Journal of
Mathematics
and Applications
JMA No 44, pp 119-122 (2021)

A Sandwich Type Hahn-Banach Theorem for Convex and Concave Functionals

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Abstract: We give a sandwich type Hahn-Banach theorem for convex and concave functionals.

AMS Subject Classification: 46A22.
Keywords and Phrases: The Hahn-Banach theorem; Convex functional; Concave functional.

The Hahn-Banach theorem is a fundamental theorem in linear functional analysis. Its sandwich form is the following, see Theorem 3.9 in [5].

Theorem 1 (Sandwich Theorem). *Let* $g: X \to \mathbb{R} \cup \{+\infty\}$, *and* $h: X \to \mathbb{R}$ *be sublinear functions on a linear space* X. *If* $-g \leqslant h$, *there exists a linear form* l *on* X *such that* $-g \leqslant l \leqslant h$.

The following Hahn-Banach extension theorem was given in [1] and [3].

Theorem 2. *Suppose* X *is a real linear space,* p *is a convex functional on* X, M *is a subspace of* X. *If* g *is a real linear functional on* M *such that* $g(x) \leqslant p(x),\ x \in M$, *then there exists a linear functional* f *on* X *such that* $f(x) \leqslant p(x),\ \forall x \in X$ *and* $f(x) = g(x),\ \forall x \in M$.

In the following we shall use 0 to denote both zero and zero vector. From Theorem 2, we have the following results.

Corollary 1. *Let* X *be a real linear space and* φ *be a convex functional on* X *such that* $\varphi(0) \geqslant 0$, *then there exists a linear functional* L *on* X *such that* $L(x) \leqslant \varphi(x)$ *for every* $x \in X$.

Proof. Let $E = \{0\}$ and $f_0(0) = 0$, The f_0 is a linear functional on E such that $f_0(x) \leqslant \varphi(x)$ for every $x \in E$. Then by Theorem 2, there exists a linear functional f on X such that $f(x) \leqslant \varphi(x)$ for every $x \in X$. □

Corollary 2. *Suppose that f_0 be a linear functional on subspace M of X, such that $\psi(x) \leqslant f_0(x)$ for every $x \in M$, where ψ is a concave function on X. Then there exists a linear functional L on X such that $L(x) = f_0(x)$ for every $x \in M$ and $\psi(x) \leqslant L(x)$ for every $x \in X$.*

Now, our main result is the following sandwich type theorem for convex and concave functionals.

Theorem 3. *Let M be a subspace in X. Suppose φ and $-\psi$ are convex functionals on X such that $\varphi(0) = \psi(0) = 0$ and $T(x) := \inf_{y\in X}\{\varphi(x+y) - \psi(y)\}$ is finite for every $x \in X$. If f_0 is a linear functional on M, then there exists an extension linear functional L on X of f_0 such that $\psi(x) \leqslant L(x) \leqslant \varphi(x)$ for every $x \in X$ if and only if $f_0(x) \leqslant T(x)$ for every $x \in M$.*

To give the proof of Theorem 3, we need the following lemmas.

Lemma 1. *Suppose φ and $-\psi$ are convex functionals on X such that $\varphi(0) = \psi(0) = 0$ and $T(x) := \inf_{y\in X}\{\varphi(x+y) - \psi(y)\}$ is finite for every $x \in X$. Let f_0 be a linear functional on a subspace M of X such that*

$$f_0(x) \leqslant T(x) \text{ for every } x \in M. \tag{1}$$

Then the following conditions are satisfied.

(i) *For every $x \in X$, $\psi(x) \leqslant \varphi(x)$;*

(ii) *For every $x \in M$, $\psi(x) \leqslant f_0(x) \leqslant \varphi(x)$.*

Proof. From (1), for every $y \in X$ and $x \in M$, $f_0(x) \leqslant \varphi(x+y) - \psi(y)$. Then, let $x = 0$, we have $\psi(y) \leqslant \varphi(y)$ for every $y \in X$. By letting $y = 0$, we see that $f_0(x) \leqslant \varphi(x) - \psi(0) \leqslant \varphi(x)$ for every $x \in M$. By letting $y = -x$, we obtain that $f_0(-y) \leqslant -\psi(y)$ or $\psi(x) \leqslant f_0(x)$ for every $x \in M$. Thus, $\psi(x) \leqslant f_0(x) \leqslant \varphi(x)$ for every $x \in M$. □

Lemma 2. *Suppose φ and $-\psi$ are convex functionals on X such that $\varphi(0) = \psi(0) = 0$ and $T(x) := \inf_{y\in X}\{\varphi(x+y) - \psi(y)\}$ is finite for every $x \in X$. Let $\psi(x) \leqslant \varphi(x)$ for every $x \in X$. Then $\psi(x) \leqslant T(x) \leqslant \varphi(x)$ for every $x \in X$, and T is a convex functional. Moreover, if L is a linear functional on X such that $\psi(x) \leqslant L(x) \leqslant \varphi(x)$ for every $x \in X$, then $L(x) \leqslant T(x)$ for every $x \in X$.*

Proof. First, we prove that T is convex. Fix $u, v \in X$. For $\alpha, \beta \geq 0$ such that $\alpha + \beta = 1$, for every $\epsilon > 0$, there exist $y, z \in X$ such that $\varphi(u+y) - \psi(y) < T(u) + \epsilon$, $\varphi(v+z) - \psi(z) < T(v) + \epsilon$, then

$$\begin{aligned}
\varphi(\alpha u + \beta v + \alpha y + \beta z) - \psi(\alpha y + \beta z)\} &\leqslant \alpha\varphi(u+y) + \beta(v+z) - \alpha\psi(y) - \beta\psi(z) \\
&\leqslant \alpha(\varphi(u+y) - \psi(y)) + \beta(\varphi(v+z) - \psi(z)) \\
&< \alpha T(u) + \beta T(v) + \epsilon.
\end{aligned}$$

Thus $T(\alpha u + \beta v) < \alpha T(u) + \beta T(v) + \epsilon$. Since ϵ is arbitrary, we obtain that $T(\alpha u + \beta v) \leqslant \alpha T(u) + \beta T(v)$. Therefore T is convex.

Since $T(x) \leqslant \varphi(x+y) - \psi(y)$, it follows that $T(x) \leqslant \varphi(x) - \psi(0)$. So $T(x) \leqslant \varphi(x)$ for every $x \in X$. Again, $T(-y) \leqslant \varphi(0) - \psi(y)$, So $T(-y) \leqslant -\psi(y)$. Since $T(0) = 0$, and by the convexity of T, $0 \leqslant T(0) \leqslant 1/2T(y) + 1/2T(-y)$, so that $-T(y) \leqslant T(-y)$. Hence, $-T(y) \leqslant T(-y) \leqslant -\psi(y)$. Thus $\psi(y) \leqslant T(y)$ for every $y \in X$. Consequently, $\psi(x) \leqslant T(x) \leqslant \varphi(x)$ for every $x \in X$.

Finally, suppose that $\psi(x) \leqslant L(x) \leqslant \varphi(x)$ for every $x \in X$. Now $\psi(u) \leqslant L(u)$, it follows that $L(u) \leqslant -\psi(-u)$. Hence, by the linearity of L we obtain that $L(u+v) \leqslant \varphi(v) - \psi(-u)$ for every $u, v \in X$. Letting $v = x + y$ and $u = -y$, we obtain $L(x) \leqslant \varphi(x+y) - \psi(y)$. Taking the infimum over all $y \in X$, we obtain that $L(x) \leqslant T(x)$ for every $x \in X$. □

Proof of Theorem 3. If a linear functional L on X is an extension of f_0 such that $\psi(x) \leqslant L(x) \leqslant \varphi(x)$ for every $x \in X$. By Lemma 2, $L(x) \leqslant T(x)$ for every $x \in X$. Since $f_0(x) = L(x)$ for each $x \in M$, so $f_0(x) \leqslant T(x)$ for every $x \in M$.

Conversely, if $f_0(x) \leqslant T(x)$ for every $x \in M$, by Lemma 1, $\psi(x) \leqslant \varphi(x)$ for all $x \in X$ and $\psi(x) \leqslant f_0(x) \leqslant \varphi(x)$ for all $x \in M$. According to Lemma 2, we see that T is a convex function. Now, by Theorem 2 there is an extension linear functional L on X such that $f_0(x) = L(x)$ for each $x \in M$ and $L(x) \leqslant T(x)$ for each $x \in X$. By Lemma 1, $\psi(x) \leqslant L(x) \leqslant \varphi(x)$ for all $x \in X$. □

By Theorem 3, we have an generalization of Theorem 1 as follows.

Theorem 4. *Suppose φ and $-\psi$ are convex functionals on X such that $\varphi(0) = \psi(0) = 0$ and $T(x) := \inf_{y \in X}\{\varphi(x+y) - \psi(y)\}$ is finite for every $x \in X$. If $\psi(x) \leqslant \varphi(x)$ for every $x \in X$, then there exists a linear functional L on X such that $\psi(x) \leqslant L(x) \leqslant \varphi(x)$ for every $x \in X$.*

Proof. Let $E = \{0\}$ and $f_0(0) = 0$, The f_0 is a linear functional on E such that $\psi(x) \leqslant f_0(x) \leqslant \varphi(x)$ for every $x \in E$. Then by Theorem 3, there exists a linear functional f on X such that $\psi(x) \leqslant f(x) \leqslant \varphi(x)$ for every $x \in X$. □

In Theorems 3 and 4, the condition $\varphi(0) = \psi(0) = 0$ is necessary. For example, in $\mathbb{R}$, let $\varphi(x) = (x+4)^2 - 4$, $\psi(x) = -e^x - 4$, then there exists no constant k such that $\varphi(x) \geqslant kx \geqslant \psi(x)$ for all $x \in \mathbb{R}$.

Remark 1. Theorem 4 partly generalises the sandwich version Hahn-Banach Theorem in [2]. Páles gave a different type Sandwich theorems in [4].

Acknowledgments

The author would like to thank the referees for their careful reading and suggestions, in particular, the reference [3] was pointed by them.

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DOI: 10.7862/rf.2021.9

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Received 24.04.2020 *Accepted 26.08.2021*