A Companion of the generalized trapezoid inequality and applications

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ABSTRACT: A sharp companion of the generalized trapezoid inequality is introduced. Applications to quadrature formula are pointed out.

1. Introduction

The following trapezoid type inequality for mappings of bounded variation was proved in [7] (see also [6]):

Theorem 1.1 *Let $f : [a, b] \to \mathbb{R}$, be a mapping of bounded variation on $[a, b]$, Then*

$$\left| \int_a^b f(t)\, dt - (b - a) \frac{f(a) + f(b)}{2} \right| \leq \frac{1}{2} (b - a) \bigvee_a^b (f). \qquad (1.1)$$

The constant $\frac{1}{2}$ is the best possible.

A generalization (1.1) for mappings of bounded variation, was considered by Cerone et al. in [6], as follows:

$$\left| (b - x) f(b) + (x - a) f(a) - \int_a^b f(t)\, dt \right| \leq \left[\frac{b - a}{2} + \left| x - \frac{a + b}{2} \right| \right] \cdot \bigvee_a^b (f) \qquad (1.2)$$

for all $x \in [a, b]$. The constant $\frac{1}{2}$ is the best possible.
In the same way, the following midpoint type inequality for mappings of bounded variation was proved in [8]:

Theorem 1.2 *Let $f : [a,b] \to \mathbb{R}$, be a mapping of bounded variation on $[a,b]$, Then*

$$\left| (b-a) f\left(\frac{a+b}{2}\right) - \int_a^b f(t)\, dt \right| \leq \frac{1}{2} (b-a) \bigvee_a^b (f). \qquad (1.3)$$

The constant $\frac{1}{2}$ is the best possible.

A weighted generalization of trapezoid inequality for mappings of bounded variation, was considered by Tseng et. al. [12]. In order to combine the midpoint and the trapezoid inequalities together Guessab and Schmeisser [13] have proved an interesting a companion of Ostrowski type inequality for r-Hölder continuous mappings. Motivated by [13], Dragomir in [14], has proved the Guessab–Schmeisser companion of Ostrowski inequality for mappings of bounded variation. Recently, in [15, 16] the authors proved a generalization of weighted Ostrowski type inequality for mappings of bounded variation and thus they deduced several trapezoid type inequalities. For recent new results regarding Ostrowski's and generalized trapezoid type inequalities see [1]–[5].

In this paper, we give a companion of (1.2) for mappings of bounded variation, Lipschitzian type and monotonic nondecreasing. Applications to quadrature formulae are given.

2. The Results

The following result holds:

Theorem 2.3 *Let $f : [a,b] \to \mathbb{R}$ be a mapping of bounded of variation on $[a,b]$. Then we have the inequality*

$$\left| (x-a)(f(a)+f(b)) + (a+b-2x) f\left(\frac{a+b}{2}\right) - \int_a^b f(t)\, dt \right|$$

$$\leq \left[\frac{b-a}{4} + \left| x - \frac{3a+b}{4} \right| \right] \cdot \bigvee_a^b (f), \quad (2.4)$$

for all $x \in \left[a, \frac{a+b}{2}\right]$. Furthermore, the constant $\frac{1}{4}$ is the best possible in the sense that it cannot be replaced by a smaller one.

Proof. Integrating by parts

$$\int_a^b K(t,x)\, df(t)\, dt = (x-a)(f(a)+f(b)) + (a+b-2x) f\left(\frac{a+b}{2}\right) - \int_a^b f(t)\, dt$$

where,

$$K(t,x) := \begin{cases} t - x, & t \in \left[a, \frac{a+b}{2}\right] \\[2mm] t - (a+b-x), & t \in \left[\frac{a+b}{2}, b\right]. \end{cases}$$

Using the fact that, for a continuous mapping $p : [a, b] \to \mathbb{R}$ and bounded variation mapping $\nu : [a, b] \to \mathbb{R}$, then one has the inequality

$$\left| \int_a^b p(t) \, d\nu(t) \right| \leq \sup_{t \in [a,b]} |p(t)| \bigvee_a^b (\nu).$$

Applying the above inequality, for $p(t) := K(t, x)$ and $\nu(t) := f(t)$, we get

$$\left| \int_a^b K(t, x) \, df(t) \, dt \right| \leq \sup_{t \in [a,b]} |K(t, x)| \cdot \bigvee_a^b (f) = \max \left\{ x - a, \frac{a+b}{2} - x \right\} \cdot \bigvee_a^b (f)$$

$$= \left[\frac{b-a}{4} + \left| x - \frac{3a+b}{4} \right| \right] \cdot \bigvee_a^b (f),$$

for all $x \in [a, \frac{a+b}{2}]$, which proves (2.4). To prove the sharpness of (2.4), assume that (2.4) holds with constant $C > 0$, i.e.,

$$\left| (x - a)(f(a) + f(b)) + (a + b - 2x) f\left(\frac{a+b}{2}\right) - \int_a^b f(t) \, dt \right|$$

$$\leq \left[C(b - a) + \left| x - \frac{3a+b}{4} \right| \right] \cdot \bigvee_a^b (f). \quad (2.5)$$

Consider the mapping $f : [a, b] \to \mathbb{R}$, given by

$$f(t) = \begin{cases} 0, & t \in (a, b) \\ \frac{1}{2}, & t = a, b. \end{cases}$$

Therefore, $\int_a^b f(t) \, dt = 0$ and $\bigvee_a^b (f) = 1$. Making of use (2.5) with $x = \frac{3a+b}{4}$, we get

$$\left| \frac{b-a}{2} \left[\frac{1}{2} + 0 \right] - 0 \right| \leq C(b - a) \cdot 1$$

which gives that, $C \geq \frac{1}{4}$, and the theorem is completely proved. ∎

Remark 2.1 *In the inequality (2.4), choose*

1. *$x = a$, then we get*

$$\left| (b - a) f\left(\frac{a+b}{2}\right) - \int_a^b f(t) \, dt \right| \leq \frac{1}{2}(b - a) \cdot \bigvee_a^b (f) \quad (2.6)$$

2. *$x = \frac{3a+b}{4}$, then we get*

$$\left| \frac{b-a}{2} \left[f\left(\frac{a+b}{2}\right) + \frac{f(a) + f(b)}{2} \right] - \int_a^b f(t) \, dt \right| \leq \frac{1}{4}(b - a) \cdot \bigvee_a^b (f) \quad (2.7)$$

3. $x = \frac{a+b}{2}$, then we get

$$\left| (b-a) \frac{f(a) + f(b)}{2} - \int_a^b f(t)\, dt \right| \leq \frac{1}{2}(b-a) \cdot \bigvee_a^b (f) \qquad (2.8)$$

Corollary 2.1 *If $f \in C^{(1)}[a,b]$. Then we have the inequality*

$$\left| (x-a)(f(a) + f(b)) + (a+b-2x) f\left(\frac{a+b}{2}\right) - \int_a^b f(t)\, dt \right|$$
$$\leq \left[\frac{b-a}{4} + \left| x - \frac{3a+b}{4} \right| \right] \cdot \| f' \|_{1,[a,b]}, \quad (2.9)$$

where $\|\cdot\|_1$ is the L_1 norm, namely $\| f' \|_{1,[a,b]} := \int_a^b |f'(t)|\, dt$.

Corollary 2.2 *If f is K-Lipschitzian on $[a,b]$ with the constant $K > 0$. Then we have the inequality*

$$\left| (x-a)(f(a) + f(b)) + (a+b-2x) f\left(\frac{a+b}{2}\right) - \int_a^b f(t)\, dt \right|$$
$$\leq K(b-a) \left[\frac{b-a}{4} + \left| x - \frac{3a+b}{4} \right| \right]. \quad (2.10)$$

Corollary 2.3 *If f is monotonic on $[a,b]$. Then we have the inequality*

$$\left| (x-a)(f(a) + f(b)) + (a+b-2x) f\left(\frac{a+b}{2}\right) - \int_a^b f(t)\, dt \right|$$
$$\leq \left[\frac{b-a}{4} + \left| x - \frac{3a+b}{4} \right| \right] \cdot |f(b) - f(a)|. \quad (2.11)$$

A refinement of (2.10), may be stated as follows:

Theorem 2.4 *Let $f : [a,b] \to \mathbb{R}$ be an L–Lipschitzian mapping on $[a,b]$. Then we have the inequality*

$$\left| (x-a)(f(a) + f(b)) + (a+b-2x) f\left(\frac{a+b}{2}\right) - \int_a^b f(t)\, dt \right|$$
$$\leq L \left[\frac{(b-a)^2}{8} + 2\left(x - \frac{3a+b}{4} \right)^2 \right], \quad (2.12)$$

for all $x \in \left[a, \frac{a+b}{2} \right]$. Furthermore, the constant $\frac{1}{8}$ is the best possible in the sense that it cannot be replaced by a smaller one.

Proof. Using the fact that, for a Riemann integrable function $p : [a, b] \to \mathbb{R}$ and L–Lipschitzian function $\nu : [a, b] \to \mathbb{R}$, then one has the inequality

$$\left| \int_a^b p(t) \, d\nu(t) \right| \leq L \int_a^b |p(t)| \, dt.$$

Applying the above inequality, for $p(t) := K(t, x)$ and $\nu(t) := f(t)$, we get

$$\left| \int_a^b K(t, x) \, df(t) \, dt \right| \leq L \int_a^b |K(t, x)| = L \left[(x - a)^2 + \left(\frac{a + b}{2} - x \right)^2 \right]$$

$$= L \left[\frac{(b - a)^2}{8} + 2 \left(x - \frac{3a + b}{4} \right)^2 \right],$$

for all $x \in [a, \frac{a+b}{2}]$, which proves (2.12). To prove the sharpness of (2.12), assume that (2.12) holds with constant $C > 0$, i.e.,

$$\left| (x - a)(f(a) + f(b)) + (a + b - 2x) f \left(\frac{a + b}{2} \right) - \int_a^b f(t) \, dt \right|$$

$$\leq \left[C(b - a) + \left| x - \frac{3a + b}{4} \right| \right] \cdot \bigvee_a^b (f). \quad (2.13)$$

Consider the mapping $f : [a, b] \to \mathbb{R}$, given by $f(t) := t - \frac{3a+b}{4}$. Therefore, f is Lipschitzian with $L = 1$ and $\int_a^b f(t) \, dt = \frac{(b-a)^2}{4}$. Making of use (2.13) with $x = \frac{3a+b}{4}$, we get

$$\frac{(b - a)^2}{8} \leq C(b - a)^2$$

which gives that, $C \geq \frac{1}{8}$, and the theorem is completely proved. ∎

Remark 2.2 *In the inequality (2.12), choose*

    1. $x = a$, then we get

$$\left| (b - a) f \left(\frac{a + b}{2} \right) - \int_a^b f(t) \, dt \right| \leq L \frac{(b - a)^2}{4} \quad (2.14)$$

    2. $x = \frac{3a+b}{4}$, then we get

$$\left| \frac{b - a}{2} \left[f \left(\frac{a + b}{2} \right) + \frac{f(a) + f(b)}{2} \right] - \int_a^b f(t) \, dt \right| \leq L \frac{(b - a)^2}{8} \quad (2.15)$$

3. $x = \frac{a+b}{2}$, then we get

$$\left| (b-a) \frac{f(a) + f(b)}{2} - \int_a^b f(t)\, dt \right| \leq L \frac{(b-a)^2}{4} \tag{2.16}$$

A refinement of (2.11), may be stated as follows:

Theorem 2.5 *Let $f : [a,b] \to \mathbb{R}$ be a monotonic non-decreasing on $[a, \frac{a+b}{2}]$ and on $[\frac{a+b}{2}, b]$. Then we have the inequality*

$$\left| (x-a)(f(a) + f(b)) + (a+b-2x) f\left(\frac{a+b}{2}\right) - \int_a^b f(t)\, dt \right|$$

$$\leq (x-a)(f(b) - f(a)) + (2x - a - b) f\left(\frac{a+b}{2}\right)$$

$$+ 2\left(\frac{3a+b}{4} - x\right)(f(a+b-x) - f(x)) \tag{2.17}$$

for all $x \in \left[a, \frac{a+b}{2}\right]$.

Proof. Using the fact that, for a monotonic non-decreasing function $\nu : [a,b] \to \mathbb{R}$ and continuous function $p : [a,b] \to \mathbb{R}$, then one has the inequality

$$\left| \int_a^b p(t)\, d\nu(t) \right| \leq \int_a^b |p(t)|\, d\nu(t).$$

Applying the above inequality, for $p(t) := K(t,x)$ and $\nu(t) := f(t)$, we get

$$\left| \int_a^b K(t,x)\, df(t)\, dt \right| \leq \int_a^b |K(t,x)|\, df(t)$$

By the integration by parts formula for the Stieltjes integral we have

$$\int_a^b |K(t,x)|\, df(t) = \int_a^{\frac{a+b}{2}} |t-x|\, df(t) + \int_{\frac{a+b}{2}}^b |t-(a+b-x)|\, df(t)$$

$$= \int_a^x (x-t)\, df(t) + \int_x^{\frac{a+b}{2}} (t-x)\, df(t)$$

$$+ \int_{\frac{a+b}{2}}^{a+b-x} (a+b-x-t)\, df(t) + \int_{a+b-x}^b (t+x-a-b)\, df(t)$$

$$= (x-t) f(t)\big|_a^x + \int_a^x f(t)\, dt + (x-t) f(t)\big|_x^{\frac{a+b}{2}} - \int_x^{\frac{a+b}{2}} f(t)\, dt$$

$$+ (a+b-x-t) f(t)\big|_{\frac{a+b}{2}}^{a+b-x} + \int_{\frac{a+b}{2}}^{a+b-x} f(t)\, dt$$

$$+ (t+x-a-b) f(t)\big|_{a+b-x}^b - \int_{a+b-x}^b f(t)\, dt$$

$$= (x-a)(f(b)-f(a)) + (2x-a-b) f\left(\frac{a+b}{2}\right)$$

$$+ \int_a^x f(t)\, dt - \int_x^{\frac{a+b}{2}} f(t)\, dt + \int_{\frac{a+b}{2}}^{a+b-x} f(t)\, dt - \int_{a+b-x}^b f(t)\, dt$$

Now, by the monotonicity property of f, we have

$$\int_a^x f(t)\, dt \le (x-a) f(x), \quad \int_x^{\frac{a+b}{2}} f(t)\, dt \ge \left(\frac{a+b}{2}-x\right) f(x),$$

and

$$\int_{\frac{a+b}{2}}^{a+b-x} f(t)\, dt \le \left(\frac{a+b}{2}-x\right) f(a+b-x),$$

$$\int_{a+b-x}^b f(t)\, dt \ge (x-a) f(a+b-x)$$

giving that

$$\int_a^b |K(t,x)|\, df(t) \le (x-a)(f(b)-f(a)) + (2x-a-b) f\left(\frac{a+b}{2}\right)$$

$$+ \int_a^x f(t)\, dt - \int_x^{\frac{a+b}{2}} f(t)\, dt + \int_{\frac{a+b}{2}}^{a+b-x} f(t)\, dt - \int_{a+b-x}^b f(t)\, dt$$

$$\le (x-a)(f(b)-f(a)) + (2x-a-b) f\left(\frac{a+b}{2}\right)$$

$$+ 2\left(\frac{3a+b}{4}-x\right)(f(a+b-x)-f(x)),$$

which is required. ∎

Remark 2.3 *In the inequality (2.17), choose*

    1. $x = a$, then we get

$$\left| (b-a) f\left(\frac{a+b}{2}\right) - \int_a^b f(t)\,dt \right|$$

$$\leq \frac{(b-a)}{2}\left[f(b) - 2f\left(\frac{a+b}{2}\right) - f(a) \right]. \quad (2.18)$$

    2. $x = \frac{3a+b}{4}$, then we get

$$\left| \frac{b-a}{2}\left[f\left(\frac{a+b}{2}\right) + \frac{f(a)+f(b)}{2} \right] - \int_a^b f(t)\,dt \right|$$

$$\leq \frac{(b-a)}{4}\left[f(b) - 2f\left(\frac{a+b}{2}\right) - f(a) \right]. \quad (2.19)$$

    3. $x = \frac{a+b}{2}$, then we get

$$\left| (b-a)\frac{f(a)+f(b)}{2} - \int_a^b f(t)\,dt \right|$$

$$\leq \frac{(b-a)}{2}\left[f(b) - 2f\left(\frac{a+b}{2}\right) - f(a) \right]. \quad (2.20)$$

3   3. Applications to Quadrature Formulae

Let $I_n : a = x_0 < x_1 < \cdots < x_n = b$ be a division of the interval $[a,b]$, $\xi_i \in [x_i, x_{i+1}]$, $h_i = x_{i+1} - x_i$, $(i = 0,1,2,\cdots,n-1)$ and $\nu(h) := \max\{h_i | i = 0,1,2,\ldots,n-1\}$.

Define the quadrature

$$T_n(f, I_n, \xi) = \sum_{i=0}^{n-1}\left[(\xi_i - x_i)(f(x_i) + f(x_{i+1})) + (x_i + x_{i+1} - 2\xi_i) f\left(\frac{x_i + x_{i+1}}{2}\right) \right].$$

In the following, we establish some upper bounds for the error approximation of $\int_a^b f(t)\,dt$ by the quadrature $T(f, I_n, \xi)$.

Theorem 4.1 *Let f be as in Theorem 2.3. Then we have*

$$\int_a^b f(t)\,dt = T_n(f, I_n, \xi_n) + R_n(f, I_n, \xi_n) \quad (4.1)$$

where, $R_n(f, I_n, \xi_n)$ satisfies the estimation

$$|R_n(f, I_n, \xi_n)| \leq \left[\frac{1}{4}\nu(h) + \sup_{i=\overline{0,n-1}}\left|\xi_i - \frac{3x_i + x_{i+1}}{4}\right|\right] \cdot \bigvee_a^b (f) \qquad (4.2)$$

$$\leq \frac{1}{2}\nu(h)\bigvee_a^b(f).$$

Proof. Applying Theorem 2.3 on the intervals $[x_i, x_{i+1}]$, $i = 1, 2, \cdots n - 1$, we get

$$\left|\left[(\xi_i - x_i)(f(x_i) + f(x_{i+1})) + (x_i + x_{i+1} - 2\xi_i)f\left(\frac{x_i + x_{i+1}}{2}\right)\right] - \int_{x_i}^{x_{i+1}} f(t)\,dt\right|$$

$$\leq \left[\frac{1}{4}h_i + \left|\xi_i - \frac{3x_i + x_{i+1}}{4}\right|\right] \cdot \bigvee_{x_i}^{x_{i+1}}(f).$$

Summing the above inequality over i from 0 to $n-1$ and using the generalized triangle inequality, we deduce that

$$\left|T(f, \xi_n, I_n) - \int_a^b f(t)\,dt\right| \leq \sum_{i=0}^{n-1}\left[\frac{1}{4}h_i + \left|\xi_i - \frac{3x_i + x_{i+1}}{4}\right|\right] \cdot \bigvee_{x_i}^{x_{i+1}}(f)$$

$$\leq \sup_{i=\overline{0,n-1}}\left[\frac{1}{4}h_i + \left|\xi_i - \frac{3x_i + x_{i+1}}{4}\right|\right] \cdot \sum_{i=0}^{n-1}\bigvee_{x_i}^{x_{i+1}}(f)$$

$$\leq \left[\frac{1}{4}\nu(h) + \sup_{i=\overline{0,n-1}}\left|\xi_i - \frac{3x_i + x_{i+1}}{4}\right|\right] \cdot \bigvee_a^b(f)$$

which completely proves the first inequality in (4.2).

For the second inequality, we observe that

$$\left|\xi_i - \frac{3x_i + x_{i+1}}{4}\right| \leq \frac{1}{4}h_i$$

it follows that

$$\sup_{i=\overline{0,n-1}}\left|\xi_i - \frac{3x_i + x_{i+1}}{4}\right| \leq \frac{1}{4}\sup_{i=\overline{0,n-1}}h_i = \frac{1}{4}\nu(h)$$

which proves the second inequality in (4.2). ∎

Theorem 4.2 *Let f be as in Theorem 2.4. Then (4.1) holds where, $R_n(f, I_n, \xi_n)$ satisfies the estimation*

$$|R_n(f, I_n, \xi_n)| \leq L\sum_{i=0}^{n-1}\left[\frac{1}{8}h_i^2 + 2\left(\xi_i - \frac{3x_i + x_{i+1}}{4}\right)^2\right] \qquad (4.3)$$

Proof. Applying Theorem 2.4 on the intervals $[x_i, x_{i+1}]$, $i = 1, 2, \cdots n - 1$, we get

$$\left| \left[(\xi_i - x_i)(f(x_i) + f(x_{i+1})) + (x_i + x_{i+1} - 2\xi_i) f\left(\frac{x_i + x_{i+1}}{2}\right) \right] - \int_{x_i}^{x_{i+1}} f(t)\, dt \right|$$
$$\leq L\left[\frac{1}{8} h_i^2 + 2\left(\xi_i - \frac{3x_i + x_{i+1}}{4} \right)^2 \right].$$

Summing the above inequality over i from 0 to $n-1$ and using the generalized triangle inequality, we deduce that

$$\left| T(f, \xi_n, I_n) - \int_a^b f(t)\, dt \right| \leq L \sum_{i=0}^{n-1} \left[\frac{1}{8} h_i^2 + 2\left(\xi_i - \frac{3x_i + x_{i+1}}{4} \right)^2 \right],$$

which completely proves the inequality in (4.3). ∎

Remark 4.1 *One may state another result for monotonic mappings by applying Theorem 2.5. We shall left the details to the interested readers.*

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