\lambda_0 \notin P$.

Denote by $x(\lambda)$ the eigenvector of the operator A , which corresponds to its eigenvalue $\lambda \in \Lambda \subseteq \mathbb{C}$, i.e. nonzero solutions in H of the equations

$$Ax(\lambda) = \lambda x(\lambda), \quad \lambda \in \Lambda.$$

If λ is not an eigenvalue of the operator A , then we assume $x(\lambda) = 0$.

Consider an analytical on Λ function

$$a(\lambda) = \sum_{n=0}^{\infty} a_n \lambda^n,$$

which would be a symbol of the abstract operator

$$a(A) = \sum_{n=0}^{\infty} a_n A^n,$$

in general, of infinite order, assuming that

$$a(A)x(\lambda) = a(\lambda)x(\lambda).$$

2. Constructing the formal solution of the problem.

In this section, we propose a method of solving the problem (1), (2).

Definition 1. We shall say that vector φ from H belongs $L \subseteq H$, if on Λ there exist depending on φ linear operator $R_\varphi(\lambda): H \rightarrow H$, $\lambda \in \Lambda$, and measure $\mu_\varphi(\lambda)$ such that

$$\varphi = \int_{\Lambda} R_\varphi(\lambda)x(\lambda)d\mu_\varphi(\lambda). \quad (4)$$

Lemma 1. On the set $\Lambda^* \times (0, h)$, $\Lambda^* = \Lambda \setminus P$, the following identity holds:

$$\left[\frac{d}{dt} - a(A) \right] \left\{ \frac{\exp[a(\lambda)t]}{\eta(\lambda)} x(\lambda) \right\} \equiv 0. \quad (5)$$

Proof. As supposed, for the operator A , arbitrary powers A^n , for $n = 2, 3, \dots$, are defined in H . Then for any $\lambda \in \Lambda^*$ and $t \in (0, h)$ we have

$$\begin{aligned} \left[\frac{d}{dt} - a(A) \right] \left\{ \frac{\exp[a(\lambda)t]}{\eta(\lambda)} x(\lambda) \right\} &= \frac{d}{dt} \left\{ \frac{\exp[a(\lambda)t]}{\eta(\lambda)} x(\lambda) \right\} - a(A) \left\{ \frac{\exp[a(\lambda)t]}{\eta(\lambda)} x(\lambda) \right\} \\ &= \frac{a(\lambda) \exp[a(\lambda)t]}{\eta(\lambda)} x(\lambda) - \frac{\exp[a(\lambda)t]}{\eta(\lambda)} a(\lambda) x(\lambda) \equiv 0. \end{aligned}$$

This completes our proof.

Theorem 1. Let in the problem (1), (2), the vector φ belong L , i.e. φ can be represented in the form (4). Then the formula

$$U(t) = \int_{\Lambda^*} R_\varphi(\lambda) \left\{ \frac{\exp[a(\lambda)t]}{\eta(\lambda)} x(\lambda) \right\} d\mu_\varphi(\lambda) \quad (6)$$

defines a formal solution of the problem (1), (2).

Proof. According to the formulas (6), we have:

$$\begin{aligned} \left[\frac{d}{dt} - a(A) \right] U(t) &= \left[\frac{d}{dt} - a(A) \right] \int_{\Lambda^*} R_\varphi(\lambda) \left\{ \frac{\exp[a(\lambda)t]}{\eta(\lambda)} x(\lambda) \right\} d\mu_\varphi(\lambda) \\ &= \int_{\Lambda^*} R_\varphi(\lambda) \left[\frac{d}{dt} - a(A) \right] \left\{ \frac{\exp[a(\lambda)t]}{\eta(\lambda)} x(\lambda) \right\} d\mu_\varphi(\lambda). \end{aligned}$$

From the identity (5) we obtain

$$\left[\frac{d}{dt} - a(A) \right] U(t) = \int_{\Lambda^*} R_\varphi(\lambda) \{0\} d\mu_\varphi(\lambda).$$

Since the operator $R_\varphi(\lambda)$ is linear, the last integral is equal to zero, i.e. $U(t)$ formally satisfies the equality (1).

We shall prove the realization of integral condition (2) using fomula (4):

$$\begin{aligned} \int_0^h U(t) dt &= \int_0^h \left(\int_{\Lambda^*} R_\varphi(\lambda) \left\{ \frac{\exp[a(\lambda)t]}{\eta(\lambda)} x(\lambda) \right\} d\mu_\varphi(\lambda) \right) dt \\ &= \int_{\Lambda^*} R_\varphi(\lambda) \left\{ \int_0^h \frac{\exp[a(\lambda)t]}{\eta(\lambda)} x(\lambda) dt \right\} d\mu_\varphi(\lambda) \\ &= \int_{\Lambda^*} R_\varphi(\lambda) \left\{ \frac{\eta(\lambda)}{\eta(\lambda)} x(\lambda) \right\} d\mu_\varphi(\lambda) = \int_{\Lambda^*} R_\varphi(\lambda) x(\lambda) d\mu_\varphi(\lambda) = \varphi. \end{aligned}$$

This completes our proof.

Remark. The formula (6) defines a solutions of the problem (1), (2) just formally, since the following equalities are not justified:

$$\begin{aligned} & \left[\frac{d}{dt} - a(A) \right] \int_{\Lambda^*} R_\varphi(\lambda) \left\{ \frac{\exp[a(\lambda)t]}{\eta(\lambda)} x(\lambda) \right\} d\mu_\varphi(\lambda) = \\ & = \int_{\Lambda^*} R_\varphi(\lambda) \left[\frac{d}{dt} - a(A) \right] \left\{ \frac{\exp[a(\lambda)t]}{\eta(\lambda)} x(\lambda) \right\} d\mu_\varphi(\lambda), \end{aligned} \quad (7)$$

$$\begin{aligned} & \int_0^h \left(\int_{\Lambda^*} R_\varphi(\lambda) \left\{ \frac{\exp[a(\lambda)t]}{\eta(\lambda)} x(\lambda) \right\} d\mu_\varphi(\lambda) \right) dt = \\ & = \int_{\Lambda^*} R_\varphi(\lambda) \left\{ \int_0^h \frac{\exp[a(\lambda)t]}{\eta(\lambda)} x(\lambda) dt \right\} d\mu_\varphi(\lambda). \end{aligned} \quad (8)$$

We do not prove the existence of the Stieltjes integrals in the equalities (7) and (8) as well.

3. Problem with integral condition for partial differential equation.

In this section, we shall give the example of using an abstract approach to solving the problem for the partial differential equation

$$\left[\frac{\partial}{\partial t} - a \left(\frac{\partial}{\partial x} \right) \right] U(t, x) = 0, \quad t \in (0; h), \quad x \in \mathbb{R}, \quad (9)$$

$$\int_0^h U(t, x) dt = \varphi(x), \quad x \in \mathbb{R}, \quad (10)$$

where $a\left(\frac{\partial}{\partial x}\right)$ is an operator generally of infinite order with entire symbol $a(\lambda) \neq \text{const}$.

The problem (9), (10) has been studied in the work [11] by means of the differential-symbol method [12, 13]. We shall represent this problem as problem (1), (2), in which $A = \frac{d}{dx}$, $\exp[\lambda x]$ is an eigenfunction of the operator A , H is a class of entire functions, $L = K_M$ is a class of quasipolynomials

$$\varphi(x) = \sum_{j=1}^m Q_j(x) \exp[\alpha_j x], \quad (11)$$

where $\alpha_j \in M \subseteq \mathbb{C}$, $\alpha_j \neq \alpha_k$ for $j \neq k$, $x \in \mathbb{R}$, $m \in \mathbb{N}$; $Q_j(x)$, $j = \overline{1, m}$, are polynomials with complex coefficients.

As a measure $\mu(\lambda)$, take the Dirac measure. From the representation (4) we obtain

$$\varphi(x) = R_\varphi(\lambda) \exp[\lambda x] \Big|_{\lambda=0},$$

from which it follows that

$$R_\varphi(\lambda) = \varphi \left(\frac{d}{d\lambda} \right).$$

Each quasipolynomial $\varphi(x)$ of the form (11) defines a differential operation $\varphi \left(\frac{d}{d\lambda} \right)$ of finite order on the class of entire functions $\Phi(\lambda)$, namely

$$\varphi \left(\frac{d}{d\lambda} \right) \Phi(\lambda) = \sum_{j=1}^m Q_j \left(\frac{d}{d\lambda} \right) \Phi(\lambda + \alpha_j),$$

in particular,

$$\varphi \left(\frac{d}{d\lambda} \right) \Phi(\lambda) \Big|_{\lambda=0} = \sum_{j=1}^m Q_j \left(\frac{d}{d\lambda} \right) \Phi(\lambda) \Big|_{\lambda=\alpha_j}.$$

From formula (6), we obtain the representation of the solution of problem (9), (10) in the form

$$U(t, x) = \varphi \left(\frac{d}{d\lambda} \right) \left\{ \frac{\exp[a(\lambda)t + \lambda x]}{\eta(\lambda)} \right\} \Big|_{\lambda=0},$$

moreover, this solution exists and is unique in appropriate class of quasipolynomials of variables t, x , if at that $\varphi \in K_M$, where $M = \mathbb{C} \setminus P$, P is the set of zeros of function (3).

Conclusions. In this work, we propose a method of solving a problem with non-homogeneous integral condition for homogeneous evolution equation with abstract operator in a linear space. The solution of the problem is represented in the form of Stieltjes integral over a certain measure. We give the example of applying this method to solving the problem with integral condition for partial differential equation.

In the future research, the subject of interest is the development of analogous method of solving the problem for nonhomogeneous evolution equation.

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Some results on 2-absorbing ideals in commutative semirings

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ABSTRACT: In this paper, we study the concepts of 2-absorbing and weakly 2-absorbing ideals in a commutative semiring with non-zero identity which is a generalization of prime ideals of a commutative semiring and prove number of results related to the same. We also use these concepts to prove some results of Q -ideals in terms of subtractive extension of ideals in a commutative semiring.

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Keywords and Phrases: *Semiring, subtractive ideal, 2-absorbing ideal, weakly 2-absorbing ideal, subtractive extension of an ideal, Q -ideal.*

1 Introduction

The semiring is an important algebraic structure which plays a prominent role in various branches of mathematics as well as in diverse areas of applied science. The concepts of semiring was first introduced by H. S. Vandiver [14] in 1934. After that several authors have applied this concept in various disciplines in many ways. The structure of prime ideals in semiring theory has gained importance and many mathematicians have exploited its usefulness in algebraic systems over the decades. Anderson and Smith[3] introduced the notion of weakly prime ideals in commutative ring. The concept of 2-absorbing and weakly 2-absorbing ideals of commutative ring with non-zero unity have been introduced by Badawi [5] and Badawi and Darani[6] respectively which are generalizations of prime and weakly prime ideals in a commutative ring. Darani[8] has explored these concepts in commutative semiring and characterized several results in terms of 2-absorbing and weakly 2-absorbing ideals in commutative semiring. Chaudhary and Bonde[7] have introduced the notion of subtractive extension of an ideal to study the ideal theory in quotient semiring.

A commutative semiring is a commutative semigroup (S, \cdot) and a commutative monoid $(S, +, 0_S)$ in which 0_S is the additive identity and $0_S \cdot x = x \cdot 0_S = 0_S$ for

all $x \in S$, both are connected by ring like distributivity. A nonempty subset I of a semiring S is called an ideal of S if $a, b \in I, r \in S, a + b \in I$ and $ra, ar \in I$. An ideal I of a semiring S is called subtractive if $a, a + b \in I, b \in S$ then $b \in I$. An ideal $(I : r)$ is defined as $(I : r) = \{x \in S : rx \in I\}$. It is easy to see that if I is a subtractive ideal of S , then $(I : r)$ is a subtractive ideal of S . Radical of an ideal I is defined as $Rad(I) = \sqrt{I} = \{a \in S : a^n \in I \text{ for some positive integer } n\}$. An element s in a semiring S is said to be nilpotent if there exists a positive integer n (depending on s) such that $s^n = 0$. $Nil(S)$ denotes the set of all nilpotent element of S . A proper ideal I of a semiring S is said to be prime (respectively, weakly prime) if $ab \in I$ (respectively, $0 \neq ab \in I$) implies $a \in I$ or $b \in I$ for some $a, b \in S$. An ideal I of a semiring S is said to be irreducible if for ideals H and K of $S, I = H \cap K$ implies that $I = H$ or $I = K$. A semiring S is said to be regular if for each $a \in S$ there exists $x \in S$ such that $a = axa$. In [11], it is proved that a semiring S is regular if and only if $HK = H \cap K$ for all left ideals K and right ideals H of S .

Throughout this paper, S will always denote a commutative semiring with identity $1 \neq 0$.

2 2-absorbing and weakly 2-absorbing ideals

In this section, we prove number of results correspond to 2-absorbing and weakly 2-absorbing ideals in commutative semirings. Recall [8] the following definitions.

Definition 2.1. A proper ideal I of a commutative semiring S is said to be a 2-absorbing ideal of S if whenever $a, b, c \in S$ and $abc \in I$, then $ab \in I$ or $ac \in I$ or $bc \in I$.

Definition 2.2. A proper ideal I of S is said to be a weakly 2-absorbing ideal of S if whenever $a, b, c \in S$ and $0 \neq abc \in I$, then $ab \in I$ or $ac \in I$ or $bc \in I$.

It is easy to see that every 2-absorbing ideal of a semiring S is a weakly 2-absorbing ideal of S but converse need not be true. For further understanding properties of 2-absorbing and weakly 2-absorbing ideals in commutative semirings, refer [8].

Theorem 2.3. Let I be a 2-absorbing ideal of S . Then $(I : r)$ is a 2-absorbing ideal of S for all $r \in S \setminus I$.

Proof. Let $r \in S \setminus I$ and let $a, b, c \in S$ be such that $abc \in (I : r)$. Then $rabc \in I$. So $ra \in I$ or $rbc \in I$ or $abc \in I$, since I is a 2-absorbing ideal of S . If either $ra \in I$ or $rbc \in I$, we are done. If $abc \in I$, then $ab \in I$ or $ac \in I$ or $bc \in I$, which implies $rab \in I$ or $rac \in I$ or $rbc \in I$. Hence $(I : r)$ is a 2-absorbing ideal of S . \square

Theorem 2.4. Let I be a 2-absorbing subtractive ideal of S with $\sqrt{I} = J$ and $J^2 \subseteq I$. If $I \neq J$ and for all $r \in J \setminus I$, then $(I : r)$ is a prime ideal of S containing I with $J \subseteq (I : r)$.

Proof. Let $uv \in (I : r)$ for some $u, v \in S$. Then $ruv \in I$. Since I is a 2-absorbing ideal of S , therefore $ru \in I$ or $rv \in I$ or $uv \in I$. If $ru \in I$ and $rv \in I$, then $u \in (I : r)$

or $v \in (I : r)$, therefore nothing to prove. If $uv \in I$ and also, $r^2 \in J^2 \subseteq I$. This gives $rv \in (I : r)$ for particular $v \in S$. We have $(r+u)v \in (I : r)$, that is, $r(r+u)v \in I$ and since I is a 2-absorbing ideal of S , therefore $rv \in I$ or $(r+u)v \in I$ or $r(r+u) \in I$. If $rv \in I$ then $v \in (I : r)$, which is required. If $(r+u)v \in I$ and $uv \in I$, then $rv \in I$ (as I is a subtractive). This gives $v \in (I : r)$, so $(I : r)$ is prime. Finally, if $r(r+u) \in I$ and since $r^2 \in J^2 \subseteq I$. This gives $ru \in I$ implies $u \in (I : r)$. Hence $(I : r)$ is a prime ideal of S . \square

Corollary 2.5. *Let I be a 2-absorbing subtractive ideal of S with $\sqrt{I} = J$ and $J^2 \subseteq I$. If $I \neq J$ and for all $r \in J \setminus I$, then $(I : r)$ is a 2-absorbing ideal of S with $J \subseteq (I : r)$.*

Theorem 2.6. *If I is a subtractive ideal of S such that $I \neq \sqrt{I}$ and \sqrt{I} is a prime ideal of S with $(\sqrt{I})^2 \subseteq I$. Then I is a 2-absorbing ideal of S if and only if $(I : r) = \{x \in S : rx \in I\}$ is a prime ideal of S for each $r \in \sqrt{I} \setminus I$.*

Proof. (\Rightarrow) One way is straight forward by above theorem.

(\Leftarrow) Conversely, let $abc \in I$ for some $a, b, c \in S$. Then, we may assume that $a \in \sqrt{I}$ (as $I \subseteq \sqrt{I}$ and \sqrt{I} is a prime ideal of S). If $a \in I$, then $ab \in I$, which gives I is a 2-absorbing ideal of S . Assume that $a \in \sqrt{I} \setminus I$. Also, $bc \in (I : a)$ and by assumption $(I : a)$ is a prime ideal of S , therefore we have either $b \in (I : a)$ or $c \in (I : a)$. This implies that either $ab \in I$ or $ac \in I$. Thus, I is a 2-absorbing ideal of S . \square

The following result is used to prove the next theorem.

Result 2.7. [12] *Let I and J be two subtractive ideals in S . Then $I \cup J$ is a subtractive ideal of S if and only if $I \cup J = I$ or $I \cup J = J$.*

Theorem 2.8. *Let I be a 2-absorbing subtractive ideal of S with $\sqrt{I} = J$. If $I \neq J$, J is a prime ideal of S and for all $r \in S \setminus J$, then $\Omega = \{(I : r) : r \in S\}$ is a totally ordered set.*

Proof. Let $r, s \in S \setminus J$. Since J is a prime ideal of S therefore $rs \in S \setminus J$. Clearly, $rs \notin I$ and $(I : r) \subseteq (I : rs)$ and $(I : s) \subseteq (I : rs)$ which implies $(I : r) \cup (I : s) \subseteq (I : rs)$. Again, let $t \in (I : rs)$. Then, $rst \in I$ which implies that either $rt \in I$ or $st \in I$, as $rs \notin I$. Thus, $(I : rs) \subseteq (I : r) \cup (I : s)$. Hence by Result 2.7, we have either $(I : rs) = (I : r)$ or $(I : rs) = (I : s)$. This implies that either $(I : r) \subseteq (I : s)$ or $(I : s) \subseteq (I : r)$. Therefore $\Omega = \{(I : r) : r \in S \setminus J\}$ is a totally ordered set.

Again, we show that $(I : s) \subseteq (I : r)$ for $r, s \in J \setminus I$. Let $r, s \in J \setminus I$. Then for any $p \in (I : r) \setminus (I : s)$ we may assume that $p \in (I : r) \setminus J$, since $J \subseteq (I : s)$. Similarly, for any $q \in (I : s) \setminus (I : r)$ we may assume that $q \in (I : s) \setminus J$. Since $p \notin J$ and $q \notin J$ therefore $pq \notin J$. Also, $p(r+s)q \in I$ and $pq \notin I$, therefore we have $p(r+s) \in I$ or $(r+s)q \in I$, which gives either $ps \in I$ or $rq \in I$. This implies $p \in (I : s)$ or $q \in (I : r)$. Therefore, in each case we get a contradiction. Hence either $(I : r) \subseteq (I : s)$ or $(I : s) \subseteq (I : r)$ for $r, s \in J \setminus I$. Thus, $\Omega = \{(I : r) : r \in S\}$ is a totally ordered set. \square

Theorem 2.9. *Let I be an irreducible subtractive ideal of S and let J be an ideal of S such that $\sqrt{I} = J$ and $J^2 \subseteq I$. Then I is 2-absorbing if and only if $(I : r) = (I : r^2)$ for all $r \in S \setminus J$.*

Proof. Let I be a 2-absorbing ideal of S . For $r \in S \setminus J$, $r^2 \notin I$ because if $r^2 \in I$, then $r \in \sqrt{I} = J$, which is a contradiction and also $(I : r) \subseteq (I : r^2)$ is obvious. So, for any $s \in (I : r^2)$ we have $r^2s \in I$. Since I is a 2-absorbing ideal of S , we have either $rs \in I$ or $r^2 \in I$. Since $r^2 \notin I$, therefore $rs \in I$, that is, $s \in (I : r)$ and thus $(I : r) = (I : r^2)$.

Conversely, let $rst \in I$ for some $r, s, t \in S$ and $rs \notin I$. We show that either $rt \in I$ or $st \in I$. From $rs \notin I$, we have $r \notin J$ or $s \notin J$. Because, if $r \in J$ and $s \in J$, then $rs \in J^2 \subseteq I$, a contradiction. Now, by assumption, we have either $(I : r) = (I : r^2)$ or $(I : s) = (I : s^2)$. If $(I : r) = (I : r^2)$ and also assume that $rt \notin I$ and $st \notin I$, then we prove the result by way of contradiction. Let $p \in (I + (rt)) \cap (I + (st))$. Then there are $p_1, p_2 \in I$ and $r_1, r_2 \in S$ such that $p = p_1 + r_1rt = p_2 + r_2st$. Thus, $pr = p_1r + r_1r^2t = p_2r + r_2rst \in I$. Since $rst \in I$, therefore $r_1r^2t \in I$ (as I is a subtractive ideal of S). This implies $r_1rt \in I$ because $(I : r) = (I : r^2)$. Hence $p = p_1 + r_1rt \in I$. This shows that $(I + rt) \cap (I + st) \subseteq I$ and thus $(I + rt) \cap (I + st) = I$, a contradiction because I is an irreducible. Thus, we have $rt \in I$ or $st \in I$ and consequently, I is a 2-absorbing ideal of S . \square

Theorem 2.10. *Let S be a regular semiring. Then every irreducible ideal I of S is 2-absorbing ideal of S .*

Proof. Let S be a regular semiring and I be an irreducible ideal of S . If $rst \in I$ and $rs \notin I$, then we have to show that $rt \in I$ or $st \in I$. On contrary, we assume that $rt \notin I$ and $st \notin I$. Then, $H = (I + (rt))$ and $K = (I + (st))$ be two ideals of S properly contain I . Since I is an irreducible, therefore $I \neq H \cap K$. Thus, there exists $p \in S$ such that $p \in (I + (rt)) \cap (I + (st)) \setminus I$. Also, by regularity of S , we have $H \cap K = HK$, therefore $p \in (I + (rt))(I + (st)) \setminus I$. Then, there are $p_1, p_2 \in I$ and $r_1, r_2 \in S$ such that $p = (p_1 + r_1rt)(p_2 + r_2st) = p_1p_2 + p_1r_2st + r_1r_2p_2 + r_1r_2st^2$. This implies that $p \in I$, which is a contradiction. Hence I is a 2-absorbing ideal of S . \square

Proposition 2.11. *Let $a \in S$ and I be an ideal of S . Then the following holds:*

- (i) *If Sa is a subtractive ideal of S and $(0 : a) \subseteq Sa$, then the ideal Sa is 2-absorbing if and only if it is weakly 2-absorbing.*
- (ii) *If I is a subtractive ideal of S and $(0 : a) \subseteq Ia$, then the ideal Ia is 2-absorbing if and only if it is weakly 2-absorbing.*

Proof. (i). Let Sa be weakly 2-absorbing ideal of S and $rst \in Sa$ for some $r, s, t \in S$. If $rst \neq 0$, then $rs \in Sa$ or $st \in Sa$ or $rt \in Sa$. Then we have done. Assume that $rst = 0$. Clearly, $r(s + a)t = rst + rat \in Sa$. If $r(s + a)t \neq 0$, then $r(s + a) \in Sa$ or $rt \in Sa$ or $(s + a)t \in Sa$ (as Sa is a weakly 2-absorbing ideal of S). Hence $rs \in Sa$ or $st \in Sa$ or $rt \in Sa$, since Sa is a subtractive ideal of S . So, assume that $r(s + a)t = 0$.

Since $rst = 0$, therefore we have $rat = 0$ and so $rt \in (0 : a) \subseteq Sa$. Thus, $rt \in Sa$. Hence Sa is a 2-absorbing ideal of S .

(ii). The proof is similar to (i) □

Theorem 2.12. ([8], Theorem 2.6) *Let S be a commutative semiring. If I is a weakly 2-absorbing subtractive ideal of S , then either $I^3 = 0$ or I is 2-absorbing.*

The above theorem is used to prove the next theorem which is a generalization of ([6], Theorem 2.7).

Theorem 2.13. *Let I be a weakly 2-absorbing subtractive ideal of S but not a 2-absorbing ideal of S . Then*

(i) *if $r \in \text{Nil}(S)$, then either $r^2 \in I$ or $r^2I = rI^2 = \{0\}$.*

(ii) *$\text{Nil}(S)^2I^2 = \{0\}$.*

Proof. (i). Let $r \in \text{Nil}(S)$. We claim that if $r^2I \neq \{0\}$. Then $r^2 \in I$. Suppose that $r^2I \neq \{0\}$. Let n be the least positive integer such that $r^n = 0$, then for $n \geq 3$ and for some $s \in I$, we have $0 \neq r^2s = r^2(s + r^{n-2}) \in I$. Since I is a weakly 2-absorbing ideal of S , we have either $r^2 \in I$ or $(rs + r^{n-1}) \in I$. If $r^2 \in I$, we have nothing to prove. Let $r^2 \notin I$. Then $(rs + r^{n-1}) \in I$, which gives $r^{n-1} \in I$ and $r^{n-1} \neq 0$, and thus $r^2 \in I$. Hence for each $r \in \text{Nil}(S)$ we have either $r^2 \in I$ or $r^2I = \{0\}$. If $s^2 \notin I$ for some $s \in \text{Nil}(S)$, then by previous argument, we have $s^2I = \{0\}$. We claim that $sI^2 = \{0\}$. Suppose that $si_1i_2 \neq 0$ for some $i_1, i_2 \in I$. Let $m \geq 3$ be the least positive integer such that $s^m = 0$. Since $s^2 \notin I, m \geq 3$ and $s^2I = \{0\}$, therefore $s(s + i_1)(s^{m-2} + i_2) = si_1i_2 \neq 0$. Since $0 \neq s(s + i_1)(s^{m-2} + i_2) \in I$ and I is a weakly 2-absorbing ideal of S , we have either $s^2 \in I$ or $0 \neq s^{m-1} \in I$ (as I is a subtractive ideal of S). Therefore, we have $s^2 \in I$, a contradiction. Hence $sI^2 = \{0\}$.

(ii). Let $a, b \in \text{Nil}(S)$. If either $a^2 \notin I$ or $b^2 \notin I$, then by part (i), we have $abI^2 = \{0\}$ and hence the result. For $a^2 \in I$ and $b^2 \in I$, then $ab(a + b) \in I$. If $0 \neq ab(a + b) \in I$ and since I is a subtractive weakly 2-absorbing ideal of S , we have $ab \in I$. So by Theorem 2.12, we have $abI^2 = \{0\}$. Again, if $0 = ab(a + b) \in I$ and $0 \neq abi \in I$ for some $i \in I$, then $0 \neq ab(a + b + i) \in I$ implies either $a(a + b + i) \in I$ or $b(a + b + i) \in I$ or $ab \in I$. In each case, we have $ab \in I$, which is a contradiction, as I is a weakly 2-absorbing and not a 2-absorbing ideal of S . Thus, we have $abI = \{0\}$ and hence $abI^2 = \{0\}$. □

Definition 2.14 ([4], Definition 1(i)). *A proper ideal I of S is called strong ideal, if for each $a \in I$ there exists $b \in I$ such that $a + b = 0$.*

Proposition 2.15. *Let S and S' be semirings, $f : S \mapsto S'$ be an epimorphism such that $f(0) = 0$ and I be a subtractive strong ideal of S . Then the following holds:*

(i). *If I is a weakly 2-absorbing ideal of S such that $\ker f \subseteq I$, then $f(I)$ is a weakly 2-absorbing ideal of S' .*

(ii). *If I is a 2-absorbing ideal of S such that $\ker f \subseteq I$, then $f(I)$ is a 2-absorbing ideal of S' .*

Proof. (i). Let $a, b, c \in S'$ be such that $0 \neq abc \in f(I)$. Then there exists $n \in I$ such that $0 \neq abc = f(n)$. Since f is an epimorphism, therefore there exist $p, q, r \in S$ such that $f(p) = a, f(q) = b, f(r) = c$. Also, since I is a strong ideal of S and $n \in I$, then there exists $m \in I$ such that $n + m = 0$. This implies $f(n + m) = 0$, that is, $f(pqr + m) = 0$, implies, $pqr + m \in \ker f \subseteq I$. So, $0 \neq pqr \in I$ (as I is subtractive) because if $pqr = 0$, then $f(n) = 0$, a contradiction. Since I is a weakly 2-absorbing ideal of S , therefore either $pq \in I$ or $qr \in I$ or $rp \in I$. Thus, $ab \in f(I)$ or $bc \in f(I)$ or $ca \in f(I)$. Therefore, $f(I)$ is a weakly 2-absorbing ideal of S' .

(ii). It follows from (i). \square

Consider $S = S_1 \times S_2$ where each S_i is a commutative semiring with unity, $i = 1, 2$ with $(a_1, a_2)(b_1, b_2) = (a_1b_1, a_2b_2)$ for all $a_1, b_1 \in S_1$ and $a_2, b_2 \in S_2$.

Proposition 2.16. *If I is a proper ideal of a semiring S_1 . Then the following statements are equivalent:*

(i). I is a 2-absorbing ideal of S_1 .

(ii). $I \times S_2$ is a 2-absorbing ideal of $S = S_1 \times S_2$.

(iii). $I \times S_2$ is a weakly 2-absorbing ideal of $S = S_1 \times S_2$.

Proof. (i) \Rightarrow (ii). Let $(a_1, a_2), (b_1, b_2), (c_1, c_2) \in S$ be such that $(a_1, a_2)(b_1, b_2)(c_1, c_2) \in I \times S_2$. Then $(a_1b_1c_1, a_2b_2c_2) \in I \times S_2$. Therefore, $a_1b_1c_1 \in I$. This gives either $a_1b_1 \in I$ or $b_1c_1 \in I$ or $c_1a_1 \in I$, since I is a 2-absorbing ideal of S_1 . If $a_1b_1 \in I$, then $(a_1, a_2)(b_1, b_2) \in I \times S_2$. Similarly, we can prove the other cases. Hence, $I \times S_2$ is a 2-absorbing ideal of S .

(ii) \Rightarrow (iii). It is obvious.

(iii) \Rightarrow (i). Let $abc \in I$ for some $a, b, c \in S_1$. Then for each $0 \neq r \in S_2$, we have $(0, 0) \neq (a, 1)(b, 1)(c, r) \in I \times S_2$. This gives, either $(a, 1)(b, 1) \in I \times S_2$ or $(b, 1)(c, r) \in I \times S_2$ or $(c, r)(a, 1) \in I \times S_2$, since $I \times S_2$ is a weakly 2-absorbing ideal of S . That is, either $ab \in I$ or $bc \in I$ or $ca \in I$. This shows that I is a 2-absorbing ideal of S_1 . \square

Definition 2.17 ([1], Definition(4)). *An ideal I of a semiring S is called a Q -ideal (partitioning ideal) if there exists a subset Q of S such that*

$$(i) \quad S = \cup\{q + I : q \in Q\}$$

$$(ii) \quad \text{If } q_1, q_2 \in Q, \text{ then } (q_1 + I) \cap (q_2 + I) \neq \emptyset \Leftrightarrow q_1 = q_2.$$

Let I be a Q -ideal of a semiring S . Then $S/I_{(Q)} = \{q + I : q \in Q\}$ forms a semiring under the following addition ' \oplus ' and multiplication ' \odot ', $(q_1 + I) \oplus (q_2 + I) = q_3 + I$ where $q_3 \in Q$ is unique such that $q_1 + q_2 + I \subseteq q_3 + I$, and $(q_1 + I) \odot (q_2 + I) = q_4 + I$ where $q_4 \in Q$ is unique such that $q_1q_2 + I \subseteq q_4 + I$. This semiring $S/I_{(Q)}$ is called the quotient semiring of S by I and denoted by $(S/I_{(Q)}, \oplus, \odot)$ or just $S/I_{(Q)}$. By definition of Q -ideal, there exists a unique $q_0 \in Q$ such that $0 + I \subseteq q_0 + I$. Then $q_0 + I$ is a zero element of $S/I_{(Q)}$. Clearly, if S is commutative then so is $S/I_{(Q)}$.

Definition 2.18 ([7], Definition(2.4)). *Let I be an ideal of a semiring S . An ideal A of S with $I \subseteq A$ is said to be subtractive extension of I if $x \in I$, $x + y \in A$, $y \in S$, then $y \in A$.*

Further, we give some characterizations of 2-absorbing and weakly 2-absorbing ideals in terms of subtractive extension of an ideal of a semiring S , which are derived from generalizations of [7].

Theorem 2.19. *Let S be a semiring, I be a Q -ideal of S and P a subtractive extension of I . Then P is 2-absorbing ideal of S if and only if $P/I_{(Q \cap P)}$ is a 2-absorbing ideal of $S/I_{(Q)}$.*

Proof. Let P be a 2-absorbing ideal of S . Suppose that $q_1 + I, q_2 + I, q_3 + I \in S/I_{(Q)}$ are such that $(q_1 + I) \odot (q_2 + I) \odot (q_3 + I) = q_4 + I \in P/I_{(Q \cap P)}$ where $q_4 \in Q \cap P$ is a unique element such that $q_1 q_2 q_3 + I \subseteq q_4 + I \in P/I_{(Q \cap P)}$. So $q_1 q_2 q_3 = q_4 + i$ for some $i \in I$. Since P is a 2-absorbing ideal of S and $q_1 q_2 q_3 \in P$, therefore $q_1 q_2 \in P$ or $q_2 q_3 \in P$ or $q_3 q_1 \in P$. Consider the case $q_1 q_2 \in P$. If $(q_1 + I) \odot (q_2 + I) = i_1 + I$ where $i_1 \in Q$ is a unique element such that $q_1 q_2 + I \subseteq i_1 + I$. So $i_1 + f = q_1 q_2 + e$ for some $e, f \in I$. Since P is a subtractive extension of I , we have $i_1 \in P$, therefore $i_1 \in Q \cap P$. Hence $P/I_{(Q \cap P)}$ is a 2-absorbing ideal of $S/I_{(Q)}$.

Conversely, if $P/I_{(Q \cap P)}$ is a 2-absorbing ideal of $S/I_{(Q)}$. Let $abc \in P$ for some $a, b, c \in S$. Since I is a Q -ideal of S , therefore there exist $q_1, q_2, q_3, q_4 \in Q$ such that $a \in q_1 + I, b \in q_2 + I, c \in q_3 + I$ and $abc \in (q_1 + I) \odot (q_2 + I) \odot (q_3 + I) = q_4 + I$. So, $abc = q_4 + i_2 \in P$ for some $i_2 \in I$. Since P is a subtractive extension of I , we have $q_4 \in P$. So $(q_1 + I) \odot (q_2 + I) \odot (q_3 + I) = q_4 + I \in P/I_{(Q \cap P)}$, which gives $(q_1 + I) \odot (q_2 + I) \in P/I_{(Q \cap P)}$ or $(q_2 + I) \odot (q_3 + I) \in P/I_{(Q \cap P)}$ or $(q_3 + I) \odot (q_1 + I) \in P/I_{(Q \cap P)}$, since $P/I_{(Q \cap P)}$ is a 2-absorbing ideal of $S/I_{(Q)}$. If $(q_1 + I) \odot (q_2 + I) \in P/I_{(Q \cap P)}$, then there exists $q_5 \in Q \cap P$ such that $ab \in (q_1 + I) \odot (q_2 + I) = q_5 + I$. This gives $ab = q_5 + i_3$ for some $i_3 \in I$. This implies $ab \in P$. Thus, P is a 2-absorbing ideal of S . \square

Corollary 2.20. *Let S be a semiring, I be a Q -ideal of S and P be subtractive ideal of S such that $I \subseteq P$. Then P is a 2-absorbing ideal of S if and only if $P/I_{(Q \cap P)}$ is a 2-absorbing ideal of $S/I_{(Q)}$.*

Note that, if $(q_1 + I) \odot (q_2 + I) \odot (q_3 + I) \neq 0$ in $S/I_{(Q)}$, then $q_1 q_2 q_3 \neq 0$ in S . Now one can easily prove the next theorem, adopting the proof of the last theorem.

Theorem 2.21. *Let S be a semiring, I a Q -ideal of S and P a subtractive extension of I . Then*

- (i) *if P is a weakly 2-absorbing ideal of S , then $P/I_{(Q \cap P)}$ is a weakly 2-absorbing ideal of $S/I_{(Q)}$.*
- (ii) *if I and $P/I_{(Q \cap P)}$ is a weakly 2-absorbing ideal of S and $S/I_{(Q)}$ respectively, then P is a weakly 2-absorbing ideal of S .*

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Approximate controllability of the impulsive semilinear heat equation¹

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ABSTRACT: In this paper we apply Rothe's Fixed Point Theorem to prove the interior approximate controllability of the following semilinear impulsive Heat Equation

$$\begin{cases} z_t = \Delta z + 1_\omega u(t, x) + f(t, z, u(t, x)), & \text{in } (0, \tau] \times \Omega, t \neq t_k \\ z = 0, & \text{on } (0, \tau) \times \partial\Omega, \\ z(0, x) = z_0(x), & x \in \Omega, \\ z(t_k^+, x) = z(t_k^-, x) + I_k(t_k, z(t_k, x), u(t_k, x)), & x \in \Omega, \end{cases}$$

where $k = 1, 2, \dots, p$, Ω is a bounded domain in \mathbb{R}^N ($N \geq 1$), $z_0 \in L_2(\Omega)$, ω is an open nonempty subset of Ω , 1_ω denotes the characteristic function of the set ω , the distributed control u belongs to $C([0, \tau]; L_2(\Omega))$ and $f, I_k \in C([0, \tau] \times \mathbb{R} \times \mathbb{R}; \mathbb{R})$, $k = 1, 2, 3, \dots, p$, such that

$$|f(t, z, u)| \leq a_0 |z|^{\alpha_0} + b_0 |u|^{\beta_0} + c_0, \quad u \in \mathbb{R}, z \in \mathbb{R}.$$

$$|I_k(t, z, u)| \leq a_k |z|^{\alpha_k} + b_k |u|^{\beta_k} + c_k, \quad k = 1, 2, 3, \dots, p, \quad u \in \mathbb{R}, z \in \mathbb{R}.$$

with $\frac{1}{2} \leq \alpha_k < 1$, $\frac{1}{2} \leq \beta_k < 1$, $k = 0, 1, 2, 3, \dots, p$. Under this condition we prove the following statement: For all open nonempty subsets ω of Ω the system is approximately controllable on $[0, \tau]$. Moreover, we could exhibit a sequence of controls steering the nonlinear system from an initial state z_0 to an ϵ neighborhood of the final state z_1 at time $\tau > 0$.

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1 Introduction

There are many practical examples of impulsive control systems, a chemical reactor system with the quantities of different chemicals serve as the states, a financial system with two state variables of the amount of money in a market and the saving rates of a central bank and the growth of a population diffusing throughout its habitat is often modeled by reaction-diffusion equation, for which much has been done under the assumption that the system parameters related to the population environment, either are constant or change continuously. However, one may easily visualize situations in nature where abrupt changes such as harvesting, disasters and instantaneous stoking may occur. This observation motivates us to study the approximate controllability of the following Semilinear Impulsive Heat Equation

$$\begin{cases} z_t = \Delta z + 1_\omega u(t, x) + f(t, z, u(t, x)), & \text{in } (0, \tau] \times \Omega, t \neq t_k \\ z = 0, & \text{on } (0, \tau) \times \partial\Omega, \\ z(0, x) = z_0(x), & x \in \Omega, \\ z(t_k^+, x) = z(t_k^-, x) + I_k(t_k, z(t_k, x), u(t_k, x)), & x \in \Omega, \end{cases} \quad (1.1)$$

where $k = 1, 2, \dots, p$, Ω is a bounded domain in \mathbb{R}^N ($N \geq 1$), $z_0 \in L_2(\Omega)$, ω is an open nonempty subset of Ω , 1_ω denotes the characteristic function of the set ω , the distributed control u belongs to $C([0, \tau]; L_2(\Omega))$ and $f, I_k \in C([0, \tau] \times \mathbb{R} \times \mathbb{R}; \mathbb{R})$, $k = 1, 2, 3, \dots, p$, such that

$$|f(t, z, u)| \leq a_0 |z|^{\alpha_0} + b_0 |u|^{\beta_0} + c_0, \quad u, z \in \mathbb{R}. \quad (1.2)$$

$$|I_k(t, z, u)| \leq a_k |z|^{\alpha_k} + b_k |u|^{\beta_k} + c_k, \quad k = 1, 2, 3, \dots, p, \quad u, z \in \mathbb{R}. \quad (1.3)$$

$$\frac{1}{2} \leq \alpha_k < 1, \frac{1}{2} \leq \beta_k < 1, \quad k = 0, 1, 2, 3, \dots, p, \quad (1.4)$$

and

$$z(t_k, x) = z(t_k^+, x) = \lim_{t \rightarrow t_k^+} z(t, x), \quad z(t_k^-, x) = \lim_{t \rightarrow t_k^-} z(t, x).$$

In almost all reference on impulsive differential equations the natural space to work in is the Banach space

$$\begin{aligned} & PC([0, \tau]; Z) \\ & = \{z : J = [0, \tau] \rightarrow Z : z \in C(J'; Z), \exists z(t_k^+, \cdot), z(t_k^-, \cdot) \text{ and } z(t_k, \cdot) = z(t_k^+, \cdot)\}, \end{aligned}$$

where $Z = L_2(\Omega)$ and $J' = [0, \tau] \setminus \{t_1, t_2, \dots, t_p\}$, endowed with the norm

$$\|z\| = \sup_{t \in [0, \tau]} |z(t, \cdot)|_Z,$$

with

$$\|z\|_Z = \sqrt{\int_{\Omega} \|z(x)\|^2 dx}, \quad \forall z \in Z = L_2(\Omega).$$

Definition 1.1 (Approximate Controllability) *The system (1.1) is said to be approximately controllable on $[0, \tau]$ if for every $z_0, z_1 \in Z = U = L_2(\Omega)$, $\varepsilon > 0$ there exists $u \in C([0, \tau]; U)$ such that the solution $z(t)$ of (1.1) corresponding to u verifies:*

$$z(0) = z_0 \text{ and } \|z(\tau) - z_1\|_Z < \varepsilon, \quad (\text{Fig.2}),$$

where

$$\|z(\tau) - z_1\|_Z = \left(\int_{\Omega} |z(\tau, x) - z_1(x)|^2 dx \right)^{\frac{1}{2}}.$$

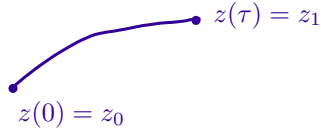


Fig.1

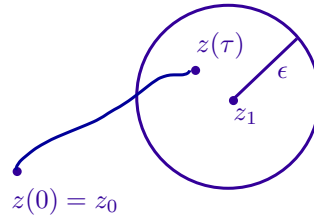


Fig.2

Definition 1.2 (Controllability to Trajectories) *The system (1.1) is said to be controllable to trajectories on $[0, \tau]$ if for every $z_0, \hat{z}_0 \in Z = U = L_2(\Omega)$ and $\hat{u} \in C([0, \tau]; U)$ there exists $u \in C([0, \tau]; U)$ such that the mild solution $z(t)$ of (1.1) corresponding to u verifies:*

$$z(\tau, z_0, u) = z(\tau, \hat{z}_0, \hat{u}), \quad (\text{Fig.3}).$$

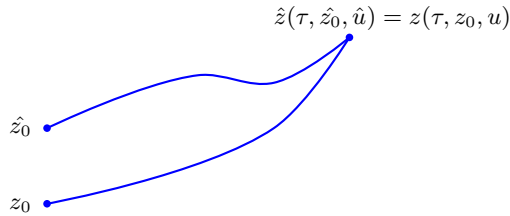


Fig.3

Definition 1.3 (Null Controllability) *The system (1.1) is said to be null controllable on $[0, \tau]$ if for every $z_0 \in Z = U = L_2(\Omega)$ there exists $C([0, \tau]; U)$ such that the mild solution $z(t)$ of (1.1) corresponding to u verifies:*

$$z(0) = z_0 \text{ and } z(\tau) = 0, \quad (\text{Fig.4}).$$



Fig.4

Remark 1.1 *It is clear that exact controllability of the system (1.1) implies approximate controllability, null controllability and controllability to trajectories of the system. But, it is well known ([2]) that due to the diffusion effect or the compactness of the semigroup generated by $-\Delta$, the heat equation can never be exactly controllable. We observe also that in the linear case controllability to trajectories and null controllability are equivalent. Nevertheless, the approximate controllability and the null controllability are in general independent. Therefore, in this paper we will be concentrated only on the study of the approximate controllability of the system (1.1).*

Recently the interior controllability of the semilinear heat equation (1.1) without impulses has been proved in [13], [14] and [15] under the following condition:

$$\sup_{(t,z,u) \in Q_\tau} |f(t, z, u) - az - cu| < \infty, \quad (1.5)$$

where $a, c \in \mathbb{R}$, with $c \neq -1$ and $Q_\tau = [0, \tau] \times \mathbb{R} \times \mathbb{R}$.

More recently, in [14], the approximate controllability of the semilinear heat equation (1.1) without impulses has been proved under the following non linear perturbation:

$$|f(t, z, u) - az| \leq c|u|^\beta + b, \quad \forall u, z \in \mathbb{R}, \quad |u|, |z| \geq R, \quad (1.6)$$

where $a, b, c \in \mathbb{R}$, $R > 0$ and $\frac{1}{2} \leq \beta < 1$. We note that, the interior approximate controllability of the linear heat equation

$$\begin{cases} z_t(t, x) = \Delta z(t, x) + 1_\omega u(t, x) & \text{in } (0, \tau] \times \Omega, \\ z = 0, & \text{on } (0, \tau) \times \partial\Omega, \\ z(0, x) = z_0(x), & x \in \Omega, \end{cases} \quad (1.7)$$

has been study by several authors, particularly by [22],[23],[24]; and in a general fashion in [12].

The controllability of Impulsive Evolution Equations has been studied recently for several authors, but most them study the exact controllability only, to mention: D.N. Chalişhajar([4]), studied the exact controllability of impulsive partial neutral functional differential equations with infinite delay, B. Radhakrishnan and K. Balachandran([19]) studied the exact controllability of semilinear impulsive integrodifferential evolution systems with nonlocal conditions and S. Selvi, M. Mallika Arjunan([20]) studied the exact controllability for impulsive differential systems with finite delay. To our knowledge, there are a few works on approximate controllability of impulsive semilinear evolution equations, to mention: Lizhen Chen and Gang Li([5])

studied the Approximate controllability of impulsive differential equations with non-local conditions, using measure of noncompactness and Monch fixed point theorem, and assuming that the nonlinear term $f(t, z)$ does not depend on the control variable.

Finally, the approximate controllability of the system (1.1) follows from the approximate controllability of (1.7), the compactness of the semigroup generated by the Laplacian operator $-\Delta$, the conditions (1.2) and (1.5) satisfied by the nonlinear term f, I_k and the following results:

Proposition 1.1 *Let (X, Σ, μ) be a measure space with $\mu(X) < \infty$ and $1 \leq q < r < \infty$. Then $L_r(\mu) \subset L_q(\mu)$ and*

$$\|f\|_q \leq \mu(X)^{\frac{r-q}{rq}} \|f\|_r, \quad f \in L_r(\mu). \tag{1.8}$$

Proof The proof of this proposition follows from Theorem I.V.6 from [3] by putting $p = \frac{r}{q} > 1$ and considering the relation

$$\int_X (|f|^q)^p d\mu = \int_X |f|^r d\mu, \quad \forall f \in L_r(\mu).$$

□

Theorem 1.1 *(Rothe's Fixed Theorem, [1],[9], [21]) Let E be a Banach space. Let $B \subset E$ be a closed convex subset such that the zero of E is contained in the interior of B . Let $\Phi : B \rightarrow E$ be a continuous mapping with $\Phi(B)$ relatively compact in E and $\Phi(\partial B) \subset B$. Then there is a point $x^* \in B$ such that $\Phi(x^*) = x^*$.*

The technique we use here to prove the approximate controllability of the linear part of equation (1.7) is based on the classical Unique Continuation for Elliptic Equations (see [18]) and the following lemma:

Lemma 1.1 *(see Lemma 3.14 from [6], pg. 62) Let $\{\alpha_j\}_{j \geq 1}$ and $\{\beta_{i,j} : i = 1, 2, \dots, m\}_{j \geq 1}$ be two sequences of real numbers such that: $\alpha_1 > \alpha_2 > \alpha_3 \dots$. Then*

$$\sum_{j=1}^{\infty} e^{\alpha_j t} \beta_{i,j} = 0, \quad \forall t \in [0, \tau], \quad i = 1, 2, \dots, m$$

iff

$$\beta_{i,j} = 0, \quad i = 1, 2, \dots, m; j = 1, 2, \dots, \infty.$$

2 Abstract Formulation of the Problem

In this section we choose a Hilbert space where system (1.1) can be written as an abstract differential equation; to this end, we consider the following results appearing in [6] pg.46, [8] pg.335 and [10] pg.147:

Let us consider the Hilbert space $Z = L_2(\Omega)$ and $0 < \lambda_1 < \lambda_2 < \dots < \lambda_j \rightarrow \infty$ the eigenvalues of $-\Delta$ with the Dirichlet homogeneous conditions, each one with finite

multiplicity γ_j equal to the dimension of the corresponding eigenspace. Then we have the following well known properties

- (i) There exists a complete orthonormal set $\{\phi_{j,k}\}$ of eigenvectors of $A = -\Delta$.
(ii) For all $z \in D(A)$ we have

$$Az = \sum_{j=1}^{\infty} \lambda_j \sum_{k=1}^{\gamma_j} \langle z, \phi_{j,k} \rangle \phi_{j,k} = \sum_{j=1}^{\infty} \lambda_j E_j z, \quad (2.1)$$

where $\langle \cdot, \cdot \rangle$ is the inner product in Z and

$$E_j z = \sum_{k=1}^{\gamma_j} \langle z, \phi_{j,k} \rangle \phi_{j,k}. \quad (2.2)$$

So, $\{E_j\}$ is a family of complete orthogonal projections in Z and $z = \sum_{j=1}^{\infty} E_j z$, $z \in Z$.

- (iii) $-A$ generates an analytic semigroup $\{T(t)\}$ given by

$$T(t)z = \sum_{j=1}^{\infty} e^{-\lambda_j t} E_j z \quad \text{and} \quad \|T(t)\| \leq e^{-\lambda_1 t}, \quad t \geq 0. \quad (2.3)$$

Consequently, system (1.1) can be written as an abstract impulsive differential equations in Z :

$$\begin{cases} z' = -Az + B_\omega u + f^e(t, z, u), & t \in (0, \tau], t \neq t_k, \quad z \in Z \\ z(0) = z_0, \\ z(t_k^+) = z(t_k^-) + I_k^e(t_k, z(t_k), u(t_k)), & k = 1, 2, 3, \dots, p. \end{cases} \quad (2.4)$$

where $u \in C([0, \tau]; U)$, $U = Z$, $B_\omega : U \rightarrow Z$, $B_\omega u = 1_\omega u$ is a bounded linear operator, $I_k^e, f^e : [0, \tau] \times Z \times U \rightarrow Z$, are defined by

$$I_k^e(t, z, u)(x) = I_k(t, z(x), u(x)), \quad f^e(t, z, u)(x) = f(t, z(x), u(x)), \quad \forall x \in \Omega, k = 1, 2, \dots, p.$$

On the other hand, from conditions (1.2) and (1.5) we get the following estimates.

Proposition 2.1 *Under the conditions (1.2)-(1.5) the functions $f^e, I_k^e : [0, \tau] \times Z \times U \rightarrow Z, k = 1, 2, 3, \dots, p$, defined above satisfy $\forall u, z \in Z = L_2(\Omega)$:*

$$\|f^e(t, z, u)\|_Z \leq \tilde{a}_0 \|z\|_Z^{\alpha_0} + \tilde{b}_0 \|u\|_Z^{\beta_0} + \tilde{c}_0 \quad (2.5)$$

$$\|I_k^e(t, z, u)\|_Z \leq \tilde{a}_k \|z\|_Z^{\alpha_k} + \tilde{b}_k \|u\|_Z^{\beta_k} + \tilde{c}_k, \quad k = 1, 2, 3, \dots, p. \quad (2.6)$$

Proof.

$$\begin{aligned} \|f^e(t, z, u)\|_Z^2 &= \int_{\Omega} |f(t, z(x), u(x))|^2 dx \\ &\leq \int_{\Omega} (a_0 |z(x)|^{\alpha_0} + b_0 |u(x)|^{\beta_0} + c_0)^2 dx \\ &\leq \int_{\Omega} (4a_0^2 |z(x)|^{2\alpha_0} + 4^2 b_0^2 |u(x)|^{2\beta_0} + 4^2 c_0^2) dx \\ &\leq 4a_0^2 \int_{\Omega} |z(x)|^{2\alpha_0} dx + 4^2 b_0^2 \int_{\Omega} |u(x)|^{2\beta_0} dx + 4^2 c_0^2 \mu(\Omega). \end{aligned}$$

Then

$$\begin{aligned} \|f^e(t, z, u)\|_Z &\leq 2a_0 \left(\int_{\Omega} |z(x)|^{2\alpha_0} dx \right)^{\frac{1}{2}} + 4b_0 \left(\int_{\Omega} |u(x)|^{2\beta_0} dx \right)^{\frac{1}{2}} + 4c_0 \sqrt{\mu(\Omega)} \\ &= 2a_0 \|z\|_{L^{2\alpha_0}}^{\alpha_0} + 4b_0 \|z\|_{L^{2\beta_0}}^{\beta_0} + 4c_0 \sqrt{\mu(\Omega)} \end{aligned}$$

Now, since $\frac{1}{2} \leq \alpha_0 < 1 \Leftrightarrow 1 \leq 2\alpha_0 < 2$ and $\frac{1}{2} \leq \beta_0 < 1 \Leftrightarrow 1 \leq 2\beta_0 < 2$ applying proposition 1.1, we obtain that:

$$\|f^e(t, z, u)\|_Z \leq 2a_0 \mu(\Omega)^{\frac{1-\alpha_0}{\alpha_0}} \|z\|_Z^{\alpha_0} + 2b_0 \mu(\Omega)^{\frac{1-\beta_0}{\beta_0}} \|u\|_Z^{\beta_0} + 4c_0 \sqrt{\mu(\Omega)}.$$

Analogously, we obtain the following estimate for $k = 1, 2, 3, \dots, p$

$$\|I_k^e(t, z, u)\|_Z \leq 2a_k \mu(\Omega)^{\frac{1-\alpha_k}{\alpha_k}} \|z\|_Z^{\alpha_k} + 2b_k \mu(\Omega)^{\frac{1-\beta_k}{\beta_k}} \|u\|_Z^{\beta_k} + 4c_k \sqrt{\mu(\Omega)},$$

which completes the proof. ■

3 Controllability of the Linear Equation without Impulses

In this section we shall present some characterization of the interior approximate controllability of the linear heat equations without impulses. To this end, we note that, for all $z_0 \in Z$ and $u \in L_2(0, \tau; U)$ the initial value problem

$$\begin{cases} z' = -Az + B_{\omega}u(t), & z \in Z, \\ z(0) = z_0, \end{cases} \quad (3.1)$$

where the control function u belongs to $L_2(0, \tau; U)$, admits only one mild solution given by

$$z(t) = T(t)z_0 + \int_0^t T(t-s)B_{\omega}u(s)ds, \quad t \in [0, \tau]. \quad (3.2)$$

Definition 3.1 For system (3.1) we define the following concept: The controllability map (for $\tau > 0$) $G : L_2(0, \tau; U) \rightarrow Z$ is given by

$$Gu = \int_0^{\tau} T(\tau-s)B_{\omega}u(s)ds. \quad (3.3)$$

whose adjoint operator $G^* : Z \rightarrow L_2(0, \tau; Z)$ is given by

$$(G^*z)(s) = B_{\omega}^*T^*(\tau-s)z, \quad \forall s \in [0, \tau], \quad \forall z \in Z. \quad (3.4)$$

Therefore, the Grammian operator $\mathcal{W} : Z \rightarrow Z$ is given

$$\mathcal{W}z = GG^*z = \int_0^{\tau} T(\tau-s)B_{\omega}B_{\omega}^*T^*(\tau-s)ds. \quad (3.5)$$

The following lemma holds in general for a linear bounded operator $G : W \rightarrow Z$ between Hilbert spaces W and Z .

Lemma 3.1 (see [6], [7] and [12]) *The equation (3.1) is approximately controllable on $[0, \tau]$ if, and only if, one of the following statements holds:*

- a) $\overline{\text{Rang}(G)} = Z$.
- b) $\text{Ker}(G^*) = \{0\}$.
- c) $\langle GG^*z, z \rangle > 0$, $z \neq 0$ in Z .
- d) $\lim_{\alpha \rightarrow 0^+} \alpha(\alpha I + GG^*)^{-1}z = 0$.
- e) $B_\omega^*T^*(t)z = 0$, $\forall t \in [0, \tau]$, $\Rightarrow z = 0$.
- f) For all $z \in Z$ we have $Gu_\alpha = z - \alpha(\alpha I + GG^*)^{-1}z$, where

$$u_\alpha = G^*(\alpha I + GG^*)^{-1}z, \quad \alpha \in (0, 1].$$

So, $\lim_{\alpha \rightarrow 0} Gu_\alpha = z$ and the error $E_\alpha z$ of this approximation is given by

$$E_\alpha z = \alpha(\alpha I + GG^*)^{-1}z, \quad \alpha \in (0, 1].$$

Remark 3.1 *The Lemma 3.1 implies that the family of linear operators $\Gamma_\alpha : Z \rightarrow L_2(0, \tau; U)$, defined for $0 < \alpha \leq 1$ by*

$$\Gamma_\alpha z = B_\omega^*T^*(\cdot)(\alpha I + GG^*)^{-1}z = G^*(\alpha I + GG^*)^{-1}z, \quad (3.6)$$

is an approximate inverse for the right of the operator G in the sense that

$$\lim_{\alpha \rightarrow 0} G\Gamma_\alpha = I, \quad (3.7)$$

in the strong topology.

Proposition 3.1 (See [15]) *If $\overline{\text{Rang}(G)} = Z$, then*

$$\sup_{\alpha > 0} \|\alpha(\alpha I + GG^*)^{-1}\| \leq 1. \quad (3.8)$$

Remark 3.2 *The proof of the following theorem follows from foregoing characterization of dense range linear operators and the classical Unique Continuation for Elliptic Equations (see [18]), and it is similar to the one given in Theorem 4.1 in [14].*

Theorem 3.1 *System (3.1) is approximately controllable on $[0, \tau]$. Moreover, a sequence of controls steering the system (3.1) from initial state z_0 to an ϵ neighborhood of the final state z_1 at time $\tau > 0$ is given by*

$$u_\alpha(t) = B_\omega^*T^*(\tau - t)(\alpha I + GG^*)^{-1}(z_1 - T(\tau)z_0),$$

and the error of this approximation E_α is given by

$$E_\alpha = \alpha(\alpha I + GG^*)^{-1}(z_1 - T(\tau)z_0).$$

Proof . It is enough to show that $\overline{\text{Rang}(G)} = Z$ or $\text{Ker}(G^*) = \{0\}$. To this end, we observe that $B_\omega = B_\omega^*$ and $T^*(t) = T(t)$. Suppose that

$$B_\omega^* T^*(t)z = 0, \quad \forall t \in [0, \tau].$$

Then,

$$\begin{aligned} B_\omega^* T^*(t)z &= \sum_{j=1}^{\infty} e^{-\lambda_j t} B_\omega^* E_j z = \sum_{j=1}^{\infty} e^{-\lambda_j t} \sum_{k=1}^{\gamma_j} \langle z, \phi_{j,k} \rangle 1_\omega \phi_{j,k} = 0. \\ &\iff \sum_{j=1}^{\infty} e^{-\lambda_j t} \sum_{k=1}^{\gamma_j} \langle z, \phi_{j,k} \rangle 1_\omega \phi_{j,k}(x) = 0, \quad \forall x \in \omega. \end{aligned}$$

Hence, from Lemma 1.1, we obtain that

$$E_j z(x) = \sum_{k=1}^{\gamma_j} \langle z, \phi_{j,k} \rangle \phi_{j,k}(x) = 0, \quad \forall x \in \omega, \quad j = 1, 2, 3, \dots$$

Now, putting $f(x) = \sum_{k=1}^{\gamma_j} \langle z, \phi_{j,k} \rangle \phi_{j,k}(x)$, $\forall x \in \Omega$, we obtain that

$$\begin{cases} (\Delta + \lambda_j I)f \equiv 0 & \text{in } \Omega, \\ f(x) = 0 & \forall x \in \omega. \end{cases}$$

Then, from the classical Unique Continuation for Elliptic Equations (see [18]), it follows that $f(x) = 0$, $\forall x \in \Omega$. So,

$$\sum_{k=1}^{\gamma_j} \langle z, \phi_{j,k} \rangle \phi_{j,k}(x) = 0, \quad \forall x \in \Omega.$$

On the other hand, $\{\phi_{j,k}\}$ is a complete orthonormal set in $Z = L_2(\Omega)$, which implies that $\langle z, \phi_{j,k} \rangle = 0$.

Therefore, $E_j z = 0$, $j = 1, 2, 3, \dots$, which implies that $z = 0$. So, $\overline{\text{Rang}(G)} = Z$. Hence, the system (3.1) is approximately controllable on $[0, \tau]$, and the remainder of the proof follows from Lemma 3.1. \square

Lemma 3.2 *Let S be any dense subspace of $L_2(0, \tau; U)$. Then, system (3.1) is approximately controllable with control $u \in L_2(0, \tau; U)$ if, and only if, it is approximately controllable with control $u \in S$. i.e.,*

$$\overline{\text{Rang}(G)} = Z \iff \overline{\text{Rang}(G|_S)} = Z,$$

where $G|_S$ is the restriction of G to S .

Proof (\Rightarrow) Suppose $\overline{\text{Rang}(G)} = Z$ and $\overline{S} = L_2(0, \tau; U)$. Then, for a given $\epsilon > 0$ and $z \in Z$ there exists $u \in L_2(0, \tau; U)$ and a sequence $\{u_n\}_{n \geq 1} \subset S$ such that

$$\|Gu - z\| < \frac{\epsilon}{2} \quad \text{and} \quad \lim_{n \rightarrow \infty} u_n = u.$$

Therefore, $\lim_{n \rightarrow \infty} Gu_n = Gu$ and $\|Gu_n - z\| < \epsilon$ for n big enough. Hence, $\overline{\text{Rang}(G|_S)} = Z$.

(\Leftarrow) This side is trivial. \square

Remark 3.3 According to the previous lemma, if the system is controllable, it is approximately controllable with control functions in the following dense spaces of $L_2(0, \tau; U)$:

$$S = C([0, \tau]; U), \quad S = C^\infty(0, \tau; U), \quad S = PC(J).$$

Moreover, the operators G , \mathcal{W} and Γ are well define in the space of continuous functions: $G : C([0, \tau]; U) \rightarrow Z$ by

$$Gu = \int_0^\tau T(\tau - s)B_\omega u(s)ds, \quad (3.9)$$

and $G^* : Z \rightarrow C([0, \tau]; U)$ by

$$(G^*z)(s) = B^*(s)T^*(\tau - s)z, \quad \forall s \in [0, \tau]. \quad \forall z \in Z. \quad (3.10)$$

Also, the Controllability Grammian operator still the same $\mathcal{W} : Z \rightarrow Z$

$$\mathcal{W}z = GG^*z = \int_0^\tau T(\tau - s)B_\omega B_\omega^*(s)T^*(\tau - s)zds. \quad (3.11)$$

Finally, the operators $\Gamma_\alpha : Z \rightarrow C([0, \tau]; U)$ defined for $0 < \alpha \leq 1$ by

$$\Gamma_\alpha z = B_\omega^*T^*(\tau - \cdot)(\alpha I + \mathcal{W})^{-1}z = G^*(\alpha I + GG^*)^{-1}z, \quad (3.12)$$

is an approximate inverse for the right of the operator G in the sense that

$$\lim_{\alpha \rightarrow 0} G\Gamma_\alpha = I. \quad (3.13)$$

4 Controllability of the Semilinear System

In this section we shall prove the main result of this paper, the interior approximate controllability of the Semilinear Impulsive Heat Equation given by (1.1), which is equivalent to prove the approximate controllability of the system (2.4). To this end, for all $z_0 \in Z$ and $u \in C([0, \tau]; U)$ the initial value problem

$$\begin{cases} z' = -Az + B_\omega u + f^e(t, z, u), & t \in (0, \tau], t \neq t_k, \quad z \in Z \\ z(0) = z_0, \\ z(t_k^+) = z(t_k^-) + I_k^e(t, z(t_k), u(t_k)), & k = 1, 2, 3, \dots, p. \end{cases} \quad (4.1)$$

admits only one mild solution given by

$$z_u(t) = T(t)z_0 + \int_0^t T(t-s)B_\omega u(s)ds \quad (4.2)$$

$$\begin{aligned} &+ \int_0^t T(t-s)f^e(s, z_u(s), u(s))ds \\ &+ \sum_{0 < t_k < t} T(t-t_k)I_k^e(t_k, z(t_k), u(t_k)), \quad t \in [0, \tau]. \end{aligned} \quad (4.3)$$

Now, we are ready to present and prove the main result of this paper, which is the interior approximate controllability of the semilinear impulsive heat equation (1.1). We shall define the operator $\mathcal{K}^\alpha : PC([0, \tau]; Z) \times C([0, \tau]; U) \rightarrow PC([0, \tau]; Z) \times C([0, \tau]; U)$ by the following formula:

$$(y, v) = (\mathcal{K}_1^\alpha(z, u), \mathcal{K}_2^\alpha(z, u)) = \mathcal{K}^\alpha(z, u)$$

where

$$\begin{aligned} y(t) &= \mathcal{K}_1^\alpha(z, u)(t) = T(t)z_0 + \int_0^t T(t-s)B_\omega(\Gamma_\alpha \mathcal{L}(z, u))(s)ds \\ &+ \int_0^t T(t-s)f^e(s, z(s), u(s))ds + \sum_{0 < t_k < t} T(t-t_k)I_k^e(t_k, z(t_k), u(t_k)), \end{aligned} \quad (4.4)$$

and

$$v(t) = \mathcal{K}_2^\alpha(z, u)(t) = (\Gamma_\alpha \mathcal{L}(z, u))(t) = B_\omega^* T^*(\tau - t)(\alpha I + \mathcal{W})^{-1} \mathcal{L}(z, u), \quad (4.5)$$

with $\mathcal{L} : PC([0, \tau]; Z) \times C([0, \tau]; U) \rightarrow Z$ is given by

$$\begin{aligned} \mathcal{L}(z, u) &= z_1 - T(\tau)z_0 - \int_0^\tau T(\tau-s)f^e(s, z(s), u(s))ds \\ &- \sum_{0 < t_k < \tau} T(\tau-t_k)I_k^e(t_k, z(t_k), u(t_k)). \end{aligned} \quad (4.6)$$

Theorem 4.1 *The nonlinear system (1.1) is approximately controllable on $[0, \tau]$. Moreover, a sequence of controls steering the system (1.1) from initial state z_0 to an ϵ -neighborhood of the final state z_1 at time $\tau > 0$ is given by*

$$u_\alpha(t) = B_\omega^* T^*(\tau - t)(\alpha I + \mathcal{W})^{-1} \mathcal{L}(z_\alpha, u_\alpha),$$

and the error of this approximation $E_\alpha z$ is given by

$$E_\alpha z = \alpha(\alpha I + \mathcal{W})^{-1} \mathcal{L}(z_\alpha, u_\alpha),$$

where

$$\begin{aligned} z_\alpha(t) &= T(t)z_0 + \int_0^t T(t-s)B_\omega u_\alpha(s)ds \\ &+ \int_0^t T(t-s)f^e(s, z_\alpha(s), u_\alpha(s))ds \\ &+ \sum_{0 < t_k < t} T(t-t_k)I_k^e(t_k, z_\alpha(t_k), u_\alpha(t_k)), \quad t \in [0, \tau]. \end{aligned}$$

Proof We shall prove this Theorem by claims. Before we note that $\|B_\omega\| = 1$ and $\|T(t)\| \leq e^{-\lambda_1 t}$, $t \geq 0$.

Claim 1. The operator \mathcal{K}^α is continuous. In fact, it is enough to prove that the operators:

$$\mathcal{K}_1^\alpha : PC([0, \tau]; Z) \times C([0, \tau]; U) \rightarrow PC([0, \tau]; Z)$$

and

$$\mathcal{K}_2^\alpha : PC([0, \tau]; Z) \times C([0, \tau]; U) \rightarrow C([0, \tau]; U),$$

define above are continuous. The continuity of \mathcal{K}_1^α follows from the continuity of the nonlinear functions $f^\alpha(t, z, u)$, $I_k^e(t, z, u)$ and the following estimate

$$\begin{aligned} \|\mathcal{K}_1^\alpha(z, u)(t) - \mathcal{K}_1^\alpha(w, v)(t)\| &\leq \int_0^t e^{-\lambda_1(t-s)} \|(\alpha I + \mathcal{W})^{-1}\| \|\mathcal{L}(z, u) - \mathcal{L}(w, v)\| ds \\ &+ \int_0^t e^{-\lambda_1(t-s)} \|f^e(s, z(s), u(s)) - f^e(s, w(s), v(s))\| ds \\ &+ \sum_{0 < t_k < t} e^{-\lambda_1(t-t_k)} \|I_k^e(t_k, z(t_k), u(t_k)) - I_k^e(t_k, w(t_k), v(t_k))\|. \end{aligned}$$

On the other hand,

$$\begin{aligned} \|\mathcal{L}(z, u) - \mathcal{L}(w, v)\| &\leq \tau \sup_{s \in [0, \tau]} \|f^e(s, z(s), u(s)) - f^e(s, w(s), v(s))\| \\ &+ \sum_{0 < t_k < \tau} e^{-\lambda_1(t-t_k)} \|I_k^e(t_k, z(t_k), u(t_k)) - I_k^e(t_k, w(t_k), v(t_k))\|. \end{aligned}$$

Therefore,

$$\begin{aligned} \|\mathcal{K}_1^\alpha(z, u) - \mathcal{K}_1^\alpha(w, v)\| &\leq L_1 \sup_{s \in [0, \tau]} \|f^e(s, z(s), u(s)) - f^e(s, w(s), v(s))\| \\ &+ L_2 \sum_{0 < t_k < \tau} \|I_k^e(t_k, z(t_k), u(t_k)) - I_k^e(t_k, w(t_k), v(t_k))\|. \end{aligned}$$

where $L_1 = \tau(\tau\|(\alpha I + \mathcal{W})^{-1}\| + 1)$ and $L_2 = (1 + \tau\|(\alpha I + \mathcal{W})^{-1}\|)$.

The continuity of the operator \mathcal{K}_2^α follows from the continuity of the operators \mathcal{L} and Γ_α define above.

Claim 2. The operator \mathcal{K}^α is compact. In fact, let D be a bounded subset of $PC(J; Z) \times C(J; U)$. It follows that $\forall (z, u) \in D$, we have

$$\begin{aligned} \|f^e(\cdot, z, u)\| &\leq L_3, & \|(\alpha I + \mathcal{W})^{-1}\mathcal{L}(z, u)\| &\leq L_4, \\ \|\mathcal{L}(z, u)\| &\leq L_5, & \|I_k^e(\cdot, z, u)\| &\leq l_k, \quad k = 1, 2, \dots, p. \end{aligned}$$

Therefore, $\mathcal{K}(D)$ is uniformly bounded.

Now, consider the following estimate:

$$\begin{aligned} \|\mathcal{K}^\alpha(z, u)(t_2) - \mathcal{K}^\alpha(z, u)(t_1)\| &= \|\mathcal{K}_1^\alpha(z, u)(t_2) - \mathcal{K}_1^\alpha(z, u)(t_1)\| \\ &+ \|\mathcal{K}_2^\alpha(z, u)(t_2) - \mathcal{K}_2^\alpha(z, u)(t_1)\|. \end{aligned}$$

Without lose of generality we assume that $0 < t_1 < t_2$. On the other hand we have:

$$\begin{aligned}
 & \|\mathcal{K}_1^\alpha(z, u)(t_2) - \mathcal{K}_1^\alpha(z, u)(t_1)\| \leq \|T(t_2) - T(t_1)\| \|z_0\| \\
 & + \int_0^{t_1} \|T(t_2 - s) - T(t_1 - s)\| \|\mathcal{L}(z, u)(s)\| ds \\
 & + \int_{t_1}^{t_2} \|T(t_2 - s)\| \|\mathcal{L}(z, u)(s)\| ds \\
 & + \int_0^{t_1} \|T(t_2 - s) - T(t_1 - s)\| \|f^e(s, z(s), u(s))\| ds \\
 & + \int_{t_1}^{t_2} \|T(t_2 - s)\| \|f^e(s, z(s), u(s))\| ds \\
 & + \sum_{0 < t_k < t_1} \|T(t_2 - t_k) - T(t_1 - t_k)\| \|I_k^e(t_k, z(t_k), u(t_k))\| \\
 & + \sum_{t_1 < t_k < t_2} \|T(t_2 - t_k) I_k^e(t_k, z(t_k), u(t_k))\|,
 \end{aligned}$$

and

$$\|\mathcal{K}_2^\alpha(z, u)(t_2) - \mathcal{K}_2^\alpha(z, u)(t_1)\| \leq \|T^*(\tau - t_2) - T^*(\tau - t_1)\| \|(\alpha I + \mathcal{W})^{-1} \mathcal{L}(z, u)\|.$$

On the other hand, since $T(t)$ is a compact operator for $t > 0$, then from [17] we know that the function $0 < t \rightarrow T(t)$ is uniformly continuous. So,

$$\lim_{|t_2 - t_1| \rightarrow 0} \|T(t_2) - T(t_1)\| = 0.$$

Consequently, if we take a sequence $\{\phi_j : j = 1, 2, \dots\}$ on $\mathcal{K}^\alpha(D)$, this sequence is uniformly bounded and equicontinuous on the interval $[0, t_1]$ and, by Arzela theorem, there is a subsequence $\{\phi_j^1 : j = 1, 2, \dots\}$ of $\{\phi_j : j = 1, 2, \dots\}$, which is uniformly convergent on $[0, t_1]$.

Consider the sequence $\{\phi_j^1 : j = 1, 2, \dots\}$ on the interval $(t_1, t_2]$. On this interval the sequence $\{\phi_j^1 : j = 1, 2, \dots\}$ is uniformly bounded and equicontinuous, and for the same reason, it has a subsequence $\{\phi_j^2\}$ uniformly convergent on $[0, t_2]$.

Continuing this process for the intervals $(t_2, t_3]$, $(t_3, t_4]$, \dots , $(t_p, \tau]$, we see that the sequence $\{\phi_j^{p+1} : j = 1, 2, \dots\}$ converges uniformly on the interval $[0, \tau]$. This means that $\overline{\mathcal{K}^\alpha(D)}$ is compact, which implies that the operator \mathcal{K}^α is compact.

Claim 3.

$$\lim_{\|(z, u)\| \rightarrow \infty} \frac{\|\mathcal{K}^\alpha(z, u)\|}{\|(z, u)\|} = 0,$$

where $\|(z, u)\| = \|z\| + \|u\|$ is the norm in the space $PC([0, \tau]; Z) \times C(0, \tau; Z)$. In fact, consider the following estimates:

$$\|\mathcal{L}(z, u)\| \leq M_1 + M_2 \{\bar{a}_0 \|z\|^{\alpha_0} + \bar{b}_0 \|u\|^{\beta_0} + \bar{c}_0\} + M_3 \sum_{0 < t_k < \tau} \{\bar{a}_k \|z\|^{\alpha_k} + \bar{b}_k \|u\|^{\beta_k} + \bar{c}_k\},$$

where

$$M_1 = \|z_1\| + e^{-\lambda_1\tau}\|z_0\|, \quad M_2 = \frac{1}{-\lambda_1}(e^{-\lambda_1\tau} - 1) \quad \text{and} \quad M_3 = e^{-\lambda_1\tau}.$$

$$\begin{aligned} \|\mathcal{K}_2^\alpha(z, u)\| \leq & M_3 M_1 \|(\alpha I + \mathcal{W})^{-1}\| + M_3 M_2 \|(\alpha I + \mathcal{W})^{-1}\| \{\bar{a}_0 \|z\|^{\alpha_0} + \bar{b}_0 \|u\|^{\beta_0} + \bar{c}_0\} \\ & + M_3 M_2 \|(\alpha I + \mathcal{W})^{-1}\| \sum_{0 < t_k < \tau} \{\bar{a}_k \|z\|^{\alpha_k} + \bar{b}_k \|u\|^{\beta_k} + \bar{c}_k\}. \end{aligned}$$

and

$$\begin{aligned} \|\mathcal{K}_1^\alpha(z, u)\| \leq & M_3 \{\|z_0\| + M_1 M_2 \|(\alpha I + \mathcal{W})^{-1}\|\} \\ & + M_2 \{1 + M_2 M_3 \|(\alpha I + \mathcal{W})^{-1}\|\} \{\bar{a}_0 \|z\|^{\alpha_0} + \bar{b}_0 \|u\|^{\beta_0} + \bar{c}_0\} \\ & + M_3 \{1 + M_2 M_3 \|(\alpha I + \mathcal{W})^{-1}\|\} \sum_{0 < t_k < \tau} \{\bar{a}_k \|z\|^{\alpha_k} + \bar{b}_k \|u\|^{\beta_k} + \bar{c}_k\}. \end{aligned}$$

Therefore,

$$\begin{aligned} \|\|\mathcal{K}^\alpha(z, u)\|\| &= \|\mathcal{K}_1^\alpha(z, u)\| + \|\mathcal{K}_2^\alpha(z, u)\| \leq M_4 \\ &+ \{M_3 M_2 \|(\alpha I + \mathcal{W})^{-1}\|\} \{1 + 2M_2\} \{\bar{a}_0 \|z\|^{\alpha_0} + \bar{b}_0 \|u\|^{\beta_0} + \bar{c}_0\} \\ &+ \{M_3 M_2 \|(\alpha I + \mathcal{W})^{-1}\|\} \{1 + M_3\} + M_3 \sum_{0 < t_k < \tau} \{\bar{a}_k \|z\|^{\alpha_k} + \bar{b}_k \|u\|^{\beta_k} + \bar{c}_k\}, \end{aligned}$$

where M_4 is given by:

$$M_4 = M_3 \{\|z_0\| + (M_2 + 1)M_1 \|(\alpha I + \mathcal{W})^{-1}\|\}.$$

Hence

$$\begin{aligned} \frac{\|\|\mathcal{K}^\alpha(z, u)\|\|}{\|\|(z, u)\|\|} &\leq \frac{M_4}{\|z\| + \|u\|} \\ &+ \{M_3 M_2 \|(\alpha I + \mathcal{W})^{-1}\|\} \{1 + M_2\} \\ &\quad \times \{\bar{a}_0 \|z\|^{\alpha_0-1} + \bar{b}_0 \|u\|^{\beta_0-1} + \frac{\bar{c}_0}{\|z\| + \|u\|}\} \\ &+ \{M_3 M_2 \|(\alpha I + \mathcal{W})^{-1}\|\} \{1 + M_3\} + M_3 \times \\ &\quad \sum_{0 < t_k < \tau} \{\bar{a}_k \|z\|^{\alpha_k-1} + \bar{b}_k \|u\|^{\beta_k-1} + \frac{\bar{c}_k}{\|z\| + \|u\|}\}, \end{aligned}$$

and

$$\lim_{\|\|(z, u)\|\| \rightarrow \infty} \frac{\|\|\mathcal{K}^\alpha(z, u)\|\|}{\|\|(z, u)\|\|} = 0. \quad (4.7)$$

Claim 4. The operator \mathcal{K}^α has a fixed point. In fact, for a fixed $0 < \rho < 1$, there exists $R > 0$ big enough such that

$$\|\|\mathcal{K}^\alpha(z, u)\|\| \leq \rho \|\|(z, u)\|\|, \quad \|\|(z, u)\|\| = R.$$

Hence, if we denote by $B(0, R)$ the ball of center zero and radius $R > 0$, we get that $\mathcal{K}^\alpha(\partial B(0, R)) \subset B(0, R)$. Since \mathcal{K}^α is compact and maps the sphere $\partial B(0, R)$ into the interior of the ball $B(0, R)$, we can apply Rothe's fixed point Theorem 1.1 to ensure the existence of a fixed point $(z_\alpha, u_\alpha) \in B(0, R) \subset PC([0, \tau]; Z) \times C([0, \tau]; U)$ such that

$$(z_\alpha, u_\alpha) = \mathcal{K}^\alpha(z_\alpha, u_\alpha). \quad (4.8)$$

Claim 5. The sequence $\{(z_\alpha, u_\alpha)\}_{\alpha \in (0,1]}$ is bounded. In fact, for the purpose of contradiction, let us assume that $\{(z_\alpha, u_\alpha)\}_{\alpha \in (0,1]}$ is unbounded. Then, there exists a subsequence $\{(z_{\alpha_n}, u_{\alpha_n})\}_{\alpha_n \in (0,1]} \subset \{(z_\alpha, u_\alpha)\}_{\alpha \in (0,1]}$ such that

$$\lim_{n \rightarrow \infty} \|(z_{\alpha_n}, u_{\alpha_n})\| = \infty.$$

On the other hand, from (4.7) we know for all $\alpha \in (0, 1]$ that

$$\lim_{n \rightarrow \infty} \frac{\|\mathcal{K}^\alpha(z_{\alpha_n}, u_{\alpha_n})\|}{\|(z_{\alpha_n}, u_{\alpha_n})\|} = 0.$$

Particularly, we have the following situation:

$$\begin{array}{ccccccc} \frac{\|\mathcal{K}^{\alpha_1}(z_{\alpha_1}, u_{\alpha_1})\|}{\|(z_{\alpha_1}, u_{\alpha_1})\|} & \frac{\|\mathcal{K}^{\alpha_1}(z_{\alpha_2}, u_{\alpha_2})\|}{\|(z_{\alpha_2}, u_{\alpha_2})\|} & \frac{\|\mathcal{K}^{\alpha_1}(z_{\alpha_3}, u_{\alpha_3})\|}{\|(z_{\alpha_3}, u_{\alpha_3})\|} & \cdots & \frac{\|\mathcal{K}^{\alpha_1}(z_{\alpha_n}, u_{\alpha_n})\|}{\|(z_{\alpha_n}, u_{\alpha_n})\|} & \rightarrow & 0. \\ \frac{\|\mathcal{K}^{\alpha_2}(z_{\alpha_1}, u_{\alpha_1})\|}{\|(z_{\alpha_1}, u_{\alpha_1})\|} & \frac{\|\mathcal{K}^{\alpha_2}(z_{\alpha_2}, u_{\alpha_2})\|}{\|(z_{\alpha_2}, u_{\alpha_2})\|} & \frac{\|\mathcal{K}^{\alpha_2}(z_{\alpha_3}, u_{\alpha_3})\|}{\|(z_{\alpha_3}, u_{\alpha_3})\|} & \cdots & \frac{\|\mathcal{K}^{\alpha_2}(z_{\alpha_n}, u_{\alpha_n})\|}{\|(z_{\alpha_n}, u_{\alpha_n})\|} & \rightarrow & 0. \\ \vdots & \vdots & \vdots & \vdots & \vdots & & \\ \vdots & \vdots & \vdots & \vdots & \vdots & & \\ \frac{\|\mathcal{K}^{\alpha_k}(z_{\alpha_1}, u_{\alpha_1})\|}{\|(z_{\alpha_1}, u_{\alpha_1})\|} & \frac{\|\mathcal{K}^{\alpha_k}(z_{\alpha_2}, u_{\alpha_2})\|}{\|(z_{\alpha_2}, u_{\alpha_2})\|} & \frac{\|\mathcal{K}^{\alpha_k}(z_{\alpha_3}, u_{\alpha_3})\|}{\|(z_{\alpha_3}, u_{\alpha_3})\|} & \cdots & \frac{\|\mathcal{K}^{\alpha_k}(z_{\alpha_n}, u_{\alpha_n})\|}{\|(z_{\alpha_n}, u_{\alpha_n})\|} & \rightarrow & 0. \end{array}$$

Now, applying Cantor's diagonalization process, we obtain that

$$\lim_{n \rightarrow \infty} \frac{\|\mathcal{K}^{\alpha_n}(z_{\alpha_n}, u_{\alpha_n})\|}{\|(z_{\alpha_n}, u_{\alpha_n})\|} = 0,$$

and from (4.8) we have that

$$\frac{\|\mathcal{K}^{\alpha_n}(z_{\alpha_n}, u_{\alpha_n})\|}{\|(z_{\alpha_n}, u_{\alpha_n})\|} = 1,$$

which is evidently a contradiction. Then, the claim is true and there exists $\gamma > 0$ such that

$$\|(z_{\alpha_n}, u_{\alpha_n})\| \leq \gamma, \quad (0 < \alpha \leq 1).$$

Therefore, without loss of generality, we can assume that the sequence $\mathcal{L}(z_\alpha, u_\alpha)$ converges to $y \in Z$. So, if

$$u_\alpha = \Gamma_\alpha \mathcal{L}(z_\alpha, u_\alpha) = G^*(\alpha I + GG^*)^{-1} \mathcal{L}(z_\alpha, u_\alpha).$$

Then,

$$\begin{aligned} Gu_\alpha &= G\Gamma_\alpha \mathcal{L}(z_\alpha, u_\alpha) = GG^*(\alpha I + GG^*)^{-1} \mathcal{L}(z_\alpha, u_\alpha) \\ &= (\alpha I + GG^* - \alpha I)(\alpha I + GG^*)^{-1} \mathcal{L}(z_\alpha, u_\alpha) \\ &= \mathcal{L}(z_\alpha, u_\alpha) - \alpha(\alpha I + GG^*)^{-1} \mathcal{L}(z_\alpha, u_\alpha). \end{aligned}$$

Hence,

$$Gu_\alpha - \mathcal{L}(z_\alpha, u_\alpha) = -\alpha(\alpha I + GG^*)^{-1} \mathcal{L}(z_\alpha, u_\alpha).$$

To conclude the proof of this Theorem, it enough to prove that

$$\lim_{\alpha \rightarrow 0} \{-\alpha(\alpha I + GG^*)^{-1}\} \mathcal{L}(z_\alpha, u_\alpha) = 0.$$

From Lemma 3.1.d) we get that

$$\begin{aligned} \lim_{\alpha \rightarrow 0} \{\alpha(\alpha I + GG^*)^{-1} \mathcal{L}(z_\alpha, u_\alpha)\} &= \lim_{\alpha \rightarrow 0} \alpha(\alpha I + GG^*)^{-1} y \\ &\quad + \lim_{\alpha \rightarrow 0} \alpha(\alpha I + GG^*)^{-1} (\mathcal{L}(z_\alpha, u_\alpha) - y) \\ &= \lim_{\alpha \rightarrow 0} -\alpha(\alpha I + GG^*)^{-1} (\mathcal{L}(z_\alpha, u_\alpha) - y) \end{aligned}$$

On the other hand, from Proposition 3.1, we get that

$$\|\alpha(\alpha I + GG^*)^{-1} (\mathcal{L}(z_\alpha, u_\alpha) - y)\| \leq \|\mathcal{L}(z_\alpha, u_\alpha) - y\|.$$

Therefore, since $\mathcal{L}(z_\alpha, u_\alpha)$ converges to y , we get that

$$\lim_{\alpha \rightarrow 0} \{-\alpha(\alpha I + GG^*)^{-1} (\mathcal{L}(z_\alpha, u_\alpha) - y)\} = 0.$$

Consequently,

$$\lim_{\alpha \rightarrow 0} \{-\alpha(\alpha I + GG^*)^{-1} \mathcal{L}(z_\alpha, u_\alpha)\} = 0.$$

Then,

$$\lim_{\alpha \rightarrow 0} \{Gu_\alpha - \mathcal{L}(z_\alpha, u_\alpha)\} = 0.$$

Therefore,

$$\begin{aligned} \lim_{\alpha \rightarrow 0} \{T(\tau)z_0 + \int_0^\tau T(\tau - s)B_\omega u_\alpha(s)ds + \int_0^\tau T(\tau - s)f^e(s, z_\alpha(s), u_\alpha(s))ds \\ + \sum_{0 < t_k < \tau} T(\tau - t_k)I_k^e(z_\alpha(t_k), u_\alpha(t_k))\} = z_1, \end{aligned}$$

and the proof of the theorem is completed. \square

As a consequence of the foregoing theorem we can prove the following characterization:

Theorem 4.2 *The Impulsive Semilinear System (1.1) is approximately controllable if for all states z_0 and a final state z_1 and $\alpha \in (0, 1]$ the operator \mathcal{K}^α given by (4.4)-(4.6) has a fixed point and the sequence $\{\mathcal{L}(z_\alpha, u_\alpha)\}_{\alpha \in (0,1]}$ converges. i.e.,*

$$(z_\alpha, u_\alpha) = \mathcal{K}^\alpha(z_\alpha, u_\alpha),$$

$$\lim_{\alpha \rightarrow 0} \mathcal{L}(z_\alpha, u_\alpha) = y \in Z.$$

5 Final Remark

Our technique is simple and can be apply to those system involving compact semi-groups like some control system governed by diffusion processes. For example, the Benjamin -Bona-Mohany Equation, the strongly damped wave equations, beam equations, etc.

Example 5.1 *The original Benjamin -Bona-Mohany Equation is a non-linear one, in [16] the authors proved the approximate controllability of the linear part of this equation, which is the fundamental base for the study of the controllability of the non linear BBM equation. So, our next work is concerned with the controllability of non linear BBM equation*

$$\begin{cases} z_t - a\Delta z_t - b\Delta z = 1_\omega u(t, x) + f(t, z, u(t)), & t \in (0, \tau), & x \in \Omega, \\ z(t, x) = 0, & t \geq 0, & x \in \partial\Omega, \\ z(0, x) = z_0(x), & x \in \Omega, \\ z(t_k^+, x) = z(t_k^-, x) + I_k(t, z(t_k, x), u(t_k, x)), & x \in \Omega, \end{cases}$$

where $a \geq 0$ and $b > 0$ are constants, $k = 1, 2, \dots, p$, Ω is a bounded domain in \mathbb{R}^N ($N \geq 1$), $z_0 \in L_2(\Omega)$, ω is an open nonempty subset of Ω , 1_ω denotes the characteristic function of the set ω , the distributed control u belongs to $C([0, \tau]; L_2(\Omega; \mathbb{R}))$ and $f, I_k \in C([0, \tau] \times \mathbb{R} \times \mathbb{R}; \mathbb{R})$, $k = 1, 2, 3, \dots, p$.

Example 5.2 *We believe that this technique can be applied to prove the interior controllability of the strongly damped wave equation with Dirichlet boundary conditions*

$$\begin{cases} w_{tt} + \eta(-\Delta)^{1/2} w_t + \gamma(-\Delta)w = 1_\omega u(t, x) + f(t, w, w_t, u(t)), & \text{in } (0, \tau) \times \Omega, \\ w = 0, & \text{in } (0, \tau) \times \partial\Omega, \\ w(0, x) = w_0(x), \quad w_t(0, x) = w_1(x), & \text{in } \Omega, \\ w(t_k^+, x) = w(t_k^-, x) + I_k^1(t, w(t_k, x), w_t(t_k, x), u(t_k, x)), & x \in \Omega, \\ w_t(t_k^+, x) = w_t(t_k^-, x) + I_k^2(t, w(t_k, x), w_t(t_k, x), u(t_k, x)), & x \in \Omega, \end{cases}$$

in the space $Z_{1/2} = D((-\Delta)^{1/2}) \times L_2(\Omega)$, $k = 1, 2, \dots, p$, Ω is a bounded domain in \mathbb{R}^N ($N \geq 1$), ω is an open nonempty subset of Ω , 1_ω denotes the characteristic function of the set ω , the distributed control $u \in C([0, \tau]; L_2(\Omega))$, η, γ are positive numbers and $f, I_k^1, I_k^2 \in C([0, \tau] \times \mathbb{R} \times \mathbb{R}; \mathbb{R})$, $k = 1, 2, 3, \dots, p$.

Example 5.3 *Another example where this technique may be applied is a partial differential equations modeling the structural damped vibrations of a string or a beam:*

$$\begin{cases} y_{tt} - 2\beta\Delta y_t + \Delta^2 y = 1_\omega u(t, x) + f(t, y, y_t, u(t)), & \text{on } (0, \tau) \times \Omega, \\ y = \Delta y = 0, & \text{on } (0, \tau) \times \partial\Omega, \\ y(0, x) = y_0(x), \quad y_t(0, x) = y_1(x), & \text{in } \Omega, \\ y(t_k^+, x) = y(t_k^-, x) + I_k^1(t, y(t_k, x), y_t(t_k, x), u(t_k, x)), & x \in \Omega, \\ y_t(t_k^+, x) = y_t(t_k^-, x) + I_k^2(t, y(t_k, x), y_t(t_k, x), u(t_k, x)), & x \in \Omega, \end{cases}$$

where Ω is a bounded domain in \mathbb{R}^n , ω is an open nonempty subset of Ω , 1_ω denotes the characteristic function of the set ω , the distributed control $u \in C([0, \tau]; L_2(\Omega))$ and $y_0 \in H^2(\Omega) \cap H_0^1, y_1 \in L_2(\Omega)$.

Moreover, our result can be formulated in a more general setting. Indeed, we can consider the following semilinear evolution equation in a general Hilbert space Z

$$\begin{cases} \dot{z} = -Az + Bu(t) + f^e(t, z, u), & z \in Z, \quad t \in (0, \tau], \\ z(0) = z_0, \\ z(t_k^+) = z(t_k^-) + I_k^e(t_k, z(t_k), u(t_k)), & k = 1, 2, 3, \dots, p. \end{cases} \quad (5.1)$$

where $u \in C([0, \tau]; U)$, $U = Z$, $B_\omega : U \rightarrow Z$, $B_\omega u = 1_\omega u$ is a bounded linear operator, $I_k^e, f^e : [0, \tau] \times Z \times U \rightarrow Z$, $A : D(A) \subset Z \rightarrow Z$ is an unbounded linear operator in Z with the following spectral decomposition:

$$Az = \sum_{j=1}^{\infty} \lambda_j \sum_{k=1}^{\gamma_j} \langle z, \phi_{j,k} \rangle \phi_{j,k},$$

with the eigenvalues $0 < \lambda_1 < \lambda_2 < \dots < \dots \lambda_n \rightarrow \infty$ of A having finite multiplicity γ_j equal to the dimension of the corresponding eigenspaces, and $\{\phi_{j,k}\}$ is a complete orthonormal set of eigenfunctions of A . The operator $-A$ generates a strongly continuous compact semigroup $\{T_A(t)\}_{t \geq 0}$ given by

$$T_A(t)z = \sum_{j=1}^{\infty} e^{-\lambda_j t} \sum_{k=1}^{\gamma_j} \langle z, \phi_{j,k} \rangle \phi_{j,k}.$$

The control $u \in C([0, \tau]; U)$, with $U = Z$, $B : Z \rightarrow Z$ is a linear and bounded operator (linear and continuous) and the functions $f^e, I_k^e : [0, \tau] \times Z \times U \rightarrow Z$ are smooth enough and

$$\|f^e(t, z, u)\|_Z \leq \tilde{a}_0 \|z\|_Z^{\alpha_0} + \tilde{b}_0 \|u\|_Z^{\beta_0} + \tilde{c}_0 \quad (5.2)$$

$$\|I_k^e(t, z, u)\|_Z \leq \tilde{a}_k \|z\|_Z^{\alpha_k} + \tilde{b}_k \|u\|_Z^{\beta_k} + \tilde{c}_k, \quad k = 1, 2, 3, \dots, p. \quad (5.3)$$

In this case the characteristic function set is a particular operator B , and the following theorem is a generalization of Theorem 4.1.

Theorem 5.1 *If vectors $B^* \phi_{j,k}$ are linearly independent in Z , then the system (5.1) is approximately controllable on $[0, \tau]$.*

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On the solutions of a class of nonlinear functional integral equations in space $C[0, a]$

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ABSTRACT: The principal aim of this paper is to give sufficient conditions for solvability of a class of some nonlinear functional integral equations in the space of continuous functions defined on interval $[0, a]$. The main tool used in our study is associated with the technique of measures of noncompactness. We give also some examples satisfying the conditions of our main theorem but not satisfying the conditions in [8].

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1 Introduction

Nonlinear integral equations are an important part of nonlinear analysis. It is caused by the fact that this theory is frequently applicable in other branches of mathematics and mathematical physics, engineering, economics, biology as well in describing problems connected with real world, [5]. The measure of noncompactness and theory of integral equations are rapidly developing with the help of tools in functional analysis, topology and fixed-point theory. Many articles in the field of functional integral equations give different conditions for the existence of the solutions of some nonlinear functional integral equations. A. Aghajani and Y. Jalilian in [1], J. Banaś and K. Sadarangani in [3], Zeqing Liu et al. in [11] and so on are some of these. The following equation has been considered in [6] :

$$x(t) = f(t, x(\alpha(t))) \int_0^1 u(t, s, x(s)) ds,$$

for $t \in [0, 1]$. K. Maleknejad et al. in [7] and [8] studied the existence of the solutions of the following equations

$$x(t) = f(t, x(\alpha(t))) \int_0^t u(t, s, x(s)) ds, \quad t \in [0, 1]$$

and

$$x(t) = g(t, x(t)) + f\left(t, \int_0^t u(t, s, x(s))ds, x(\alpha(t))\right), \quad t \in [0, a],$$

respectively. Then, İ. Özdemir et al. dealt with the following equation in [9] and [10]

$$x(t) = g(t, x(\beta(t))) + f(t, x(\alpha(t))) \int_0^{\varphi(t)} u(t, s, x(\gamma(s)))ds, \quad t \in [0, a].$$

In this paper, we consider the following nonlinear functional integral equation:

$$x(t) = g(t, x(\alpha(t))) + f\left(t, \int_0^{\varphi(t)} u(t, s, x(\gamma(s)))ds, x(\beta(t))\right) \quad (1)$$

for $t \in [0, a]$. Note that the mentioned equation has rather general form and contains as particular cases a lot of nonlinear integral equations of Volterra type.

In next section, we present some definitions and preliminaries results about the concept of measure of noncompactness. In final section, we give our main result concerning with the solvability of the integral equation (1) by applying Darbo fixed point theorem associated with the measure of noncompactness defined by J. Banaś and K. Goebel [2] and finally we present some examples to show that our result is applicable.

2 Notations, definitions and auxiliary facts

In this section, we give some notations, definitions and results which will be needed further on. Assume that $(E, \|\cdot\|)$ is an infinite Banach space with zero element θ . We write $B(x, r)$ to denote the closed ball centered at x with radius r and especially, we write B_r instead of $B(\theta, r)$. If X is a subset of E then the symbols \overline{X} and $\text{Conv } X$ stand for the closure and the convex closure of X , respectively. Moreover, let \mathfrak{M}_E indicates the family of all nonempty bounded subsets of E and \mathfrak{N}_E indicates the its subfamily of all relatively compact sets. Finally, the standard algebraic operations on sets are denoted by λX and $X + Y$, respectively.

We use the following definition of the measure of noncompactness, given in [2].

Definition 1 A mapping $\mu : \mathfrak{M}_E \rightarrow \mathbb{R}_+ = [0, +\infty)$ is said to be a measure of noncompactness in E if it satisfies the following conditions:

1. The family $\ker \mu = \{X \in \mathfrak{M}_E : \mu(X) = 0\}$ is nonempty and $\ker \mu \subset \mathfrak{N}_E$.
2. $X \subset Y \Rightarrow \mu(X) \leq \mu(Y)$.
3. $\mu(X) = \mu(\overline{X}) = \mu(\text{Conv } X)$.

4. $\mu(\lambda X + (1 - \lambda)Y) \leq \lambda\mu(X) + (1 - \lambda)\mu(Y)$ for $\lambda \in [0, 1]$.
5. If (X_n) is a sequence of closed sets from \mathfrak{M}_E such that $X_{n+1} \subset X_n$ ($n = 1, 2, \dots$) and if $\lim_{n \rightarrow \infty} \mu(X_n) = 0$, then the intersection set $\bigcap_{n=1}^{\infty} X_n$ is nonempty.

Now, let us suppose that M is nonempty subset of a Banach space E and $T : M \rightarrow E$ is a continuous operator which transforms bounded sets onto bounded ones. We say that T satisfies the Darbo condition (with a constant $k \geq 0$) with respect to measure of noncompactness μ if for any bounded subset X of M the inequality

$$\mu(TX) \leq k\mu(X)$$

holds. If T satisfies the Darbo condition with $k < 1$, then it is said to be a contraction with respect to μ , [4]. Now, we introduce the following Darbo type fixed point theorem.

Theorem 2 *Let C be a nonempty, closed, bounded and convex subset of the Banach space E , μ be a measure of noncompactness defined in E and let $F : C \rightarrow C$ be a continuous mapping. Assume that there exists a constant $k \in [0, 1)$ such that*

$$\mu(FX) \leq k\mu(X) \tag{2}$$

for any nonempty subset X of C . Then F has a fixed point in set C , [2].

As is known the family of all real valued and continuous functions defined on interval $[0, a]$ is a Banach space with the standart norm

$$\|x\| = \max \{|x(t)| : t \in [0, a]\}.$$

Let X be a fixed subset of $\mathfrak{M}_{C[0,a]}$. For $\varepsilon > 0$ and $x \in X$, by $\omega(x, \varepsilon)$ we denote the modulus of continuity of function x , i.e.,

$$\omega(x, \varepsilon) = \sup \{|x(t_1) - x(t_2)| : t_1, t_2 \in [0, a] \text{ and } |t_1 - t_2| \leq \varepsilon\}.$$

Furthermore let $\omega(X, \varepsilon)$ and $\omega_0(X)$ are defined by

$$\omega(X, \varepsilon) = \sup \{\omega(x, \varepsilon) : x \in X\},$$

and

$$\omega_0(X) = \lim_{\varepsilon \rightarrow 0} \omega(X, \varepsilon). \tag{3}$$

The authors have shown in [2] that function ω_0 is a measure of noncompactness in space $C[0, a]$.

3 The main result

First of all we write I to denote interval $[0, a]$ throughout this section. We study functional integral equation (1) with the following hypotheses.

- (a) Functions $\alpha, \beta : I \rightarrow I$, $\varphi : I \rightarrow \mathbb{R}_+$ and $\gamma : [0, C] \rightarrow I$ are continuous.
 (b) $g : I \times \mathbb{R} \rightarrow \mathbb{R}$ is continuous and there exists nonnegative constant k such that

$$|g(t, x_1) - g(t, x_2)| \leq k |x_1 - x_2|$$

for all $t \in I$ and $x_1, x_2 \in \mathbb{R}$.

- (c) $f : I \times \mathbb{R} \times \mathbb{R} \rightarrow \mathbb{R}$ is continuous and there exist nonnegative constants l and q such that

$$\begin{aligned} |f(t, x_1, y) - f(t, x_2, y)| &\leq l |x_1 - x_2|, \\ |f(t, x, y_1) - f(t, x, y_2)| &\leq q |y_1 - y_2| \end{aligned}$$

for all $t \in I$ and $x_1, x_2, y_1, y_2, x, y \in \mathbb{R}$.

- (d) $u : I \times [0, C] \times \mathbb{R} \rightarrow \mathbb{R}$ is continuous and there exist positive constants m, n and p such that

$$|u(t, s, x)| \leq m + n |x|^p$$

for all $t \in I$ and $s \in [0, C]$, $x \in \mathbb{R}$.

- (e) The inequality

$$M + N + Cl(m + n) + k + q < 1$$

holds, where C, M and N are the positive constants such that $\varphi(t) \leq C$, $|g(t, 0)| \leq M$ and $|f(t, 0, 0)| \leq N$ for all $t \in I$.

Theorem 3 Under assumptions (a) – (e) Eq.(1) has at least one solution in space $C[0, a]$.

Proof. We define the continuous function $h : [0, 1] \rightarrow \mathbb{R}$ such that

$$h(r) = (k + q - 1)r + Cnlr^p + Clm + M + N,$$

where p is the constant given in assumption (d). Then $h(0) > 0$ and $h(1) < 0$ by assumption (e). Continuity of h guarantees that there exists number $r_0 \in (0, 1)$ such that $h(r_0) = 0$. Now, we will prove that Eq.(1) has at least one solution $x = x(t)$ belonging to $B_{r_0} \subset C[0, a]$. We define operator T by

$$(Tx)(t) = g(t, x(\alpha(t))) + f\left(t, \int_0^{\varphi(t)} u(t, s, x(\gamma(s)))ds, x(\beta(t))\right), \quad x \in C[0, a].$$

Using the conditions of Theorem 3, we infer that Tx is continuous on I . For any $x \in B_{r_0}$, we have

$$\begin{aligned}
 |(Tx)(t)| &= \left| g(t, x(\alpha(t))) + f \left(t, \int_0^{\varphi(t)} u(t, s, x(\gamma(s))) ds, x(\beta(t)) \right) \right| \\
 &\leq |g(t, x(\alpha(t))) - g(t, 0)| + |g(t, 0)| \\
 &\quad + \left| f \left(t, \int_0^{\varphi(t)} u(t, s, x(\gamma(s))) ds, x(\beta(t)) \right) - f(t, 0, x(\beta(t))) \right| \\
 &\quad + |f(t, 0, x(\beta(t))) - f(t, 0, 0)| + |f(t, 0, 0)| \\
 &\leq k|x(\alpha(t))| + M + l \left| \int_0^{\varphi(t)} u(t, s, x(\gamma(s))) ds \right| + q|x(\beta(t))| + N \\
 &\leq k\|x\| + M + Cl(m + n\|x\|^p) + q\|x\| + N \\
 &\leq kr_0 + M + Cl(m + n(r_0)^p) + qr_0 + N \\
 &= h(r_0) + r_0 \\
 &= r_0.
 \end{aligned}$$

This result shows that operator T transforms ball B_{r_0} into itself. Now, we will prove that operator $T : B_{r_0} \rightarrow B_{r_0}$ is continuous. To do this, consider $\varepsilon > 0$ and any $x, y \in B_{r_0}$ such that $\|x - y\| \leq \varepsilon$. Then, we obtain the following inequalities by taking into account the assumptions of Theorem 3.

$$\begin{aligned}
 & |(Tx)(t) - (Ty)(t)| \\
 &= \left| g(t, x(\alpha(t))) + f \left(t, \int_0^{\varphi(t)} u(t, s, x(\gamma(s))) ds, x(\beta(t)) \right) \right. \\
 &\quad \left. - g(t, y(\alpha(t))) - f \left(t, \int_0^{\varphi(t)} u(t, s, y(\gamma(s))) ds, y(\beta(t)) \right) \right| \\
 &\leq |g(t, x(\alpha(t))) - g(t, y(\alpha(t)))| \\
 &\quad + \left| f \left(t, \int_0^{\varphi(t)} u(t, s, x(\gamma(s))) ds, x(\beta(t)) \right) - f \left(t, \int_0^{\varphi(t)} u(t, s, y(\gamma(s))) ds, x(\beta(t)) \right) \right| \\
 &\quad + \left| f \left(t, \int_0^{\varphi(t)} u(t, s, y(\gamma(s))) ds, x(\beta(t)) \right) - f \left(t, \int_0^{\varphi(t)} u(t, s, y(\gamma(s))) ds, y(\beta(t)) \right) \right| \\
 &\leq k|x(\alpha(t)) - y(\alpha(t))| + l \int_0^{\varphi(t)} |u(t, s, x(\gamma(s))) - u(t, s, y(\gamma(s)))| ds \\
 &\quad + q|x(\beta(t)) - y(\beta(t))| \\
 &\leq (k + q)\|x - y\| + Cl\omega_{u_3}(I, \varepsilon) \\
 &\leq (k + q)\varepsilon + Cl\omega_{u_3}(I, \varepsilon),
 \end{aligned} \tag{4}$$

where

$$\omega_{u_3}(I, \varepsilon) = \sup \{ |u(t, s, x) - u(t, s, y)| : t \in I, s \in [0, C], x, y \in [-r_0, r_0] \text{ and } |x - y| \leq \varepsilon \}.$$

On the other hand, from the uniform continuity of function $u = u(t, s, x)$ on set $I \times [0, C] \times [-r_0, r_0]$, we derive that $\omega_{u_3}(I, \varepsilon) \rightarrow 0$ as $\varepsilon \rightarrow 0$. Hence, estimate (4) proves that operator T is continuous on B_{r_0} . Moreover, we show that operator T satisfies (2) with respect to measure of noncompactness ω_0 given by (3). To do this, we choose a fixed arbitrary $\varepsilon > 0$. Let us consider $x \in X$ and $t_1, t_2 \in I$ with $|t_1 - t_2| \leq \varepsilon$, for any nonempty subset X of B_{r_0} . Then,

$$\begin{aligned} & |(Tx)(t_1) - (Tx)(t_2)| \\ &= \left| g(t_1, x(\alpha(t_1))) + f \left(t_1, \int_0^{\varphi(t_1)} u(t_1, s, x(\gamma(s))) ds, x(\beta(t_1)) \right) \right. \\ & \quad \left. - g(t_2, x(\alpha(t_2))) - f \left(t_2, \int_0^{\varphi(t_2)} u(t_2, s, x(\gamma(s))) ds, x(\beta(t_2)) \right) \right| \\ &\leq |g(t_1, x(\alpha(t_1))) - g(t_2, x(\alpha(t_1)))| + |g(t_2, x(\alpha(t_1))) - g(t_2, x(\alpha(t_2)))| \\ & \quad + \left| f \left(t_1, \int_0^{\varphi(t_1)} u(t_1, s, x(\gamma(s))) ds, x(\beta(t_1)) \right) \right. \\ & \quad \left. - f \left(t_1, \int_0^{\varphi(t_2)} u(t_1, s, x(\gamma(s))) ds, x(\beta(t_1)) \right) \right| \\ & \quad + \left| f \left(t_1, \int_0^{\varphi(t_2)} u(t_1, s, x(\gamma(s))) ds, x(\beta(t_1)) \right) \right. \\ & \quad \left. - f \left(t_1, \int_0^{\varphi(t_2)} u(t_2, s, x(\gamma(s))) ds, x(\beta(t_1)) \right) \right| \\ & \quad + \left| f \left(t_1, \int_0^{\varphi(t_2)} u(t_2, s, x(\gamma(s))) ds, x(\beta(t_1)) \right) \right. \\ & \quad \left. - f \left(t_1, \int_0^{\varphi(t_2)} u(t_2, s, x(\gamma(s))) ds, x(\beta(t_2)) \right) \right| \\ & \quad + \left| f \left(t_1, \int_0^{\varphi(t_2)} u(t_2, s, x(\gamma(s))) ds, x(\beta(t_2)) \right) \right. \\ & \quad \left. - f \left(t_2, \int_0^{\varphi(t_2)} u(t_2, s, x(\gamma(s))) ds, x(\beta(t_2)) \right) \right| \end{aligned}$$

$$\begin{aligned}
 &\leq \omega_g(I, \varepsilon) + k|x(\alpha(t_1)) - x(\alpha(t_2))| \\
 &\quad + l \left| \int_0^{\varphi(t_1)} u(t_1, s, x(\gamma(s))) ds - \int_0^{\varphi(t_2)} u(t_1, s, x(\gamma(s))) ds \right| \\
 &\quad + l \int_0^{\varphi(t_2)} |u(t_1, s, x(\gamma(s))) - u(t_2, s, x(\gamma(s)))| ds + q|x(\beta(t_1)) - x(\beta(t_2))| \quad (5) \\
 &\quad + \omega_f(I, \varepsilon) \\
 &\leq \omega_g(I, \varepsilon) + k\omega(x, \omega(\alpha, \varepsilon)) + l \left| - \int_{\varphi(t_1)}^{\varphi(t_2)} u(t_1, s, x(\gamma(s))) ds \right| + Cl\omega_{u_1}(I, \varepsilon) \\
 &\quad + q\omega(x, \omega(\beta, \varepsilon)) + \omega_f(I, \varepsilon) \\
 &\leq \omega_g(I, \varepsilon) + k\omega(x, \omega(\alpha, \varepsilon)) + l\omega(\varphi, \varepsilon)(m + n(r_0)^p) \\
 &\quad + Cl\omega_{u_1}(I, \varepsilon) + q\omega(x, \omega(\beta, \varepsilon)) + \omega_f(I, \varepsilon), \quad (6)
 \end{aligned}$$

where

$$\begin{aligned}
 \omega_g(I, \varepsilon) &= \sup \{|g(t, x) - g(t', x)| : t, t' \in I, x \in [-r_0, r_0] \text{ and } |t - t'| \leq \varepsilon\}, \\
 \omega_{u_1}(I, \varepsilon) &= \sup \{|u(t, s, x) - u(t', s, x)| : \\
 &\quad t, t' \in I, s \in [0, C], x \in [-r_0, r_0] \text{ and } |t - t'| \leq \varepsilon\}, \\
 \omega_f(I, \varepsilon) &= \sup \{|f(t, s, x) - f(t', s, x)| : \\
 &\quad t, t' \in I, s \in [-A, A], x \in [-r_0, r_0] \text{ and } |t - t'| \leq \varepsilon\}
 \end{aligned}$$

and $A = C(m + n(r_0)^p)$. Also,

$$\omega(\alpha_i, \varepsilon) = \sup \{|\alpha_i(t) - \alpha_i(t')| : t, t' \in I \text{ and } |t - t'| \leq \varepsilon\},$$

for $i = 1, 2, 3, 4$ such that $\alpha_1 = \alpha$, $\alpha_2 = \beta$, $\alpha_3 = \varphi$ and $\alpha_4 = x$. Thus, by using estimate (6) we get

$$\begin{aligned}
 \omega(TX, \varepsilon) &\leq \omega_g(I, \varepsilon) + k\omega(X, \omega(\alpha, \varepsilon)) + l\omega(\varphi, \varepsilon)(m + n(r_0)^p) \\
 &\quad + Cl\omega_{u_1}(I, \varepsilon) + q\omega(X, \omega(\beta, \varepsilon)) + \omega_f(I, \varepsilon). \quad (7)
 \end{aligned}$$

Since functions α , β and φ are uniformly continuous on set I by condition **(a)**, we deduce that $\omega(\alpha, \varepsilon) \rightarrow 0$, $\omega(\beta, \varepsilon) \rightarrow 0$ and $\omega(\varphi, \varepsilon) \rightarrow 0$ as $\varepsilon \rightarrow 0$. Similarly, we have $\omega_g(I, \varepsilon) \rightarrow 0$, $\omega_f(I, \varepsilon) \rightarrow 0$ and $\omega_{u_1}(I, \varepsilon) \rightarrow 0$ as $\varepsilon \rightarrow 0$ since the functions g , f and u are uniformly continuous on sets $I \times [-r_0, r_0]$, $I \times [-A, A] \times [-r_0, r_0]$ and $I \times [0, C] \times [-r_0, r_0]$, respectively. Hence, (7) yields that

$$\omega_0(TX) \leq (k + q)\omega_0(X).$$

Thus, since $k + q < 1$ from condition **(e)**, we get that operator T is a contraction on ball B_{r_0} with respect to measure of noncompactness ω_0 . Therefore, Theorem 2 gives that operator T has at least one fixed point in B_{r_0} . Consequently, nonlinear functional integral equation (1) has at least one continuous solution in $B_{r_0} \subset C[0, a]$. This step completes the proof of Theorem 3. ■

4 Examples

In this section, we shall discuss some examples to illustrate the applicability of Theorem 3.

Example 4 We examine the nonlinear functional integral equation having the form

$$x(t) = \frac{2 + x(t^2)}{56 + t^3} + \frac{2^t + t^2}{21} + \frac{x(\sqrt{t}) + 1}{9 + t^4} + \frac{2}{10 + t} \int_0^t \frac{\cos t + \sqrt{|x(s^2)|}}{2 + \ln(t + 1) + s^2 t^3} ds, \quad (8)$$

for $t \in I = [0, 1]$. Put

$$\begin{aligned} \beta(t) &= \sqrt{t}, \quad \varphi(t) = t, \quad \alpha(t) = t^2, \quad \gamma(s) = s^2, \\ g(t, x) &= \frac{2 + x}{56 + t^3}, \quad u(t, s, x) = \frac{\cos t + \sqrt{|x|}}{2 + \ln(t + 1) + s^2 t^3}, \\ f(t, v, z) &= \frac{2^t + t^2}{21} + \frac{z + 1}{9 + t^4} + \frac{2v}{10 + t} \end{aligned}$$

and

$$k = \frac{1}{56}, \quad M = \frac{1}{28}, \quad l = \frac{1}{5}, \quad q = \frac{1}{9}, \quad N = \frac{17}{70}, \quad C = 1, \quad m = n = p = \frac{1}{2}.$$

It can be easily seen that conditions (d) and (e) are verified. On the other hand, it is easy to verify that the other assumptions of Theorem 3 hold. Therefore, Theorem 3 guarantees that Eq.(8) has at least one solution $x = x(t) \in C[0, 1]$.

Example 5 Let us consider the nonlinear functional integral equation of the form

$$x(t) = g(t, x(t)) + f\left(t, \int_0^t u(t, s, x(s)) ds, x(\beta(t))\right), \quad (9)$$

where g, f, u and β are the functions in Example 4. Since the conditions of Theorem 3 hold, Eq.(9) has at least one solution $x = x(t) \in C[0, 1]$ from Theorem 3.

Since

$$|u(t, s, x)| = \left| \frac{\cos t + \sqrt{|x|}}{2 + \ln(t + 1) + s^2 t^3} \right| \leq \frac{1}{2} + \frac{1}{2} |x|^{\frac{1}{2}}$$

for all $t, s \in [0, 1]$ and $x \in \mathbb{R}$, condition (H3) in [8] doesn't hold. Hence, the result presented in [8] is inapplicable to integral Eq.(9).

Example 6 Consider the following nonlinear functional integral equation:

$$\begin{aligned} x(t) &= \frac{1 + x(\sqrt{t})}{32 + t} + \frac{\cos(\sqrt{1 + t^2})}{8} + \frac{x(t^2)}{8 + t^2} \\ &+ \frac{4}{16 + t} \int_0^{t^2} \frac{\exp(-t) + x(s^2)}{1 + t^2 + s \sin^2(1 + x^2(s^2))} ds. \end{aligned} \quad (10)$$

We will look for solvability of this equation in space $C[0, 1]$. Put

$$\begin{aligned}\alpha(t) &= \sqrt{t}, \quad \varphi(t) = \beta(t) = t^2, \quad \gamma(s) = s^2, \\ g(t, x) &= \frac{1+x}{32+t}, \quad u(t, s, x) = \frac{\exp(-t) + x}{1+t^2 + s \sin^2(1+x^2)}, \\ f(t, v, z) &= \frac{\cos(\sqrt{1+t^2})}{8} + \frac{z}{8+t^2} + \frac{4v}{16+t}\end{aligned}$$

and

$$k = M = \frac{1}{32}, \quad l = \frac{1}{4}, \quad q = N = \frac{1}{8}, \quad C = m = n = p = 1.$$

One can see easily that conditions (d) and (e) of Theorem 3 are verified. On the other hand, it is easy to verify that the other assumptions of Theorem 3 hold. Therefore, Theorem 3 guarantees that Eq.(10) has at least one solution $x = x(t) \in C[0, 1]$.

Example 7 Let us consider the nonlinear functional integral equation given as

$$x(t) = g(t, x(t)) + f\left(t, \int_0^t u(t, s, x(s))ds, x(\beta(t))\right), \quad (11)$$

where g, f, u and β are the functions in Example 6. It is clear that the conditions of Theorem 3 satisfy. So, Eq.(11) has at least one solution $x = x(t) \in C[0, 1]$ by Theorem 3.

Since

$$\kappa = \frac{1}{4}, \quad \lambda = \frac{1}{8}, \quad a = n = 1$$

and $\kappa > \frac{1-\lambda}{2+2an}$ in condition (H4), the result in [8] is inapplicable to integral Eq.(11).

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Some seminormed difference sequence spaces defined by a Musielak-Orlicz function over n -normed spaces

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ABSTRACT: In the present paper we introduced some seminormed difference sequence spaces combining lacunary sequences and Musielak-Orlicz function $\mathcal{M} = (M_k)$ over n -normed spaces and examine some topological properties and inclusion relations between resulting sequence spaces.

AMS Subject Classification: *40A05, 46A45.*

Keywords and Phrases: *Paranorm space, Difference sequence spaces, Orlicz function, Musielak-Orlicz function, n -normed spaces*

1 Introduction and preliminaries

The concept of 2-normed spaces was initially developed by Gähler [6] in the mid of 1960's, while that of n -normed spaces one can see in Misiak [17]. Since then, many others have studied this concept and obtained various results, see Gunawan ([7], [8]) and Gunawan and Mashadi [9] and many others. Let $n \in \mathbb{N}$ and X be a linear space over the field \mathbb{K} , where \mathbb{K} is field of real or complex numbers of dimension d , where $d \geq n \geq 2$. A real valued function $\|\cdot, \dots, \cdot\|$ on X^n satisfying the following four conditions:

1. $\|x_1, x_2, \dots, x_n\| = 0$ if and only if x_1, x_2, \dots, x_n are linearly dependent in X ;
2. $\|x_1, x_2, \dots, x_n\|$ is invariant under permutation;
3. $\|\alpha x_1, x_2, \dots, x_n\| = |\alpha| \|x_1, x_2, \dots, x_n\|$ for any $\alpha \in \mathbb{K}$, and
4. $\|x + x', x_2, \dots, x_n\| \leq \|x, x_2, \dots, x_n\| + \|x', x_2, \dots, x_n\|$

is called a n -norm on X , and the pair $(X, \|\cdot, \dots, \cdot\|)$ is called a n -normed space over the field \mathbb{K} .

For example, we may take $X = \mathbb{R}^n$ being equipped with the Euclidean n -norm

$\|x_1, x_2, \dots, x_n\|_E$ = the volume of the n -dimensional paralleliped spanned by the vectors x_1, x_2, \dots, x_n which may be given explicitly by the formula

$$\|x_1, x_2, \dots, x_n\|_E = |\det(x_{ij})|,$$

where $x_i = (x_{i1}, x_{i2}, \dots, x_{in}) \in \mathbb{R}^n$ for each $i = 1, 2, \dots, n$. Let $(X, \|\cdot, \dots, \cdot\|)$ be an n -normed space of dimension $d \geq n \geq 2$ and $\{a_1, a_2, \dots, a_n\}$ be linearly independent set in X . Then the following function $\|\cdot, \dots, \cdot\|_\infty$ on X^{n-1} defined by

$$\|x_1, x_2, \dots, x_{n-1}\|_\infty = \max\{\|x_1, x_2, \dots, x_{n-1}, a_i\| : i = 1, 2, \dots, n\}$$

defines an $(n-1)$ -norm on X with respect to $\{a_1, a_2, \dots, a_n\}$.

A sequence (x_k) in a n -normed space $(X, \|\cdot, \dots, \cdot\|)$ is said to converge to some $L \in X$ if

$$\lim_{k \rightarrow \infty} \|x_k - L, z_1, \dots, z_{n-1}\| = 0 \text{ for every } z_1, \dots, z_{n-1} \in X.$$

A sequence (x_k) in a n -normed space $(X, \|\cdot, \dots, \cdot\|)$ is said to be Cauchy if

$$\lim_{k, i \rightarrow \infty} \|x_k - x_i, z_1, \dots, z_{n-1}\| = 0 \text{ for every } z_1, \dots, z_{n-1} \in X.$$

If every Cauchy sequence in X converges to some $L \in X$, then X is said to be complete with respect to the n -norm. Any complete n -normed space is said to be n -Banach space.

An Orlicz function M is a function, which is continuous, non-decreasing and convex with $M(0) = 0$, $M(x) > 0$ for $x > 0$ and $M(x) \rightarrow \infty$ as $x \rightarrow \infty$.

Lindenstrauss and Tzafriri [12] used the idea of Orlicz function to define the following sequence space. Let w be the space of all real or complex sequences $x = (x_k)$, then

$$\ell_M = \left\{ x \in w : \sum_{k=1}^{\infty} M\left(\frac{|x_k|}{\rho}\right) < \infty \right\}$$

which is called as an Orlicz sequence space. The space ℓ_M is a Banach space with the norm

$$\|x\| = \inf \left\{ \rho > 0 : \sum_{k=1}^{\infty} M\left(\frac{|x_k|}{\rho}\right) \leq 1 \right\}.$$

It is shown in [12] that every Orlicz sequence space ℓ_M contains a subspace isomorphic to ℓ_p ($p \geq 1$). The Δ_2 -condition is equivalent to $M(Lx) \leq kLM(x)$ for all values of $x \geq 0$, and for $L > 1$. A sequence $\mathcal{M} = (M_k)$ of Orlicz functions is called a Musielak-Orlicz function see ([16], [20]). A sequence $\mathcal{N} = (N_k)$ is defined by

$$N_k(v) = \sup\{|v|u - (M_k) : u \geq 0\}, \quad k = 1, 2, \dots$$

is called the complementary function of a Musielak-Orlicz function \mathcal{M} . For a given Musielak-Orlicz function \mathcal{M} , the Musielak-Orlicz sequence space $t_{\mathcal{M}}$ and its subspace $h_{\mathcal{M}}$ are defined as follows

$$t_{\mathcal{M}} = \left\{ x \in w : I_{\mathcal{M}}(cx) < \infty \text{ for some } c > 0 \right\},$$

$$h_{\mathcal{M}} = \left\{ x \in w : I_{\mathcal{M}}(cx) < \infty \text{ for all } c > 0 \right\},$$

where $I_{\mathcal{M}}$ is a convex modular defined by

$$I_{\mathcal{M}}(x) = \sum_{k=1}^{\infty} (M_k)(x_k), x = (x_k) \in t_{\mathcal{M}}.$$

We consider $t_{\mathcal{M}}$ equipped with the Luxemburg norm

$$\|x\| = \inf \left\{ k > 0 : I_{\mathcal{M}}\left(\frac{x}{k}\right) \leq 1 \right\}$$

or equipped with the Orlicz norm

$$\|x\|^0 = \inf \left\{ \frac{1}{k} \left(1 + I_{\mathcal{M}}(kx) \right) : k > 0 \right\}.$$

Let ℓ_{∞} , c and c_0 denotes the sequence spaces of bounded, convergent and null sequences $x = (x_k)$ respectively. A sequence $x = (x_k) \in \ell_{\infty}$ is said to be almost convergent if all Banach limits of $x = (x_k)$ coincide. In [13], it was shown that

$$\hat{c} = \left\{ x = (x_k) : \lim_{n \rightarrow \infty} \frac{1}{n} \sum_{k=1}^n x_{k+s} \text{ exists, uniformly in } s \right\}.$$

In ([14], [15]) Maddox defined strongly almost convergent sequences. Recall that a sequence $x = (x_k)$ is strongly almost convergent if there is a number L such that

$$\lim_{n \rightarrow \infty} \frac{1}{n} \sum_{k=1}^n |x_{k+s} - L| = 0, \text{ uniformly in } s.$$

By a lacunary sequence $\theta = (i_r)$, $r = 0, 1, 2, \dots$, where $i_0 = 0$, we shall mean an increasing sequence of non-negative integers $g_r = (i_r - i_{r-1}) \rightarrow \infty$ ($r \rightarrow \infty$). The intervals determined by θ are denoted by $I_r = (i_{r-1}, i_r]$ and the ratio i_r/i_{r-1} will be denoted by q_r . The space of lacunary strongly convergent sequences N_{θ} was defined by Freedman et. al [5] as follows:

$$N_{\theta} = \left\{ x = (x_k) : \lim_{r \rightarrow \infty} \frac{1}{g_r} \sum_{k \in I_r} |x_k - L| = 0 \text{ for some } L \right\}.$$

The notion of difference sequence spaces was introduced by Kızmaz [11], who studied the difference sequence spaces $l_{\infty}(\Delta)$, $c(\Delta)$ and $c_0(\Delta)$. The notion was further generalized by Et and Çolak [4] by introducing the spaces $l_{\infty}(\Delta^n)$, $c(\Delta^n)$ and $c_0(\Delta^n)$. Let m, n be non-negative integers, then for $Z = c, c_0$ and l_{∞} , we have sequence spaces

$$Z(\Delta_n^m) = \{ x = (x_k) \in w : (\Delta_n^m x_k) \in Z \}$$

for $Z = c, c_0$ and l_{∞} where $\Delta_n^m x = (\Delta_n^m x_k) = (\Delta_n^{m-1} x_k - \Delta_n^{m-1} x_{k-1})$ and $\Delta^0 x_k = x_k$ for all $k \in \mathbb{N}$, which is equivalent to the following binomial representation

$$\Delta_n^m x_k = \sum_{v=0}^m (-1)^v \binom{m}{v} x_{k+nv}.$$

Taking $n = 1$, we get the spaces $l_\infty(\Delta^m)$, $c(\Delta^m)$ and $c_0(\Delta^m)$ studied by Et and Çolak [4]. Taking $m = 1$, $n = 1$, we get the spaces $l_\infty(\Delta)$, $c(\Delta)$ and $c_0(\Delta)$ studied by Kızılmaz [11]. Let X be a linear metric space. A function $p : X \rightarrow \mathbb{R}$ is called paranorm, if

1. $p(x) \geq 0$ for all $x \in X$,
2. $p(-x) = p(x)$ for all $x \in X$,
3. $p(x + y) \leq p(x) + p(y)$ for all $x, y \in X$,
4. if (λ_n) is a sequence of scalars with $\lambda_n \rightarrow \lambda$ as $n \rightarrow \infty$ and (x_n) is a sequence of vectors with $p(x_n - x) \rightarrow 0$ as $n \rightarrow \infty$, then $p(\lambda_n x_n - \lambda x) \rightarrow 0$ as $n \rightarrow \infty$.

A paranorm p for which $p(x) = 0$ implies $x = 0$ is called total paranorm and the pair (X, p) is called a total paranormed space. It is well known that the metric of any linear metric space is given by some total paranorm (see [26], Theorem 10.4.2, pp. 183). For more details about sequence spaces see ([1], [2], [3], [18], [19], [21], [22], [23], [24], [25]) and references therein.

Let M be an Orlicz function and $p = (p_k)$ be any sequence of strictly positive real numbers. Güngör and Et [10] defined the following sequence spaces:

$$[c, M, p](\Delta^m) = \left\{ x = (x_k) : \lim_{n \rightarrow \infty} \frac{1}{n} \sum_{k=1}^n \left[M \left(\frac{|\Delta^m x_{k+s} - L|}{\rho} \right) \right]^{p_k} = 0, \right.$$

uniformly in s , for some $\rho > 0$ and $L > 0$ \},

$$[c, M, p]_0(\Delta^m) = \left\{ x = (x_k) : \lim_{n \rightarrow \infty} \frac{1}{n} \sum_{k=1}^n \left[M \left(\frac{|\Delta^m x_{k+s}|}{\rho} \right) \right]^{p_k} = 0, \right.$$

uniformly in s , for some $\rho > 0$ \},

$$[c, M, p]_\infty(\Delta^m) = \left\{ x = (x_k) : \sup_{n, s} \frac{1}{n} \sum_{k=1}^n \left[M \left(\frac{|\Delta^m x_{k+s}|}{\rho} \right) \right]^{p_k} < \infty \text{ for some } \rho > 0 \right\}.$$

Let $\mathcal{M} = (M_k)$ be a Musielak-Orlicz function and X be a seminormed space, seminormed by $q = (q_k)$. Let $p = (p_k)$ be any bounded sequence of positive real numbers and $u = (u_k)$ be any sequence of strictly positive real numbers. In this paper we define the following sequence spaces:

$$[c, \mathcal{M}, p, \|\cdot, \dots, \cdot\|]^\theta(\Delta_n^m, u, q) =$$

$$\left\{ x = (x_k) \in w(n-X) : \lim_{r \rightarrow \infty} \frac{1}{g_r} \sum_{k \in I_r} \left[M_k \left(q_k \left(\left\| \frac{u_k \Delta_n^m x_{k+s} - L}{\rho}, z_1, \dots, z_{n-1} \right\| \right) \right) \right]^{p_k} = 0, \right.$$

uniformly in s , $z_1, \dots, z_{n-1} \in X$ for some L and $\rho > 0$ \},

$$\begin{aligned}
& [c, \mathcal{M}, p, \|\cdot, \dots, \cdot\|]_0^\theta(\Delta_n^m, u, q) = \\
& \left\{ x = (x_k) \in w(n-X) : \lim_{r \rightarrow \infty} \frac{1}{g_r} \sum_{k \in I_r} \left[M_k \left(q_k \left(\left\| \frac{u_k \Delta_n^m x_{k+s}}{\rho}, z_1, \dots, z_{n-1} \right\| \right) \right) \right]^{p_k} = 0, \right. \\
& \quad \left. \text{uniformly in } s, z_1, \dots, z_{n-1} \in X \text{ for some } \rho > 0 \right\}, \\
& [c, \mathcal{M}, p, \|\cdot, \dots, \cdot\|]_\infty^\theta(\Delta_n^m, u, q) = \\
& \left\{ x = (x_k) \in w(n-X) : \sup_{r,s} \frac{1}{g_r} \sum_{k=1}^n \left[M_k \left(q_k \left(\left\| \frac{u_k \Delta_n^m x_{k+s}}{\rho}, z_1, \dots, z_{n-1} \right\| \right) \right) \right]^{p_k} < \infty, \right. \\
& \quad \left. \text{uniformly in } s, z_1, \dots, z_{n-1} \in X \text{ for some } \rho > 0 \right\}.
\end{aligned}$$

When, $\mathcal{M}(x) = x$, we get

$$\begin{aligned}
& [c, p, \|\cdot, \dots, \cdot\|]^\theta(\Delta_n^m, u, q) = \\
& \left\{ x = (x_k) \in w(n-X) : \lim_{r \rightarrow \infty} \frac{1}{g_r} \sum_{k \in I_r} \left(q_k \left(\left\| \frac{u_k \Delta_n^m x_{k+s} - L}{\rho}, z_1, \dots, z_{n-1} \right\| \right) \right)^{p_k} = 0, \right. \\
& \quad \left. \text{uniformly in } s, z_1, \dots, z_{n-1} \in X \text{ for some } L \text{ and } \rho > 0 \right\}, \\
& [c, p, \|\cdot, \dots, \cdot\|]_0^\theta(\Delta_n^m, u, q) = \\
& \left\{ x = (x_k) \in w(n-X) : \lim_{r \rightarrow \infty} \frac{1}{g_r} \sum_{k \in I_r} \left(q_k \left(\left\| u_k \frac{\Delta_n^m x_{k+s}}{\rho}, z_1, \dots, z_{n-1} \right\| \right) \right)^{p_k} = 0, \right. \\
& \quad \left. \text{uniformly in } s, z_1, \dots, z_{n-1} \in X \text{ for some } \rho > 0 \right\}, \\
& [c, p, \|\cdot, \dots, \cdot\|]_\infty^\theta(\Delta_n^m, u, q) = \\
& \left\{ x = (x_k) \in w(n-X) : \sup_{r,s} \frac{1}{g_r} \sum_{k=1}^n \left(q_k \left(\left\| \frac{u_k \Delta_n^m x_{k+s}}{\rho}, z_1, \dots, z_{n-1} \right\| \right) \right)^{p_k} < \infty, \right. \\
& \quad \left. z_1, \dots, z_{n-1} \in X \text{ for some } \rho > 0 \right\}.
\end{aligned}$$

If we take $p_k = 1$ for all k , then we get

$$\begin{aligned}
& [c, \mathcal{M}, \|\cdot, \dots, \cdot\|]^\theta(\Delta_n^m, u, q) = \\
& \left\{ x = (x_k) \in w(n-X) : \lim_{r \rightarrow \infty} \frac{1}{g_r} \sum_{k \in I_r} \left[M_k \left(q_k \left(\left\| \frac{u_k \Delta_n^m x_{k+s} - L}{\rho}, z_1, \dots, z_{n-1} \right\| \right) \right) \right] = 0, \right. \\
& \quad \left. \text{uniformly in } s, z_1, \dots, z_{n-1} \in X \text{ for some } L \text{ and } \rho > 0 \right\},
\end{aligned}$$

$$[c, \mathcal{M}, \|\cdot, \dots, \cdot\|]_0^\theta(\Delta_n^m, u, q) =$$

$$\left\{ x = (x_k) \in w(n-X) : \lim_{r \rightarrow \infty} \frac{1}{g_r} \sum_{k \in I_r} \left[M_k \left(q_k \left(\left\| \frac{u_k \Delta_n^m x_{k+s}}{\rho}, z_1, \dots, z_{n-1} \right\| \right) \right) \right] = 0, \right.$$

$$\left. \text{uniformly in } s, z_1, \dots, z_{n-1} \in X \text{ for some } \rho > 0 \right\},$$

$$[c, \mathcal{M}, \|\cdot, \dots, \cdot\|]_\infty^\theta(\Delta_n^m, u, q) =$$

$$\left\{ x = (x_k) \in w(n-X) : \sup_{r,s} \frac{1}{g_r} \sum_{k=1}^n \left[M_k \left(q_k \left(\left\| \frac{u_k \Delta_n^m x_{k+s}}{\rho}, z_1, \dots, z_{n-1} \right\| \right) \right) \right] < \infty, \right.$$

$$\left. z_1, \dots, z_{n-1} \in X \text{ for some } \rho > 0 \right\}.$$

The following inequality will be used throughout the paper. If $0 \leq p_k \leq \sup p_k = H$, $D = \max(1, 2^{H-1})$ then

$$|a_k + b_k|^{p_k} \leq D\{|a_k|^{p_k} + |b_k|^{p_k}\} \quad (1.1)$$

for all k and $a_k, b_k \in \mathbb{C}$. Also $|a|^{p_k} \leq \max(1, |a|^H)$ for all $a \in \mathbb{C}$.

The main aim of this paper is to study some seminormed difference sequence spaces defined by a Musielak-Orlicz function over n -normed space. We also make an effort to study some topological properties and prove some inclusion relations between these spaces.

2 Main Results

Theorem 2.1 *Let $\mathcal{M} = (M_k)$ be a Musielak-Orlicz function, $p = (p_k)$ be a bounded sequence of positive real numbers and $u = (u_k)$ be any sequence of strictly positive real numbers. Then the spaces $[c, \mathcal{M}, p, \|\cdot, \dots, \cdot\|]^\theta(\Delta_n^m, u, q)$, $[c, \mathcal{M}, p, \|\cdot, \dots, \cdot\|]_0^\theta(\Delta_n^m, u, q)$ and $[c, \mathcal{M}, p, \|\cdot, \dots, \cdot\|]_\infty^\theta(\Delta_n^m, u, q)$ are linear over the field of complex numbers \mathbb{C} .*

Proof. Let $x = (x_k)$, $y = (y_k) \in [c, \mathcal{M}, p, \|\cdot, \dots, \cdot\|]_0^\theta(\Delta_n^m, u, q)$ and $\alpha, \beta \in \mathbb{C}$. Then there exists positive numbers ρ_1 and ρ_2 such that

$$\lim_{r \rightarrow \infty} \frac{1}{g_r} \sum_{k \in I_r} \left[M_k \left(q_k \left(\left\| \frac{u_k \Delta_n^m x_{k+s}}{\rho_1}, z_1, \dots, z_{n-1} \right\| \right) \right) \right]^{p_k} = 0, \text{ uniformly in } s,$$

and

$$\lim_{r \rightarrow \infty} \frac{1}{g_r} \sum_{k \in I_r} \left[M_k \left(q_k \left(\left\| \frac{u_k \Delta_n^m x_{k+s}}{\rho_2}, z_1, \dots, z_{n-1} \right\| \right) \right) \right]^{p_k} = 0, \text{ uniformly in } s.$$

Let $\rho_3 = \max(2|\alpha|\rho_1, 2|\beta|\rho_2)$. Since $\mathcal{M} = (M_k)$ is non-decreasing convex function, by using inequality (1.1), we have

$$\begin{aligned} & \frac{1}{g_r} \sum_{k \in I_r} \left[M_k \left(q_k \left(\left\| \frac{u_k \Delta_n^m (\alpha x_{k+s} + \beta y_{k+s})}{\rho_3}, z_1, \dots, z_{n-1} \right\| \right) \right) \right]^{p_k} \\ & \leq D \frac{1}{g_r} \sum_{k \in I_r} \frac{1}{2^{p_k}} \left[M_k \left(q_k \left(\left\| \frac{u_k \Delta_n^m (x_{k+s})}{\rho_1}, z_1, \dots, z_{n-1} \right\| \right) \right) \right]^{p_k} \\ & \quad + D \frac{1}{g_r} \sum_{k \in I_r} \frac{1}{2^{p_k}} \left[M_k \left(q_k \left(\left\| \frac{u_k \Delta_n^m (y_{k+s})}{\rho_2}, z_1, \dots, z_{n-1} \right\| \right) \right) \right]^{p_k} \\ & \leq D \frac{1}{g_r} \sum_{k \in I_r} \left[M_k \left(q_k \left(\left\| \frac{u_k \Delta_n^m (x_{k+s})}{\rho_1}, z_1, \dots, z_{n-1} \right\| \right) \right) \right]^{p_k} \\ & \quad + D \frac{1}{g_r} \sum_{k \in I_r} \left[M_k \left(q_k \left(\left\| \frac{u_k \Delta_n^m (y_{k+s})}{\rho_1}, z_1, \dots, z_{n-1} \right\| \right) \right) \right]^{p_k} \\ & \longrightarrow 0 \text{ as } r \longrightarrow \infty, \text{ uniformly in } s. \end{aligned}$$

Thus, we have $\alpha x + \beta y \in [c, \mathcal{M}, p, \|\cdot, \dots, \cdot\|]_0^\theta(\Delta_n^m, u, q)$.

Hence $[c, \mathcal{M}, p, \|\cdot, \dots, \cdot\|]_0^\theta(\Delta_n^m, u, q)$ is a linear space. Similarly, we can prove that $[c, \mathcal{M}, p, \|\cdot, \dots, \cdot\|]^\theta(\Delta_n^m, u, q)$ and $[c, \mathcal{M}, p, \|\cdot, \dots, \cdot\|]_\infty^\theta(\Delta_n^m, u, q)$ are linear spaces. ■

Theorem 2.2 For any Musielak-Orlicz function $\mathcal{M} = (M_k)$, $p = (p_k)$ be a bounded sequence of positive real numbers and $u = (u_k)$ be any sequence of strictly positive real numbers, the space $[c, \mathcal{M}, p, \|\cdot, \dots, \cdot\|]_0^\theta(\Delta_n^m, u, q)$ is a topological linear space paranormed by

$$g(x) = \inf \left\{ \rho^{\frac{pr}{K}} : \left(\frac{1}{g_r} \sum_{k \in I_r} \left[M_k \left(q_k \left(\left\| \frac{u_k \Delta_n^m x_{k+s}}{\rho}, z_1, \dots, z_{n-1} \right\| \right) \right) \right]^{p_k} \right)^{\frac{1}{K}} \leq 1, r, s \in \mathbb{N} \right\},$$

where $K = \max(1, \sup_k p_k < \infty)$.

Proof. Clearly $g(x) \geq 0$ for $x = (x_k) \in [c, \mathcal{M}, \|\cdot, \dots, \cdot\|]_0^\theta(\Delta_n^m, u, q)$. Since $M_k(0) = 0$, we get $g(0) = 0$. Again, if $g(x) = 0$, then

$$\inf \left\{ \rho^{\frac{pr}{K}} : \left(\frac{1}{g_r} \sum_{k \in I_r} \left[M_k \left(q_k \left(\left\| \frac{u_k \Delta_n^m x_{k+s}}{\rho}, z_1, \dots, z_{n-1} \right\| \right) \right) \right]^{p_k} \right)^{\frac{1}{K}} \leq 1, r, s \in \mathbb{N} \right\} = 0.$$

This implies that for a given $\epsilon > 0$, there exists some $\rho_\epsilon (0 < \rho_\epsilon < \epsilon)$ such that

$$\left(\frac{1}{g_r} \sum_{k \in I_r} \left[M_k \left(q_k \left(\left\| \frac{u_k \Delta_n^m x_{k+s}}{\rho_\epsilon}, z_1, \dots, z_{n-1} \right\| \right) \right) \right]^{p_k} \right)^{\frac{1}{K}} \leq 1.$$

Thus

$$\begin{aligned}
& \left(\frac{1}{g_r} \sum_{k \in I_r} \left[M_k \left(q_k \left(\left\| \frac{u_k \Delta_n^m x_{k+s}}{\epsilon}, z_1, \dots, z_{n-1} \right\| \right) \right)^{p_k} \right]^{\frac{1}{K}} \\
& \leq \left(\frac{1}{g_r} \sum_{k \in I_r} \left[M_k \left(q_k \left(\left\| \frac{\Delta_n^m x_{k+s}}{\rho_\epsilon}, z_1, \dots, z_{n-1} \right\| \right) \right)^{p_k} \right]^{\frac{1}{K}} \\
& \leq 1,
\end{aligned}$$

for each r and s . Suppose that $x_k \neq 0$ for each $k \in N$. This implies that $\Delta_n^m x_{k+s} \neq 0$, for each $k, s \in N$. Let $\epsilon \rightarrow 0$, then $q_k \left(\left\| \frac{u_k \Delta_n^m x_{k+s}}{\epsilon}, z_1, \dots, z_{n-1} \right\| \right) \rightarrow \infty$. It follows that

$$\left(\frac{1}{g_r} \sum_{k \in I_r} \left[M_k \left(q_k \left(\left\| \frac{u_k \Delta_n^m x_{k+s}}{\epsilon}, z_1, \dots, z_{n-1} \right\| \right) \right)^{p_k} \right]^{\frac{1}{K}} \rightarrow \infty,$$

which is a contradiction. Therefore, $\Delta_n^m x_{k+s} = 0$ for each k and s and thus $x_k = 0$ for each $k \in N$. Let $\rho_1 > 0$ and $\rho_2 > 0$ be such that

$$\left(\frac{1}{g_r} \sum_{k \in I_r} \left[M_k \left(q_k \left(\left\| \frac{u_k \Delta_n^m x_{k+s}}{\rho_1}, z_1, \dots, z_{n-1} \right\| \right) \right)^{p_k} \right]^{\frac{1}{K}} \leq 1$$

and

$$\left(\frac{1}{g_r} \sum_{k \in I_r} \left[M_k \left(q_k \left(\left\| \frac{u_k \Delta_n^m x_{k+s}}{\rho_2}, z_1, \dots, z_{n-1} \right\| \right) \right)^{p_k} \right]^{\frac{1}{K}} \leq 1$$

for each r and s . Let $\rho = \rho_1 + \rho_2$. Then, by Minkowski's inequality, we have

$$\begin{aligned}
& \left(\frac{1}{g_r} \sum_{k \in I_r} \left[M_k \left(q_k \left(\left\| \frac{u_k \Delta_n^m (x_{k+s} + y_{k+s})}{\rho}, z_1, \dots, z_{n-1} \right\| \right) \right)^{p_k} \right]^{\frac{1}{K}} \\
& \leq \left(\sum_{k \in I_r} \left[\frac{\rho_1}{\rho_1 + \rho_2} M_k \left(q_k \left(\left\| \frac{u_k \Delta_n^m (x_{k+s})}{\rho_1}, z_1, \dots, z_{n-1} \right\| \right) \right) \right. \right. \\
& \quad \left. \left. + \frac{\rho_2}{\rho_1 + \rho_2} M_k \left(q_k \left(\left\| \frac{u_k \Delta_n^m (y_{k+s})}{\rho_2}, z_1, \dots, z_{n-1} \right\| \right) \right) \right]^{p_k} \right)^{\frac{1}{K}} \\
& \leq \left(\frac{\rho_1}{\rho_1 + \rho_2} \right) \left(\frac{1}{g_r} \sum_{k \in I_r} \left[M_k \left(q_k \left(\left\| \frac{u_k \Delta_n^m (x_{k+s})}{\rho_1}, z_1, \dots, z_{n-1} \right\| \right) \right)^{p_k} \right]^{\frac{1}{K}} \right. \\
& \quad \left. + \left(\frac{\rho_2}{\rho_1 + \rho_2} \right) \left(\frac{1}{g_r} \sum_{k \in I_r} \left[M_k \left(q_k \left(\left\| \frac{u_k \Delta_n^m (y_{k+s})}{\rho_2}, z_1, \dots, z_{n-1} \right\| \right) \right)^{p_k} \right]^{\frac{1}{K}} \right) \right) \\
& \leq 1
\end{aligned}$$

Since $\rho's$ are non-negative, so we have

$$\begin{aligned}
& g(x+y) \\
&= \inf \left\{ \rho^{\frac{pr}{K}} : \left(\frac{1}{g_r} \sum_{k \in I_r} \left[M_k \left(q_k \left(\left\| \frac{u_k \Delta_n^m (x_{k+s} + y_{k+s})}{\rho} \right\|, z_1, \dots, z_{n-1} \right) \right) \right]^{p_k} \right)^{\frac{1}{K}} \leq 1, r, s \in \mathbb{N} \right\}, \\
&\leq \inf \left\{ \rho_1^{\frac{pr}{K}} : \left(\frac{1}{g_r} \sum_{k \in I_r} \left[M_k \left(q_k \left(\left\| \frac{u_k \Delta_n^m (x_{k+s})}{\rho_1} \right\|, z_1, \dots, z_{n-1} \right) \right) \right]^{p_k} \right)^{\frac{1}{K}} \leq 1, r, s \in \mathbb{N} \right\} \\
&+ \inf \left\{ \rho_2^{\frac{pr}{K}} : \left(\frac{1}{g_r} \sum_{k \in I_r} \left[M_k \left(q_k \left(\left\| \frac{\Delta_n^m (y_{k+s})}{\rho_2} \right\|, z_1, \dots, z_{n-1} \right) \right) \right]^{p_k} \right)^{\frac{1}{K}} \leq 1, r, s \in \mathbb{N} \right\}.
\end{aligned}$$

Therefore,

$$g(x+y) \leq g(x) + g(y).$$

Finally, we prove that the scalar multiplication is continuous. Let λ be any complex number. By definition,

$$g(\lambda x) = \inf \left\{ \rho^{\frac{pr}{K}} : \left(\frac{1}{g_r} \sum_{k \in I_r} \left[M_k \left(q_k \left(\left\| \frac{u_k \Delta_n^m \lambda x_{k+s}}{\rho} \right\|, z_1, \dots, z_{n-1} \right) \right) \right]^{p_k} \right)^{\frac{1}{K}} \leq 1, r, s \in \mathbb{N} \right\}.$$

Then

$$g(\lambda x) = \inf \left\{ (|\lambda|t)^{\frac{pr}{K}} : \left(\frac{1}{g_r} \sum_{k \in I_r} \left[M_k \left(q_k \left(\left\| \frac{\Delta_n^m x_{k+s}}{t} \right\|, z_1, \dots, z_{n-1} \right) \right) \right]^{p_k} \right)^{\frac{1}{K}} \leq 1, r, s \in \mathbb{N} \right\},$$

where $t = \frac{\rho}{|\lambda|}$. Since $|\lambda|^{pr} \leq \max(1, |\lambda|^{\sup pr})$, we have $g(\lambda x) \leq \max(1, |\lambda|^{\sup pr})$

$$\inf \left\{ t^{\frac{pr}{K}} : \left(\frac{1}{g_r} \sum_{k \in I_r} \left[M_k \left(q_k \left(\left\| \frac{u_k \Delta_n^m x_{k+s}}{t} \right\|, z_1, \dots, z_{n-1} \right) \right) \right]^{p_k} \right)^{\frac{1}{K}} \leq 1, r, s \in \mathbb{N} \right\}.$$

So, the fact that scalar multiplication is continuous follows from the above inequality. This completes the proof of the theorem. ■

Theorem 2.3 Let $\mathcal{M} = (M_k)$ be a Musielak-Orlicz function. If $\sup_k [M_k(x)]^{p_k} < \infty$ for all fixed $x > 0$, then $[c, \mathcal{M}, p, \|\cdot, \dots, \cdot\|]_0^\theta(\Delta_n^m, u, q) \subset [c, \mathcal{M}, p, \|\cdot, \dots, \cdot\|]_\infty^\theta(\Delta_n^m, u, q)$.

Proof. Let $x = (x_k) \in [c, \mathcal{M}, p, \|\cdot, \dots, \cdot\|]_0^\theta(\Delta_n^m, u, q)$. There exists some positive ρ_1 such that

$$\lim_{r \rightarrow \infty} \frac{1}{g_r} \sum_{k \in I_r} \left[M_k \left(q_k \left(\left\| \frac{\Delta_n^m x_{k+s}}{\rho_1} \right\|, z_1, \dots, z_{n-1} \right) \right) \right]^{p_k} = 0, \text{ uniformly in } s.$$

Define $\rho = 2\rho_1$. Since $\mathcal{M} = (M_k)$ is non-decreasing and convex, by using inequality(1.1), we have

$$\begin{aligned}
& \sup_{r,s} \frac{1}{g_r} \sum_{k \in I_r} \left[M_k \left(q_k \left(\left\| \frac{u_k \Delta_n^m x_{k+s}}{\rho}, z_1, \dots, z_{n-1} \right\| \right) \right) \right]^{p_k} \\
& \leq D \sup_{r,s} \frac{1}{g_r} \sum_{k \in I_r} \left[\frac{1}{2^{p_k}} M_k \left(q_k \left(\left\| \frac{u_k \Delta_n^m x_{k+s} - L}{\rho_1}, z_1, \dots, z_{n-1} \right\| \right) \right) \right]^{p_k} \\
& \quad + D \sup_{r,s} \frac{1}{g_r} \sum_{k \in I_r} \left[\frac{1}{2^{p_k}} M_k \left(q_k \left(\left\| \frac{L}{\rho_1}, z_1, \dots, z_{n-1} \right\| \right) \right) \right]^{p_k} \\
& \leq D \sup_{r,s} \frac{1}{g_r} \sum_{k \in I_r} \left[M_k \left(q_k \left(\left\| \frac{u_k \Delta_n^m x_{k+s} - L}{\rho_1}, z_1, \dots, z_{n-1} \right\| \right) \right) \right]^{p_k} \\
& \quad + D \sup_{r,s} \frac{1}{g_r} \sum_{k \in I_r} \left[M_k \left(q_k \left(\left\| \frac{L}{\rho_1}, z_1, \dots, z_{n-1} \right\| \right) \right) \right]^{p_k} \\
& < \infty.
\end{aligned}$$

Hence $x = (x_k) \in [c, \mathcal{M}, p, \|\cdot, \dots, \cdot\|]_{\infty}^{\theta}(\Delta_n^m, u, q)$. ■

Theorem 2.4 If $0 < \inf p_k = h \leq p_k \leq \sup p_k = H < \infty$ and $\mathcal{M} = (M_k)$, $\mathcal{M}' = (M'_k)$ be two Musielak-Orlicz functions satisfying Δ_2 -condition, then we have

- (i) $[c, \mathcal{M}', p, \|\cdot, \dots, \cdot\|]_0^{\theta}(\Delta_n^m, u, q) \subset [c, \mathcal{M} \circ \mathcal{M}', p, \|\cdot, \dots, \cdot\|]_0^{\theta}(\Delta_n^m, u, q)$,
- (ii) $[c, \mathcal{M}', p, \|\cdot, \dots, \cdot\|]^{\theta}(\Delta_n^m, u, q) \subset [c, \mathcal{M} \circ \mathcal{M}', p, \|\cdot, \dots, \cdot\|]^{\theta}(\Delta_n^m, u, q)$,
- (iii) $[c, \mathcal{M}', p, \|\cdot, \dots, \cdot\|]_{\infty}^{\theta}(\Delta_n^m, u, q) \subset [c, \mathcal{M} \circ \mathcal{M}', p, \|\cdot, \dots, \cdot\|]_{\infty}^{\theta}(\Delta_n^m, u, q)$.

Proof. Let $x = (x_k) \in [c, \mathcal{M}', p, \|\cdot, \dots, \cdot\|]_{\infty}^{\theta}(\Delta_n^m, u, q)$. Then we have

$$\lim_{r \rightarrow \infty} \frac{1}{g_r} \sum_{k \in I_r} \left[M'_k \left(q_k \left(\left\| \frac{u_k \Delta_n^m x_{k+s} - L}{\rho}, z_1, \dots, z_{n-1} \right\| \right) \right) \right]^{p_k} = 0,$$

uniformly in s for some L .

Let $\epsilon > 0$ and choose δ with $0 < \delta < 1$ such that $M_k(t) < \epsilon$ for $0 \leq t \leq \delta$. Let

$$y_{k,s} = M'_k \left(q_k \left(\left\| \frac{u_k \Delta_n^m x_{k+s} - L}{\rho}, z_1, \dots, z_{n-1} \right\| \right) \right) \text{ for all } k, s \in \mathbb{N}.$$

We can write

$$\frac{1}{g_r} \sum_{k \in I_r} [M_k(y_{k,s})]^{p_k} = \frac{1}{g_r} \sum_{k \in I_r, y_{k,s} \leq \delta} [M_k(y_{k,s})]^{p_k} + \frac{1}{g_r} \sum_{k \in I_r, y_{k,s} > \delta} [M_k(y_{k,s})]^{p_k}.$$

Since $\mathcal{M} = (M_k)$ satisfies Δ_2 -condition, we have

$$\begin{aligned}
\frac{1}{g_r} \sum_{k \in I_r, y_{k,s} \leq \delta} [M_k(y_{k,s})]^{p_k} & \leq [M_k(1)]^H \frac{1}{g_r} \sum_{k \in I_r, y_{k,s} \leq \delta} [M_k(y_{k,s})]^{p_k} \\
& \leq [M_k(2)]^H \frac{1}{g_r} \sum_{k \in I_r, y_{k,s} \leq \delta} [M_k(y_{k,s})]^{p_k} \quad (2.1)
\end{aligned}$$

For $y_{k,s} > \delta$

$$y_{k,s} < \frac{y_{k,s}}{\delta} < 1 + \frac{y_{k,s}}{\delta}.$$

Since $\mathcal{M} = (M_k)$ is non-decreasing and convex, it follows that

$$M_k(y_{k,s}) < M_k\left(1 + \frac{y_{k,s}}{\delta}\right) < \frac{1}{2}M_k(2) + \frac{1}{2}M_k\left(\frac{2y_{k,s}}{\delta}\right).$$

Since (M_k) satisfies Δ_2 -condition, we can write

$$M_k(y_{k,s}) < \frac{1}{2}T\frac{y_{k,s}}{\delta}M_k(2) + \frac{1}{2}T\frac{y_{k,s}}{\delta}M_k(2) = T\frac{y_{k,s}}{\delta}M_k(2).$$

Hence,

$$1g_r \sum_{k \in I_r, y_{k,s} > \delta} [M_k(y_{k,s})]^{p_k} \leq \max\left(1, \left(\frac{TM_k(2)}{\delta}\right)^H\right) \frac{1}{g_r} \sum_{k \in I_r, y_{k,s} > \delta} [(y_{k,s})]^{p_k} \quad (2.2)$$

from equations (2.1) and (2.2), we have

$$x = (x_k) \in [c, \mathcal{M} \circ \mathcal{M}', p, \|\cdot, \dots, \cdot\|]_0^\theta(\Delta_n^m, u, q).$$

This completes the proof of (i). Similarly, we can prove that

$$[c, \mathcal{M}', p, \|\cdot, \dots, \cdot\|]_0^\theta(\Delta_n^m, u, q) \subset [c, \mathcal{M} \circ \mathcal{M}', \|\cdot, \dots, \cdot\|]_0^\theta(\Delta_n^m, u, q)$$

and

$$[c, \mathcal{M}', p, \|\cdot, \dots, \cdot\|]_\infty^\theta(\Delta_n^m, u, q) \subset [c, \mathcal{M} \circ \mathcal{M}', p, \|\cdot, \dots, \cdot\|]_\infty^\theta(\Delta_n^m, u, q).$$

■

Corollary 2.5 *If $0 < \inf p_k = h \leq p_k \leq \sup p_k = H < \infty$ and $\mathcal{M} = (M_k)$ be Musielak-Orlicz function satisfying Δ_2 -condition, then we have*

$$[c, p, \|\cdot, \dots, \cdot\|]_0^\theta(\Delta_n^m, u, q) \subset [c, \mathcal{M}, p, \|\cdot, \dots, \cdot\|]_0^\theta(\Delta_n^m, u, q)$$

and

$$[c, p, \|\cdot, \dots, \cdot\|]_\infty^\theta(\Delta_n^m, u, q) \subset [c, \mathcal{M}, p, \|\cdot, \dots, \cdot\|]_\infty^\theta(\Delta_n^m, u, q).$$

Proof. Taking $\mathcal{M}'(x) = x$ in the above theorem, we get the required result. ■

Theorem 2.6 *If $\mathcal{M} = (M_k)$ be the Musielak-Orlicz function, then the following statements are equivalent:*

- (i) $[c, p, \|\cdot, \dots, \cdot\|]_\infty^\theta(\Delta_n^m, u, q) \subset [c, \mathcal{M}, p, \|\cdot, \dots, \cdot\|]_\infty^\theta(\Delta_n^m, u, q)$,
- (ii) $[c, p, \|\cdot, \dots, \cdot\|]_0^\theta(\Delta_n^m, u, q) \subset [c, \mathcal{M}, p, \|\cdot, \dots, \cdot\|]_0^\theta(\Delta_n^m, u, q)$,
- (iii) $\sup_r \frac{1}{g_r} \sum_{k \in I_r} [M_k(\frac{t}{\rho})]^{p_k} < \infty$ ($t, \rho > 0$).

Proof. (i) \Rightarrow (ii) The proof is obvious in view of the fact that

$$[c, p, \|\cdot, \dots, \cdot\|]_0^\theta(\Delta_n^m, u, q) \subset [c, p, \|\cdot, \dots, \cdot\|]_\infty^\theta(\Delta_n^m, u, q).$$

(ii) \Rightarrow (iii) Let $[c, p, \|\cdot, \dots, \cdot\|]_0^\theta(\Delta_n^m, u, q) \subset [c, \mathcal{M}, p, \|\cdot, \dots, \cdot\|]_\infty^\theta(\Delta_n^m, u, q)$. Suppose that (iii) does not hold. Then for some $t, \rho > 0$

$$\sup_r \frac{1}{g_r} \sum_{k \in I_r} [M_k(\frac{t}{\rho})]^{p_k} = \infty$$

and therefore we can find a subinterval $I_{r(j)}$ of the set of interval I_r such that

$$1g_{r(j)} \sum_{k \in I_{r(j)}} \left[M_k\left(\frac{j^{-1}}{\rho}\right) \right]^{p_k} > j, \quad j = 1, 2, \quad (2.3)$$

Define the sequence $x = (x_k)$ by

$$\Delta^m x_{k+s} = \begin{cases} j^{-1}, & k \in I_{r(j)} \\ 0, & k \notin I_{r(j)} \end{cases} \quad \text{for all } s \in \mathbb{N}.$$

Then $x = (x_k) \in [c, p, \|\cdot, \dots, \cdot\|]_0^\theta(\Delta_n^m, u, q)$ but by equation(2.3), $x = (x_k) \notin [c, \mathcal{M}, p, \|\cdot, \dots, \cdot\|]_\infty^\theta(\Delta_n^m, u, q)$, which contradicts (ii). Hence (iii) must hold.

(iii) \Rightarrow (i) Let (iii) hold and $x = (x_k) \in [c, p, \|\cdot, \dots, \cdot\|]_\infty^\theta(\Delta_n^m, u, q)$. Suppose that $x = (x_k) \notin [c, \mathcal{M}, p, \|\cdot, \dots, \cdot\|]_\infty^\theta(\Delta_n^m, u, q)$. Then

$$\sup_{r,s} \frac{1}{g_r} \sum_{k \in I_r} \left[M_k\left(\left\| \frac{\Delta^m x_{k+s}}{\rho}, z_1, \dots, z_{n-1} \right\| \right) \right]^{p_k} = \infty. \quad (2.4)$$

Let $t = q_k \left(\|u_k \Delta^m x_{k+s}, z_1, \dots, z_{n-1}\| \right)$ for each k and fixed s , then by equations(2.4)

$$\sup_r \frac{1}{g_r} \sum_{k \in I_r} \left[M_k\left(\frac{t}{\rho}\right) \right] = \infty,$$

which contradicts (iii). Hence (i) must hold. ■

Theorem 2.7 Let $1 \leq p_k \leq \sup p_k < \infty$ and $\mathcal{M} = (M_k)$ be a Musielak Orlicz function. Then the following statements are equivalent:

- (i) $[c, \mathcal{M}, p, \|\cdot, \dots, \cdot\|]_0^\theta(\Delta_n^m, u, q) \subset [c, p, \|\cdot, \dots, \cdot\|]_0^\theta(\Delta_n^m, u, q)$,
- (ii) $[c, \mathcal{M}, p, \|\cdot, \dots, \cdot\|]_0^\theta(\Delta_n^m, u, q) \subset [c, p, \|\cdot, \dots, \cdot\|]_\infty^\theta(\Delta_n^m, u, q)$,
- (iii) $\inf_r \frac{1}{g_r} \sum_{k \in I_r} \left[M_k\left(\frac{t}{\rho}\right) \right]^{p_k} > 0$ ($t, \rho > 0$).

Proof. (i) \Rightarrow (ii) It is trivial.

(ii) \Rightarrow (iii) Let (ii) hold. Suppose that (iii) does not hold. Then

$$\inf_r \frac{1}{g_r} \sum_{k \in I_r} \left[M_k\left(\frac{t}{\rho}\right) \right]^{p_k} = 0 \quad (t, \rho > 0),$$

so we can find a subinterval $I_{r(j)}$ of the set of interval I_r such that

$$\frac{1}{g_{r(j)}} \sum_{k \in I_{r(j)}} \left[M_k \left(\frac{j}{\rho} \right) \right]^{p_k} < j^{-1}, \quad j = 1, 2, \quad (2.5)$$

Define the sequence $x = (x_k)$ by

$$\Delta^m x_{k+s} = \begin{cases} j, & k \in I_{r(j)} \\ 0, & k \notin I_{r(j)} \end{cases} \text{ for all } s \in \mathbb{N}.$$

Thus by equation(2.5), $x = (x_k) \in [c, \mathcal{M}, p, \|\cdot, \dots, \cdot\|]_0^\theta(\Delta_n^m, u, q)$, hence $x = (x_k) \notin [c, p, \|\cdot, \dots, \cdot\|]_\infty^\theta(\Delta_n^m, u, q)$, which contradicts (ii). Hence (iii) must hold.

(iii) \Rightarrow (i) Let (iii) hold and suppose that $x = (x_k) \in [c, \mathcal{M}, p, \|\cdot, \dots, \cdot\|]_0^\theta(\Delta_n^m, u, q)$, i.e,

$$\lim_{r \rightarrow \infty} \frac{1}{g_r} \sum_{k \in I_r} \left[M_k \left(q_k \left(\left\| \frac{u_k \Delta_n^m x_{k+s}}{\rho}, z_1, \dots, z_{n-1} \right\| \right) \right) \right]^{p_k} = 0, \quad (2.6)$$

uniformly in s , for some $\rho > 0$.

Again, suppose that $x = (x_k) \notin [c, p, \|\cdot, \dots, \cdot\|]_0^\theta(\Delta_n^m, u, q)$. Then, for some number $\epsilon > 0$ and a subinterval $I_{r(j)}$ of the set of interval I_r , we have

$$\|u_k \Delta_n^m x_{k+s}, z_1, \dots, z_{n-1}\| \geq \epsilon$$

for all $k \in \mathbb{N}$ and some $s \geq s_0$. Then, from the properties of the Orlicz function, we can write

$$M_k \left(q_k \left(\left\| \frac{u_k \Delta_n^m x_{k+s}}{\rho}, z_1, \dots, z_{n-1} \right\| \right) \right)^{p_k} \geq M_k \left(\frac{\epsilon}{\rho} \right)^{p_k}$$

and consequently by (2.6)

$$\lim_{r \rightarrow \infty} \frac{1}{g_r} \sum_{k \in I_r} \left[M_k \left(\frac{\epsilon}{\rho} \right) \right]^{p_k} = 0,$$

which contradicts (iii). Hence (i) must hold. ■

Theorem 2.8 Let $0 < p_k \leq q_k$ for all $k \in \mathbb{N}$ and $\left(\frac{q_k}{p_k} \right)$ be bounded. Then,

$$[c, \mathcal{M}, q, \|\cdot, \dots, \cdot\|]^\theta(\Delta_n^m, u, q) \subset [c, \mathcal{M}, p, \|\cdot, \dots, \cdot\|]^\theta(\Delta_n^m, u, q).$$

Proof. Let $x \in [c, \mathcal{M}, q, \|\cdot, \dots, \cdot\|]^\theta(\Delta_n^m, u, q)$. Write

$$t_k = \left[M_k \left(q_k \left(\left\| u_k \frac{\Delta_n^m x_{k+s} - L}{\rho}, z_1, \dots, z_{n-1} \right\| \right) \right) \right]^{q_k}$$

and $\mu_k = \frac{p_k}{q_k}$ for all $k \in \mathbb{N}$. Then $0 < \mu_k \leq 1$ for $k \in \mathbb{N}$. Take $0 < \mu < \mu_k$ for $k \in \mathbb{N}$. Define the sequences (u_k) and (v_k) as follows: For $t_k \geq 1$, let $u_k = t_k$ and $v_k = 0$ and for $t_k < 1$, let $u_k = 0$ and $v_k = t_k$. Then clearly for all $k \in \mathbb{N}$, we have

$$t_k = u_k + v_k, \quad t_k^{\mu_k} = u_k^{\mu_k} + v_k^{\mu_k}$$

Now it follows that $u_k^{\mu_k} \leq u_k \leq t_k$ and $v_k^{\mu_k} \leq v_k^{\mu}$. Therefore,

$$\begin{aligned} \frac{1}{g_r} \sum_{k \in I_r} t_k^{\mu_k} &= \frac{1}{g_r} \sum_{k \in I_r} (u_k^{\mu_k} + v_k^{\mu_k}) \\ &\leq \frac{1}{g_r} \sum_{k \in I_r} t_k + \frac{1}{g_r} \sum_{k \in I_r} v_k^{\mu}. \end{aligned}$$

Now for each k ,

$$\begin{aligned} \frac{1}{g_r} \sum_{k \in I_r} v_k^{\mu} &= \sum_{k \in I_r} \left(\frac{1}{g_r} v_k \right)^{\mu} \left(\frac{1}{g_r} \right)^{1-\mu} \\ &\leq \left(\sum_{k \in I_r} \left[\left(\frac{1}{g_r} v_k \right)^{\mu} \right]^{\frac{1}{\mu}} \right)^{\mu} \left(\sum_{k \in I_r} \left[\left(\frac{1}{g_r} \right)^{1-\mu} \right]^{\frac{1}{1-\mu}} \right)^{1-\mu} \\ &= \left(\frac{1}{g_r} \sum_{k \in I_r} v_k \right)^{\mu} \end{aligned}$$

and so

$$\frac{1}{g_r} \sum_{k \in I_r} t_k^{\mu_k} \leq \frac{1}{g_r} \sum_{k \in I_r} t_k + \left(\frac{1}{g_r} \sum_{k \in I_r} v_k \right)^{\mu}.$$

Hence $x \in [c, \mathcal{M}, p, \|\cdot, \dots, \cdot\|]^{\theta}(\Delta_n^m, u, q)$. ■

Theorem 2.9 (a) If $0 < \inf p_k \leq p_k \leq 1$ for all $k \in \mathbb{N}$, then

$$[c, \mathcal{M}, p, \|\cdot, \dots, \cdot\|]^{\theta}(\Delta_n^m, u, q) \subset [c, \mathcal{M}, \|\cdot, \dots, \cdot\|]^{\theta}(\Delta_n^m, u, q).$$

(b) If $1 \leq p_k \leq \sup p_k < \infty$ for all $k \in \mathbb{N}$. Then

$$[c, \mathcal{M}, \|\cdot, \dots, \cdot\|]^{\theta}(\Delta_n^m, u, q) \subset [c, \mathcal{M}, p, \|\cdot, \dots, \cdot\|]^{\theta}(\Delta_n^m, u, q).$$

Proof. (a) Let $x \in [c, \mathcal{M}, p, \|\cdot, \dots, \cdot\|]^{\theta}(\Delta_n^m, u, q)$, then

$$\lim_{r \rightarrow \infty} \frac{1}{g_r} \sum_{k \in I_r} \left[M_k \left(q_k \left(\left\| \frac{u_k \Delta_n^m x_{k+s} - L}{\rho}, z_1, \dots, z_{n-1} \right\| \right) \right)^{p_k} \right] = 0.$$

Since $0 < \inf p_k \leq p_k \leq 1$. This implies that

$$\begin{aligned} & \lim_{r \rightarrow \infty} \frac{1}{g_r} \sum_{k \in I_r} \left[M_k \left(q_k \left(\left\| \frac{u_k \Delta_n^m x_{k+s} - L}{\rho}, z_1, \dots, z_{n-1} \right\| \right) \right) \right] \\ & \leq \lim_{r \rightarrow \infty} \frac{1}{g_r} \sum_{k \in I_r} \left[M_k \left(q_k \left(\left\| \frac{u_k \Delta_n^m x_{k+s} - L}{\rho}, z_1, \dots, z_{n-1} \right\| \right) \right) \right]^{p_k}, \end{aligned}$$

therefore, $\lim_{r \rightarrow \infty} \frac{1}{g_r} \sum_{k \in I_r} \left[M_k \left(q_k \left(\left\| \frac{u_k \Delta_n^m x_{k+s} - L}{\rho}, z_1, \dots, z_{n-1} \right\| \right) \right) \right] = 0$. This shows that $x \in [c, \mathcal{M}, \|\cdot, \dots, \cdot\|]^\theta(\Delta_n^m, u, q)$. Therefore,

$$[c, \mathcal{M}, p, \|\cdot, \dots, \cdot\|]^\theta(\Delta_n^m, u, q) \subset [c, \mathcal{M}, \|\cdot, \dots, \cdot\|]^\theta(\Delta_n^m, u, q).$$

This completes the proof.

(b) Let $p_k \geq 1$ for each k and $\sup p_k < \infty$. Let $x \in [c, p, \|\cdot, \dots, \cdot\|]^\theta(\Delta_n^m, u, q)$. Then for each $\epsilon > 0$ there exists a positive integer N such that

$$\lim_{r \rightarrow \infty} \frac{1}{g_r} \sum_{k \in I_r} \left[M_k \left(q_k \left(\left\| \frac{u_k \Delta_n^m x_{k+s} - L}{\rho}, z_1, \dots, z_{n-1} \right\| \right) \right) \right]^{p_k} = 0 < 1.$$

Since $1 \leq p_k \leq \sup p_k < \infty$, we have

$$\begin{aligned} & \lim_{r \rightarrow \infty} \frac{1}{g_r} \sum_{k \in I_r} \left[M_k \left(q_k \left(\left\| \frac{u_k \Delta_n^m x_{k+s} - L}{\rho}, z_1, \dots, z_{n-1} \right\| \right) \right) \right]^{p_k} \\ & \leq \lim_{r \rightarrow \infty} \frac{1}{g_r} \sum_{k \in I_r} \left[M_k \left(q_k \left(\left\| \frac{u_k \Delta_n^m x_{k+s} - L}{\rho}, z_1, \dots, z_{n-1} \right\| \right) \right) \right] \\ & = 0 \\ & < 1. \end{aligned}$$

Therefore $x \in [c, \mathcal{M}, p, \|\cdot, \dots, \cdot\|]^\theta(\Delta_n^m, u, q)$. ■

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The random of lacunary statistical on χ^2 over p -metric spaces defined by Musielak

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ABSTRACT: Mursaleen introduced the concepts of statistical convergence in random 2-normed spaces. Recently Mohiuddine and Aiyup defined the notion of lacunary statistical convergence and lacunary statistical Cauchy in random 2-normed spaces. In this paper, we define and study the notion of lacunary statistical convergence and lacunary of statistical Cauchy sequences in random on χ^2 over p - metric spaces defined by Musielak and prove some theorems which generalizes Mohiuddine and Aiyup results.

AMS Subject Classification: *analytic sequence, double sequences, χ^2 space, Musielak - modulus function, Random p - metric space, Lacunary sequence, Statistical convergence*

Keywords and Phrases: *40A05,40C05,40D05*

1 Introduction

The concept of statistical convergence play a vital role not only in pure mathematics but also in other branches of science involving mathematics, especially in information theory, computer science, biological science, dynamical systems, geographic information systems, population modeling, and motion planning in robotics.

The notion of statistical convergence was introduced by Fast and Schoenberg independently. Over the years and under different names statistical convergence has been discussed in the theory of fourier analysis, ergodic theory and number theory. Later on it was further investigated by Fridy , Šalát , Çakalli , Maio and Kocinac , Miller , Maddox , Leindler , Mursaleen and Alotaibi , Mursaleen and Edely , and many others. In the recent years, generalizations of statistical convergence have appeared in the study of strong integral summability and the structure of ideals of bounded continuous functions on Stone-Čech compactification of the natural numbers. Moreover statistical convergence is closely related to the concept of convergence in probability.

Throughout w, χ and Λ denote the classes of all, gai and analytic scalar valued

single sequences, respectively.

We write w^2 for the set of all complex sequences (x_{mn}) , where $m, n \in \mathbb{N}$, the set of positive integers. Then, w^2 is a linear space under the coordinate wise addition and scalar multiplication.

Some initial works on double sequence spaces is found in Bromwich [2]. Later on, they were investigated by Hardy [3], Moricz [6], Moricz and Rhoades [7], Basarir and Solankan [1], Tripathy [8], Turkmenoglu [9], and many others.

We procure the following sets of double sequences:

$$\begin{aligned}\mathcal{M}_u(t) &:= \left\{ (x_{mn}) \in w^2 : \sup_{m,n \in \mathbb{N}} |x_{mn}|^{t_{mn}} < \infty \right\}, \\ \mathcal{C}_p(t) &:= \left\{ (x_{mn}) \in w^2 : p - \lim_{m,n \rightarrow \infty} |x_{mn} - l|^{t_{mn}} = 1 \text{ for some } l \in \mathbb{C} \right\}, \\ \mathcal{C}_{0p}(t) &:= \left\{ (x_{mn}) \in w^2 : p - \lim_{m,n \rightarrow \infty} |x_{mn}|^{t_{mn}} = 1 \right\}, \\ \mathcal{L}_u(t) &:= \left\{ (x_{mn}) \in w^2 : \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} |x_{mn}|^{t_{mn}} < \infty \right\}, \\ \mathcal{C}_{bp}(t) &:= \mathcal{C}_p(t) \cap \mathcal{M}_u(t) \text{ and } \mathcal{C}_{0bp}(t) = \mathcal{C}_{0p}(t) \cap \mathcal{M}_u(t); \end{aligned}$$

where $t = (t_{mn})$ is the sequence of strictly positive reals t_{mn} for all $m, n \in \mathbb{N}$ and $p - \lim_{m,n \rightarrow \infty}$ denotes the limit in the Pringsheim's sense. In the case $t_{mn} = 1$ for all $m, n \in \mathbb{N}$; $\mathcal{M}_u(t)$, $\mathcal{C}_p(t)$, $\mathcal{C}_{0p}(t)$, $\mathcal{L}_u(t)$, $\mathcal{C}_{bp}(t)$ and $\mathcal{C}_{0bp}(t)$ reduce to the sets \mathcal{M}_u , \mathcal{C}_p , \mathcal{C}_{0p} , \mathcal{L}_u , \mathcal{C}_{bp} and \mathcal{C}_{0bp} , respectively. Now, we may summarize the knowledge given in some document related to the double sequence spaces. Gökhan and Colak [11,12] have proved that $\mathcal{M}_u(t)$ and $\mathcal{C}_p(t)$, $\mathcal{C}_{bp}(t)$ are complete paranormed spaces of double sequences and gave the α -, β -, γ - duals of the spaces $\mathcal{M}_u(t)$ and $\mathcal{C}_{bp}(t)$. Quite recently, in her PhD thesis, Zelter [13] has essentially studied both the theory of topological double sequence spaces and the theory of summability of double sequences. Mursaleen and Edely [14] have independently introduced the statistical convergence and Cauchy for double sequences and given the relation between statistical convergent and strongly Cesàro summable double sequences. Altay and Basar [15] have defined the spaces \mathcal{BS} , $\mathcal{BS}(t)$, \mathcal{CS}_p , \mathcal{CS}_{bp} , \mathcal{CS}_r and \mathcal{BV} of double sequences consisting of all double series whose sequence of partial sums are in the spaces \mathcal{M}_u , $\mathcal{M}_u(t)$, \mathcal{C}_p , \mathcal{C}_{bp} , \mathcal{C}_r and \mathcal{L}_u , respectively, and also examined some properties of those sequence spaces and determined the α - duals of the spaces \mathcal{BS} , \mathcal{BV} , \mathcal{CS}_{bp} and the $\beta(\vartheta)$ - duals of the spaces \mathcal{CS}_{bp} and \mathcal{CS}_r of double series. Basar and Sever [16] have introduced the Banach space \mathcal{L}_q of double sequences corresponding to the well-known space ℓ_q of single sequences and examined some properties of the space \mathcal{L}_q . Quite recently Subramanian and Misra [17] have studied the space $\chi_M^2(p, q, u)$ of double sequences and gave some inclusion relations.

The class of sequences which are strongly Cesàro summable with respect to a modulus was introduced by Maddox [5] as an extension of the definition of strongly Cesàro summable sequences. Cannon [18] further extended this definition to a definition of strong A - summability with respect to a modulus where $A = (a_{n,k})$ is

a nonnegative regular matrix and established some connections between strong A -summability, strong A - summability with respect to a modulus, and A - statistical convergence. In [19] the notion of convergence of double sequences was presented by A. Pringsheim. Also, in [20]-[21], and [22] the four dimensional matrix transformation $(Ax)_{k,\ell} = \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} a_{k\ell}^{mn} x_{mn}$ was studied extensively by Robison and Hamilton.

We need the following inequality in the sequel of the paper. For $a, b, \geq 0$ and $0 < p < 1$, we have

$$(a + b)^p \leq a^p + b^p \tag{1.1}$$

The double series $\sum_{m,n=1}^{\infty} x_{mn}$ is called convergent if and only if the double sequence (s_{mn}) is convergent, where $s_{mn} = \sum_{i,j=1}^{m,n} x_{ij} (m, n \in \mathbb{N})$.

A sequence $x = (x_{mn})$ is said to be double analytic if $\sup_{mn} |x_{mn}|^{1/m+n} < \infty$. The vector space of all double analytic sequences will be denoted by Λ^2 . A sequence $x = (x_{mn})$ is called double gai sequence if $((m + n)! |x_{mn}|)^{1/m+n} \rightarrow 0$ as $m, n \rightarrow \infty$. The double gai sequences will be denoted by χ^2 . Let $\phi = \{all\ finitesequences\}$.

Consider a double sequence $x = (x_{ij})$. The $(m, n)^{th}$ section $x^{[m,n]}$ of the sequence is defined by $x^{[m,n]} = \sum_{i,j=0}^{m,n} x_{ij} \mathfrak{S}_{ij}$ for all $m, n \in \mathbb{N}$; where \mathfrak{S}_{ij} denotes the double sequence whose only non zero term is a $\frac{1}{(i+j)!}$ in the $(i, j)^{th}$ place for each $i, j \in \mathbb{N}$.

Let M and Φ are mutually complementary modulus functions. Then, we have:

(i) For all $u, y \geq 0$,

$$uy \leq M(u) + \Phi(y), (Young's\ inequality)[See[10]] \tag{1.2}$$

(ii) For all $u \geq 0$,

$$u\eta(u) = M(u) + \Phi(\eta(u)). \tag{1.3}$$

(iii) For all $u \geq 0$, and $0 < \lambda < 1$,

$$M(\lambda u) \leq \lambda M(u) \tag{1.4}$$

Lindenstrauss and Tzafriri [4] used the idea of Orlicz function to construct Orlicz sequence space

$$\ell_M = \left\{ x \in w : \sum_{k=1}^{\infty} M\left(\frac{|x_k|}{\rho}\right) < \infty, \text{ for some } \rho > 0 \right\},$$

The space ℓ_M with the norm

$$\|x\| = inf \left\{ \rho > 0 : \sum_{k=1}^{\infty} M\left(\frac{|x_k|}{\rho}\right) \leq 1 \right\},$$

becomes a Banach space which is called an Orlicz sequence space. For $M(t) = t^p (1 \leq p < \infty)$, the spaces ℓ_M coincide with the classical sequence space ℓ_p .

A sequence $f = (f_{mn})$ of modulus function is called a Musielak-modulus function. A sequence $g = (g_{mn})$ defined by

$$g_{mn}(v) = \sup \{ |v|u - (f_{mn})(u) : u \geq 0 \}, m, n = 1, 2, \dots$$

is called the complementary function of a Musielak-modulus function f . For a given Musielak modulus function f , the Musielak-modulus sequence space t_f is defined as follows

$$t_f = \left\{ x \in w^2 : I_f(|x_{mn}|)^{1/m+n} \rightarrow 0 \text{ as } m, n \rightarrow \infty \right\},$$

where I_f is a convex modular defined by

$$I_f(x) = \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} f_{mn}(|x_{mn}|)^{1/m+n}, x = (x_{mn}) \in t_f.$$

We consider t_f equipped with the Luxemburg metric

$$d(x, y) = \sup_{mn} \left\{ \inf \left(\sum_{m=1}^{\infty} \sum_{n=1}^{\infty} f_{mn} \left(\frac{|x_{mn}|^{1/m+n}}{mn} \right) \right) \leq 1 \right\}$$

If X is a sequence space, we give the following definitions:

- (i) X' = the continuous dual of X ;
- (ii) $X^\alpha = \{ a = (a_{mn}) : \sum_{m,n=1}^{\infty} |a_{mn}x_{mn}| < \infty, \text{ for each } x \in X \}$;
- (iii) $X^\beta = \{ a = (a_{mn}) : \sum_{m,n=1}^{\infty} a_{mn}x_{mn} \text{ is convergent, for each } x \in X \}$;
- (iv) $X^\gamma = \{ a = (a_{mn}) : \sup_{mn} \geq 1 \left| \sum_{m,n=1}^{M,N} a_{mn}x_{mn} \right| < \infty, \text{ for each } x \in X \}$;
- (v) let X be an FK-space $\supset \phi$; then $X^f = \{ f(\mathfrak{S}_{mn}) : f \in X' \}$;
- (vi) $X^\delta = \{ a = (a_{mn}) : \sup_{mn} |a_{mn}x_{mn}|^{1/m+n} < \infty, \text{ for each } x \in X \}$;

$X^\alpha, X^\beta, X^\gamma$ are called α - (or Köthe - Toeplitz) dual of X , β - (or generalized - Köthe - Toeplitz) dual of X , γ - dual of X , δ - dual of X respectively. X^α is defined by Gupta and Kamptan [10]. It is clear that $X^\alpha \subset X^\beta$ and $X^\alpha \subset X^\gamma$, but $X^\beta \subset X^\gamma$ does not hold, since the sequence of partial sums of a double convergent series need not to be bounded.

The notion of difference sequence spaces (for single sequences) was introduced by Kizmaz as follows

$$Z(\Delta) = \{ x = (x_k) \in w : (\Delta x_k) \in Z \}$$

for $Z = c, c_0$ and ℓ_∞ , where $\Delta x_k = x_k - x_{k+1}$ for all $k \in \mathbb{N}$.

Here c, c_0 and ℓ_∞ denote the classes of convergent, null and bounded scalar valued single sequences respectively. The difference sequence space bv_p of the classical space ℓ_p is introduced and studied in the case $1 \leq p \leq \infty$ by Başar and Altay and in the case $0 < p < 1$ by Altay and Başar in [15]. The spaces $c(\Delta), c_0(\Delta), \ell_\infty(\Delta)$ and bv_p are Banach spaces normed by

$$\|x\| = |x_1| + \sup_{k \geq 1} |\Delta x_k| \text{ and } \|x\|_{bv_p} = \left(\sum_{k=1}^{\infty} |x_k|^p \right)^{1/p}, (1 \leq p < \infty).$$

Later on the notion was further investigated by many others. We now introduce the following difference double sequence spaces defined by

$$Z(\Delta) = \{x = (x_{mn}) \in w^2 : (\Delta x_{mn}) \in Z\}$$

where $Z = \Lambda^2, \chi^2$ and $\Delta x_{mn} = (x_{mn} - x_{mn+1}) - (x_{m+1n} - x_{m+1n+1}) = x_{mn} - x_{mn+1} - x_{m+1n} + x_{m+1n+1}$ for all $m, n \in \mathbb{N}$. The generalized difference double notion has the following representation: $\Delta^m x_{mn} = \Delta^{m-1} x_{mn} - \Delta^{m-1} x_{mn+1} - \Delta^{m-1} x_{m+1n} + \Delta^{m-1} x_{m+1n+1}$, and also this generalized difference double notion has the following binomial representation:

$$\Delta^m x_{mn} = \sum_{i=0}^m \sum_{j=0}^m (-1)^{i+j} \binom{m}{i} \binom{m}{j} x_{m+i, n+j}.$$

2 Definition and Preliminaries

Let $n \in \mathbb{N}$ and X be a real vector space of dimension w , where $n \leq w$. A real valued function $d_p(x_1, \dots, x_n) = \|(d_1(x_1), \dots, d_n(x_n))\|_p$ on X satisfying the following four conditions:

- (i) $\|(d_1(x_1), \dots, d_n(x_n))\|_p = 0$ if and only if $d_1(x_1), \dots, d_n(x_n)$ are linearly dependent,
 - (ii) $\|(d_1(x_1), \dots, d_n(x_n))\|_p$ is invariant under permutation,
 - (iii) $\|(\alpha d_1(x_1), \dots, d_n(x_n))\|_p = |\alpha| \|(d_1(x_1), \dots, d_n(x_n))\|_p, \alpha \in \mathbb{R}$
 - (iv) $d_p((x_1, y_1), (x_2, y_2), \dots, (x_n, y_n)) = (d_X(x_1, x_2, \dots, x_n)^p + d_Y(y_1, y_2, \dots, y_n)^p)^{1/p}$ for $1 \leq p < \infty$; (or)
 - (v) $d((x_1, y_1), (x_2, y_2), \dots, (x_n, y_n)) := \sup \{d_X(x_1, x_2, \dots, x_n), d_Y(y_1, y_2, \dots, y_n)\}$,
- for $x_1, x_2, \dots, x_n \in X, y_1, y_2, \dots, y_n \in Y$ is called the p product metric of the Cartesian product of n metric spaces is the p norm of the n -vector of the norms of the n sub-spaces.

A trivial example of p product metric of n metric space is the p norm space is $X = \mathbb{R}$ equipped with the following Euclidean metric in the product space is the p norm:

$$\begin{aligned} \|(d_1(x_1), \dots, d_n(x_n))\|_E &= \sup(|\det(d_{mn}(x_{mn}))|) \\ &= \sup \left(\begin{vmatrix} d_{11}(x_{11}) & d_{12}(x_{12}) & \dots & d_{1n}(x_{1n}) \\ d_{21}(x_{21}) & d_{22}(x_{22}) & \dots & d_{2n}(x_{2n}) \\ \vdots & \vdots & \ddots & \vdots \\ d_{n1}(x_{n1}) & d_{n2}(x_{n2}) & \dots & d_{nn}(x_{nn}) \end{vmatrix} \right) \end{aligned}$$

where $x_i = (x_{i1}, \dots, x_{in}) \in \mathbb{R}^n$ for each $i = 1, 2, \dots, n$.

If every Cauchy sequence in X converges to some $L \in X$, then X is said to be complete with respect to the p - metric. Any complete p - metric space is said to be p - Banach metric space.

Let X be a linear metric space. A function $w : X \rightarrow \mathbb{R}$ is called paranorm, if

- (1) $w(x) \geq 0$, for all $x \in X$;
- (2) $w(-x) = w(x)$, for all $x \in X$;
- (3) $w(x + y) \leq w(x) + w(y)$, for all $x, y \in X$;
- (4) If (σ_{mn}) is a sequence of scalars with $\sigma_{mn} \rightarrow \sigma$ as $m, n \rightarrow \infty$ and (x_{mn}) is a sequence of vectors with $w(x_{mn} - x) \rightarrow 0$ as $m, n \rightarrow \infty$, then $w(\sigma_{mn}x_{mn} - \sigma x) \rightarrow 0$ as $m, n \rightarrow \infty$.

A paranorm w for which $w(x) = 0$ implies $x = 0$ is called total paranorm and the pair (X, w) is called a total paranormed space. It is well known that the metric of any linear metric space is given by some total paranorm (see [23], Theorem 10.4.2, p.183).

By the convergence of a double sequence we mean the convergence on the Pringsheim sense that is, a double sequence $x = (x_{mn})$ has Pringsheim limit L (denoted by $P\text{-}\lim x = L$) provided that given $\epsilon > 0$ there exists $n \in \mathbb{N}$ such that $|x_{mn} - L| < \epsilon$. We shall write more briefly as P -convergent.

The double sequence $\theta_{rs} = \{(m_r, n_s)\}$ is called double lacunary sequence if there exist two increasing of integers such that

$$m_0 = 0, \varphi_r = m_r - m_{r-1} \rightarrow \infty \text{ as } r \rightarrow \infty \text{ and}$$

$$n_0 = 0, \varphi_s = n_s - n_{s-1} \rightarrow \infty \text{ as } s \rightarrow \infty.$$

Notations: $m_{rs} = m_r n_s, h_{rs} = \varphi_r \bar{\varphi}_s, \theta_{rs}$ is determined by

$$I_{rs} = \{(m, n) : m_{r-1} < m \leq m_r \text{ and } n_{s-1} < n \leq n_s\},$$

$$q_r = \frac{m_r}{m_{r-1}}, \bar{q}_s = \frac{n_s}{n_{s-1}} \text{ and } q_{rs} = q_r \bar{q}_s.$$

The notion of λ -double gai and double analytic sequences as follows: Let $\lambda = (\lambda_{mn})_{m,n=0}^\infty$ be a strictly increasing sequences of positive real numbers tending to infinity, that is

$$0 < \lambda_{00} < \lambda_{11} < \dots \text{ and } \lambda_{mn} \rightarrow \infty \text{ as } m, n \rightarrow \infty$$

and said that a sequence $x = (x_{mn}) \in w^2$ is λ -convergent to 0, called a the λ -limit of x , if $\mu_{mn}(x) \rightarrow 0$ as $m, n \rightarrow \infty$, where

$$\mu_{mn}(x) = \frac{1}{\varphi_{rs}} \sum_{m \in I_{rs}} \sum_{n \in I_{rs}} (\Delta^{m-1} \lambda_{m,n} - \Delta^{m-1} \lambda_{m,n+1} - \Delta^{m-1} \lambda_{m+1,n} + \Delta^{m-1} \lambda_{m+1,n+1}) |x_{mn}|^{1/m+n}.$$

The sequence $x = (x_{mn}) \in w^2$ is λ -double analytic if $\sup_{uv} |\mu_{mn}(x)| < \infty$. If $\lim_{mn} x_{mn} = 0$ in the ordinary sense of convergence, then

$$\lim_{mn} \left(\frac{1}{\varphi_{rs}} \sum_{m \in I_{rs}} \sum_{n \in I_{rs}} (\Delta^{m-1} \lambda_{m,n} - \Delta^{m-1} \lambda_{m,n+1} - \Delta^{m-1} \lambda_{m+1,n} + \Delta^{m-1} \lambda_{m+1,n+1}) \right. \\ \left. ((m+n)! |x_{mn} - 0|)^{1/m+n} \right) = 0.$$

This implies that

$$\lim_{mn} |\mu_{mn}(x) - 0| = \lim_{mn} \left| \left(\frac{1}{\varphi_{rs}} \sum_{m \in I_{rs}} \sum_{n \in I_{rs}} (\Delta^{m-1} \lambda_{m,n} - \Delta^{m-1} \lambda_{m,n+1} - \Delta^{m-1} \lambda_{m+1,n} + \Delta^{m-1} \lambda_{m+1,n+1}) ((m+n)! \|x_{mn} - 0\|)^{1/m+n} \right) \right| = 0.$$

which yields that $\lim_{uv} \mu_{mn}(x) = 0$ and hence $x = (x_{mn}) \in w^2$ is λ -convergent to 0.

Let I^2 - be an admissible ideal of $2^{\mathbb{N} \times \mathbb{N}}$, θ_{rs} be a double lacunary sequence, $f = (f_{mn})$ be a Musielak-modulus function and $(X, \|(d(x_1), d(x_2), \dots, d(x_{n-1}))\|_p)$ be a p -metric space, $q = (q_{mn})$ be double analytic sequence of strictly positive real numbers. By $w^2(p-X)$ we denote the space of all sequences defined over $(X, \|(d(x_1), d(x_2), \dots, d(x_{n-1}))\|_p)$. The following inequality will be used throughout the paper. If $0 \leq q_{mn} \leq \sup q_{mn} = H, K = \max(1, 2^{H-1})$ then

$$|a_{mn} + b_{mn}|^{q_{mn}} \leq K \{|a_{mn}|^{q_{mn}} + |b_{mn}|^{q_{mn}}\} \tag{2.1}$$

for all m, n and $a_{mn}, b_{mn} \in \mathbb{C}$. Also $|a|^{q_{mn}} \leq \max(1, |a|^H)$ for all $a \in \mathbb{C}$.

In the present paper we define the following sequence spaces:

$$\begin{aligned} & \left[\chi_{f\mu}^{2q}, \|(d(x_1), d(x_2), \dots, d(x_{n-1}))\|_p \right]_{\theta_{rs}}^{I^2} \\ &= \left\{ r, s \in I_{rs} : \left[f_{mn} \left(\|\mu_{mn}(x), (d(x_1), d(x_2), \dots, d(x_{n-1}))\|_p \right) \right]^{q_{mn}} \geq \epsilon \right\} \\ &\in I^2 \end{aligned}$$

$$\begin{aligned} & \left[\Lambda_{f\mu}^{2q}, \|(d(x_1), d(x_2), \dots, d(x_{n-1}))\|_p \right]_{\theta_{rs}}^{I^2} \\ &= \left\{ r, s \in I_{rs} : \left[f_{mn} \left(\|\mu_{mn}(x), (d(x_1), d(x_2), \dots, d(x_{n-1}))\|_p \right) \right]^{q_{mn}} \geq K \right\} \\ &\in I^2, \end{aligned}$$

If we take $f_{mn}(x) = x$, we get

$$\begin{aligned} & \left[\chi_{f\mu}^{2q}, \|(d(x_1), d(x_2), \dots, d(x_{n-1}))\|_p \right]_{\theta_{rs}}^{I^2} \\ &= \left\{ r, s \in I_{rs} : \left[\left(\|\mu_{mn}(x), (d(x_1), d(x_2), \dots, d(x_{n-1}))\|_p \right) \right]^{q_{mn}} \geq \epsilon \right\} \\ &\in I^2, \end{aligned}$$

$$\begin{aligned} & \left[\Lambda_{f\mu}^{2q}, \|(d(x_1), d(x_2), \dots, d(x_{n-1}))\|_p \right]_{\theta_{rs}}^{I^2} \\ &= \left\{ r, s \in I_{rs} : \left[\left(\|\mu_{mn}(x), (d(x_1), d(x_2), \dots, d(x_{n-1}))\|_p \right) \right]^{q_{mn}} \geq K \right\} \\ &\in I^2, \end{aligned}$$

If we take $q = (q_{mn}) = 1$, we get

$$\begin{aligned} & \left[\chi_{f\mu}^2, \|(d(x_1), d(x_2), \dots, d(x_{n-1}))\|_p^\varphi \right]_{\theta_{rs}}^I \\ &= \left\{ r, s \in I_{rs} : \left[f_{mn} \left(\|\mu_{mn}(x), (d(x_1), d(x_2), \dots, d(x_{n-1}))\|_p \right) \right] \geq \epsilon \right\} \\ &\in I^2 \\ & , \left[\Lambda_{f\mu}^2, \|(d(x_1), d(x_2), \dots, d(x_{n-1}))\|_p^\varphi \right]_{\theta_{rs}}^{I^2} \\ &= \left\{ r, s \in I_{rs} : \left[f_{mn} \left(\|\mu_{mn}(x), (d(x_1), d(x_2), \dots, d(x_{n-1}))\|_p \right) \right] \geq K \right\} \\ &\in I^2, \end{aligned}$$

In the present paper we plan to study some topological properties and inclusion relation between the above defined sequence spaces.

$$\left[\chi_{f\mu}^{2q}, \|(d(x_1), d(x_2), \dots, d(x_{n-1}))\|_p^\varphi \right]_{\theta_{rs}}^{I^2}$$

and

$$\left[\Lambda_{f\mu}^{2q}, \|(d(x_1), d(x_2), \dots, d(x_{n-1}))\|_p^\varphi \right]_{\theta_{rs}}^{I^2}$$

which we shall discuss in this paper.

3 Main Results

3.1 Theorem

Let $f = (f_{mn})$ be a Musielak-modulus function, $q = (q_{mn})$ be a double analytic sequence of strictly positive real numbers, the sequence spaces

$$\left[\chi_{f\mu}^{2q}, \|(d(x_1), d(x_2), \dots, d(x_{n-1}))\|_p^\varphi \right]_{\theta_{rs}}^{I^2}$$

and $\left[\Lambda_{f\mu}^{2q}, \|(d(x_1), d(x_2), \dots, d(x_{n-1}))\|_p^\varphi \right]_{\theta_{rs}}^{I^2}$ are linear spaces.

Proof: It is routine verification. Therefore the proof is omitted.

3.2 Theorem

Let $f = (f_{mn})$ be a Musielak-modulus function, $q = (q_{mn})$ be a double analytic sequence of strictly positive real numbers, the sequence space

$$\left[\chi_{f\mu}^{2q}, \|(d(x_1), d(x_2), \dots, d(x_{n-1}))\|_p^\varphi \right]_{\theta_{rs}}^{I^2}$$

is a paranormed space with respect to the paranorm defined by

$$g(x) = \inf \left\{ \left[f_{mn} \left(\|\mu_{mn}(x), (d(x_1), d(x_2), \dots, d(x_{n-1}))\|_p \right) \right]^{q_{mn}} \leq 1 \right\}.$$

Proof: Clearly $g(x) \geq 0$ for

$$x = (x_{mn}) \in \left[\chi_{f\mu}^{2q}, \|(d(x_1), d(x_2), \dots, d(x_{n-1}))\|_p \right]_{\theta_{rs}}^{I^2}$$

Since $f_{mn}(0) = 0$, we get $g(0) = 0$.

Conversely, suppose that $g(x) = 0$, then

$$\inf \left\{ \left[f_{mn} \left(\|\mu_{mn}(x), (d(x_1), d(x_2), \dots, d(x_{n-1}))\|_p \right) \right]^{q_{mn}} \leq 1 \right\} = 0.$$

Suppose that $\mu_{mn}(x) \neq 0$ for each $m, n \in \mathbb{N}$. Then

$$\|\mu_{mn}(x), (d(x_1), d(x_2), \dots, d(x_{n-1}))\|_p^\varphi \rightarrow \infty.$$

It follows that

$$\left(\left[f_{mn} \left(\|\mu_{mn}(x), (d(x_1), d(x_2), \dots, d(x_{n-1}))\|_p \right) \right]^{q_{mn}} \right)^{1/H} \rightarrow \infty$$

which is a contradiction. Therefore $\mu_{mn}(x) = 0$. Let

$$\left(\left[f_{mn} \left(\|\mu_{mn}(x), (d(x_1), d(x_2), \dots, d(x_{n-1}))\|_p \right) \right]^{q_{mn}} \right)^{1/H} \leq 1$$

and

$$\left(\left[f_{mn} \left(\|\mu_{mn}(y), (d(x_1), d(x_2), \dots, d(x_{n-1}))\|_p \right) \right]^{q_{mn}} \right)^{1/H} \leq 1$$

Then by using Minkowski's inequality, we have

$$\begin{aligned} & \left(\left[f_{mn} \left(\|\mu_{mn}(x+y), (d(x_1), d(x_2), \dots, d(x_{n-1}))\|_p \right) \right]^{q_{mn}} \right)^{1/H} \\ & \leq \left(\left[f_{mn} \left(\|\mu_{mn}(x), (d(x_1), d(x_2), \dots, d(x_{n-1}))\|_p \right) \right]^{q_{mn}} \right)^{1/H} \\ & \quad + \left(\left[f_{mn} \left(\|\mu_{mn}(y), (d(x_1), d(x_2), \dots, d(x_{n-1}))\|_p \right) \right]^{q_{mn}} \right)^{1/H}. \end{aligned}$$

So we have

$$\begin{aligned} g(x+y) &= \inf \left\{ \left[f_{mn} \left(\|\mu_{mn}(x+y), (d(x_1), d(x_2), \dots, d(x_{n-1}))\|_p \right) \right]^{q_{mn}} \leq 1 \right\} \\ &\leq \inf \left\{ \left[f_{mn} \left(\|\mu_{mn}(x), (d(x_1), d(x_2), \dots, d(x_{n-1}))\|_p \right) \right]^{q_{mn}} \leq 1 \right\} \\ &\quad + \inf \left\{ \left[f_{mn} \left(\|\mu_{mn}(y), (d(x_1), d(x_2), \dots, d(x_{n-1}))\|_p \right) \right]^{q_{mn}} \leq 1 \right\} \end{aligned}$$

Therefore,

$$g(x + y) \leq g(x) + g(y).$$

Finally, to prove that the scalar multiplication is continuous. Let λ be any complex number. By definition,

$$g(\lambda x) = \inf \left\{ \left[f_{mn} \left(\|\mu_{mn}(\lambda x), (d(x_1), d(x_2), \dots, d(x_{n-1}))\|_p \right) \right]^{q_{mn}} \leq 1 \right\}.$$

Then

$$g(\lambda x) = \inf \left\{ ((|\lambda| t)^{q_{mn}/H} : \left[f_{mn} \left(\|\mu_{mn}(\lambda x), (d(x_1), d(x_2), \dots, d(x_{n-1}))\|_p \right) \right]^{q_{mn}} \leq 1 \right\}$$

where $t = \frac{1}{|\lambda|}$. Since $|\lambda|^{q_{mn}} \leq \max(1, |\lambda|^{supq_{mn}})$, we have

$$g(\lambda x) \leq \max(1, |\lambda|^{supq_{mn}})$$

$$\inf \left\{ t^{q_{mn}/H} : \left[f_{mn} \left(\|\mu_{mn}(\lambda x), (d(x_1), d(x_2), \dots, d(x_{n-1}))\|_p \right) \right]^{q_{mn}} \leq 1 \right\}$$

This completes the proof.

3.3 Theorem

(i) If the Musielak modulus function (f_{mn}) satisfies Δ_2 - condition, then

$$\begin{aligned} & \left[\chi_{f\mu}^{2q}, \|\mu_{mn}(x), (d(x_1), d(x_2), \dots, d(x_{n-1}))\|_p \right]_{\theta_{rs}}^{I^{2\alpha}} \\ &= \left[\chi_g^{2q\mu}, \|\mu_{uv}(x), (d(x_1), d(x_2), \dots, d(x_{n-1}))\|_p \right]_{\theta_{rs}}^{I^2}. \end{aligned}$$

(ii) If the Musielak modulus function (g_{mn}) satisfies Δ_2 - condition, then

$$\begin{aligned} & \left[\chi_g^{2q\mu}, \|\mu_{mn}(x), (d(x_1), d(x_2), \dots, d(x_{n-1}))\|_p \right]_{\theta_{rs}}^{I^{2\alpha}} \\ &= \left[\chi_{f\mu}^{2q}, \|\mu_{mn}(x), (d(x_1), d(x_2), \dots, d(x_{n-1}))\|_p \right]_{\theta_{rs}}^{I^2} \end{aligned}$$

Proof: Let the Musielak modulus function (f_{mn}) satisfies Δ_2 - condition, we get

$$\begin{aligned} & \left[\chi_g^{2q\mu}, \|\mu_{mn}(x), (d(x_1), d(x_2), \dots, d(x_{n-1}))\|_p \right]_{\theta_{rs}}^{I^2} \subset \quad (3.1) \\ & \left[\chi_{f\mu}^{2q}, \|\mu_{mn}(x), (d(x_1), d(x_2), \dots, d(x_{n-1}))\|_p \right]_{\theta_{rs}}^{I^{2\alpha}} \end{aligned}$$

To prove the inclusion

$$\begin{aligned} & \left[\chi_{f\mu}^{2q}, \|\mu_{mn}(x), (d(x_1), d(x_2), \dots, d(x_{n-1}))\|_p \right]_{\theta_{rs}}^{I^{2\alpha}} \\ & \subset \left[\chi_g^{2q\mu}, \|\mu_{mn}(x), (d(x_1), d(x_2), \dots, d(x_{n-1}))\|_p \right]_{\theta_{rs}}^{I^2}, \end{aligned}$$

let

$$a \in \left[\chi_{f\mu}^{2q}, \|\mu_{mn}(x), (d(x_1), d(x_2), \dots, d(x_{n-1}))\|_p^\varphi \right]_{\theta_{rs}}^{I^{2\alpha}}.$$

Then for all $\{x_{mn}\}$ with

$$(x_{mn}) \in \left[\chi_{f\mu}^{2q}, \|\mu_{mn}(x), (d(x_1), d(x_2), \dots, d(x_{n-1}))\|_p^\varphi \right]_{\theta_{rs}}^{I^2}$$

we have

$$\sum_{m=1}^{\infty} \sum_{n=1}^{\infty} |x_{mn} a_{mn}| < \infty. \tag{3.2}$$

Since the Musielak modulus function (f_{mn}) satisfies Δ_2 - condition, then

$$(y_{mn}) \in \left[\chi_{f\mu}^{2q}, \|\mu_{mn}(x), (d(x_1), d(x_2), \dots, d(x_{n-1}))\|_p^\varphi \right]_{\theta_{rs}}^{I^2},$$

we get

$$\sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \left| \frac{\varphi_{rs} y_{mn} a_{mn}}{\Delta^m \lambda_{mn} (m+n)!} \right| < \infty.$$

by (3.2). Thus

$$\begin{aligned} (\varphi_{rs} a_{mn}) &\in \left[\chi_{f\mu}^{2q}, \|\mu_{mn}(x), (d(x_1), d(x_2), \dots, d(x_{n-1}))\|_p^\varphi \right]_{\theta_{rs}}^{I^2} \\ &= \left[\chi_g^{2q\mu}, \|\mu_{mn}(x), (d(x_1), d(x_2), \dots, d(x_{n-1}))\|_p^\varphi \right]_{\theta_{rs}}^{I^2} \end{aligned}$$

and hence

$$(a_{mn}) \in \left[\chi_g^{2q\mu}, \|\mu_{mn}(x), (d(x_1), d(x_2), \dots, d(x_{n-1}))\|_p^\varphi \right]_{\theta_{rs}}^{I^2}.$$

This gives that

$$\begin{aligned} &\left[\chi_{f\mu}^{2q}, \|\mu_{mn}(x), (d(x_1), d(x_2), \dots, d(x_{n-1}))\|_p^\varphi \right]_{\theta_{rs}}^{I^{2\alpha}} \\ &\subset \left[\chi_g^{2q\mu}, \|\mu_{mn}(x), (d(x_1), d(x_2), \dots, d(x_{n-1}))\|_p^\varphi \right]_{\theta_{rs}}^{I^2} \end{aligned}$$

we are granted with (3.1) and (3.3)

$$\begin{aligned} &\left[\chi_{f\mu}^{2q}, \|\mu_{mn}(x), (d(x_1), d(x_2), \dots, d(x_{n-1}))\|_p^\varphi \right]_{\theta_{rs}}^{I^{2\alpha}} \\ &= \left[\chi_g^{2q\mu}, \|\mu_{mn}(x), (d(x_1), d(x_2), \dots, d(x_{n-1}))\|_p^\varphi \right]_{\theta_{rs}}^{I^2} \end{aligned}$$

(ii) Similarly, one can prove that

$$\begin{aligned} & \left[\chi_g^{2q\mu}, \|\mu_{mn}(x), (d(x_1), d(x_2), \dots, d(x_{n-1}))\|_p^\varphi \right]_{\theta_{rs}}^{I^{2\alpha}} \\ & \subset \left[\chi_{f\mu}^{2q}, \|\mu_{mn}(x), (d(x_1), d(x_2), \dots, d(x_{n-1}))\|_p^\varphi \right]_{\theta_{rs}}^{I^2} \end{aligned}$$

if the Musielak modulus function (g_{mn}) satisfies Δ_2 -condition.

3.4 Proposition

If $0 < q_{mn} < p_{mn} < \infty$ for each m and n , then

$$\begin{aligned} & \left[\Lambda_{f\mu}^{2q}, \|\mu_{mn}(x), (d(x_1), d(x_2), \dots, d(x_{n-1}))\|_p^\varphi \right]_{\theta_{rs}}^{I^2} \\ & \subseteq \left[\Lambda_{f\mu}^{2p}, \|\mu_{mn}(x), (d(x_1), d(x_2), \dots, d(x_{n-1}))\|_p^\varphi \right]_{\theta_{rs}}^{I^2} \end{aligned}$$

Proof: The proof is standard, so we omit it.

3.5 Proposition

(i) If $0 < inf q_{mn} \leq q_{mn} < 1$ then

$$\begin{aligned} & \left[\Lambda_{f\mu}^{2q}, \|\mu_{mn}(x), (d(x_1), d(x_2), \dots, d(x_{n-1}))\|_p^\varphi \right]_{\theta_{rs}}^{I^2} \\ & \subset \left[\Lambda_{f\mu}^2, \|\mu_{mn}(x), (d(x_1), d(x_2), \dots, d(x_{n-1}))\|_p^\varphi \right]_{\theta_{rs}}^{I^2}. \end{aligned}$$

(ii) If $1 \leq q_{mn} \leq sup q_{mn} < \infty$, then

$$\begin{aligned} & \left[\Lambda_{f\mu}^2, \|\mu_{mn}(x), (d(x_1), d(x_2), \dots, d(x_{n-1}))\|_p^\varphi \right]_{\theta_{rs}}^{I^2} \\ & \subset \left[\Lambda_{f\mu}^{2q}, \|\mu_{mn}(x), (d(x_1), d(x_2), \dots, d(x_{n-1}))\|_p^\varphi \right]_{\theta_{rs}}^{I^2} \end{aligned}$$

Proof: The proof is standard, so we omit it.

3.6 Proposition

Let $f' = (f'_{mn})$ and $f'' = (f''_{mn})$ are sequences of Musielak functions, we have

$$\begin{aligned} & \left[\Lambda_{f'\mu}^{2q}, \|\mu_{mn}(x), (d(x_1), d(x_2), \dots, d(x_{n-1}))\|_p^\varphi \right]_{\theta_{rs}}^{I^2} \\ & \quad \cap \left[\Lambda_{f''\mu}^{2q}, \|\mu_{mn}(x), (d(x_1), d(x_2), \dots, d(x_{n-1}))\|_p^\varphi \right]_{\theta_{rs}}^{I^2} \\ & \subseteq \left[\Lambda_{f'+f''\mu}^{2q}, \|\mu_{mn}(x), (d(x_1), d(x_2), \dots, d(x_{n-1}))\|_p^\varphi \right]_{\theta_{rs}}^{I^2} \end{aligned}$$

Proof: The proof is easy so we omit it.

3.7 Proposition

For any sequence of Musielak functions $f = (f_{mn})$ and $q = (q_{mn})$ be double analytic sequence of strictly positive real numbers. Then

$$\begin{aligned} & \left[\chi_{f\mu}^{2q}, \|\mu_{mn}(x), (d(x_1), d(x_2), \dots, d(x_{n-1}))\|_p^\varphi \right]_{\theta_{rs}}^{I^2} \\ & \subset \left[\Lambda_{f\mu}^{2q}, \|\mu_{mn}(x), (d(x_1), d(x_2), \dots, d(x_{n-1}))\|_p^\varphi \right]_{\theta_{rs}}^{I^2}. \end{aligned}$$

Proof: The proof is easy so we omit it.

3.8 Proposition

The sequence space $\left[\Lambda_{f\mu}^{2q}, \|\mu_{mn}(x), (d(x_1), d(x_2), \dots, d(x_{n-1}))\|_p^\varphi \right]_{\theta_{rs}}^{I^2}$ is solid

Proof: Let $x = (x_{mn}) \in \left[\Lambda_{f\mu}^{2q}, \|\mu_{mn}(x), (d(x_1), d(x_2), \dots, d(x_{n-1}))\|_p^\varphi \right]_{\theta_{rs}}^{I^2}$, (i.e)

$$\sup_{mn} \left[\Lambda_{f\mu}^{2q}, \|\mu_{mn}(x), (d(x_1), d(x_2), \dots, d(x_{n-1}))\|_p^\varphi \right]_{\theta_{rs}}^{I^2} < \infty.$$

Let (α_{mn}) be double sequence of scalars such that $|\alpha_{mn}| \leq 1$ for all $m, n \in \mathbb{N} \times \mathbb{N}$. Then we get

$$\begin{aligned} & \sup_{mn} \left[\Lambda_{f\mu}^{2q}, \|\mu_{mn}(\alpha x), (d(x_1), d(x_2), \dots, d(x_{n-1}))\|_p^\varphi \right]_{\theta_{rs}}^{I^2} \\ & \leq \sup_{mn} \left[\Lambda_{f\mu}^{2q}, \|\mu_{mn}(x), (d(x_1), d(x_2), \dots, d(x_{n-1}))\|_p^\varphi \right]_{\theta_{rs}}^{I^2}. \end{aligned}$$

This completes the proof.

3.9 Proposition

The sequence space $\left[\Lambda_{f\mu}^{2q}, \|\mu_{mn}(x), (d(x_1), d(x_2), \dots, d(x_{n-1}))\|_p^{\varphi} \right]_{\theta_{rs}}^{I^2}$ is monotone

Proof: The proof follows from Proposition 3.8.

3.10 Proposition

If $f = (f_{mn})$ be any Musielak function. Then

$$\begin{aligned} & \left[\Lambda_{f\mu}^{2q}, \|\mu_{mn}(x), (d(x_1), d(x_2), \dots, d(x_{n-1}))\|_p^{\varphi^*} \right]_{\theta_{rs}}^{I^2} \\ & \subset \left[\Lambda_{f\mu}^{2q}, \|\mu_{mn}(x), (d(x_1), d(x_2), \dots, d(x_{n-1}))\|_p^{\varphi^{**}} \right]_{\theta_{rs}}^{I^2} \end{aligned}$$

if and only if $\sup_{r,s \geq 1} \frac{\varphi_{rs}^*}{\varphi_{rs}^{**}} < \infty$.

Proof: Let

$$x \in \left[\Lambda_{f\mu}^{2q}, \|\mu_{mn}(x), (d(x_1), d(x_2), \dots, d(x_{n-1}))\|_p^{\varphi^*} \right]_{\theta_{rs}}^{I^2}$$

and

$$N = \sup_{r,s \geq 1} \frac{\varphi_{rs}^*}{\varphi_{rs}^{**}} < \infty.$$

Then we get

$$\begin{aligned} & \left[\Lambda_{f\mu}^{2q}, \|\mu_{mn}(x), (d(x_1), d(x_2), \dots, d(x_{n-1}))\|_p^{\varphi^{**}} \right]_{\theta_{rs}}^{I^2} \\ & = \mathbb{N} \left[\Lambda_{f\mu}^{2q}, \|\mu_{mn}(x), (d(x_1), d(x_2), \dots, d(x_{n-1}))\|_p^{\varphi^*} \right]_{\theta_{rs}}^{I^2} = 0. \end{aligned}$$

Thus $x \in \left[\Lambda_{f\mu}^{2q}, \|\mu_{mn}(x), (d(x_1), d(x_2), \dots, d(x_{n-1}))\|_p^{\varphi^{**}} \right]_{\theta_{rs}}^{I^2}$.

Conversely, suppose that

$$\begin{aligned} & \left[\Lambda_{f\mu}^{2q}, \|\mu_{mn}(x), (d(x_1), d(x_2), \dots, d(x_{n-1}))\|_p^{\varphi^*} \right]_{N\theta}^I \\ & \subset \left[\Lambda_{f\mu}^{2qu}, \|\mu_{mn}(x), (d(x_1), d(x_2), \dots, d(x_{n-1}))\|_p^{\varphi^{**}} \right]_{\theta_{rs}}^{I^2} \end{aligned}$$

and

$$x \in \left[\Lambda_{f\mu}^{2q}, \|\mu_{mn}(x), (d(x_1), d(x_2), \dots, d(x_{n-1}))\|_p^{\varphi^*} \right]_{\theta_{rs}}^{I^2}.$$

Then

$$\left[\Lambda_{f\mu}^{2q}, \|\mu_{mn}(x), (d(x_1), d(x_2), \dots, d(x_{n-1}))\|_p^{\varphi^*} \right]_{\theta_{rs}}^{I^2} < \epsilon,$$

for every $\epsilon > 0$. Suppose that $\sup_{r,s \geq 1} \frac{\varphi_{rs}^*}{\varphi_{rs}^{**}} = \infty$, then there exists a sequence of members (rs_{jk}) such that $\lim_{j,k \rightarrow \infty} \frac{\varphi_{jk}^*}{\varphi_{jk}^{**}} = \infty$. Hence, we have

$$\left[\Lambda_{f\mu}^{2q}, \|\mu_{mn}(x), (d(x_1), d(x_2), \dots, d(x_{n-1}))\|_p^{\varphi_{rs}^*} \right]_{\theta_{rs}}^{I^2} = \infty.$$

Therefore

$$x \notin \left[\Lambda_{f\mu}^{2q}, \|\mu_{mn}(x), (d(x_1), d(x_2), \dots, d(x_{n-1}))\|_p^{\varphi^{**}} \right]_{\theta_{rs}}^{I^2},$$

which is a contradiction. This completes the proof.

3.11 Proposition

If $f = (f_{mn})$ be any Musielak function. Then

$$\begin{aligned} & \left[\Lambda_{f\mu}^{2q}, \|\mu_{mn}(x), (d(x_1), d(x_2), \dots, d(x_{n-1}))\|_p^{\varphi^*} \right]_{\theta_{rs}}^{I^2} \\ &= \left[\Lambda_{f\mu}^{2q}, \|\mu_{mn}(x), (d(x_1), d(x_2), \dots, d(x_{n-1}))\|_p^{\varphi^{**}} \right]_{\theta_{rs}}^{I^2} \end{aligned}$$

if and only if

$$\sup_{r,s \geq 1} \frac{\varphi_{rs}^*}{\varphi_{rs}^{**}} < \infty, \quad \sup_{r,s \geq 1} \frac{\varphi_{rs}^{**}}{\varphi_{rs}^*} > \infty.$$

Proof: It is easy to prove so we omit.

3.12 Proposition

The sequence space

$$\left[\chi_{f\mu}^{2q}, \|\mu_{mn}(x), (d(x_1), d(x_2), \dots, d(x_{n-1}))\|_p^{\varphi} \right]_{\theta_{rs}}^{I^2}$$

is not solid

Proof: The result follows from the following example.

Example: Consider

$$x = (x_{mn}) = \begin{pmatrix} 1 & 1 & \dots & 1 \\ 1 & 1 & \dots & 1 \\ \cdot & & & \\ \cdot & & & \\ \cdot & & & \\ 1 & 1 & \dots & 1 \end{pmatrix} \in \left[\chi_{f\mu}^{2q}, \|\mu_{mn}(x), (d(x_1), d(x_2), \dots, d(x_{n-1}))\|_p^{\varphi} \right]_{\theta_{rs}}^{I^2}.$$

Let

$$\alpha_{mn} = \begin{pmatrix} -1^{m+n} & -1^{m+n} & \dots & -1^{m+n} \\ -1^{m+n} & -1^{m+n} & \dots & -1^{m+n} \\ \vdots & & & \\ -1^{m+n} & -1^{m+n} & \dots & -1^{m+n} \end{pmatrix},$$

for all $m, n \in \mathbb{N}$. Then

$$\alpha_{mn} x_{mn} \notin \left[\chi_{f\mu}^{2q}, \|\mu_{mn}(x), (d(x_1), d(x_2), \dots, d(x_{n-1}))\|_p^\varphi \right]_{\theta_{rs}}^{I^2}.$$

Hence

$$\left[\chi_{f\mu}^{2q}, \|\mu_{mn}(x), (d(x_1), d(x_2), \dots, d(x_{n-1}))\|_p^\varphi \right]_{\theta_{rs}}^{I^2}$$

is not solid.

3.13 Proposition

The sequence space $\left[\chi_{f\mu}^{2q}, \|\mu_{mn}(x), (d(x_1), d(x_2), \dots, d(x_{n-1}))\|_p^\varphi \right]_{\theta_{rs}}^{I^2}$ is not monotone

Proof: The proof follows from Proposition 3.12.

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Structure of solutions of nonautonomous optimal control problems in metric spaces

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ABSTRACT: We establish turnpike results for a nonautonomous discrete-time optimal control system describing a model of economic dynamics.

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Keywords and Phrases: *Compact metric space, good program, infinite horizon problem, overtaking optimal program, turnpike property*

1 Introduction

The study of the existence, the structure and properties of (approximate) solutions of optimal control problems defined on infinite intervals and on sufficiently large intervals has recently been a rapidly growing area of research [4-8, 10, 11, 14, 15, 16, 18-20, 22, 23, 27, 30]. These problems arise in engineering [1, 32], in models of economic growth [2, 9, 12, 17, 21, 24, 25, 27-29, 31], in infinite discrete models of solid-state physics related to dislocations in one-dimensional crystals [3, 26] and in the theory of thermodynamical equilibrium for materials [13, 16].

In this paper we study the structure of approximate solutions of nonautonomous discrete-time optimal control systems arising in economic dynamics which are determined by sequences of lower semicontinuous objective functions.

For each nonempty set Y denote by $\mathcal{B}(Y)$ the set of all bounded functions $f : Y \rightarrow \mathbb{R}^1$ and for each $f \in \mathcal{B}(Y)$ set

$$\|f\| = \sup\{|f(y)| : y \in Y\}.$$

For each nonempty compact metric space Y denote by $C(Y)$ the set of all continuous functions $f : Y \rightarrow \mathbb{R}^1$.

Let (X, ρ) be a compact metric space with the metric ρ . The set $X \times X$ is equipped with the metric ρ_1 defined by

$$\rho_1((x_1, x_2), (y_1, y_2)) = \rho(x_1, y_1) + \rho(x_2, y_2), \quad (x_1, x_2), (y_1, y_2) \in X \times X.$$

For each integer $t \geq 0$ let Ω_t be a nonempty closed subset of the metric space $X \times X$.

Let $T \geq 0$ be an integer. A sequence $\{x_t\}_{t=0}^{\infty} \subset X$ is called a program if $(x_t, x_{t+1}) \in \Omega_t$ for all integers $t \geq 0$.

Let T_1, T_2 be integers such that $0 \leq T_1 < T_2$. A sequence $\{x_t\}_{t=T_1}^{T_2} \subset X$ is called a program if $(x_t, x_{t+1}) \in \Omega_t$ for all integers t satisfying $T_1 \leq t < T_2$.

We assume that there exists a program $\{x_t\}_{t=0}^{\infty}$. Denote by \mathcal{M} the set of all sequences of functions $\{f_t\}_{t=0}^{\infty}$ such that for each integer $t \geq 0$

$$f_t \in \mathcal{B}(\Omega_t) \tag{1.1}$$

and that

$$\sup\{\|f_t\| : t = 0, 1, \dots\} < \infty. \tag{1.2}$$

For each pair of sequences $\{f_t\}_{t=0}^{\infty}, \{g_t\}_{t=0}^{\infty} \in \mathcal{M}$ set

$$d(\{f_t\}_{t=0}^{\infty}, \{g_t\}_{t=0}^{\infty}) = \sup\{\|f_t - g_t\| : t = 0, 1, \dots\}. \tag{1.3}$$

It is easy to see that $d : \mathcal{M} \times \mathcal{M} \rightarrow [0, \infty)$ is a metric on \mathcal{M} and that the metric space (\mathcal{M}, d) is complete.

Let $\{f_t\}_{t=0}^{\infty} \in \mathcal{M}$. We consider the following optimization problems

$$\sum_{t=T_1}^{T_2-1} f_t(x_t, x_{t+1}) \rightarrow \min \text{ s. t. } \{x_t\}_{t=T_1}^{T_2} \text{ is a program,}$$

$$\sum_{t=T_1}^{T_2-1} f_t(x_t, x_{t+1}) \rightarrow \min \text{ s. t. } \{x_t\}_{t=T_1}^{T_2} \text{ is a program and } x_{T_1} = y,$$

$$\sum_{t=T_1}^{T_2-1} f_t(x_t, x_{t+1}) \rightarrow \min \text{ s. t. } \{x_t\}_{t=T_1}^{T_2} \text{ is a program and } x_{T_1} = y, x_{T_2} = z,$$

where $y, z \in X$ and integers T_1, T_2 satisfy $0 \leq T_1 < T_2$.

The interest in these discrete-time optimal problems stems from the study of various optimization problems which can be reduced to this framework, e. g., continuous-time control systems which are represented by ordinary differential equations whose cost integrand contains a discounting factor [12], the study of the discrete Frenkel-Kontorova model related to dislocations in one-dimensional crystals [3, 26] and the analysis of a long slender bar of a polymeric material under tension in [13, 16]. Similar optimization problems are also considered in mathematical economics [9, 17, 24, 28, 29, 31].

For each $y, z \in X$ and each pair of integers T_1, T_2 satisfying $0 \leq T_1 < T_2$ set

$$U(\{f_t\}_{t=0}^\infty, T_1, T_2) = \inf\left\{\sum_{t=T_1}^{T_2-1} f_t(x_t, x_{t+1}) : \{x_t\}_{t=T_1}^{T_2} \text{ is a program}\right\}, \quad (1.4)$$

$$U(\{f_t\}_{t=0}^\infty, T_1, T_2, y) = \inf\left\{\sum_{t=T_1}^{T_2-1} f_t(x_t, x_{t+1}) : \{x_t\}_{t=T_1}^{T_2} \text{ is a program and } x_{T_1} = y\right\}, \quad (1.5)$$

$$U(\{f_t\}_{t=0}^\infty, T_1, T_2, y, z) = \inf\left\{\sum_{t=T_1}^{T_2-1} f_t(x_t, x_{t+1}) : \{x_t\}_{t=T_1}^{T_2} \text{ is a program and } x_{T_1} = y, x_{T_2} = z\right\}. \quad (1.6)$$

Here we assume that the infimum over empty set is ∞ .

Denote by \mathcal{M}_{reg} the set of all sequences of functions $\{f_i\}_{i=0}^\infty \in \mathcal{M}$ for which there exist a program $\{x_t^f\}_{t=0}^\infty$ and constants $c_f > 0$, $\gamma_f > 0$ such that the following conditions hold:

- (C1) the function f_t is lower semicontinuous for all integers $t \geq 0$;
- (C2) for each pair of integers $T_1 \geq 0$, $T_2 > T_1$,

$$\sum_{t=T_1}^{T_2-1} f_t(x_t^f, x_{t+1}^f) \leq U(\{f_t\}_{t=0}^\infty, T_1, T_2) + c_f;$$

- (C3) for each $\epsilon > 0$ there exists $\delta > 0$ such that for each integer $t \geq 0$ and each $(x, y) \in \Omega_t$ satisfying $\rho(x, x_t^f) \leq \delta$, $\rho(y, x_{t+1}^f) \leq \delta$ we have

$$|f_t(x_t^f, x_{t+1}^f) - f_t(x, y)| \leq \epsilon;$$

- (C4) for each integer $t \geq 0$, each $(x_t, x_{t+1}) \in \Omega_t$ satisfying $\rho(x_t, x_t^f) \leq \gamma_f$ and each $(x'_{t+1}, x'_{t+2}) \in \Omega_{t+1}$ satisfying $\rho(x'_{t+2}, x_{t+2}^f) \leq \gamma_f$ there is $x \in X$ such that

$$(x_t, x) \in \Omega_t, (x, x'_{t+2}) \in \Omega_{t+1};$$

moreover, for each $\epsilon > 0$ there exists $\delta \in (0, \gamma_f)$ such that for each integer $t \geq 0$, each $(x_t, x_{t+1}) \in \Omega_t$ and each $(x'_{t+1}, x'_{t+2}) \in \Omega_{t+1}$ satisfying $\rho(x_t, x_t^f) \leq \delta$ and $\rho(x'_{t+2}, x_{t+2}^f) \leq \delta$ there is $x \in X$ such that

$$(x_t, x) \in \Omega_t, (x, x'_{t+2}) \in \Omega_{t+1}, \rho(x, x_{t+1}^f) \leq \epsilon.$$

Denote by $\bar{\mathcal{M}}_{reg}$ the closure of \mathcal{M}_{reg} in (\mathcal{M}, d) . Denote by $\mathcal{M}_{c,reg}$ the set of all sequences $\{f_i\}_{i=0}^\infty \in \mathcal{M}_{reg}$ such that $f_i \in C(\Omega_i)$ for all integers $i \geq 0$ and by $\bar{\mathcal{M}}_{c,reg}$ the closure of $\mathcal{M}_{c,reg}$ in (\mathcal{M}, d) .

We study the optimization problems stated above with the sequence of objective functions $\{f_i\}_{i=0}^\infty \in \mathcal{M}_{reg}$. Our study is based on the relation between these finite

horizon problems and the corresponding infinite horizon optimization problem determined by $\{f_i\}_{i=0}^\infty$. Note that the condition (C2) means that the program $\{x_i^f\}_{i=0}^\infty$ is an approximate solution of this infinite horizon problem.

We are interested in turnpike properties of approximate solutions of our optimization problems, which are independent of the length of the interval $T_2 - T_1$, for all sufficiently large intervals. To have these properties means that the approximate solutions of the problems are determined mainly by the objective functions, and are essentially independent of the choice of interval and endpoint conditions, except in regions close to the endpoints. Turnpike properties are well known in mathematical economics. The term was first coined by Samuelson in 1948 (see [26]) where he showed that an efficient expanding economy would spend most of the time in the vicinity of a balanced equilibrium path (also called a von Neumann path).

The paper is organized as follows. In Section 2 we present turnpike results and show the existence of optimal solutions over infinite horizon established in [31]. Our main results (Theorems 3.1 and 3.2) are stated in Section 3. Section 4 contains an example. Our auxiliary results are proved in Section 5. Section 6 contains the proof of Theorem 3.1 while Theorem 3.2 is proved in Section 7.

2 Preliminaries

Let $\{f_i\}_{i=0}^\infty \in \mathcal{M}_{reg}$, a program $\{x_i^f\}_{i=0}^\infty$, $c_f > 0$ and $\gamma_f > 0$ be such that (C1)-(C4) hold.

In [31] we proved the following useful result.

Proposition 2.1 *Let $S \geq 0$ be an integer and $\{x_i\}_{i=S}^\infty$ be a program. Then either the sequence $\{\sum_{i=S}^{T-1} f_i(x_i, x_{i+1}) - \sum_{i=S}^{T-1} f_i(x_i^f, x_{i+1}^f)\}_{T=S+1}^\infty$ is bounded or*

$$\lim_{T \rightarrow \infty} \left[\sum_{i=S}^{T-1} f_i(x_i, x_{i+1}) - \sum_{i=S}^{T-1} f_i(x_i^f, x_{i+1}^f) \right] = \infty.$$

A program $\{x_t\}_{t=S}^\infty$, where $S \geq 0$ is an integer, is called $(\{f_i\}_{i=0}^\infty)$ -good if the sequence

$$\left\{ \sum_{i=S}^{T-1} f_i(x_i, x_{i+1}) - \sum_{i=S}^{T-1} f_i(x_i^f, x_{i+1}^f) \right\}_{T=S+1}^\infty$$

is bounded [9, 27-29, 31].

We say that the sequence $\{f_i\}_{i=0}^\infty$ possesses an asymptotic turnpike property (or briefly (ATP)) [31] with $\{x_i^f\}_{i=0}^\infty$ being the turnpike if for each integer $S \geq 0$ and each $(\{f_i\}_{i=0}^\infty)$ -good program $\{x_i\}_{i=S}^\infty$,

$$\lim_{i \rightarrow \infty} \rho(x_i, x_i^f) = 0.$$

We say that the sequence $\{f_i\}_{i=0}^\infty$ possesses a turnpike property (or briefly (TP)) [31] if for each $\epsilon > 0$ and each $M > 0$ there exist $\delta > 0$ and a natural number L such

that for each pair of integers $T_1 \geq 0$, $T_2 \geq T_1 + 2L$ and each program $\{x_t\}_{t=T_1}^{T_2}$ which satisfies

$$\sum_{i=T_1}^{T_2-1} f_i(x_i, x_{i+1}) \leq \min\{U(\{f_i\}_{i=0}^\infty, T_1, T_2, x_{T_1}, x_{T_2}) + \delta, U(\{f_i\}_{i=0}^\infty, T_1, T_2) + M\},$$

the inequality $\rho(x_i, x_i^f) \leq \epsilon$ holds for all integers $i = T_1 + L, \dots, T_2 - L$.

The sequence $\{x_i^f\}_{i=0}^\infty$ is called the turnpike of $\{f_i\}_{i=0}^\infty$.

In [31] we proved the following results (see Theorems 2.1-2.4).

Theorem 2.1 *The sequence $\{f_i\}_{i=0}^\infty$ possesses the turnpike property if and only if $\{f_i\}_{i=0}^\infty$ possesses (ATP) and the following property:*

(P) *For each $\epsilon > 0$ and each $M > 0$ there exist $\delta > 0$ and a natural number L such that for each integer $T \geq 0$ and each program $\{x_t\}_{t=T}^{T+L}$ which satisfies*

$$\begin{aligned} & \sum_{i=T}^{T+L-1} f_i(x_i, x_{i+1}) \\ & \leq \min\{U(\{f_i\}_{i=0}^\infty, T, T+L, x_T, x_{T+L}) + \delta, U(\{f_i\}_{i=0}^\infty, T, T+L) + M\} \end{aligned}$$

there is an integer $j \in \{T, \dots, T+L\}$ for which $\rho(x_j, x_j^f) \leq \epsilon$.

The property (P) means that if a natural number L is large enough and a program $\{x_t\}_{t=T}^{T+L}$ is an approximate solution of the corresponding finite horizon problem, then there is $j \in \{T, \dots, T+L\}$ such that x_j is close to x_j^f .

We denote by $\text{Card}(A)$ the cardinality of the set A .

Theorem 2.2 *Assume that the sequence $\{f_i\}_{i=0}^\infty$ possesses (ATP) and the property (P), $\epsilon > 0$ and $M > 0$. Then there exists a natural number L such that for each pair of integers $T_1 \geq 0$, $T_2 > T_1 + L$ and each program $\{x_t\}_{t=T_1}^{T_2}$ which satisfies*

$$\sum_{t=T_1}^{T_2-1} f_t(x_t, x_{t+1}) \leq U(\{f_i\}_{i=0}^\infty, T_1, T_2) + M$$

the following inequality holds:

$$\text{Card}(\{t \in \{T_1, \dots, T_2\} : \rho(x_t, x_t^f) > \epsilon\}) \leq L.$$

Let $S \geq 0$ be an integer. A program $\{x_t\}_{t=S}^\infty$ is called $(\{f_i\}_{i=0}^\infty)$ -minimal [3, 26, 31] if for each integer $T > S$,

$$\sum_{t=S}^{T-1} f_t(x_t, x_{t+1}) = U(\{f_i\}_{i=0}^\infty, S, T, x_S, x_T).$$

A program $\{x_t\}_{t=S}^\infty$ is called $(\{f_i\}_{i=0}^\infty)$ -overtaking optimal [12, 27, 31] if for each program $\{x'_t\}_{t=S}^\infty$ satisfying $x_S = x'_S$,

$$\limsup_{T \rightarrow \infty} \left(\sum_{t=S}^{T-1} f_t(x_t, x_{t+1}) - \sum_{t=S}^{T-1} f_t(x'_t, x'_{t+1}) \right) \leq 0.$$

Theorem 2.3 Assume that the sequence $\{f_i\}_{i=0}^\infty$ possesses (ATP), $z \in X$, $S \geq 0$ is an integer and that there exists an $(\{f_i\}_{i=0}^\infty)$ -good program $\{x_t\}_{t=S}^\infty$ satisfying $x_S = z$. Then there exists an $(\{f_i\}_{i=0}^\infty)$ -overtaking optimal program $\{x_t^*\}_{t=S}^\infty$ satisfying $x_S^* = z$.

Theorem 2.4 Assume that the sequence $\{f_i\}_{i=0}^\infty$ possesses (ATP), $z \in X$, $S \geq 0$ is an integer and that there exists an $(\{f_i\}_{i=0}^\infty)$ -good program $\{\bar{x}_t\}_{t=S}^\infty$ satisfying $\bar{x}_S = z$. Let a program $\{x_t\}_{t=S}^\infty$ satisfy $x_S = z$. Then the following properties are equivalent.

- (i) $\{x_t\}_{t=S}^\infty$ is an $(\{f_i\}_{i=0}^\infty)$ -overtaking optimal program;
- (ii) the program $\{x_t\}_{t=S}^\infty$ is $(\{f_i\}_{i=0}^\infty)$ -minimal and $(\{f_i\}_{i=0}^\infty)$ -good;
- (iii) the program $\{x_t\}_{t=S}^\infty$ is $(\{f_i\}_{i=0}^\infty)$ -minimal and satisfies $\lim_{t \rightarrow \infty} \rho(x_t, x_t^f) = 0$.

3 Main results

Let $\{f_i\}_{i=0}^\infty \in \mathcal{M}_{reg}$, a program $\{x_i^f\}_{i=0}^\infty$, $c_f > 0$ and $\gamma_f > 0$ be such that (C1)-(C4) hold.

We say that the sequence $\{f_i\}_{i=0}^\infty$ possesses a strong asymptotic turnpike property (or briefly (SATP)) with $\{x_i^f\}_{i=0}^\infty$ being the turnpike if for each integer $S \geq 0$ and each $(\{f_i\}_{i=0}^\infty)$ -good program $\{x_i\}_{i=S}^\infty$,

$$\sum_{i=0}^{\infty} \rho(x_i, x_i^f) < \infty.$$

Clearly, (SATP) implies (ATP).

We say that the sequence $\{f_i\}_{i=0}^\infty$ possesses a strong turnpike property (or briefly (STP)) if for each $\epsilon > 0$ and each $M > 0$ there exist $\delta > 0$ and a natural number L such that for each pair of integers $T_1 \geq 0$, $T_2 \geq T_1 + 2L$ and each program $\{x_t\}_{t=T_1}^{T_2}$ which satisfies

$$\sum_{i=T_1}^{T_2-1} f_i(x_i, x_{i+1}) \leq \min\{U(\{f_i\}_{i=0}^\infty, T_1, T_2, x_{T_1}, x_{T_2}) + \delta, U(\{f_i\}_{i=0}^\infty, T_1, T_2) + M\},$$

the inequality $\sum_{i=T_1+L}^{T_2-L} \rho(x_i, x_i^f) \leq \epsilon$ holds.

The sequence $\{x_i^f\}_{i=0}^\infty$ is called the turnpike of $\{f_i\}_{i=0}^\infty$.

Clearly, (STP) implies (TP).

In this paper we prove the following two results which are extensions of Theorems 2.1 and 2.2 respectively.

Theorem 3.1 *The sequence $\{f_i\}_{i=0}^\infty$ possesses the strong turnpike property if and only if $\{f_i\}_{i=0}^\infty$ possesses (SATP) and the property (P).*

Theorem 3.2 *Assume that the sequence $\{f_i\}_{i=0}^\infty$ possesses (SATP) and the property (P), and $M > 0$. Then there exist a natural number L and $M_0 > 0$ such that for each pair of integers $T_1 \geq 0$, $T_2 > T_1 + L$ and each program $\{x_t\}_{t=T_1}^{T_2}$ which satisfies*

$$\sum_{t=T_1}^{T_2-1} f_t(x_t, x_{t+1}) \leq U(\{f_i\}_{i=0}^\infty, T_1, T_2) + M$$

the following inequality holds:

$$\sum_{i=T_1}^{T_2} \rho(x_i, x_i^f) \leq M_0.$$

4 An example

Let $\{f_i\}_{i=0}^\infty \in \mathcal{M}_{reg}$, a program $\{x_i^f\}_{i=0}^\infty$, $c_f > 0$ and $\gamma_f > 0$ be such that (C1)-(C4) hold.

Now we show that $\{f_i\}_{i=0}^\infty$ is approximated by elements of \mathcal{M}_{reg} possessing (STP). For each $r \in (0, 1)$ and all integers $i \geq 0$ set

$$f_i^{(r)}(x, y) = f_i(x, y) + r\rho(x, x_i^f), \quad (x, y) \in \Omega_i. \quad (4.1)$$

Clearly, $\{f_i^{(r)}\}_{i=0}^\infty \in \mathcal{M}_{reg}$ for all $r \in (0, 1)$ and $\lim_{r \rightarrow 0^+} d(\{f_i^{(r)}\}_{i=0}^\infty, \{f_i\}_{i=0}^\infty) = 0$.

Proposition 4.1 *Let $r \in (0, 1)$. Then $\{f_i^{(r)}\}_{i=0}^\infty$ possesses (STP) with $\{x_i^f\}_{i=0}^\infty$ being the turnpike.*

Proof. By Proposition 2.6 of [31], $\{f_i^{(r)}\}_{i=0}^\infty$ possesses (TP) with $\{x_i^f\}_{i=0}^\infty$ being the turnpike. It follows from Theorem 2.1 that $\{f_i^{(r)}\}_{i=0}^\infty$ has the property (P). In view of Theorem 3.1 it is sufficient to show that $\{f_i^{(r)}\}_{i=0}^\infty$ possesses (SATP).

Assume that $S \geq 0$ is an integer and that a program $\{x_i\}_{i=S}^\infty$ is $(\{f_i^{(r)}\}_{i=0}^\infty)$ -good. Then there is $c_1 > 0$ such that

$$\left| \sum_{t=S}^{T-1} f_t^{(r)}(x_t, x_{t+1}) - \sum_{t=S}^{T-1} f_t^{(r)}(x_t^f, x_{t+1}^f) \right| \leq c_1 \text{ for all integers } T > S. \quad (4.2)$$

By Proposition 2.1, (4.1) and (4.2), $\sum_{t=S}^\infty \rho(x_t, x_t^f) < \infty$. Thus (SATP) holds. Proposition 4.1 is proved.

5 Auxiliary results

We use the notation, definitions and assumptions introduced in Sections 1-3.

The following two results were obtained in [31].

Lemma 5.1 *Let an integer $S \geq 0$ and a program $\{x_i\}_{i=S}^\infty$ be $(\{f_i\}_{i=0}^\infty)$ -good. Then there is a number $c > 0$ such that for each pair of integers $T_1 \geq S$ and $T_2 > T_1$,*

$$\sum_{i=T_1}^{T_2-1} f_i(x_i, x_{i+1}) \leq U(\{f_i\}_{i=0}^\infty, T_1, T_2) + c$$

and the following property holds:

for each $\epsilon > 0$ there exists a natural number L such that for each integer $T_1 \geq L$ and each integer $T_2 > T_1$,

$$\sum_{i=T_1}^{T_2-1} f_i(x_i, x_{i+1}) \leq U(\{f_i\}_{i=0}^\infty, T_1, T_2, x_{T_1}, x_{T_2}) + \epsilon.$$

Lemma 5.2 *Let $\epsilon > 0$. Then there exists $\delta > 0$ such that for each pair of integers $T_1 > 0$, $T_2 > T_1 + 2$ and each program $\{x_i\}_{i=T_1}^{T_2}$ satisfying*

$$\rho(x_{T_1+1}, x_{T_1+1}^f) \leq \delta, \quad \rho(x_{T_2-1}, x_{T_2-1}^f) \leq \delta,$$

$$\sum_{i=T_1+1}^{T_2-2} f_i(x_i, x_{i+1}) \leq U(\{f_i\}_{i=0}^\infty, T_1 + 1, T_2 - 1, x_{T_1+1}, x_{T_2-1}) + \delta$$

there exists a program $\{\tilde{x}_i\}_{i=T_1-1}^{T_2+1}$ such that

$$\tilde{x}_{T_1-1} = x_{T_1-1}^f, \quad \tilde{x}_{T_2+1} = x_{T_2+1}^f, \quad \tilde{x}_i = x_i, \quad i = T_1 + 1, \dots, T_2 - 1$$

and that the following inequality holds:

$$\sum_{i=T_1-1}^{T_2} f_i(\tilde{x}_i, \tilde{x}_{i+1}) \leq \sum_{i=T_1-1}^{T_2} f_i(x_i^f, x_{i+1}^f) + \epsilon.$$

Lemma 5.3 *Assume that $\{f_i\}_{i=0}^\infty$ possesses (SATP) and let $\epsilon > 0$. Then there exist $\delta > 0$ and a natural number L such that for each pair of integers $T_2 > T_1 \geq L$ and each program $\{x_i\}_{i=T_1}^{T_2}$ satisfying*

$$x_{T_1} = x_{T_1}^f, \quad x_{T_2} = x_{T_2}^f, \quad \sum_{i=T_1}^{T_2-1} f_i(x_i, x_{i+1}) \leq \sum_{i=T_1}^{T_2-1} f_i(x_i^f, x_{i+1}^f) + \delta$$

the inequality $\sum_{t=T_1}^{T_2} \rho(x_t, x_t^f) \leq \epsilon$ holds.

Proof. Assume that the lemma is not true. Then there exist sequences of natural numbers $\{T_k\}_{k=1}^\infty, \{S_k\}_{k=1}^\infty$ such that for each natural number k ,

$$T_k < S_k < T_{k+1}$$

and there exists a program $\{x_i^{(k)}\}_{i=T_k}^{S_k}$ such that

$$x_{T_k}^{(k)} = x_{T_k}^f, \quad x_{S_k}^{(k)} = x_{S_k}^f, \quad (5.1)$$

$$\sum_{i=T_k}^{S_k-1} f_i(x_i^{(k)}, x_{i+1}^{(k)}) \leq \sum_{i=T_k}^{S_k-1} f_i(x_i^f, x_{i+1}^f) + 2^{-k}, \quad (5.2)$$

$$\sum_{i=T_k}^{S_k} \rho(x_i^{(k)}, x_i^f) > \epsilon. \quad (5.3)$$

Define a sequence $\{x_i\}_{i=0}^\infty \subset X$ as follows: for each integer $k \geq 1$,

$$x_i = x_i^{(k)}, \quad i = T_k, \dots, S_k, \quad (5.4)$$

$$x_i = x_i^f \text{ for all integers } i \geq 0 \text{ such that } i \notin \cup_{k=1}^\infty \{T_k, \dots, S_k\}. \quad (5.5)$$

By (5.1), (5.4) and (5.5) $\{x_i\}_{i=0}^\infty$ is a well-defined program. By (5.2), (5.4) and (5.5) for each integer $p \geq 1$,

$$\sum_{i=0}^{S_p} f_i(x_i, x_{i+1}) \leq \sum_{i=0}^{S_p} f_i(x_i^f, x_{i+1}^f) + \sum_{i=1}^p 2^{-i}.$$

Combined with Proposition 2.1 this implies that the program $\{x_i\}_{i=0}^\infty$ is $(\{f_i\}_{i=0}^\infty)$ -good. In view of (SATP),

$$\sum_{i=0}^\infty \rho(x_i, x_i^f) < \infty.$$

On the other hand, it follows from (5.3), (5.4) and (5.5) that

$$\sum_{i=0}^\infty \rho(x_i, x_i^f) \geq \sum_{k=0}^\infty \left(\sum_{i=T_k}^{S_k} \rho(x_i^{(k)}, x_i^f) \right) \geq \sum_{k=0}^\infty \epsilon = \infty.$$

The contradiction we have reached completes the proof of Lemma 5.3.

Lemma 5.4 *Assume that $\{f_i\}_{i=0}^\infty$ possesses (SATP) and let $\epsilon > 0$. Then there exist $\delta > 0$ and a natural number L such that for each pair of integers T_1, T_2 satisfying $T_1 > L, T_2 > T_1 + 2$ and each program $\{x_i\}_{i=T_1}^{T_2}$ satisfying*

$$\rho(x_{T_1+1}, x_{T_1+1}^f) \leq \delta, \quad \rho(x_{T_2-1}, x_{T_2-1}^f) \leq \delta, \quad (5.6)$$

$$\sum_{i=T_1+1}^{T_2-2} f_i(x_i, x_{i+1}) \leq U(\{f_i\}_{i=0}^{\infty}, T_1 + 1, T_2 - 1, x_{T_1+1}, x_{T_2-1}) + \delta \quad (5.7)$$

the following inequality holds:

$$\sum_{t=T_1+1}^{T_2-1} \rho(x_t, x_t^f) \leq \epsilon.$$

Proof. By Lemma 5.3, there exist $\delta_1 > 0$ and a natural number L such that for each pair of integers $S_2 > S_1 \geq L$ and each program $\{x_i\}_{i=S_1}^{S_2}$ satisfying

$$x_{S_i} = x_{S_i}^f, \quad i = 1, 2, \quad \sum_{i=S_1}^{S_2-1} f_i(x_i, x_{i+1}) \leq \sum_{i=S_1}^{S_2-1} f_i(x_i^f, x_{i+1}^f) + \delta_1 \quad (5.8)$$

we have

$$\sum_{i=S_1}^{S_2} \rho(x_i, x_i^f) \leq \epsilon. \quad (5.9)$$

By Lemma 5.2 there exist $\delta > 0$ such that for each pair of integers $T_1 > 0$, $T_2 > T_1 + 2$ and each program $\{x_i\}_{i=T_1}^{T_2}$ satisfying

$$\rho(x_{T_1+1}, x_{T_1+1}^f) \leq \delta, \quad \rho(x_{T_2-1}, x_{T_2-1}^f) \leq \delta,$$

$$\sum_{i=T_1+1}^{T_2-2} f_i(x_i, x_{i+1}) \leq U(\{f_i\}_{i=0}^{\infty}, T_1 + 1, T_2 - 1, x_{T_1+1}, x_{T_2-1}) + \delta$$

there exists a program $\{\tilde{x}_i\}_{i=T_1-1}^{T_2+1}$ such that

$$\tilde{x}_{T_1-1} = x_{T_1-1}^f, \quad \tilde{x}_{T_2+1} = x_{T_2+1}^f, \quad \tilde{x}_i = x_i, \quad i = T_1 + 1, \dots, T_2 - 1, \quad (5.10)$$

$$\sum_{i=T_1-1}^{T_2} f_i(\tilde{x}_i, \tilde{x}_{i+1}) \leq \sum_{i=T_1-1}^{T_2} f_i(x_i^f, x_{i+1}^f) + \delta_1. \quad (5.11)$$

Assume that an integer $T_1 > L$, an integer $T_2 > T_1 + 2$ and a program $\{x_i\}_{i=T_1}^{T_2}$ satisfies (5.6) and (5.7). By (5.6), (5.7) and the choice of δ , there exists a program $\{\tilde{x}_i\}_{i=T_1-1}^{T_2+1}$ which satisfies (5.10), (5.11). By (5.10), (5.11), the choice of δ_1 (see (5.8), (5.9)),

$$\sum_{i=T_1-1}^{T_2+1} \rho(\tilde{x}_i, x_i^f) \leq \epsilon.$$

Together with (5.10) this implies that

$$\sum_{i=T_1+1}^{T_2-1} \rho(x_i, x_i^f) \leq \epsilon.$$

Lemma 5.4 is proved.

6 Proof of Theorem 3.1

Assume that (STP) holds. Then (TP) holds and in view of Theorem 2.1, (ATP) and the the property (P) hold. Therefore

$$\lim_{t \rightarrow \infty} \rho(x_t, x_t^f) = 0 \quad (6.1)$$

for each integer $S \geq 0$ and each $(\{f_i\}_{i=0}^\infty)$ -good program $\{x_i\}_{i=S}^\infty$.

Let us show that (SATP) holds. Assume that $S \geq 0$ is an integer and a program $\{x_i\}_{i=S}^\infty$ is $(\{f_i\}_{i=0}^\infty)$ -good. Then (6.1) is true. By Lemma 5.1, there is $c > 0$ such that for all integers $T_1 \geq S$, $T_2 > T_1$,

$$\sum_{i=T_1}^{T_2-1} f_i(x_i, x_{i+1}) \leq U(\{f_i\}_{i=0}^\infty, T_1, T_2) + c. \quad (6.2)$$

By (STP), there exist $\delta > 0$ and a natural number L_0 such that for each pair of integers $T_1 \geq 0$, $T_2 \geq T_1 + 2L_0$ and each program $\{z_t\}_{t=T_1}^{T_2}$ which satisfies

$$\sum_{i=T_1}^{T_2-1} f_i(z_i, z_{i+1}) \leq \min\{U(\{f_i\}_{i=0}^\infty, T_1, T_2, z_{T_1}, z_{T_2}) + \delta, U(\{f_i\}_{i=0}^\infty, T_1, T_2) + c\} \quad (6.3)$$

we have

$$\sum_{i=T_1+L_0}^{T_2-L_0} \rho(z_i, x_i^f) \leq 1. \quad (6.4)$$

By Lemma 5.1, there exists a natural number $L_1 > S$ such that for each integer $T_1 \geq L_1$ and each integer $T_2 > T_1$,

$$\sum_{i=T_1}^{T_2-1} f_i(x_i, x_{i+1}) \leq U(\{f_i\}_{i=0}^\infty, T_1, T_2, x_{T_1}, x_{T_2}) + \delta. \quad (6.5)$$

Assume that integers

$$T_1 \geq L_1, \quad T_2 \geq T_1 + 2L_0. \quad (6.6)$$

Then (6.2) and (6.5) hold. In view of (6.2), (6.5), (6.6) and the choice of δ, L_0 ,

$$\sum_{i=T_1+L_0}^{T_2-L_0} \rho(x_i, x_i^f) \leq 1. \quad (6.7)$$

Since (6.7) holds for any pair of integers T_1, T_2 satisfying (6.6) we conclude that

$$\sum_{i=L_1+L_0}^{\infty} \rho(x_i, x_i^f) \leq 1.$$

This implies that $\sum_{i=0}^{\infty} \rho(x_i, x_i^f) < \infty$ and that (SATP) holds. Thus we have shown that (STP) implies (SATP) and the property (P).

Assume that (SATP) and the property (P) hold.

Let $\epsilon > 0$ and $M > 0$. By Lemma 5.4 there exist $\delta_0 > 0$ and a natural number L_0 such that for each pair of integers S_1, S_2 satisfying $S_1 > L_0, S_2 > S_1 + 2$ and each program $\{x_i\}_{i=S_1}^{S_2}$ satisfying

$$\rho(x_{S_1+1}, x_{S_1+1}^f) \leq \delta_0, \quad \rho(x_{S_2-1}, x_{S_2-1}^f) \leq \delta_0,$$

$$\sum_{i=S_1+1}^{S_2-2} f_i(x_i, x_{i+1}) \leq U(\{f_i\}_{i=0}^\infty, S_1 + 1, S_2 - 1, x_{S_1+1}, x_{S_2-1}) + \delta_0 \quad (6.8)$$

we have

$$\sum_{i=S_1+1}^{S_2-2} \rho(x_i, x_i^f) \leq \epsilon. \quad (6.9)$$

By the property (P) there exist

$$\delta \in (0, \delta_0) \quad (6.10)$$

and a natural number L_1 such that for each integer $T \geq 0$ and each program $\{x_t\}_{t=T}^{T+L_1}$ which satisfies

$$\begin{aligned} & \sum_{i=T}^{T+L_1-1} f_i(x_i, x_{i+1}) \\ & \leq \min\{U(\{f_i\}_{i=0}^\infty, T, T+L_1, x_T, x_{T+L_1}) + \delta, U(\{f_i\}_{i=0}^\infty, T, T+L_1) + 3c_f + M\}, \end{aligned} \quad (6.11)$$

there is an integer j such that

$$j \in \{T, \dots, T + L_1\}, \quad \rho(x_j, x_j^f) \leq \delta_0. \quad (6.12)$$

Choose a natural number

$$L \geq 4L_0 + 4L_1. \quad (6.13)$$

Assume that a pair of integers $T_1 \geq 0, T_2 \geq T_1 + 2L$ and that a program $\{x_t\}_{t=T_1}^{T_2}$ satisfies

$$\sum_{i=T_1}^{T_2-1} f_i(x_i, x_{i+1}) \leq \min\{U(\{f_i\}_{i=0}^\infty, T_1, T_2, x_{T_1}, x_{T_2}) + \delta, U(\{f_i\}_{i=0}^\infty, T_1, T_2) + M\}. \quad (6.14)$$

In order to complete the proof of the theorem it is sufficient to show that

$$\sum_{i=T_1+L}^{T_2-L} \rho(x_i, x_i^f) \leq \epsilon. \quad (6.15)$$

Let integers S_1, S_2 satisfy $T_1 < S_1 < S_2 < T_2$. By (6.14) and (C2),

$$\begin{aligned}
\sum_{i=S_1}^{S_2-1} f_i(x_i, x_{i+1}) &= \sum_{i=T_1}^{T_2-1} f_i(x_i, x_{i+1}) - \sum_{i=T_1}^{S_1-1} f_i(x_i, x_{i+1}) - \sum_{i=S_2}^{T_2-1} f_i(x_i, x_{i+1}) \\
&\leq U(\{f_i\}_{i=0}^{\infty}, T_1, T_2) + M - U(\{f_i\}_{i=0}^{\infty}, T_1, S_1) - U(\{f_i\}_{i=0}^{\infty}, S_2, T_2) \\
&\leq \sum_{i=T_1}^{T_2-1} f_i(x_i^f, x_{i+1}^f) + M - \sum_{i=T_1}^{S_1-1} f_i(x_i^f, x_{i+1}^f) + c_f - \sum_{i=S_2}^{T_2-1} f_i(x_i^f, x_{i+1}^f) + c_f \\
&= \sum_{i=S_1}^{S_2-1} f_i(x_i^f, x_{i+1}^f) + 2c_f + M \leq U(\{f_i\}_{i=0}^{\infty}, S_1, S_2) + 3c_f + M.
\end{aligned}$$

Thus

$$\sum_{i=S_1}^{S_2-1} f_i(x_i, x_{i+1}) \leq U(\{f_i\}_{i=0}^{\infty}, S_1, S_2) + 3c_f + M \quad (6.16)$$

for all pairs of integers S_1, S_2 satisfying $T_1 < S_1 < S_2 < T_2$.

By (6.13), (6.14), (6.16), the choice of δ (see (6.10)-(6.12)) there exist integers

$$\tau_1 \in \{L_1 + T_1 + 2L_0, \dots, T_1 + 2L_0 + 2L_1\}, \tau_2 \in \{T_2 - 2L_1, \dots, T_2 - L_1\} \quad (6.17)$$

such that

$$\rho(x_{\tau_i}, x_{\tau_i}^f) \leq \delta_0, \quad i = 1, 2. \quad (6.18)$$

By (6.13) and (6.17),

$$\tau_2 - \tau_1 \geq 2L_0 + L. \quad (6.19)$$

By (6.14) and (6.17),

$$\sum_{i=\tau_1}^{\tau_2-1} f_i(x_i, x_{i+1}) \leq U(\{f_i\}_{i=0}^{\infty}, \tau_1, \tau_2, x_{\tau_1}, x_{\tau_2}) + \delta. \quad (6.20)$$

By (6.19), (6.20), (6.17), (6.18), (6.10) and the choice of L_0 and δ_0 (see (6.7)-(6.9)),

$$\sum_{i=\tau_1}^{\tau_2} \rho(x_i, x_i^f) \leq \epsilon.$$

Together with (6.13) and (6.17) this implies (6.15). Theorem 3.1 is proved.

7 Proof of Theorem 3.2

Set

$$D_0 = \sup\{\rho(z_1, z_2) : z_1, z_2 \in X\}. \quad (7.1)$$

We suppose that the sum over empty set is zero. By (SATP), the property (P) and Theorem 3.1, $\{f_i\}_{i=0}^{\infty}$ possesses (STP). By (STP) there exist $\delta \in (0, 1)$ and a natural number L_0 such that the following property holds:

(a) for each pair of integers $\tau_1 \geq 0$, $\tau_2 \geq \tau_1 + 2L_0$ and each program $\{x_t\}_{t=\tau_1}^{\tau_2}$ which satisfies

$$\sum_{t=\tau_1}^{\tau_2-1} f_t(x_t, x_{t+1}) \leq \min\{U(\{f_i\}_{i=0}^{\infty}, \tau_1, \tau_2, x_{\tau_1}, x_{\tau_2}) + \delta, U(\{f_i\}_{i=0}^{\infty}, \tau_1, \tau_2) + 2M + 4c_f\}$$

the inequality

$$\sum_{i=\tau_1+L_0}^{\tau_2-L_0} \rho(x_i, x_i^f) \leq 1$$

holds.

Choose a natural number

$$L > (4L_0 + 3)(\delta^{-1}M + 1) \quad (7.2)$$

and

$$M_0 > 1 + (\delta^{-1}M + 2)(1 + 2D_0(2L_0 + 1)). \quad (7.3)$$

Assume that integers $T_1 \geq 0$, $T_2 > T_1 + L$ and that a program $\{x_t\}_{t=T_1}^{T_2}$ satisfies

$$\sum_{t=T_1}^{T_2-1} f_t(x_t, x_{t+1}) \leq U(\{f_i\}_{i=0}^{\infty}, T_1, T_2) + M. \quad (7.4)$$

Set

$$t_0 = T_1. \quad (7.5)$$

By induction we define a finite strictly increasing sequence of integers $\{t_i\}_{i=0}^q \subset [T_1, T_2]$ where q is a natural number such that:

$$t_q = T_2; \quad (7.6)$$

(b) for each integer i satisfying $0 \leq i < q - 1$,

$$\sum_{t=t_i}^{t_{i+1}-1} f_t(x_t, x_{t+1}) > U(\{f_i\}_{i=0}^{\infty}, t_i, t_{i+1}, x_{t_i}, x_{t_{i+1}}) + \delta; \quad (7.7)$$

(c) if an integer i satisfies $0 \leq i \leq q - 1$ and (7.7), then

$$t_{i+1} > t_i + 1 \text{ and } \sum_{t=t_i}^{t_{i+1}-2} f_t(x_t, x_{t+1}) \leq U(\{f_j\}_{j=0}^{\infty}, t_i, t_{i+1} - 1, x_{t_i}, x_{t_{i+1}-1}) + \delta. \quad (7.8)$$

Assume that an integer $p \geq 0$ and we have already defined a strictly increasing sequence of integers $\{t_i\}_{i=0}^p \subset [T_1, T_2]$ such that $t_p < T_2$ and that for each integer i satisfying $0 \leq i < p$, (7.7) and (7.8) hold. (Note that for $p = 0$ our assumption holds.) We define t_{p+1} .

There are two cases:

$$\sum_{t=t_p}^{T_2-1} f_t(x_t, x_{t+1}) \leq U(\{f_i\}_{i=0}^{\infty}, t_p, T_2, x_{t_p}, x_{T_2}) + \delta; \quad (7.9)$$

$$\sum_{t=t_p}^{T_2-1} f_t(x_t, x_{t+1}) > U(\{f_i\}_{i=0}^{\infty}, t_p, T_2, x_{t_p}, x_{T_2}) + \delta. \quad (7.10)$$

Assume that (7.9) holds. Then we set $q = p + 1$, $t_q = T_2$, the construction of the sequence is completed and the properties (b), (c) hold.

Assume that (7.10) holds. Set

$$t_{p+1} = \min\{S \in \{t_p + 1, \dots, T_2\} :$$

$$\sum_{t=t_p}^{S-1} f_t(x_t, x_{t+1}) > U(\{f_i\}_{i=0}^{\infty}, t_p, S, x_{t_p}, x_S) + \delta\}. \quad (7.11)$$

Clearly, t_{p+1} is well-defined. If $t_{p+1} = T_2$, then we set $q = p + 1$, the construction is completed and it is not difficult to see that (b) and (c) hold.

Assume that $t_{p+1} < T_2$. Then it is easy to see that the assumption made for p is also true for $p + 1$.

Clearly our construction is completed after a final number of steps and let $t_q = T_2$ be its last element, where q is a natural number. It follows from the construction that the properties (b) and (c) hold.

By (7.4) and the property (b)

$$\begin{aligned} M &\geq \sum_{t=T_1}^{T_2-1} f_t(x_t, x_{t+1}) - U(\{f_i\}_{i=0}^{\infty}, T_1, T_2) \\ &\geq \sum_{t=T_1}^{T_2-1} f_t(x_t, x_{t+1}) - U(\{f_i\}_{i=0}^{\infty}, T_1, T_2, x_{T_1}, x_{T_2}) \\ &\geq \sum_{i=0}^{q-1} \left\{ \sum_{t=t_i}^{t_{i+1}-1} f_t(x_t, x_{t+1}) - U(\{f_j\}_{j=0}^{\infty}, t_i, t_{i+1}, x_{t_i}, x_{t_{i+1}}) \right\} : \\ &\quad \text{is an integer, } 0 \leq i < q - 1 \geq \delta(q - 1), \\ &\quad q \leq \delta^{-1}M + 1. \end{aligned} \quad (7.12)$$

Set

$$A = \{i \in \{0, \dots, q - 1\} : t_{i+1} - t_i > 2L_0\}. \quad (7.13)$$

Let

$$j \in A. \quad (7.14)$$

By (b), (c) and (7.13) and (7.14),

$$\sum_{t=t_j}^{t_{j+1}-2} f_t(x_t, x_{t+1}) \leq U(\{f_i\}_{i=0}^{\infty}, t_j, t_{j+1} - 1, x_{t_j}, x_{t_{j+1}-1}) + \delta. \quad (7.15)$$

By (7.4), (7.13), (7.14) and (C2),

$$\begin{aligned} \sum_{t=t_j}^{t_{j+1}-2} f_t(x_t, x_{t+1}) &= \sum_{t=T_1}^{T_2-1} f_t(x_t, x_{t+1}) \\ &\quad - \sum \{f_t(x_t, x_{t+1}) : t \text{ is an integer, } T_1 \leq t < t_j\} - \sum_{t=t_{j+1}-1}^{T_2-1} f_t(x_t, x_{t+1}) \\ &\leq \sum_{t=T_1}^{T_2-1} f_t(x_t^f, x_{t+1}^f) + M + c_f \\ &\quad - \sum \{f_t(x_t^f, x_{t+1}^f) : t \text{ is an integer, } T_1 \leq t < t_j\} + c_f - \sum_{t=t_{j+1}-1}^{T_2-1} f_t(x_t^f, x_{t+1}^f) \\ &= \sum_{t=t_j}^{t_{j+1}-2} f_t(x_t^f, x_{t+1}^f) + M + 2c_f \leq U(\{f_i\}_{i=0}^{\infty}, t_j, t_{j+1} - 1) + M + 3c_f. \end{aligned} \quad (7.16)$$

By (7.13), (7.14), (7.15), (7.16) and property (a),

$$\sum_{t=t_j+L_0}^{t_{j+1}-1-L_0} \rho(x_t, x_t^f) \leq 1 \quad (7.17)$$

for all $j \in A$. By (7.5), (7.6), (7.13), (7.1), (7.17), (7.12) and (7.3),

$$\begin{aligned} \sum_{t=0}^T \rho(x_t, x_t^f) &\leq D_0 + \sum_{j=0}^{q-1} \left(\sum_{t=t_j}^{t_{j+1}-1} \rho(x_t, x_t^f) \right) \\ &= D_0 + \sum_{j \in A} \left(\sum_{t=t_j}^{t_{j+1}-1} \rho(x_t, x_t^f) \right) \\ &\quad + \sum \left\{ \sum_{t=t_j}^{t_{j+1}-1} \rho(x_t, x_t^f) : j \in \{0, \dots, q-1\} \setminus A \right\} \\ &\leq D_0 + \sum_{j \in A} (L_0 D_0 + 1 + D_0(L_0 + 1)) + q D_0(2L_0 + 1) \\ &\leq D_0 + q(1 + 2D_0(2L_0 + 1)) \leq (\delta^{-1}M + 2)(1 + 2D_0(2L_0 + 1)) < M_0. \end{aligned}$$

Theorem 3.2 is proved. *

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Application of the multi-step differential transform method to solve a fractional human T-cell lymphotropic virus I (HTLV-I) infection of CD4⁺ T-cells

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ABSTRACT: Human T-cell Lymphotropic Virus I (HTLV-I) infection of CD4⁺ T-Cells is one of the causes of health problems and continues to be one of the significant health challenges. In this article, a multi-step differential transform method is implemented to give approximate solutions of fractional model of HTLV-I infection of CD4⁺ T-cells. Numerical results are compared to those obtained by the fourth-order Runge-Kutta method in the case of integer-order derivatives. The suggested method is efficient as the Runge-Kutta method. Some plots are presented to show the reliability and simplicity of the method.

AMS Subject Classification: *11Y35, 65L05*

Keywords and Phrases: *Fractional differential equations; Multi-step differential transform method; Human T-cell Lymphotropic Virus Infection of CD4+ T-Cells; Numerical solution*

1 Introduction

Human T-cell lymphotropic virus Type I (HTLV-I) infection is associated with member of the exogenous human retroviruses that have a tropism for T lymphocytes. HTLV-I belongs to the delta-type retroviruses, which also include bovine leukemia virus; human T-cell leukemia virus Type II (HTLV-II), and simian T-cell leukemia virus. Human T-cell lymphotropic virus (HTLV) is a infection with HTLV-I is now a global epidemic, affecting 10 million to 20 million people. This virus has been linked to life-threatening, incurable diseases:

- a) Adult T-cell leukemia (ATL).
- b) HTLV-I-associated myelopathy/tropical spastic paraparesis.

These syndromes are important causes of mortality and morbidity in the areas where HTLV-I is endemic, mainly in the tropics and subtropics. Mathematical models have proven valuable in understanding the dynamics of medical systems. Dynamic of HTLV-I infection of CD4⁺ T-cells is examined by [1, 2, 3, 4, 5, 6]. The components of the basic four-component model are the concentration of healthy CD4⁺ T-cells at time t , the concentration of latently infected CD4⁺ T-cells, the concentration of actively infected CD4⁺ T-cells and the concentration of leukemic cells at time t respectively, they are denoted by $T(t)$, $I(t)$, $V(t)$ and $L(t)$. These quantities satisfy:

$$\begin{aligned}\frac{dT(t)}{dt} &= \lambda - \mu_T T(t) - \kappa V(t)T(t), \\ \frac{dI(t)}{dt} &= \kappa_1 V(t)T(t) - (\mu_L + \omega)I(t), \\ \frac{dV(t)}{dt} &= \omega I(t) - (\mu_A + \rho)V(t), \\ \frac{dL(t)}{dt} &= \rho V(t) + \beta L(t)\left(1 - \frac{L(t)}{L_{\max}}\right) - \mu_M L(t).\end{aligned}\tag{1}$$

With the initial conditions:

$$T(0) = T_0, \quad I(0) = I_0, \quad V(0) = V_0, \quad L(0) = L_0.\tag{2}$$

The parameters λ, μ_T, κ and κ_1 are the source of CD4⁺T-cells from precursors, the natural death rate of CD4⁺ T-cells, the rate at which uninfected cells are contacted by actively infected cells, the rate of infection of T-cells with virus from actively infected cells, respectively. μ_L, μ_A and μ_M are blanket death terms for latently infected, actively infected and leukemic cells. ω and ρ represent the rates at which latently infected and actively infected cells become actively infected and leukemic, respectively. The rate β determines the speed at which the saturation level for leukemia cells is reached. T_{\max} is the maximal value that adult T-cell leukemia can reach. All parameters are assumed to be positive constants. In recent years, there has been a great deal of interest in fractional diffusion equations. These equations arise in viscous flows [7], biological models [8], evolution equations [9], reaction equations [10] and so on. Fractional derivatives provide an excellent instrument for the description of memory and hereditary properties of various materials and processes. Half-order derivatives and integrals prove to be more useful for the formulation of certain electrochemical problems than the classical models [11]. A great deal of effort has been expended over the last 10 years or so in attempting to find robust and stable numerical and analytical methods for solving fractional differential equations of physical interest. Our motivation for this work is to obtain the approximate solution of the fractional model of HTLV-I infection of CD4⁺ T-Cells using the multi-step differential transform method (MSDTM). This method is only a simple modification of the differential transform method (DTM) [12, 13, 14, 15], in which it is treated as an algorithm in a sequence of small intervals (i.e. time step) for finding accurate approximate solutions to the corresponding systems. The approximate solutions obtained by using DTM are valid only for a short time. While the ones obtained by using the MSDTM [16] are more

valid and accurate during a long time, and are in good agreement with the RK4-5 numerical solution when the order of the derivative ($\alpha = 1$). The rest of the paper is organized as follows. Section 2 gives an idea about the fractional calculus theory. In Section 3, we describe the MSDTM of the fractional order model of HTLV-I infection of CD4⁺ T-Cells. Numerical simulations are presented graphically in Section 4. Finally, the conclusions are given in Section 5.

2 Fractional calculus

Fractional calculus has been extensively applied in different fields. Many mathematicians and applied researchers have tried to model real processes using the fractional calculus. Jesus, Machado and Cunha [17] analyzed the fractional order dynamics in botanical electrical impedances. In biology, it has been deduced that the membranes of cells of biological organism have fractional order electrical conductance [18] and then are classified in groups of non-integer order models. Fractional order ordinary differential equations are naturally related to systems with memory which exists in most biological systems. Also, they are closely related to fractals, which are abundant in biological systems. We first give the definition of fractional-order integration and fractional-order differentiation [19, 20, 21]. There are several approaches to the generalization of the notion of differentiation to fractional orders e.g. Riemann-Liouville, Caputo and generalized functions approach. For the concept of fractional derivative, we will adopt Caputo's definition, which is a modification of the Riemann-Liouville definition and has the advantage of dealing properly with initial value problems.

Definition 2.1 A real function $f(x)$, $x > 0$, is said to be in the space C_α , $\alpha \in R$ if it can be written as $f(x) = x^p f_1(x)$, for some $p > \alpha$ where $f_1(x)$ is continuous in $[0, \infty)$, and it is said to be in the space C_α^m if $f^{(m)} \in C_\alpha$, $m \in N$.

Definition 2.2 The fractional integral of order $\alpha > 0$ of a function $f : R^+ \rightarrow R$ is given by

$$\begin{aligned} J^\alpha f(x) &= \frac{1}{\Gamma(\alpha)} \int_0^x (x-t)^{\alpha-1} f(t) dt, \quad \alpha > 0, \quad x > 0, \\ J^0 f(x) &= f(x). \end{aligned} \quad (3)$$

Here we only need the following properties: For $f \in C_\alpha$, $\alpha, \beta > 0$, $c \in R$ and $\gamma > -1$, we have

$$\begin{aligned} (J^\alpha J^\beta) f(x) &= J^{\alpha+\beta} f(x) = (J^\beta J^\alpha) f(x), \\ J^\alpha x^\gamma &= \frac{x^{\gamma+\alpha}}{\Gamma(\alpha)} B_\tau(\alpha, \gamma+1), \end{aligned} \quad (4)$$

where $B_\tau(\alpha, \gamma + 1)$ is the incomplete beta function which is defined as

$$B_\tau(\alpha, \gamma + 1) = \int_0^\tau t^{\alpha-1}(1-t)^\gamma dt, \quad (5)$$

The Riemann–Liouville derivative has certain disadvantages when trying to model real world phenomena with fractional differential equations. Therefore, we shall introduce a modified fractional differential operator D^α proposed by Caputo in his work on the theory of viscoelasticity.

Definition 2.3 *The Caputo fractional derivative of $f(x)$ of order $\alpha > 0$ is defined as*

$$D^\alpha f(x) = (J^{m-\alpha})f^{(m)}(x) = \frac{1}{\Gamma(m-\alpha)} \int_0^x \frac{f^{(m)}(t)}{(x-t)^{\alpha+1-m}} dt, \quad (6)$$

for $m-1 < \alpha \leq m$, $m \in N$, $x \geq 0$, $f(x) \in C_{-1}^m$. The Caputo fractional derivative was investigated by many authors, for $m-1 < \alpha \leq m$, $f(x) \in C_\alpha^m$ and $\alpha \geq -1$, we have

$$(J^\alpha D^\alpha)f(x) = J^m D^m f(x) = f(x) - \sum_{k=0}^{m-1} f^{(k)}(0) \frac{x^k}{k!}. \quad (7)$$

For more mathematical properties of fractional derivatives and integrals one can back to the mentioned references.

3 MSDTM Algorithm

This paper attempts to find numerical solution for a general class of fractional order model of HTLV-I infection of CD4⁺ T-cells. Therefore, the paper summarizes specific techniques for MSDTM, as well as the applications of Caputo fractional calculus. The fractional order differential equations (FOD) are used because they are naturally related to systems with memory since the definition of fractional derivative involves an integration which is non local operator (as it is defined on an interval), so fractional derivative is a non local operator. Also, they are closely related to fractals which are abundant in biological systems. It has been shown that the approximated solutions obtained using DTM are not valid for large t for some systems [12, 13, 14, 15]. Therefore, we use the MSDTM to solve the following fractional order model of HTLV-I infection of CD4⁺ T-Cells of order $0 < \alpha \leq 1$:

$$\begin{aligned} D^\alpha T(t) &= \lambda - \mu_T T(t) - \kappa V(t)T(t), \\ D^\alpha I(t) &= \kappa_1 V(t)T(t) - (\mu_L + \omega)I(t), \\ D^\alpha V(t) &= \omega I(t) - (\mu_A + \rho)V(t), \\ D^\alpha L(t) &= \rho V(t) + \beta L(t)\left(1 - \frac{L(t)}{L_{\max}}\right) - \mu_M L(t). \end{aligned} \quad (8)$$

With the initial conditions:

$$T(0) = 1000, I(0) = 250, V(0) = 1.5, L(0) = 0. \tag{9}$$

The method is a simple modification of the DTM, in which it is treated as an algorithm in a sequence of small intervals (i.e. time step) for finding accurate approximate solutions to the corresponding systems (8). This MSDTM offers accurate solutions over a longer time frame (more stable) compared to the standard DTM. Using the theorems given in [12] and taking the differential transform for the system (8) with respect to time t gives

$$\begin{aligned} T^*(k+1) &= \frac{\Gamma(\alpha k + 1)}{\Gamma(\alpha(k+1) + 1)} \left[\lambda \delta(k) - \mu_T T^*(k) - \kappa \sum_{l=0}^k V^*(l)T^*(k-l) \right], \\ I^*(k+1) &= \frac{\Gamma(\alpha k + 1)}{\Gamma(\alpha(k+1) + 1)} \left[\kappa_1 \sum_{l=0}^k V^*(l)T^*(k-l) - (\mu_L + \omega)I^*(k) \right], \\ V^*(k+1) &= \frac{\Gamma(\alpha k + 1)}{\Gamma(\alpha(k+1) + 1)} [\omega I^*(k) - (\mu_A + \rho)V^*(k)], \\ L^*(k+1) &= \frac{\Gamma(\alpha k + 1)}{\Gamma(\alpha(k+1) + 1)} \left[\rho V^*(k) + \beta L^*(k) - \frac{\beta}{L_{\max}} \sum_{l=0}^k L^*(l)L^*(k-l) - \mu_M L^*(k) \right]. \end{aligned} \tag{10}$$

where $T^*(k)$, $I^*(k)$, $V^*(k)$ and $L^*(k)$ are the differential transformations of $T(t)$, $I(t)$, $V(t)$ and $L(t)$, respectively. The differential transform of the initial conditions are given by $T^*(0) = 1000$, $I^*(0) = 250$, $V^*(0) = 1.5$ and $L^*(0) = 0$. In view of the differential inverse transform, the differential transform series solution for System (8) can be obtained as

$$\begin{aligned} T(t) &= \sum_{n=0}^N T^*(n) t^{\alpha n}, \quad I(t) = \sum_{n=0}^N I^*(n) t^{\alpha n}, \\ V(t) &= \sum_{n=0}^N V^*(n) t^{\alpha n}, \quad L(t) = \sum_{n=0}^N L^*(n) t^{\alpha n}. \end{aligned} \tag{11}$$

Now, according to the MSDTM, the series solution for the system (8) is suggested to be

$$T(t) = \begin{cases} \sum_{n=0}^K T_1^*(n) t^{\alpha n}, & t \in [0, t_1], \\ \sum_{n=0}^K T_2^*(n) (t - t_1)^{\alpha n}, & t \in [t_1, t_2], \\ \vdots \\ \sum_{n=0}^K T_m^*(n) (t - t_{m-1})^{\alpha n}, & t \in [t_{m-1}, t_m], \end{cases} \tag{12}$$

$$I(t) = \begin{cases} \sum_{n=0}^K I_1^*(n) t^{\alpha n}, & t \in [0, t_1], \\ \sum_{n=0}^K I_2^*(n) (t - t_1)^{\alpha n}, & t \in [t_1, t_2], \\ \vdots \\ \sum_{n=0}^K I_m^*(n) (t - t_{m-1})^{\alpha n}, & t \in [t_{m-1}, t_m], \end{cases} \quad (13)$$

$$V(t) = \begin{cases} \sum_{n=0}^K V_1^*(n) t^{\alpha n}, & t \in [0, t_1], \\ \sum_{n=0}^K V_2^*(n) (t - t_1)^{\alpha n}, & t \in [t_1, t_2], \\ \vdots \\ \sum_{n=0}^K V_m^*(n) (t - t_{m-1})^{\alpha n}, & t \in [t_{m-1}, t_m], \end{cases} \quad (14)$$

$$L(t) = \begin{cases} \sum_{n=0}^K L_1^*(n) t^{\alpha n}, & t \in [0, t_1], \\ \sum_{n=0}^K L_2^*(n) (t - t_1)^{\alpha n}, & t \in [t_1, t_2], \\ \vdots \\ \sum_{n=0}^K L_m^*(n) (t - t_{m-1})^{\alpha n}, & t \in [t_{m-1}, t_m], \end{cases} \quad (15)$$

where $T_i^*(n)$, $I_i^*(n)$, $V_i^*(n)$ and $L_i^*(n)$ for $i = 1, 2, \dots, m$ satisfy the following recurrence relations

$$\begin{aligned} T_i^*(k+1) &= \frac{\Gamma(\alpha k + 1)}{\Gamma(\alpha(k+1) + 1)} \left[\lambda \delta(k) - \mu_T T_i^*(k) - \kappa \sum_{l=0}^k V_i^*(l) T_i^*(k-l) \right], \\ I_i^*(k+1) &= \frac{\Gamma(\alpha k + 1)}{\Gamma(\alpha(k+1) + 1)} \left[\kappa_1 \sum_{l=0}^k V_i^*(l) T_i^*(k-l) - (\mu_L + \omega) I_i^*(k) \right], \\ V_i^*(k+1) &= \frac{\Gamma(\alpha k + 1)}{\Gamma(\alpha(k+1) + 1)} [\omega I_i^*(k) - (\mu_A + \rho) V_i^*(k)], \\ L_i^*(k+1) &= \frac{\Gamma(\alpha k + 1)}{\Gamma(\alpha(k+1) + 1)} \left[\rho V_i^*(k) + \beta L_i^*(k) - \frac{\beta}{L_{\max}} \sum_{l=0}^k L_i^*(l) L_i^*(k-l) - \mu_M L_i^*(k) \right], \end{aligned} \quad (16)$$

such that $T_i^*(0) = T_{i-1}^*(0)$, $I_i^*(0) = I_{i-1}^*(0)$, $V_i^*(0) = V_{i-1}^*(0)$ and $L_i^*(0) = L_{i-1}^*(0)$. Finally, if we start with $T_o^*(0) = 1000$, $I_0^*(0) = 250$, $V_0^*(0) = 1.5$ and $L_0^*(0) = 0$, using the recurrence relation given in System (16) then we can obtain the multi-step solution given in Systems (12)-(15).

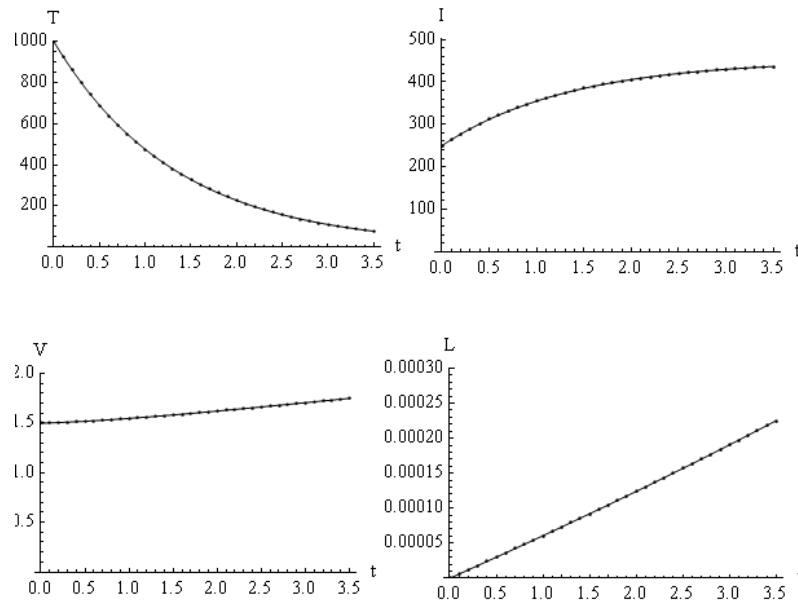


Figure 1: Plots of the components of lymphotropic virus I (HTLV) infection of $CD4^+$ T-cells model. Solid line: MSDTM solution, Dotted line: Runge-Kutta method solution

4 Numerical results

In this work, we propose the MSDTM, a reliable modification of the DTM that improves the convergence of the series solution. The method provides immediate and visible symbolic terms of analytic solutions, as well as numerical approximate solutions to both linear and nonlinear differential equations. Figure 1 shows the approximate solutions obtained using the MSDTM and the fourth-order Runge–Kutta method of the concentration of healthy $CD4^+$ T-cells at time t , the concentration of latently infected $CD4^+$ T-cells, the concentration of actively infected $CD4^+$ T-cells and the concentration of leukemic cells when $\alpha = 1$ and the step size $\Delta t = 0.1$. We assumed that all parameters are positive in mm^3/day as follows: $\lambda = 6$, $\mu_T = 0.6$, $\mu_L = 0.006$, $\mu_A = 0.05$, $\mu_M = 0.0005$, $\omega = 0.0004$, $\rho = 0.00004$, $\beta = 0.0003$, $T_{max} = 2200$ and $\kappa = \kappa_1 = 0.1$. It can be seen that the results obtained based on MSDTM match the results of the Runge–Kutta method very well, which implies that the MSDTM can predict the behaviour of these variables accurately for the region under consideration. Next, interstet to show how the concentrations of healthy $CD4^+$ T-cells, latently infected $CD4^+$ T-cells, actively infected $CD4^+$ T-cells and leukemic cells depend upon the magnitude of the order of fractional derivatives. We fix the parameters and perform a numerical simulation for different values of α . Simulation results are presented in Figure 2. It it is clear that these solutions continuously depend

on the fractional derivatives.

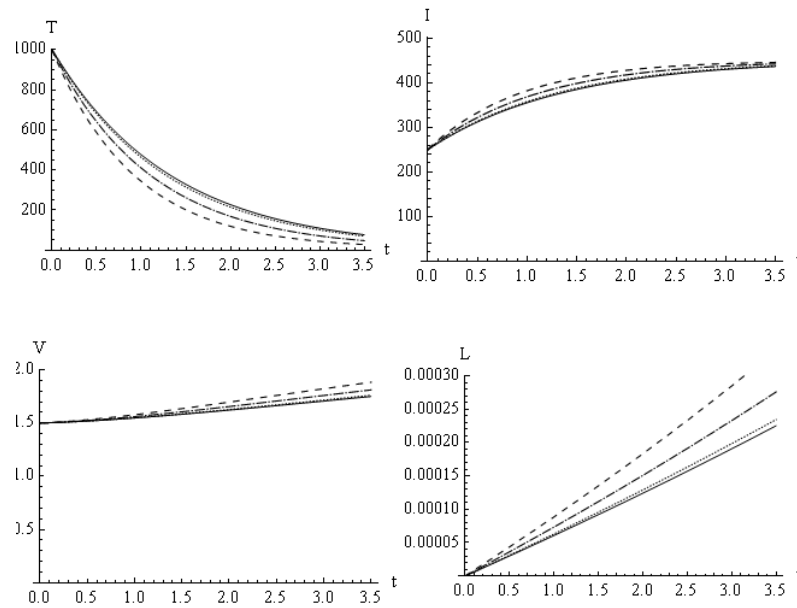


Figure 2: Plots of the components of lymphotropic virus I (HTLV) infection of $CD4^+$ T-cells model using MSDTM. Solid line: ($\alpha = 1$), Dotted line: ($\alpha = 0.99$), Dashed line: ($\alpha = 0.95$), Dash-dotted line: ($\alpha = 0.9$).

5 Conclusions

In this paper we employed the multi-step differential transform method in order to solve the fractional model of human T-cell lymphotropic virus I (HTLV-I) infection of $CD4^+$ T-cells. Comparisons of the results obtained by using the MSDTM with that obtained by the classical Runge–Kutta method in the integer case reveal that the approximate solutions obtained by DTM are only valid for a small time, while the ones obtained by MSDTM are highly accurate and valid for a long time to nonlinear systems of differential equations. The reliability of the method and the reduction in the size of the computational domain give this method a wider applicability. It is of interest to note here that time fractional derivatives change the solutions, also we usually get in standard System (1). The concentration of healthy $CD4^+$ T-cells $T(t)$, the concentration of latently infected $CD4^+$ T-cells $I(t)$, the concentration of actively infected $CD4^+$ T-cells $V(t)$ and the concentration of leukemic cells $L(t)$ have been obtained, therefore when $\alpha \rightarrow 1$ the solution of the fractional model (8) $D^\alpha T(t)$, $D^\alpha I(t)$, $D^\alpha V(t)$, $D^\alpha L(t)$ reduce to the standard solution $T(t)$, $I(t)$, $V(t)$, $L(t)$ (see fig. 2). The recent appearance of nonlinear fractional

differential equations as models in science and engineering makes it necessary to investigate the method of solutions for such equations. Consequently, the proposed method for the considered model verifies that it is a useful tool for these kind of models. The obtained results demonstrate the reliability of the algorithm and its wider applicability to fractional nonlinear evolution equations.

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