

Location of Zeros of Lacunary-type Polynomials

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ABSTRACT: In this paper, we present some interesting results concerning the location of zeros of Lacunary-type of polynomial in the complex plane. By relaxing the hypothesis and putting less restrictive conditions on the coefficients of the polynomial, our results generalize and refines some classical results.

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1. Introduction

The study of the zeros of a polynomial dates from about the time when the geometric representation of complex numbers was introduced into mathematics. The first contributors of the subject were Gauss and Cauchy. Since the days of Gauss and Cauchy many mathematicians have contributed to the further growth of the subject. The classical results of Cauchy [3], concerning the bounds for the moduli of the zeros of a polynomial states

Theorem A. $P(z) := \sum_{j=0}^n a_j z^j$ is a polynomial of degree n and $M = \max_{0 \leq j \leq n-1} \left| \frac{a_j}{a_n} \right|$, then all the zeros of $P(z)$ lie in

$$|z| \leq 1 + M.$$

There exists several generalizations and improvements of this result (for reference see [12] and [13]). As an improvement of this result, Joyal, Labelle and Rahman [8]

proved that, if $B = \max_{0 \leq j \leq n-1} |a_j|$, then all the zeros of the polynomial $P(z) := z^n + \sum_{j=0}^{n-1} a_j z^j$ are contained in the circle

$$|z| \leq \frac{1}{2} \left\{ 1 + |a_{n-1}| + \{(1 - |a_{n-1}|)^2 + 4B\}^{\frac{1}{2}} \right\}. \quad (1.0)$$

Next we state the following elegant result which is commonly known as Eneström-Kakeya Theorem in the theory of distribution of zeros of polynomials.

Theorem B. Let $P(z) := \sum_{j=0}^n a_j z^j$ be a polynomial of degree n such that

$$a_n \geq a_{n-1} \geq \dots \geq a_1 \geq a_0 > 0,$$

then all the zeros of $P(z)$ lie in $|z| \leq 1$.

Theorem B was proved by Eneström [4], independently by Kakeya [9] and Hurwitz [7]. Applying this result to the polynomial $z^n P(\frac{1}{z})$, one gets equivalent form of Eneström-Kakeya Theorem which states that

Theorem C. If $P(z) := \sum_{j=0}^n a_j z^j$ is a polynomial of degree n such that

$$a_0 \geq a_1 \geq \dots \geq a_{n-1} \geq a_n > 0,$$

then all the zeros of $P(z)$ lie in $|z| \geq 1$.

Applying the above results to the polynomial $P(tz)$, the following more general result is immediate:

Theorem D. Let $P(z) := \sum_{j=0}^n a_j z^j$ be a polynomial of degree n such that

$$a_n t^n \geq a_{n-1} t^{n-1} \geq \dots \geq a_1 t \geq a_0 > 0,$$

then all the zeros of $P(z)$ lie in $|z| \leq t$ and in case

$$0 < a_n \leq a_{n-1} t^{n-1} \leq \dots \leq a_1 t^{n-1} \leq a_0 t^n,$$

then $P(z)$ has all zeros in $|z| \geq \frac{1}{t}$.

Now consider the class of polynomials

$$P(z) := a_0 + a_1 z + \dots + a_\mu z^\mu + a_{n_1} z^{n_1} + a_{n_2} z^{n_2} + \dots + a_{n_k} z^{n_k}, \quad (1.1)$$

$$0 < n_0 = \mu < n_1 < n_2 < \dots < n_k, \quad a_0 a_\mu a_{n_1} a_{n_2} \dots a_{n_k} \neq 0.$$

Here the coefficients a_j , $0 \leq j \leq \mu$, are fixed, the coefficients a_{n_j} , $j = 1, 2, \dots, k$ are arbitrary and the remaining coefficients a_j are zero.

Landau ([10] and [11]) Initiated the study of polynomials of this form in 1906-7.

He considered the case $\mu = 1$, $k = 1$ or 2 and proved that every trinomial

$$a_0 + a_1z + a_nz^n, \quad a_1a_n \neq 0, \quad n \geq 2$$

has at least one zero in the circle $|z| \leq 2 \left| \frac{a_0}{a_1} \right|$ and every quadrinomial

$$a_0 + a_1z + a_mz^m + a_nz^n, \quad a_1a_ma_n \neq 0, \quad 2 \leq m < n,$$

has at least one zero in the circle $|z| \leq \frac{17}{3} \left| \frac{a_0}{a_1} \right|$. Thus for these cases Landau proved the existence of a circle $|z| = R(a_0, a_1)$ containing at least one zero of $P(z)$. He also raised the question as to whether or not a circle with this same property existed in the case $\mu = 1$ and k arbitrary.

An affirmative reply was given in 1907 by Allardice [1] who proved that when $\mu = 1$, at least one zero of $P(z)$ lies in the circle

$$|z| \leq \left| \frac{a_0}{a_1} \right| \prod_{j=1}^k \frac{n_j}{n_j - 1}$$

and by Fejér ([6], [5]) who proved that, when $a_1 = a_2 = \dots = a_{\mu-1} = 0$, at least one zero of $P(z)$ lies in the circle

$$|z| \leq \left\{ \left| \frac{a_0}{a_\mu} \right| \prod_{j=1}^k \frac{n_j}{n_j - \mu} \right\}^{\frac{1}{\mu}}.$$

Another result which is instructive is the one due to Van Vleck [14], who proved that the polynomial

$$P(z) := 1 + a_rz^r + a_{r+1}z^{r+1} + \dots + a_nz^n, \quad r < n, \quad a_r \neq 0$$

has at least r zeros in the disk $|z| \leq \left[\frac{C(n,r)}{|a_r|} \right]^{\frac{1}{r}}$.

Recently Aziz and Rather [2] proved the following result for Lacunary-type of polynomials.

Theorem E. For any given positive number t , all the zeros of the polynomial

$$P(z) := a_nz^n + a_\mu z^\mu + a_{\mu-1}z^{\mu-1} + \dots + a_1z + a_0, \quad \mu < n, \quad a_0a_\mu a_n \neq 0$$

lie in the circle

$$|z| \leq \max \left\{ N_{p,t}^{\frac{1}{n-\mu}}, N_{p,t}^{\frac{1}{\mu}} \right\},$$

where

$$N_{p,t} = (\mu + 1)^{\frac{1}{q}} \left\{ \sum_{j=0}^{\mu} \left| \frac{a_j}{a_n t^{n-j}} \right|^p \right\}^{\frac{1}{p}},$$

$p > 1, q > 1$ with $p^{-1} + q^{-1} = 1$.

In this paper, we consider the following Lacunary-type of polynomials of type (1.1) with $k = 1$ and μ arbitrary

$$P(z) := a_n z^n + a_\mu z^\mu + a_{\mu-1} z^{\mu-1} + \dots + a_1 z + a_0, \quad \mu < n, \quad a_0 a_\mu a_n \neq 0, \quad (1.2)$$

and prove some results concerning the bounds for the zeros of polynomials of this form.

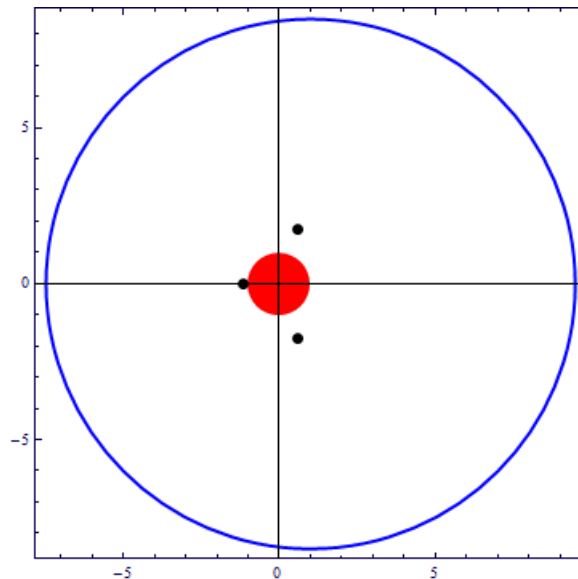
2. Main Results

Theorem 1. *Let $P(z)$ be a polynomial of type (1.2) which does not vanish in $|z| < t$, where $t > 0$, then for $p > 0, q > 0, p^{-1} + q^{-1} = 1$, all the zeros of $P(z)$ lie in*

$$|z - t| \leq A = (\mu + 2)^{\frac{1}{q}} \left\{ \sum_{j=0}^{\mu+1} \left| \frac{t a_j - a_{j-1}}{a_n t^{n-j}} \right|^p \right\}^{\frac{1}{p}}, \quad (2.1)$$

where $a_{-1} = a_{\mu+1} = 0$.

Theorem 1 states that if no zero of a polynomial $P(z)$ of type (1.2) lie in $|z| < t, t > 0$, then all its zeros will lie in the region between the circles $|z| < t$ and $|z - t| \leq A$, where A is defined above. As an example we take a polynomial $P(z) = z^3 + 2z + 3$. Here $\mu = 1$ and $n = 3$. We make use of *Wolfram Mathematica* to visualize the zeros of the above polynomial in a specific region (figure 1). The zeros of the polynomial are $(-1.17951, 0); (0.589755, -1.74454); (0.589755, 1.74454)$. Take $t = 1$, clearly $P(z)$ does not vanish in $|z| < 1$. Therefore it follows from Theorem 1 with $p = 2, q = 2$ that all the zeros of $P(z)$ lie in $|z - 1| \leq 8.48528$.



Thus it is clear from figure 1 that all the zeros of the above polynomial lie in the unshaded region between the circles $|z| < 1$ and $|z - 1| \leq 8.48528$.

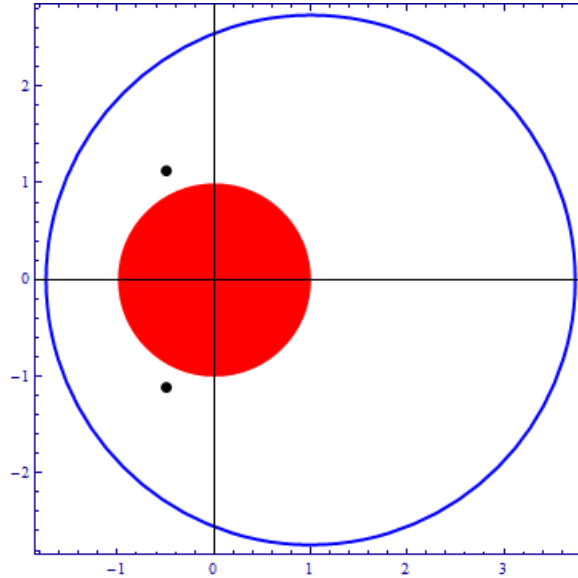
For $\mu = n - 1$, the Lacunary polynomial $P(z)$ in Theorem 1 reduces to a simple polynomial of degree n and yields the following result.

Corollary 1. Let $P(z) := \sum_{j=0}^n a_j z^j$, be a polynomial of degree n which does not vanish in $|z| < t$ where $t > 0$, then for $p > 0$, $q > 0$, $p^{-1} + q^{-1} = 1$, all the zeros of $P(z)$ lie in

$$|z - t| \leq (n + 1)^{\frac{1}{q}} \left\{ \sum_{j=0}^n \left| \frac{ta_j - a_{j-1}}{a_n t^{n-j}} \right|^p \right\}^{\frac{1}{p}}, \tag{2.2}$$

where $a_{-1} = 0$.

As an example, we take $P(z) = 2z^2 + 2z + 3$, having zeros $(-0.5, -1.11803)$; $(-0.5, 1.11803)$. Also take $t = 1$. Clearly these zeros does not lie in $|z| < 1$. Moreover from Corollary 1, it is clear that all the zeros lie in the region between the circles $|z| < 1$ and $|z - 1| < 2.738$ as is clear from figure 2.



For $\mu = 1$, we have the following result for trinomial of degree n .

Corollary 2. Let $P(z) = a_n z^n + a_1 z + a_0$, be a trinomial of degree n which does not vanish in $|z| < t$ where $t > 0$, then for $p > 0$, $q > 0$, $p^{-1} + q^{-1} = 1$, all the zeros of $P(z)$ lie in

$$|z - t| \leq 3^{\frac{1}{q}} \left\{ \sum_{j=0}^2 \left| \frac{ta_j - a_{j-1}}{a_n t^{n-j}} \right|^p \right\}^{\frac{1}{p}}. \tag{2.3}$$

Letting $q \rightarrow \infty$ so that $p = 1$ in Theorem 1, we get the following result.

Corollary 3. *Let $P(z)$ be a polynomial of type (1.2) which does not vanish in $|z| < t$, where $t > 0$, then all the zeros of $P(z)$ lie in the circle*

$$|z - t| \leq \sum_{j=0}^{\mu+1} \left| \frac{ta_j - a_{j-1}}{a_n t^{n-j}} \right|. \quad (2.4)$$

In particular for $t = 1$, we have the following Corollary by restricting the coefficients of the polynomial.

Corollary 4. *Let $P(z)$ be a polynomial of type (1.2) with real coefficients, which does not vanish in $|z| < 1$ and*

$$a_\mu \geq a_{\mu-1} \geq \cdots \geq a_1 \geq a_0 > 0, \quad a_n > 0,$$

then all the zeros of the polynomial lie in the circle

$$|z - 1| \leq 2 \frac{a_\mu}{a_n}. \quad (2.5)$$

In other words all the zeros of the polynomial $P(z)$ which does not vanish in $|z| < 1$ lie in the region

$$\left\{ z : 1 \leq |z| \cap |z - 1| \leq 2 \frac{a_\mu}{a_n} \right\}.$$

If we reverse the monotonicity of the coefficients of the polynomial, we get the following result.

Corollary 5. *Let $P(z)$ be a polynomial of type (1.2) with real coefficients, which does not vanish in $|z| < 1$ and*

$$a_0 \geq a_1 \geq \cdots \geq a_\mu > 0, \quad a_n > 0,$$

then all the zeros of $P(z)$ lie in the circle

$$|z - 1| \leq 2 \frac{a_0}{a_n}. \quad (2.6)$$

In other words if $P(z)$ does not vanish in $|z| < 1$, then all the zeros of $P(z)$ lie in the region

$$\left\{ z : 1 \leq |z| \cap |z - 1| \leq 2 \frac{a_0}{a_n} \right\}.$$

Again we make use of WOLFRAM MATHEMATICA to show that the bounds obtained in our results are sharper than the prior ones. For this we take the following

Lacunary polynomial of type (1.2)

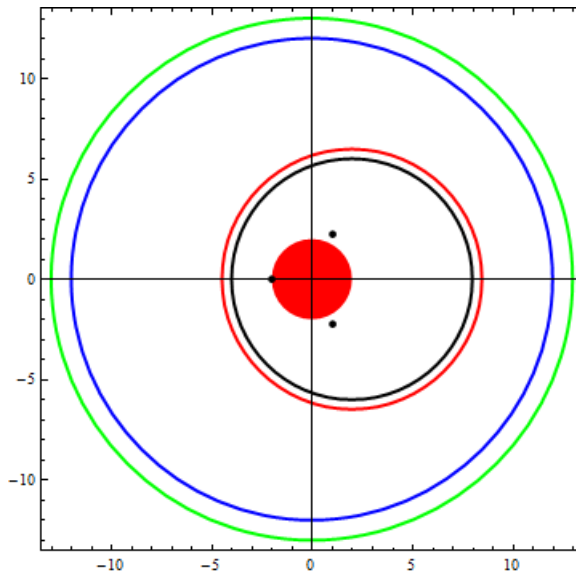
$$P(z) := z^3 + 2z + 12.$$

Its zeros are given below

$(-2), (1 - 2.23607i), (1 + 2.23607i)$. The bounds for the zeros of the above Lacunary polynomial $P(z)$ obtained by using different results are given in the following table.

| Bounds for the zeros of $P(z) := 2z^8 + z + 5$ by using different results | | |
|---|---------------------------------|---------------------------------------|
| S.No. | Theorems | Bounds |
| 1 | Landau's Theorem for Trinomials | $ z \leq 12$ |
| 2 | Theorem A | $ z \leq 13$ |
| 3 | Theorem C | $ z \geq 1$ |
| 4 | Corollary 2 | $1 \leq z \cap z - 2 \leq 6.4807$ |
| 5 | Corollary 3 | $1 \leq z \cap z - 2 \leq 6$ |

In order to visualize the above bounds we draw the following figure in which circles of different colours represents different bounds obtained by various results.



3. Proof of Theorem

Proof of Theorem 1. Consider the polynomial

$$\begin{aligned} F(z) &= (t - z)P(z) = (t - z)(a_n z^n + a_\mu z^\mu + \dots + a_1 z + a_0) \\ &= -a_n z^{n+1} + ta_n z^n - a_\mu z^{\mu+1} + (ta_\mu - a_{\mu-1})z^\mu \\ &\quad + (ta_{\mu-1} - a_{\mu-2})z^{\mu-1} + \dots + (ta_1 - a_0)z + ta_0. \end{aligned}$$

This gives

$$|F(z)| = |-a_n z^{n+1} + ta_n z^n - a_\mu z^{\mu+1} + (ta_\mu - a_{\mu-1})z^\mu + (ta_{\mu-1} - a_{\mu-2})z^{\mu-1} \\ + \dots + (ta_1 - a_0)z + ta_0| \geq |a_n||z^n| \left[|z-t| - \sum_{j=0}^{\mu+1} \left| \frac{ta_j - a_{j-1}}{a_n} \right| \frac{1}{|z|^{n-j}} \right].$$

Since $p > 0$, $q > 0$ and $p^{-1} + q^{-1} = 1$, therefore we have by Hölder's inequality for $|z| \geq t$,

$$|F(z)| \geq |a_n||z^n| \left[|z-t| - (\mu+2)^{\frac{1}{q}} \left\{ \sum_{j=0}^{\mu+1} \left| \frac{ta_j - a_{j-1}}{a_n t^{n-j}} \right|^p \right\}^{\frac{1}{p}} \right] > 0$$

if

$$|z-t| > (\mu+2)^{\frac{1}{q}} \left\{ \sum_{j=0}^{\mu+1} \left| \frac{ta_j - a_{j-1}}{a_n t^{n-j}} \right|^p \right\}^{\frac{1}{p}}.$$

This shows that for $|z| \geq t$, $F(z)$ does not vanish in

$$|z-t| > (\mu+2)^{\frac{1}{q}} \left\{ \sum_{j=0}^{\mu+1} \left| \frac{ta_j - a_{j-1}}{a_n t^{n-j}} \right|^p \right\}^{\frac{1}{p}}.$$

Hence, we conclude that those zeros of $F(z)$ and therefore $P(z)$ whose modulus is greater than t lie in

$$|z-t| \leq (\mu+2)^{\frac{1}{q}} \left\{ \sum_{j=0}^{\mu+1} \left| \frac{ta_j - a_{j-1}}{a_n t^{n-j}} \right|^p \right\}^{\frac{1}{p}}.$$

This completes proof of Theorem 1. □

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