

Journal of Mathematics and Applications

vol. 41 (2018)

e-ISSN 2300-9926

Issued with the consent of the Rector

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Publishing House of Rzeszów University of Technology
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The printed version of JMA is an original version.

p-ISSN 1733-6775

e-ISSN 2300-9926

Publisher: Publishing House of Rzeszów University of Technology,
12 Powstańców Warszawy Ave., 35-959 Rzeszów (e-mail: oficyna@prz.edu.pl)
<http://oficyna.prz.edu.pl/en/>

Editorial Office: Rzeszów University of Technology, Faculty of Mathematics and Applied Physics, P.O. BOX 85
8 Powstańców Warszawy Ave., 35-959 Rzeszów (e-mail: jma@prz.edu.pl)
<http://jma.prz.edu.pl/en/>

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On Maximum Induced Matching Numbers of Special Grids

Tayo Charles Adefokun and Deborah Olayide Ajayi

ABSTRACT: A subset M of the edge set of a graph G is an induced matching of G if given any two edges $e_1, e_2 \in M$, none of the vertices on e_1 is adjacent to any of the vertices on e_2 . Suppose that $Max(G)$, a positive integer, denotes the maximum size of M in G , then, M is the maximum induced matching of G and $Max(G)$ is the maximum induced matching number of G . In this work, we obtain upper bounds for the maximum induced matching number of grid $G = G_{n,m}$, $n \geq 9, m \equiv 3 \pmod{4}, m \geq 7$, and nm odd.

AMS Subject Classification: 05C70, 05C15.

Keywords and Phrases: Induced matching; Grid; Maximum induced matching number; Strong matching number.

1. Introduction

For a graph G , let $V(G), E(G)$ be vertex and edge sets respectively and let $e \in E(G)$. We define $e = uv$, where $u, v \in V(G)$ and the respective order and size of $V(G)$ and $E(G)$ are $|V(G)|$ and $|E(G)|$. For some $M \subseteq E(G)$, M is an induced matching of G if for all $e_1 = u_i v_j$ and $e_2 = v_i v_j$ in M , $u_k v_l \notin M$, where k and l are from $\{i, j\}$. Induced matching, a variant of the matching problem, was introduced in 1982 by Stockmeyer and Vazirani [10] and has also been studied under the names strong matching [7] and “risk free” marriage problem [8]. It has found theoretical and practical applications in a lot of areas including network problems and cryptology [3]. For more on induced matching and its applications, see [2], [3], [4], [5] and [11].

The size $|M|$ of an induced matching M of G is a positive integer and translates as the maximum induced matching number $Max(G)$ (or strong matching number) of

G if $|M|$ is maximum. Obtaining $Max(G)$ is NP -hard, even for regular bipartite graphs [4]. However, $Max(G)$ of some graphs have been found in polynomial time such as the cases in [3], [6].

A grid $G_{n,m}$ is the Cartesian product of two paths P_n and P_m , resulting in n -rows and m -columns. Marinescu-Ghemaci in [9], obtained the $Max(G)$ for $G_{n,m}$, grid where both n, m are even; either of n and m is even and for quite a number of grids $G_{n,m}$ where nm is odd, which is called the odd grid in [1]. Marinescu-Ghemaci [9] also gave useful lower and upper bounds and conjectured that the $Max(G)$ of grids can be found in polynomial time and also by combining the maximum induced numbers of partitions of odd grids, Marinescu-Ghemaci confirmed that for any odd grid $G \equiv G_{n,m}$, $Max(G) \leq \lfloor \frac{nm+1}{4} \rfloor$. This bound was improved on in [1] for the case where $n \geq 9$ and $m \equiv 1 \pmod{4}$.

In this paper, the Marinescu-Ghemaci's bound for the case where $n \geq 9$ and $m \equiv 3 \pmod{4}$ is considered and more compact values are obtained. The results in this work, combined with some of the results in [9], confirm the maximum induced matching numbers of certain graphs, whose lower bounds were established in [9].

2. Definitions and Preliminary Results

Grid, $G_{n,m}$, as defined in this work, is the Cartesian product of paths P_n and P_m with $V(P_n) = \{u_1, u_2, \dots, u_n\}$ and $V(P_m) = \{v_1, v_2, \dots, v_m\}$. We adopt the following notations which are similar to those in [1]:

$$V_i = \{u_1v_i, u_2v_i, \dots, u_nv_i\} \subset V(G_{n,m}), \quad i \in [1, m],$$

$$U_i = \{u_iv_1, u_iv_2, \dots, u_iv_m\} \subset V(G_{n,m}), \quad i \in [1, n].$$

For edge set $E(G_{n,m})$ of $G_{n,m}$, if $(u_iv_j, u_kv_j) \in E(G_{n,m})$ and $(u_iv_j, u_iv_k) \in E(G_{n,m})$, we write $u_{(i,k)}v_j \in E(G_{n,m})$ and $u_iv_{(j,k)} \in E(G_{n,m})$ respectively.

A saturated vertex v is any vertex on some edge in M , otherwise, v is unsaturated, cf. [1]. We define v as saturable if it can be saturated relative to the nearest saturated vertex. Any vertex that is at least distant-2 from the nearest saturated vertex is saturable. By this definition, therefore, it is clear that a saturated vertex is at first saturable. However, not every saturable vertex is saturated. The set of all saturable vertices on a graph G is denoted by $V_{sb}(G)$ while the set of saturated vertices is $V_{st}(G)$. Clearly, $|V_{st}(G)|$ is even and $V_{st}(G) \subseteq V_{sb}(G)$. Free saturable vertex set (FSV) is the set of saturable vertices which can not be on any members of M . In other words, $v \in FSV$ is a saturable vertex of graph G , which is not adjacent to some saturable vertex $u \in G$. Note that $FSV = V_{sb} \setminus V_{st}$. Let G be a $G_{n,m}$ grid. We define $G^{[k]}$ as a $G_{n,k}$ subgraph of G induced by $\{V_{i+1}, V_{i+2}, \dots, V_{i+k}\}$. An unsaturated vertex $v \in G$ is unsaturable if $v \notin FSV$ and $v \notin V_{sb}(G)$. Furthermore, for positive integers a and b , $a < b$, $[a, b] := \{a, a+1, \dots, b\}$.

The following results from [9] on G , a $G_{n,m}$ grid, are useful in this work:

Lemma 2.1. *Let $m, n \geq 2$ be two positive integers and let G be a $G_{n,m}$ grid. Then,*

- (a) If $m \equiv 2 \pmod{4}$ and n odd then $|V_{sb}(G)| = \frac{mn+2}{2}$; and $|V_{sb}(G)| = \frac{mn}{2}$ otherwise;
 (b) for $m \geq 3$, m odd, $|V_{sb}(G)| = \frac{nm+1}{2}$, for $n \in \{3, 5\}$.

Theorem 2.2. Let G be a $G_{n,m}$ grid with $2 \leq n \leq m$. Then,

- (a) if n even and m even or odd, then $Max(G) = \lceil \frac{mn}{4} \rceil$;
 (b) if $n \in \{3, 5\}$ then for
 (i) $m \equiv 1 \pmod{4}$, $Max(G) = \frac{n(m-1)}{4} + 1$,
 (ii) $m \equiv 3 \pmod{4}$, $Max(G) = \frac{n(m-1)+2}{4}$.

The following theorem is the statement of the bound given by Marinescu-Ghemaci [9].

Theorem 2.3. Let G be a $G_{n,m}$ grid, $m, n \geq 2$, mn odd. Then $Max(G) \leq \lfloor \frac{mn+1}{4} \rfloor$.

3. Maximum Induced Matching Number of Odd Grids

The following lemma and the remark describe the importance of the saturation status of certain vertices in $G_{5,p}$ grid, where $p \equiv 2 \pmod{4}$.

Lemma 3.1. Let G be a $G_{n,m}$ grid and let $\{V_{i+1}, V_{i+2}, \dots, V_{i+p}\} \subset G$ induce $G^{|p|}$, a $G_{5,p}$ subgrid of G , where $p \equiv 2 \pmod{4}$. Suppose that M_1 , is an induced matching of $G^{|p|}$ and that for $u_3v_{i+1} \in V_{i+1} \subset V(G^{|p|})$, $u_3v_{i+1} \notin V_{st}(G^{|p|})$. Then, $V_{st}(G^{|p|}) \leq 10k + 4$, for positive integer k , where $p = 4k + 2$ and M_1 is not a maximum induced matching of $G^{|p|}$.

Proof. For a positive integer k , let $p = 4k + 2$, $G^{|2|}$ and $G^{|p-2|}$ be partitions of G_1 , induced by $\{V_{i+1}, V_{i+2}\}$ and $\{V_{i+3}, V_{i+4}, \dots, V_{i+p}\}$, respectively. Since u_3v_{i+1} is not saturated in $G^{|2|}$, it easy to check that $|V_{sb}(G^{|2|})| = 5$. From [9], $|V_{sb}(G^{|p-2|})| = |V_{st}(G^{|p-2|})| = 10k$. Thus $|V_{sb}(G^{|p|})| \leq |V_{sb}(G^{|2|})| + |V_{sb}(G^{|p-2|})| \leq 10k + 5$ and therefore, $|V_{st}(G^{|p|})| \leq 10k + 4$ since $|V_{st}(G)|$ is even, for any graph G . This is a contradiction since by [9], $|V_{st}(G^{|p|})| = 10k + 6$. \square

Remark 3.2. It should be noted that M_1 in Lemma 3.1 will still not be a maximum induced matching of $G^{|p|}$ if for the vertex set $A = \{u_1v_{i+1}, u_5v_{i+1}, u_1v_{i+p}, u_3v_{i+p}, u_5v_{i+p}\} \subset V(G^{|p|})$, any member of A is unsaturated.

Lemma 3.3. Suppose $u_{(1,2)}v_i, u_5v_{(i-1,i)} \in M$ or $u_{(1,2)}v_i, u_5v_{(i,i+1)} \in M$, where M is an induced matching of G , a $G_{5,m}$ grid, $m \equiv 3 \pmod{4}$, $m \geq 23$ and $1 < i < m$, $i \notin \{4, m-3\}$. Then M is not a maximum induced matching of G .

Proof. Let G be partitioned into $G^{|m(1)|}$ and $G^{|m(2)|}$, which are induced respectively by $A = \{V_1, V_2, \dots, V_i\}$ and $B = \{V_{i+1}, V_{i+2}, \dots, V_m\}$. Suppose that M is a maximum induced matching of G .

Case 1: $i \equiv 1 \pmod{4}$.

Let $m = 4k + 3$ and set $i = 4t + 1$, where $k \geq 5$ and $t > 0$. Then, $|m(1)| \equiv 1 \pmod{4}$ and $|m(2)| \equiv 2 \pmod{4}$. Since u_1v_i, u_2v_i, u_5v_i and u_5v_{i-1} are saturated vertices in V_i and V_{i-1} , then the only FSV member on V_{i-1} is u_3v_{i-1} . Suppose that u_3v_{i-1} remains unsaturated. Let $G^{|m(3)|} \subset G^{|m(1)|}$ be induced by $\{V_1, V_2, \dots, V_{i-2}\}$, where $|m(3)| \equiv 3 \pmod{4}$. By [9], $|V_{st}(G^{|m(3)|})| = 10t - 4$. Thus, $|V_{st}(G^{|m(1)|})| \leq 10t$. Suppose that u_3v_{i-1} is saturated, then, $u_3v_{(i-1, i-2)} \in M$. Thus, $u_3v_{i-3} \in V_{i-3} \subset G^{|m(4)|}$ is unsaturable, where $G^{|m(4)|}$ is $G^{|m(3)|} \setminus V_{i-2}$. Note that $|m(4)| \equiv 2 \pmod{4}$. From Lemma 3.1, therefore, $|V_{st}(G^{|m(4)|})| \leq 10t - 6$ and thus, $|V_{st}G^{|m(1)|}| \leq 10t - 6 + 6 = 10t$. Now, since u_1v_i, u_2v_i and u_5v_i are saturated vertices in V_i , then, $u_3v_{i+1}, u_4v_{i+1} \in V(G^{|m(2)|})$ are saturable vertices in $G^{|m(2)|}$.

Claim: Edge $u_{(3,4)}v_{i+1}$ belongs to M .

Reason: Suppose that both u_3v_{i+1} and u_4v_{i+1} are not saturated, then V_{i+1} contains no saturable vertices. Let $G^{|m(2)|} \setminus \{V_{i+1}\} = G^{|m(5)|}$, where $|m(5)| \equiv 1 \pmod{4}$. Thus, $|V_{st}(G)| \leq |V_{st}G^{|m(1)|}| + |V_{st}(G^{|m(5)|})| = 10k + 2$, which is less than the required saturated vertices by 4 and hence the claim. Now, $u_{(3,4)}v_{i+1}$ belongs to M . Clearly for $G^{|m(5)|}$ defined above, $|V_{sb}(G^{|m(5)|})| = 10(k - t) + 3$ and suppose $u_3v_{i+1}, u_4v_{i+1} \in V_{st}(G)$, then $|V_{st}(G)| \leq 10k + 5$. In fact, $|V_{st}(G)| = 10k + 4$. Thus establishing the first part of the case that with $u_{(1,2)}v_i, u_5v_{(i-1, i)} \in M$, $M \neq \text{Max}(G)$.

For the second part of the case, suppose that $u_{(1,2)}v_i, u_5v_{(i, i+1)} \in M$. Let $G^{|n(1)|} = G^{|m(1)|} \setminus \{V_i\}$ and $G^{|n(2)|} = G^{|m(2)|} \cup \{V_i\}$. Now, $|n(1)| \equiv 0 \pmod{4}$ and $|n(2)| \equiv 3 \pmod{4}$. Consequently, $|V_{st}(G^{|n(2)|})| = 10(k - t) + 6$. Now, on $V_{i-1} \subset G^{|n(1)|}$, only vertices u_3v_{i-1} and u_4v_{i-1} are saturable. Suppose they are both not saturated after all. Let $G^{|n(3)|} \subset G^{|n(1)|}$ be induced by $\{V_1, V_2, \dots, V_{i-2}\}$, where $|n(3)| \equiv 3 \pmod{4}$. $|V_{st}(G^{|n(3)|})| = 10t - 4$. Thus $|V_{st}(G)| = 10k + 2$. Therefore, M requires four saturated vertices to be a maximum induced matching of G . Now, $|V_{sb}(G^{|n(3)|})| = 10t - 2$, and thus, $V(G^{|n(3)|})$ contains two extra FSV vertices, say, v_1, v_2 which are not adjacent. Thus, the maximum number of saturable vertices from the vertex set $\{v_1, v_2, u_3v_{i-1}, u_4v_{i-1}\}$ is 2. Therefore, $|V_{st}(G)| \leq 10k + 4$, which is a contradiction.

Case 2: $i \equiv 2 \pmod{4}$.

Let $G^{|p(1)|}$ and $G^{|p(2)|}$ be partitions of G induced by $\{V_1, V_2, \dots, V_i\}$ and $\{V_{i+1}, V_{i+2}, \dots, V_m\}$, with $m = 4k + 3$ and $i = 4t + 2$. Let $u_{(1,2)}v_i$ and $u_5v_{(i-1, i)} \in M$. Since $u_{(1,2)}v_i$ belongs in M of G , then u_3v_i cannot be saturated. Thus, $|V_{st}(G^{|p(2)|})| \geq 10(k - t) + 2$ for M to be maximal. It can be seen that $|p(2)| \equiv 1 \pmod{4}$. Now, u_3v_{i+1} and u_4v_{i+1} are saturable vertices in V_{i+1} . Suppose both of them are not saturated, then for $G^{|p(3)|}$ induced by $\{V_{i+2}, V_{i+3}, \dots, V_m\}$, where $|p(3)| \equiv 0 \pmod{4}$, $|V_{st}(G^{|p(3)|})| \leq 10(k - t)$. Thus u_3v_{i+1} and u_4v_{i+1} are saturable vertices and in fact, $u_{(3,4)}v_{i+1} \in M$. On V_{i+2} , therefore, there exists three saturable vertices u_1v_{i+1}, u_2v_{i+2} and u_5v_{i+5} . Suppose none of these three vertices are saturated. Then, $|V_{st}(G^{|p(3)|})| \leq |V_{st}(G^{|p(4)|})| + 2$, with $G^{|p(4)|}$ induced by $\{V_{i+3}, \dots, V_m\}$ and $|p(4)| \equiv 3 \pmod{4}$ and thus, $|V_{st}(G^{|p(2)|})| \leq 10(t - k) - 2$. Therefore it requires extra four saturated vertices

for M to be maximum. There exist two other saturable vertices, $v_1, v_2 \in V(G^{|p(4)|})$ (since $V_{st}(G^{|p(4)|}) = 10(k-t) - 4$ and $V_{sb}(G^{|p(4)|}) = 10(k-t) - 2$). Clearly, v_1, v_2 are not adjacent, else they would have formed an edge in M . Suppose $v_1, v_2 \in V_{i+3}$. For v_1 and v_2 to be saturated, they have to be u_5v_{i+3} and one of u_1v_{i+3} and u_2v_{i+3} . Thus, $u_5v_{i+3} \in M$ and one of $u_1v_{(i+2,i+3)}$ $u_2v_{(i+2,i+3)}$ or $u_{(1,2)}v_{i+2}$ belongs to M . Let $G^{|p(5)|}$ be induced by $\{V_{i+4}, \dots, V_m\}$, where $|p(5)| \equiv 2 \pmod{4}$. Now, since $v_5v_{(i+2,i+3)} \in M$, then $u_5v_{i+5} \in V_{i+4}$ is unsaturable and therefore, by Remark 3.2, $|V_{st}(G^{|p(5)|})| = 10(k-t-1) + 4$ and thus, $|V_{st}(G^{|p(2)|})| = 10(k-t)$, which is less than required. The case of $u_5v_{(i,i+1)} \in M$ is the same as the case of $u_5v_{(i-1,i)} \in M$ for $i \equiv 2 \pmod{4}$.

Case 3: $i \equiv 0 \pmod{4}$, $i \geq 6$ or $i \leq m-5$, with $u_{(1,2)}v_i, u_5v_{(i-1,i)} \in M$. Let $G^{|r(1)|}$ and $G^{|r(2)|}$ be partitions of G which are induced respectively by $\{V_1, V_2, \dots, V_i\}$ and $\{V_{i+1}, V_{i+2}, \dots, V_m\}$. Since $i \equiv 0 \pmod{4}$, then $|r(1)| \equiv 0 \pmod{4}$, while $|r(2)| \equiv 3 \pmod{4}$. Also, $u_5v_{(i-1,i)} \in M$, implies u_5v_{i-1} is unsaturable. Since $i-2 \equiv 2 \pmod{4}$, then by Lemma 3.1 and Remark 3.2, $|V_{st}(G^{|r(1)|})| \leq 10t - 2$, implying that for M to be maximal, $|V_{st}(G^{|r(2)|})| \geq 10(k-t) + 8$. It can be seen that V_{i+1} has two only saturable vertices u_3v_{i+1}, u_4v_{i+2} left. It should also be noted that if any of u_3v_{i+1} and u_4v_{i+2} is saturated, then u_3v_{i+3} can not be saturated in $G^{|r(3)|}$, a subgrid of $G^{|r(2)|}$ induced by $\{V_{i+2}, V_{i+3}, \dots, V_m\}$, with $|r(3)| \equiv 2 \pmod{4}$. Thus suppose $u_3v_{i+1}, u_4v_{i+2} \in V_{st}(G)$, then $|V_{st}(G)| \leq 10(k-t) + 4$. Likewise, if $u_3v_{i+1}, u_4v_{i+2} \notin V_{st}(G)$, $|V_{st}(G)| \leq 10t - 2 + 10(k-t) + 6$. The case of $u_5v_{(i,i+1)} \in M$ follows the same argument as the case of $u_5v_{(i-1,i)} \in M$. \square

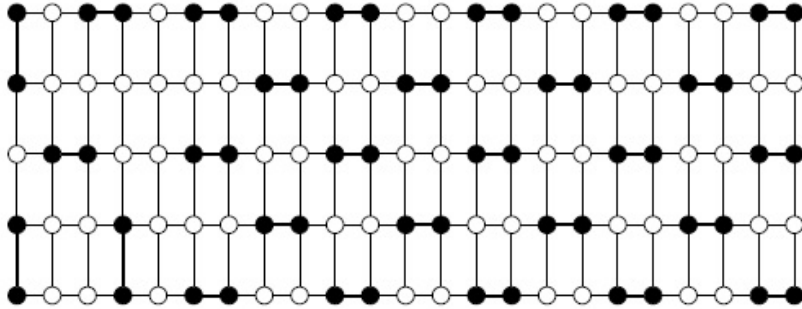


Figure 1: A Grid $G \equiv G_{5,23}$ with $Max(G) = 28$, $u_{(1,2)}v_1, u_{(1,2)}v_4 \in M$ of G

Remark 3.4.

- (a) In the case of $i \equiv 0 \pmod{4}$ in Lemma 3.3, M remains a maximum induced matching when $i = 4$ or when $i = m - 3$ as seen in Figure 1 of $Max(G) = 28$ of $G_{5,23}$.

- (b) It should be noted that the case of $i \equiv 3 \pmod{4}$ has been taken care of by the case of $i \equiv 1 \pmod{4}$ by ‘flipping’ the grid from right to left or vice versa.
- (c) From Lemma 3.3, we note that if for some induced matching M of $G_{5,m}$, $m \equiv 3 \pmod{4}$, $u_{(1,2)}v_i$ and $u_5v_{(i-1,i)}$ (or $u_5v_{(i,i+2)}$) $\in M$, then M is not a maximal induced matching of G for any $1 < i < m$.

Next we investigate some induced matching M of $G_{5,m}$ if it contains $u_{(1,2)}v_i$ and $u_{(4,5)}v_i$.

Lemma 3.5. *Suppose $G = G_{5,m}$, where $m \geq 23$ and $m \equiv 3 \pmod{4}$. Let $u_{(1,2)}v_i, u_{(4,5)}v_i \in M$, an induced matching of G and $1 < i < m$, $i \not\equiv 0 \pmod{4}$ then M is not a maximum induced matching of G .*

Proof. Let M be an induced matching of $G = G_{5,m}$. Suppose that $i \equiv 2 \pmod{4}$. Let $G^{[m(1)]}$ and $G^{[m(2)]}$ be partitions of G induced by $\{V_1, V_2, \dots, V_i\}$ and $\{V_{i+1}, V_{i+2}, \dots, V_m\}$. Since $u_{(1,2)}v_i, u_{(4,5)}v_i \in M$, then, u_3v_i is unsaturated. Let $i = 4t + 2$, for some positive integer t , by Lemma 3.3, $|V_{st}(G^{[m(1)]})| = 10t + 4$. Now, only u_3v_{i+1} is saturable on V_{i+1} . Let $G^{[m(3)]} \subset G^{[m(2)]}$, induced by $\{V_{i+2}, \dots, V_m\}$. Clearly $|m(3)| = |m(2)| - 1 = 4(k - t)$. Therefore, $|V_{st}(G^{[m(3)]} \cup u_3v_i)| \leq 10(k - t) + 1$, which, in fact, is $10(k - t)$. Thus, $|V_{st}(G)| = 10k + 4$.

Now, suppose $i \equiv 1 \pmod{4}$. Let $G^{[n(1)]}$ be induced by $\{V_1, V_2, \dots, V_i\}$ and let $G^{[n(2)]}$ be induced by $\{V_{i+1}, V_{i+2}, \dots, V_m\}$. Since $|n(1)| = 4t + 1$, it is easy to see that $|n(2)| \equiv 2 \pmod{4}$ and hence, $|n(2)| = 4(k - t) + 2$.

Claim: For M to be maximum, both u_3v_{i-1} and u_3v_{i+1} must be saturated.

Reason: Suppose, say u_3v_{i-1} is not saturated. Then, no vertex on V_{i-1} is saturable. Now, let $\{V_1, V_2, \dots, V_{i-2}\}$ induce grid $G^{[n(3)]}$, with $|n(3)| \equiv 3 \pmod{4}$. Then, $|V_{st}(G^{[n(3)]})| = 10t - 4$, and thus, $|G^{[n(1)]}| = 10t$. Also, let $G^{[n(4)]}$ be induced by $\{V_{i+2}, V_{i+3}, \dots, V_m\}$. Since $|n(4)| = 4(k - t) + 1$, then for $G^{[n(4)]} + u_5v_{i+1}$, $|V_{sb}[(G^{[n(4)]}) \cup u_3v_{i+1}]| = 10(k - t) + 4$. Therefore, $|V_{st}(G)| \leq 10k + 4$. Now suppose $u_3v_{(i-2,i-1)} \in M$ and let $G^{[n(5)]}$ be induced by $\{V_1, V_2, \dots, V_{i-3}\}$, with $|n(5)| \equiv 2 \pmod{4}$. By Lemma 3.1, $|V_{st}(G^{[n(5)]})| = 10t - 6$. Thus, $|V_{st}(G^{[n(1)]})| = 10t$ and therefore, $|V_{st}(G)| \leq 10k + 4$, which is less than required number by at least 2. Hence, $M \neq \text{Max}(G)$. \square

Remark 3.6. Like in Remark 3.4, for $i \equiv 0 \pmod{4}$, it can be seen that $u_{(1,2)}v_1, u_{(1,2)}v_4$ or $u_{(1,2)}v_{m-3}, u_{(1,2)}v_m$ can be in M if M is a maximum induced matching of G . Also given $i \equiv 0 \pmod{4}$ and $4 < i < m - 3$, for at most one i in $[4, m - 3]$ for which $u_{(1,2)}v_i$ can be a member of maximal M .

Next we investigate the maximality of the induced matching of $G = G_{5,m}$, $m \equiv 3 \pmod{4}$.

Lemma 3.7. *Let $u_{(1,2)}v_i, u_4v_{(i-1,i)} \in M$ or $u_{(1,2)}v_i, u_4v_{(i,i+1)} \in M$, where M is an induced matching of G , a $G_{5,m}$ grid, $m \equiv 3 \pmod{4}$, $m \geq 23$ and $1 < i < m$, $i \not\equiv 0 \pmod{4}$. Then M is not a maximum induced matching of G .*

Proof. Case 1: $i \equiv 1 \pmod 4$.

Suppose that $m = 4k + 3$ and $i = 4t + 1$, $t \geq 1$. Let $G^{|m(1)|}$ and $G^{|m(2)|}$ be two partitions of G , induced by $\{V_1, V_2, \dots, V_i\}$ and $\{V_{i+1}, V_{i+2}, \dots, V_m\}$, respectively. Since $u_{(1,2)}v_i, u_4v_{(i-1,i)} \in M$, then there is no other saturated vertex on both of V_{i-1} and V_i . Let $G^{|m(3)|} \subset G^{|m(1)|}$ be a grid induced by $\{V_1, V_2, \dots, V_{i-2}\}$. Now, $n(3) \equiv 3 \pmod 4$. Therefore, $|V_{st}(G^{|m(3)|})| = 10t - 4$ and hence, $|V_{st}(G^{|m(1)|})| = 10t$. Now, $|m(2)| \equiv 2 \pmod 4$, since $u_{(1,2)}v_i \in M$, then $u_1v_{i+1} \in V_{i+1}$ is unsaturable. From a previous result, $|V_{st}(G^{|n(2)|})| = 10(k - t) + 4$ and thus, $|V_{st}(G)| = 10k + 4$. For $u_4v_{(i,i+1)} \in M$, let $G^{|n(1)|}$ and $G^{|n(2)|}$ be induced by $G^{|m(1)|} \setminus V_i$ and $G^{|m(2)|} \cup V_i$. Then, $|n(1)| \equiv 0 \pmod 4$ and $|n(2)| = 4(k - t) + 3$. It can be seen that on V_{i-1} , only u_3v_{i-1} and u_5v_{i-1} are saturable vertices.

Claim: Vertices u_3v_{i-1} and u_5v_{i-1} are not saturable for M to be maximal.

Reason: Suppose without loss of generality, that any of u_3v_{i-1} and u_5v_{i-1} is saturated, say u_5v_{i-1} . Then $u_5v_{(i-2,i-1)} \in M$. This implies that v_5v_{i-3} is not saturable in V_{i-3} . Now $\{V_1, V_2, \dots, V_{i-3}\}$ induces a grid $G^{|n(4)|}$ and $|n(4)| \equiv 2 \pmod 4$. Then, $|V_{st}(G^{|m(4)|})| = 10t - 6$ and thus, $|V_{st}(G^{|n(1)|})| = 10t - 4$. Now, since $|n(2)| = 4(k - t) + 3$, $|V_{st}(G^{|m(2)|})| = 10(k - t) + 6$ and therefore, $|V_{st}(G)| = 10k + 2$.

Case 2: $i \equiv 2 \pmod 4$.

Let $G^{|n(1)|}$ and $G^{|n(2)|}$ be two partitions of G , induced by $\{V_1, V_2, \dots, V_i\}$ and $\{V_{i+1}, V_{i+2}, \dots, V_m\}$ respectively. Since $u_{(1,2)}v_i$ and $u_4v_{(i-1,i)} \in M$, vertex $u_5v_i \in V_{sb}(G^{|n(1)|})$, and therefore, $|V_{st}G^{|n(1)|}| = 10t + 4$, where $|n(1)| = 4t + 2$. Also, only u_3v_{i+1} and u_5v_{i+1} are saturable on V_{i+1} . Suppose without loss of generality, that both u_3v_{i+1} and u_5v_{i+1} are saturated and thus, $u_3v_{(i+1,i+2)}, u_5v_{(i+1,i+2)} \in M$. Now, suppose that $G^{|n(4)|}$ is induced by $\{V_{i+3}, V_{i+4}, \dots, V_m\}$, with $|n(4)| = 4(k - t - 1) + 3$. By following the techniques employed earlier, it can be shown that $|V_{st}(G)| \leq |V_{st}(G^{|n(1)|})| + |V_{st}(G^{|n(2)|})| \leq 10k + 4$. The $u_4v_{(i,i+4)}$ case, has the same proof as the $u_4v_{(i-1,i)}$ case. \square

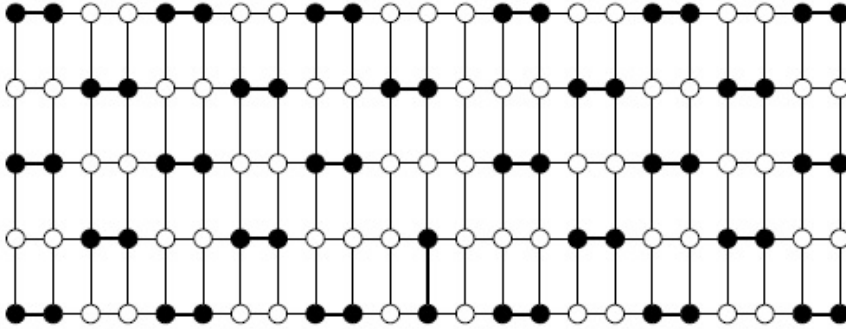


Figure 2: A $G \equiv G_{5,23}$ Grid with $Max(G) = 28$, $u_{1,2}v_i \in M, i \equiv 0 \pmod 4$

Remark 3.8.

- (a) There can be only one edge $u_{(1,2)}v_i \in M$ for which M is the maximum induced matching of $G_{5,m}$, if M contains $u_{(1,2)}v_i$ and $u_4v_{(i-1,i)}$ (or $u_4v_{(i,i+1)}$), and in this case, $i \equiv 0 \pmod{4}$ as shown in Figure 2.
- (b) It should be noted that the proof of the case $i \equiv 1 \pmod{4}$ in Lemma 3.7 will hold for $i \equiv 3 \pmod{4}$ by flipping the grid from right to left.

The previous results and remarks yield the following conclusion.

Corollary 3.9. *Suppose that $m \geq 23$ and M is the maximum induced matching of G , some $G_{5,m}$ grid. Then, if for at most some positive integer i , $1 < i < m$, $u_{(1,2)}v_i \in M$, then, $i \equiv 0 \pmod{4}$.*

Lemma 3.10. *Let M be a matching of $G_{5,m}$ with $m \equiv 3 \pmod{4}$ and let $u_{(1,2)}v_i, u_{(1,2)}v_j \in M$, $1 < i < j < m$, such that $i \equiv 0 \pmod{4}$ and $j \equiv 0 \pmod{4}$, then M is not a maximum induced matching of G .*

The claim in Lemma 3.10 can easily be proved using earlier techniques and Lemma 3.1 and Remark 3.2.

Remark 3.11. It should be noted from the previous results and from Corollary 3.9 that if M is the maximum induced matching of $G_{5,m}$, $m \equiv 3 \pmod{4}$, then at most, M contains two edges of the form $u_{(1,2)}v_i, u_{(1,2)}v_j$ and j can only be 4 when $i = 1$ or i can only be $m - 3$ when $j = m$.

Theorem 3.12. *Let M be the maximum induced matching of G , a $G_{5,m}$ grid, with $m \geq 7$, $m = 4k + 3$ and $k \geq 1$. Let M contain $u_{(1,2)}v_1$ and $u_{(1,2)}v_4$ (or $u_{(1,2)}v_{m-3}$ and $u_{(1,2)}v_m$). Then there are at least $2k + 2$ saturated vertices on $U_1 \subset G$.*

Proof. For $u_{(1,2)}v_1$ and $u_{(1,2)}v_4$ to be in M , either $u_{(4,5)}v_4 \in M$ or $u_5v_{(3,4)} \in M$. Now, let $\{V_6, V_7, \dots, V_m\}$ induce $G^{|m(1)|} \subset G$. Clearly, $|m(1)| \equiv 2 \pmod{4}$ and $|V_{st}(G^{|m(1)|})| = 10k - 4$.

Let $G^{|m(1)|} \setminus \{u_1v_6, u_1v_7, \dots, u_1v_m\}$ induce $G^{|m(2)|} \subset G^{|m(1)|}$. Then, $G^{|m(2)|}$ is a $G_{4,m-5}$ subgraph of $G^{|m(1)|}$. Now, $|V_{st}(G^{|m(2)|})| \leq 8k - 4$. Thus for $V(U_1) \subset V(G^{|m(1)|})$, $|V(U)| \geq 2k$. Thus, U_1 contains at least $2k + 2$ (i.e. $\frac{m-1}{2}$) saturated vertices. \square

Next we investigate $G_{3,m}$, where $m \equiv 3 \pmod{4}$.

Lemma 3.13. *Suppose that G is a $G_{3,m}$ grid with $m \equiv 3 \pmod{4}$ and M is an induced matching of $G_{3,m}$, with $\{u_{(1,2)}v_i, u_{(1,2)}v_{i+2}, u_{(1,2)}v_j, u_{(1,2)}v_{j+2}\} \in M$ and $i + 2 \geq j$. Then M is not a maximum induced matching of G .*

Proof. Suppose $i + 2 \geq j$. Since $m = 4k + 3$, $|V_{sb}(G)| = 6k + 5$ and $|V_{st}(G)| = 6k + 4$. Thus, G contains at most one FSV vertex. Now from the conditions in the hypothesis, it is clear that u_3v_{i+1} and u_3v_{j+1} are FSV members in G , which is a contradiction. Same argument hold if $i + 2 = j$ since both u_3v_{i+1} and u_3v_{i+3} are FSV vertexes in G . \square

Remark 3.14. Suppose that G_n is $G_{3,n}$, a subgrid of $G_{3,m}$ and induced by $\{V_{i+1}, V_{i+2}, \dots, V_{i+n}\}$ and G' is a subgraph of G , with $G' = G_n + \{u_3v_i, u_3v_{i+n+1}\}$, then the following are easy to verify. For

- (a) $n \equiv 0 \pmod{4}$, $|V_{st}(G')| \leq |V_{sb}(G_n)| + 2$.
- (b) $n \equiv 1 \pmod{4}$, $|V_{st}(G')| \leq |V_{sb}(G_n)| + 2$.
- (c) $n \equiv 2 \pmod{4}$, $|V_{st}(G')| = |V_{sb}(G_n)|$.
- (d) $n \equiv 3 \pmod{4}$, $|V_{st}(G')| \leq |V_{sb}(G_n)| + 1$.

Lemma 3.15. Let $u_{(1,2)}v_j, u_{(1,2)}v_{j+3}, u_{(1,2)}v_k, u_{(1,2)}v_{k+3}, u_{(1,2)}v_l, u_{(1,2)}v_{l+3}$ be in M an induced matching of G a $G_{3,m}$ grid and $m \equiv 3 \pmod{4}$. Then M is not maximum induced matching of G .

Proof. Case 1: $j + 3 = k$ and $l = k + 3$.

Suppose $m = 4p + 3$ and $G^{l(m(1))}$ is a subgraph of G , induced by $\{V_{j-1}, V_j, \dots, V_{i+4}\}$. Then $|m(1)| = 12$ and $u_3v_{j-1}, u_3v_{i+4} \in FSV$. For one of u_3v_{j-1} and u_3v_{i+4} to be relevant for M to be a maximum induced matching of G , say u_3v_{j-1} , then for $G^{l(m(2))}$, induced by $\{V_1, V_2, \dots, V_{j-2}\}$, $|V_{sb}(G^{l(m(2))})|$ must be odd, which can only be if $j-2 \equiv 3 \pmod{4}$. Suppose $j-2 \equiv 3 \pmod{4}$, then $|V_{st}(G^{l(m(2))}) + u_3v_{j-1}| \leq |V_{sb}(G^{l(m(2))})| + 1 = 6q + 6$, where $|m(2)| = 4q + 3$, for $q \geq 1$, since $|m(1)| = 12$ and $|n(2)| \equiv 3 \pmod{4}$. Now let $G^{l(m(3))} = G^{l(m(1))} \cup G^{l(m(2))}$, where $|m(3)| = |m(1)| + |m(2)| \equiv 3 \pmod{4}$ and $G^{l(m(4))} \subset G$ be defined as a subgrid of G induced by $\{V_{i+5}, V_{i+6}, \dots, V_m\}$. Clearly, $|m(4)| \equiv 0 \pmod{4}$. Since $|V_{sb}(G^{l(m(4))})| = |V_{st}(G^{l(m(4))})|$, which is even, then $|V_{st}(G^{l(m(4))} \cup u_3v_{i+4})| = |V_{st}(G^{l(m(4))})| = 6p - 6q - 18$. It can be seen that $|V_{st}(G^{l(m(1))}) \setminus \{u_3v_{j-1}, u_3v_{l+4}\}| = 14$. Therefore, $|V_{st}(G)| \leq 6p + 2$ instead of $6p + 4$, and hence a contradiction.

Case 2: $j + 3 < k$ and $k + 3 < l$.

As in Case 1 and without loss of generality, let $j-2 \equiv 3 \pmod{4}$ and let $G^{l(m(2))}$ still be induced by $\{V_1, V_2, \dots, V_{j-2}\}$. Also, let $G^{l(m(4))}$ be induced by $\{V_{l+5}, V_{l+6}, \dots, V_m\}$, and set $|m(4)| \equiv 3 \pmod{4}$. Thus, u_3v_{j-1} and u_3v_{i+4} are both relevant for M to be a maximum induced matching of G , $|V_{st}(G^{l(m(2))} \cup V_{j-1})| = |V_{sb}(G^{l(m(2))})| + 1$ and $|V_{st}(G^{l(m(4))} \cup V_{l+4})| = |V_{sb}(G^{l(m(4))})| + 1$. Set $G^{l(m(2))} \cup V_{j-1} = G^{l(m(2^+))}$ and $G^{l(m(4))} \cup V_{l+4} = G^{l(m(4^+)})$ also let $\{V_j, V_{j+1}, V_{j+2}, V_{j+3}\}$ and $\{V_i, V_{i+1}, V_{i+2}, V_{i+3}\}$ induce $G^{l(m(5))}$ and $G^{l(m(6))}$, respectively. Furthermore, let $G^{l(m(5^+))} = G^{l(m(5))} \cup V_{j+4}$ and $G^{l(m(6^+))}$ contain, say, h columns of V_i in all, where $h \equiv 2 \pmod{4}$. Therefore, for $G^{l(m(7))} = G \setminus \{G^{l(m(2^+))} \cup G^{l(m(4^+))} \cup G^{l(m(5^+))} \cup G^{l(m(6^+))}\}$, $|m(7)| = m - h = b \equiv 1 \pmod{4}$. Let $b = 4a + 1$, for some positive integer a and let $G^{l(m(4))} \subset G^{l(m(7))}$, where $G^{l(m(7))}$ is induced by $\{V_k, V_{k+1}, V_{k+2}, V_{k+3}\}$. Certainly, $u_3v_{k-1}, u_3v_{k+4}, u_3v_{j+4}, u_3v_{l-1} \in V_{sb}(G)$. Now, let $G^{l(4)} \subset G^{l(m(4))}$ be induced by $\{V_k, V_{k+1}, V_{k+2}, V_{k+3}\}$ and $G^{l(4^+)} \subset G^{l(m(4))}$ be induced by $G^{l(4)} \cup \{V_{k-1}, V_{k+4}\}$, with $|4^+| = 6$. So, $b - 6 \equiv 3 \pmod{4}$, which is odd and thus can only be the sum of an even and an odd positive integer. Therefore, let $G^{l(m(8))}$ and $G^{l(m(9))}$ be induced by $\{V_{j+5}, V_{j+6}, \dots, V_{k-2}\}$ and $\{V_{j+5}, V_{j+6}, \dots, V_{i-2}\}$, respectively, with $|m(8)| + |m(9)| = b$. Suppose thus, that $|m(8)| \equiv 0 \pmod{4}$, then, $|m(9)| \equiv 3 \pmod{4}$ and suppose $|m(8)| \equiv 1 \pmod{4}$, then $|m(9)| \equiv 2 \pmod{4}$. For $|m(8)| \equiv 0 \pmod{4}$, let

$G^{|m(10)|} = G^{|m(2^+)|+|m(5^+)|}$ be $G^{|m(2^+)|} \cup G^{|m(5^+)|}$ and $G^{|m(11)|} = G^{|m(6^+)|+|m(4^+)|}$ be $G^{|m(6^+)|} \cup G^{|m(4^+)|}$, where $|m(2^+)| + |m(5^+)| = 4q + 9$ and $|m(4^+)| + |m(6^+)| = 4r + 9$, where $|m(4)| = 4r + 3$. Therefore, as defined, $b = |m(7)| = 4p - 4q - 4r - 15$ and thus $b - 6 = 4(p - q - r - 6) + 3$. Set $p - q - r - 6 = f$. Now, for $|m(8)|$ and $|m(9)|$, if $|m(8)| = 4g$, for some positive integer g , then $|m(9)| = 4(f - g) + 3$. The maximal values of the subgrid of G is: $|V_{st}(G)| \leq |V_{st}(G^{|m(2^+)|} \cup G^{|m(5^+)|})| + |V_{st}(G^{|m(8)|} + \{u_3v_{j+4}, u_3v_{k-1}\})| + |V_{st}(G^{|m(4^+)|})| + |V_{st}(G^{|m(9)|} + \{u_3v_{k+4}, u_3v_{l-1}\})| + |V_{st}(G^{|m(6^+)|} \cup G^{|m(4^+)|})| \leq 6p + 2$, which is less than $6p + 4$ and hence a contradiction. For $|m(8)| \equiv 1 \pmod{4}$, and $|m(9)| \equiv 2 \pmod{4}$, we have $|m(8)| = 4g + 1$ and hence $|m(9)| = 4(f - g) + 2$ and $|V_{st}(G^{|m(9)|} \cup \{u_3v_{k+4}, u_3v_{l-1}\})| = 6(f - g) + 4$ and thus, $|V_{st}(G)| \leq 6p + 2$.

Case 3: $j + 3 = k$ or $k + 3 = i$.

Suppose as in Case 2, $j - 2 \equiv 3 \pmod{4}$ and $m - (i + 4) \equiv 3 \pmod{4}$. Let $G^{|n(1)|} \subset G$, a $G_{3,9}$ subgrid of G be induced by $\{V_{j-1}, v_j, \dots, V_{j+7}\}$. Then for $G^{|n(2)|} = G^{|m(2^+)|} \cup G^{|n(1)|}$, $|n(2)| = |m(2)| + |n(1)|$, $|n(2)| \equiv 0 \pmod{4}$. Likewise, suppose $\{V_{i-1}, V_i, \dots, V_m\}$ induces $G^{|n(3)|}$, for which $|n(3)| \equiv 1 \pmod{4}$. If $|n(2)|$ and $|n(3)|$ are $4q$ and $4r + 1$ respectively, then $|n(4)| \equiv 2 \pmod{4}$. So far, $G^{|n(4)|}$, is induced by $\{V_{i+8}, V_{i+9}, \dots, V_{l-2}\}$ and by Remark 3.14, $|V_{st}(G^{|n(4)|}) + \{u_3v_{j+7}, u_3v_{l-1}\}| = |V_{sb}(G^{|n(4)|})|$. By a summation similar to the one at the end of Case 2, $|V_{st}(G)| \leq |V_{st}G^{|n(2)|}| + |V_{st}(G^{|n(4)|})| + |V_{st}(G^{|n(3)|})| \leq 6p + 2$. \square

Remark 3.16.

- (a) By following the technique employed in Lemma 3.15, it can be established that given $u_{(1,2)}v_i, u_{(1,2)}v_{i+2} \in M$ and $u_{(1,2)}v_j, u_{(1,2)}v_{j+2} \in M$ of G , a $G_{3,m}$ grid, $m \equiv 3 \pmod{4}$, $i + 2 \leq j$, then M is not a maximum induced matching of G .
- (b) Let M be an induced matching of G , a $G_{3,m}$ grid, and i be some fixed positive integer. Suppose $u_{(1,2)}v, i + 8(n) \in M$, for all non-negative integer n for which $1 \leq i + 8(n) \leq m$. Let M be the maximum induced matching of G . Then,
 - (i) if $i > 1$, then $i - 1$ is either 2, 3, 4 or 6;
 - (ii) if $i + 8(n) < m$, for the maximum value of n , then $m - (i + 8(n))$ is either 2, 3, 4 or 6.

Based on the results so far, we note that if M is the maximum induced matching of G , a $G_{3,m}$ grid, $m \equiv 3 \pmod{4}$, $m \geq 11$, the maximum number of edges of the type $u_{(1,2)}v_k$ that is contained in M , k , a positive integer, is $k + 2$ when $m = 8k + 3$ and $k + 3$ when $m = 8k + 7$.

It can be easily established that for H that is a $G_{k,m}$ grid, with $k \equiv 0 \pmod{4}$ and $m \equiv 3 \pmod{4}$, which is induced by $\{U_1, U_2, \dots, U_k\}$, if M_1 is a maximum induced matching of H , then, the least saturated vertices in U_k is $\frac{m-1}{2}$. The next result describes the positions of the members of M_1 in $E(H)$ if U_k contains $\frac{m-1}{2}$ saturated vertices.

Lemma 3.17. *Let H be a $G_{k,m}$ grid with $k \equiv 0 \pmod{4}$ and $m \equiv 3 \pmod{4}$ and let U_k contain the least possible, $\frac{m-1}{2}$, saturated vertices for which N remains maximum induced matching of H . Then, for any adjacent vertices $v', v'' \in U_k$, edge $v'v'' \notin M$.*

Proof. Induced by $\{U_1, U_2, \dots, U_{k-2}\}$ and $\{U_{k-1}, U_k\}$ respectively, let $G_1^{[m]}$ and $G_2^{[m]}$ be partitions of H with $k-2 \equiv 2 \pmod{4}$. It can be seen that $|V_{st}(G_1^{[m]})| = |V_{sb}(G_1^{[m]})| = \frac{km-2m+2}{2}$. Since $|V_{st}(H)| = \frac{km}{2}$, then $|V_{st}(G_2^{[m]})| \leq m-1$. Now, let $G_3^{[m]}$ be a $G_{1,m}$ subgrid (a P_m path) of H , induced by U_k . By the hypothesis, U_k contains maximum of $\frac{m-1}{2}$ saturated vertices. Now, let $u_k v_i, u_k v_{i+1}$ be adjacent and saturated vertices of $G_3^{[m]}$. Then there are $\frac{m-5}{2}$ other saturated vertices on $G_3^{[m]}$. Without loss of generality, suppose that each of the remaining $\frac{m-5}{2}$ saturated vertices in $G_3^{[m]}$ is adjacent to some saturated vertex in U_{k-1} . Now, suppose $u_{k-1} v_j$ is a saturable vertex in U_{k-1} and that $v \in V(H)$, such that $u_{k-1} v_j v \in M_1$. Now, $v \notin U_k$, since all the saturable vertices in U_k is saturated. Likewise, suppose $v \in U_{k-1}$ and then $u_{k-1} v_j v \in E(G_4^{[m]})$, where $G_4^{[m]}$ is a $G_{1,m}$ subgraph of H induced by U_{k-1} . Then, clearly, at least one of $u_{k-1} v_j$ and v is adjacent to a saturated vertex in $V_{st}(G_1^{[m]})$. Also, suppose that $v \in U_{k-2}$, since $|V_{sb}(G_1^{[m]})| = |V_{st}(G_1^{[m]})|$, then $|V_{st}(G_1^{[m]})| = |V_{st}(G_1^{[m]} + u_{k-1} u_j)|$. Hence $v \in FSV$ in $G_1^{[m]}$. Therefore, $|V_{st}H| \leq |V_{st}G_1^{[m]}| + |V_{st}G_2^{[m]}| \leq \frac{km-4}{2}$, which is a contradiction since $|V_{st}(H)| = \frac{km}{2}$, by [9]. \square

Remark 3.18. The implication of Lemma 3.17 is that for a grid $H' \subset H$, which is induced by $\{U_1, U_2, \dots, U_{k-2}\} \subset V(H)$, $k-2 \equiv 2 \pmod{4}$, suppose U_k contains the least possible number of saturated vertices, $\frac{m-1}{2}$, then $u_k v_2, u_k v_4, \dots, u_k v_{m-1}$ are saturated as shown in the example in Figure 3, for which $k = 4$ and $m = 7$.

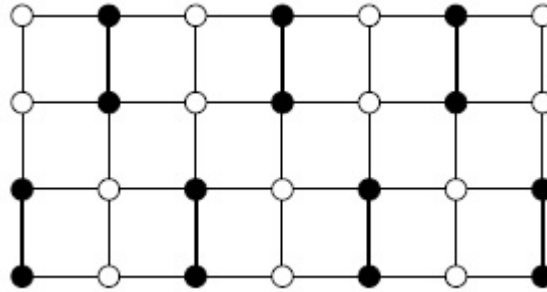


Figure 3: A $G_{4,7}$ Grid with $Max(G) = 7$

Lemma 3.19. Let G be a $G_{3,m}$ with an induced matching M and $G^{(9)}$, induced by $\{V_i, V_{i+2}, \dots, V_{i+8}\}$ be a $G_{3,9}$ subgrid of G . Suppose that $M' \subset M$ is an induced matching of $G^{(9)}$ such that $u_{(1,2)} v_i, u_{(1,2)} v_{i+8} \in M'$. No other edge $u_{(1,2)} v_{i+t}, 1 < t < i+7$ is contained in M' . Then for $G'^{(9)} \subset G^{(9)}$, defined as $G^{(9)} \setminus U_1$, $|V_{sb}(G'^{(9)})| \leq 8$.

Proof. Let $G^{(7)} = G^{(9)} \setminus \{\{u_1 v_{i+1}, u_i v_{i+2}, \dots, u_1 v_{i+7}\}, V_i, V_{i+8}\}$. It can be seen that $G^{(7)}$ is a $G_{2,7}$ subgrid of $G^{(9)}$. Clearly also, $G^{(7)} \subset G'^{(9)}$. Since

$u_{(1,2)}v_i, u_{(1,2)}v_{i+8} \in M'$, then, u_2v_{i+1} and u_2v_{i+7} can not be saturated. Let G_y be subgraph of $G^{(7)}$, defined as $G^{(7)} \setminus \{u_2v_{i+1}, u_2v_{i+7}\}$. Now, $|V(G_y)| = 12$ and $|V_{sb}(G_y)|$ can be seen to be at most 6. Thus $|V_{sb}(G^{(9)})| = |V_{sb}(G_y)| + 2 = 8$, since u_2v_i and u_2v_{i+8} are saturated in M' . \square

Remark 3.20. For $U_1 \subset G^{(9)}$ as defined in Lemma 3.19, U_1 contains at least 6 saturated vertices, implying that M' contains two edges whose four vertices are from U_1 .

Corollary 3.21. Let G be a $G_{3,m}$ grid with $m \geq 11$ and $m \equiv 3 \pmod 4$. If M' is a maximum induced matching of G . Then M' contains at least $2k'$ edges from U_1 , where $m = 8k' + 3$ or $m = 8k' + 7$.

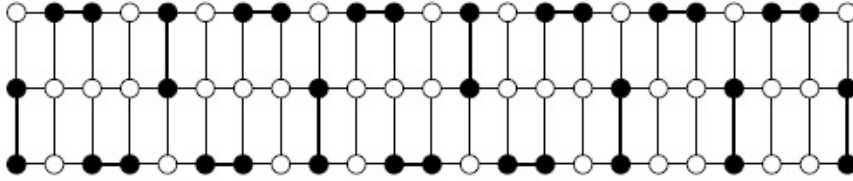


Figure 4: A $G \equiv G_{3,23}$ Grid with $Max(G) = 17$

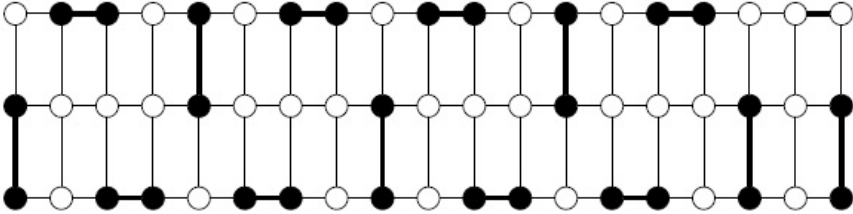


Figure 5: A $G \equiv G_{3,19}$ Grid with $Max(G) = 14$

Theorem 3.22. Let G be a $G_{n,m}$ grid, with $m \geq 23$. Then for $n \equiv 1 \pmod 4$, $Max(G) \leq \lfloor \frac{2mn-m-3}{8} \rfloor$.

Proof. For $n \equiv 1 \pmod 4$, $n - 5 \equiv 0 \pmod 4$. Let G_1 and G_2 be partitions of G induced by $\{U_1, U_2, \dots, U_{n-5}\}$ and $\{U_{n-4}, U_{n-3}, U_{n-2}, U_{n-1}, U_n\}$ respectively. Also, let M', M'' be maximum induced matching of G_1 and G_2 respectively.

Suppose, U_{n-5} contains at least $\frac{m-1}{2}$ saturated vertices, the least U_{n-5} can contain for M' to remain maximum induced matching of G_1 . By Theorem 3.12, $U_1 \subset G_2$ (the U_{n-4} of G) contains at least $2k + 2$ saturated vertices with $k = \frac{m-3}{4}$. Following the proof of Theorem 3.12, it is shown that M'' contains $\frac{m-3}{4}$ edges of $U_1 \subset G_2$ and either

of $u_{(1,2)}v_4$ and $u_{(1,2)}v_{m-3}$. Now, with $G = G' \cup G''$ and hence, $|M| \leq |M'| + |M''|$, it is obvious therefore, that for each edge $u_\alpha u_\beta \in U_{n-4}$ contained in M'' , either u_α or u_β is adjacent to a saturated vertex on U_{n-5} and also, $u_{n-4}v_4$ (or $u_{n-4}v_{m-3}$) is adjacent to saturated $u_{n-5}v_4$ (or to saturated $u_{n-4}v_{m-3}$). Hence, $|V_{st}(G)| \leq \frac{2mn-m-7}{4}$ and thus, $Max(G) \leq \lfloor \frac{2mn-m-7}{8} \rfloor$. \square

Theorem 3.23. *Let G be a $G_{n,m}$ grid with $n \equiv 3 \pmod{4}$ and $m \equiv 3 \pmod{4}$, $m \geq 11$. Then $Max(G) \leq \lfloor \frac{2mn-m+1}{8} \rfloor$ and $Max(G) \leq \lfloor \frac{2mn-m+5}{8} \rfloor$ for $m = 8k' + 3$ and $m = 8k' + 7$ respectively.*

Proof. The proof follows similar techniques as in Theorem 3.22. \square

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DOI: 10.7862/rf.2018.1

Tayo Charles Adefokun

email: tayoadefokun@crawforduniversity.edu.ng

ORCID: 0000-0001-9236-6664

Department of Computer and Mathematical Sciences

Crawford University

NIGERIA

Deborah Olayide Ajayi

email: olayide.ajayi@ui.edu.ng

adelaideajayi@yahoo.com

ORCID: 0000-0002-3306-9851

Department of Mathematics

University of Ibadan

Ibadan

NIGERIA

Received 09.11.2017

Accepted 12.08.2018

On the Existence of Solutions of a Perturbed Functional Integral Equation in the Space of Lebesgue Integrable Functions on \mathbb{R}_+

Waad Al Sayed and Mohamed Abdalla Darwish

ABSTRACT: In this paper, we investigate and study the existence of solutions for perturbed functional integral equations of convolution type using Darbo's fixed point theorem, which is associated with the measure of noncompactness in the space of Lebesgue integrable functions on \mathbb{R}_+ . Finally, we offer an example to demonstrate that our abstract result is applicable.

AMS Subject Classification: 45G10, 45M99, 47H09.

Keywords and Phrases: Existence; Convolution; The space of Lebesgue integrable functions; Measure of noncompactness.

1. Introduction

It is well known that functional integral equations of different types find numerous applications in modeling real world problems which appear in physics, engineering, biology, etc, see for example [1, 3, 6, 13, 14, 16, 17, 20]. Apart from that, integral equations are often investigated in monographs and research papers (cf. [5, 11, 15, 17, 23, 24]) and the references cited therein.

In [5], the authors discussed the solvability of the Urysohn integral equation

$$x(t) = f(t) + \int_0^{\infty} u(t, s, x(s)) ds,$$

while the authors in [3] studied the existence of integrable solutions of the following integral equation

$$x(t) = f_1 \left(t, \int_0^t k(t, s) f_2(s, x(s)) ds \right).$$

In [2], the authors studied the solvability of the functional integral equation

$$x(t) = f \left(t, x(\alpha(t)), \int_0^{\beta(t)} g(t, s, x(\gamma(s))) ds \right), t \geq 0$$

in the space $BC(\mathbb{R}_+)$ (the space of all continuous and bounded functions on \mathbb{R}_+). The authors in [4] studied the nonlinear integral equation

$$x(t) = p(t) + \int_0^t v(t, s, x(s)) ds, t \geq 0$$

by using a combination of the technique of weak noncompactness and the classical Schauder fixed point principle. Also, Banaś and Knap [7] discussed the solvability of the equations considered in the space of Lebesgue integrable functions using the technique of measures of weak noncompactness and the fixed point theorem due to Emmanuel [19].

In addition in [22], the authors study the functional integral equation of convolution type

$$x(t) = f(t, x(t)) + \int_0^\infty k(t-s)Q(s)ds$$

using a new construction of a measure of noncompactness in $L^p(\mathbb{R}_+)$.

Motivated by the work [22], in this paper, we will study the existence of solutions to the following more general functional integral equation

$$x(t) = f_1(t, x(t)) + f_2 \left(t, \int_0^\infty k(t-s)Q(x)(s)ds \right), t \in \mathbb{R}_+. \quad (1.1)$$

Throughout $f_1, f_2 : \mathbb{R}_+ \times \mathbb{R} \rightarrow \mathbb{R}$, $k \in L^1(\mathbb{R})$ and Q is an operator which acts continuously from the space $L^p(\mathbb{R}_+)$ onto itself.

2. Notation and Auxiliary Facts

We will collect in this section some definitions and basic results which will be used further on throughout the paper.

First, we denote by $L^p(\mathbb{R}_+)$ the space of Lebesgue integrable functions on \mathbb{R}_+ equipped with the standard norm $\|x\|_p^p = \int_0^\infty |x(t)|^p dt$.

Theorem 2.1. ([10, 21]) *Let \mathcal{F} be a bounded set in $L^p(\mathbb{R}^N)$ with $1 \leq p < \infty$. Then, \mathcal{F} has a compact closure in $L^p(\mathbb{R}^N)$ if and only if $\lim_{h \rightarrow 0} \|\tau_h f - f\|_p = 0$ uniformly in $f \in \mathcal{F}$, where $\tau_h f(x) = f(x+h)$ for all $x \in \mathbb{R}^N$.*

In addition, for $\epsilon > 0$, there is a bounded and measurable subset Ω of \mathbb{R}^N such that $\|f\|_{L^p(\mathbb{R}^N \setminus \Omega)} < \epsilon$ for all $f \in \mathcal{F}$.

Corollary 2.2. Let \mathcal{F} be a bounded set in $L^p(\mathbb{R}_+)$ with $1 \leq p < \infty$. The closure of \mathcal{F} in $L^p(\mathbb{R}_+)$ is compact if and only if $\lim_{h \rightarrow 0} \left(\int_0^\infty |f(x) - f(x+h)|^p dx \right)^{\frac{1}{p}} = 0$ uniformly in $f \in \mathcal{F}$.

In addition, for $\epsilon > 0$, there is a constant $T > 0$ such that $\left(\int_T^\infty |f(x)|^p dx \right)^{\frac{1}{p}} < \epsilon$, for all $f \in \mathcal{F}$.

Next, we recall some basic facts concerning measures of noncompactness, [8, 9]. Let us assume that E is Banach space with norm $\|\cdot\|$ and zero element θ . Denote by \mathfrak{M}_E the family of all nonempty and bounded subsets of E and by \mathfrak{N}_E its subfamilies consisting of all relatively compact sets. For a subset X of \mathbb{R} , the symbol \bar{X} stands for the closure of X and the symbol $\text{co}X$ denotes the convex closed hull of X . By $B(x, r)$, we mean the ball centered at x and of radius r .

Definition 2.3. A mapping $\mu : \mathfrak{M}_E \rightarrow \mathbb{R}_+$ is said to be a measure of noncompactness in E if it satisfies the following conditions:

- 1) The family of kernel of μ defined by $\ker \mu = \{X \in \mathfrak{M}_E : \mu(X) = 0\}$ is nonempty and $\ker \mu \subset \mathfrak{N}_E$.
- 2) $X \subset Y \Rightarrow \mu(X) \leq \mu(Y)$.
- 3) $\mu(\bar{X}) = \mu(\text{co}X) = \mu(X)$.
- 4) $\mu(\lambda X + (1 - \lambda)Y) \leq \lambda \mu(X) + (1 - \lambda) \mu(Y)$ for $0 \leq \lambda \leq 1$.
- 5) If $X_n \in \mathfrak{M}_E$, $X_n = \bar{X}_n$, $X_{n+1} \subset X_n$ for $n = 1, 2, 3, \dots$ and if $\lim_{n \rightarrow \infty} \mu(X_n) = 0$ then $\bigcap_{n=1}^\infty X_n \neq \emptyset$.

In the following, we fix $\emptyset \neq X \subset L^p(\mathbb{R}_+)$ bounded, $\epsilon > 0$, and $T > 0$. For arbitrary function $x \in X$, we let

$$w(x, \epsilon) = \sup \left\{ \left(\int_0^\infty |x(t+h) - x(t)|^p dt \right)^{\frac{1}{p}} : |h| < \epsilon \right\},$$

$$w(X, \epsilon) = \sup \{w(x, \epsilon) : x \in X\}$$

and

$$w_0(X) = \lim_{\epsilon \rightarrow 0} w(X, \epsilon).$$

Also, let

$$d_T(X) = \sup \left\{ \left(\int_T^\infty |x(s)|^p ds \right)^{\frac{1}{p}} : x \in X \right\}$$

and

$$d(X) = \lim_{T \rightarrow \infty} d_T(X).$$

Then, the function $\mu : \mathcal{M}_{L^p(\mathbb{R}_+)} \rightarrow \mathbb{R}$ given by $\mu(X) = w_0(X) + d(X)$ is a measure of noncompactness on $L^p(\mathbb{R}_+)$, [22].

In the end of this section, we state Darbo's fixed point theorem which play an important role in carrying out the proof of our main result.

Theorem 2.4. [12] *Let Ω be a nonempty, bounded, closed, and convex subset of a Banach space E , and let $F : \Omega \rightarrow \Omega$ be a continuous mapping such that a constant $k \in [0, 1)$ exists with the property*

$$\mu(FX) \leq k\mu(X),$$

for any nonempty subset X of Ω . Then F has a fixed point in the set Ω .

3. Main Results

In this section, we study the existence of solutions to Eq.(1.1) in the space $L^p(\mathbb{R}_+)$.

We consider equation (1.1) under the following assumptions:

(a₀) $f_i(\cdot, 0) \in L^p(\mathbb{R}_+)$, $i = 1, 2$.

(a₁) The functions $f_i : \mathbb{R}_+ \times \mathbb{R} \rightarrow \mathbb{R}$, $i = 1, 2$, satisfy Carathéodory conditions and there exist constant $\lambda_i \in [0, 1)$ and $a_i \in L^p(\mathbb{R}_+)$ such that

$$|f_i(t, x) - f_i(s, y)| \leq |a_i(t) - a_i(s)| + \lambda_i(|x - y|)$$

for almost all $t, s \in \mathbb{R}_+$ and $x, y \in \mathbb{R}$.

(a₂) $k \in L^1(\mathbb{R})$.

Notice that, under this hypothesis, the linear operator $K : L^p(\mathbb{R}_+) \rightarrow L^p(\mathbb{R}_+)$ is given by $(Kx)(t) = \int_0^\infty k(t-s)x(s) ds$ and it is a continuous operator and $\|Kx\|_p \leq \|k\|_{L^1(\mathbb{R})} \|x\|_p$.

(a₃) The operator Q maps continuously the space $L^p(\mathbb{R}_+)$ onto itself and there exists a constant $b \in \mathbb{R}_+$ such that $\lambda_1 + \lambda_2 b \|k\|_{L^p(\mathbb{R})} < 1$ and $\|Qx\|_{L^p[T, \infty)} \leq b \|x\|_{L^p[T, \infty)}$ for any $x \in L^p(\mathbb{R}_+)$ and $T \in \mathbb{R}_+$.

Now, we are in a position to present our main result.

Theorem 3.1. *Under the assumptions (a₀) – (a₃), Eq.(1.1) has at least one solution $x \in L^p(\mathbb{R}_+)$.*

Proof: First of all, we define the operator $F : L^p(\mathbb{R}_+) \rightarrow L^p(\mathbb{R}_+)$, by

$$F(x)(t) = f_1(t, x(t)) + f_2 \left(t, \int_0^\infty k(t-s)Q(x)(s)ds \right). \quad (3.1)$$

It is clear that Fx is measurable for any $x \in L^p(\mathbb{R}_+)$, thanks to Carathéodory conditions. Next, claim that $Fx \in L^p(\mathbb{R}_+)$ for any $x \in L^p(\mathbb{R}_+)$. To establish this claim, we use the assumptions $(a_0) - (a_3)$, for a.e. $t \in \mathbb{R}_+$, then, we have

$$\begin{aligned} |F(x)(t)| &\leq |f_1(t, x) - f_1(t, 0)| + |f_1(t, 0)| \\ &\quad + \left| f_2 \left(t, \int_0^\infty k(t-s)Q(x)(s)ds \right) - f_2(t, 0) \right| + |f_2(t, 0)| \\ &\leq \lambda_1 \|x\|_p + \|f_1(\cdot, 0)\|_p + \|f_2(\cdot, 0)\|_p + \lambda_2 b \|k\|_{L^1(\mathbb{R})} \|x\|_p, \end{aligned}$$

where we have used Young's inequality. Therefore, we obtain

$$\|F(x)\|_p \leq \lambda_1 \|x\|_p + \|f_1(\cdot, 0)\|_p + \|f_2(\cdot, 0)\|_p + \lambda_2 b \|k\|_{L^1(\mathbb{R})} \|x\|_p. \quad (3.2)$$

Hence, $F(x) \in L^p(\mathbb{R}_+)$ and F is well defined. Moreover, from (3.2), we have $F(\overline{B}_{r_0}) \subset \overline{B}_{r_0}$, where $r_0 = \frac{\|f_1(\cdot, 0)\|_p + \|f_2(\cdot, 0)\|_p}{1 - \lambda_1 - \lambda_2 b \|k\|_{L^1(\mathbb{R})}}$. Also, F is continuous in $L^p(\mathbb{R}_+)$ because $f_1(t, \cdot)$, $f_2(t, \cdot)$, K and Q are continuous for a.e. $t \in \mathbb{R}_+$.

Further, we will show that $w_0(FX) \leq (\lambda_1 + \lambda_2 b \|k\|_{L^1(\mathbb{R})}) w_0(X)$ for any set $\emptyset \neq X \subset \overline{B}_{r_0}$. For, we fix an arbitrary $\epsilon > 0$ and we choose $x \in X$ and $t, h \in \mathbb{R}_+$ with $|h| \leq \epsilon$. Then, we have

$$\begin{aligned} &|(F(x)(t) - (F(x))(t+h))| \\ &\leq |f_1(t, x(t)) - f_1(t+h, x(t))| + |f_1(t+h, x(t)) - f_1(t+h, x(t+h))| \\ &\quad + \left| f_2 \left(t, \int_0^\infty k(t-s)Q(x)(s)ds \right) - f_2 \left(t+h, \int_0^\infty k(t-s)Q(x)(s)ds \right) \right| \\ &\quad + \left| f_2 \left(t+h, \int_0^\infty k(t-s)Q(x)(s)ds \right) - f_2 \left(t+h, \int_0^\infty k(t+h-s)Q(x)(s)ds \right) \right| \\ &\leq |a_1(t) - a_1(t+h)| + \lambda_1 |x(t) - x(t+h)| + |a_2(t) - a_2(t+h)| \\ &\quad + \lambda_2 \left| \int_0^\infty (k(t-s) - k(t+h-s))Q(x)(s)ds \right|. \end{aligned}$$

Therefore,

$$\begin{aligned} &\left(\int_0^\infty |(F(x)(t) - (F(x))(t+h))^p dt \right)^{\frac{1}{p}} \\ &\leq \left(\int_0^\infty |a_1(t) - a_1(t+h)|^p dt \right)^{\frac{1}{p}} + \lambda_1 \left(\int_0^\infty |x(t) - x(t+h)|^p dt \right)^{\frac{1}{p}} \\ &\quad + \left(\int_0^\infty |a_2(t) - a_2(t+h)|^p dt \right)^{\frac{1}{p}} \end{aligned}$$

$$\begin{aligned}
& +\lambda_2 \left(\int_0^\infty \left| \int_0^\infty (k(t-s) - k(t+h-s))Q(x)(s)ds \right|^p dt \right)^{\frac{1}{p}} \\
& \leq \left(\int_0^\infty |a_1(t) - a_1(t+h)|^p dt \right)^{\frac{1}{p}} + \lambda_1 \left(\int_0^\infty |x(t) - x(t+h)|^p dt \right)^{\frac{1}{p}} \\
& \quad + \left(\int_0^\infty |a_2(t) - a_2(t+h)|^p dt \right)^{\frac{1}{p}} + \lambda_2 \|Qx\|_p \int_{\mathbb{R}} |k(t) - k(t+h)| dt \\
& \leq w(a_1, \epsilon) + \lambda_1 w(x, \epsilon) + w(a_2, \epsilon) + \lambda_2 \|Qx\|_p \|k - \tau_h k\|_{L^1(\mathbb{R})}.
\end{aligned}$$

From the above inequalities, we get

$$w(FX, \epsilon) \leq w(a_1, \epsilon) + \lambda_1 w(X, \epsilon) + w(a_2, \epsilon) + \lambda_2 b r_0 \|k - \tau_h k\|_{L^1(\mathbb{R})}.$$

Since $\{a_1\}$ and $\{a_2\}$ are compact sets in $L^p(\mathbb{R}_+)$ and $\{k\}$ is a compact set in $L^1(\mathbb{R})$, we have $w(a_1, \epsilon) \rightarrow 0$, $w(a_2, \epsilon) \rightarrow 0$ and $\|k - \tau_h k\|_{L^1(\mathbb{R})} \rightarrow 0$ as $\epsilon \rightarrow 0$. Then, we get

$$w_0(FX) \leq \lambda_1 w_0(X) \leq (\lambda_1 + \lambda_2 b \|k\|_{L^1(\mathbb{R})}) w_0(X). \quad (3.3)$$

In the following, we fix an arbitrary number $T > 0$. Then, for an arbitrary function $x \in X$, we obtain

$$\begin{aligned}
& \left(\int_T^\infty |F(x)(t)|^p dt \right)^{\frac{1}{p}} \\
& \leq \left(\int_T^\infty |f_1(t, x) - f_1(t, 0)|^p dt \right)^{\frac{1}{p}} + \left(\int_T^\infty |f_1(t, 0)|^p dt \right)^{\frac{1}{p}} \\
& \quad + \left(\int_T^\infty |f_2(t, \int_0^\infty k(t-s)Q(x)(s)ds) - f_2(t, 0)|^p dt \right)^{\frac{1}{p}} + \left(\int_T^\infty |f_2(t, 0)|^p dt \right)^{\frac{1}{p}} \\
& \leq \lambda_1 \left(\int_T^\infty |x(t)|^p dt \right)^{\frac{1}{p}} + \left(\int_T^\infty |f_1(t, 0)|^p dt \right)^{\frac{1}{p}} \\
& \quad + \lambda_2 b \|k\|_{L^1(\mathbb{R})} \left(\int_T^\infty |x(t)|^p dt \right)^{\frac{1}{p}} + \left(\int_T^\infty |f_2(t, 0)|^p dt \right)^{\frac{1}{p}}.
\end{aligned}$$

Since $\{f_1(t, 0)\}$ and $\{f_2(t, 0)\}$ are compact in $L^p(\mathbb{R}_+)$, then, as $T \rightarrow 0$, we obtain $\left(\int_T^\infty |f_1(t, 0)|^p dt \right)^{\frac{1}{p}} \rightarrow 0$ and $\left(\int_T^\infty |f_2(t, 0)|^p dt \right)^{\frac{1}{p}} \rightarrow 0$. Therefore,

$$d(FX) \leq (\lambda_1 + \lambda_2 b \|k\|_{L^1(\mathbb{R})}) d(X). \quad (3.4)$$

From (3.3) and (3.4), we get

$$\mu(FX) \leq (\lambda_1 + \lambda_2 b \|k\|_{L^1(\mathbb{R})}) \mu(X). \quad (3.5)$$

By (3.5) and Theorem 2.4, we deduce that the operator F has a fixed point x in B_{r_0} and consequently, Eq.(1.1) has at least one solution in $L^p(\mathbb{R}_+)$. \square

4. Example

Consider the functional integral equation

$$x(t) = \frac{t}{t^3 + 1} + \frac{1}{4} \ln(1 + x^2) + \frac{3}{4} \int_0^\infty (t - s)e^{-(t-s)} |x(s)| ds. \quad (4.1)$$

In our example, the functions $f_1(t, x)$ and $f_2(t, x)$ are given by

$$f_1(t, x) = \frac{t}{t^3 + 1} + \frac{1}{4} \ln(1 + x^2)$$

and

$$f_2(t, x) = \frac{3}{4}x.$$

It is clear that for $i = 1, 2$, $f_i : \mathbb{R}_+ \times \mathbb{R} \rightarrow \mathbb{R}$ satisfies assumption (a_0) . In fact we have $a_1(t) = \frac{t}{t^3 + 1}$, $\lambda_1 = \frac{1}{4}$, $a_2(t) = 0$ and $\lambda_2 = \frac{3}{4}$. Indeed by using the Mean Value Theorem, we have

$$|f_1(t, x) - f_1(s, y)| \leq \left| \frac{t}{t^3 + 1} - \frac{s}{s^3 + 1} \right| + \frac{1}{4}|x - y|.$$

Furthermore we have

$$|f_2(t, x) - f_2(s, y)| \leq \frac{3}{4}|x - y|.$$

It is easy to see that assumption (a_1) is satisfied.

In our example, the function $k(t)$ takes the form

$$k(t) = te^{-t}.$$

In fact assumption (a_2) is satisfied and by [3] $\|k\|_{L^1(\mathbb{R}_+)} \leq \frac{1}{\sqrt{e}}$.

In our example, the operator Q is defined by

$$(Qx)(t) = [|x(t)|].$$

Q satisfies assumption (a_3) and we have if $b = 1$

$$\lambda_1 + \lambda_2 b \|k\|_{L^1(\mathbb{R}_+)} \leq \frac{1}{4} + \frac{3}{4\sqrt{e}} \leq 1.$$

Now, by Theorem 3.1, our functional integral equation (4.1) has a solution belonging to $L^1(\mathbb{R}_+)$.

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DOI: 10.7862/rf.2018.2

Waad Al Sayed

email: wsaid@fbsu.edu.sa

ORCID: 0000-0002-1767-9780

College of Sciences and Humanities

Fahad Bin Sultan University

Tabuk

SAUDI ARABIA

Mohamed Abdalla Darwish

email: madarwish@sci.dmu.edu.eg

ORCID: 0000-0002-4245-4364

Department of Mathematics, Faculty of Science

Damanhour University

Damanhour

EGYPT

Received 10.01.2018

Accepted 15.05.2018

On a Cubic Integral Equation of Urysohn Type with Linear Perturbation of Second Kind

*Hamed Kamal Awad, Mohamed Abdalla Darwish and
Mohamed M.A. Metwali*

ABSTRACT: In this paper, we concern by a very general cubic integral equation and we prove that this equation has a solution in $C[0, 1]$. We apply the measure of noncompactness introduced by Banaś and Olszowy and Darbo's fixed point theorem to establish the proof of our main result.

AMS Subject Classification: 45G10, 45M99, 47H09.

Keywords and Phrases: Cubic integral equation; Darbo's fixed point theorem; Monotonicity measure of noncompactness.

1. Introduction

Cubic integral equations have several useful applications in modeling numerous problems and events of the real world (cf. [3, 8, 9, 12, 13, 18, 19]).

In this paper we consider the cubic Urysohn integral equation with linear perturbation of second kind

$$x(\tau) = \phi(\tau) + \varphi(\tau, x(\tau)) + x^2(\tau) \int_0^1 u(\tau, s, (\Lambda x)(s)) ds, \quad \tau \in I = [0, 1]. \quad (1.1)$$

In the above equation, we consider $\phi : I \rightarrow \mathbb{R}$, $\varphi : I \times \mathbb{R} \rightarrow \mathbb{R}$, $u : I \times I \times \mathbb{R} \rightarrow \mathbb{R}$ are given functions and $\Lambda : C(I) \rightarrow C(I)$ is an operator verifies special assumption which will state in Section 3.

Eq.(1.1) is of interest since it contains many includes several integral equations studied earlier as special cases, see [1, 2, 6, 7, 10, 11, 14, 15, 16, 20, 21, 22] and references therein. By using the measure of noncompactness related to monotonicity associated with fixed point theorem due to Darbo, we show that Eq.(1.1) has at least one solution in $C(I)$ which is nondecreasing on the interval I .

2. Auxiliary Facts and Results

In this section, we present some definitions and results which we will use further on.

Let $(E, \|\cdot\|)$ be a real Banach space with zero element 0. Let $B(x, r)$ be the closed ball centered at x with radius r . We denote by B_r the closed ball $B(0, r)$. Next, let X be a subset of E , we denote by \overline{X} and $\text{Conv}X$ the closure and convex closure of X , respectively. We use the symbols λX and $X + Y$ for the usual algebraic operations on the sets. Moreover, the symbol \mathfrak{M}_E stands for the family of all nonempty and bounded subsets of E and the symbol \mathfrak{N}_E stands for its subfamily consisting of all relatively compact subsets.

Now, we state the definition of a measure of noncompactness [4]:

Definition 2.1. A function $\mu : \mathfrak{M}_E \rightarrow \mathbb{R}_+$ is called a measure of noncompactness in E if it verifies the following assumptions:

- (1) The family $\ker\mu \neq \emptyset$ and $\ker\mu \subset \mathfrak{N}_E$, where $\ker\mu = \{X \in \mathfrak{M}_E : \mu(X) = 0\}$.
- (2) $\mu(X) \leq \mu(Y)$, if $X \subset Y$.
- (3) $\mu(\overline{X}) = \mu(X)$ and $\mu(\text{Conv}X) = \mu(X)$.
- (4) $\mu(\lambda X + (1 - \lambda)Y) \leq \lambda\mu(X) + (1 - \lambda)\mu(Y)$, $0 \leq \lambda \leq 1$.
- (5) If $X_n \in \mathfrak{M}_E$, $X_n = \overline{X}_n$, $X_{n+1} \subset X_n$ for $n = 1, 2, 3, \dots$ and $\lim_{n \rightarrow \infty} \mu(X_n) = 0$, then $\bigcap_{n=1}^{\infty} X_n \neq \emptyset$.

Notice that $\ker\mu$ is said to be the kernel of the measure of noncompactness μ .

In the following, we will work in the Banach space $C(I)$ of all real functions defined and continuous on $I = [0, 1]$ equipped with the standard norm $\|x\| = \max\{|x(\tau)| : \tau \in I\}$. We recall the measure of noncompactness in $C(I)$ which we will need in the next section (see [5]).

Let $\emptyset \neq X \subset C(I)$. For $x \in X$ and $\varepsilon \geq 0$ we denote by $\omega(x, \varepsilon)$ the modulus of continuity of the function x as follows

$$\omega(x, \varepsilon) = \sup\{|x(\tau) - x(t)| : \tau, t \in I, |\tau - t| \leq \varepsilon\}.$$

Next, we put $\omega(X, \varepsilon) = \sup\{\omega(x, \varepsilon) : x \in X\}$ and $\omega_0(X) = \lim_{\varepsilon \rightarrow 0} \omega(X, \varepsilon)$. Moreover, we define

$$d(x) = \sup\{|x(\tau) - x(t)| - [x(\tau) - x(t)] : \tau, t \in I, \tau \geq t\}$$

and

$$d(X) = \sup\{d(x) : x \in X\}.$$

Notice that $d(X) = 0$ if and only if all functions belonging to X are nondecreasing on I .

Finally, we define the function μ on the family $\mathfrak{M}_{C(I)}$ as follows

$$\mu(X) = \omega_0(X) + d(X).$$

Notice that the function μ is a measure of noncompactness in $C(I)$ [5].

We present a fixed point theorem due to Darbo [17] which we will need in the proof of our main result. First, we make use of the following definition.

Definition 2.2. Let $\emptyset \neq M$ be a subset of a Banach space E and let $\mathfrak{P} : M \rightarrow E$ be a continuous mapping which maps bounded sets onto bounded sets. The operator \mathfrak{P} satisfies the Darbo condition (with a constant $\kappa \geq 0$) with respect to a measure of noncompactness μ if for any bounded subset X of M we have

$$\mu(\mathfrak{P}X) \leq \kappa\mu(X).$$

If \mathfrak{P} verifies the Darbo condition with $\kappa < 1$ then it is a contraction operator with respect to μ .

Theorem 2.3. Let $\emptyset \neq \Omega$ be a closed, bounded and convex subset of the space E and let $\mathfrak{P} : \Omega \rightarrow \Omega$ be a contraction mapping with respect to the measure of noncompactness μ .

Then \mathfrak{P} has a fixed point in the set Ω .

Notice that the assumptions of the above theorem gives us that the set $\text{Fix}\mathfrak{P}$ of all fixed points of \mathfrak{P} belongs to Ω is an element of $\ker\mu$ [4].

3. The Main Result

We consider Eq.(1.1) and assume that the following assumptions are verified:

(a₁) The function $\phi : I \rightarrow \mathbb{R}$ is continuous, nonnegative and nondecreasing on I .

(a₂) The function $\varphi : I \times \mathbb{R} \rightarrow \mathbb{R}$ is continuous, $\varphi : I \times \mathbb{R}_+ \rightarrow \mathbb{R}_+$ and

$$\exists c \geq 0 : |\varphi(\tau, x_1) - \varphi(\tau, x_2)| \leq c|x_1 - x_2| \quad \forall (x_1, x_2) \in \mathbb{R}^2 \ \& \ \tau \in I.$$

(a₃) The superposition operator Φ generated by the function φ satisfies for any nonnegative function x the condition $d(\Phi x) \leq cd(x)$, where c is the same c appears in assumption (a₂).

(a₄) The function $u : I \times I \times \mathbb{R} \rightarrow \mathbb{R}$ is a continuous, $u : I \times I \times \mathbb{R}_+ \rightarrow \mathbb{R}_+$ and for arbitrary fixed $t \in I$ and $x \in \mathbb{R}$ the function $\tau \rightarrow u(\tau, t, x)$ is nondecreasing on I . Moreover,

$$\exists \Psi : \mathbb{R}_+ \rightarrow \mathbb{R}_+ \text{ (nondecreasing)} : |u(\tau, t, x)| \leq \Psi(|x|) \quad \forall (\tau, t) \in I^2 \ \& \ x \in \mathbb{R}.$$

(a₅) The operator $\Lambda : C(I) \rightarrow C(I)$ is continuous and

$$\exists \psi : \mathbb{R}_+ \rightarrow \mathbb{R}_+ \text{ (nondecreasing)} : |(\Lambda x)(\tau)| \leq \psi(\|x\|) \text{ for any } \tau \in I, \ x \in C(I).$$

Moreover, for every nonnegative function $x \in C(I)$, the function Λx is nonnegative and nondecreasing on I .

(a₆) The inequality

$$\|\phi\| + cr + \varphi^* + r^2\Psi(\psi(r)) \leq r \quad (3.1)$$

has a positive solution r_0 such that $c+2r_0\Psi(\psi(r_0)) < 1$, where $\varphi^* = \max_{0 \leq \tau \leq 1} \varphi(\tau, 0)$.

Under the above assumptions, we state our main result as follows.

Theorem 3.1. *Let the assumptions (a₁) – (a₆) be verified, then the cubic Urysohn integral equation (1.1) has at least one solution $x \in C(I)$ which is nondecreasing on I .*

Proof. Let \mathfrak{F} be an operator defined on $C(I)$ by

$$(\mathfrak{F}x)(\tau) = \phi(\tau) + \varphi(\tau, x(\tau)) + x^2(\tau)(\mathcal{U}x)(t), \quad (3.2)$$

where \mathcal{U} is the Urysohn integral operator

$$(\mathcal{U}x)(\tau) = \int_0^1 u(\tau, t, (\Lambda x)(t)) dt. \quad (3.3)$$

For better readability, we will write the proof in seven steps.

Step 1: \mathfrak{F} maps the space $C(I)$ into itself.

Notice that for a given $x \in C(I)$, according to assumptions (a₁) – (a₅), we have $\mathfrak{F}x \in C(I)$. Therefore, the operator \mathfrak{F} maps $C(I)$ into itself.

Step 2: \mathfrak{F} maps the ball B_{r_0} into itself.

For all $\tau \in I$, we have

$$\begin{aligned} |(\mathfrak{F}x)(\tau)| &\leq \left| \phi(\tau) + \varphi(\tau, x(\tau)) + x^2(\tau) \int_0^1 u(\tau, t, (\Lambda x)(t)) dt \right| \\ &\leq |\phi(\tau)| + |\varphi(\tau, x(\tau)) - \varphi(\tau, 0)| + |\varphi(\tau, 0)| \\ &\quad + |x^2(\tau)| \int_0^1 |u(\tau, t, (\Lambda x)(t))| dt \\ &\leq \|\phi\| + c\|x\| + \varphi^* + \|x\|^2\Psi(\psi(\|x\|)) \int_0^1 ds \\ &= \|\phi\| + c\|x\| + \varphi^* + \|x\|^2\Psi(\psi(\|x\|)). \end{aligned}$$

From the above estimate, we get

$$\|\mathfrak{F}x\| \leq \|\phi\| + c\|x\| + \varphi^* + \|x\|^2\Psi(\psi(\|x\|)).$$

Therefore, if we have $\|x\| \leq r_0$, we obtain

$$\|\mathfrak{F}x\| \leq \|\phi\| + cr_0 + \varphi^* + r_0^2\Psi(\psi(r_0)) \leq r_0,$$

in view of the assumption (a₆). Consequently, the operator \mathfrak{F} maps the ball B_{r_0} into itself.

Further, let $B_{r_0}^+$ be the subset of B_{r_0} given by

$$B_{r_0}^+ = \{x \in B_{r_0} : x(\tau) \geq 0, \text{ for } \tau \in I\}.$$

Notice that, the set $\emptyset \neq B_{r_0}^+$ is closed, bounded and convex.

Step 3: \mathfrak{F} maps continuously the ball $B_{r_0}^+$ into itself.

In view of the above facts about $B_{r_0}^+$ and assumptions $(a_1) - (a_4)$, we infer that \mathfrak{F} maps the set $B_{r_0}^+$ into itself.

Step 4: The operator \mathfrak{F} is continuous on $B_{r_0}^+$.

To establish this, let us fix arbitrarily $\varepsilon > 0$ and $y \in B_{r_0}^+$. By assumption (a_4) , we can find $\delta > 0$ such that for arbitrary $x \in B_{r_0}^+$ with $\|x - y\| \leq \delta$ we have that $\|\Lambda x - \Lambda y\| \leq \varepsilon$. Indeed, for each $\tau \in I$ we have

$$\begin{aligned} & |(\mathfrak{F}x)(\tau) - (\mathfrak{F}y)(\tau)| \\ & \leq |\varphi(\tau, x(\tau)) - \varphi(\tau, y(\tau))| \\ & \quad + \left| x^2(\tau) \int_0^1 u(\tau, t, (\Lambda x)(t)) dt - y^2(\tau) \int_0^1 u(\tau, t, (\Lambda y)(t)) dt \right| \\ & \leq c|x(\tau) - y(\tau)| + \left| x^2(\tau) \int_0^1 u(\tau, t, (\Lambda x)(t)) dt - y^2(\tau) \int_0^1 u(\tau, t, (\Lambda x)(t)) dt \right| \\ & \quad + \left| y^2(\tau) \int_0^1 u(\tau, t, (\Lambda x)(t)) dt - y^2(\tau) \int_0^1 u(\tau, t, (\Lambda y)(t)) dt \right| \\ & \leq c|x(\tau) - y(\tau)| + |x^2(\tau) - y^2(\tau)| \int_0^1 |u(\tau, t, (\Lambda x)(t))| dt \\ & \quad + |y^2(\tau)| \int_0^1 |u(\tau, t, (\Lambda x)(t)) - u(\tau, t, (\Lambda y)(t))| dt. \end{aligned}$$

Therefore, we have

$$\|\mathfrak{F}x - \mathfrak{F}y\| \leq c\|x - y\| + 2r_0\Psi(\psi(r_0))\|x - y\| + r_0^2\omega^*(u, \varepsilon), \quad (3.4)$$

where we denoted

$$\omega^*(u, \varepsilon) = \sup\{|u(\tau, t, x) - u(\tau, t, y)| : \tau, t \in I, x, y \in [0, \psi(r_0)], |x - y| \leq \varepsilon\}.$$

From assumption (a_4) we infer that $\omega^*(u, \varepsilon) \rightarrow 0$ as $\varepsilon \rightarrow 0$ and therefore, the operator \mathfrak{F} is continuous in $B_{r_0}^+$.

Step 5: An estimate of \mathfrak{F} with respect to the term related to continuity ω_0 .

Let $\emptyset \neq X \subset B_{r_0}^+$, fix an arbitrarily number $\varepsilon > 0$ and choose $x \in X$ and $\tau_1, \tau_2 \in I$ such that $|\tau_2 - \tau_1| \leq \varepsilon$. Without restriction of the generality, we may assume that $\tau_1 \leq \tau_2$. In the view of our assumptions, we have

$$\begin{aligned}
& |(\mathfrak{F}x)(\tau_2) - (\mathfrak{F}x)(\tau_1)| \\
& \leq |\phi(\tau_2) - \phi(\tau_1)| + |\varphi(\tau_2, x(\tau_2)) - \varphi(\tau_1, x(\tau_1))| \\
& \quad + |x^2(\tau_2) (\mathcal{U}x)(\tau_2) - x^2(\tau_2) (\mathcal{U}x)(\tau_1)| \\
& \quad + |x^2(\tau_2) (\mathcal{U}x)(\tau_1) - x^2(\tau_1) (\mathcal{U}x)(\tau_1)| \\
& \leq \omega(\phi, \varepsilon) + |\varphi(\tau_2, x(\tau_2)) - \varphi(\tau_1, x(\tau_2))| + |\varphi(\tau_1, x(\tau_2)) - \varphi(\tau_1, x(\tau_1))| \\
& \quad + |x^2(\tau_2)| |(\mathcal{U}x)(\tau_2) - (\mathcal{U}x)(\tau_1)| + |x^2(\tau_2) - x^2(\tau_1)| |(\mathcal{U}x)(\tau_1)| \\
& \leq \omega(\phi, \varepsilon) + \gamma_{r_0}(\varphi, \varepsilon) + c \omega(x, \varepsilon) + |x(\tau_2)|^2 |(\mathcal{U}x)(\tau_2) - (\mathcal{U}x)(\tau_1)| \\
& \quad + |x(\tau_2) - x(\tau_1)| |x(\tau_2) + x(\tau_1)| |(\mathcal{U}x)(\tau_1)| \\
& \leq \omega(\phi, \varepsilon) + \gamma_{r_0}(\varphi, \varepsilon) + c \omega(x, \varepsilon) \\
& \quad + \|x\|^2 \int_0^1 |u(\tau_2, t, (\Lambda x)(t)) - u(\tau_1, t, (\Lambda x)(t))| dt + 2\|x\| \omega(x, \varepsilon) \Psi(\psi(\|x\|)) \\
& \leq \omega(\phi, \varepsilon) + \gamma_{r_0}(\varphi, \varepsilon) + c \omega(x, \varepsilon) + \|x\|^2 \omega_{\psi(\|x\|)}(u, \varepsilon) + 2\|x\| \omega(x, \varepsilon) \Psi(\psi(\|x\|)),
\end{aligned}$$

where we denoted

$$\gamma_{r_0}(\varphi, \varepsilon) = \sup \{ |\varphi(\tau_2, x) - \varphi(\tau_1, x)| : \tau_1, \tau_2 \in I, x \in [0, r_0], |\tau_2 - \tau_1| \leq \varepsilon \}$$

and

$$\omega_b(u, \varepsilon) = \sup \{ |u(\tau_2, t, y) - u(\tau_1, t, y)| : t, \tau_1, \tau_2 \in I, y \in [0, b], |\tau_2 - \tau_1| \leq \varepsilon \}.$$

Hence,

$$\omega(\mathfrak{F}x, \varepsilon) \leq \omega(\phi, \varepsilon) + \gamma_{r_0}(\varphi, \varepsilon) + c \omega(x, \varepsilon) + r_0^2 \omega_{\psi(r_0)}(u, \varepsilon) + 2r_0 \omega(x, \varepsilon) \Psi(\psi(r_0)).$$

Consequently,

$$\omega(\mathfrak{F}X, \varepsilon) \leq \omega(\phi, \varepsilon) + \gamma_{r_0}(\varphi, \varepsilon) + (c + 2r_0 \Psi(\psi(r_0))) \omega(X, \varepsilon) + r_0^2 \omega_{\psi(r_0)}(u, \varepsilon).$$

Since the function ϕ is continuous on I , the function φ is uniformly continuous on $I \times [0, r_0]$ and the function u is uniformly continuous the set $I \times I \times [0, \psi(r_0)]$, then we obtain

$$\omega_0(\mathfrak{F}X) \leq (c + 2r_0 \Psi(\psi(r_0))) \omega_0(X). \quad (3.5)$$

Step 6: *An estimate of \mathfrak{F} with respect to the term related to monotonicity d.*

Fix an arbitrary $x \in X$ and $\tau_1, \tau_2 \in I$ with $\tau_2 > \tau_1$. Then, taking into account our assumption, we get

$$\begin{aligned}
& |(\mathfrak{F}x)(\tau_2) - (\mathfrak{F}x)(\tau_1)| - ((\mathfrak{F}x)(\tau_2) - (\mathfrak{F}x)(\tau_1)) \\
& = \left| \phi(\tau_2) + \varphi(\tau_2, x(\tau_2)) + x^2(\tau_2) \int_0^1 u(\tau_2, t, (\Lambda x)(t)) dt \right.
\end{aligned}$$

$$\begin{aligned}
 & \left| -\phi(\tau_1) - \varphi(\tau_1, x(\tau_1)) - x^2(\tau_1) \int_0^1 u(\tau_1, t, (\Lambda x)(t)) dt \right. \\
 & \quad - \left(\phi(\tau_2) + \varphi(\tau_2, x(\tau_2)) + x^2(\tau_2) \int_0^1 u(\tau_2, t, (\Lambda x)(t)) dt \right. \\
 & \quad \left. \left. - \phi(\tau_1) - \varphi(\tau_1, x(\tau_1)) - x^2(\tau_1) \int_0^1 u(\tau_1, t, (\Lambda x)(t)) dt \right) \right. \\
 \leq & \left[|\phi(\tau_2) - \phi(\tau_1)| - (\phi(\tau_2) - \phi(\tau_1)) \right] \\
 & + \left[|\varphi(\tau_2, x(\tau_2)) - \varphi(\tau_1, x(\tau_1))| - (\varphi(\tau_2, x(\tau_2)) - \varphi(\tau_1, x(\tau_1))) \right] \\
 & + \left| x^2(\tau_2) \int_0^1 u(\tau_2, t, (\Lambda x)(t)) dt - x^2(\tau_1) \int_0^1 u(\tau_2, t, (\Lambda x)(t)) dt \right| \\
 & + \left| x^2(\tau_1) \int_0^1 u(\tau_2, t, (\Lambda x)(t)) dt - x^2(\tau_1) \int_0^1 u(\tau_1, t, (\Lambda x)(t)) dt \right| \\
 & - \left(x^2(\tau_2) \int_0^1 u(\tau_2, t, (\Lambda x)(t)) dt - x^2(\tau_1) \int_0^1 u(\tau_2, t, (\Lambda x)(t)) dt \right) \\
 & - \left(x^2(\tau_1) \int_0^1 u(\tau_2, t, (\Lambda x)(t)) dt - x^2(\tau_1) \int_0^1 u(\tau_1, t, (\Lambda x)(t)) dt \right) \\
 \leq & |\varphi(\tau_2, x(\tau_2)) - \varphi(\tau_1, x(\tau_1))| - (\varphi(\tau_2, x(\tau_2)) - \varphi(\tau_1, x(\tau_1))) \\
 & + \left[|x^2(\tau_2) - x^2(\tau_1)| - (x^2(\tau_2) - x^2(\tau_1)) \right] \int_0^1 u(\tau_2, t, (\Lambda x)(t)) dt \\
 & + x^2(\tau_1) \left[\left| \int_0^1 u(\tau_2, t, (\Lambda x)(t)) dt - \int_0^1 u(\tau_1, t, (\Lambda x)(t)) dt \right| \right. \\
 & \quad \left. - \left(\int_0^1 u(\tau_2, t, (\Lambda x)(t)) dt - \int_0^1 u(\tau_1, t, (\Lambda x)(t)) dt \right) \right] \\
 \leq & d(\Phi x) + 2\|x\| \Psi(\psi(\|x\|))d(x).
 \end{aligned}$$

The above estimate gives us that

$$d(\mathfrak{F}x) \leq cd(x) + 2r_0\Psi(\psi(r_0))d(x),$$

and consequently,

$$d(\mathfrak{F}X) \leq (c + 2r_0\Psi(\psi(r_0)))d(X). \quad (3.6)$$

Step 7: \mathfrak{F} is a contraction with respect to the measure of noncompactness μ .

By adding (3.5) and (3.6), we get

$$\omega_0(\mathfrak{F}X) + d(\mathfrak{F}X) \leq (c + 2r_0\Psi(\psi(r_0)))\omega_0(X) + (c + 2r_0\Psi(\psi(r_0)))d(X)$$

or

$$\mu(\mathfrak{F}X) \leq (c + 2r_0\Psi(\psi(r_0)))\mu(X).$$

Since $c + 2r_0\Psi(\psi(r_0)) < 1$, then the operator \mathfrak{F} is contraction with respect to the measure of noncompactness μ .

Finally, Theorem 2.3 guarantees that Eq.(1.1) has at least one solution $x \in C(I)$ which is nondecreasing on I . This completes the proof. \square

4. Example

Let us consider the cubic Urysohn integral equation

$$x(\tau) = \frac{\sqrt{\tau}}{8} + \frac{\tau x(\tau)}{1 + \tau^2} + \frac{x^2(\tau)}{4} \int_0^1 \arctan\left(\frac{\tau \int_0^t s x^2(s) ds}{1 + t^2}\right) dt. \quad (4.1)$$

Here, $\phi(\tau) = \frac{\sqrt{\tau}}{8}$ and this function verifies assumption (a_1) and $\|\phi\| = 1/8$. Also, $\varphi(\tau, x) = \frac{\tau x}{1 + \tau^2}$ and this function verifies assumption (a_2) with

$$|\varphi(\tau, x) - \varphi(\tau, y)| \leq \frac{1}{2}|x - y| \quad \forall t \in I \ \& \ (x, y) \in \mathbb{R}^2.$$

Moreover, the function φ verifies assumption (a_3) . Indeed, for arbitrary nonnegative function $x \in C(I)$ and $\tau_1, \tau_2 \in I$ with $\tau_1 \leq \tau_2$, we have

$$\begin{aligned} d(\Phi x) &= |(\Phi x)(\tau_2) - (\Phi x)(\tau_1)| - ((\Phi x)(\tau_2) - (\Phi x)(\tau_1)) \\ &= |\varphi(\tau_2, x(\tau_2)) - \varphi(\tau_1, x(\tau_1))| - (\varphi(\tau_2, x(\tau_2)) - \varphi(\tau_1, x(\tau_1))) \\ &= \left| \frac{\tau_2}{1 + \tau_2^2} x(\tau_2) - \frac{\tau_1}{1 + \tau_1^2} x(\tau_1) \right| - \left(\frac{\tau_2}{1 + \tau_2^2} x(\tau_2) - \frac{\tau_1}{1 + \tau_1^2} x(\tau_1) \right) \\ &\leq \frac{\tau_2}{1 + \tau_2^2} |x(\tau_2) - x(\tau_1)| + \left| \frac{\tau_2}{1 + \tau_2^2} - \frac{\tau_1}{1 + \tau_1^2} \right| x(\tau_1) \\ &\quad - \frac{\tau_2}{1 + \tau_2^2} (x(\tau_2) - x(\tau_1)) - \left(\frac{\tau_2}{1 + \tau_2^2} - \frac{\tau_1}{1 + \tau_1^2} \right) x(\tau_1) \\ &= \frac{\tau_2}{1 + \tau_2^2} [|x(\tau_2) - x(\tau_1)| - (x(\tau_2) - x(\tau_1))] \\ &= \frac{\tau_2}{1 + \tau_2^2} d(x) \leq \frac{1}{2} d(x). \end{aligned}$$

The function $u(\tau, t, x) = \arctan \frac{\tau x}{1 + t^2}$ satisfies assumption (a_4) , we have $|u(\tau, t, x)| \leq |x|$ which means $\Psi(r) = r$. Moreover, the operator $(\Lambda x)(\tau) = \int_0^\tau t x^2(t) dt$ verifies assumption (a_5) with $\psi(r) = r^2$.

Therefore, the inequality (3.1) has the form $\frac{1}{8} + \frac{\tau}{2} + r^4 \leq r$ or $\frac{1}{4} + r + 2r^4 \leq 2r$. This inequality admits $r_0 = 1/2$ as a positive solution. Moreover,

$$c + 2r_0\Psi(\psi(r_0)) = \frac{1}{2} + \frac{1}{4} = \frac{3}{4} < 1.$$

Consequently, Theorem 3.1 guarantees that equation (4.1) has a continuous nondecreasing solution.

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DOI: 10.7862/rf.2018.3

Hamed Kamal Awad

email: hamedk66@sci.dmu.edu.eg
ORCID: 0000-0003-4866-3164
Department of Mathematics, Faculty of Science
Damanhour University
Damanhour
EGYPT

Mohamed Abdalla Darwish

email: dr.madarwish@gmail.com
ORCID: 0000-0002-4245-4364
Department of Mathematics, Faculty of Science
Damanhour University
Damanhour
EGYPT

Mohamed M.A. Metwali

email: m.metwali@yahoo.com
ORCID: 0000-0003-1091-8619
Department of Mathematics, Faculty of Science
Damanhour University
Damanhour
EGYPT

Received 09.11.2018

Accepted 11.12.2018

Nonlinear Fractional Differential Equations with Non-Instantaneous Impulses in Banach Spaces

Mouffak Benchohra and Mehdi Slimane

ABSTRACT: This paper is devoted to study the existence of solutions for a class of initial value problems for non-instantaneous impulsive fractional differential equations involving the Caputo fractional derivative in a Banach space. The arguments are based upon Mönch's fixed point theorem and the technique of measures of noncompactness.

AMS Subject Classification: 26A33, 34A37, 34G20.

Keywords and Phrases: Initial value problem; Impulses; Caputo fractional derivative; Measure of noncompactness; Fixed point; Banach space.

1. Introduction

The theory of fractional differential equations is an important branch of differential equation theory, which has an extensive physical, chemical, biological, and engineering background, and hence has been emerging as an important area of investigation in the last few decades; see the monographs of Abbas *et al.* [3, 4], Kilbas *et al.* [18], Podlubny [23], and Zhou [25], and the references therein.

On the other hand, the theory of impulsive differential equations has undergone rapid development over the years and played a very important role in modern applied mathematical models of real processes rising in phenomena studied in physics, population dynamics, chemical technology, biotechnology and economics; see for instance the monographs by Bainov and Simeonov [12], Benchohra *et al.* [13], Lakshmikantham *et al.* [19], and Samoilenko and Perestyuk [24] and references therein. Moreover,

impulsive differential equations present a natural framework for mathematical modeling of several real-world problems. In pharmacotherapy, instantaneous impulses cannot describe the dynamics of certain evolution processes. For example, when one considers the hemodynamic equilibrium of a person, the introduction of the drugs in the bloodstream and the consequent absorption for the body are a gradual and continuous process. In [1, 2, 5, 16, 22] the authors studied some new classes of abstract impulsive differential equations with not instantaneous impulses.

However, the theory for fractional differential equations in Banach spaces has yet been sufficiently developed. Recently, Benchohra *et al.* [14] applied the measure of noncompactness to a class of Caputo fractional differential equations of order $r \in (0, 1]$ in a Banach space. Let E be a Banach space with norm $\|\cdot\|$.

In this paper, we study the following initial value problem (IVP for short), for fractional order differential equations

$${}^c D^r y(t) = f(t, y(t)), \quad \text{for a.e. } t \in (s_k, t_{k+1}], \quad k = 0, \dots, m, \quad 0 < r \leq 1, \quad (1)$$

$$y(t) = g_k(t, y(t)), \quad t \in (t_k, s_k], \quad k = 1, \dots, m, \quad (2)$$

$$y(0) = y_0, \quad (3)$$

where ${}^c D^r$ is the Caputo fractional derivative, $f : J \times E \rightarrow E$, $g_k : (t_k, s_k] \times E \rightarrow E$, $k = 1, \dots, m$, are given functions, $J = [0, T]$ and $y_0 \in E$, $0 = s_0 < t_1 < s_1 < \dots < t_m < s_m < t_{m+1} = T$.

To our knowledge no paper has been considered for non-instantaneous impulsive fractional differential equations in abstract spaces. This paper fills the gap in the literature. To investigate the existence of solutions of the problem above, we use Mönch's fixed point theorem combined with the technique of measures of noncompactness, which is an important method for seeking solutions of differential equations. See Akhmerov *et al.* [7], Álvarez [8], Banaś *et al.* [9, 10, 11], Guo *et al.* [15], Mönch [20], Mönch and Von Harten [21].

2. Preliminaries

In this section, we first state the following definitions, lemmas and some notation. By $C(J, E)$ we denote the Banach space of all continuous functions from J into E with the norm

$$\|y\|_\infty = \sup\{\|y(t)\| : t \in J\}.$$

Let $L^1(J, E)$ be the Banach space of measurable functions $y : J \rightarrow E$ which are Bochner integrable, equipped with the norm

$$\|y\|_{L^1} = \int_0^T \|y(t)\| dt.$$

$PC(J, E) = \{y : J \rightarrow E : y \in C((t_k, t_{k+1}], E), k = 0, \dots, m \text{ and there exist } y(t_k^-)$
 and $y(t_k^+), k = 1, \dots, m \text{ with } y(t_k^-) = y(t_k)\}$.

$PC(J, E)$ is a Banach space with the norm

$$\|y\|_{PC} = \sup_{t \in J} \|y(t)\|.$$

Set

$$J' = J \setminus \cup_{k=1}^m (t_k, s_k].$$

Moreover, for a given set V of functions $v : J \rightarrow E$, let us denote by

$$V(t) = \{v(t), v \in V\}, t \in J$$

and

$$V(J) = \{v(t), v \in V, t \in J\}.$$

Now let us recall some fundamental facts of the notion of Kuratowski measure of noncompactness.

Definition 2.1. ([9]). Let X be a Banach space and Ω_X the bounded subsets of X . The Kuratowski measure of noncompactness is the map $\alpha : \Omega_X \rightarrow [0, \infty]$ defined by

$$\alpha(B) = \inf\{\epsilon > 0 : B \subseteq \cup_{i=1}^n B_i \text{ and } \text{diam}(B_i) \leq \epsilon\}; \text{ here } B \in \Omega_X.$$

Properties: The Kuratowski measure of noncompactness satisfies the following properties (for more details see [9])

- (a) $\alpha(B) = 0 \Leftrightarrow \overline{B}$ is compact (B is relatively compact).
- (b) $\alpha(B) = \alpha(\overline{B})$.
- (c) $A \subset B \Rightarrow \alpha(A) \leq \alpha(B)$.
- (d) $\alpha(A + B) \leq \alpha(A) + \alpha(B)$.
- (e) $\alpha(cB) = |c|\alpha(B); c \in \mathbb{R}$.
- (f) $\alpha(\text{conv}B) = \alpha(B)$.

For completeness we recall the definition of Caputo derivative of fractional order.

Definition 2.2. ([18]). The fractional (arbitrary) order integral of the function $h \in L^1([0, T], E)$ of order $r \in \mathbb{R}_+$ is defined by

$$I^r h(t) = \frac{1}{\Gamma(r)} \int_0^t (t-s)^{r-1} h(s) ds, \text{ for a.e. } t \in [0, T],$$

where Γ is the Euler gamma function defined by $\Gamma(r) = \int_0^\infty t^{r-1} e^{-t} dt, r > 0$.

Definition 2.3. ([18]). For a function $h \in AC^n(J, E)$, the Caputo fractional-order derivative of order r of h is defined by

$$({}^c D_0^r h)(t) = \frac{1}{\Gamma(n-r)} \int_0^t (t-s)^{n-r-1} h^{(n)}(s) ds, \quad \text{for a.e. } t \in [0, T],$$

where $n = [r] + 1$.

We need the following auxiliary lemmas ([18]).

Lemma 2.4. Let $r > 0$ and $h \in AC^n(J, E)$. Then the differential equation

$${}^c D_0^r h(t) = 0, \quad \text{for a.e. } t \in J$$

has solutions $h(t) = c_0 + c_1 t + c_2 t^2 + \dots + c_{n-1} t^{n-1}$, $c_i \in \mathbb{R}$, $i = 0, 1, 2, \dots, n-1$, $n = [r] + 1$.

Lemma 2.5. Let $r > 0$ and $h \in AC^n(J, E)$. Then

$$I^r {}^c D_0^r h(t) = h(t) + c_0 + c_1 t + c_2 t^2 + \dots + c_{n-1} t^{n-1}, \quad \text{for a.e. } t \in J$$

for some $c_i \in \mathbb{R}$, $i = 0, 1, 2, \dots, n-1$, $n = [r] + 1$.

Definition 2.6. A map is said to be Carathéodory if

i $t \rightarrow f(t, u)$ is measurable for each $u \in E$.

ii $u \rightarrow F(t, u)$ is continuous for almost all $t \in J$.

For our purpose we will only need the following fixed point theorem, and the important Lemma.

Theorem 2.7. ([6, 20]) (Mönch's fixed point theorem). Let D be a bounded, closed and convex subset of a Banach space such that $0 \in D$, and let N be a continuous mapping of D into itself. If the implication

$$V = \overline{\text{conv}}N(V) \text{ or } V = N(V) \cup \{0\} \Rightarrow \alpha(V) = 0$$

holds for every subset V of D , then N has a fixed point.

Lemma 2.8. ([15]) If $V \subset C(J; E)$ is a bounded and equicontinuous set, then

(i) the function $t \rightarrow \alpha(V(t))$ is continuous on J , and

$$\alpha_c(V) = \sup_{0 \leq t \leq T} \alpha(V(t)).$$

(ii) $\alpha(\int_0^T x(s) ds : x \in V) \leq \int_0^T \alpha(V(s)) ds$,

where

$$V(s) = \{x(s) : x \in V\}, \quad s \in J.$$

3. Existence of Solutions

First of all, we define what we mean by a solution of the IVP (1)-(3).

Definition 3.1. A function $y \in PC(J, E) \cap AC(J', E)$ is said to be a solution of (1)-(3) if y satisfies $y(0) = y_0$, ${}^cD^r y(t) = f(t, y(t))$, for a.e. $t \in (s_k, t_{k+1}]$, and each $k = 0, \dots, m$, and $y(t) = g_k(t, y(t))$, for all $t \in (t_k, s_k]$, and every $k = 1, \dots, m$,

To prove the existence of solutions to (1)-(3), we need the following auxiliary lemmas.

Lemma 3.2. Let $0 < r \leq 1$ and let $h : J \rightarrow E$ be integrable. Then linear problem

$${}^cD^r y(t) = h(t), \text{ for each } t \in J_k := (s_k, t_{k+1}], \quad k = 0, \dots, m, \quad (4)$$

$$y(t) = g_k(t), \text{ for each } t \in J'_k := (t_k, s_k] \quad k = 1, \dots, m, \quad (5)$$

$$y(0) = y_0 \quad (6)$$

has a unique solution which is given by:

$$y(t) = \begin{cases} y_0 + \frac{1}{\Gamma(r)} \int_0^t (t-s)^{r-1} h(s) ds, & \text{if } t \in [0, t_1], \\ g_k(t), & \text{if } t \in J'_k \quad k = 1, \dots, m, \\ g_k(s_k) + \frac{1}{\Gamma(r)} \int_{s_k}^t (t-s)^{r-1} h(s) ds, & \text{if } t \in J_k \quad k = 1, \dots, m. \end{cases} \quad (7)$$

Proof. Assume that y satisfies (4)-(6).

If $t \in [0, t_1]$ then

$${}^cD^r y(t) = h(t).$$

Lemma 2.5 implies

$$y(t) = y_0 + \frac{1}{\Gamma(r)} \int_0^t (t-s)^{r-1} h(s) ds.$$

If $t \in J'_1 = (t_1, s_1]$ we have $y(t) = g_1(t)$.

If $t \in J_1 = (s_1, t_2]$, then Lemma 2.5 implies

$$\begin{aligned} y(t) &= y(s_1^+) + \frac{1}{\Gamma(r)} \int_{s_1}^t (t-s)^{r-1} h(s) ds \\ &= g_1(s_1) + \frac{1}{\Gamma(r)} \int_{s_1}^t (t-s)^{r-1} h(s) ds. \end{aligned}$$

If $t \in J'_2 = (t_2, s_2]$ we have $y(t) = g_2(t)$.

If $t \in J_2 = (s_2, t_3]$ then again Lemma 2.5 implies

$$\begin{aligned} y(t) &= y(s_2^+) + \frac{1}{\Gamma(r)} \int_{s_2}^t (t-s)^{r-1} h(s) ds \\ &= g_2(s_2) + \frac{1}{\Gamma(r)} \int_{s_2}^t (t-s)^{r-1} h(s) ds. \end{aligned}$$

If $t \in J'_k = (t_k, s_k]$ we have $y(t) = g_k(t)$.

If $t \in J_k = (s_k, t_{k+1}]$ then Lemma 2.5 implies

$$\begin{aligned} y(t) &= y(s_k^+) + \frac{1}{\Gamma(r)} \int_{s_k}^t (t-s)^{r-1} h(s) ds \\ &= g_k(s_k) + \frac{1}{\Gamma(r)} \int_{s_k}^t (t-s)^{r-1} h(s) ds. \end{aligned}$$

Conversely, assume that y satisfies equation (7).

If $t \in [0, t_1]$, then $y(0) = y_0$ and, using the fact that ${}^c D^r$ is the left inverse of I^r , we get

$${}^c D^r y(t) = h(t), \text{ for each } t \in (0, t_1].$$

If $t \in J_k := (s_k, t_{k+1}]$, $k = 1, \dots, m$, and using the fact that ${}^c D^r C = 0$, where C is a constant, we get

$${}^c D^r y(t) = h(t), \text{ for each } t \in J_k := (s_k, t_{k+1}], \quad k = 1, \dots, m.$$

Also, we have easily that

$$y(t) = g_k(t), \text{ for each } t \in J'_k := (t_k, s_k], \quad k = 1, \dots, m.$$

□

We are now in a position to state and prove our existence result for the problem (1)–(3) based on Mönch's fixed point. Let us list some conditions on the functions involved in the IVP (1)–(3).

(H1) The function $f : J \times E \rightarrow E$ satisfies the Carathéodory conditions.

(H2) There exists $p \in C(J, \mathbb{R}_+)$ such that

$$\|f(t, y)\| \leq p(t)\|y\| \text{ for any } y \in E \text{ and } t \in J.$$

(H3) g_k are uniformly continuous functions and there exists $c_k \in C(J, \mathbb{R}_+)$ such that

$$\|g_k(t, y)\| \leq c_k(t)\|y\|, \text{ for each } y \in E \text{ and } t \in J, \quad k = 1, \dots, m.$$

(H4) For each bounded set $B \subset E$ we have

$$\alpha(g_k(t, B)) \leq c_k(t)\alpha(B), \quad t \in J.$$

(H5) For each bounded set $B \subset E$ we have

$$\alpha(f(t, B)) \leq p(t)\alpha(B), \quad t \in J.$$

Let

$$p^* = \sup_{t \in J} p(t), \quad c^* = \max_{k=1, \dots, m} (\sup_{t \in J} (c_k(t))).$$

Theorem 3.3. *Assume that assumptions (H1)-(H5) hold. If*

$$\frac{p^* T^r}{\Gamma(r+1)} + c^* < 1, \quad (8)$$

then the IVP (1)–(3) has at least one solution J .

Proof. Transform the problem (1)–(3) into a fixed point problem. Consider the operator $N : PC(J, E) \rightarrow PC(J, E)$ defined by

$$N(y)(t) = \begin{cases} y_0 + \frac{1}{\Gamma(r)} \int_0^t (t-s)^{r-1} f(s, y(s)) ds, & \text{if } t \in [0, t_1], \\ g_k(t, y(t)), & \text{if } t \in J'_k := (t_k, s_k], \\ g_k(s_k, y(s_k)) + \frac{1}{\Gamma(r)} \int_{s_k}^t (t-s)^{r-1} f(s, y(s)) ds, & \text{if } t \in J_k := (s_k, t_{k+1}]. \end{cases} \quad (9)$$

Clearly, the fixed points of operator N are solutions of problem (1)–(3).

Let

$$r_0 \geq \frac{\|y_0\|}{1 - \frac{p^* T^r}{\Gamma(r+1)} - c^*}, \quad (10)$$

and consider the set

$$D_{r_0} = \{y \in PC(J, E) : \|y\|_\infty \leq r_0\}.$$

Clearly, the subset D_{r_0} is closed, bounded and convex. We shall show that N satisfies the assumptions of Theorem 2.7. The proof will be given in a couple of steps.

Step 1: N is continuous.

Let $\{u_n\}$ be a sequence such that $u_n \rightarrow u$ in $PC(J, E)$. Then

for $t \in J_k$, we have

$$\begin{aligned} \|N(y_n)(t) - N(y)(t)\| &\leq \|g_k(t, y_n(t)) - g_k(t, y(t))\| \\ &\quad + \frac{1}{\Gamma(r)} \int_{s_k}^t (t-s)^{r-1} \|f(s, y_n(s)) - f(s, y(s))\| ds, \end{aligned}$$

for $t \in [0, t_1]$, we have

$$\|N(y_n)(t) - N(y)(t)\| \leq \frac{1}{\Gamma(r)} \int_0^t (t-s)^{r-1} \|f(s, y_n(s)) - f(s, y(s))\| ds,$$

and for $t \in J'_k$, we have

$$\|N(u_n)(t) - N(u)(t)\| \leq \|g_k(t, y_n(t)) - g_k(t, y(t))\|.$$

Since g_k is continuous and f is of Carathéodory type, the Lebesgue dominated convergence theorem implies

$$\|N(u_n) - N(u)\|_\infty \rightarrow 0 \text{ as } n \rightarrow \infty.$$

Consequently, N is continuous.

Step 2: N maps D_{r_0} into itself.

For each $y \in D_{r_0}$, by (H2), (H3) and (10) we have for each $t \in J$,

$$\begin{aligned} \|N(y)(t)\| &\leq \|g_k(t, y(t))\| + \|y_0\| + \frac{1}{\Gamma(r)} \int_0^t (t-s)^{r-1} \|f(s, y(s))\| ds \\ &\leq c_k \|y(t)\| + \|y_0\| + \frac{1}{\Gamma(r)} \int_{s_k}^{t_{k+1}} (t-s)^{r-1} p(s) \|y(s)\| ds \\ &\leq \|y_0\| + r_0 \left(\frac{p^* T^r}{\Gamma(r+1)} + c^* \right) \\ &\leq r_0. \end{aligned}$$

Step 3: $N(D_{r_0})$ is bounded and equicontinuous.

By Step 2, it is obvious that $N(D_{r_0}) \subset PC(J, E)$ is bounded.

For the equicontinuous of $N(D_{r_0})$, let $\tau_1, \tau_2 \in J$, $\tau_1 < \tau_2$ and $y \in D_{r_0}$. Then, for $\tau_1, \tau_2 \in J_k$, we have

$$\begin{aligned} \|N(y)(\tau_2) - N(y)(\tau_1)\| &= \frac{1}{\Gamma(r)} \int_{\tau_1}^{\tau_2} |(\tau_2 - s)^{r-1} - (\tau_1 - s)^{r-1}| \|f(s, y(s))\| ds \\ &\leq 2 \frac{r_0 p^*}{\Gamma(r+1)} [\tau_2^r - \tau_1^r], \end{aligned}$$

for $\tau_1, \tau_2 \in [0, t_1]$, we have

$$\begin{aligned} \|N(y)(\tau_2) - N(y)(\tau_1)\| &= \frac{1}{\Gamma(r)} \int_{\tau_1}^{\tau_2} |(\tau_2 - s)^{r-1} - (\tau_1 - s)^{r-1}| \|f(s, y(s))\| ds. \\ &\leq 2 \frac{r_0 p^*}{\Gamma(r+1)} [\tau_2^r - \tau_1^r], \end{aligned}$$

and for $\tau_1, \tau_2 \in J'_k$, we have

$$\|N(y)(\tau_2) - N(y)(\tau_1)\| = \|g_k(\tau_2, y(\tau_2)) - g_k(\tau_1, y(\tau_1))\|.$$

As $\tau_1 \rightarrow \tau_2$, the right-hand side of the above inequality tends to zero.

Now let V be a subset of D_{r_0} such that $V \subset \overline{\text{conv}}(N(V) \cup \{0\})$. Then V is bounded and equicontinuous and therefore the function $t \rightarrow v(t) = \alpha(V(t))$ is continuous on J . By (H4), (H5), Lemma 2.8 and the properties of the measure α we have for each $t \in J$

$$\begin{aligned} v(t) &\leq \alpha(N(V)(t) \cup \{0\}) \\ &\leq \alpha(N(V)(t)). \end{aligned}$$

If $t \in J_k$,

$$\begin{aligned} v(t) &\leq \alpha(g_k(s_k, V(s_k))) + \frac{1}{\Gamma(r)} \int_{s_k}^t (t-s)^{r-1} f(s, V(s)) ds \\ &\leq c_k(t) \alpha(V(s)) + \frac{1}{\Gamma(r)} \int_{s_k}^t (t-s)^{r-1} p(t) \alpha(V(s)) ds \\ &\leq c_k(t) v(s) + \frac{1}{\Gamma(r)} \int_{s_k}^t (t-s)^{r-1} p(t) v(s) ds \\ &\leq \|v\|_\infty \left(c^* + \frac{p^* T^r}{\Gamma(r+1)} \right), \end{aligned}$$

if $t \in [0, t_1]$

$$\begin{aligned} v(t) &\leq \alpha\left(\frac{1}{\Gamma(r)} \int_0^t (t-s)^{r-1} f(s, V(s)) ds\right) \\ &\leq \frac{1}{\Gamma(r)} \int_0^t (t-s)^{r-1} p(t) \alpha(V(s)) ds \\ &\leq \frac{1}{\Gamma(r)} \int_0^t (t-s)^{r-1} p(t) v(s) ds \\ &\leq \|v\|_\infty \left(\frac{p^* T^r}{\Gamma(r+1)} \right) \\ &\leq \|v\|_\infty \left(c^* + \frac{p^* T^r}{\Gamma(r+1)} \right), \end{aligned}$$

if $t \in J'_k$

$$\begin{aligned} v(t) &\leq \alpha(g_k(s_k, V(s_k))) \\ &\leq c_k(t) \alpha(V(s)) \\ &\leq c_k(t) v(s) \\ &\leq \|v\|_\infty c^* \\ &\leq \|v\|_\infty \left(c^* + \frac{p^* T^r}{\Gamma(r+1)} \right). \end{aligned}$$

This means that

$$\|v\|_\infty \left[1 - \left(c^* + \frac{p^* T^r}{\Gamma(r+1)} \right) \right] \leq 0.$$

By (8) it follows that $\|v\|_\infty = 0$; that is, $v(t) = 0$ for each $t \in J$, and then $V(t)$ is relatively compact in E . In view of the Ascoli–Arzela theorem, V is relatively compact in D_{r_0} . Applying now Theorem 2.7 we conclude that N has a fixed point which is a solution of the problem (1)-(3). \square

4. An Example

Let us consider the following infinite system of impulsive fractional initial value problem,

$${}^c D^{\frac{1}{2}} y_n(t) = \frac{1}{9+n+e^t} \ln(1+|y_n(t)|), \text{ for a.e. } t \in \left(0, \frac{1}{3}\right] \cup \left(\frac{1}{2}, 1\right], \quad (11)$$

$$y_n(t) = \frac{1}{4+n+e^t} \sin |y_n(t)|, \quad t \in \left(\frac{1}{3}, \frac{1}{2}\right], \quad (12)$$

$$y_n(0) = 0. \quad (13)$$

Set

$$E = l^1 = \{y = (y_1, y_2, \dots, y_n, \dots), \sum_{n=1}^{\infty} |y_n| < \infty\},$$

E is a Banach space with the norm

$$\|y\| = \sum_{n=1}^{\infty} |y_n|.$$

Let

$$f(t, y) = (f_1(t, y), f_2(t, y), \dots, f_n(t, y), \dots),$$

$$f_n(t, y) = \frac{\ln(1+|y_n(t)|)}{9+n+e^t},$$

and

$$g_1(t, y) = (g_{1_1}(t, y), g_{1_2}(t, y), \dots, g_{1_n}(t, y), \dots),$$

$$g_{1_n}(t, y) = \frac{\sin |y_n(t)|}{4+n+e^t}.$$

Clearly conditions (H2) and (H3) hold with

$$p(t) = \frac{1}{9+e^t}, \text{ and } c_1(t) = \frac{1}{4+e^t}.$$

We shall check that condition (8) is satisfied with $r = \frac{1}{2}$, $T = 1$, $P^* = \frac{1}{10}$ and $c^* = \frac{1}{5}$.
Indeed

$$\left(\frac{p^* T^r}{\Gamma(r+1)} + c^* \right) = \frac{1}{5\sqrt{\pi}} + \frac{1}{5} < 1.$$

Then by Theorem 3.3 the problem (11)-(13) has at least one solution.

Acknowledgement

The authors are grateful to the referee for the helpful remarks.

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DOI: 10.7862/rf.2018.4

Mouffak Benchohra

email: benchohra@yahoo.com

ORCID: 0000-0003-3063-9449

Laboratory of Mathematics

Djillali Liabes University of Sidi-Bel-Abbès

P.O. Box 89, Sidi Bel-Abbès 22000

ALGERIA

Mehdi Slimane

email: m1slim@yahoo.com

ORCID: 0000-0002-1556-3709

Laboratory of Mathematics

Djillali Liabes University of Sidi-Bel-Abbès

P.O. Box 89, Sidi Bel-Abbès 22000

ALGERIA

Received 28.12.2017

Accepted 23.04.2018

A Characterization of Weakly $J(n)$ -Rings

Peter V. Danchev

ABSTRACT: A ring R is called a $J(n)$ -ring if there exists a natural number $n \geq 1$ such that for each element $r \in R$ the equality $r^{n+1} = r$ holds and a *weakly $J(n)$ -ring* if there exists a natural number $n \geq 1$ such that for each element $r \in R$ the equalities $r^{n+1} = r$ or $r^{n+1} = -r$ hold.

We completely describe both classes of these rings R for any n , thus considerably extending some well-known results in the subject, especially that of V. Perić in Publ. Inst. Math. Beograd (1983) as well as, in particular, the classical description of Boolean rings when $n = 1$.

AMS Subject Classification: 16D60, 16S34, 16U60.

Keywords and Phrases: Boolean rings; Idempotents; Units; Nilpotents; Jacobson radical; $J(n)$ -rings.

1. Introduction and Background Material

Throughout, all rings R examined in the current paper shall be assumed associative, containing the identity element 1 which possibly differs from the zero element 0. Standardly, $U(R)$ denotes the set of all invertible elements of R , $Id(R)$ the set of all idempotent elements of R and $Nil(R)$ the set of all nilpotent elements of R . Traditionally, $J(R)$ denotes the Jacobson radical of R . All other notions and notations, not explicitly defined herein, are well-established in the existing literature. About the specific terminology, specifically that of a *PI-ring*, let us recall that it is a ring whose elements satisfy a polynomial identity with coefficients in \mathbb{Z} , the ring of all integers, and at least one coefficient has to be invertible, that is, ± 1 . In particular, commutative rings are always PI-rings.

The following concept is rather well-known.

Definition 1.1. A ring R is called *Boolean* if $x^2 = x$ for each $x \in R$, that is, $R = Id(R)$.

These rings have a complete characterization as (the subring of) the direct product of family of copies of the two element field \mathbb{Z}_2 (see, e.g., [2]). Hence, Boolean rings are themselves commutative.

To consider some other generalizations, for a fixed prime p , a p -ring is a ring R in which $a^p = a$ and $pa = 0$ for all $a \in R$. Thus any Boolean ring is simply a 2-ring. It is known that a ring is a p -ring if, and only if, it is a subdirect product of fields of order p (cf. [15]). On the other hand, for a prime p and a positive integer k , a p^k -ring is a ring R in which $a^{p^k} = a$ and $pa = 0$ for all $a \in R$. The structure of p^k -rings has been described in [1]. A ring R is said to be *periodic* if, for each $a \in R$, there is a positive integer $n(a)$ such that $a^{n(a)+1} = a$. Every periodic ring is commutative by a fundamental result due to Jacobson from [12]. If such a natural number $n(a)$ does not depend on the choice of the element a , that is it could be fixed, we thus come to the following concept.

Definition 1.2. Let $n \geq 1$ be a natural number. We shall say that the ring R is a $J(n)$ -ring if, for every $x \in R$, the equation $x^{n+1} = x$ holds.

The special case when $n = 1$ gives the famous Boolean rings; notice also that $x = x^2$ always implies that $x = x^j$ for all $j \in \mathbb{N}$. Likewise, these rings are obviously perfect, i.e., $R = R^{n+1}$. Moreover, it was proved in [14] that a ring is a $J(n)$ -ring if, and only if, it is the direct sum of finitely many p^k -rings. Hence, with the aforementioned result in [1] at hand, the structural characterization of $J(n)$ -rings can be assumed for totally exhausted. On the other side, they are also somewhat studied in [10], but without any concrete full description given.

However, the complete description of $J(n)$ -rings was given in [17]. There was proved that R is a $J(n)$ -ring if, and only if, R is a subdirect product of fields \mathbb{F}_{p^k} , where p is a prime and k is an integer such that $p^k - 1$ divides n . Nevertheless, we will give here a new more convenient and attractive for further applications description of their structure in terms of the simple p -element fields \mathbb{Z}_p , where p is a prime, and the fields \mathbb{F}_q of $q = p^k$ elements, where $k \in \mathbb{N}$. So, the objective of this article is to do that by using an elementary algebraic approach. We refer also to [3] and [13] for more account to that topic.

In order to substantially enlarge the above explorations, we shall be concerned here and with giving up the full characterization up to isomorphism of weakly $J(n)$ -rings, that are, rings whose elements satisfy the polynomial identities $x^{n+1} - x = 0$ or $x^{n+1} + x = 0$.

We thus come to the following new concept:

Definition 1.3. Suppose that $n \geq 1$ be a natural number. We shall say that the ring R is a *weakly $J(n)$ -ring* if, for every $x \in R$, the equations $x^{n+1} = x$ or $x^{n+1} = -x$ hold.

It is pretty obvious that subrings and homomorphic images of (weakly) $J(n)$ -rings are again (weakly) $J(n)$ -rings. Some concrete folklore examples are these:

- " $n = 1$ ": A ring R is a $J(2)$ -ring if, and only if, R can be embedded as a subring of the direct product of family of copies of the fields \mathbb{Z}_2 and \mathbb{Z}_3 .

As usual, \mathbb{F}_4 denotes the field of characteristic 2 consisting of four elements constructed as follows: It is well known that in the polynomial ring $\mathbb{Z}_2[x]$ the polynomial $1 + x + x^2$ is irreducible over \mathbb{Z}_2 and hence $\mathbb{Z}_2[x]/(x^2 + x + 1) \cong \mathbb{Z}_2(\theta)$ is a field of 4 elements, denoting it by \mathbb{F}_4 , where $\theta \notin \mathbb{Z}_2$ is a solution of the equation $x^3 = 1$. In fact, the elements in \mathbb{F}_4 are $\{0, 1, \theta, \theta^2\}$ taking into account that $\theta + 1 = \theta^{-1} = \theta^2$ and that $x^3 - 1 = (x^2 + x + 1)(x - 1)$ whence $x^3 = 1$ has the set $\{1 = \theta^0, \theta, \theta^2\}$ as solutions. Thus \mathbb{F}_4 is the splitting field of these two polynomials. Note that under such a construction the equality $\mathbb{F}_2 = \mathbb{Z}_2$ is true.

• " $n = 2$ ": A ring R is a $J(3)$ -ring if, and only if, R can be embedded as a subring of the direct product of family of copies of the fields \mathbb{Z}_2 and \mathbb{F}_4 .

• " $n = 3$ ": A ring R is a $J(4)$ -ring if, and only if, R can be embedded as a subring of the direct product of family of copies of the fields \mathbb{Z}_2 , \mathbb{Z}_3 and \mathbb{Z}_5 .

In what follows in the subsequent section, we shall provide a full characterizing of both $J(n)$ -rings and weakly $J(n)$ -rings.

2. The Main Results

We start in this section with the following technicality by treating the general case of $J(n)$ -rings, as our purpose is to give a new more transparent and conceptual proof of the characterization result for these rings than that in [17]. In doing that, we need the following technical claim.

Lemma 2.1. *Let $n \in \mathbb{N}$ and let R be a ring whose elements satisfy the identity $x^{n+1} = x$, while $x^{k+1} \neq x$ for some x , provided $k < n$ and $k \in \mathbb{N}$, that is, for every $k < n$ there exists x in R for which x^{k+1} is not equal to x . The next three items are true:*

1. R is reduced.
2. $J(R) = 0$.
3. If R is primitive, then $n = p^m - 1$ for some $m \in \mathbb{N}$ and R is a field with p^m elements.

Proof. Items (1) and (2) are rather obvious, which follow directly from the condition $x^{n+1} = x$, so we omit their verification. The third item is an immediate consequence of the fact that R is a PI-ring and of the well-known Kaplansky's theorem by using the method presented in detail in [8]. \square

We are now proceed by proving with the following basic statement, which somewhat improves on the aforementioned characterizing result from [17] concerning $J(n)$ -rings.

Theorem 2.2. *Suppose that $n \in \mathbb{N}$. Then, for a ring R , the following two conditions are equivalent:*

1. R is a $J(n)$ -ring.
2. R is a subdirect product of finite fields $\mathbb{F}_{p_k^{m_k}}$ for some primes p_k and integers m_k , $k \in \mathbb{N}$, where $(p_k^{m_k} - 1)/n$ for each k .

Proof. "(1) \Rightarrow (2)". With Lemma 2.1 at hand, R is a subdirect products of finite fields F_i satisfying the equality $x^{n+1} = x$. Let us fix such a field F with p^m elements. It is then well known that $U(F)$ is a cyclic group of order $p^m - 1$ which satisfies the identity $x^n = 1$. Thus $p^m - 1$ divides n .

"(2) \Rightarrow (1)". Letting R be a subdirect product of the fields F_i , we then easily see that each field satisfies $x^{n+1} = x$, thus R will also satisfy this identity. \square

By the same token, we can derive the following consequence for $J(n)$ -rings in the presence of minimality of the existing natural number n .

Corollary 2.3. *Suppose $n \in \mathbb{N}$. Then, for a ring R , the following two conditions are tantamount:*

1. R satisfies the equation $x^{n+1} = x$ with n minimal possible.
2. R is a subdirect product of finite fields $\mathbb{F}_{p_k^{m_k}}$ for some primes p_k and integers m_k , $k \in \mathbb{N}$, where $(p_k^{m_k} - 1)/n$ for each k and $n = \text{LCM}(p_k^{m_k} - 1 \mid k \in \mathbb{N})$.

For a convenience of the reader, let us recall that \mathbb{F}_q is the finite field with q elements with q a prime power. We are now ready to prepare our chief result which completely settles the question when an arbitrary ring is weakly $J(n)$ and which states as follows:

Theorem 2.4. *Let $n \geq 1$ be a natural and let R be a ring. Then R is a weakly $J(n)$ -ring if, and only if, R is a $J(n)$ -ring, or either $R = \mathbb{F}_q$ or $R = P \times \mathbb{F}_q$, where $q - 1$ divides $2n$ but not n , and P is a $J(n)$ -ring of characteristic 2.*

Proof. In one direction, if $R = P \times \mathbb{F}_q$, where P is the zero ring or a $J(n)$ -ring of characteristic 2, for any pair $(x, y) \in R$ we indeed have $(x, y)^{n+1} = \pm(x, y)$ depending on the fact whether $y \in \mathbb{F}_q$ is a square or not.

In the other direction, we first observe that $x^{2n+1} = x$ for all $x \in R$. By the famous Jacobson's theorem for commutativity, R is really commutative. Moreover, by the usage of classical arguments, R is the direct product of characteristic p rings, where p is one of the finitely many primes, for which $p - 1$ divides $2n$. Then, at least one of these rings, say K , contains an element a with $a^{n+1} \neq a$. Certainly, $\text{char}(K) = p > 2$. If foremost $R \neq K$, then R is of the form $K \times P$, where P is a non-zero ring. Consider the element $(a, 1)$ in R . Its $n + 1$ -th power is $(a^{n+1}, 1)$, and is equal to $\pm(a, 1)$. Since a^{n+1} differs from a , it must be that $(a^{n+1}, 1) = -(a, 1) = (-a, -1)$ whence $1 = -1$ in P and, therefore, it follows that P is necessarily of characteristic 2.

So, it suffices to show that $K = \mathbb{F}_q$, where q has the properties indicated in the statement of the theorem. To that goal, since the element a satisfied the inequality $a^{n+1} \neq a$, we must have $a^{n+1} = -a$. Besides, since every prime ideal of K is maximal and since the nil-radical of K is trivial, there exists a maximal ideal I of K for which $a^{n+1} = -a$ modulo I . Then the field K/I is finite of characteristic p . We denote it by \mathbb{F}_q . Now, every element $y \in K/I$ satisfies $y^{2n+1} = y$. Since \mathbb{F}_q^* is a cyclic group of order $q - 1$, $q - 1$ divides $2n$. Since $a^{n+1} = -a$ and the characteristic of \mathbb{F}_q is not 2, the number n is not divisible by $q - 1$.

If now K admits a second maximal ideal M , then I and M are co-prime, so that $(K/I) \times (K/M)$ is a quotient of K . Consider its element $(a, 1)$. Then $(a, 1)^{n+1} = (-a, 1)$ which is not equal to $\pm(a, 1)$ – a contradiction. Thus I is the only maximal ideal of K . This means that K is a local ring, and hence equal to its residue field $K/I = \mathbb{F}_q$. The last claim is a direct consequence of the fact that $x^{n+1} = \pm x$ for all $x \in K$. \square

Remark 2.5. It is worthwhile noticing that the cases $n = 1$ are settled in [9] and [4]; $n = 2$ in [5]; and $n = 3$ in [6]. Moreover, by virtue of the main Theorem 2.4, in accordance with Theorem 2.2, or with the main result from [17], the study of the structure of weakly $J(n)$ -rings is completely exhausted.

We shall now be involved with some applications by giving up a slight generalization of $J(n)$ -rings for $n \in \mathbb{N}$ to rings with elements satisfying the equation $x^n = \zeta x$, where ζ is a (primitive) d -th root of unity for some positive integer d (compare with the slightly weaker version stated in Problem 3 listed below). Precisely, the following assertion is true:

Theorem 2.6. *Let R be a commutative ring and let $f(X) = \sum_{n \geq 0} a_n X^n$ be a polynomial in $R[X]$. Suppose also that the polynomial $\tilde{f}(X) = \sum_{n \geq 0} a_n a_0^{n-1} X^n$ has a root in R and that $a_n a_0^{n-1} = 1$ for $n = 0$. Then, for all $y \in R$ and each ideal $I \subseteq \text{Ann}(y)$, the annihilator of y , we have*

$$f(y) \equiv 0 \pmod{I} \iff \exists \zeta \in R : \zeta \equiv y \pmod{I} \bigwedge f(\zeta) = 0.$$

Proof. The implication " \Leftarrow " being elementary, we will be concentrated on the reverse one " \Rightarrow ". To that goal, let $\lambda \in R$ be a zero of $\tilde{f}(X)$. We therefore readily check that $\zeta = y + \lambda f(y)$ is a zero of $f(X)$. Keeping in mind that $y f(y) = 0$, we deduce that

$$f(y + \lambda f(y)) = \sum_{n \geq 0} a_n (y + \lambda f(y))^n = f(y) + \sum_{n \geq 1} a_n \lambda^n f(y)^n = f(y) + \sum_{n \geq 1} a_n \lambda^n f(y) a_0^{n-1}.$$

This is obviously equal to

$$f(y) \sum_{n \geq 0} a_n a_0^{n-1} \lambda^n = f(y) \tilde{f}(\lambda) = 0,$$

as required. \square

As a valuable consequence, one derives the following.

Corollary 2.7. *Let R be a ring and let d, n be positive integers. Then, for all $x \in R$, we have*

$$x^{d(n-1)+1} = x \iff \exists \varsigma \in R : \varsigma^d = 1 \bigwedge x^n = \varsigma x.$$

Proof. Let $x \in R$. Since for any $\varsigma \in R$ and for $i = d, \dots, 1$, we obtain

$$x^{i(n-1)+1} = \varsigma x x^{i(n-1)+1-n} = \varsigma x^{(i-1)(n-1)+1},$$

the implication " \Leftarrow " is clear. To prove the converse implication " \Rightarrow ", we may with no harm in generality replace R by the subring generated by 1 and x . In particular, we can assume even that R is commutative.

Now, put $I = \text{Ann}(x)$. Consequently, the statement of the corollary is equivalent to

$$x^{d(n-1)} \equiv 1 \pmod{I} \iff \exists \varsigma \in R : \varsigma^d = 1 \bigwedge \varsigma \equiv x^{n-1} \pmod{I}.$$

Next put $f(X) = X^d - 1 \in R[X]$ and $y = x^{n-1}$; thus $f(\varsigma) = 0$. So, the statement of the corollary is amounting to

$$f(y) \equiv 0 \pmod{I} \iff \exists \varsigma \in R : f(\varsigma) = 0 \bigwedge \varsigma \equiv y \pmod{I}.$$

Note that the ideal I is pretty obviously contained in $\text{Ann}(y)$. Since $\lambda = -1$ is a zero of $\tilde{f}(X) = (-1)^{d-1}X^d + 1$, the condition of the previous theorem is satisfied and thus the corollary follows after all. \square

It is worthwhile noticing that Theorem 2.6 perhaps can be considerably extended in the non-commutative case as follows (actually, its formulation smells a little like the classical well-known Hensel's lemma):

Let R be a ring and let $f(X) \in R[X]$ be a polynomial with coefficients in the center of R . Suppose also that $f(X)$ has an invertible root in the center of R and that $f(0)$ inverts in R . Then, for all $y \in R$ and every ideal $I \subset \text{Ann}(y)$, the annihilator of y , we have

$$f(y) \equiv 0 \pmod{I} \iff \exists \varsigma \in R : \varsigma \equiv y \pmod{I} \bigwedge f(\varsigma) = 0.$$

Some idea for an eventual proof could be the following: The right-to-left part being self-evident, we will deal with the left-to-right one. To that purpose, let $\varepsilon \in R$ be an invertible central root of $f(X)$. Put

$$h(X) = \frac{\varepsilon f(X)}{f(0)}.$$

Notice that $h(0) = \varepsilon$ inverts in R . Thus $h(y)$ lies in I . We easily check that $\varsigma = y+h(y)$ satisfies the equality $f(\varsigma) = 0$. Writing $f(X) = \sum_{n \geq 0} a_n X^n$ and bearing in mind that $yh(y) = 0$, one infers that

$$f(y + h(y)) = \sum_{n \geq 0} a_n (y + h(y))^n = f(y) + \sum_{n \geq 1} a_n h(y)^n.$$

We however see that $h(y)^n = h(0)^{n-1}h(y)$ for every $n \geq 1$. Therefore, we get that

$$0 = f(y + h(y)) = f(y) + h(y) \sum_{n \geq 1} a_n h(0)^{n-1} = f(y) - \frac{h(y)}{h(0)} f(0) + \frac{h(y)}{h(0)} f(h(0)).$$

Since $h(0) = \varepsilon$ and $f(\varepsilon) = 0$, the last expression on the right must be zero, and thus $f(y) = \frac{h(y)}{h(0)} f(0)$ is in I , which demonstrably riches us that we are done.

3. Concluding Discussion and Open Questions

We call a ring R π -*simply presented* if, for any $a \in R$, there exists an integer $n = n(a) \geq 2$ such that either $a^n = a$ or $a^n = 0$. If such a natural n is fixed, and so it does not depend on a , the ring R is just called n -*simply presented*.

It is rather clear that 2-simply presented rings R are just the Boolean ones. In fact, if $u \in U(R)$, then $u^2 = u$ and hence $u = 1$. This means that $U(R) = \{1\}$ whence $Nil(R) = \{0\}$. Consequently, for every $r \in R$, it must be that $r^2 = r$, as required. It is worthwhile noticing that this could also be deduced from [7].

Recall also that (see, e.g., [16]) a ring is *strongly clean* if every its element is the sum of a unit and an idempotent which commute each to other. Thus the following is true: *Any n -simply presented ring is strongly clean with nil Jacobson radical*. In fact, if R is such a ring and $a \in R$ with $a^n = 0$, one represents $a = (a-1)+1 \in U(R)+Id(R)$. If now $a^n = a$, one checks that $a = a^2b = ba^2$, where $b = a^{n-1} + a^{n-2} - 1$. Therefore, a is strongly regular element and, thereby, it follows from [16] that such an element a is strongly clean. Finally, this enables us that R is strongly clean. Also, it is pretty easy to find that $U(R)$ is torsion having $U^{n-1}(R) = \{1\}$ and thus, in view of [8], one infers that R is strongly m -torsion clean for some $m \leq n$. We, consequently, again appeal to [8] to get some expected subdirect isomorphism.

About the nil property of $J(R)$, choose an arbitrary $z \in J(R)$. If $z^n = 0$, we are set. If, however, $z^n = z$, then $z(1 - z^{n-1}) = 0$ which allows us to conclude that $z = 0$ since $1 - z^{n-1} \in U(R)$.

We close the work with the following three problems of interest and importance.

Problem 1. Describe n -simply presented rings for all $n \in \mathbb{N}$ as well as π -simply presented rings.

Problem 2. For an arbitrary natural $n \geq 1$ and a ring R such that its characteristic is the prime number $n + 1$, does it follow that R is a $J(n)$ -ring if, and only if, each element of R is a sum of n idempotents? Equivalently, is such a ring R a $J(n)$ -ring exactly when $R = Id(R) + \cdots + Id(R)$ (where the sum is taken n -times)?

We conjecture that the answer is "yes", provided that $n + 1$ runs over some special primes by noticing in this way that if $n = 1$, we just identify Boolean rings, and that if $n = 2$, the conjecture holds in the affirmative in accordance with [11, Theorem 1].

Problem 3. Describe those rings whose elements satisfy the polynomial identity $x^{n+1} - vx = 0$, where $n \in \mathbb{N}$ and $v^2 = 1$.

It is pretty obvious that weakly $J(n)$ -rings are a partial case of these rings, when $v = \pm 1$.

Acknowledgement

The author is very grateful to René Schoof for his proposal of an amended version of the proof of the main result presented here and his numerous valuable comments and proposals leading to an improvement of the exposition. He is also indebted to the expert referees for their constructive insightful suggestions made.

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DOI: 10.7862/rf.2018.5

Peter V. Danchev

email: danchev@math.bas.bg; pvdanchev@yahoo.com

ORCID: 0000-0002-2016-2336

Institute of Mathematics and Informatics

Bulgarian Academy of Sciences

”Acad. G. Bonchev” str., bl. 8, 1113 Sofia

BULGARIA

Received 29.06.2018

Accepted 06.09.2018

On Population Dynamics with Campaign on Contraception as Control Strategy

Virtue U. Ekhosuehi and Fidelis O. Chete

ABSTRACT: This work considers a population divided into two groups according to the adoption of contraception. The campaign in favour of contraception is modelled as a bounded optimal control problem within the framework of the logistic and the Malthusian models of population dynamics. The control is the fraction of non-adopters successfully educated on contraception. The objective is to maximise the number of non-adopters successfully educated on contraception over time. The optimisation problem is solved using the Pontryagin's maximum principle and the parameters of the model are estimated using the method of least squares.

AMS Subject Classification: 49J15, 92D25.

Keywords and Phrases: Contraception; Logistic model; Malthusian model; Optimal control model; Population.

1. Introduction

The subject of population expansion and control has received considerable attention in the literature (for instance, [23, 36]). The need to control the rate of population expansion has led to the introduction of several programmes on the use of contraceptives in many developing countries [21]. For example, in Nigeria (with which the authors are acquainted), the 'Get it together' campaign has been introduced to sensitise the masses on the use of contraceptives. The commonly used contraceptives include condom, diaphragm, vaginal cream/foaming tablets, oral contraceptives or pills, Intra-Uterine Device (IUD), implant and sterilization. These contraceptives are

accepted globally for birth control [9, 21]. There is no need to stress the different contraceptives and how it has been accepted worldwide as this can be found elsewhere [9, 26, 35, 36]. The practice of birth control in Japan, Russia, Puerto Rico, China, India and Cameroon has been reported in the literature [26]. The campaign on birth control is usually inexpensive for the social crusaders¹ (that is, the birth control advocates). This is because fertility control are subsidized in both high- and low-income countries [28]. On the whole condoms are distributed inexpensively and IUD insertion costs little [8]. This ought to be enough motivation for many adults to adopt the use of contraceptives. Regardless of the campaign to create awareness on use of contraceptives some individuals still hold on to their belief and may get involved in unprotected sex so much so that it may result to unplanned births, sexually transmitted diseases and child abandonment. Early research has shown that factors such as fear of the unknown effects of contraceptives, spouse's disapproval, religious and cultural beliefs, inadequate information and poor service of family planning clinics, may be barriers to use of contraceptives [21]. The difficulty in getting the population to accept the use of contraceptives is a problem, particularly in rural areas of developing countries. This problem is the motive for the continuous research on awareness creation on birth control with the use of contraceptives [21].

This paper considers a system made up of individuals that have attained the reproductive (or child-bearing) age and focuses on the use of a method of contraception (e.g., condoms). The study is aimed at deriving the optimal number of non-adopters that should be successfully educated on contraception using optimal control theory. Models based on optimal control theory are well-known in the literature [1, 5, 6]. The increased application of optimal control theory in ecology and natural resource management has been discussed as well [29]. In this present study, the state variables, which are the adopters and the non-adopters of the use of contraceptives, coexist. The control variable is the fraction of non-adopters successfully educated on contraception (i.e., the new adopters). This control is used as a proxy for the campaign effort. The use of a fraction of the population as a control is not novel as it has earlier been considered [23]. Before delving into the mathematical formulation of the population dynamics, we provide a review on methods of birth control and population models in Section 2. The model formulation is given in Section 3. Section 4 is concerned with a numerical illustration of the population dynamics, and Section 5 concludes the paper.

2. Related Works

Birth control is crucial to reducing population expansion [23] and poverty [36] in developing economies. The gains of birth control, *inter alia*, include: a smaller population, higher Gross National Product (GNP) per head and reduction in the ratio of dependent children to work-age population [8]. The methods of birth control are found in the literature [21, 32]. These include: the long-term methods such as Intra-Uterine Device (IUD), sterilization for both male and female and implant; the hor-

¹The crusaders in a developing country like Nigeria are mainly physicians, social workers and non-governmental organisations.

monal methods such as oral contraceptives, patch and ring; and the barrier methods such as the condoms, diaphragm and spermicides. It has been found that men who tend to assign contraceptive responsibility to women have more negative attitudes towards male contraceptive use [35]. The use of contraceptives among women and the factors that influence their use have been examined [4]. The factors include: being in a relationship, number of sex partners, pregnancy status, sexual activity status, age and social class. The prevalence of contraceptive use among women of reproductive age in Calabar, Nigeria has earlier been studied [7]. Lack of information as one of the factors that hinders the use of modern methods of birth control has been identified in Nigeria [21]. Essentially birth control is important in order to attain a steady-state growth rate of the population [23].

Population studies have gained prominence in the literature. Some of the early works on population dynamics are found in [16, 24]. Research on the relationship between population and economic growth has also been carried out [3]. In the study of pre-industrial societies, the Malthusian model of population dynamics occupies a central position in the analysis of the demographic change [2, 14, 19, 27]. A competing model of population dynamics to the Malthusian model is the logistic model. The Malthusian model is well-suited for populations that are not limited by space, while the logistic model is the standard model for single-species population growth [34]. The logistic equation, wherein the instantaneous birth rate per individual and the carrying capacity of the system are the parameters, is a more realistic model in terms of the birth and death processes of population growth [10]. The logistic curve provides reliable projections of the total population provided that there is a relationship among births, deaths and migration [13]. The work of [13] has been generalized [22]. It has been found that the population size of the logistic model with varying carrying capacity will eventually be gamma-distributed [25] and that population densities may exhibit oscillatory behaviour owing to seasonality [15].

Solutions to population models can be either exact or numerical. In [18] exact solutions to a quasi-linear first-order differential equation that models the growth of a single population subject to the logistic growth was found. However, in [11], numerical solutions based on the central finite difference method to the first-order hyperbolic equation of age and time variables which describes the one-sex models of population dynamics was provided. The existence of equilibrium solutions of a nonlinear structured population model and the local asymptotic stability of the equilibria has been proved [33].

Demographic and environmental variability and the possibility of extinction of a population may be modelled correctly in stochastic population models [37]. The structure variables include chronological age of each individual and the population size. In the stochastic population model it is possible to approximate the model as a diffusion [12]. In this case the population is at risk of extinction and the stochastic nature is caused by demographic and environmental fluctuations. The distribution of the extinction times in the stochastic logistic population model wherein the lifespan of any population can be described has been investigated [20]. In another study an alternative approach to the forecasting ability of the logistic population model was illustrated by modifying the assumption of the homoscedasticity of the error term

[17]. Later on, a method that can be used to fit a population model in the presence of observation error was described [31]. This study improves on the existing population models in the literature [2, 10, 14, 19, 27] by integrating the population dynamics and the effect of birth control campaign in the same dynamical system. The method of estimation of parameters for the state-transition equations is similar to the one found in [31].

3. Model Formulation

In this section, we complete the statement of the problem alongside with the underlying assumptions and provide the solution.

3.1. Model Development

Consider a system which consists of individuals of reproductive (or child-bearing) age. Individuals in the system are assumed to be divided into two mutually exclusive compartments: non-adopters ($x(t)$) and adopters ($y(t)$). We assume that the babies and the people in child-bearing age are distinguishable. This assumption is necessary because the transition from non-adopter to adopter is only applicable to people in child-bearing age. Only a portion of $x(t)$ can transfer to adopters and the new-born babies cannot be adopter or non-adopter in less than a legally allowable child-bearing age (say, 15 years). We assume that the adopters and the non-adopters coexist in the system and their interaction precludes personal issues such as the use of contraceptives. This assumption is consistent with the setting in rural communities in developing countries like Nigeria, where sex education is seen as either a taboo or immoral [21]. The non-adopters may change their opinion due to a re-orientation campaign on the use of contraceptives provided by birth control crusaders (e.g. physicians, social workers and non-governmental organisations) by whatever means. We assume that the cost of the campaign, which includes the cost of consultation with physicians and social workers on the use of contraceptives, is negligible. This is because fertility control is subsidized in both high- and low-income countries [8, 28]. As a result cost is not considered within the model formulation.

Let $\theta(t)$ be the fraction of 'non-adopters successfully educated on contraception' (new adopters hereinafter). Then $\theta(t)x(t)$ is the number of new adopters attributed to the re-orientation campaign on the use of contraceptives. The changes in the total population are induced by two effects: maturity (the attainability of reproductive age) and attrition. On attaining the reproductive age, the new member of the system may be either adopter or non-adopter. The loss in population may be attributed to attrition such as death, emigration, or attaining menopause. It is reasonable to assume that the population dynamics of the birth-control adopters and non-adopters are different. Consequent upon this, the dynamics of the system is assumed to follow the population growth models below

$$\frac{dx(t)}{dt} = \gamma_1 x(t) - \gamma_2 x^2(t) - \theta(t)x(t), \quad x(0) = x_0, \quad (3.1)$$

and

$$\frac{dy(t)}{dt} = \beta_1 y(t) + \theta(t)x(t), \quad y(0) = y_0, \quad (3.2)$$

where γ_1 is the intrinsic growth rate of non-adopters and β_1 is the intrinsic growth rate of adopters. The term $\gamma_2 x^2(t)$ in equation (3.1) is used to model the loss of non-adopters induced by the non-use of contraceptives. This is realistic because ill health, which is one of the consequences of increased family size [9], is worse-off for the poorest (most of whom are in rural areas) that are most ignorant and apathetic on use of contraceptives [8]. The effect of interaction between the adopters and the non-adopters is not considered. Equations (3.1) and (3.2) are analogous to the well-known logistic model and Malthusian model of population dynamics, respectively. In practice, equations (3.1) and (3.2) may be subjected to statistical analysis to ascertain their significance as an appropriate model for the two population compartments. We assume that the non-adopters may not all accept and practise contraception no matter the campaign owing to their religious beliefs. For this reason, the control $\theta(t)$ is assumed to satisfy the relation, $0 \leq \theta(t) < 1$.

Let $\{t : 0 < t \leq T\}$ be a fixed time horizon. Since efforts are made to increase the number of adopters, we define the objective function to be

$$\max_{\theta(t)} \int_0^T \theta(t)x(t)dt. \quad (3.3)$$

The optimal control problem posed by the objective function (3.3) and the state transition equations (3.1) and (3.2) together with the initial conditions and the bounds for the control is thus:

$$\max_{\theta(t)} \int_0^T \theta(t)x(t)dt,$$

subject to

$$\frac{dx(t)}{dt} = \gamma_1 x(t) - \gamma_2 x^2(t) - \theta(t)x(t),$$

$$\frac{dy(t)}{dt} = \beta_1 y(t) + \theta(t)x(t),$$

$$x(0) = x_0, \quad y(0) = y_0, \quad 0 \leq \theta(t) < 1, \quad t \in (0, T].$$

This model set-up is a bounded optimal control problem with the bounds being the closed-open interval $0 \leq \theta(t) < 1$.

3.2. Model Solution

To solve the bounded control problem, we employ the Pontryagin's maximum principle. The analysis of our solution is as follows.

We compute the control function, $\theta(t)$, by assuming that its value is at the lower bound or it is in the interior. Suppose the total population is a variable, then the

Hamiltonian, H , with arguments given as $(x(t), y(t), \theta(t), \lambda_1(t), \lambda_2(t))$, for the problem is

$$H = \theta(t)x(t) + \lambda_1(t)(\gamma_1x(t) - \gamma_2x^2(t) - \theta(t)x(t)) + \lambda_2(t)(\beta_1y(t) + \theta(t)x(t)), \quad (3.4)$$

where $\lambda_j(t), j = 1, 2$, is a multiplier function, which defines the marginal valuation of the productive capacity of the respective state variables. The influence equations for the state variables $x(t)$ and $y(t)$ are obtained as

$$\frac{d\lambda_1(t)}{dt} = -\frac{\partial H}{\partial x(t)} = -(\theta(t) + \lambda_1(t)(\gamma_1 - 2\gamma_2x(t) - \theta(t)) + \lambda_2(t)\theta(t)), \quad (3.5)$$

and

$$\frac{d\lambda_2(t)}{dt} = -\frac{\partial H}{\partial y(t)} = -\beta_1\lambda_2(t). \quad (3.6)$$

Thus,

$$\lambda_2(t) = \varphi \exp(-\beta_1t), \quad (3.7)$$

where φ is a constant. Equation (3.7) implies that the marginal value of the adopters decays exponentially with time. The Lagrangian function, L , for the Hamiltonian subject to the control bounds, $0 \leq \theta(t) < 1$, is

$$L = \theta(t)x(t) + \lambda_1(t)(\gamma_1x(t) - \gamma_2x^2(t) - \theta(t)x(t)) + \lambda_2(t)(\beta_1y(t) + \theta(t)x(t)) + \rho_1\theta(t) + \rho_2(1 - \theta(t)), \quad (3.8)$$

where ρ_1 and ρ_2 are the Lagrangian multipliers when the total population is a variable. The necessary conditions for $\theta(t)$ to maximise the bounded control problem are

$$\frac{\partial L}{\partial \theta(t)} = x(t) - \lambda_1(t)x(t) + \lambda_2(t)x(t) + \rho_1 - \rho_2 = 0, \rho_1 \geq 0, \rho_1\theta(t) = 0, \rho_2 = 0. \quad (3.9)$$

Without campaign on contraception, that is, $\theta(t) = 0$, we use equation (3.9) to get

$$\frac{d\lambda_1(t)}{dt} \geq \frac{d\lambda_2(t)}{dt}. \quad (3.10)$$

In this case, the optimal population, $x^*(t) + y^*(t)$, is obtained from the transition equations (3.1) and (3.2) by setting $\theta(t) = 0$. Thus the optimal sub-populations are found by solving the respective state transition equations. We obtain

$$x^*(t) = \frac{x_0 \exp(\gamma_1 t)}{\left(1 - \frac{\gamma_2}{\gamma_1} x_0 (1 - \exp(\gamma_1 t))\right)}, \quad (3.11)$$

and

$$y^*(t) = y_0 \exp(\beta_1 t). \quad (3.12)$$

We use the symbol * to denote the optimal value. To increase the adopters, there is a need for awareness campaign. Such a campaign is effective when at least one

non-adopter accepts and practises contraception. With campaign on contraception $\theta(t)$ lies in the open interval $(0, 1)$. In this open interval, we obtain from equation (3.9) that

$$\frac{d\lambda_1(t)}{dt} = \frac{d\lambda_2(t)}{dt}. \quad (3.13)$$

The optimal sub-populations, $x^*(t)$ and $y^*(t)$, are obtained using equation (3.7) to simplify the influence equation (3.5) and then the solution is substituted into the transition equations (3.1) and (3.2). We therefore obtain

$$x^*(t) = \frac{\gamma_1}{2\gamma_2} - \frac{\beta_1\varphi \exp(-\beta_1 t)}{2\gamma_2(1 + \varphi \exp(-\beta_1 t))}, \quad (3.14)$$

$$y^*(t) = \exp(\beta_1 t) \left(y_0 + \vartheta_1(1 - \exp(-\beta_1 t)) + \vartheta_2 \left(\ln(\zeta(t))^2 - \wp(t) \right) \right), \quad (3.15)$$

and

$$\theta^*(t) = \gamma_1 - \gamma_2 x^*(t) - \frac{x'(t)}{x^*(t)}, \quad (3.16)$$

where

$$\begin{aligned} \varphi &= \frac{\gamma_1 - 2\gamma_2 x_0}{\beta_1 - \gamma_1 + 2\gamma_2 x_0}, \\ \vartheta_1 &= \frac{\gamma_1^2}{4\gamma_2\beta_1}, \quad \vartheta_2 = \frac{\beta_1(\varphi + 1)}{2\gamma_2\varphi^2}, \quad \zeta(t) = \frac{1 + \varphi}{1 + \varphi \exp(-\beta_1 t)}, \\ \wp(t) &= \frac{\varphi(1 - \exp(-\beta_1 t))(1 + (1 + \varphi)(1 + \varphi \exp(-\beta_1 t)))}{(1 + \varphi)(1 + \varphi \exp(-\beta_1 t))}, \end{aligned}$$

and

$$x'(t) = \frac{\beta_1^2 \varphi \exp(-\beta_1 t)}{2\gamma_2(1 + \varphi \exp(-\beta_1 t))^2}.$$

The optimal solutions (3.14) – (3.16) are feasible, provided that $\theta(t) \in (0, 1)$.

On the other hand, if the total population is fixed, say N , then

$$\frac{dx(t)}{dt} + \frac{dy(t)}{dt} = 0,$$

so that $\beta_1 y(t) = -(\gamma_1 x(t) - \gamma_2 x^2(t))$. In this case, equation (3.4) becomes

$$H = \theta(t)x(t) + \lambda(t)(\gamma_1 x(t) - \gamma_2 x^2(t) - \theta(t)x(t)), \quad (3.17)$$

where $\lambda(t) = \lambda_1(t) - \lambda_2(t)$. The influence equations for the state variable $x(t)$ is

$$\frac{d\lambda(t)}{dt} = -(\theta(t) + \lambda(t)(\gamma_1 - 2\gamma_2 x(t) - \theta(t))). \quad (3.18)$$

The Lagrangian function, L , for the Hamiltonian subject to the control bounds, $0 \leq \theta(t) < 1$, would be

$$L = \theta(t)x(t) + \lambda(t) (\gamma_1 x(t) - \gamma_2 x^2(t) - \theta(t)x(t)) + w_1 \theta(t) + w_2 (1 - \theta(t)), \quad (3.19)$$

where w_1 and w_2 are the Lagrangian multipliers when the total population is fixed. With $\theta(t) = 0$, we obtain the same result as in equation (3.11). For $0 < \theta(t) < 1$ and $x(t) \neq 0$, we obtain $\lambda(t) = 1$,

$$x(t) = \frac{\gamma_1}{2\gamma_2}, \quad (3.20)$$

$$y(t) = N - \frac{\gamma_1}{2\gamma_2}, \quad (3.21)$$

and

$$\theta(t) = \frac{\gamma_1}{2}. \quad (3.22)$$

3.3. Estimation of the Parameters

In most cases, data on population are available at discrete periods, so that the discrete-time model may be used to approximate the continuous-time process. Suppose historical data are available for $t = 1, 2, \dots, \eta$. Then we estimate the parameters of the model by setting $\theta(t) = 0$ and applying the method of least squares. By so doing, we use the difference equation

$$x_t - x_{t-1} = \gamma_1 x_{t-1} - \gamma_2 x_{t-1}^2 + \text{error}, \quad (3.23)$$

as the discrete-time analogue of the logistic model. Thereafter, we apply the least squares method to get

$$\hat{\gamma}_1 = \left([1 \quad 0] ([\mathbf{X}_{-1} \quad \mathbf{X}_{-1}^2])' [\mathbf{X}_{-1} \quad \mathbf{X}_{-1}^2] \right)^{-1} [\mathbf{X}_{-1} \quad \mathbf{X}_{-1}^2]' \mathbf{X} - 1, \quad (3.24)$$

and

$$\hat{\gamma}_2 = \left([0 \quad -1] ([\mathbf{X}_{-1} \quad \mathbf{X}_{-1}^2])' [\mathbf{X}_{-1} \quad \mathbf{X}_{-1}^2] \right)^{-1} [\mathbf{X}_{-1} \quad \mathbf{X}_{-1}^2]' \mathbf{X}, \quad (3.25)$$

where \mathbf{X} is an $\eta \times 1$ vector of x_t , \mathbf{X}_{-1} is an $\eta \times 1$ vector of the one period lagged number of non-adopters, x_{t-1} , and \mathbf{X}_{-1}^2 is an $\eta \times 1$ vector of the squares of the one period lagged number of non-adopters, x_{t-1}^2 . We use the hat over the parameters to denote an estimate. Our approach towards obtaining the estimators for γ_1 and γ_2 is similar to the Solow method [31], except that the first-order derivative $dx(t)/dt$ is replaced by the first-order difference $x_t - x_{t-1}$ instead of the current value x_t as in [31].

Similarly, we obtain

$$\hat{\beta}_1 = [0 \quad 1] ([\mathbf{e} \quad \mathbf{\Omega}])' [\mathbf{e} \quad \mathbf{\Omega}]^{-1} [\mathbf{e} \quad \mathbf{\Omega}]' \mathbf{\Gamma}, \quad (3.26)$$

where $\mathbf{\Gamma}$ is an $\eta \times 1$ vector of $\ln y_t$, $\mathbf{\Omega}$ is an $\eta \times 1$ vector of time instants and \mathbf{e} is an $\eta \times 1$ vector of ones.

4. Numerical Illustration

The model defined by equations (3.14) – (3.16) is illustrated using values tabulated in Table 1 with the population size at the current period and the proportion of adopters given as 4.85×10^6 and 0.070, respectively.

Table 1: The population size over time

t (in years)	1	2	3	4	5	6	7
Population $\times 10^6$	2.00	2.28	2.65	3.12	3.45	4.19	4.33
Proportion of adopters	0.043	0.041	0.038	0.040	0.045	0.049	0.045

The parameters of the model are estimated using the least squares estimators in equations (3.24) – (3.26) as $\hat{\gamma}_1 = 0.2893$, $\hat{\gamma}_2 = 5.1278 \times 10^{-8}$ and $\hat{\beta}_1 = 0.1593$, respectively. These parameter estimates as well as the model are statistically significant at the 5% level from the output of the MATLAB program (see Appendix). Using the parameter estimates, numerical simulations for the optimal sub-populations and the optimal fraction of new adopters are carried out and depicted in Figure 1.

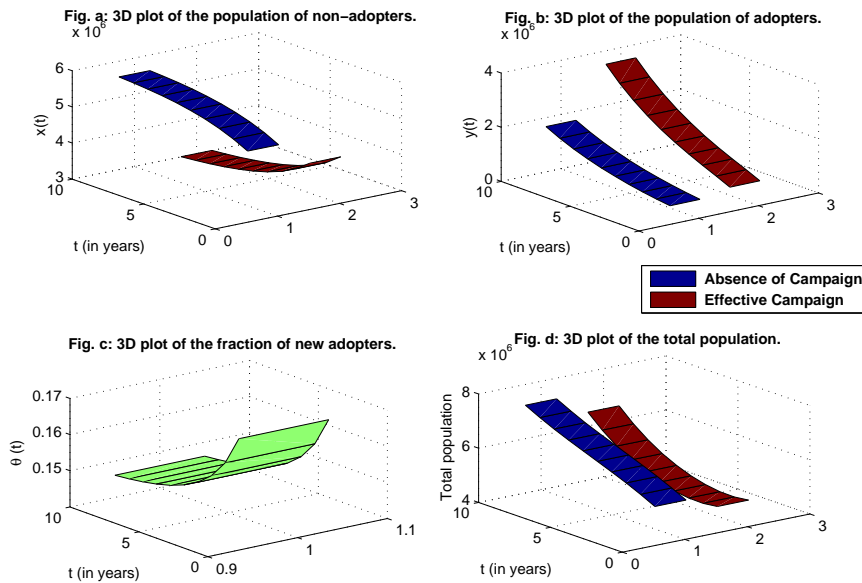


Figure 1: 3D plot of the population dynamics and the control

These simulations are performed in the MATLAB environment (see Appendix for the MATLAB source code). The simulations show that in the absence of campaign, the non-adopters and the adopters would continue to grow and the total population would rise rapidly. It is further shown that with the campaign, the non-adopters would

reduce drastically and the adopters would increase tremendously. On the whole, the campaign is able to achieve a reduction in the total population, even though the fraction of new adopters decreases steadily. These simulations therefore suggest that the campaign on contraception is a way to improve on the use of contraceptives. Since contraception could be used as a means of population control, the model proposed in this paper is a way out of reducing population expansion.

5. Conclusion

This study has provided an insight into population dynamics under birth control campaign. The approach is to develop a continuous-time optimal control model to serve as an alternative to the discrete-time approach as in [23]. The fundamentals of optimal control theory and the Malthusian and logistic models of population dynamics have been used as theoretical underpinnings. The method of least squares has been employed to provide the parameter estimates. Our approach to describing population defined by two sub-populations according to the use of contraceptives is very inspiring. Nonetheless, further work may be undertaken so as to incorporate the interaction between the adopters and the non-adopters. One of the innovations of this study is to integrate population dynamics and the effect of birth control campaign in the same dynamical system, by adding the term $\theta(t)x(t)$ in the formula for both types of population dynamics, as in equations (3.1) and (3.2). This setting may be improved upon. This can be achieved by taking into consideration the time lags of the two processes as well as the delay in adopting the use of contraceptives. The time lags of the two processes, that is the population dynamics by birth and death and the transition from non-adopter to adopter, may be different. The population dynamics by birth and death takes some decades, while the transition from non-adopter to adopter happens in shorter time period, only several months or a few years. Incorporating these variables will go a long way towards refining the model as the most likely approach may involve systems of delay differential equations. Finally in the absence of subsidy, the core check and balance of driving force in the birth control campaign would be cost. In this case a well-defined cost function needs to be figured out and added as a part of the objective function.

Acknowledgement

The authors would like to thank the anonymous reviewers for their valuable comments which have greatly improved the quality of the earlier manuscript.

Appendix

```
clc
P=[2; 2.28; 2.65; 3.12; 3.45; 4.19; 4.33]*10^6;
Xlag=(ones(length(P),1)-[0.043; 0.041; 0.038; 0.040; 0.045; 0.049; 0.045]).*P;
```



```

Ylag=P-Xlag;
P0=4.85*10^6; x0=(1-0.07)*P0; y0=(P0-x0);

X=[Xlag(2:length(P),1); x0];
Y=[Ylag(2:length(P),1); y0];
T=[1:length(P)]';

g1=[1 0]*(inv([Xlag Xlag.^2]'*[Xlag Xlag.^2])*[Xlag Xlag.^2]'*X)-1,

g2=[0 -1]*(inv([Xlag Xlag.^2]'*[Xlag Xlag.^2])*[Xlag Xlag.^2]'*X),

b1=[0 1]*(inv([ones(length(P),1) T]'*[ones(length(P),1) T])*[ones(length(P),1)
T]'*log(Ylag)),

%t-test for the significance of parameters.
I=eye(length(P)); N=length(P); p=2; s=inv([Xlag Xlag.^2]'*[Xlag Xlag.^2]);
betahat1=(inv([Xlag Xlag.^2]'*[Xlag Xlag.^2])*[Xlag Xlag.^2]'*X);
se=sqrt((X'*(I-[Xlag Xlag.^2]*s*[Xlag Xlag.^2]')*X)/(N-p)),
covbeta=(se^2)*s,
tcal0=(g1+1)/sqrt(covbeta(1,1)),
tcal1=-g2/sqrt(covbeta(2,2)),
tTab1=2.02;

%Decision rule.
if abs(tcal0)>2.02
    disp('Reject H0: the constant term, g1, is significant at 5% level')
else
    if abs(tcal0)<2.02
        disp('We do not reject H0: the constant term, g1, is not significant at 5%
level')
    end
end

if abs(tcal1)>2.02
    disp('Reject H0: the constant term, g2, is significant at 5% level')
else
    if abs(tcal1)< 2.02
        disp('We do not reject H0: the constant term, g2, is not significant at 5%
level')
    end
end

Rsquare=(betahat1'*[Xlag Xlag.^2]'*[Xlag Xlag.^2]*betahat1-N*(mean(X))^2) ...
/(X'*X-N*(mean(X))^2),

Fcal=(N-2)*Rsquare/(1-Rsquare),
Ftab=5.59;

if Fcal>Ftab

```

```

disp('Reject H0: the model 1 is significant at 5% level')
else
  if Fcal<Ftab
    disp('We do not reject H0: the model 1 is not significant at 5% level')
  end
end

%t-test for the significance of parameter beta.
I=eye(length(P)); N=length(P); p=2;
s2=inv([ones(length(P),1) T]'*[ones(length(P),1) T]);
betahat2=(inv([ones(length(P),1) T]'*[ones(length(P),1) T])* ...
[ones(length(P),1) T]'*log(Ylag)));
se2=sqrt((log(Ylag)'*(I-[ones(length(P),1) T]*s2*[ones(length(P),1) T]'))* ...
log(Ylag))/(N-p)),
covbeta2=(se2^2)*s2,
tcal2=b1/sqrt(covbeta2(2,2)),

%Decision rule.
if abs(tcal2)>2.02
  disp('Reject H0: the constant term, b1, is significant at 5% level')
else
  if abs(tcal2)<2.02
    disp('We do not reject H0: the constant term, b1, is not significant at 5%
level')
  end
end

Rsquare=(betahat2'*[ones(length(P),1) T]'*[ones(length(P),1) T]*betahat2-N*...
(mean(log(Ylag)))^2)/(log(Ylag)'*log(Ylag)-N*(mean(log(Ylag)))^2),

Fcal=(N-2)*Rsquare/(1-Rsquare),
Ftab=5.59;

if Fcal>Ftab
  disp('Reject H0: the model 2 is significant at 5% level')
else
  if Fcal<Ftab
    disp('We do not reject H0: the model 2 is not significant at 5% level')
  end
end

v=(g1-2*g2*x0)/(b1-g1+2*g2*x0);

n=10;
for t=1:n;
  x1(t)=(x0*exp(g1*t))/(1-(g2*x0/g1)*(1-exp(g1*t)));
  y1(t)=y0*exp(b1*t);
  x(t)=(g1/(2*g2))-(b1*v*exp(-b1*t))/(2*g2*(1+v*exp(-b1*t)));

```

```

    th(t)=g1-g2*((g1/(2*g2))-(b1*v*exp(-b1*t))/(2*g2*(1+v*exp(-b1*t))))...
-((b1^2)*v*exp(-b1*t))/((2*g2*(1+v*exp(-b1*t))^2)*((g1/(2*g2))...
-(b1*v*exp(-b1*t))/(2*g2*(1+v*exp(-b1*t)))));
    z1=(v*(1-exp(-b1*t)))*(1+(1+v)*(1+v*exp(-b1*t)));
    z2=(1+v)*(1+v*exp(-b1*t));
    k1=b1*(v+1)/(2*g2*v^2); k0=g2^2/(4*g2*b1); m=((1+v)/(1+v*exp(-b1*t)))^2;
y(t)=exp(b1*t)*(y0+k0*(1-exp(-b1*t))+k1*(log(((1+v)/(1+v*exp(-b1*t)))^2)...
-((v*(1-exp(-b1*t)))*(1+(1+v)*(1+v*exp(-b1*t)))))/((1+v)*(1+v*exp(-b1*t)))));
end

clf
subplot(2,2,1)
t=1:n;
ribbon(t',[x1' x'],0.5)
zlabel('x(t)')
ylabel('t (in years)')
title ('Fig. a: 3D plot of the population of non-adopters.')

subplot(2,2,2)
t=1:n;
ribbon(t',[y1' y'],0.5)
zlabel('y(t)')
ylabel('t (in years)')
title ('Fig. b: 3D plot of the population of adopters.')

subplot(2,2,3)
t=1:n;
ribbon(t',th',0.1)
zlabel('theta (t)')
ylabel('t (in years)')
title ('Fig. c: 3D plot of the fraction of new adopters.')

subplot(2,2,4)
t=1:n;
ribbon(t',[(x1'+y1'), (x'+y')],0.5)
zlabel('Total population')
ylabel('t (in years)')
legend('Absence of Campaign','Effective Campaign')
title ('Fig. d: 3D plot of the total population.')

```

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DOI: 10.7862/rf.2018.6

Virtue U. Ekhosuehi

email: virtue.ekhosuehi@uniben.edu

ORCID: 0000-0002-7796-1657

Department of Mathematics

University of Benin

Benin City, Edo State

NIGERIA

Fidelis O. Chete

email: odichet@yahoo.com

ORCID: 0000-0001-8445-2570

Department of Computer Science

University of Benin

Benin City, Edo State

NIGERIA

Received 05.10.2017

Accepted 13.04.2018

Some Triple Difference Rough Cesàro and Lacunary Statistical Sequence Spaces

Ayhan Esi and Nagarajan Subramanian

ABSTRACT: We generalized the concepts in probability of rough Cesàro and lacunary statistical by introducing the difference operator Δ_γ^α of fractional order, where α is a proper fraction and $\gamma = (\gamma_{mnk})$ is any fixed sequence of nonzero real or complex numbers. We study some properties of this operator involving lacunary sequence θ and arbitrary sequence $p = (p_{rst})$ of strictly positive real numbers and investigate the topological structures of related with triple difference sequence spaces.

The main focus of the present paper is to generalized rough Cesàro and lacunary statistical of triple difference sequence spaces and investigate their topological structures as well as some inclusion concerning the operator Δ_γ^α .

AMS Subject Classification: 40F05, 40J05, 40G05.

Keywords and Phrases: Analytic sequence; Musielak-Orlicz function; Triple sequences; Chi sequence; Cesàro summable; Lacunary statistical convergence.

Introduction

A triple sequence (real or complex) can be defined as a function $x : \mathbb{N} \times \mathbb{N} \times \mathbb{N} \rightarrow \mathbb{R} (\mathbb{C})$, where \mathbb{N}, \mathbb{R} and \mathbb{C} denote the set of natural numbers, real numbers and complex numbers respectively. The different types of notions of triple sequence was introduced and investigated at the initial by Sahiner et al. [10, 11], Esi et al. [1-3], Dutta et al. [4], Subramanian et al. [12-15], Debnath et al. [5] and many others.

A triple sequence $x = (x_{mnk})$ is said to be triple analytic if

$$\sup_{m,n,k} |x_{mnk}|^{\frac{1}{m+n+k}} < \infty.$$

The space of all triple analytic sequences are usually denoted by Λ^3 . A triple sequence $x = (x_{mnk})$ is called triple gai sequence if

$$((m+n+k)!|x_{mnk}|)^{\frac{1}{m+n+k}} \rightarrow 0 \text{ as } m, n, k \rightarrow \infty.$$

The notion of difference sequence spaces (for single sequences) was introduced by Kizmaz [6] as follows

$$Z(\Delta) = \{x = (x_k) \in w : (\Delta x_k) \in Z\}$$

for $Z = c, c_0$ and ℓ_∞ , where $\Delta x_k = x_k - x_{k+1}$ for all $k \in \mathbb{N}$.

The difference triple sequence space was introduced by Debnath et al. (see [5]) and is defined as

$$\begin{aligned} \Delta x_{mnk} &= x_{mnk} - x_{m,n+1,k} - x_{m,n,k+1} + x_{m,n+1,k+1} \\ &\quad - x_{m+1,n,k} + x_{m+1,n+1,k} + x_{m+1,n,k+1} - x_{m+1,n+1,k+1} \end{aligned}$$

and $\Delta^0 x_{mnk} = \langle x_{mnk} \rangle$.

1. Some New Difference Triple Sequence Spaces with Fractional Order

Let $\Gamma(\alpha)$ denote the Euler gamma function of a real number α . Using the definition $\Gamma(\alpha)$ with $\alpha \notin \{0, -1, -2, -3, \dots\}$ can be expressed as an improper integral as follows: $\Gamma(\alpha) = \int_0^\infty e^{-x} x^{\alpha-1} dx$, where α is a positive proper fraction. We have defined the generalized fractional triple sequence spaces of difference operator

$$\Delta_\gamma^\alpha(x_{mnk}) = \sum_{u=0}^\infty \sum_{v=0}^\infty \sum_{w=0}^\infty \frac{(-1)^{u+v+w} \Gamma(\alpha+1)}{(u+v+w)! \Gamma(\alpha - (u+v+w) + 1)} x_{m+u, n+v, k+w}. \quad (1.1)$$

In particular, we have

$$\begin{aligned} (i) \quad \Delta^{\frac{1}{2}}(x_{mnk}) &= x_{mnk} - \frac{1}{16} x_{m+1, n+1, k+1} - \dots; \\ (ii) \quad \Delta^{-\frac{1}{2}}(x_{mnk}) &= x_{mnk} + \frac{5}{16} x_{m+1, n+1, k+1} + \dots; \\ (iii) \quad \Delta^{\frac{2}{3}}(x_{mnk}) &= x_{mnk} - \frac{4}{81} x_{m+1, n+1, k+1} - \dots. \end{aligned}$$

Now we determine the new classes of triple difference sequence spaces $\Delta_\gamma^\alpha(x)$ as follows:

$$\Delta_\gamma^\alpha(x) = \{x : (x_{mnk}) \in w^3 : (\Delta_\gamma^\alpha x) \in X\}, \quad (1.2)$$

where

$$\Delta_\gamma^\alpha(x_{mnk}) = \sum_{u=0}^\infty \sum_{v=0}^\infty \sum_{w=0}^\infty \frac{(-1)^{u+v+w} \Gamma(\alpha+1)}{(u+v+w)! \Gamma(\alpha - (u+v+w) + 1)} x_{m+u, n+v, k+w}$$

and

$$\begin{aligned} X \in \chi_f^{3\Delta}(x) &= \chi_f^3(\Delta_\gamma^\alpha x_{mnk}) = \mu_{mnk}(\Delta_\gamma^\alpha x) \\ &= \left[f_{mnk} \left(((m+n+k)! |\Delta_\gamma^\alpha|)^{\frac{1}{m+n+k}}, \bar{0} \right) \right]. \end{aligned}$$

Proposition 1.1.

- (i) For a proper fraction α $\Delta^\alpha : W \times W \times W \rightarrow W \times W \times W$ defined by equation of (2.1) is a linear operator;
- (ii) For $\alpha, \beta > 0$, $\Delta^\alpha (\Delta^\beta (x_{mnk})) = \Delta^{\alpha+\beta} (x_{mnk})$ and $\Delta^\alpha (\Delta^{-\alpha} (x_{mnk})) = x_{mnk}$.

Proof: Omitted.

Proposition 1.2. For a proper fraction α and f be an Musielak-Orlicz function, if $\chi_f^3(x)$ is a linear space, then $\chi_f^{3\Delta^\alpha}(x)$ is also a linear space.

Proof: Omitted.

2. Definitions and Preliminaries

Throughout the article $w^3, \chi^3(\Delta), \Lambda^3(\Delta)$ denote the spaces of all, triple gai difference sequence spaces and triple analytic difference sequence spaces respectively. Subramanian et al. (see [12]) introduced by a triple entire sequence spaces, triple analytic sequences spaces and triple gai sequence spaces. The triple sequence spaces of $\chi^3(\Delta), \Lambda^3(\Delta)$ are defined as follows:

$$\begin{aligned} \chi^3(\Delta) &= \left\{ x \in w^3 : ((m+n+k)! |\Delta x_{mnk}|)^{1/m+n+k} \rightarrow 0 \text{ as } m, n, k \rightarrow \infty \right\}, \\ \Lambda^3(\Delta) &= \left\{ x \in w^3 : \sup_{m,n,k} |\Delta x_{mnk}|^{1/m+n+k} < \infty \right\}. \end{aligned}$$

Definition 2.1. An Orlicz function ([see [7]) is a function $M : [0, \infty) \rightarrow [0, \infty)$ which is continuous, non-decreasing and convex with $M(0) = 0, M(x) > 0$, for $x > 0$ and $M(x) \rightarrow \infty$ as $x \rightarrow \infty$. If convexity of Orlicz function M is replaced by $M(x+y) \leq M(x) + M(y)$, then this function is called modulus function.

Lindenstrauss and Tzafriri ([8]) used the idea of Orlicz function to construct Orlicz sequence space.

A sequence $g = (g_{mn})$ defined by

$$g_{mn}(v) = \sup \{ |v| u - (f_{mnk})(u) : u \geq 0 \}, \quad m, n, k = 1, 2, \dots$$

is called the complementary function of a Musielak-Orlicz function f . For a given Musielak-Orlicz function f , (see [9]) the Musielak-Orlicz sequence space t_f is defined as follows

$$t_f = \left\{ x \in w^3 : I_f(|x_{mnk}|)^{1/m+n+k} \rightarrow 0 \text{ as } m, n, k \rightarrow \infty \right\},$$

where I_f is a convex modular defined by

$$I_f(x) = \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \sum_{k=1}^{\infty} f_{mnk} (|x_{mnk}|)^{1/m+n+k}, x = (x_{mnk}) \in t_f.$$

We consider t_f equipped with the Luxemburg metric

$$d(x, y) = \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \sum_{k=1}^{\infty} f_{mnk} \left(\frac{|x_{mnk}|^{1/m+n+k}}{mnk} \right)$$

is an extended real number.

Definition 2.2. Let α be a proper fraction. A triple difference sequence spaces of $\Delta_\gamma^\alpha x = (\Delta_\gamma^\alpha x_{mnk})$ is said to be Δ_γ^α strong Cesàro summable to $\bar{0}$ if

$$\lim_{uvw \rightarrow \infty} \frac{1}{uvw} \sum_{m=1}^u \sum_{n=1}^v \sum_{k=1}^w |\Delta_\gamma^\alpha x_{mnk}, \bar{0}| = 0.$$

In this we write $\Delta_\gamma^\alpha x_{mnk} \xrightarrow{[C,1,1,1]} \Delta_\gamma^\alpha x_{mnk}$. The set of all Δ_γ^α strong Cesàro summable triple sequence spaces is denoted by $[C, 1, 1, 1]$.

Definition 2.3. Let α be a proper fraction and β be a nonnegative real number. A triple difference sequence spaces of $\Delta_\gamma^\alpha x = (\Delta_\gamma^\alpha x_{mnk})$ is said to be Δ_γ^α rough strong Cesàro summable in probability to a random variable $\Delta_\gamma^\alpha x : W \times W \times W \rightarrow \mathbb{R} \times \mathbb{R} \times \mathbb{R}$ with respect to the roughness of degree β if for each $\epsilon > 0$,

$$\lim_{uvw \rightarrow \infty} \frac{1}{uvw} \sum_{m=1}^u \sum_{n=1}^v \sum_{k=1}^w P(|\Delta_\gamma^\alpha x_{mnk}, \bar{0}| \geq \beta + \epsilon) = 0.$$

In this case we write $\Delta_\gamma^\alpha x_{mnk} \xrightarrow{\beta [C,1,1,1]^{P\Delta}} \Delta_\gamma^\alpha x_{mnk}$. The class of all $\beta \Delta_\gamma^\alpha$ -strong Cesàro summable triple sequence spaces of random variables in probability and it will be denoted by $\beta [C, 1, 1, 1]^{P\Delta}$.

3. Rough Cesàro Summable of Triple of Δ_γ^α

In this section by using the operator Δ_γ^α , we introduce some new triple difference sequence spaces of rough Cesàro summable involving lacunary sequences θ and arbitrary sequence $p = (p_{rst})$ of strictly positive real numbers.

If α be a proper fraction and β be nonnegative real number. A triple difference sequence spaces of $\Delta_\gamma^\alpha X = (\Delta_\gamma^\alpha x_{mnk})$ is said to be Δ_γ^α -rough strong Cesàro summable in probability to a random variable $\Delta_\gamma^\alpha X : W \times W \times W \rightarrow \mathbb{R} \times \mathbb{R} \times \mathbb{R}$ with respect to the roughness of degree β if for each $\epsilon > 0$ then define the triple difference sequence spaces as follows:

(i)

$$C(\Delta_\gamma^\alpha, p)_\theta = \sum_{r=1}^{\infty} \sum_{s=1}^{\infty} \sum_{t=1}^{\infty} P \left(f_{mnk} \left[\left| \frac{1}{h_{rst}} \sum_{(mnk) \in I_{rst}} \Delta_\gamma^\alpha X \right|^{p_{rst}} \right] \geq \beta + \epsilon \right) < \infty.$$

In this case we write $C(\Delta_\gamma^\alpha, p)_\theta \xrightarrow{[C,1,1,1]^{P\Delta}} C(\Delta_\gamma^\alpha, p)_\theta$. The class of all $\beta C(\Delta_\gamma^\alpha, p)_\theta$ – rough strong Cesàro summable triple sequence spaces of random variables in probability and it will be denoted by $\beta[C, 1, 1, 1]^{P\Delta}$.

(ii)

$$C[\Delta_\gamma^\alpha, p]_\theta = \sum_{r=1}^{\infty} \sum_{s=1}^{\infty} \sum_{t=1}^{\infty} P \left(\frac{1}{h_{rst}} \sum_{(mnk) \in I_{rst}} f_{mnk} [|\Delta_\gamma^\alpha X|^{p_{rst}}] \geq \beta + \epsilon \right) < \infty.$$

In this case we write $C[\Delta_\gamma^\alpha, p]_\theta \xrightarrow{[C,1,1,1]^{P\Delta}} C[\Delta_\gamma^\alpha, p]_\theta$. The class of all $\beta C[\Delta_\gamma^\alpha, p]_\theta$ – rough strong Cesàro summable triple sequence spaces of random variables in probability.

(iii)

$$C_\Lambda(\Delta_\gamma^\alpha, p)_\theta = P \left(f_{mnk} \left[\left| \frac{1}{h_{rst}} \sum_{(mnk) \in I_{rst}} \Delta_\gamma^\alpha X \right|^{p_{rst}} \right] \geq \beta + \epsilon \right) < \infty.$$

In this case we write $C_\Lambda(\Delta_\gamma^\alpha, p)_\theta \xrightarrow{[C,1,1,1]^{P\Delta}} C_\Lambda(\Delta_\gamma^\alpha, p)_\theta$. The class of all $\beta C_\Lambda(\Delta_\gamma^\alpha, p)_\theta$ – rough strong Cesàro summable triple sequence spaces of random variables in probability.

(iv)

$$C_\Lambda[\Delta_\gamma^\alpha, p]_\theta = \frac{1}{h_{rst}} \sum_{(mnk) \in I_{rst}} P(f_{mnk} [|\Delta_\gamma^\alpha X|^{p_{rst}}] \geq \beta + \epsilon) < \infty.$$

In this case we write $C_\Lambda[\Delta_\gamma^\alpha, p]_\theta \xrightarrow{[C,1,1,1]^{P\Delta}} C_\Lambda[\Delta_\gamma^\alpha, p]_\theta$. The class of all $\beta C_\Lambda[\Delta_\gamma^\alpha, p]_\theta$ – rough strong Cesàro summable triple sequence spaces of random variables in probability.

(v)

$$N(\Delta_\gamma^\alpha, p)_\theta = \lim_{rst \rightarrow \infty} \frac{1}{h_{rst}} \sum_{(mnk) \in I_{rst}} P(f_{mnk} [|\Delta_\gamma^\alpha X, \bar{0}|^{p_{rst}}] \geq \beta + \epsilon) = 0.$$

In this case we write $N(\Delta_\gamma^\alpha, p)_\theta \xrightarrow{[C,1,1,1]^{P\Delta}} N(\Delta_\gamma^\alpha, p)_\theta$. The class of all $\beta N(\Delta_\gamma^\alpha, p)_\theta$ – rough strong Cesàro summable triple sequence spaces of random variables in probability.

Theorem 3.1. *If α be a proper fraction, β be nonnegative real number, f be an Musielak-Orlicz function and (p_{rst}) is a triple difference analytic sequence then the sequence spaces $C(\Delta_\gamma^\alpha, p)_\theta$, $C[\Delta_\gamma^\alpha, p]_\theta$, $C_\Lambda(\Delta_\gamma^\alpha, p)_\theta$, $C_\Lambda[\Delta_\gamma^\alpha, p]_\theta$ and $N(\Delta_\gamma^\alpha, p)_\theta$ are linear spaces.*

Proof: Because the linearity may be proved in a similar way for each of the sets of triple sequences, hence it is omitted.

Theorem 3.2. *If α be a proper fraction, β be nonnegative real number, f be an Musielak-Orlicz function and (p_{rst}) , for all $r, s, t \in \mathbb{N}$, then the triple difference sequence spaces $C[\Delta_\gamma^\alpha, p]_\theta$ is a BK-space with the Luxemburg metric is defined by*

$$d(x, y)_1 = \sum_{u=0}^{\infty} \sum_{v=0}^{\infty} \sum_{w=0}^{\infty} f_{mnk} \left[\frac{\gamma_{uvw} x_{uvw}}{uvw} \right] \\ + \lim_{uvw \rightarrow \infty} \frac{1}{uvw} \sum_{r=1}^{\infty} \sum_{s=1}^{\infty} \sum_{t=1}^{\infty} f_{mnk} \left[P \left(\frac{1}{h_{rst}} \sum_{(m,n,k) \in I_{rst}} |\Delta_\gamma^\alpha x|^p \right) \geq \beta + \epsilon \right]^{1/p}, \quad 1 \leq p.$$

Also if $p_{rst} = 1$ for all $(r, s, t) \in \mathbb{N}$, then the triple difference spaces $C_\Lambda[\Delta_\gamma^\alpha, p]_\theta$ and $N(\Delta_\gamma^\alpha, p)_\theta$ are BK-spaces with the Luxemburg metric is defined by

$$d(x, y)_2 = \sum_{u=0}^{\infty} \sum_{v=0}^{\infty} \sum_{w=0}^{\infty} f_{mnk} \left[\frac{\gamma_{uvw} x_{uvw}}{uvw} \right] \\ + \lim_{uvw \rightarrow \infty} \frac{1}{uvw} \frac{1}{h_{rst}} \sum_{(m,n,k) \in I_{rst}} f_{mnk} [P(|\Delta_\gamma^\alpha x|) \geq \beta + \epsilon].$$

Proof. We give the proof for the space $C_\Lambda[\Delta_\gamma^\alpha, p]_\theta$ and that of others followed by using similar techniques.

Suppose (x^n) is a Cauchy sequence in $C_\Lambda[\Delta_\gamma^\alpha, p]_\theta$, where $x^n = (x_{ij\ell})^n$ and $x^m = (x_{ij\ell}^m)$ are two elements in $C_\Lambda[\Delta_\gamma^\alpha, p]_\theta$. Then there exists a positive integer $n_0(\epsilon)$ such that $|x^n - x^m| \rightarrow 0$ as $m, n \rightarrow \infty$ for all $m, n \geq n_0(\epsilon)$ and for each $i, j, \ell \in \mathbb{N}$. Therefore

$$\begin{bmatrix} x_{uvw}^{11} & x_{uvw}^{12} & \dots & \dots \\ x_{uvw}^{21} & x_{uvw}^{22} & \dots & \dots \\ \vdots & \vdots & \ddots & \ddots \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} \Delta_\gamma^\alpha x_{ij\ell}^{11} & \Delta_\gamma^\alpha x_{ij\ell}^{12} & \dots & \dots \\ \Delta_\gamma^\alpha x_{ij\ell}^{21} & \Delta_\gamma^\alpha x_{ij\ell}^{22} & \dots & \dots \\ \vdots & \vdots & \ddots & \ddots \end{bmatrix}$$

are Cauchy sequences in complex field \mathbb{C} and $C_\Lambda[\Delta_\gamma^\alpha, p]_\theta$ respectively. By using the completeness of \mathbb{C} and $C_\Lambda[\Delta_\gamma^\alpha, p]_\theta$ we have that they are convergent and suppose

that $x_{ij\ell}^n \rightarrow x_{ij\ell}$ in \mathbb{C} and $(\Delta_\gamma^\alpha x_{ij\ell}^n) \rightarrow y_{ij\ell}$ in $C_\Lambda [\Delta_\gamma^\alpha, p]_\theta$ for each $i, j, \ell \in \mathbb{N}$ as $n \rightarrow \infty$. Then we can find a triple sequence space of $(x_{ij\ell})$ such that $y_{ij\ell} = \Delta_\gamma^\alpha x_{ij\ell}$ for $i, j, \ell \in \mathbb{N}$. These $x_{ij\ell}^n$ can be interpreted as

$$x_{ij\ell} = \frac{1}{\gamma_{ij\ell}} \sum_{u=1}^{i-m} \sum_{v=1}^{j-n} \sum_{w=1}^{\ell-k} \Delta_\gamma^\alpha y_{uvw}$$

$$= \frac{1}{\gamma_{ij\ell}} \sum_{u=1}^i \sum_{v=1}^j \sum_{w=1}^\ell \Delta_\gamma^\alpha y_{u-m, v-n, w-k}, (y_{1-m, 1-n, 1-k} = y_{2-m, 2-n, 2-k} = \dots = y_{000} = 0).$$

for sufficiently large (i, j, ℓ) ; that is,

$$(\Delta_\gamma^\alpha x^n) = \begin{bmatrix} \Delta_\gamma^\alpha x_{ij\ell}^{11} & \Delta_\gamma^\alpha x_{ij\ell}^{12} & \dots & \dots \\ \Delta_\gamma^\alpha x_{ij\ell}^{21} & \Delta_\gamma^\alpha x_{ij\ell}^{22} & \dots & \dots \\ \vdots & \vdots & \vdots & \vdots \end{bmatrix}$$

converges to $(\Delta_\gamma^\alpha x_{ij\ell})$ for each $i, j, \ell \in \mathbb{N}$ as $n \rightarrow \infty$. Thus $|x^m - x|_2 \rightarrow 0$ as $m \rightarrow \infty$. Since $C_\Lambda [\Delta_\gamma^\alpha, p]_\theta$ is a Banach Luxemburg metric with continuous coordinates, that is $|x^n - x|_2 \rightarrow 0$ implies $|x_{ij\ell}^n - x_{ij\ell}| \rightarrow 0$ for each $i, j, \ell \in \mathbb{N}$ as $n \rightarrow \infty$, this shows that $C_\Lambda [\Delta_\gamma^\alpha, p]_\theta$ is a BK-space. \square

Theorem 3.3. *If α be a proper fraction, β be nonnegative real number, f be an Musielak-Orlicz function and (p_{rst}) , for all $r, s, t \in \mathbb{N}$, then the triple difference sequence space $C(\Delta_\gamma^\alpha, p)_\theta$ is a BK-space with the Luxemburg metric is defined by*

$$d(x, y)_3 = \sum_{u=0}^\infty \sum_{v=0}^\infty \sum_{w=0}^\infty f_{mnk} \left[\frac{\gamma_{uvw} x_{uvw}}{uvw} \right]$$

$$+ \lim_{uvw \rightarrow \infty} \frac{1}{uvw} \sum_{r=1}^\infty \sum_{s=1}^\infty \sum_{t=1}^\infty f_{mnk} \left[P \left(\left| \frac{1}{h_{rst}} \sum_{(m,n,k) \in I_{rst}} \Delta_\gamma^\alpha x \right|^p \right) \geq \beta + \epsilon \right]^{1/p}, \quad 1 \leq p.$$

Also if $p_{rst} = 1$ for all $(r, s, t) \in \mathbb{N}$, then the triple difference spaces $C_\Lambda(\Delta_\gamma^\alpha, p)_\theta$ is a BK-spaces with the Luxemburg metric is defined by

$$d(x, y)_4 = \sum_{u=0}^\infty \sum_{v=0}^\infty \sum_{w=0}^\infty f_{mnk} \left[\frac{\gamma_{uvw} X_{uvw}}{uvw} \right]$$

$$+ \lim_{uvw \rightarrow \infty} \frac{1}{uvw} f_{mnk} \left[P \left(\left| \frac{1}{h_{rst}} \sum_{(m,n,k) \in I_{rst}} \Delta_\gamma^\alpha x \right| \right) \geq \beta + \epsilon \right].$$

Proof: The proof follows from Theorem 4.2.

Now, we can present the following theorem, determining some inclusion relations without proof, since it is a routine verification.

Theorem 3.4. *Let α, ξ be two positive proper fractions $\alpha > \xi > 0$ and β be two nonnegative real number, f be an Musielak-Orlicz function and $(p_{rst}) = p$, for each $r, s, t \in \mathbb{N}$ be given. Then the following inclusions are satisfied:*

- (i) $C(\Delta_{\gamma}^{\xi}, p)_{\theta} \subset C(\Delta_{\gamma}^{\alpha}, p)_{\theta}$;
- (ii) $C[\Delta_{\gamma}^{\xi}, p]_{\theta} \subset C[\Delta_{\gamma}^{\alpha}, p]_{\theta}$;
- (iii) $C(\Delta_{\gamma}^{\alpha}, p)_{\theta} \subset C(\Delta_{\gamma}^{\alpha}, q)_{\theta}$, $0 < p < q$.

4. Rough Lacunary Statistical Convergence of Triple of Δ_{γ}^{α}

In this section by using the operator Δ_{γ}^{α} , we introduce some new triple difference sequence spaces involving rough lacunary statistical sequences spaces and arbitrary sequence $p = (p_{rst})$ of strictly positive real numbers.

Definition 4.1. The triple sequence $\theta_{i,\ell,j} = \{(m_i, n_{\ell}, k_j)\}$ is called triple lacunary if there exist three increasing sequences of integers such that

$$\begin{aligned} m_0 &= 0, h_i = m_i - m_{r-1} \rightarrow \infty \text{ as } i \rightarrow \infty \text{ and} \\ n_0 &= 0, \bar{h}_{\ell} = n_{\ell} - n_{\ell-1} \rightarrow \infty \text{ as } \ell \rightarrow \infty, \\ k_0 &= 0, \bar{h}_j = k_j - k_{j-1} \rightarrow \infty \text{ as } j \rightarrow \infty. \end{aligned}$$

Let $m_{i,\ell,j} = m_i n_{\ell} k_j$, $h_{i,\ell,j} = h_i \bar{h}_{\ell} \bar{h}_j$, and $\theta_{i,\ell,j}$ is determine by

$$\begin{aligned} I_{i,\ell,j} &= \{(m, n, k) : m_{i-1} < m < m_i \text{ and } n_{\ell-1} < n \leq n_{\ell} \text{ and } k_{j-1} < k \leq k_j\}, \\ q_i &= \frac{m_i}{m_{i-1}}, \bar{q}_{\ell} = \frac{n_{\ell}}{n_{\ell-1}}, \bar{q}_j = \frac{k_j}{k_{j-1}}. \end{aligned}$$

Definition 4.2. Let α be a proper fraction, f be an Musielak-Orlicz function and $\theta = \{m_r n_s k_t\}_{(rst) \in \mathbb{N} \cup 0}$ be the triple difference lacunary sequence spaces of $(\Delta_{\gamma}^{\alpha} X_{mnk})$ is said to be Δ_{γ}^{α} -lacunary statistically convergent to a number $\bar{0}$ if for any $\epsilon > 0$,

$$\lim_{rst \rightarrow \infty} \frac{1}{h_{rst}} |\{(m, n, k) \in I_{rst} : f_{mnk} [|\Delta_{\gamma}^{\alpha} X_{mnk}, \bar{0}|] \geq \epsilon\}| = 0,$$

where

$$\begin{aligned} I_{r,s,t} &= \{(m, n, k) : m_{r-1} < m < m_r \text{ and } n_{s-1} < n \leq n_s \text{ and } k_{t-1} < k \leq k_t\}, \\ q_r &= \frac{m_r}{m_{r-1}}, \bar{q}_s = \frac{n_s}{n_{s-1}}, \bar{q}_t = \frac{k_t}{k_{t-1}}. \end{aligned}$$

In this case write $\Delta_{\gamma}^{\alpha} X \rightarrow^{S_{\theta}} \Delta_{\gamma}^{\alpha} x$.

Definition 4.3. If α be a proper fraction, β be nonnegative real number, f be an Musielak-Orlicz function and $\theta = \{m_r n_s k_t\}_{(r,s,t) \in \mathbb{N}^3 \cup (0,0,0)}$ be the triple difference sequence spaces of lacunary. A number X is said to be $\Delta_\gamma^\alpha - N_\theta$ -convergent to a real number $\bar{0}$ if for every $\epsilon > 0$,

$$\lim_{rst \rightarrow \infty} \frac{1}{h_{rst}} \sum_{m \in I_r} \sum_{n \in I_s} \sum_{k \in I_t} f_{mnk} [|\Delta_\gamma^\alpha X_{mnk}, \bar{0}|] = 0.$$

In this case we write $\Delta_\gamma^\alpha X_{mnk} \rightarrow^{N_\theta} \bar{0}$.

Definition 4.4. Let α be a proper fraction, β be nonnegative real number, f be an Musielak-Orlicz function and arbitrary sequence $p = (p_{rst})$ of strictly positive real numbers. A triple difference sequence spaces of random variables is said to be Δ_γ^α -rough lacunary statistically convergent in probability to $\Delta_\gamma^\alpha X : W \times W \times W \rightarrow \mathbb{R} \times \mathbb{R} \times \mathbb{R}$ with respect to the roughness of degree β if for any $\epsilon, \delta > 0$,

$$\lim_{rst \rightarrow \infty} \frac{1}{h_{rst}} |\{(m, n, k) \in I_{rst} : P([f_{mnk} (|\Delta_\gamma^\alpha(x_{mnk})|)]^{p_{rst}} \geq \beta + \epsilon) \geq \delta\}| = 0$$

and we write $\Delta_\gamma^\alpha X_{mnk} \rightarrow_{\beta}^{S^P} \bar{0}$. It will be denoted by βS_θ^P .

Definition 4.5. Let α be a proper fraction, β be nonnegative real number, f be an Musielak-Orlicz function and arbitrary sequence $p = (p_{rst})$ of strictly positive real numbers. A triple difference sequence spaces of random variables is said to be Δ_γ^α -rough N_θ -convergent in probability to $\Delta_\gamma^\alpha X : W \times W \times W \rightarrow \mathbb{R} \times \mathbb{R} \times \mathbb{R}$ with respect to the roughness of degree β if for any $\epsilon > 0$,

$$\lim_{rst \rightarrow \infty} \frac{1}{h_{rst}} \sum_{m \in I_r} \sum_{n \in I_s} \sum_{k \in I_t} |\{P([f_{mnk} (|\Delta_\gamma^\alpha X_{mnk}|)]^{p_{rst}} \geq \beta + \epsilon)\}| = 0,$$

and we write $\Delta_\gamma^\alpha X_{mnk} \rightarrow_{\beta}^{N_\theta^P} \Delta_\gamma^\alpha X$. The class of all $\beta - N_\theta$ -convergent triple difference sequence spaces of random variables in probability will be denoted by βN_θ^P .

Definition 4.6. Let α be a proper fraction, β be nonnegative real number, f be an Musielak-Orlicz function and arbitrary sequence $p = (p_{rst})$ of strictly positive real numbers. A triple difference sequence spaces of random variables is said to be Δ_γ^α -rough lacunary statistically Cauchy if there exists a number $N = N(\epsilon)$ in probability to $\Delta_\gamma^\alpha X : W \times W \times W \rightarrow \mathbb{R} \times \mathbb{R} \times \mathbb{R}$ with respect to the roughness of degree β if for any $\epsilon, \delta > 0$,

$$\lim_{rst \rightarrow \infty} \frac{1}{h_{rst}} |\{(m, n, k) \in I_{rst} : P([f_{mnk} (|\Delta_\gamma^\alpha(x_{mnk} - x_N)|)]^{p_{rst}} \geq \beta + \epsilon) \geq \delta\}| = 0.$$

Theorem 4.1. Let α be a proper fraction, β be nonnegative real number, f be an Musielak-Orlicz function and arbitrary sequence $p = (p_{rst})$ of strictly positive real numbers, $0 < p < \infty$.

(i) If $(x_{mnk}) \rightarrow (N(\Delta_\gamma^\alpha, p)_\theta)$ for $p_{rst} = p$ then $(x_{mnk}) \rightarrow (\Delta_\gamma^\alpha(S_\theta))$.

(ii) If $x \in (\Delta_\gamma^\alpha(S_\theta))$, then $(x_{mnk}) \rightarrow (N(\Delta_\gamma^\alpha, p)_\theta)$.

Proof. Let $x = (x_{mnk}) \in (N(\Delta_\gamma^\alpha, p)_\theta)$ and $\epsilon > 0$,

$$|\{P([f_{mnk}(|\Delta_\gamma^\alpha X_{mnk}|)]^{p_{rst}} \geq \beta + \epsilon)\}| = 0.$$

We have

$$\begin{aligned} & \frac{1}{h_{rst}} \sum_{(mnk) \in I_{rst}} |\{P([f_{mnk}(|\Delta_\gamma^\alpha X_{mnk}|)]^{p_{rst}} \geq \beta + \epsilon)\}| \\ & \geq \frac{1}{h_{rst}} |\{(m, n, k) \in I_{rst} : P([f_{mnk}(|\Delta_\gamma^\alpha(x_{mnk})|)]^{p_{rst}} \geq \beta + \epsilon) \geq \delta\}| \left(\frac{\beta + \epsilon}{\delta}\right)^p. \end{aligned}$$

So we observe by passing to limit as $r, s, t \rightarrow \infty$,

$$\begin{aligned} & \lim_{rst \rightarrow \infty} \frac{1}{h_{rst}} |\{(m, n, k) \in I_{rst} : P([f_{mnk}(|\Delta_\gamma^\alpha(x_{mnk})|)]^{p_{rst}} \geq \beta + \epsilon) \geq \delta\}| \\ & \leq \left(\frac{\delta}{\alpha + \epsilon}\right)^p P\left(\lim_{rst \rightarrow \infty} \frac{1}{h_{rst}} \sum_{(m, n, k) \in I_{rst}} |\Delta_\gamma^\alpha x_{mnk}|^p\right) = 0, \end{aligned}$$

which implies that $x_{mnk} \rightarrow (\Delta_\gamma^\alpha(S_\theta))$.

Suppose that $x \in \Delta_\gamma^\alpha(\Lambda^3)$ and $(x_{mnk}) \rightarrow (\Delta_\gamma^\alpha(S))$. Then it is obvious that $(\Delta_\gamma^\alpha x) \in \Lambda^3$ and

$$\frac{1}{h_{rst}} |\{(m, n, k) \in I_{rst} : P([f_{mnk}(|\Delta_\gamma^\alpha(x_{mnk})|)]^{p_{rst}} \geq \beta + \epsilon) \geq \delta\}| \rightarrow 0$$

as $r, s, t \rightarrow \infty$. Let $\epsilon > 0$ be given and there exists $u_0 v_0 w_0 \in \mathbb{N}$ such that

$$\begin{aligned} & \left| \left\{ (m, n, k) \in I_{rst} : P\left([f_{mnk}(|\Delta_\gamma^\alpha(x_{mnk})|)]^{p_{rst}} \geq \beta + \frac{\epsilon}{2}\right) \geq \frac{\delta}{2} \right\} \right| \\ & \leq \frac{\epsilon}{2(d(\Delta_\gamma^\alpha x, y))_{\Lambda^3}} + \frac{\delta}{2}, \end{aligned}$$

where $\sum_{u=1}^\infty \sum_{v=1}^\infty \sum_{w=1}^\infty |\gamma_{uvw} x_{uvw}| = 0$, for all $r \geq u_0, s \geq v_0, t \geq w_0$. Further more, we can write $|\Delta_\gamma^\alpha x_{mnk}| \leq d(\Delta_\gamma^\alpha x_{mnk}, y)_{\Delta_\gamma^\alpha} \leq d(\Delta_\gamma^\alpha x, y)_{\Lambda^3} = d(x, y)_{\Delta_\gamma^\alpha x}$. For $r, s, t \geq u_0, v_0, w_0$

$$\begin{aligned} & \frac{1}{h_{rst}} \sum_{(mnk) \in I_{rst}} P([f_{mnk}(|\Delta_\gamma^\alpha X_{mnk}|)]^p) = \frac{1}{h_{rst}} P\left(\sum_{(mnk) \in I_{rst}} [f_{mnk}(|\Delta_\gamma^\alpha X_{mnk}|)]^p\right) \\ & + \frac{1}{h_{rst}} P\left(\sum_{(mnk) \notin I_{rst}} [f_{mnk}(|\Delta_\gamma^\alpha X_{mnk}|)]^p\right) < \frac{1}{h_{rst}} P\left(h_{rst} \left(\frac{\epsilon}{2} + \frac{\delta}{2}\right)\right) \\ & + h_{rst} \frac{\epsilon d(x, y)_{\Delta_\gamma^\alpha X}^p}{2 d(x, y)_{\Delta_\gamma^\alpha X}^p} + \frac{\delta}{2} = \epsilon + \delta. \end{aligned}$$

Hence $(x_{mnk}) \rightarrow (N(\Delta_\gamma^\alpha, p)_\theta)$. □

Corollary 4.1. *If α be a proper fraction, β be nonnegative real number, f be an Musielak-Orlicz function and arbitrary sequence $p = (p_{rst})$ of strictly positive real numbers then the following statements are hold:*

$$(i) S \cap \Lambda^3 \subset \Delta_\gamma^\alpha(S_\theta) \cap \Delta_\gamma^\alpha(\Lambda^3);$$

$$(ii) \Delta_\gamma^\alpha(S_\theta) \cap \Delta_\gamma^\alpha(\Lambda^3) = \Delta_\gamma^\alpha(w_p^3).$$

Theorem 4.2. *Let α be a proper fraction, β be nonnegative real number, f be an Musielak-Orlicz function and arbitrary sequence $p = (p_{rst})$ of strictly positive real numbers. If $x = (x_{mnk})$ is a Δ_γ^α -triple difference rough lacunary statistically convergent sequence, then x is a Δ_γ^α -triple difference rough lacunary statistically Cauchy sequence.*

Proof. Assume that $(x_{mnk}) \rightarrow (\Delta_\gamma^\alpha(S_\theta))$ and $\epsilon, \delta > 0$. Then

$$\frac{1}{\delta} \left| \left\{ (m, n, k) \in I_{rst} : P \left([f_{mnk} (|\Delta_\gamma^\alpha x_{mnk}|)]^{p_{rst}} \geq \beta + \frac{\epsilon}{2} \right) \right\} \right|$$

for almost all m, n, k and if we select N , then

$$\frac{1}{\delta} \left| \left\{ (m, n, k) \in I_{rst} : P \left([f_{mnk} (|\Delta_\gamma^\alpha x_N|)]^{p_{rst}} \geq \beta + \frac{\epsilon}{2} \right) \right\} \right|$$

holds. Now, we have

$$\begin{aligned} & \left| \left\{ (m, n, k) \in I_{rst} : P \left([f_{mnk} (|\Delta_\gamma^\alpha (x_{mnk} - x_N)|)]^{p_{rst}} \right) \right\} \right| \\ & \leq \frac{1}{\delta} \left| \left\{ (m, n, k) \in I_{rst} : P \left([f_{mnk} (|\Delta_\gamma^\alpha x_{mnk}|)]^{p_{rst}} \geq \beta + \frac{\epsilon}{2} \right) \right\} \right| \\ & + \frac{1}{\delta} \left| \left\{ (m, n, k) \in I_{rst} : P \left([f_{mnk} (|\Delta_\gamma^\alpha x_N|)]^{p_{rst}} \geq \beta + \frac{\epsilon}{2} \right) \right\} \right| < \frac{1}{\delta} (\beta + \epsilon) = \epsilon, \end{aligned}$$

for almost all m, n, k . Hence (x_{mnk}) is a Δ_γ^α -rough lacunary statistically Cauchy. □

Theorem 4.3. *If α be a proper fraction, β be nonnegative real number, f be an Musielak-Orlicz function and arbitrary sequence $p = (p_{rst})$ of strictly positive real numbers and $0 < p < \infty$, then $N(\Delta_\gamma^\alpha, p)_\theta \subset \Delta_\gamma^\alpha(S_\theta)$.*

Proof. Suppose that $x = (x_{mnk}) \in N(\Delta_\gamma^\alpha, p)_\theta$ and

$$\left| \left\{ (m, n, k) \in I_{rst} : P \left([f_{mnk} (|\Delta_\gamma^\alpha x_{mnk}|)]^p \geq \beta + \epsilon \right) \right\} \right|.$$

Therefore we have

$$\begin{aligned} & \frac{1}{h_{rst}} \sum_{(mnk) \in I_{rst}} P \left([f_{mnk} (|\Delta_\gamma^\alpha x_{mnk}|)]^p \right) \geq \frac{1}{h_{rst}} \sum_{(mnk) \in I_{rst}} (\beta + \epsilon)^p \\ & \geq \frac{1}{h_{rst}} \left| \left\{ (m, n, k) \in I_{rst} : P \left([f_{mnk} (|\Delta_\gamma^\alpha x_{mnk}|)]^p \geq \beta + \epsilon \right) \right\} \right| (\beta + \epsilon)^p. \end{aligned}$$

So we observe by passing to limit as $r, s, t \rightarrow \infty$,

$$\begin{aligned} & \lim_{rst \rightarrow \infty} \frac{1}{h_{rst}} |\{(m, n, k) \in I_{rst} : P([f_{mnk}(|\Delta_\gamma^\alpha(x_{mnk})|)]^p \geq \beta + \epsilon) \geq \delta\}| \\ & < \frac{1}{(\beta + \epsilon)^p} \left(P \left(\lim_{rst \rightarrow \infty} \frac{1}{h_{rst}} \sum_{(m,n,k) \in I_{rst}} [f_{mnk}(|\Delta_\gamma^\alpha(x_{mnk})|)]^p \right) \right) = 0 \end{aligned}$$

implies that $x \in \Delta_\gamma^\alpha(S_\theta)$. Hence $N(\Delta_\gamma^\alpha, p)_\theta \subset \Delta_\gamma^\alpha(S_\theta)$. \square

Competing Interests: The authors declare that there is not any conflict of interests regarding the publication of this manuscript.

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DOI: 10.7862/rf.2018.7

Ayhan Esi

email: aesi23@hotmail.com
ORCID: 0000-0003-3137-3865
Department of Mathematics
Adiyaman University
02040, Adiyaman
TURKEY

Nagarajan Subramanian

email: nsmaths@yahoo.com
ORCID: 0000-0002-5895-673X
Department of Mathematics
SASTRA University
Thanjavur-613 401
INDIA

Received 02.07.2018

Accepted 21.10.2018

On the Exponential Stability of a Neutral Differential Equation of First Order

Melek Gözen and Cemil Tunç

ABSTRACT: In this work, we establish some assumptions that guaranteeing the global exponential stability (GES) of the zero solution of a neutral differential equation (NDE). We aim to extend and improves some results found in the literature.

AMS Subject Classification: 34K20, 93D09, 93D20.

Keywords and Phrases: (GES); (NDE); Time-varying delays.

1. Introduction

In [1], sufficient conditions for solutions of the (NDEs) form

$$\frac{d}{dt}(x(t) + c(t)x(t - \tau)) + p(t)x(t) + q(t)x(t - \sigma) = 0 \quad (1)$$

to tend zero as $t \rightarrow \infty$ are established.

In [4, 8, 10, 15, 17], it was considered a (NDE),

$$\frac{d}{dt}(x(t) + px(t - \tau)) = -\alpha x(t) + b \tanh(x(t - \sigma)) = 0 \quad (2)$$

and the asymptotic stability (AS) of solutions are investigated.

In addition, some qualitative behaviors of solutions of equation (2) or some different models of that (NDE) were investigated in the relevant literature; for example, (S), (AS), (ES) in [2, 6, 9, 11, 14, 16, 17-25], (GES) in [3], asymptotic behaviors in [13], oscillation and non-oscillation in [5, 7] and so on.

In this paper, we deal with the following (NDE) with different variable delays:

$$\frac{d}{dt} \left[x(t) + \sum_{i=1}^2 p_i(t)x(t - \tau_i(t)) \right] + a(t)h(x(t)) - \sum_{i=1}^2 b_i(t) \tanh x(t - \sigma_i(t)) = 0, \quad (3)$$

for $t \geq 0$ where $a_i, b_i : [0, \infty) \rightarrow [0, \infty)$ are continuously differentiable functions and $\sum_{i=1}^2 a_i^2(t) \leq 1$. The functions $\tau_i(\cdot) : [0, \infty) \rightarrow [0, \tau_i]$, ($\tau_i > 0$) and $\sigma_i(\cdot) : [0, \infty) \rightarrow [0, \sigma_i]$, ($\sigma_i > 0$) are bounded and continuously differentiable, and the functions h , p_1 and p_2 are continuous with $h(0) = 0$. Let $r_i = \max\{\tau_i, \sigma_i\} > 0$, ($i = 1, 2$). Let $\mu_1, \mu_2, \mu_3, \mu_4 \in (0, 1)$ be positive constants such that $\tau_1'(t) \leq \mu_1$, $\tau_2'(t) \leq \mu_2$, $\sigma_1'(t) \leq \mu_3$ and $\sigma_2'(t) \leq \mu_4$. For each solution of (NDE) (3), we suppose that

$$x_0(\theta) = \phi(\theta), \theta \in [-r_i, 0], \quad \text{where } \phi \in C([-r_i, 0]; R).$$

2. Stability Result

Our stability result is given below.

Theorem. *Let $K, \alpha_0, \alpha_1, \alpha_2, \alpha_3$ and α_4 be positive constants. The zero solution of (NDE)(3) is global exponential stable if the following matrix inequalities hold:*

$$\Omega = \begin{bmatrix} \Omega_{11} & \Omega_{12} & \Omega_{13} & \Omega_{14} & \Omega_{15} & 0 \\ * & \Omega_{22} & \Omega_{23} & \Omega_{24} & \Omega_{25} & 0 \\ * & * & \Omega_{33} & \Omega_{34} & \Omega_{35} & 0 \\ * & * & * & \Omega_{44} & 0 & 0 \\ * & * & * & * & \Omega_{55} & 0 \\ * & * & * & * & * & \Omega_{66} \end{bmatrix} < 0,$$

$$\Delta = \begin{bmatrix} \Delta_{11} & \Delta_{12} & \Delta_{13} & \Delta_{14} & \Delta_{15} & 0 \\ * & \Delta_{22} & \Delta_{23} & \Delta_{24} & \Delta_{25} & 0 \\ * & * & \Delta_{33} & \Delta_{34} & \Delta_{35} & 0 \\ * & * & * & \Delta_{44} & 0 & 0 \\ * & * & * & * & \Delta_{55} & 0 \\ * & * & * & * & * & \Delta_{66} \end{bmatrix} < 0, \quad (4)$$

where

$$h_1(x) = \begin{cases} h(x)x^{-1}, & x \neq 0 \\ h'(0), & x = 0 \end{cases}$$

and

$$\Omega_{11} = 2K\alpha_0\lambda_1 - 2a(t)\alpha_0 \frac{h(x)}{x} \lambda_1 + \alpha_1\lambda_1 \sum_{i=1}^2 e^{2K\tau_i} + \alpha_3\lambda_1 \sum_{i=1}^2 e^{2K\sigma_i}$$

$$\begin{aligned}
 & + \frac{\alpha_2 \lambda_1}{2K} \sum_{i=1}^2 (e^{2K\tau_i} - 1) + \frac{\alpha_4 \lambda_1}{2K} \sum_{i=1}^2 (e^{2K\sigma_i} - 1), \\
 \Omega_{12} & = 2\lambda_1 K \alpha_0 p_1(t) - \lambda_1 \alpha_0 a(t) \frac{h(x)}{x} p_1(t), \\
 \Omega_{13} & = 2\lambda_1 K \alpha_0 p_2(t) - \lambda_1 \alpha_0 a(t) \frac{h(x)}{x} p_2(t), \\
 \Omega_{14} & = \lambda_1 \alpha_0 b_1(t), \\
 \Omega_{15} & = \lambda_1 \alpha_0 b_2(t), \\
 \Omega_{22} & = 2\lambda_1 K \alpha_0 p_1^2(t) - \alpha_1 \lambda_1 (1 - \mu_1), \\
 \Omega_{23} & = 2\lambda_1 K \alpha_0 p_1(t) p_2(t), \\
 \Omega_{24} & = \lambda_1 \alpha_0 p_1(t) b_1(t), \\
 \Omega_{25} & = \lambda_1 \alpha_0 p_1(t) b_2(t), \\
 \Omega_{33} & = 2\lambda_1 K \alpha_0 p_2^2(t) - \alpha_1 \lambda_1 (1 - \mu_2), \\
 \Omega_{34} & = \lambda_1 \alpha_0 p_2(t) b_1(t), \\
 \Omega_{35} & = \lambda_1 \alpha_0 p_2(t) b_2(t), \\
 \Omega_{44} & = -\alpha_3 \lambda_1 (1 - \mu_3), \\
 \Omega_{55} & = -\alpha_3 \lambda_1 (1 - \mu_4), \\
 \Omega_{66} & = -\alpha_2 \tau_i, \\
 \Delta_{11} & = 2K \alpha_0 \lambda_2 - 2a(t) \alpha_0 \frac{h(x)}{x} \lambda_2 + \alpha_1 \lambda_2 \sum_{i=1}^2 e^{2K\tau_i} + \alpha_3 \lambda_2 \sum_{i=1}^2 e^{2K\sigma_i}, \\
 & + \frac{\alpha_2 \lambda_2}{2K} \sum_{i=1}^2 (e^{2K\tau_i} - 1) + \frac{\alpha_4 \lambda_2}{2K} \sum_{i=1}^2 (e^{2K\sigma_i} - 1), \\
 \Delta_{12} & = 2\lambda_2 K \alpha_0 p_1(t) - \lambda_2 \alpha_0 a(t) \frac{h(x)}{x} p_1(t), \\
 \Delta_{13} & = 2\lambda_2 K \alpha_0 p_2(t) - \lambda_2 \alpha_0 a(t) \frac{h(x)}{x} p_2(t), \\
 \Delta_{14} & = \lambda_2 \alpha_0 b_1(t), \\
 \Delta_{15} & = \lambda_2 \alpha_0 b_2(t), \\
 \Delta_{22} & = 2\lambda_2 K \alpha_0 p_1^2(t) - \alpha_1 \lambda_2 (1 - \mu_1), \\
 \Delta_{23} & = 2\lambda_2 K \alpha_0 p_1(t) p_2(t), \\
 \Delta_{24} & = \lambda_2 \alpha_0 p_1(t) b_1(t), \\
 \Delta_{25} & = \lambda_2 \alpha_0 p_1(t) b_2(t), \\
 \Delta_{33} & = 2\lambda_2 K \alpha_0 p_2^2(t) - \alpha_1 \lambda_2 (1 - \mu_2), \\
 \Delta_{34} & = \lambda_2 \alpha_0 p_2(t) b_1(t), \\
 \Delta_{35} & = \lambda_2 \alpha_0 p_2(t) b_2(t), \\
 \Delta_{44} & = -\alpha_3 \lambda_2 (1 - \mu_3), \\
 \Delta_{55} & = -\alpha_3 \lambda_2 (1 - \mu_4), \\
 \Delta_{66} & = -\alpha_4 \sigma_i, \\
 \lambda_1 & = \frac{1}{2} \frac{\tau_i}{\tau_i + \sigma_i}, \quad \lambda_2 = \frac{1}{2} \frac{\sigma_i}{\tau_i + \sigma_i}, \quad (i = 1, 2).
 \end{aligned}$$

Proof. Choose an auxiliary functional, that is, Lyapunov functional (LF) by

$$\begin{aligned}
V(\cdot) = V(t, x) &= e^{2Kt} \alpha_0 \left[x(t) + \sum_{i=1}^2 p_i(t) x(t - \tau_i(t)) \right]^2 \\
&+ \alpha_1 \sum_{i=1}^2 \int_{t-\tau_i(t)}^t e^{2K(s+\tau_i)} x^2(s) ds \\
&+ \alpha_2 \sum_{i=1}^2 \int_{-\tau_i}^0 \int_{t+\theta}^t e^{2K(s-\theta)} x^2(s) ds d\theta \\
&+ \alpha_3 \sum_{i=1}^2 \int_{t-\sigma_i(t)}^t e^{2K(s+\sigma_i)} \tanh^2 x(s) ds \\
&+ \alpha_4 \sum_{i=1}^2 \int_{-\sigma_i}^0 \int_{t+\theta}^t e^{2K(s-\theta)} \tanh^2 x(s) ds d\theta,
\end{aligned}$$

where $\alpha_i \in \mathfrak{R}$, ($i = 0, 1, \dots, 4$), $\alpha_i > 0$, and we choose them later.

The calculation of derivative of (LF) $V(\cdot)$ with respect to the (NDE) (3) gives that

$$\begin{aligned}
\frac{dV(\cdot)}{dt} &= 2K e^{2Kt} \alpha_0 \left[x(t) + \sum_{i=1}^2 p_i(t) x(t - \tau_i(t)) \right]^2 \\
&+ 2e^{2Kt} \alpha_0 \left[x(t) + \sum_{i=1}^2 p_i(t) x(t - \tau_i(t)) \right] \\
&\times \left[-a(t)h(x(t)) + \sum_{i=1}^2 b_i(t) \tanh x(t - \sigma_i(t)) \right] \\
&+ \alpha_1 \sum_{i=1}^2 e^{2K(t+\tau_i)} x^2(t) - \alpha_1 \sum_{i=1}^2 (1 - \tau_i'(t)) e^{2K(t-\tau_i(t)+\tau_i)} x^2(t - \tau_i(t)) \\
&- \frac{\alpha_2}{2K} \sum_{i=1}^2 [e^{2Kt} - e^{2K(t+\tau_i)}] x^2(t) - \alpha_2 e^{2Kt} \sum_{i=1}^2 \int_{t-\tau_i}^t x^2(s) ds \\
&+ \alpha_3 \sum_{i=1}^2 e^{2K(t+\sigma_i)} \tanh^2 x(t) \\
&- \alpha_3 \sum_{i=1}^2 (1 - \sigma_i') e^{2K(t-\sigma_i(t)+\sigma_i)} \tanh^2 x(t - \sigma_i(t))
\end{aligned}$$

$$\begin{aligned}
 & -\frac{\alpha_4}{2K} \sum_{i=1}^2 [e^{2Kt} - e^{2K(t+\sigma_i)}] \tanh^2 x(t) - \alpha_4 e^{2Kt} \sum_{i=1}^2 \int_{t-\sigma_i}^t \tanh^2 x(s) ds \\
 & = 2Ke^{2Kt} \alpha_0 [x^2(t) + 2x(t)p_1(t)x(t - \tau_1(t)) + 2x(t)p_2(t)x(t - \tau_2(t))] \\
 & + p_1^2(t)x^2(t - \tau_1(t)) + p_2^2(t)x^2(t - \tau_2(t)) \\
 & + 2p_1(t)p_2(t)x(t - \tau_1(t))x(t - \tau_2(t)) \\
 & + 2e^{2Kt} \alpha_0 \left[-\alpha(t) \frac{h(x)}{x} x^2(t) + x(t)b_1(t) \tanh x(t - \sigma_1(t)) \right. \\
 & + x(t)b_2(t) \tanh x(t - \sigma_2(t)) - a(t) \frac{h(x)}{x} x(t)p_1(t)x(t - \tau_1(t)) \\
 & - a(t) \frac{h(x)}{x} x(t)p_2(t)x(t - \tau_2(t)) \\
 & + (p_1(t)x(t - \tau_1(t)) + p_2(t)x(t - \tau_2(t)))(b_1(t) \tanh x(t - \sigma_1(t)) \\
 & \left. + b_2(t) \tanh x(t - \sigma_2(t))) \right] \\
 & + \alpha_1 \sum_{i=1}^2 e^{2K(t+\tau_i)} x^2(t) - \alpha_1 (1 - \tau_1'(t)) e^{2K(t-\tau_1(t)+\tau_1)} x^2(t - \tau_1(t)) \\
 & - \alpha_1 (1 - \tau_2'(t)) e^{2K(t-\tau_2(t)+\tau_2)} x^2(t - \tau_2(t)) \\
 & + \frac{\alpha_2}{2K} e^{2Kt} \sum_{i=1}^2 [e^{2K\tau_i} - 1] x^2(t) - \alpha_2 e^{2Kt} \sum_{i=1}^2 \int_{t-\tau_i}^t x^2(s) ds \\
 & + \alpha_3 \sum_{i=1}^2 e^{2K(t+\sigma_i)} \tanh^2 x(t) \\
 & - \alpha_3 (1 - \sigma_1'(t)) e^{2K(t-\sigma_1(t)+\sigma_1)} \tanh^2 x(t - \sigma_1(t)) \\
 & - \alpha_3 (1 - \sigma_2'(t)) e^{2K(t-\sigma_2(t)+\sigma_2)} \tanh^2 x(t - \sigma_2(t)) \\
 & + \frac{\alpha_4}{2K} e^{2Kt} \sum_{i=1}^2 [e^{2K\sigma_i} - 1] \tanh^2 x(t) - \alpha_4 e^{2Kt} \sum_{i=1}^2 \int_{t-\sigma_i}^t \tanh^2 x(s) ds.
 \end{aligned}$$

The assumptions of the theorem implies

$$\begin{aligned}
 & -\alpha_1 (1 - \tau_1'(t)) e^{2K(\tau_1 - \tau_1(t))} \leq -\alpha_1 (1 - \mu_1) \\
 & -\alpha_1 (1 - \tau_2'(t)) e^{2K(\tau_2 - \tau_2(t))} \leq -\alpha_1 (1 - \mu_2) \\
 & -\alpha_3 (1 - \sigma_1'(t)) e^{2K(\sigma_1 - \sigma_1(t))} \leq -\alpha_3 (1 - \mu_3)
 \end{aligned}$$

and

$$-\alpha_3 (1 - \sigma_2'(t)) e^{2K(\sigma_2 - \sigma_2(t))} \leq -\alpha_3 (1 - \mu_4).$$

Then,

$$\begin{aligned}
\frac{dV(\cdot)}{dt} &\leq 2Ke^{2Kt}\alpha_0[x^2(t) + 2x(t)p_1(t)x(t - \tau_1(t)) + 2x(t)p_2(t)x(t - \tau_2(t))] \\
&\quad + p_1^2(t)x^2(t - \tau_1(t)) + p_2^2(t)x^2(t - \tau_2(t)) \\
&\quad + 2p_1(t)p_2(t)x(t - \tau_1(t))x(t - \tau_2(t)) \\
&\quad + 2e^{2Kt}\alpha_0\left[-\alpha(t)\frac{h(x)}{x}x^2(t) + x(t)b_1(t)\tanh x(t - \sigma_1(t))\right. \\
&\quad + x(t)b_2(t)\tanh x(t - \sigma_2(t)) - a(t)\frac{h(x)}{x}x(t)p_1(t)x(t - \tau_1(t)) \\
&\quad - a(t)\frac{h(x)}{x}x(t)p_2(t)x(t - \tau_2(t)) \\
&\quad + p_1(t)b_1(t)x(t - \tau_1(t))\tanh x(t - \sigma_1(t)) \\
&\quad + p_1(t)b_2(t)x(t - \tau_1(t))\tanh x(t - \sigma_2(t)) \\
&\quad + p_2(t)b_1(t)x(t - \tau_2(t))\tanh x(t - \sigma_1(t)) \\
&\quad + p_2(t)b_2(t)x(t - \tau_2(t))\tanh x(t - \sigma_2(t)) \\
&\quad + \alpha_1\sum_{i=1}^2 e^{2K(t+\tau_i)}x^2(t) - \alpha_1e^{2Kt}(1 - \mu_1)x^2(t - \tau_1(t)) \\
&\quad - \alpha_1e^{2Kt}(1 - \mu_2)x^2(t - \tau_2(t)) \\
&\quad + \frac{\alpha_2}{2K}e^{2Kt}\sum_{i=1}^2[e^{2K\tau_i} - 1]x^2(t) - \alpha_2e^{2Kt}\sum_{i=1}^2\int_{t-\tau_i}^t x^2(s)ds \\
&\quad + \alpha_3\sum_{i=1}^2 e^{2K(t+\sigma_i)}\tanh^2 x(t) - \alpha_3e^{2Kt}(1 - \mu_3)\tanh^2 x(t - \sigma_1(t)) \\
&\quad - \alpha_3e^{2Kt}(1 - \mu_4)\tanh^2 x(t - \sigma_2(t)) \\
&\quad + \frac{\alpha_4}{2K}e^{2Kt}\sum_{i=1}^2[e^{2K\sigma_i} - 1]\tanh^2 x(t) \\
&\quad - \alpha_4e^{2Kt}\sum_{i=1}^2\int_{t-\sigma_i}^t \tanh^2 x(s)ds.
\end{aligned}$$

Since

$$\tanh^2 x \leq x^2,$$

then

$$\begin{aligned}
\frac{dV(\cdot)}{dt} &\leq e^{2Kt}\left\{2K\alpha_0 - 2\alpha_0a(t)\frac{h(x)}{x} + \alpha_1\sum_{i=1}^2 e^{2K\tau_i} + \frac{\alpha_2}{2K}\sum_{i=1}^2(e^{2K\tau_i} - 1)\right. \\
&\quad \left.+ \alpha_3\sum_{i=1}^2 e^{2K\sigma_i} + \frac{\alpha_4}{2K}\sum_{i=1}^2(e^{2K\sigma_i} - 1)\right\}x^2(t)
\end{aligned}$$

$$\begin{aligned}
 &+ [4K\alpha_0 p_1(t) - 2\alpha_0 a(t) \frac{h(x)}{x} p_1(t)] x(t)(t - \tau_1(t)) \\
 &+ [4K\alpha_0 p_2(t) - 2\alpha_0 a(t) \frac{h(x)}{x} p_2(t)] x(t)(t - \tau_2(t)) \\
 &+ 2\alpha_0 b_1(t) x(t) \tanh x(t - \sigma_1(t)) \\
 &+ 2\alpha_0 b_2(t) x(t) \tanh x(t - \sigma_2(t)) \\
 &+ [2K\alpha_0 p_1^2(t) - \alpha_1(1 - \mu_1)] x^2(t - \tau_1(t)) \\
 &+ 4K\alpha_0 p_1(t) p_2(t) x(t - \tau_1(t)) x(t - \tau_2(t)) \\
 &+ 2\alpha_0 p_1(t) b_1(t) x(t - \tau_1(t)) \tanh x(t - \sigma_1(t)) \\
 &+ 2\alpha_0 p_1(t) b_2(t) x(t - \tau_1(t)) \tanh x(t - \sigma_2(t)) \\
 &+ [2K\alpha_0 p_2^2(t) - \alpha_1(1 - \mu_2)] x^2(t - \tau_2(t)) \\
 &+ 2\alpha_0 p_2(t) b_1(t) x(t - \tau_2(t)) \tanh x(t - \sigma_1(t)) \\
 &+ 2\alpha_0 p_2(t) b_2(t) x(t - \tau_2(t)) \tanh x(t - \sigma_2(t)) \\
 &- \alpha_3(1 - \mu_3) \tanh^2 x(t - \sigma_1(t)) \\
 &- \alpha_3(1 - \mu_4) \tanh^2 x(t - \sigma_2(t)) \\
 &- \alpha_2 \sum_{i=1}^2 \int_{t-\tau_i}^t x^2(s) ds \\
 &- \alpha_4 \sum_{i=1}^2 \int_{t-\sigma_i}^t \tanh^2 x(s) ds \}.
 \end{aligned}$$

Then,

$$\frac{dV(\cdot)}{dt} \leq \sum_{i=1}^2 \frac{1}{\tau_i} \int_{t-\tau_i}^t \xi_1^T(t, s) \Omega \xi_1(t, s) ds + \sum_{i=1}^2 \frac{1}{\sigma_i} \int_{t-\sigma_i}^t \xi_2^T(t, s) \Delta \xi_2(t, s) ds,$$

where

$$\xi_1(t, s) = [x(t), x(t - \tau_1(t)), x(t - \tau_2(t)), \tanh x(t - \sigma_1(t)), \tanh x(t - \sigma_2(t)), x(s)]^T$$

and

$$\xi_2(t, s) = [x(t), x(t - \tau_1(t)), x(t - \tau_2(t)), \tanh x(t - \sigma_1(t)), \tanh x(t - \sigma_2(t)), \tanh x(s)]^T.$$

From (4), we have $\frac{dV(\cdot)}{dt} < 0$, which implies that $V(\cdot) \leq V(0, x(0))$. In view of the (LF) $V(\cdot)$, we find

$$V(0, x(0)) = \alpha_0 [x(0) + \sum_{i=1}^2 p_i(0) x(-\tau_i(0))]^2 + \alpha_1 \sum_{i=1}^2 \int_{-\tau_i(0)}^0 e^{2K(s+\tau_i)} x^2(s) ds$$

$$\begin{aligned}
& + \alpha_2 \sum_{i=1}^2 \int_{-\tau_i}^0 \int_{\theta}^0 e^{2K(s-\theta)} x^2(s) ds d\theta + \alpha_3 \sum_{i=1}^2 \int_{-\sigma_i(0)}^0 e^{2K(s+\sigma_i)} \tanh^2 x(s) ds \\
& + \alpha_4 \sum_{i=1}^2 \int_{-\sigma_i}^0 \int_{\theta}^0 e^{2K(s-\theta)} \tanh^2 x(s) ds d\theta.
\end{aligned}$$

It is also obvious that

$$\begin{aligned}
\alpha_0 [x(0) + \sum_{i=1}^2 p_i(0)x(-\tau_i(0))]^2 & = \alpha_0 [x^2(0) + 2x(0) \sum_{i=1}^2 p_i(0)x(-\tau_i(0)) \\
& + (\sum_{i=1}^2 p_i(0)x(-\tau_i(0)))^2] \\
& = \alpha_0 [x^2(0) + 2x(0) \sum_{i=1}^2 p_i(0)x(-\tau_i(0)) \\
& + p_1^2(0)x^2(-\tau_1(0)) \\
& + 2p_1(0)x(-\tau_1(0))p_2(0)x(-\tau_2(0)) \\
& + p_2^2(0)x^2(-\tau_2(0))].
\end{aligned}$$

If we use the inequality

$$2|xy| \leq x^2 + y^2,$$

then

$$\begin{aligned}
\alpha_0 [x(0) + \sum_{i=1}^2 p_i(0)x(-\tau_i(0))]^2 & \leq \alpha_0 [x^2(0) + 2x^2(0) + p_1^2(0)x^2(-\tau_1(0)) \\
& + p_2^2(0)x^2(-\tau_2(0)) + p_1^2(0)x^2(-\tau_1(0)) \\
& + p_1^2(0)x^2(-\tau_1(0)) + p_2^2(0)x^2(-\tau_2(0)) \\
& + p_2^2(0)x^2(-\tau_2(0))] \\
& = \alpha_0 [3x^2(0) + 3p_1^2(0)x^2(-\tau_1(0)) \\
& + 3p_2^2(0)x^2(-\tau_2(0))].
\end{aligned}$$

In view of the assumption $\sum_{i=1}^2 p_i^2(t) \leq 1$, it follows that

$$\begin{aligned}
\alpha_0 [x(0) + \sum_{i=1}^2 p_i(0)x(-\tau_i(0))]^2 & \leq \alpha_0 [3x^2(0) + 3x^2(-\tau_1(0)) + 3x^2(-\tau_2(0))] \\
& \leq 9\alpha_0 \sup_{\theta \in [-r_i, 0]} |\phi(\theta)|^2, \\
\alpha_1 \sum_{i=1}^2 \int_{-\tau_i(0)}^0 e^{2K(s+\tau_i)} x^2(s) ds & \leq \alpha_1 \sum_{i=1}^2 e^{2K\tau_i} \int_{-\tau_i(0)}^0 \sup_{t \in [-\tau_i(0), 0]} e^{2Kt} x^2(t) ds
\end{aligned}$$

$$\begin{aligned}
 &= \alpha_1 \sum_{i=1}^2 e^{2K\tau_i} \sup_{t \in [-\tau_i(0), 0]} e^{2Kt} x^2(t) \tau_i(0) \\
 &\leq \alpha_1 \sum_{i=1}^2 e^{2Kr_i r_i} \sup_{t \in [-\tau_i(0), 0]} e^{2Kt} x^2(t) \\
 &\leq \alpha_1 \sum_{i=1}^2 e^{2Kr_i r_i} \sup_{\theta \in [-r_i, 0]} |\phi(\theta)|^2, \\
 \alpha_2 \sum_{i=1}^2 \int_{-\tau_i}^0 \int_{\theta}^0 e^{2K(s-\theta)} x^2(s) ds d\theta &\leq \alpha_2 \sum_{i=1}^2 \int_{-\tau_i}^0 \left[\sup_{s \in [\theta, 0]} x^2(s) \int_{\theta}^0 e^{2K(s-\theta)} ds \right] d\theta \\
 &= \alpha_2 \sum_{i=1}^2 \int_{-\tau_i}^0 \sup_{s \in [-\theta, 0]} x^2(s) \left[\frac{1}{2K} e^{-2K\theta} - \frac{1}{2K} \right] d\theta \\
 &\leq \frac{1}{2K} \alpha_2 \sum_{i=1}^2 \int_{-\tau_i}^0 \sup_{s \in [-\theta, 0]} x^2(s) e^{-2K\theta} d\theta \\
 &\leq \alpha_2 \sum_{i=1}^2 \sup_{\theta \in [-r_i, 0]} |\phi(\theta)|^2 \left[-\frac{1}{4K^2} + \frac{1}{4K^2} e^{2K\tau_i} \right] \\
 &\leq \frac{1}{4K^2} \alpha_2 \sum_{i=1}^2 e^{2Kr_i} \sup_{\theta \in [-r_i, 0]} |\phi(\theta)|^2, \\
 \alpha_3 \sum_{i=1}^2 \int_{-\sigma_i(0)}^0 e^{2K(s+\sigma_i)} \tanh^2 x(s) ds &\leq \alpha_3 \sum_{i=1}^2 e^{2K\sigma_i} \int_{-\sigma_i(0)}^0 e^{2Ks} x^2(s) ds \\
 &\leq \alpha_3 \sum_{i=1}^2 e^{2K\sigma_i} \int_{-\sigma_i(0)}^0 \sup_{t \in [-\sigma_i(0), 0]} e^{2Kt} x^2(t) ds \\
 &= \alpha_3 \sum_{i=1}^2 e^{2K\sigma_i} \sup_{t \in [-\sigma_i(0), 0]} e^{2Kt} x^2(t) \sigma_i(0) \\
 &\leq \alpha_3 \sum_{i=1}^2 e^{2Kr_i r_i} \sup_{t \in [-\sigma_i(0), 0]} e^{2Kt} x^2(t), \\
 \alpha_4 \sum_{i=1}^2 \int_{-\sigma_i}^0 \int_{\theta}^0 e^{2K(s-\theta)} \tanh^2 x(s) ds d\theta &\leq \alpha_4 \sum_{i=1}^2 \int_{-\sigma_i}^0 \left[\sup_{s \in [\theta, 0]} x^2(s) \int_{\theta}^0 e^{2K(s-\theta)} ds \right] d\theta \\
 &= \alpha_4 \sum_{i=1}^2 \int_{-\sigma_i}^0 \sup_{s \in [-\theta, 0]} x^2(s) \left[\frac{1}{2K} e^{-2K\theta} - \frac{1}{2K} \right] d\theta \\
 &\leq \frac{1}{2K} \alpha_4 \sum_{i=1}^2 \int_{-\sigma_i}^0 \sup_{s \in [-\theta, 0]} x^2(s) e^{-2K\theta} d\theta
 \end{aligned}$$

$$\begin{aligned}
&\leq \alpha_4 \sum_{i=1}^2 \sup_{\theta \in [-r_i, 0]} |\phi(\theta)|^2 \left[-\frac{1}{4K^2} + \frac{1}{4K^2} e^{2K\sigma_i} \right] \\
&\leq \frac{1}{4K^2} \alpha_4 \sum_{i=1}^2 e^{2Kr_i} \sup_{\theta \in [-r_i, 0]} |\phi(\theta)|^2.
\end{aligned}$$

Hence,

$$\begin{aligned}
V(0, x(0)) &= \alpha_0 [x(0) + \sum_{i=1}^2 p_i(0)x(-\tau_i(0))]^2 + \alpha_1 \sum_{i=1}^2 \int_{-\tau_i(0)}^0 e^{2K(s+\tau_i)} x^2(s) ds \\
&\quad + \alpha_2 \sum_{i=1}^2 \int_{-\tau_i}^0 \int_{\theta}^0 e^{2K(s-\theta)} x^2(s) ds d\theta + \alpha_3 \sum_{i=1}^2 \int_{-\sigma_i(0)}^0 e^{2K(s+\sigma_i)} \tanh^2 x(s) ds \\
&\quad + \alpha_4 \sum_{i=1}^2 \int_{-\sigma_i}^0 \int_{\theta}^0 e^{2K(s-\theta)} \tanh^2 x(s) ds d\theta \\
&\leq [9\alpha_0 + (\alpha_1 + \alpha_3) \sum_{i=1}^2 r_i e^{2Kr_i} + (\alpha_2 + \alpha_4) \frac{1}{4K^2} \sum_{i=1}^2 e^{2Kr_i}] \sum_{i=1}^2 \sup_{\theta \in [r_i, 0]} |\phi(\theta)|^2 \\
&\equiv M.
\end{aligned}$$

We can now write

$$\left| x + \sum_{i=1}^2 p_i(t)x(-\tau_i(t)) \right|^2 \leq M_1 e^{-2kt},$$

where $M_1 = \frac{M}{\alpha_0} > 0$. For $\forall \varepsilon \in (0, \min\{2K, -\frac{2}{r_i} \log |p_i(t)|\})$ and $v > 0$, the inequality $xy \leq vx^2 + \frac{1}{v}y^2$ for any $x, y \in R$ implies that

$$\begin{aligned}
e^{\varepsilon t} |x|^2 &\leq (1+v)e^{\varepsilon t} \left| x(t) + \sum_{i=1}^2 p_i(t)x(t-\tau_i(t)) \right|^2 + \frac{1+v}{v} e^{\varepsilon t} \sum_{i=1}^2 |p_i(t)x(t-\tau_i(t))|^2 \\
&\leq (1+v)M_1 + \frac{1+v}{v} \sum_{i=1}^2 |p_i(t)|^2 |x(t-\tau_i(t))|^2 e^{\varepsilon r_i} e^{\varepsilon(t-\tau_i(t))}.
\end{aligned}$$

And from $\forall \varepsilon \in (0, \min\{2K, -\frac{2}{r_i} \log |p_i(t)|\})$, we have $\sum_{i=1}^2 |p_i(t)|^2 e^{\varepsilon r_i} < 1$. Thus, if we choose $v > 0$ sufficiently large, then it follows that

$$\gamma = \frac{\sum_{i=1}^2 |p_i(t)|^2 (1+v) e^{\varepsilon r_i}}{v} < 1.$$

Therefore,

$$e^{\varepsilon t}|x|^2 \leq (1 + v)M_1 + \gamma \sum_{i=1}^2 |x(t - \tau_i(t))|^2 e^{\varepsilon(t - \tau_i(t))} \quad (\forall T \geq 0),$$

$$\sup_{0 \leq t \leq T} \{e^{\varepsilon t}|x(t)|^2\} \leq (1 + v)M_1 + \gamma \sup_{\theta \in [r_i, 0]} |\varphi(\theta)|^2 + \gamma \sup_{0 \leq t \leq T} \{e^{\varepsilon t}|x(t)|^2\}, \quad (i = 1, 2).$$

Consequently, we obtain

$$\sup_{0 \leq t \leq T} \{e^{\varepsilon t}|x|^2\} \leq \frac{(1 + v)M_1 + \gamma \sup_{\theta \in [r_i, 0]} |\varphi(\theta)|^2}{1 - \gamma}, \quad (i = 1, 2).$$

When $T \rightarrow +\infty$, we can find that

$$\sup_{0 \leq t \leq \infty} \{e^{\varepsilon t}|x(t)|^2\} \leq \frac{(1 + v)M_1 + \gamma \sup_{\theta \in [r_i, 0]} |\varphi(\theta)|^2}{1 - \gamma}, \quad (i = 1, 2).$$

Thus,

$$|x| \leq M_2 e^{-\alpha t},$$

where

$$M_2 = \sqrt{\frac{(1 + v)M_1 + \gamma \sup_{\theta \in [r_i, 0]} |\varphi(\theta)|^2}{1 - \gamma}} > 0, \alpha = \frac{\varepsilon}{2} > 0, \quad (i = 1, 2).$$

This ends the proof. □

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DOI: 10.7862/rf.2018.8

Melek Gözen

email: melekgozen2013@gmail.com
ORCID: 0000-0002-7487-9869
Department of Business Administration
Management Faculty
Van Yuzuncu Yil University
65080, Erciş
TURKEY

Cemil Tuğ

email: cemtunc@yahoo.com
ORCID: 0000-0003-2909-8753
Department of Mathematics
Faculty of Sciences
Van Yuzuncu Yil University
65080, Van
TURKEY

Received 07.09.2017

Accepted 13.04.2018

Existence and Convergence Results for Caputo Fractional Volterra Integro-Differential Equations

Ahmed A. Hamoud*, M.Sh. Bani Issa, Kirtiwant P. Ghadle
and Mohammed Abdulghani

ABSTRACT: In this article, homotopy analysis method is successfully applied to find the approximate solution of Caputo fractional Volterra integro-differential equation. The reliability of the method and reduction in the size of the computational work give this method a wider applicability. Also, the behavior of the solution can be formally determined by analytical approximate. Moreover, we proved the existence and convergence of the solution. Finally, an example is included to demonstrate the validity and applicability of the proposed technique.

AMS Subject Classification: 65H20, 26A33, 35C10.

Keywords and Phrases: Homotopy analysis method; Caputo fractional derivative; Volterra integro-differential equation; Approximate solution.

1. Introduction

In this paper, we consider Caputo fractional Volterra integro-differential equation of the form:

$${}^c D^\alpha u(x) = g(x) + \int_0^x K(x,t)F(u(t))dt, \quad (1.1)$$

with the initial condition

$$u^{(i)}(0) = \delta_i, \quad i = 0, 1, 2, \dots, n-1, \quad (1.2)$$

where ${}^c D^\alpha$ is the Caputo's fractional derivative, $n-1 < \alpha \leq n$, $n \in \mathbb{N}$ and $u : J \rightarrow \mathbb{R}$, where $J = [0, 1]$ is the continuous function which has to be determined, $g : J \rightarrow \mathbb{R}$ and $K : J \times J \rightarrow \mathbb{R}$, are continuous functions. $F : \mathbb{R} \rightarrow \mathbb{R}$, is Lipschitz continuous function.

The fractional integro-differential equations have attracted much more interest of mathematicians and physicists which provides an efficiency for the description of many practical dynamical arising in engineering and scientific disciplines such as, physics, biology, electrochemistry, chemistry, economy, electromagnetic, control theory and viscoelasticity [2, 5, 8, 7, 9, 10, 17, 18, 20]. In recent years, many authors focus on the development of numerical and analytical techniques for fractional integro-differential equations. For instance, we can remember the following works. An application of fractional derivatives was first given in 1823 by Abel [1] who applied the fractional calculus in the solution of an integral equation that arises in the formulation of the Tautochrone problem, Al-Samadi and Gumah [3] applied the homotopy analysis method for fractional SEIR epidemic model, Zurigat et al. [23] applied HAM for system of fractional integro-differential equations, Yang and Hou [20] applied the Laplace decomposition method to solve the fractional integro-differential equations, Mittal and Nigam [18] applied the Adomian decomposition method to approximate solutions for fractional integro-differential equations, and Ma and Huang [17] applied hybrid collocation method to study integro-differential equations of fractional order. Moreover, properties of the fractional integro-differential equations have been studied by several authors [11, 12, 21, 23]. The homotopy analysis method (HAM) that was first proposed by Liao [14, 15, 16], is implemented to derive analytic approximate solutions of fractional integro-differential equations (FIDEs) and convergence of HAM for this kind of equations is considered. Unlike all other analytical methods, HAM adjusts and controls the convergence region of the series solution via an auxiliary parameter \hbar .

The main objective of the present paper is to study the behavior of the solution that can be formally determined by analytical approximated method as the homotopy analysis method. Moreover, we proved the existence and convergence of the solution of the Caputo fractional Volterra integro-differential equation.

The rest of the paper is organized as follows: In Section 2, some preliminaries and basic definitions related to fractional calculus are recalled. In Section 3, homotopy analysis method is constructed for solving Caputo fractional Volterra integro-differential equations. In Section 4, the existence and convergence of the solution have been proved. In Section 5, the analytical example is presented to illustrate the accuracy of this method. Finally, we will give a report on our paper and a brief conclusion is given in Section 6.

2. Preliminaries

The mathematical definitions of fractional derivative and fractional integration are the subject of several different approaches. The most frequently used definitions of the fractional calculus involves the Riemann-Liouville fractional derivative, Caputo

derivative [13, 19, 22].

Definition 2.1. (Riemann-Liouville fractional integral). The Riemann-Liouville fractional integral of order $\alpha > 0$ of a function f is defined as

$$\begin{aligned} J^\alpha f(x) &= \frac{1}{\Gamma(\alpha)} \int_0^x (x-t)^{\alpha-1} f(t) dt, & x > 0, \quad \alpha \in \mathbb{R}^+, \\ J^0 f(x) &= f(x), \end{aligned} \tag{2.1}$$

where \mathbb{R}^+ is the set of positive real numbers.

Definition 2.2. (Caputo fractional derivative). The fractional derivative of $f(x)$ in the Caputo sense is defined by

$$\begin{aligned} {}^c D_x^\alpha f(x) &= J^{m-\alpha} D^m f(x) \\ &= \begin{cases} \frac{1}{\Gamma(m-\alpha)} \int_0^x (x-t)^{m-\alpha-1} \frac{d^m f(t)}{dt^m} dt, & m-1 < \alpha < m, \\ \frac{d^m f(x)}{dx^m}, & \alpha = m, \quad m \in \mathbb{N}, \end{cases} \end{aligned} \tag{2.2}$$

where the parameter α is the order of the derivative and is allowed to be real or even complex. In this paper, only real and positive α will be considered.

Hence, we have the following properties:

1. $J^\alpha J^\nu f = J^{\alpha+\nu} f, \quad \alpha, \nu > 0,$
2. $J^\alpha x^\beta = \frac{\Gamma(\beta+1)}{\Gamma(\beta+\alpha+1)} x^{\beta+\alpha},$
3. $J^\alpha D^\alpha f(x) = f(x) - \sum_{k=0}^{m-1} f^{(k)}(0^+) \frac{x^k}{k!}, \quad x > 0, \quad m-1 < \alpha \leq m.$

Definition 2.3. (Riemann-Liouville fractional derivative). The Riemann Liouville fractional derivative of order $\alpha > 0$ is normally defined as

$$D^\alpha f(x) = D^m J^{m-\alpha} f(x), \quad m-1 < \alpha \leq m, \quad m \in \mathbb{N}. \tag{2.3}$$

Theorem 2.4. [22] (Banach contraction principle). Let (X, d) be a complete metric space, then each contraction mapping $T : X \rightarrow X$ has a unique fixed point x of T in X i.e. $Tx = x$.

3. Homotopy Analysis Method (HAM)

Consider,

$$N[u] = 0,$$

where N is a nonlinear operator, $u(x)$ is unknown function and x is an independent variable. Let $u_0(x)$ denote an initial guess of the exact solution $u(x)$, $\hbar \neq 0$ an auxiliary parameter, $H_1(x) \neq 0$ an auxiliary function, and L an auxiliary linear operator with the property $L[s(x)] = 0$ when $s(x) = 0$. Then using $q \in [0, 1]$ as an

embedding parameter, we can construct a homotopy when consider, $N[u] = 0$, as follows [4, 6, 14, 15, 21]:

$$(1 - q)L[\phi(x; q) - u_0(x)] - q\hbar H_1(x)N[\phi(x; q)] = \hat{H}[\phi(x; q); u_0(x), H_1(x), \hbar, q]. \quad (3.1)$$

It should be emphasized that we have great freedom to choose the initial guess $u_0(x)$, the auxiliary linear operator L , the non-zero auxiliary parameter \hbar , and the auxiliary function $H_1(x)$. Enforcing the homotopy Eq.(3.1) to be zero, i.e.,

$$\hat{H}_1[\phi(x; q); u_0(x), H_1(x), \hbar, q] = 0, \quad (3.2)$$

we have the so-called zero-order deformation equation

$$(1 - q)L[\phi(x; q) - u_0(x)] = q\hbar H_1(x)N[\phi(x; q)], \quad (3.3)$$

when $q = 0$, the zero-order deformation Eq.(3.3) becomes

$$\phi(x; 0) = u_0(x), \quad (3.4)$$

and when $q = 1$, since $\hbar \neq 0$ and $H_1(x) \neq 0$, the zero-order deformation Eq.(3.3) is equivalent to

$$\phi(x; 1) = u(x). \quad (3.5)$$

Thus, according to Eqs.(3.4) and (3.5), as the embedding parameter q increases from 0 to 1, $\phi(x; q)$ varies continuously from the initial approximation $u_0(x)$ to the exact solution $u(x)$. Such a kind of continuous variation is called deformation in homotopy [14, 23]. Due to Taylor's theorem, $\phi(x; q)$ can be expanded in a power series of q as follows

$$\phi(x; q) = u_0(x) + \sum_{m=1}^{\infty} u_m(x)q^m, \quad (3.6)$$

where

$$u_m(x) = \frac{1}{m!} \frac{\partial^m \phi(x; q)}{\partial q^m} \Big|_{q=0}. \quad (3.7)$$

Let the initial guess $u_0(x)$, the auxiliary linear parameter L , the nonzero auxiliary parameter \hbar and the auxiliary function $H_1(x)$ be properly chosen so that the power series (3.6) of $\phi(x; q)$ converges at $q = 1$, then, we have under these assumptions the solution series

$$u(x) = \phi(x; 1) = u_0(x) + \sum_{m=1}^{\infty} u_m(x). \quad (3.8)$$

From Eq.(3.6), we can write Eq.(3.3) as follows:

$$\begin{aligned} (1 - q)L[\phi(x; q) - u_0(x)] &= (1 - q)L\left[\sum_{m=1}^{\infty} u_m(x)q^m\right] \\ &= q\hbar H_1(x)N[\phi(x; q)], \end{aligned} \quad (3.9)$$

then

$$L\left[\sum_{m=1}^{\infty} u_m(x)q^m\right] - qL\left[\sum_{m=1}^{\infty} u_m(x)q^m\right] = q\hbar H_1(x)N[\phi(x; q)]. \tag{3.10}$$

By differentiating Eq.(3.10) m times with respect to q , we obtain

$$\begin{aligned} \left\{L\left[\sum_{m=1}^{\infty} u_m(x)q^m\right] - qL\left[\sum_{m=1}^{\infty} u_m(x)q^m\right]\right\}^{(m)} &= q\hbar H_1(x)N[\phi(x; q)]^{(m)} \\ &= m!L[u_m(x) - u_{m-1}(x)] \\ &= \hbar H_1(x)m \frac{\partial^{m-1}N[\phi(x; q)]}{\partial q^{m-1}}\Big|_{q=0}. \end{aligned}$$

Therefore,

$$L[u_m(x) - \chi_m u_{m-1}(x)] = \hbar H_1(x)\mathfrak{R}_m(\overrightarrow{u_{m-1}}(x)), \tag{3.11}$$

where

$$\mathfrak{R}_m(\overrightarrow{u_{m-1}}(x)) = \frac{1}{(m-1)!} \frac{\partial^{m-1}N[\phi(x; q)]}{\partial q^{m-1}}\Big|_{q=0}, \tag{3.12}$$

and

$$\chi_m = \begin{cases} 0, & m \leq 1, \\ 1, & m > 1. \end{cases}$$

Note that the high-order deformation Eq.(3.11) is governing the linear operator L , and the term $\mathfrak{R}_m(\overrightarrow{u_{m-1}}(x))$ can be expressed simply by Eq.(3.12) for any nonlinear operator N .

HAM applied to fractional Volterra integro-differential equation

We consider Caputo fractional Volterra integro-differential equation given by (1.1), with the initial condition (1.2). We can define

$$N[\phi(x; q)] = {}^c D^\alpha \phi(x; q) - g(x) - \int_0^x K(x, t)F(\phi(t; q))dt.$$

Now we construct the zero-order deformation equation

$$(1 - q)^c D^\alpha [\phi(x; q) - u_0(x)] = q\hbar N[\phi(x; q)], \tag{3.13}$$

subject to the following initial conditions

$$u_0(x) = \phi(0; q) = u_0 = \sum_{k=0}^{n-1} \delta_k \frac{x^k}{k!}, \tag{3.14}$$

where $q \in [0, 1]$ is the embedding parameter, $\hbar \neq 0$ is an auxiliary parameter, $u_0(x)$ is an initial guess of the solution $u(x)$ and $\phi(x; q)$ is an unknown function on the independent variables x and q . Also we suppose that

$${}^c D^\alpha(C) = 0, \quad (3.15)$$

where C is an integral constant. When the parameter q increases from 0 to 1, then the homotopy solution $\phi(x; q)$ varies from $u_0(x)$ to solution $u(x)$ of the original equation (1.1). Using the parameter q , $\phi(x; q)$ can be expanded in Taylor series as follows:

$$\phi(x; q) = u_0(x) + \sum_{m=1}^{\infty} u_m(x)q^m, \quad (3.16)$$

where $u_m(x)$ define as (3.7).

Assuming that the auxiliary parameter \hbar is properly selected so that the above series is convergent when $q = 1$, then the solution $u(x)$ can be given by

$$u(x) = u_0(x) + \sum_{m=1}^{\infty} u_m(x). \quad (3.17)$$

Differentiating (3.13) and the initial condition (3.14) m times with respect to q , then setting $q = 0$, and finally dividing them by $m!$, we get the m^{th} -order deformation equation

$${}^c D^\alpha [u_m(x) - \chi_m u_{m-1}(x)] = \hbar \mathfrak{R}_m(\overrightarrow{u_{m-1}}(x)), \quad (3.18)$$

subject to the following initial conditions,

$$u_m(0) = 0, \quad (3.19)$$

where

$$\begin{aligned} \mathfrak{R}_m(\overrightarrow{u_{m-1}}(x)) &= \frac{1}{(m-1)!} \frac{\partial^{m-1} N[\phi(x; q)]}{\partial q^{m-1}} \Big|_{q=0} \\ &= {}^c D^\alpha u_{m-1}(x) - \int_0^x K(x, t) F(u_{m-1}(t)) dt - (1 - \chi_m)g(x), \end{aligned}$$

and

$$\overrightarrow{u_m} = u_0, u_1, \dots, u_m.$$

Applying the operator J^α to both sides of the linear m -order deformation (3.18)

$$u_m(x) = (\chi_m + \hbar)u_{m-1}(x) - \hbar J^\alpha \left[\int_0^x K(x, t) F(u_{m-1}(t)) dt + (1 - \chi_m)g(x) \right].$$

4. Main Results

In this section, we shall give an existence and uniqueness results of Eq. (1.1), with the initial condition (1.2) and prove it. Before starting and proving the main results, we introduce the following hypotheses:

(H1) There exists a constant $L_F > 0$ such that, for any $u_1, u_2 \in C(J, \mathbb{R})$

$$|F(u_1(x)) - F(u_2(x))| \leq L_F |u_1 - u_2|.$$

(H2) There exists a function $K^* \in C(D, \mathbb{R}^+)$, the set of all positive function continuous on $D = \{(x, t) \in \mathbb{R} \times \mathbb{R} : 0 \leq t \leq x \leq 1\}$ such that

$$K^* = \sup_{x \in [0,1]} \int_0^x |K(x, t)| dt < \infty.$$

(H3) The function $g : J \rightarrow \mathbb{R}$ is continuous.

Lemma 4.1. *If $u_0(x) \in C(J, \mathbb{R})$, then $u(x) \in C(J, \mathbb{R}^+)$ is a solution of the problem (1.1) – (1.2) iff u satisfies*

$$\begin{aligned} u(x) = & u_0 + \frac{1}{\Gamma(\alpha)} \int_0^x (x - s)^{\alpha-1} g(s) ds \\ & + \frac{1}{\Gamma(\alpha)} \int_0^x (x - s)^{\alpha-1} \left(\int_0^s K(s, \tau) F(u(\tau)) d\tau \right) ds, \end{aligned}$$

for $x \in J$, and $u_0 = \sum_{k=0}^{n-1} \delta_k \frac{x^k}{k!}$.

Now, we will study the existence and uniqueness result of the solution based on the Banach contraction principle.

Theorem 4.2. *Assume that (H1)–(H3) hold. If*

$$\left(\frac{K^* L_F}{\Gamma(\alpha + 1)} \right) < 1, \tag{4.1}$$

then there exists a unique solution $u(x) \in C(J)$ to (1.1) – (1.2).

Proof. By Lemma 4.1. we know that a function u is a solution to (1.1) – (1.2) iff u satisfies

$$\begin{aligned} u(x) = & u_0 + \frac{1}{\Gamma(\alpha)} \int_0^x (x - s)^{\alpha-1} g(s) ds + \frac{1}{\Gamma(\alpha)} \int_0^x (x - s)^{\alpha-1} \\ & \times \left(\int_0^s K(s, \tau) F(u(\tau)) d\tau \right) ds. \end{aligned}$$

Let the operator $T : C(J, \mathbb{R}) \rightarrow C(J, \mathbb{R})$ be defined by

$$\begin{aligned} (Tu)(x) &= u_0 + \frac{1}{\Gamma(\alpha)} \int_0^x (x-s)^{\alpha-1} g(s) ds \\ &\quad + \frac{1}{\Gamma(\alpha)} \int_0^x (x-s)^{\alpha-1} \left(\int_0^s K(s, \tau) F(u(\tau)) d\tau \right) ds. \end{aligned}$$

Firstly, we prove that the operator T is completely continuous. We can see that, if $u \in C(J, \mathbb{R})$ is a fixed point of T , then u is a solution of (1.1) – (1.2).

Now we prove T has a fixed point u in $C(J, \mathbb{R})$. For that, let $u_1, u_2 \in C(J, \mathbb{R})$ and for any $x \in [0, 1]$ such that

$$\begin{aligned} u_1(x) &= u_0 + \frac{1}{\Gamma(\alpha)} \int_0^x (x-s)^{\alpha-1} g(s) ds \\ &\quad + \frac{1}{\Gamma(\alpha)} \int_0^x (x-s)^{\alpha-1} \left(\int_0^s K(s, \tau) F(u_1(\tau)) d\tau \right) ds, \end{aligned}$$

and

$$\begin{aligned} u_2(x) &= u_0 + \frac{1}{\Gamma(\alpha)} \int_0^x (x-s)^{\alpha-1} g(s) ds \\ &\quad + \frac{1}{\Gamma(\alpha)} \int_0^x (x-s)^{\alpha-1} \left(\int_0^s K(s, \tau) F(u_2(\tau)) d\tau \right) ds. \end{aligned}$$

Consequently, we get

$$\begin{aligned} & |(Tu_1)(x) - (Tu_2)(x)| \\ & \leq \frac{1}{\Gamma(\alpha)} \int_0^x (x-s)^{\alpha-1} \left(\int_0^s |K(s, \tau)| |F(u_1(\tau)) - F(u_2(\tau))| d\tau \right) ds \\ & \leq \frac{K^* L_F}{\Gamma(\alpha + 1)} |u_1(x) - u_2(x)| \\ & = \left(\frac{K^* L_F}{\Gamma(\alpha + 1)} \right) |u_1(x) - u_2(x)|. \end{aligned}$$

From the inequality (4.1) we have

$$\|Tu_1 - Tu_2\|_\infty \leq \left(\frac{K^* L_F}{\Gamma(\alpha + 1)} \right) \|u_1 - u_2\|_\infty.$$

This means that T is contraction map. By the Banach contraction principle, we can conclude that T has a unique fixed point u in $C(J, \mathbb{R})$. \square

Now, we will study the convergence theorem of the solutions based on the HAM.

Theorem 4.3. *If the series solution $u(x) = \sum_{m=0}^{\infty} u_m(x)$ obtained by the m -order deformation is convergent, then it converges to the exact solution of the fractional Volterra integro-differential equation (1.1) – (1.2).*

Proof. We assume $\sum_{m=0}^{\infty} u_m(x)$ converge to $u(x)$ then

$$\lim_{m \rightarrow \infty} u_m(x) = 0.$$

We can write

$$\begin{aligned} \sum_{m=1}^n {}^c D^\alpha [u_m(x) - \chi_m u_{m-1}(x)] &= {}^c D^\alpha u_1(x) + ({}^c D^\alpha u_2(x) - {}^c D^\alpha u_1(x)) \\ &\quad + ({}^c D^\alpha u_3(x) - {}^c D^\alpha u_2(x)) + \dots \\ &\quad + ({}^c D^\alpha u_n(x) - {}^c D^\alpha u_{n-1}(x)) \\ &= {}^c D^\alpha u_n(x). \end{aligned} \tag{4.2}$$

Hence, from Eq.(4.2)

$$\lim_{n \rightarrow \infty} u_n(x) = 0. \tag{4.3}$$

So, using Eq.(4.3), we have

$$\sum_{m=1}^{\infty} {}^c D^\alpha [u_m(x) - \chi_m u_{m-1}(x)] = \sum_{m=1}^{\infty} [{}^c D^\alpha u_m(x) - \chi_m {}^c D^\alpha u_{m-1}(x)] = 0.$$

Therefore from Eq.(4.3), we can obtain that

$$\sum_{m=1}^{\infty} {}^c D^\alpha [u_m(x) - \chi_m u_{m-1}(x)] = \hbar \sum_{m=1}^{\infty} \mathfrak{R}_{m-1}(\overrightarrow{u_{m-1}}(x)) = 0.$$

Since $\hbar \neq 0$ and we have

$$\sum_{m=1}^{\infty} \mathfrak{R}_{m-1}(\overrightarrow{u_{m-1}}(x)) = 0. \tag{4.4}$$

By substituting $\mathfrak{R}_{m-1}(\overrightarrow{u_{m-1}}(x))$ into the relation (4.4) and simplifying it, we have

$$\begin{aligned} \mathfrak{R}_{m-1}(\overrightarrow{u_{m-1}}(x)) &= \sum_{m=1}^{\infty} [{}^c D^\alpha u_{m-1}(x) - \int_0^x K(x,t)F(u_{m-1}(t))dt - (1 - \chi_m)g(x)] \\ &= {}^c D^\alpha \left(\sum_{m=1}^{\infty} u_{m-1}(x) \right) - \int_0^x K(x,t) \left[\sum_{m=1}^{\infty} F(u_{m-1}(t)) \right] dt \\ &\quad - \sum_{m=1}^{\infty} (1 - \chi_m)g(x) \\ &= {}^c D^\alpha u(x) - \int_0^x K(x,t)F(u(t))dt - g(x). \end{aligned}$$

From Eq.(4.4) and Eq.(4.5), we have

$${}^c D^\alpha u(x) = g(x) + \int_0^x K(x,t)F(u(t))dt,$$

therefore, $u(x)$ must be the exact solution of Eq.(1.1) and the proof is complete. \square

5. Illustrative Example

In this section, we present the analytical technique based on HAM to solve Caputo fractional Volterra integro-differential equations.

Example 1. Let us consider Caputo fractional Volterra integro-differential equation:

$${}^c D^{0.5}[u(x)] = \frac{32 - 3\sqrt{\pi}}{12\sqrt{\pi}} x^{1.5} + \int_0^x \frac{t}{x^{2.5}} u(t) dt, \quad (5.1)$$

with the initial condition

$$u(0) = 0.$$

From (3.13), (5.1) can be written as

$$N[\phi(x; q)] = {}^c D^{0.5}\phi(x; q) - \frac{32 - 3\sqrt{\pi}}{12\sqrt{\pi}} x^{1.5} - \int_0^x \frac{t}{x^{2.5}} \phi(t; q) dt.$$

Now, using the m^{th} -order deformation equation (3.18) and initial conditions (3.19), and recursive equation (3.20) we can write

$$u_m(x) = (\chi_m + \hbar)u_{m-1}(x) - \hbar J^{0.5}[(1 - \chi_m) \frac{32 - 3\sqrt{\pi}}{12\sqrt{\pi}} x^{1.5} + \int_0^x \frac{t}{x^{2.5}} u_{m-1}(t) dt].$$

Then,

$$\begin{aligned} u_0(x) &= 0, \\ u_1(x) &= \hbar \left(\frac{3\sqrt{\pi}}{32} - 1 \right) x^2, \\ u_2(x) &= \hbar \left(1 - \hbar \left(\frac{3\sqrt{\pi}}{32} - 1 \right) \right) \left(\frac{3\sqrt{\pi}}{32} - 1 \right) x^2, \\ u_3(x) &= \hbar \left(1 - \hbar \left(\frac{3\sqrt{\pi}}{32} - 1 \right) \right)^2 \left(\frac{3\sqrt{\pi}}{32} - 1 \right) x^2, \\ &\vdots \\ &\vdots \\ u_n(x) &= \hbar \left(1 - \hbar \left(\frac{3\sqrt{\pi}}{32} - 1 \right) \right)^{n-1} \left(\frac{3\sqrt{\pi}}{32} - 1 \right) x^2, \\ &\vdots \\ &\vdots \end{aligned}$$

thus the HAM series solution can be written as

$$u_m(x) = \sum_{n=0}^m u_n(x) = \hbar \left(\frac{3\sqrt{\pi}}{32} - 1 \right) \left[1 + \left(1 - \hbar \left(\frac{3\sqrt{\pi}}{32} - 1 \right) \right) + \dots + \left(1 - \hbar \left(\frac{3\sqrt{\pi}}{32} - 1 \right) \right)^{m-1} \right] x^2.$$

The exact solution of (5.1) when $\frac{64}{3\sqrt{\pi}-32} < \hbar < 0$ is

$$\begin{aligned} u(x) &= \sum_{n=0}^{\infty} u_n(x) \\ &= \hbar \left(\frac{3\sqrt{\pi}}{32} - 1 \right) \left[1 + (1 - \hbar \left(\frac{3\sqrt{\pi}}{32} - 1 \right)) + (1 - \hbar \left(\frac{3\sqrt{\pi}}{32} - 1 \right))^2 + \dots \right] x^2 \\ &= \hbar \left(\frac{3\sqrt{\pi}}{32} - 1 \right) \left(\frac{1}{1 - (1 - \hbar \left(\frac{3\sqrt{\pi}}{32} - 1 \right))} \right) x^2 = x^2. \end{aligned}$$

6. Conclusions

Homotopy analysis method is successfully applied to derive approximate analytical solutions for fractional Volterra integro-differential equations. Also, we proved the existence and convergence of the solution. Moreover, the obtained results show that we can control of the convergence district of homotopy analysis technique by control the auxiliary parameter \hbar . The convergence theorem and the illustrative example establish the precision and efficiency of the proposed technique.

Acknowledgements

The authors present their very grateful thanks to the editor and anonymous referees for their valuable suggestions and comments on improving this paper.

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DOI: 10.7862/rf.2018.9

Ahmed A. Hamoud*

email: drahmed985@yahoo.com

ORCID: 0000-0002-8877-7337

Department of Mathematics

Dr. Babasaheb Ambedkar Marathwada University

Aurangabad-431 004

INDIA

Department of Mathematics

Taiz University

Taiz

YEMEN

(* Corresponding author)

M.Sh. Bani Issa

email: moh.smarh@yahoo.com

ORCID: 0000-0001-5388-9331

Department of Mathematics

P.E.T. Research Foundation Mandya

University of Mysore

570401

INDIA

Kirtiwant P. Ghadle

email: drkp.ghadle@gmail.com

ORCID: 0000-0003-3205-5498

Department of Mathematics

Dr. Babasaheb Ambedkar Marathwada University

Aurangabad-431 004

INDIA

Mohammed Abdulghani

email: mhmed.ghnee@yahoo.com

ORCID: 0000-0003-0113-2649

Ministry of Education

Directorate General of Education Ninawa

IRAQ

Received 07.04.2018

Accepted 29.10.2018

The Real and Complex Convexity

Abidi Jamel

ABSTRACT: We prove that the holomorphic differential equation $\varphi''(\varphi+c) = \gamma(\varphi')^2$ ($\varphi : \mathbb{C} \rightarrow \mathbb{C}$ be a holomorphic function and $(\gamma, c) \in \mathbb{C}^2$) plays a classical role on many problems of real and complex convexity. The condition exactly $\gamma \in \{1, \frac{s-1}{s}/s \in \mathbb{N} \setminus \{0\}\}$ (independently of the constant c) is of great importance in this paper.

On the other hand, let $n \geq 1$, $(A_1, A_2) \in \mathbb{C}^2$, and $g_1, g_2 : \mathbb{C}^n \rightarrow \mathbb{C}$ be two analytic functions. Put $u(z, w) = |A_1w - g_1(z)|^2 + |A_2w - g_2(z)|^2$, $v(z, w) = |A_1w - \overline{g_1}(z)|^2 + |A_2w - \overline{g_2}(z)|^2$, for $(z, w) \in \mathbb{C}^n \times \mathbb{C}$. We prove that u is strictly plurisubharmonic and convex on $\mathbb{C}^n \times \mathbb{C}$ if and only if $n = 1$, $(A_1, A_2) \in \mathbb{C}^2 \setminus \{0\}$ and the functions g_1 and g_2 have a classical representation form described in the present paper.

Now v is convex and strictly psh on $\mathbb{C}^n \times \mathbb{C}$ if and only if $(A_1, A_2) \in \mathbb{C}^2 \setminus \{0\}$, $n \in \{1, 2\}$ and g_1, g_2 have several representations investigated in this paper.

AMS Subject Classification: 32A10, 32A60, 32F17, 32U05, 32W50.

Keywords and Phrases: Analytic; Convex and plurisubharmonic functions; Harmonic function; Inequalities; Holomorphic differential equation; Strictly; Polynomials.

1. Introduction

It is not difficult to prove that if $g : D \rightarrow \mathbb{C}$ be a function (not necessarily holomorphic) such that v is convex over $D \times \mathbb{C}$, then g is an affine function, where D is a convex domain of \mathbb{C}^n , $n \geq 1$ and $v(z, w) = |w - g(z)|^2$, for $(z, w) \in D \times \mathbb{C}$.

But if we consider the case of 2 functions, the problem is difficult. However if $g_1, g_2 : \mathbb{C}^n \rightarrow \mathbb{C}$ be 2 holomorphic functions, $v_1(z, w) = |A_1w - g_1(z)|^2 + |A_2w - g_2(z)|^2$, $v_2(z, w) = v_1(\overline{z}, w)$, for $(z, w) \in \mathbb{C}^n \times \mathbb{C}$ and $A_1, A_2 \in \mathbb{C}$.

We have the questions:

- Find exactly all the conditions described by g_1 and g_2 such that v_1 is convex over $\mathbb{C}^n \times \mathbb{C}$?
- Find exactly all the conditions described by g_1 and g_2 such that v_1 (respectively v_2) is convex and not strictly psh over $\mathbb{C}^n \times \mathbb{C}$?
- Find exactly all the conditions described by g_1 and g_2 such that v_1 (respectively v_2) is convex and strictly psh over $\mathbb{C}^n \times \mathbb{C}$?

Several questions can be studied in this situation.

The class of convex and strictly psh functions is a good family for the study and has several applications in complex analysis, convex analysis in several complex variables, harmonic analysis (representation theory), physics, mechanics and others. For example, the importance of my study of this last class is to discover the existence of an infinite family of convex and strictly psh functions but not strictly convex (or not strictly convex in all Euclidean open ball of the domain of definition) on the above form. It follows that the exact characterization of the (convex and strictly psh) functions of the form $|A_1w - g_1(z)|^2 + |A_2w - g_2(z)|^2$ describe the existence of an important family of holomorphic functions (which is fundamental for the study). Note that if n increases, the problem is difficult if we consider several absolute values.

Using this paper, we can answer to the following question.

Characterize all the holomorphic not constant functions $f_1, f_2 : \mathbb{C}^n \rightarrow \mathbb{C}$ and all the holomorphic not constant functions $F_1, F_2 : \mathbb{C}^m \rightarrow \mathbb{C}$, such that u is convex (respectively convex and strictly psh) over $\mathbb{C}^n \times \mathbb{C}^m$, where $n, m \geq 1$ and

$$u(z, w) = |f_1(z) - F_1(w)|^2 + |f_2(z) - F_2(w)|^2$$

for each $(z, w) \in \mathbb{C}^n \times \mathbb{C}^m$.

Now, for example, given $g_1, g_2 : \mathbb{C}^n \rightarrow \mathbb{C}$ be two analytic functions, $n \geq 1$ and $A_1, A_2 \in \mathbb{C} \setminus \{0\}$. Define $u(z, w) = |A_1w - g_1(z)|^2 + |A_2w - g_2(z)|^2$, for $(z, w) \in \mathbb{C}^n \times \mathbb{C}$. We prove that u is convex and strictly plurisubharmonic on $\mathbb{C}^n \times \mathbb{C}$ if and only if $n = 1$, g_1 and g_2 satisfies

$$\begin{cases} g_1(z) = A_1(az + b) + \overline{A_2}(cz + d) \\ g_2(z) = A_2(az + b) - \overline{A_1}(cz + d) \end{cases}$$

(for each $z \in \mathbb{C}$ with $a, b, d \in \mathbb{C}$ and $c \in \mathbb{C} \setminus \{0\}$), or

$$\begin{cases} g_1(z) = A_1(a_1z + b_1) + \overline{A_2}e^{(c_1z + d_1)} \\ g_2(z) = A_2(a_1z + b_1) - \overline{A_1}e^{(c_1z + d_1)} \end{cases}$$

(for each $z \in \mathbb{C}$, where $a_1, b_1, d_1 \in \mathbb{C}$ and $c_1 \in \mathbb{C} \setminus \{0\}$).

However, the number of the absolute values implies that $n = 1$. The great differences between the classes of functions (convex and strictly psh) and strictly convex is one of the purpose of this paper.

Moreover, if we replace \mathbb{C}^n by a convex domain bounded on \mathbb{C}^n , the above result is not true.

We show extension results of ([3], Corollaire 17), which is the following.

Let $\alpha, \beta \in \mathbb{C}$, ($\alpha \neq \beta$) and $g : \mathbb{C}^n \rightarrow \mathbb{C}$ be analytic. Using holomorphic differential equations, we prove that $|g + \alpha|$ and $|g + \beta|$ are convex functions over \mathbb{C}^n if and only if g is an affine function on \mathbb{C}^n .

Observe that the complex structure plays a key role in this situation. For example, let $\varphi(z) = x_1^2 + 1$, for $z = (z_1, \dots, z_n) \in \mathbb{C}^n$, $z_1 = (x_1 + iy_1) \in \mathbb{C}$, where $x_1, y_1 \in \mathbb{R}$. Then φ is real analytic on \mathbb{C}^n . $|\varphi + 0| = |\varphi|$ and $|\varphi + 1|$ are convex functions on \mathbb{C}^n . But φ is not affine on \mathbb{C}^n .

Let U be a domain of \mathbb{R}^d , ($d \geq 2$). We denote by $sh(U)$ the subharmonic functions on U and m_d the Lebesgue measure on \mathbb{R}^d . Let $f : U \rightarrow \mathbb{C}$ be a function. $|f|$ is the modulus of f , $Re(f)$ is the real part of f . $supp(f)$ is the support of f . For $N \geq 1$ and $h = (h_1, \dots, h_N)$, where $h_1, \dots, h_N : U \rightarrow \mathbb{C}$, $\|h\| = (|h_1|^2 + \dots + |h_N|^2)^{\frac{1}{2}}$.

Let $g : D \rightarrow \mathbb{C}$ be an analytic function, D is a domain of \mathbb{C} . We denote by $g^{(0)} = g, g^{(1)} = g'$ is the holomorphic derivative of g over D . $g^{(2)} = g'', g^{(3)} = g'''$. In general $g^{(m)} = \frac{\partial^m g}{\partial z^m}$ is the holomorphic derivative of g of order m , for all $m \in \mathbb{N}$.

Let $z \in \mathbb{C}^n, z = (z_1, \dots, z_n), n \geq 1$. For $n \geq 2$ and $j \in \{1, \dots, n\}$, we write $z = (z_j, Z_j) = (z_1, \dots, z_{j-1}, z_j, z_{j+1}, \dots, z_n)$ where $Z_j = (z_1, \dots, z_{j-1}, z_{j+1}, \dots, z_n) \in \mathbb{C}^{n-1}$. If $\xi = (\xi_1, \dots, \xi_n) \in \mathbb{C}^n$, we denote $\langle z/\xi \rangle = z_1 \xi_1 + \dots + z_n \xi_n$ and $B(\xi, r) = \{\zeta \in \mathbb{C}^n / \|\zeta - \xi\| < r\}$ for $r > 0$, where $\sqrt{\langle \xi/\xi \rangle} = \|\xi\|$ is the Euclidean norm of ξ .

$C(U) = \{\varphi : U \rightarrow \mathbb{C}/\varphi \text{ is continuous on } U\}$.

$C^k(U) = \{\varphi : U \rightarrow \mathbb{C}/\varphi \text{ is of class } C^k \text{ on } U\}$ and $C_c^k(U) = \{\varphi : U \rightarrow \mathbb{C}/\varphi \in C^k(U) \text{ and have a compact support on } U\}$, $k \in \mathbb{N} \cup \{\infty\}$ and $k \geq 1$.

Let $\varphi : U \rightarrow \mathbb{C}$ be a function of class C^2 . $\Delta(\varphi)$ is the Laplacian of φ .

Let D be a domain of $\mathbb{C}^n, (n \geq 1)$. $psh(D)$ and $prh(D)$ are respectively the class of plurisubharmonic and pluriharmonic functions on D .

Definition 1. Let $\varphi : D \rightarrow \mathbb{R}$ be a function of class C^2 and $a \in D$. We say that φ is strictly plurisubharmonic at a if $\sum_{j,k=1}^n \frac{\partial^2 \varphi}{\partial z_j \partial \bar{z}_k}(a) \alpha_j \bar{\alpha}_k > 0$, for all $\alpha = (\alpha_1, \dots, \alpha_n) \in \mathbb{C}^n \setminus \{0\}$.

Moreover, we say that φ is strictly plurisubharmonic on D if φ is strictly psh at every point $a \in D$.

For all $a \in \mathbb{C}, |a|$ is the modulus of a . $Re(a)$ is the real part of a . $D(a, r) = \{z \in \mathbb{C} / |z - a| < r\}$ and $\partial D(a, r) = \{z \in \mathbb{C} / |z - a| = r\}$, for $r > 0$.

For p an analytic polynomial over \mathbb{C} , $deg(p)$ is the degree of p .

For the study of properties and extension problems of analytic and plurisubharmonic functions we cite the references [1], [4], [5], [6], [7], [8], [10], [13], [14], [15], [16], [19], [20], [21], [24], [25], [26], [27], [29], [30], [32], [34], [35] and [12]. Several properties of analytic functions and their graphs are obtained in [12] and [13].

The class of n -harmonic functions is introduced by Rudin in [33]. There are many investigations of plurisubharmonic functions in [2], [18], [22], [23], [28], [29], [31], [11] and [9]. Good references for the study of convex functions in complex convex domains are [17], [21] and [35].

2. A Fundamental Properties over \mathbb{C}^n

The following 4 lemmas (Lemma 1, Lemma 2, Lemma 3 and Lemma 4) are fundamental in this paper. Convex and plurisubharmonic functions are connected by the

Lemma 1. *Let $u : \mathbb{C}^n \rightarrow \mathbb{R}$ be a continuous function, $n \geq 1$. Put $v(z, w) = u(w - \bar{z})$, for $(z, w) \in \mathbb{C}^n \times \mathbb{C}^n$. For $z = (z_1, \dots, z_n), \alpha = (\alpha_1, \dots, \alpha_n) \in \mathbb{C}^n$ and $1 \leq j \leq n$, we write $z_j = (x_j + ix_{j+n})$ and $\alpha_j = (b_j + ib_{j+n})$, where $x_j, x_{j+n}, b_j, b_{j+n} \in \mathbb{R}$.*

The following conditions are equivalent

- (a) u is convex on \mathbb{C}^n ;
- (b) v is psh on $\mathbb{C}^n \times \mathbb{C}^n$;
- (c) For all $\varphi \in C_c^\infty(\mathbb{C}^n)$, $\varphi \geq 0$, we have

$$\begin{aligned} \frac{1}{2} \sum_{j,k=1}^{2n} \int u(z) \frac{\partial^2 \varphi}{\partial x_j \partial x_k}(z) b_j b_k dm_{2n}(z) &= \operatorname{Re} \left(\sum_{j,k=1}^n \int u(z) \frac{\partial^2 \varphi}{\partial z_j \partial z_k}(z) \alpha_j \alpha_k dm_{2n}(z) \right) \\ &+ \sum_{j,k=1}^n \int u(z) \frac{\partial^2 \varphi}{\partial z_j \partial \bar{z}_k}(z) \alpha_j \bar{\alpha}_k dm_{2n}(z) \geq 0 \end{aligned}$$

for each $\alpha = (\alpha_1, \dots, \alpha_n) \in \mathbb{C}^n$;

- (d) For all $\varphi \in C_c^\infty(\mathbb{C}^n)$, $\varphi \geq 0$, we have

$$\begin{aligned} \operatorname{Re} \left(\sum_{j,k=1}^n \int u(z) \frac{\partial^2 \varphi}{\partial z_j \partial z_k}(z) \alpha_j \alpha_k dm_{2n}(z) \right) &\leq \frac{1}{4} \sum_{j,k=1}^{2n} \int u(z) \frac{\partial^2 \varphi}{\partial x_j \partial x_k}(z) b_j b_k dm_{2n}(z) \\ &\leq \sum_{j,k=1}^n \int u(z) \frac{\partial^2 \varphi}{\partial z_j \partial \bar{z}_k}(z) \alpha_j \bar{\alpha}_k dm_{2n}(z) \end{aligned}$$

for each $\alpha = (\alpha_1, \dots, \alpha_n) \in \mathbb{C}^n$. (This is an important property in real and complex analysis);

- (e) $\left| \sum_{j,k=1}^n \int u(z) \frac{\partial^2 \varphi}{\partial z_j \partial z_k}(z) \alpha_j \alpha_k dm_{2n}(z) \right| \leq \sum_{j,k=1}^n \int u(z) \frac{\partial^2 \varphi}{\partial z_j \partial \bar{z}_k}(z) \alpha_j \bar{\alpha}_k dm_{2n}(z)$,
for each $\alpha = (\alpha_1, \dots, \alpha_n) \in \mathbb{C}^n$, for each $\varphi \in C_c^\infty(\mathbb{C}^n)$, $\varphi \geq 0$.

Proof. (a) implies (b) is evident.

(b) implies (a).

Case 1. $n = 1$.

Let $\rho : \mathbb{C} \rightarrow \mathbb{R}_+$, ρ is a radial C^∞ function, $\operatorname{supp}(\rho) \subset D(0, 1)$ and $\int \rho(\xi) dm_2(\xi) = 1$.

For all $\delta > 0$, we define ρ_δ by $\rho_\delta(\xi) = \frac{1}{\delta^2} \rho(\frac{\xi}{\delta})$, for $\xi \in \mathbb{C}$.

Observe that $v(z, \cdot)$ is sh and continuous on \mathbb{C} .

Fix $\delta > 0$ and $z \in \mathbb{C}$. We have

$$\begin{aligned} v(z, \cdot) * \rho_\delta(w) &= \int v(z, w - \xi) \rho_\delta(\xi) dm_2(\xi) = \int u(w - \xi - \bar{z}) \rho_\delta(\xi) dm_2(\xi) \\ &= \varphi_\delta(w - \bar{z}) = \psi_\delta(z, w), \end{aligned}$$

where $\varphi_\delta(\zeta) = \int u(\zeta - \xi) \rho_\delta(\xi) dm_2(\xi) = u * \rho_\delta(\zeta)$, for $\zeta \in \mathbb{C}$.

Therefore the function φ_δ is C^∞ on \mathbb{C} . Consequently, ψ_δ is C^∞ on \mathbb{C}^2 .

Let $A(z, w, \xi) = v(z, w - \xi) \rho_\delta(\xi)$, for $z, w, \xi \in \mathbb{C}$. Since u is continuous on \mathbb{C} , then A is continuous on \mathbb{C}^3 . Note that the function $A(\cdot, \cdot, \xi)$ is psh on \mathbb{C}^2 , for each $\xi \in \mathbb{C}$. Since ρ_δ have a support compact, then by ([32], p.75), ψ_δ is psh on \mathbb{C}^2 .

Consequently, ψ_δ is C^∞ and psh over \mathbb{C}^2 .

By ([3], Lemme 3 p. 339), the function φ_δ is convex over \mathbb{C} . Thus $u * \rho_\delta$ is a convex function on \mathbb{C} , for all $\delta > 0$. The sequence of functions $(u * \rho_{\frac{1}{j}})$, (for $j \in \mathbb{N} \setminus \{0\}$), converges to the function u uniformly over all compact subset of \mathbb{C} because u is continuous. Therefore, u is convex on \mathbb{C} .

Case 2. $n \geq 2$. This proof is similar to the Case 1.

(a) implies (c) is well known.

(c) implies (a).

Let $j \in \{1, \dots, 2n\}$. If $b_j = 1$ and $b_k = 0$, for all $k \neq j$, then

$$\int u(z) \frac{\partial^2 \varphi}{\partial x_j^2}(z) dm_{2n}(z) \geq 0.$$

It follows that

$$\sum_{j=1}^{2n} \int u(z) \frac{\partial^2 \varphi}{\partial x_j^2}(z) dm_{2n}(z) = \int u(z) \Delta \varphi(z) dm_{2n}(z) \geq 0,$$

for all $\varphi \in C_c^\infty(\mathbb{C}^n)$, $\varphi \geq 0$.

Therefore $u = v$ on $\mathbb{C}^n \setminus E$, where v is a subharmonic function on \mathbb{C}^n and E is a borelien subset of \mathbb{C}^n with $m_{2n}(E) = 0$.

Now, assume that u is not subharmonic on \mathbb{C}^n . Then there exists $z_0 \in \mathbb{C}^n$ and $r > 0$ such that

$$u(z_0) > \frac{1}{m_{2n}(B(z_0, r))} \int_{B(z_0, r)} u(\xi) dm_{2n}(\xi).$$

Since

$$\int_{B(z_0, r)} u(\xi) dm_{2n}(\xi) = \int_{B(z_0, r)} v(\xi) dm_{2n}(\xi),$$

it follows that $u(z_0) > v(z_0)$ and consequently, $v(z_0) - u(z_0) < 0$.

Since u is continuous on \mathbb{C}^n , then $(v - u)$ is an upper semi-continuous function on \mathbb{C}^n . Therefore, there exists $\eta \in]0, r[$ such that $(v - u) < 0$ on $B(z_0, \eta)$. Since $m_{2n}(B(z_0, \eta)) > 0$ and $u = v$ on $\mathbb{C}^n \setminus E$, we have a contradiction.

The rest of the proof of this lemma is similar to the two above proofs. \square

Remark 1. The constant $\frac{1}{4}$ is the good constant for the two inequalities in the assertion (d) at Lemma 1.

Let D be a not empty convex domain of \mathbb{C}^n , $n \geq 1$ and $s \in \mathbb{N} \setminus \{0, 1\}$. There does not exist a constant $c > 0$ such that for all $u : D \rightarrow \mathbb{R}$ be a function of class C^s and convex on D , we have

$$\frac{1}{c} \left| \sum_{j,k=1}^n \frac{\partial^2 u}{\partial z_j \partial z_k}(z) \alpha_j \alpha_k \right| \leq \sum_{j,k=1}^{2n} \frac{\partial^2 u}{\partial x_j \partial x_k}(z) b_j b_k \leq c \left| \sum_{j,k=1}^n \frac{\partial^2 u}{\partial z_j \partial z_k}(z) \alpha_j \alpha_k \right|,$$

$\forall z = (z_1, \dots, z_n) \in D$, $\forall \alpha = (\alpha_1, \dots, \alpha_n) \in \mathbb{C}^n$, $z_j = (x_j + ix_{j+n})$, $\alpha_j = (b_j + ib_{j+n})$, $(x_j, x_{j+n}, b_j, b_{j+n} \in \mathbb{R})$, $1 \leq j \leq n$.

Lemma 2. Let $a, b, c \in \mathbb{C}$. We have

- (A) $(a\alpha\bar{\alpha} + b\beta\bar{\beta} + 2\operatorname{Re}(c\alpha\bar{\beta})) \geq 0$, for all $(\alpha, \beta) \in \mathbb{C}^2$ if and only if $(a \geq 0, b \geq 0$ and $|c|^2 \leq ab)$.
- (B) $(a\alpha\bar{\alpha} + b\beta\bar{\beta} + 2\operatorname{Re}(c\alpha\bar{\beta})) > 0$, for all $(\alpha, \beta) \in \mathbb{C}^2 \setminus \{0\}$ if and only if $(a > 0, b > 0$ and $|c|^2 < ab)$.

Proof. See ([3], Lemme 9, p. 354).

Lemma 3. Let $u : G \rightarrow \mathbb{R}$ and $h : D \rightarrow \mathbb{C}$, G is a convex domain of \mathbb{C}^n , D is a domain of \mathbb{C}^n , $n \geq 1$. Suppose that u is a function of class C^2 on G and h is a pluriharmonic (prh) function over D . Then we have

- (A) The Levi hermitian form of $|h|^2$ is

$$\begin{aligned} L(|h|^2)(z)(\alpha) &= \sum_{j,k=1}^n \frac{\partial^2 (|h|^2)}{\partial z_j \partial \bar{z}_k}(z) \alpha_j \bar{\alpha}_k \\ &= \left| \sum_{j=1}^n \frac{\partial h}{\partial z_j}(z) \alpha_j \right|^2 + \left| \sum_{j=1}^n \frac{\partial \bar{h}}{\partial \bar{z}_j}(z) \alpha_j \right|^2 \end{aligned}$$

for each $z = (z_1, \dots, z_n) \in D$, for all $\alpha = (\alpha_1, \dots, \alpha_n) \in \mathbb{C}^n$.
We can also study the case where h is n -harmonic on D .

- (B) u is convex on G if and only if

$$\left| \sum_{j,k=1}^n \frac{\partial^2 u}{\partial z_j \partial z_k}(z) \alpha_j \alpha_k \right| \leq \sum_{j,k=1}^n \frac{\partial^2 u}{\partial z_j \partial \bar{z}_k}(z) \alpha_j \bar{\alpha}_k$$

for each $z \in G$ and all $\alpha = (\alpha_1, \dots, \alpha_n) \in \mathbb{C}^n$.
 u is strictly convex on G if and only if

$$\left| \sum_{j,k=1}^n \frac{\partial^2 u}{\partial z_j \partial z_k}(z) \alpha_j \alpha_k \right| < \sum_{j,k=1}^n \frac{\partial^2 u}{\partial z_j \partial \bar{z}_k}(z) \alpha_j \bar{\alpha}_k$$

for each $z \in G$ and every $\alpha = (\alpha_1, \dots, \alpha_n) \in \mathbb{C}^n \setminus \{0\}$.

Proof. Let $z = (z_1, \dots, z_n) \in D$, $\alpha = (\alpha_1, \dots, \alpha_n) \in \mathbb{C}^n$.

$\forall j, k \in \{1, \dots, n\}$, since h is prh on D then

$$\frac{\partial^2(\|h\|^2)}{\partial z_j \partial \bar{z}_k}(z) = \frac{\partial h}{\partial z_j}(z) \frac{\partial(\bar{h})}{\partial \bar{z}_k}(z) + \frac{\partial h}{\partial \bar{z}_k}(z) \frac{\partial(\bar{h})}{\partial z_j}(z).$$

Therefore,

$$\begin{aligned} \sum_{j,k=1}^n \frac{\partial^2(\|h\|^2)}{\partial z_j \partial \bar{z}_k}(z) \alpha_j \bar{\alpha}_k &= \sum_{j,k=1}^n \frac{\partial h}{\partial z_j}(z) \alpha_j \frac{\partial(\bar{h})}{\partial \bar{z}_k}(z) \bar{\alpha}_k + \sum_{j,k=1}^n \frac{\partial h}{\partial \bar{z}_k}(z) \bar{\alpha}_k \frac{\partial(\bar{h})}{\partial z_j}(z) \alpha_j \\ &= \left(\sum_{j=1}^n \frac{\partial h}{\partial z_j}(z) \alpha_j \right) \overline{\left(\sum_{k=1}^n \frac{\partial(\bar{h})}{\partial \bar{z}_k}(z) \bar{\alpha}_k \right)} + \left(\sum_{j=1}^n \frac{\partial h}{\partial \bar{z}_k}(z) \bar{\alpha}_k \right) \overline{\left(\sum_{k=1}^n \frac{\partial(\bar{h})}{\partial z_j}(z) \alpha_j \right)} \\ &= \left| \sum_{j=1}^n \frac{\partial h}{\partial z_j}(z) \alpha_j \right|^2 + \left| \sum_{j=1}^n \frac{\partial \bar{h}}{\partial z_j}(z) \alpha_j \right|^2. \end{aligned}$$

□

The following lemma plays a classical role on several problems of complex analysis. Several fundamental properties of pluripotential theory deduced by this lemma was obtained in this paper.

Lemma 4. Let $f_1, \dots, f_N, g_1, \dots, g_N : D \rightarrow \mathbb{C}$, D is a domain of \mathbb{C}^n , $n, N \geq 1$.

Put $f = (f_1, \dots, f_N)$, $g = (g_1, \dots, g_N)$ and assume that $f_1, \dots, f_N, g_1, \dots, g_N$ are holomorphic functions on D . Let $u : D \rightarrow \mathbb{R}$ be a function of class C^2 . Then $(\|f\|^2 + \|g\|^2)$ and $(\|f + \bar{g}\|^2)$ have the same hermitian Levi form over D . In particular $(u + \|f\|^2 + \|g\|^2)$ is strictly psh on D if and only if $(u + \|f + \bar{g}\|^2)$ is strictly psh on D .

Proof. $\|f + \bar{g}\|^2 = \sum_{j=1}^N |f_j + \bar{g}_j|^2 = \|f\|^2 + \|g\|^2 + \sum_{j=1}^N \bar{f}_j \bar{g}_j + \sum_{j=1}^N f_j g_j$ on D .

Observe that $\sum_{j=1}^N (f_j g_j + \bar{f}_j \bar{g}_j) = 2\text{Re}(\sum_{j=1}^N f_j g_j)$ is a pluriharmonic (prh) function on

D . Consequently, the Levi hermitian form of the function $\sum_{j=1}^N (f_j g_j + \bar{f}_j \bar{g}_j)$ is equal zero on $D \times \mathbb{C}^n$. It follows that $\|f + \bar{g}\|^2$ and $(\|f\|^2 + \|g\|^2)$ have the same hermitian Levi form on D . □

Now we choose a proof which is classical in complex analysis of the following.

Theorem 1. Let $g_1, g_2 : \mathbb{C}^n \rightarrow \mathbb{C}$ be two analytic functions, $n \geq 1$ and $A_1, A_2 \in \mathbb{C} \setminus \{0\}$. Put

$$u_{(a,b)}(z) = |A_1(\langle z/a \rangle + b) - g_1(z)|^2 + |A_2(\langle z/a \rangle + b) - g_2(z)|^2 = u(z),$$

for $z \in \mathbb{C}^n$, $a \in \mathbb{C}^n$ and $b \in \mathbb{C}$.

The following conditions are equivalent

(A) $u_{(a,b)}$ is convex on \mathbb{C}^n , for all $a \in \mathbb{C}^n$ and $b \in \mathbb{C}$;

(B)

$$\begin{cases} g_1(z) = A_1(\langle z/a_1 \rangle + b_1) + \overline{A_2}(\langle z/c_1 \rangle + d_1)^m \\ g_2(z) = A_2(\langle z/a_1 \rangle + b_1) - \overline{A_1}(\langle z/c_1 \rangle + d_1)^m \end{cases}$$

(for each $z \in \mathbb{C}^n$ with $a_1, c_1 \in \mathbb{C}^n, b_1, d_1 \in \mathbb{C}, m \in \mathbb{N}$), or

$$\begin{cases} g_1(z) = A_1(\langle z/a_2 \rangle + b_2) + \overline{A_2}e^{\langle z/c_2 \rangle + d_2} \\ g_2(z) = A_2(\langle z/a_2 \rangle + b_2) - \overline{A_1}e^{\langle z/c_2 \rangle + d_2} \end{cases}$$

(for each $z \in \mathbb{C}^n$, where $a_2, c_2 \in \mathbb{C}^n, b_2, d_2 \in \mathbb{C}$).

Proof. Case 1. $n = 1$.

(A) implies (B). For $a, b \in \mathbb{C}$, $u_{(\bar{a},b)}$ is a function of class C^∞ on \mathbb{C}^2 . Therefore we have

$$\left| \frac{\partial^2 u_{(\bar{a},b)}(z)}{\partial z^2} \right| \leq \frac{\partial^2 u_{(\bar{a},b)}(z)}{\partial z \partial \bar{z}}, \quad \forall z \in \mathbb{C}, \forall (a, b) \in \mathbb{C}^2.$$

Fix $z \in \mathbb{C}$. Then

$$\begin{aligned} & \left| g_1''(z) \overline{[A_1(az + b) - g_1(z)]} + g_2''(z) \overline{[A_2(az + b) - g_2(z)]} \right| \\ & \leq |A_1 a - g_1'(z)|^2 + |A_2 a - g_2'(z)|^2, \end{aligned}$$

for all $a, b \in \mathbb{C}$.

State 1. Take $a = 0$. Then

$$\left| -g_1''(z) \overline{g_1(z)} - g_2''(z) \overline{g_2(z)} + \bar{b}(\overline{A_1 g_1''(z)} + \overline{A_2 g_2''(z)}) \right| \leq |g_1'(z)|^2 + |g_2'(z)|^2,$$

for all $b \in \mathbb{C}$.

If $(\overline{A_1 g_1''(z)} + \overline{A_2 g_2''(z)}) \neq 0$. Then the subset \mathbb{C} is bounded. A contradiction.

It follows that $(\overline{A_1 g_1''} + \overline{A_2 g_2''}) = 0$ over \mathbb{C} . Consequently, $(\overline{A_1 g_1} + \overline{A_2 g_2})$ is an affine function on \mathbb{C} .

State 2. For all $a \in \mathbb{C}$, we have

$$\begin{aligned} & \left| g_1''(z) \overline{[A_1 a z - g_1(z)]} + g_2''(z) \overline{[A_2 a z - g_2(z)]} \right| \\ & \leq |A_1 a - g_1'(z)|^2 + |A_2 a - g_2'(z)|^2, \quad \forall z \in \mathbb{C}. \end{aligned}$$

It follows that

$$\left| g_1''(z) \overline{g_1(z)} + g_2''(z) \overline{g_2(z)} \right| \leq |A_1 a - g_1'(z)|^2 + |A_2 a - g_2'(z)|^2$$

for each $z \in \mathbb{C}$. We have

$$\begin{aligned} & (|A_1|^2 + |A_2|^2) |a|^2 - 2\operatorname{Re}[\bar{a}(\overline{A_1}g_1'(z) + \overline{A_2}g_2'(z))] + |g_1'(z)|^2 + |g_2'(z)|^2 \\ & - |g_1''(z)\overline{g_1(z)} + g_2''(z)\overline{g_2(z)}| \geq 0, \forall z \in \mathbb{C}, \quad \forall a \in \mathbb{C}. \end{aligned}$$

Now observe that

$$\begin{aligned} & (|A_1|^2 + |A_2|^2) |a|^2 - 2\operatorname{Re}[\bar{a}(\overline{A_1}g_1'(z) + \overline{A_2}g_2'(z))] + |g_1'(z)|^2 + |g_2'(z)|^2 \\ & - |g_1''(z)\overline{g_1(z)} + g_2''(z)\overline{g_2(z)}| = |a\sqrt{|A_1|^2 + |A_2|^2}|^2 \\ & - \frac{1}{\sqrt{|A_1|^2 + |A_2|^2}} |\overline{A_1}g_1'(z) + \overline{A_2}g_2'(z)|^2 + \frac{-1}{|A_1|^2 + |A_2|^2} |\overline{A_1}g_1'(z) + \overline{A_2}g_2'(z)|^2 \\ & + |g_1'(z)|^2 + |g_2'(z)|^2 - |g_1''(z)\overline{g_1(z)} + g_2''(z)\overline{g_2(z)}| \geq 0 \end{aligned}$$

for each $a \in \mathbb{C}$.

For $a = \frac{1}{|A_1|^2 + |A_2|^2} (\overline{A_1}g_1'(z) + \overline{A_2}g_2'(z))$, we have

$$\begin{aligned} & \frac{|A_2|^2}{|A_1|^2 + |A_2|^2} |g_1'(z)|^2 + \frac{|A_1|^2}{|A_1|^2 + |A_2|^2} |g_2'(z)|^2 \\ & - \frac{2}{|A_1|^2 + |A_2|^2} \operatorname{Re}[\overline{A_1}A_2g_1'(z)\overline{g_2'(z)}] - |g_1''(z)\overline{g_1(z)} + g_2''(z)\overline{g_2(z)}| \geq 0. \end{aligned}$$

Thus

$$\frac{1}{|A_1|^2 + |A_2|^2} |A_2g_1'(z) - A_1g_2'(z)|^2 - |g_1''(z)\overline{g_1(z)} + g_2''(z)\overline{g_2(z)}| \geq 0$$

for each $z \in \mathbb{C}$. Put $A = \frac{A_1}{A_2} \in \mathbb{C} \setminus \{0\}$.

$\overline{A_1}g_1'' + \overline{A_2}g_2'' = 0$ on \mathbb{C} and then $g_2'' = -\overline{A}g_1''$ over \mathbb{C} .

Therefore we have

(1)

$$\frac{1}{(|A_1|^2 + |A_2|^2)} |A_2g_1'(z) - A_1g_2'(z)|^2 \geq |g_1''(z)(g_1(z) - Ag_2(z))|$$

for each $z \in \mathbb{C}$.

Since $g_1'' = -\frac{1}{A}g_2''$, then

(2)

$$\begin{aligned} & \frac{1}{(|A_1|^2 + |A_2|^2)} |A_2g_1'(z) - A_1g_2'(z)|^2 \geq \\ & \left| \frac{1}{A}g_2''(z)(g_1(z) - Ag_2(z)) \right| = \left| \frac{-1}{A}g_2''(z)(g_1(z) - Ag_2(z)) \right| \end{aligned}$$

for every $z \in \mathbb{C}$.

(1) implies that

$$|g_1''(z)(g_1(z) - A_2g_2(z))| \leq \frac{1}{(|A_1|^2 + |A_2|^2)} |A_2g_1'(z) - A_1g_2'(z)|^2$$

for each $z \in \mathbb{C}$.

Then

$$\begin{aligned} |g_1''(z)(g_1(z) - \frac{A_1}{A_2}g_2(z))| &= \frac{1}{|A_2|^2} |A_2g_1''(z)(A_2g_1(z) - A_1g_2(z))| \\ &\leq \frac{1}{(|A_1|^2 + |A_2|^2)} |A_2g_1'(z) - A_1g_2'(z)|^2 \end{aligned}$$

for each $z \in \mathbb{C}$.

Then we obtain the inequality

(3)

$$|A_2g_1''(z)(A_2g_1(z) - A_1g_2(z))| \leq \frac{|A_2|^2}{(|A_1|^2 + |A_2|^2)} |A_2g_1'(z) - A_1g_2'(z)|^2$$

for every $z \in \mathbb{C}$.

Now (2) implies the following inequality

(4)

$$|-A_1g_1''(z)(A_2g_1'(z) - A_1g_2'(z))| \leq \frac{|A_1|^2}{(|A_1|^2 + |A_2|^2)} |A_2g_1'(z) - A_1g_2'(z)|^2$$

for every $z \in \mathbb{C}$.

The sum between the inequalities (3) and (4) implies that

$$\begin{aligned} &|A_2g_1''(z)(A_2g_1(z) - A_1g_2(z))| + |-A_1g_2''(z)(A_2g_1(z) - A_1g_2(z))| \\ &\leq \frac{(|A_1|^2 + |A_2|^2)}{(|A_1|^2 + |A_2|^2)} |A_2g_1'(z) - A_1g_2'(z)|^2 = |A_2g_1'(z) - A_1g_2'(z)|^2 \end{aligned}$$

for each $z \in \mathbb{C}$.

By the triangle inequality we have

$$|(A_2g_1''(z) - A_1g_2''(z))(A_2g_1(z) - A_1g_2(z))| \leq |A_2g_1'(z) - A_1g_2'(z)|^2$$

for each $z \in \mathbb{C}$.

Now put $\varphi(z) = A_2g_1(z) - A_1g_2(z)$, for $z \in \mathbb{C}$.

Note that $\varphi : \mathbb{C} \rightarrow \mathbb{C}$, φ is holomorphic over \mathbb{C} . φ satisfy the holomorphic differential inequality $|\varphi''\varphi| \leq |\varphi'|^2$ on \mathbb{C} . Then $\varphi''\varphi = \gamma(\varphi')^2$, where $\gamma \in \mathbb{C}$, $|\gamma| \leq 1$.

By ([3], Corollaire 14, p. 361; Théorème 22, p. 362) exactly $\gamma \in \{1, \frac{t-1}{t}/t \in \mathbb{N} \setminus \{0\}\}$.

Therefore $\varphi(z) = (az + b)^s$ for all $z \in \mathbb{C}$, where $a, b \in \mathbb{C}$ and $s \in \mathbb{N}$, or $\varphi(z) = e^{(cz+d)}$, for all $z \in \mathbb{C}$, where $c, d \in \mathbb{C}$.

Step 1. $\varphi(z) = (az + b)^s$, for all $z \in \mathbb{C}$. Then $A_2g_1(z) - A_1g_2(z) = (az + b)^s$.

Now since $\overline{A_1g_1''(z)} + \overline{A_2g_2''(z)} = 0$, then $\overline{A_1g_1(z)} + \overline{A_2g_2(z)} = a_1z + b_1$, for all $z \in \mathbb{C}$, where $a_1, b_1 \in \mathbb{C}$. We have the system

$$\begin{cases} A_2g_1(z) - A_1g_2(z) = (az + b)^s \\ \overline{A_1g_1(z)} + \overline{A_2g_2(z)} = a_1z + b_1 \end{cases}$$

for each $z \in \mathbb{C}$.

It follows that $(|A_2|^2 + |A_1|^2)g_1(z) = \overline{A_2}(az + b)^s + A_1(a_1z + b_1)$, and then

$$\begin{cases} g_1(z) = A_1(a_2z + b_2) + \overline{A_2}(a_3z + b_3)^s \\ g_2(z) = A_2(a_2z + b_2) - \overline{A_1}(a_3z + b_3)^s \end{cases}$$

for each $z \in \mathbb{C}$, where $a_2, b_2, a_3, b_3 \in \mathbb{C}$ and $s \in \mathbb{N}$.

Step 2. $\varphi(z) = e^{(cz+d)}$, for all $z \in \mathbb{C}$.

Then we have by the Step 1, the system

$$\begin{cases} A_2g_1(z) - A_1g_2(z) = e^{(cz+d)} \\ \overline{A_1g_1(z)} + \overline{A_2g_2(z)} = a_1z + b_1 \end{cases}$$

for all $z \in \mathbb{C}$, with $(a_1, b_1 \in \mathbb{C})$.

Then

$$\begin{cases} g_1(z) = A_1(c_1z + d_1) + \overline{A_2}e^{(c_2z+d_2)} \\ g_2(z) = A_2(c_1z + d_1) - \overline{A_1}e^{(c_2z+d_2)} \end{cases}$$

for each $z \in \mathbb{C}$, where $c_1, d_1, c_2, d_2 \in \mathbb{C}$.

(B) implies (A) is evident.

Case 2. $n \geq 2$.

For $z = (z_1, \dots, z_n) \in \mathbb{C}^n$, we write $z = (z_1, Z_1)$, $Z_1 \in \mathbb{C}^{n-1}$, $z_1 \in \mathbb{C}$.

We can prove that $(\overline{A_1g_1} + \overline{A_2g_2})$ is an affine function on \mathbb{C}^n .

$$\overline{A_1g_1(z)} + \overline{A_2g_2(z)} = \langle z/a_0 \rangle + b_0, \quad a_0 \in \mathbb{C}^n, \quad b_0 \in \mathbb{C}.$$

Consider the functions $g_1(\cdot, Z_1)$, $g_2(\cdot, Z_1)$ and we use the problem of fibration as follows. By the Case 1, we have

$$\begin{cases} g_1(z) = A_1[\alpha(Z_1)z_1 + \beta(Z_1)] + \overline{A_2}\varphi(z) \\ g_2(z) = A_2[\alpha(Z_1)z_1 + \beta(Z_1)] - \overline{A_1}\varphi(z), \end{cases}$$

where $\alpha, \beta : \mathbb{C}^{n-1} \rightarrow \mathbb{C}$ and $\varphi : \mathbb{C}^n \rightarrow \mathbb{C}$.

$$A_2g_1(z) - A_1g_2(z) = (|A_1|^2 + |A_2|^2)\varphi(z).$$

Then φ is analytic on \mathbb{C}^n . Consequently,

$$(\overline{A_1g_1(z)} + \overline{A_2g_2(z)}) = (|A_1|^2 + |A_2|^2)[\alpha(Z_1)z_1 + \beta(Z_1)] = \langle z/a_0 \rangle + b_0$$

for each $z \in \mathbb{C}^n$.

Then α and β are analytic functions. α is constant and β is an affine function on \mathbb{C}^{n-1} . Then $\alpha(Z_1)z_1 + \beta(Z_1) = \langle z/\lambda \rangle + \mu$, $\lambda \in \mathbb{C}^n$, $\mu \in \mathbb{C}$ ($z = (z_1, Z_1) \in \mathbb{C}^n$). It follows that $|\varphi|^2$ is convex on \mathbb{C}^n . By ([3], Théorème 20, p. 358), the proof is complete. \square

Theorem 2. Let $g_1, g_2 : \mathbb{C}^n \rightarrow \mathbb{C}$ be two analytic functions, $n \geq 1$ and $A_1, A_2 \in \mathbb{C} \setminus \{0\}$. For all $a \in \mathbb{C}^n$ and $b \in \mathbb{C}$, define

$$u_{(a,b)}(z) = |A_1(\langle z/a \rangle + b) - g_1(z)|^2 + |A_2(\langle z/a \rangle + b) - g_2(z)|^2,$$

$$u_{(a,b,c_1,c_2)}(z) = |A_1(\langle z/a \rangle + b) - g_1(z) + c_1|^2 + |A_2(\langle z/a \rangle + b) - g_2(z) + c_2|^2,$$

for each $z \in \mathbb{C}^n$.

The following assertions are equivalent

- (A) $u_{(a,b)}$ is strictly convex on \mathbb{C}^n , for each $(a, b) \in \mathbb{C}^n \times \mathbb{C}$;
- (B) $n = 1$ and g_1, g_2 are affine functions on \mathbb{C} with the condition $(A_1 g_1' \neq A_2 g_2')$;
- (C) There exists $c_1, c_2 \in \mathbb{C}$ such that $u_{(a,b,c_1,c_2)}$ is strictly convex on \mathbb{C}^n , for every $(a, b) \in \mathbb{C}^n \times \mathbb{C}$.

Proof. (A) implies (B).

Since $u_{(a,b)}$ is strictly convex on \mathbb{C}^n , for each $(a, b) \in \mathbb{C}^n \times \mathbb{C}$, then by Theorem 1, we have

$$\begin{cases} g_1(z) = A_1(\langle z/a_1 \rangle + b_1) + \overline{A_2}(\langle z/c_1 \rangle + d_1)^m \\ g_2(z) = A_2(\langle z/a_1 \rangle + b_1) - \overline{A_1}(\langle z/c_1 \rangle + d_1)^m \end{cases}$$

(for each $z \in \mathbb{C}^n$, where $a_1, c_1 \in \mathbb{C}^n$, $b_1, d_1 \in \mathbb{C}$, $m \in \mathbb{N}$), or

$$\begin{cases} g_1(z) = A_1(\langle z/a_2 \rangle + b_2) + \overline{A_2}e^{(\langle z/c_2 \rangle + d_2)} \\ g_2(z) = A_2(\langle z/a_2 \rangle + b_2) - \overline{A_1}e^{(\langle z/c_2 \rangle + d_2)} \end{cases}$$

(for each $z \in \mathbb{C}^n$, where $a_2, c_2 \in \mathbb{C}^n$, $b_2, d_2 \in \mathbb{C}$).

Case 1.

$$\begin{cases} g_1(z) = A_1(\langle z/a_1 \rangle + b_1) + \overline{A_2}(\langle z/c_1 \rangle + d_1)^m \\ g_2(z) = A_2(\langle z/a_1 \rangle + b_1) - \overline{A_1}(\langle z/c_1 \rangle + d_1)^m \end{cases}$$

for each $z \in \mathbb{C}^n$.

$$\begin{aligned} u_{(a,b)}(z) = & |A_1(\langle z/a \rangle + b - \langle z/a_1 \rangle - b_1) - \overline{A_2}(\langle z/c_1 \rangle + d_1)^m|^2 \\ & + |A_2(\langle z/a \rangle + b - \langle z/a_1 \rangle - b_1) + \overline{A_1}(\langle z/c_1 \rangle + d_1)^m|^2, \end{aligned}$$

where $(a, b) \in \mathbb{C}^n \times \mathbb{C}$.

Choose now $a = a_1$ and $b = b_1$. It follows that

$$u(z) = |\langle z/c_1 \rangle + d_1|^{2m} (|A_1|^2 + |A_2|^2)$$

and u is strictly convex on \mathbb{C}^n .

Thus v is strictly convex on \mathbb{C}^n , where $v(z) = |\langle z/c_1 \rangle|^{2m}$, for $z \in \mathbb{C}^n$. But v is strictly convex on \mathbb{C}^n if and only if $m = 1$, $n = 1$ and $c_1 \in \mathbb{C} \setminus \{0\}$.

$$g_1(z) = A_1(a_1z + b_1) + \overline{A_2}(c_1z + d_1) = \alpha_1z + \beta_1,$$

$$g_2(z) = A_2(a_1z + b_1) - \overline{A_1}(c_1z + d_1) = \alpha_2z + \beta_2,$$

for $z \in \mathbb{C}$, with $\alpha_1, \beta_1, \alpha_2, \beta_2 \in \mathbb{C}$ and $(\alpha_1 \neq 0$ or $\alpha_2 \neq 0)$.

In this case $A_1g'_2 = A_1(A_2a_1 - \overline{A_1}c_1)$, $A_2g'_1 = A_2(A_1a_1 + \overline{A_2}c_1)$.

$A_1g'_2 \neq A_2g'_1$, because $-|A_1|^2c_1 \neq |A_2|^2c_1$.

Case 2.

$$\begin{cases} g_1(z) = A_1(\langle z/a_2 \rangle + b_2) + \overline{A_2}e^{(\langle z/c_2 \rangle + d_2)} \\ g_2(z) = A_2(\langle z/a_2 \rangle + b_2) - \overline{A_1}e^{(\langle z/c_2 \rangle + d_2)} \end{cases}$$

for each $z \in \mathbb{C}$. For $(a, b) \in \mathbb{C}^n \times \mathbb{C}$,

$$\begin{aligned} u_{(a,b)}(z) &= |A_1(\langle z/a \rangle + b - \langle z/a_2 \rangle - b_2) - \overline{A_2}e^{(\langle z/c_2 \rangle + d_2)}|^2 \\ &\quad + |A_2(\langle z/a \rangle + b - \langle z/a_2 \rangle - b_2) + \overline{A_1}e^{(\langle z/c_2 \rangle + d_2)}|^2. \end{aligned}$$

Choose now $a = a_2$ and $b = b_2$. It follows that

$$u(z) = |e^{(\langle z/c_2 \rangle + d_2)}|^2 (|A_1|^2 + |A_2|^2)$$

and u is strictly convex on \mathbb{C}^n . Thus φ is strictly convex on \mathbb{C}^n , where $\varphi(z) = |e^{\langle z/c_2 \rangle}|^2$, for all $z \in \mathbb{C}^n$. But now observe that φ is not strictly convex at all point of \mathbb{C}^n , for all $n \geq 1$. Therefore this case is impossible.

(B) implies (A) is evident.

(B) implies (C). Note that if

$$u_{(a,b,c_1,c_2)}(z) = |A_1(az + b) - g_1(z) + c_1|^2 + |A_2(az + b) - g_2(z) + c_2|^2,$$

$a, b, c_1, c_2 \in \mathbb{C}$, we now prove that

$$0 < |A_1a - g'_1|^2 + |A_2a - g'_2|^2, \quad \text{for each } a \in \mathbb{C}.$$

If $a = \frac{g'_1}{A_1} \in \mathbb{C}$ (g_1 is an affine function), then $a \neq \frac{g'_2}{A_2}$, because if $a = \frac{g'_2}{A_2}$, then $\frac{g'_1}{A_1} = \frac{g'_2}{A_2}$ and therefore $A_2g'_1 = A_1g'_2$. A contradiction.

Consequently, $|A_1a - g'_1|^2 + |A_2a - g'_2|^2 > 0$, for every $a \in \mathbb{C}$. It follows that $u_{(a,b,c_1,c_2)}$ is strictly convex on \mathbb{C} , for all $(a, b, c_1, c_2) \in \mathbb{C}^4$.

(C) implies (B). By the proof of the assertion (A) implies (B), we have

$$\begin{cases} g_1(z) - c_1 = A_1(\langle z/\alpha_1 \rangle + \beta_1) + \overline{A_2}(\langle z/\alpha_2 \rangle + \beta_2)^m \\ g_2(z) - c_2 = A_2(\langle z/\alpha_1 \rangle + \beta_1) - \overline{A_1}(\langle z/\alpha_2 \rangle + \beta_2)^m \end{cases}$$

(for each $z \in \mathbb{C}^n$, where $\alpha_1, \alpha_2 \in \mathbb{C}^n, \beta_1, \beta_2 \in \mathbb{C}, m \in \mathbb{N}$), or

$$\begin{cases} g_1(z) - c_1 = A_1(\langle z/\gamma_1 \rangle + \delta_1) + \overline{A_2}e^{\langle z/\gamma_2 \rangle + \delta_2} \\ g_2(z) - c_2 = A_2(\langle z/\gamma_1 \rangle + \delta_1) - \overline{A_1}e^{\langle z/\gamma_2 \rangle + \delta_2} \end{cases}$$

(for each $z \in \mathbb{C}^n$, where $\gamma_1, \gamma_2 \in \mathbb{C}^n, \delta_1, \delta_2 \in \mathbb{C}$).

Case 1.

$$\begin{cases} g_1(z) - c_1 = A_1(\langle z/\alpha_1 \rangle + \beta_1) + \overline{A_2}(\langle z/\alpha_2 \rangle + \beta_2)^m \\ g_2(z) - c_2 = A_2(\langle z/\alpha_1 \rangle + \beta_1) - \overline{A_1}(\langle z/\alpha_2 \rangle + \beta_2)^m \end{cases}$$

for each $z \in \mathbb{C}^n$.

$$u_{(a,b,c_1,c_2)}(z) = |A_1(\langle z/a \rangle + b - \langle z/\alpha_1 \rangle - \beta_1) + \overline{A_2}(\langle z/\alpha_2 \rangle + \beta_2)^m|^2 \\ + |A_2(\langle z/a \rangle + b - \langle z/\alpha_1 \rangle - \beta_1) - \overline{A_1}(\langle z/\alpha_2 \rangle + \beta_2)^m|^2$$

for each $z \in \mathbb{C}^n$.

Take $a = \alpha_1, b = \beta_1$, then we have

$$u_{(a,b,c_1,c_2)} = (|A_1|^2 + |A_2|^2) |\langle z/\alpha_2 \rangle + \beta_2|^{2m}.$$

Therefore $u_{(a,b,c_1,c_2)}$ is strictly convex on \mathbb{C}^n if and only if $m = n = 1$ and $\alpha_2 \neq 0$.

Therefore $(g_1 - c_1)$ and $(g_2 - c_2)$ are affine functions and consequently, g_1 and g_2 are affine functions.

$$g_1(z) = \lambda_1 z + \mu_1 = A_1(\alpha_1 z + \beta_1) + \overline{A_2}(\alpha_2 z + \beta_2) + c_1,$$

$$g_2(z) = \lambda_2 z + \mu_2 = A_2(\alpha_1 z + \beta_1) - \overline{A_1}(\alpha_2 z + \beta_2) + c_2,$$

where $\lambda_1, \mu_1, \lambda_2, \mu_2 \in \mathbb{C}$. Then $(A_1 g'_2 \neq A_2 g'_1)$.

Case 2.

$$\begin{cases} g_1(z) - c_1 = A_1(\langle z/\gamma_1 \rangle + \delta_1) + \overline{A_2}e^{\langle z/\gamma_2 \rangle + \delta_2} \\ g_2(z) - c_2 = A_2(\langle z/\gamma_1 \rangle + \delta_1) - \overline{A_1}e^{\langle z/\gamma_2 \rangle + \delta_2} \end{cases}$$

for each $z \in \mathbb{C}^n$.

We prove that this case is impossible. □

Using the holomorphic differential equation $k''(k+c) = \gamma(k')^2$ ($k : \mathbb{C} \rightarrow \mathbb{C}, (\gamma, c) \in \mathbb{C}^2, k$ is holomorphic on \mathbb{C}), we prove

Theorem 3. Let $(A_1, A_2) \in \mathbb{C}^2 \setminus \{0\}$ and $n \geq 1$. Given two analytic functions $g_1, g_2 : \mathbb{C}^n \rightarrow \mathbb{C}$. Put $u_{(a,b)}(z) = |A_1(\langle z/a \rangle + b) - g_1(z)|^2 + |A_2(\langle z/a \rangle + b) - g_2(z)|^2$, for $z \in \mathbb{C}^n, (a, b) \in \mathbb{C}^n \times \mathbb{C}$.

The following conditions are equivalent

- (A) $u_{(a,b)}$ is strictly convex on \mathbb{C}^n , for each $(a, b) \in \mathbb{C}^n \times \mathbb{C}$;
- (B) $n = 1, g_1, g_2$ are affine functions on \mathbb{C} and we have the following 3 cases.
 - $A_2 = 0, A_1 \neq 0$. Then $g'_2 \neq 0$.
 - $A_1 = 0, A_2 \neq 0$. Then $g'_1 \neq 0$.
 - $A_1 \neq 0$ and $A_2 \neq 0$. Then $A_2 g'_1 \neq A_1 g'_2$.

Proof. If $A_1 \neq 0$ and $A_2 \neq 0$, we use the above Theorem 2.

Now suppose that $A_2 = 0$ and $A_1 \neq 0$. For $(a, b) \in \mathbb{C}^n \times \mathbb{C}$, $u_{(a,b)}$ is C^∞ and strictly convex on \mathbb{C}^n . Therefore we have

$$\left| \sum_{j,k=1}^n \frac{\partial^2 u_{(a,b)}}{\partial z_j \partial z_k}(z) \alpha_j \alpha_k \right| < \sum_{j,k=1}^n \frac{\partial^2 u_{(a,b)}}{\partial z_j \partial \bar{z}_k}(z) \alpha_j \bar{\alpha}_k$$

for each $z = (z_1, \dots, z_n) \in \mathbb{C}^n$, for every $\alpha = (\alpha_1, \dots, \alpha_n) \in \mathbb{C}^n \setminus \{0\}$.

It follows that for $z = (z_1, \dots, z_n)$ fixed on \mathbb{C}^n , for $a \in \mathbb{C}^n$ fixed and $\alpha = (\alpha_1, \dots, \alpha_n) \in \mathbb{C}^n \setminus \{0\}$ fixed, we have the inequality

$$\begin{aligned} (S) \quad & \left| \overline{g_1}(z) \sum_{j,k=1}^n \frac{\partial^2 g_1}{\partial z_j \partial z_k}(z) \alpha_j \alpha_k + \overline{g_2}(z) \sum_{j,k=1}^n \frac{\partial^2 g_2}{\partial z_j \partial z_k}(z) \alpha_j \alpha_k \right. \\ & \left. - \overline{A_1}(\langle z/a \rangle + b) \sum_{j,k=1}^n \frac{\partial^2 g_1}{\partial z_j \partial z_k}(z) \alpha_j \alpha_k \right| < \left| A_1 \langle \alpha/a \rangle - \sum_{j=1}^n \frac{\partial g_1}{\partial z_j}(z) \alpha_j \right|^2 \\ & + \left| \sum_{j=1}^n \frac{\partial g_2}{\partial z_j}(z) \alpha_j \right|^2 \end{aligned}$$

for each $b \in \mathbb{C}$.

Observe that the right expression of the above strict inequality (S) is independent of b . Therefore if $\sum_{j,k=1}^n \frac{\partial^2 g_1}{\partial z_j \partial z_k}(z) \alpha_j \alpha_k \neq 0$, then the subset \mathbb{C} is bounded.

A contradiction. It follows that

$$\sum_{j,k=1}^n \frac{\partial^2 g_1}{\partial z_j \partial z_k}(z) \alpha_j \alpha_k = 0, \text{ for every } z = (z_1, \dots, z_n) \in \mathbb{C}^n \text{ and } \alpha = (\alpha_1, \dots, \alpha_n) \in \mathbb{C}^n.$$

Since g_1 is a holomorphic function over \mathbb{C}^n , then g_1 is an affine function on \mathbb{C}^n .

Choose $(a_0, b_0) \in \mathbb{C}^n \times \mathbb{C}$ such that $A_1(\langle z/a_0 \rangle + b_0) = g_1(z)$, for all $z \in \mathbb{C}^n$. Therefore $u_{(a_0,b_0)}(z) = |g_2(z)|^2$, for each $z \in \mathbb{C}^n$. Consequently, $|g_2|^2$ is strictly convex on \mathbb{C}^n . Then, $n = 1$. In particular $|g_2|^2$ is convex on \mathbb{C} . By ([3], Théorème 20, p. 358) we have $g_2(z) = (\lambda z + \delta)^s$, (for all $z \in \mathbb{C}$, where $\lambda, \delta \in \mathbb{C}$, $s \in \mathbb{N}$), or $g_2(z) = e^{(\lambda_1 z + \delta_1)}$, (for all $z \in \mathbb{C}$, with $\lambda_1, \delta_1 \in \mathbb{C}$).

Case 1. $g_2(z) = (\lambda z + \delta)^s$, for all $z \in \mathbb{C}$.

We have $|g_2''(z)g_2(z)| < |g_2'(z)|^2$, for each $z \in \mathbb{C}$. Then $\lambda \neq 0$ and $s = 1$.

$$g_2'(z) = \lambda \neq 0, \quad (z \in \mathbb{C}).$$

Case 2. $g_2(z) = e^{(\lambda_1 z + \delta_1)}$, for each $z \in \mathbb{C}$.

$|g_2|^2$ is a function of class C^∞ on \mathbb{C} . We prove that $|g_2|^2$ is not strictly convex at all point of \mathbb{C} . Therefore this case is impossible. \square

Corollary 1. Let $g_1, g_2 : \mathbb{C}^n \rightarrow \mathbb{C}$ be two analytic functions. For $a \in \mathbb{C}^n$, $b, c \in \mathbb{C}$, put

$$u_{(a,b,c)}(z) = |\langle z/a \rangle + b - g_1(z) + c|^2 + |\langle z/a \rangle + b - g_2(z)|^2$$

for $z \in \mathbb{C}^n$. The following conditions are equivalent

- (A) $u_{(a,b,c)}$ is convex on \mathbb{C}^n , for each $(a, b, c) \in \mathbb{C}^n \times \mathbb{C} \times \mathbb{C}$;
- (B) g_1 and g_2 are affine functions on \mathbb{C}^n .

Question. Let $(A_1, A_2) \in \mathbb{C}^2 \setminus \{0\}$ and $n \geq 1$. Find exactly all the analytic functions $g_1, g_2 : \mathbb{C}^n \rightarrow \mathbb{C}$ such that v is convex and u is strictly $(n+1)$ -sh on $\mathbb{C}^n \times \mathbb{C}$, where $v(z, w) = |A_1 w - g_1(z)|^2 + |A_2 w - g_2(z)|^2$ and $u(z, w) = v(z, w) + v(\bar{z}, w)$, for $(z, w) \in \mathbb{C}^n \times \mathbb{C}$?

The case of the conjugate of holomorphic functions

Theorem 4. Let $g_1, g_2 : \mathbb{C}^n \rightarrow \mathbb{C}$ be two analytic functions, where $n \geq 1$. Given $(A_1, A_2) \in (\mathbb{C} \setminus \{0\})^2$ and $u(z, w) = |A_1 w - \overline{g_1}(z)|^2 + |A_2 w - \overline{g_2}(z)|^2$, for $(z, w) \in \mathbb{C}^n \times \mathbb{C}$. The following assertions are equivalent

- (A) u is convex on $\mathbb{C}^n \times \mathbb{C}$;
- (B) We have the two following fundamental representations.

$$\begin{cases} g_1(z) = \overline{A_1}(\langle z/a \rangle + b) + A_2(\langle z/c \rangle + d)^m \\ g_2(z) = \overline{A_2}(\langle z/a \rangle + b) - A_1(\langle z/c \rangle + d)^m \end{cases}$$

(for each $z \in \mathbb{C}^n$, where $a, c \in \mathbb{C}^n, b, d \in \mathbb{C}, m \in \mathbb{N}$), or

$$\begin{cases} g_1(z) = \overline{A_1}(\langle z/\lambda \rangle + \mu) + A_2 e^{(\langle z/\gamma \rangle + \delta)} \\ g_2(z) = \overline{A_2}(\langle z/\lambda \rangle + \mu) - A_1 e^{(\langle z/\gamma \rangle + \delta)} \end{cases}$$

(for each $z \in \mathbb{C}^n$, where $\lambda, \gamma \in \mathbb{C}^n, \mu, \delta \in \mathbb{C}$).

Proof. Let $T(z, w) = (z, \bar{w})$, for $(z, w) \in \mathbb{C}^n \times \mathbb{C}$. T is an \mathbb{R} -linear bijective transformation over $\mathbb{C}^n \times \mathbb{C}$. Therefore, $v = u \circ T$ is convex on $\mathbb{C}^n \times \mathbb{C}$. But

$$v(z, w) = |A_1 \bar{w} - \overline{g_1}(z)|^2 + |A_2 \bar{w} - \overline{g_2}(z)|^2 = |\overline{A_1} w - g_1(z)|^2 + |\overline{A_2} w - g_2(z)|^2$$

for $(z, w) \in \mathbb{C}^n \times \mathbb{C}$. By the Theorem 1, we conclude the proof. \square

Example. Let $g(z) = z^2 + 2, z \in \mathbb{C}$. Put $g_1 = g, g_2 = -g$. Then g_1 and g_2 are analytic functions on \mathbb{C} . Let $D = D(2i, \frac{1}{4})$. Define $u(z, w) = |w - g_1(z)|^2 + |w - g_2(z)|^2, v(z, w) = |w - \overline{g_1}(z)|^2 + |w - \overline{g_2}(z)|^2, (z, w) \in \mathbb{C}^2$. Then $u(z, w) = v(z, w) = 2(|w|^2 + |g(z)|^2), (z, w) \in \mathbb{C}^2$. We have u is strictly convex on $D \times \mathbb{C}$. But we can not write g_1 and g_2 on the form of the above theorem.

Now let $(A_1, A_2) \in \mathbb{C}^2 \setminus \{0\}$. Define $u_1(z, w) = |A_1 w - k_1(z)|^2 + |A_2 w - k_2(z)|^2, v_1(z, w) = |A_1 w - \overline{k_3}(z)|^2 + |A_2 w - \overline{k_4}(z)|^2$, for $(z, w) \in D \times \mathbb{C}$, where $k_1 = \overline{A_2} g, k_2 = -\overline{A_1} g, k_3 = A_2 g, k_4 = -A_1 g$. Note that k_1, k_2, k_3 and k_4 are analytic functions on D . We have $u_1(z, w) = v_1(z, w) = (|A_1|^2 + |A_2|^2)(|w|^2 + |g(z)|^2)$, for $(z, w) \in D \times \mathbb{C}$. Then u_1 and v_1 are functions strictly convex on $D \times \mathbb{C}$, but k_1, k_2, k_3 and k_4 are not affine functions on D .

It follows that in all bounded not empty convex domain of \mathbb{C}^n ($n \geq 1$), the above theorem is false.

Theorem 5. Let $g_1, g_2 : \mathbb{C}^n \rightarrow \mathbb{C}$ be two analytic functions, where $n \geq 1$. Let $(A_1, A_2) \in (\mathbb{C} \setminus \{0\})^2$ and define $v(z, w) = |A_1 w - \overline{g_1(z)}|^2 + |A_2 w - \overline{g_2(z)}|^2$, for $(z, w) \in \mathbb{C}^n \times \mathbb{C}$. The following assertions are equivalent

(A) v is convex and strictly psh on $\mathbb{C}^n \times \mathbb{C}$;

(B) $n \in \{1, 2\}$ and we have the following cases:

If $n = 1$, then

$$\begin{cases} g_1(z) = \overline{A_1}(az + b) + A_2(cz + d)^m \\ g_2(z) = \overline{A_2}(az + b) - A_1(cz + d)^m \end{cases}$$

(for each $z \in \mathbb{C}$, where $a, b, c, d \in \mathbb{C}, m \in \mathbb{N}$ with $(m = 0, a \neq 0), (m = 1, (a, c) \neq (0, 0)), (m \geq 2, a \neq 0)$), or

$$\begin{cases} g_1(z) = \overline{A_1}(\lambda z + \mu) + A_2 e^{(\gamma z + \delta)} \\ g_2(z) = \overline{A_2}(\lambda z + \mu) - A_1 e^{(\gamma z + \delta)} \end{cases}$$

(for each $z \in \mathbb{C}$, where $\lambda, \mu, \gamma, \delta \in \mathbb{C}, (\lambda, \gamma) \neq (0, 0)$).

If $n = 2$, then

$$\begin{cases} g_1(z) = \overline{A_1}(\langle z/a \rangle + b) + A_2(\langle z/c \rangle + d) \\ g_2(z) = \overline{A_2}(\langle z/a \rangle + b) - A_1(\langle z/c \rangle + d) \end{cases}$$

(for each $z \in \mathbb{C}^2$, where $a, c \in \mathbb{C}^2, b, d \in \mathbb{C}$ with the determinant $\det(a, c) \neq 0$), or

$$\begin{cases} g_1(z) = \overline{A_1}(\langle z/\lambda \rangle + \mu) + A_2 e^{(\langle z/\gamma \rangle + \delta)} \\ g_2(z) = \overline{A_2}(\langle z/\lambda \rangle + \mu) - A_1 e^{(\langle z/\gamma \rangle + \delta)} \end{cases}$$

(for each $z \in \mathbb{C}^2$, where $\lambda, \gamma \in \mathbb{C}^2, \mu, \delta \in \mathbb{C}$ with the determinant $\det(\lambda, \gamma) \neq 0$).

Proof. Let $T : \mathbb{C}^n \times \mathbb{C} \rightarrow \mathbb{C}^n \times \mathbb{C}, T(z, w) = (z, \overline{w})$, for $(z, w) \in \mathbb{C}^n \times \mathbb{C}$.

T is an \mathbb{R} linear bijective transformation on $\mathbb{C}^n \times \mathbb{C}$. Then $voT = u$ is convex on $\mathbb{C}^n \times \mathbb{C}$. $u(z, w) = |\overline{A_1}w - g_1(z)|^2 + |\overline{A_2}w - g_2(z)|^2$, for $(z, w) \in \mathbb{C}^n \times \mathbb{C}$.

It follows that

$$\begin{cases} g_1(z) = \overline{A_1}(\langle z/a \rangle + b) + A_2(\langle z/c \rangle + d)^m \\ g_2(z) = \overline{A_2}(\langle z/a \rangle + b) - A_1(\langle z/c \rangle + d)^m \end{cases}$$

(for each $z \in \mathbb{C}^n$, where $a, c \in \mathbb{C}^n, b, d \in \mathbb{C}, m \in \mathbb{N}$), or

$$\begin{cases} g_1(z) = \overline{A_1}(\langle z/\lambda \rangle + \mu) + A_2 e^{(\langle z/\gamma \rangle + \delta)} \\ g_2(z) = \overline{A_2}(\langle z/\lambda \rangle + \mu) - A_1 e^{(\langle z/\gamma \rangle + \delta)} \end{cases}$$

(for each $z \in \mathbb{C}^n$, where $\lambda, \gamma \in \mathbb{C}^n, \mu, \delta \in \mathbb{C}$).

Case 1.

$$\begin{cases} g_1(z) = \overline{A_1}(\langle z/a \rangle + b) + A_2(\langle z/c \rangle + d)^m \\ g_2(z) = \overline{A_2}(\langle z/a \rangle + b) - A_1(\langle z/c \rangle + d)^m \end{cases}$$

for each $z \in \mathbb{C}^n$. We have

$$\begin{aligned}
v(z, w) &= |A_1(w - \overline{\langle z/a \rangle} - \bar{b}) - \overline{A_2(\langle z/c \rangle + d)}|^m|^2 \\
&\quad + |A_2(w - \overline{\langle z/a \rangle} - \bar{b}) + \overline{A_1(\langle z/c \rangle + d)}|^m|^2 \\
&= (|A_1|^2 + |A_2|^2)(|w - \overline{\langle z/a \rangle} - \bar{b}|^2 + |\langle z/c \rangle + d|^{2m}),
\end{aligned}$$

for $(z, w) \in \mathbb{C}^n \times \mathbb{C}$.

Let $v_1(z, w) = |w - \overline{\langle z/a \rangle} - \bar{b}|^2 + |\langle z/c \rangle + d|^{2m}$, $(z, w) \in \mathbb{C}^n \times \mathbb{C}$.
 v and v_1 are functions of class C^∞ on $\mathbb{C}^n \times \mathbb{C}$. Note that v is strictly psh on $\mathbb{C}^n \times \mathbb{C}$ if and only if v_1 is strictly psh on $\mathbb{C}^n \times \mathbb{C}$. By Lemma 4, v_1 is strictly psh on $\mathbb{C}^n \times \mathbb{C}$ if and only if v_2 is strictly psh on $\mathbb{C}^n \times \mathbb{C}$, where

$$v_2(z, w) = |w|^2 + |\langle z/a \rangle + b|^2 + |\langle z/c \rangle + d|^{2m}$$

for each $(z, w) \in \mathbb{C}^n \times \mathbb{C}$ (v_2 is a function of class C^∞ on $\mathbb{C}^n \times \mathbb{C}$).

But the Levi hermitian form of v_2 is

$$L(v_2)(z, w)(\alpha, \beta) = |\beta|^2 + |\langle \alpha/a \rangle|^2 + m^2 |\langle \alpha/c \rangle|^2 |\langle z/c \rangle + d|^{2m-2},$$

for each $(z, w) \in \mathbb{C}^n \times \mathbb{C}$ and all $(\alpha, \beta) \in \mathbb{C}^n \times \mathbb{C}$.

We have $(L(v_2)(z, w)(\alpha, \beta) > 0, \forall (z, w) \in \mathbb{C}^n \times \mathbb{C}, \forall (\alpha, \beta) \in \mathbb{C}^n \times \mathbb{C} \setminus \{0\})$ if and only if $(\varphi_2(z, \alpha) > 0, \forall z \in \mathbb{C}^n, \forall \alpha \in \mathbb{C}^n \setminus \{0\})$, where

$$\varphi_2(\xi, \delta) = |\langle \delta/a \rangle|^2 + m^2 |\langle \delta/c \rangle|^2 |\langle \xi/c \rangle + d|^{2m-2}$$

for $(\xi, \delta) \in \mathbb{C}^n \times \mathbb{C}^n$.

Step 1. $m = 0$.

Then $|\langle \alpha/a \rangle| > 0$, for each $\alpha \in \mathbb{C}^n \setminus \{0\}$. Thus $n = 1$ and $a \in \mathbb{C} \setminus \{0\}$.

In this case we have

$$\begin{cases} g_1(z) = \overline{A_1}(az + b) + A_2 \\ g_2(z) = \overline{A_2}(az + b) - A_1 \end{cases}$$

for each $z \in \mathbb{C}$.

Step 2. $m = 1$.

Let $\varphi_3(\alpha) = \varphi_2(z, \alpha) = |\langle \alpha/a \rangle|^2 + |\langle \alpha/c \rangle|^2$, for $(z, \alpha) \in \mathbb{C}^n \times \mathbb{C}$. Now since we have $\varphi_2(z, \alpha) > 0$, for each $z \in \mathbb{C}^n$, and $\alpha \in \mathbb{C}^n \setminus \{0\}$. Then $\varphi_3(\alpha) = |\langle \alpha/a \rangle|^2 + |\langle \alpha/c \rangle|^2 > 0$, for every $\alpha \in \mathbb{C}^n \setminus \{0\}$.

Put $a = (\overline{a_1}, \dots, \overline{a_n})$, $c = (\overline{c_1}, \dots, \overline{c_n})$. Let $\alpha = (\alpha_1, \dots, \alpha_n) \in \mathbb{C}^n$. We have $\varphi_3(\alpha) = 0$ if and only if $\alpha = 0$. But $\varphi_3(\alpha) = 0$ is equivalent with $\langle \alpha/a \rangle = 0$ and $\langle \alpha/c \rangle = 0$. Therefore

$$\begin{cases} \alpha_1 a_1 + \dots + \alpha_n a_n = 0 \\ \alpha_1 c_1 + \dots + \alpha_n c_n = 0. \end{cases}$$

Then $\alpha_1(a_1, c_1) + \dots + \alpha_n(a_n, c_n) = (0, 0) \in \mathbb{C}^2$ (\mathbb{C}^2 is considered a complex vector space of dimension 2) if and only if $\alpha_1 = \dots = \alpha_n = 0$. Then the set of vectors $\{(a_1, c_1), \dots, (a_n, c_n)\}$ is a free family of n vectors of \mathbb{C}^2 . Therefore $n \leq 2$.

State 1. $n = 1$.

$$|\langle \alpha/a \rangle|^2 + |\langle \alpha/c \rangle|^2 = |\alpha a|^2 + |\alpha c|^2 > 0,$$

for each $\alpha \in \mathbb{C} \setminus \{0\}$. Then $(a, c) \neq (0, 0)$. Therefore

$$\begin{cases} g_1(z) = \overline{A_1}(\overline{a}z + b) + A_2(\overline{c}z + d) \\ g_2(z) = \overline{A_2}(\overline{a}z + b) - A_1(\overline{c}z + d) \end{cases}$$

for each $z \in \mathbb{C}$. We have

$$v_1(z, w) = |w - a\overline{z} - \overline{b}|^2 + |\overline{c}z + d|^2$$

and

$$v_2(z, w) = |w|^2 + |\overline{a}z + b|^2 + |\overline{c}z + d|^2.$$

v_2 is strictly psh on \mathbb{C}^2 because $|a|^2 + |c|^2 > 0$.

State 2. $n = 2$.

In this case $\{(a_1, c_1), (a_2, c_2)\}$ is a basis of the \mathbb{C} -vector space \mathbb{C}^2 . It follows that $\{(a_1, a_2), (c_1, c_2)\}$ is a basis of \mathbb{C}^2 and consequently, $\{(\overline{a_1}, \overline{a_2}), (\overline{c_1}, \overline{c_2})\} = \{a, c\}$ is a basis of \mathbb{C}^2 . Then the determinant $\det(a, c) \neq 0$.

In this case we have

$$\begin{cases} g_1(z) = \overline{A_1}(\langle z/a \rangle + b) + A_2(\langle z/c \rangle + d) \\ g_2(z) = \overline{A_2}(\langle z/a \rangle + b) - A_1(\langle z/c \rangle + d) \end{cases}$$

(for each $z \in \mathbb{C}^2$, where $a, c \in \mathbb{C}^2$, $b, d \in \mathbb{C}$ with the determinant $\det(a, c) \neq 0$).

Step 3. $m \geq 2$.

$$\varphi_2(z, \alpha) = |\langle \alpha/a \rangle|^2 + m^2 |\langle \alpha/c \rangle|^2 |\langle z/c \rangle + d|^{2m-2}, \quad z, \alpha \in \mathbb{C}^n.$$

State 1. $c = 0$.

Then $\varphi_2(z, \alpha) = |\langle \alpha/a \rangle|^2 > 0$, for every $\alpha \in \mathbb{C}^n \setminus \{0\}$.

It follows that $n = 1$. Consequently, $a \neq 0$. In this case we have

$$\begin{cases} g_1(z) = \overline{A_1}(\overline{a}z + b) + A_2 d^m \\ g_2(z) = \overline{A_2}(\overline{a}z + b) - A_1 d^m \end{cases}$$

(for each $z \in \mathbb{C}$, where $a \in \mathbb{C} \setminus \{0\}$, $b, d \in \mathbb{C}$ and $m \in \mathbb{N}$, $m \geq 2$).

State 2. $c \neq 0$.

There exists $z_0 \in \mathbb{C}^n$ such that $|\langle z_0/c \rangle + d| = 0$.

Since $(2m - 2) \geq 2$, then $|\langle z_0/c \rangle + d|^{2m-2} = 0$. It follows that $\varphi_2(z_0, \alpha) = |\langle \alpha/a \rangle|^2 > 0$, for each $\alpha \in \mathbb{C}^n \setminus \{0\}$.

Then $n = 1$ and $a \in \mathbb{C} \setminus \{0\}$. In this case

$$\begin{cases} g_1(z) = \overline{A_1}(\overline{a}z + b) + A_2(cz + d)^m \\ g_2(z) = \overline{A_2}(\overline{a}z + b) - A_1(cz + d)^m \end{cases}$$

(for each $z \in \mathbb{C}$, where $a \in \mathbb{C} \setminus \{0\}$, $c \in \mathbb{C} \setminus \{0\}$, $b, d \in \mathbb{C}$ and $m \in \mathbb{N}$, $m \geq 2$).
Consequently, for $m \geq 2$ and independently of c , we have in all this step 3, $n = 1$ and

$$\begin{cases} g_1(z) = \overline{A_1}(\overline{az + b}) + A_2(cz + d)^m \\ g_2(z) = \overline{A_2}(\overline{az + b}) - A_1(cz + d)^m \end{cases}$$

(for each $z \in \mathbb{C}$, where $a \in \mathbb{C} \setminus \{0\}$, $b, c, d \in \mathbb{C}$ and $m \in \mathbb{N}$, $m \geq 2$).

Case 2.

$$\begin{cases} g_1(z) = \overline{A_1}(\langle z/\lambda \rangle + \mu) + A_2 e^{\langle z/\gamma \rangle + \delta} \\ g_2(z) = \overline{A_2}(\langle z/\lambda \rangle + \mu) - A_1 e^{\langle z/\gamma \rangle + \delta} \end{cases}$$

for all $z \in \mathbb{C}^n$.

$$v(z, w) = (|A_1|^2 + |A_2|^2)(|w - \overline{\langle z/\lambda \rangle} - \overline{\mu}|^2 + |e^{\langle z/\gamma \rangle + \delta}|^2),$$

for $(z, w) \in \mathbb{C}^n \times \mathbb{C}$.

Let $u_1(z, w) = |w - \overline{\langle z/\lambda \rangle} - \overline{\mu}|^2 + |e^{\langle z/\gamma \rangle + \delta}|^2$, for $(z, w) \in \mathbb{C}^n \times \mathbb{C}$.
 v and u_1 are functions of class C^∞ on $\mathbb{C}^n \times \mathbb{C}$. We have v is strictly psh on $\mathbb{C}^n \times \mathbb{C}$ if and only if u_1 is strictly psh on $\mathbb{C}^n \times \mathbb{C}$.

Now define

$$u_2(z, w) = |w|^2 + |\langle z/\lambda \rangle + \mu|^2 + |e^{\langle z/\gamma \rangle + \delta}|^2,$$

for $(z, w) \in \mathbb{C}^n \times \mathbb{C}$. u_2 is a function of class C^∞ on $\mathbb{C}^n \times \mathbb{C}$. By Lemma 4, we have u_1 is strictly psh on $\mathbb{C}^n \times \mathbb{C}$ if and only if u_2 is strictly psh on $\mathbb{C}^n \times \mathbb{C}$.

The Levi hermitian form of u_2 is

$$L(u_2)(z, w)(\alpha, \beta) = |\beta|^2 + |\langle \alpha/\lambda \rangle|^2 + |\langle \alpha/\gamma \rangle|^2 |e^{\langle z/\gamma \rangle + \delta}|^2,$$

for each $(z, w) \in \mathbb{C}^n \times \mathbb{C}$, for all $(\alpha, \beta) \in \mathbb{C}^n \times \mathbb{C}$. We have

$$(L(u_2)(z, w)(\alpha, \beta) > 0, \forall (z, w) \in \mathbb{C}^n \times \mathbb{C}, \forall (\alpha, \beta) \in \mathbb{C}^n \times \mathbb{C} \setminus \{(0, 0)\})$$

if and only if

$$(\varphi_1(z, \alpha) = |\langle \alpha/\lambda \rangle|^2 + |\langle \alpha/\gamma \rangle|^2 |e^{\langle z/\gamma \rangle + \delta}|^2 > 0, \forall z \in \mathbb{C}^n, \forall \alpha \in \mathbb{C}^n \setminus \{0\}).$$

Now observe that $(\varphi_1(z, \alpha) > 0, \forall z \in \mathbb{C}^n, \forall \alpha \in \mathbb{C}^n \setminus \{0\})$ if and only if $(\theta(z, \alpha) = |\langle \alpha/\lambda \rangle|^2 + |\langle \alpha/\gamma \rangle|^2 > 0, \forall \alpha \in \mathbb{C}^n \setminus \{0\})$. But θ is independent of $z \in \mathbb{C}^n$. Therefore, u is strictly psh on $\mathbb{C}^n \times \mathbb{C}$ if and only if $\varphi(\alpha) = |\langle \alpha/\lambda \rangle|^2 + |\langle \alpha/\gamma \rangle|^2 > 0$, for all $\alpha \in \mathbb{C}^n \setminus \{0\}$.

By the same method of the Case 1, we prove that $n \leq 2$.

Step 1. $n = 1$. Then $(|\lambda|^2 + |\gamma|^2) > 0$.

Step 2. $n = 2$. Then by the same algebraic method developed in the Case 1, we prove that the determinant $\det(\lambda, \gamma) \neq 0$.

The proof is now finished. \square

The complete characterization

Theorem 6. Let $g_1, g_2 : \mathbb{C}^n \rightarrow \mathbb{C}$ be two analytic functions, $n \geq 1$. Given $(A_1, A_2) \in \mathbb{C}^2 \setminus \{(0, 0)\}$. Put $v(z, w) = |A_1 w - \overline{g_1}(z)|^2 + |A_2 w - \overline{g_2}(z)|^2$, for $(z, w) \in \mathbb{C}^n \times \mathbb{C}$. The following conditions are equivalent

- (A) v is convex and strictly psh on $\mathbb{C}^n \times \mathbb{C}$;
- (B) $n \in \{1, 2\}$ and we have the following three cases.
 If $A_1, A_2 \in \mathbb{C} \setminus \{0\}$, this situation is studied in the above theorem.
 If $A_1 \neq 0, A_2 = 0$, then g_1 is affine on \mathbb{C}^n , $|g_2|^2$ is convex on \mathbb{C}^n and $(|g_1|^2 + |g_2|^2)$ is strictly psh on \mathbb{C}^n .
 If $A_1 = 0, A_2 \neq 0$, then g_2 is affine, $|g_1|^2$ is convex on \mathbb{C}^n and $(|g_1|^2 + |g_2|^2)$ is strictly psh on \mathbb{C}^n .

Corollary 2. Let $g_1, g_2 : \mathbb{C}^n \rightarrow \mathbb{C}$ be two analytic functions, $n \geq 1$. Given $(A_1, A_2) \in \mathbb{C}^2$. Put $v(z, w) = |A_1 w - \overline{g_1}(z)|^2 + |A_2 w - \overline{g_2}(z)|^2$, for $(z, w) \in \mathbb{C}^n \times \mathbb{C}$. The following conditions are equivalent

- (A) v is convex strictly psh and not strictly convex on $\mathbb{C}^n \times \mathbb{C}$;
- (B) $n \in \{1, 2\}$, $(A_1, A_2) \in \mathbb{C}^2 \setminus \{0\}$ and we have

$$\begin{cases} g_1(z) = \overline{A_1}(\langle z/\lambda \rangle + \mu) + A_2(\langle z/\lambda_1 \rangle + \mu_1)^s \\ g_2(z) = \overline{A_2}(\langle z/\lambda \rangle + \mu) - A_1(\langle z/\lambda_1 \rangle + \mu_1)^s \end{cases}$$

(for each $z \in \mathbb{C}^n$, where $\lambda, \lambda_1 \in \mathbb{C}^n, \mu, \mu_1 \in \mathbb{C}, s \in \mathbb{N}$ with $(s = 0, n = 1, \lambda = 0)$, or $(s = 1, \lambda_1 = 0, n = 1, \lambda \neq 0)$, or $(s \geq 2, n = 1, \lambda \neq 0)$), or

$$\begin{cases} g_1(z) = \overline{A_1}(\langle z/\lambda_2 \rangle + \mu_2) + A_2 e^{(\langle z/\lambda_3 \rangle + \mu_3)} \\ g_2(z) = \overline{A_2}(\langle z/\lambda_2 \rangle + \mu_2) - A_1 e^{(\langle z/\lambda_3 \rangle + \mu_3)} \end{cases}$$

(for each $z \in \mathbb{C}^n$, where $\lambda_2, \lambda_3 \in \mathbb{C}^n, \mu_2, \mu_3 \in \mathbb{C}$, with $(n = 1, \lambda_2 \neq 0)$, or $(n = 1, \lambda_3 \neq 0)$, or $(n = 2, \text{the determinant } \det(\lambda_2, \lambda_3) \neq 0)$).

Corollary 3. Let $g_1, g_2 : \mathbb{C}^n \rightarrow \mathbb{C}$ be two analytic functions, $n \geq 1$. Given $(A_1, A_2) \in \mathbb{C}^2$. Put $v(z, w) = |A_1 w - \overline{g_1}(z)|^2 + |A_2 w - \overline{g_2}(z)|^2$, for $(z, w) \in \mathbb{C}^n \times \mathbb{C}$. The following conditions are equivalent

- (A) v is convex strictly psh and not strictly convex at all point of $\mathbb{C}^n \times \mathbb{C}$;
- (B) $n \in \{1, 2\}$, $(A_1, A_2) \in \mathbb{C}^2 \setminus \{0\}$ and we have

$$\begin{cases} g_1(z) = \overline{A_1}(\langle z/\lambda \rangle + \mu) + A_2(\langle z/\lambda_1 \rangle + \mu_1)^s \\ g_2(z) = \overline{A_2}(\langle z/\lambda \rangle + \mu) - A_1(\langle z/\lambda_1 \rangle + \mu_1)^s \end{cases}$$

(for each $z \in \mathbb{C}^n$, where $\lambda, \lambda_1 \in \mathbb{C}^n, \mu, \mu_1 \in \mathbb{C}, s \in \mathbb{N}$ with $(n = 1, s = 0, \lambda \neq 0)$, or $(n = 1, s \in \mathbb{N}, \lambda_1 = 0, \lambda \neq 0)$), or

$$\begin{cases} g_1(z) = \overline{A_1}(\langle z/\lambda_2 \rangle + \mu_2) + A_2 e^{(\langle z/\lambda_3 \rangle + \mu_3)} \\ g_2(z) = \overline{A_2}(\langle z/\lambda_2 \rangle + \mu_2) - A_1 e^{(\langle z/\lambda_3 \rangle + \mu_3)} \end{cases}$$

(for each $z \in \mathbb{C}^n$, where $\lambda_2, \lambda_3 \in \mathbb{C}^n, \mu_2, \mu_3 \in \mathbb{C}$, with ($n = 1, \lambda_3 \neq 0, \lambda_2 = 0$), or ($n = 2$ and the determinant $\det(\lambda_2, \lambda_3) \neq 0$)).

In fact we have the following.

Theorem 7. Let $n \geq 1$ and consider two holomorphic functions $g_1, g_2 : \mathbb{C}^n \rightarrow \mathbb{C}$. Given $(A_1, A_2) \in (\mathbb{C} \setminus \{0\})^2$. Let

$$\begin{aligned} u(z, w) = & |A_1 w - \overline{g_1}(z)|^2 + |A_2 w - \overline{g_2}(z)|^2, v(z, w) = u(z, w) + |\overline{A_1} w - g_1(z)|^2 \\ & + |\overline{A_2} w - g_2(z)|^2, v_1(z, w) = |\overline{A_1} w - g_1(z)|^2 + |\overline{A_2} w - g_2(z)|^2 \\ & + |\overline{A_1} w - \overline{g_1}(z)|^2 + |\overline{A_2} w - \overline{g_2}(z)|^2, \end{aligned}$$

for $(z, w) \in \mathbb{C}^n \times \mathbb{C}$. The following conditions are equivalent

(A) u is convex on $\mathbb{C}^n \times \mathbb{C}$ and v is strictly psh on $\mathbb{C}^n \times \mathbb{C}$;

(B) u is convex on $\mathbb{C}^n \times \mathbb{C}$ and v_1 is strictly psh on $\mathbb{C}^n \times \mathbb{C}$;

(C) $n \in \{1, 2\}$ and we have the following two cases.

(I)

$$\begin{cases} g_1(z) = \overline{A_1}(\langle z/\lambda \rangle + \mu) + A_2(\langle z/\lambda_1 \rangle + \mu_1)^s \\ g_2(z) = \overline{A_2}(\langle z/\lambda \rangle + \mu) - A_1(\langle z/\lambda_1 \rangle + \mu_1)^s \end{cases}$$

(for each $z \in \mathbb{C}^n$, where $\lambda, \lambda_1 \in \mathbb{C}^n, \mu, \mu_1 \in \mathbb{C}$, $s \in \mathbb{N}$, with ($n = 1, \lambda \neq 0$), or ($n = 1, \lambda_1 \neq 0, s = 1$), or ($n = 2, s = 1$ and the determinant $\det(\lambda, \lambda_1) \neq 0$)).

(II)

$$\begin{cases} g_1(z) = \overline{A_1}(\langle z/\lambda_2 \rangle + \mu_2) + A_2 e^{\langle z/\lambda_3 \rangle + \mu_3} \\ g_2(z) = \overline{A_2}(\langle z/\lambda_2 \rangle + \mu_2) - A_1 e^{\langle z/\lambda_3 \rangle + \mu_3} \end{cases}$$

(for each $z \in \mathbb{C}^n$, where $\lambda_2, \lambda_3 \in \mathbb{C}^n, \mu_2, \mu_3 \in \mathbb{C}$, with ($n = 1, \lambda_2 \neq 0$), or ($n = 1, \lambda_3 \neq 0$), or ($n = 2$ and the determinant $\det(\lambda_2, \lambda_3) \neq 0$)).

Proof. This proof is similar to the proof of Theorem 4.

Now we can answer to the following question.

Question. Let $n \geq 1$ and $A_1, A_2 \in \mathbb{C} \setminus \{0\}$. Find all the functions $f_1, f_2 : \mathbb{C}^n \rightarrow \mathbb{C}$ such that $f_1 \in C(\mathbb{C}^n)$ and

$$\begin{cases} \varphi_1 \text{ is psh on } \mathbb{C}^n \times \mathbb{C} \\ \varphi_2 \text{ is convex on } \mathbb{C}^n \times \mathbb{C}, \end{cases}$$

or (for example)

$$\begin{cases} \varphi_1 \text{ is psh on } \mathbb{C}^n \times \mathbb{C} \\ \varphi_2 \text{ is convex and strictly psh on } \mathbb{C}^n \times \mathbb{C}, \text{ but not strictly convex on all} \\ \text{not empty open ball of } \mathbb{C}^n \times \mathbb{C}, \end{cases}$$

where

$$\begin{aligned} \varphi_1(z, w) &= \log |A_1w - f_1(z)| + \log |A_2w - f_2(z)|, \\ \varphi_2(z, w) &= |A_1w - \overline{f_1(z)}|^2 + |A_2w - \overline{f_2(z)}|^2, \end{aligned}$$

for $(z, w) \in \mathbb{C}^n \times \mathbb{C}$.

Using algebraic methods, we can prove the following theorem:

Theorem 8. *Let $n \geq 1$ and $(A_1, A_2) \in \mathbb{C}^2$. Given $g_1, g_2 : \mathbb{C}^n \rightarrow \mathbb{C}$ be two analytic functions. Put $u(z, w) = |A_1w - g_1(z)|^2 + |A_2w - \overline{g_2(z)}|^2$, $(z, w) \in \mathbb{C}^n \times \mathbb{C}$. The following conditions are equivalent*

- (A) u is convex and strictly psh on $\mathbb{C}^n \times \mathbb{C}$;
- (B) $n \in \{1, 2\}$, $(A_1, A_2) \in \mathbb{C}^2 \setminus \{(0, 0)\}$ and we have the following three situations.
 - (1) $A_1 \neq 0$ and $A_2 = 0$. Then $n = 1$, g_1 is affine, $|g_2|^2$ is convex and strictly sh on \mathbb{C} .
 - (2) $A_1 = 0$ and $A_2 \neq 0$. Then $n \in \{1, 2\}$, $|g_1|^2$ is convex on \mathbb{C}^n , g_2 is affine on \mathbb{C}^n and $(|g_1|^2 + |g_2|^2)$ is strictly psh on \mathbb{C}^n .
 - (3) $A_1, A_2 \in \mathbb{C} \setminus \{0\}$. Then $n \in \{1, 2\}$, g_1 and g_2 are affine functions on \mathbb{C}^n and $(|g_1|^2 + |g_2|^2)$ is strictly psh on \mathbb{C}^n .

3. A Classical Complex Analysis Problem

Let $n, N \geq 1$ and $(A_1, B_1, \dots, A_N, B_N \in \mathbb{C} \setminus \{0\})$. For $f_1, g_1, \dots, f_N, g_N : \mathbb{C}^n \rightarrow \mathbb{C}$, define

$$\begin{aligned} u_1(z, w) &= |A_1w - f_1(z)|^2 + |B_1w - g_1(z)|^2, \\ v_1(z, w) &= |A_1w - \overline{f_1(z)}|^2 + |B_1w - \overline{g_1(z)}|^2, \dots, \\ u_N(z, w) &= |A_Nw - f_N(z)|^2 + |B_Nw - g_N(z)|^2, \\ v_N(z, w) &= |A_Nw - \overline{f_N(z)}|^2 + |B_Nw - \overline{g_N(z)}|^2, \end{aligned}$$

$u = (u_1 + \dots + u_N)$ and $v = (v_1 + \dots + v_N)$, for $(z, w) \in \mathbb{C}^n \times \mathbb{C}$. Define

$$\begin{aligned} \varphi_1(z, w) &= \log |A_1w - f_1(z)| + \log |B_1w - g_1(z)|, \dots, \\ \varphi_N(z, w) &= \log |A_Nw - f_N(z)| + \log |B_Nw - g_N(z)|, \quad (z, w) \in \mathbb{C}^n \times \mathbb{C}. \end{aligned}$$

Question. Find all the functions $f_1, g_1, \dots, f_N, g_N : \mathbb{C}^n \rightarrow \mathbb{C}$ such that f_1, \dots, f_N are continuous functions on \mathbb{C}^n and

$$\left\{ \begin{array}{l} u_1 \text{ is convex on } \mathbb{C}^n \times \mathbb{C} \text{ and} \\ \varphi_1 \text{ is psh on } \mathbb{C}^n \times \mathbb{C} \\ \cdot \\ \cdot \\ u_N \text{ is convex on } \mathbb{C}^n \times \mathbb{C} \text{ and} \\ \varphi_N \text{ is psh on } \mathbb{C}^n \times \mathbb{C}; \text{ and} \end{array} \right.$$

the function u is strictly psh on $\mathbb{C}^n \times \mathbb{C}$?

Question. Find exactly all the functions $f_1, g_1, \dots, f_N, g_N : \mathbb{C}^n \rightarrow \mathbb{C}$ such that f_1, \dots, f_N are continuous functions on \mathbb{C}^n , and

$$\left\{ \begin{array}{l} v_1 \text{ is convex on } \mathbb{C}^n \times \mathbb{C} \text{ and} \\ \varphi_1 \text{ is psh on } \mathbb{C}^n \times \mathbb{C} \\ \cdot \\ \cdot \\ v_N \text{ is convex on } \mathbb{C}^n \times \mathbb{C} \text{ and} \\ \varphi_N \text{ is psh on } \mathbb{C}^n \times \mathbb{C}; \text{ and} \end{array} \right.$$

v is strictly psh on $\mathbb{C}^n \times \mathbb{C}$?

Theorem 9. Let $n \geq 1, n + 1 = 2q, q \in \mathbb{N}$. Let $A_1, B_1, \dots, A_q, B_q \in \mathbb{C} \setminus \{0\}$ and $f_1, g_1, \dots, f_q, g_q : \mathbb{C}^n \rightarrow \mathbb{C}$ be $2q$ analytic functions. Define

$$\begin{aligned} u_1(z, w) &= |A_1 w - f_1(z)|^2 + |B_1 w - g_1(z)|^2, \dots, \\ u_q(z, w) &= |A_q w - f_q(z)|^2 + |B_q w - g_q(z)|^2 \end{aligned}$$

and $u = (u_1 + \dots + u_q)$, for $(z, w) \in \mathbb{C}^n \times \mathbb{C}$.

The following conditions are equivalent

- (A) u_1, \dots, u_q are convex functions on $\mathbb{C}^n \times \mathbb{C}$ and u is strictly psh on $\mathbb{C}^n \times \mathbb{C}$;
- (B) For each $j \in \{1, \dots, q\}$, we have

$$\begin{cases} f_j(z) = A_j \langle z/\lambda_j \rangle + \mu_j + \overline{B_j} \varphi_j(z) \\ g_j(z) = B_j \langle z/\lambda_j \rangle + \mu_j - \overline{A_j} \varphi_j(z) \end{cases}$$

for each $z \in \mathbb{C}^n$, with $\lambda_j \in \mathbb{C}^n, \mu_j \in \mathbb{C}, \varphi_j : \mathbb{C}^n \rightarrow \mathbb{C}$ be a holomorphic function, $|\varphi_j|^2$ is a convex function on \mathbb{C}^n and

$$(\lambda_1 - \lambda_2, \dots, \lambda_1 - \lambda_q, \overline{\frac{\partial \varphi_1}{\partial z_1}}(a), \dots, \overline{\frac{\partial \varphi_1}{\partial z_n}}(a), \dots, \overline{\frac{\partial \varphi_q}{\partial z_1}}(a), \dots, \overline{\frac{\partial \varphi_q}{\partial z_n}}(a))$$

is a basis of \mathbb{C}^n , for all $a \in \mathbb{C}^n$.

(We can also study the problem u is strictly psh on $\mathbb{C}^n \times \mathbb{C}$ and not strictly convex on all not empty open ball of $\mathbb{C}^n \times \mathbb{C}, \dots$).

Proof. (A) implies (B). Let $j \in \{1, \dots, q\}$. By Theorem 1, we have

$$\begin{cases} f_j(z) = A_j \langle z/\lambda_j \rangle + \mu_j + \overline{B_j} \varphi_j(z) \\ g_j(z) = B_j \langle z/\lambda_j \rangle + \mu_j - \overline{A_j} \varphi_j(z) \end{cases}$$

$\varphi_j : \mathbb{C}^n \rightarrow \mathbb{C}, \varphi_j$ is analytic and $|\varphi_j|^2$ is convex on \mathbb{C}^n .

In fact $\varphi_j(z) = \langle z/\gamma_j \rangle + \delta_j^{s_j}$, (for all $z \in \mathbb{C}^n$, where $\gamma_j \in \mathbb{C}^n, \delta_j \in \mathbb{C}, s_j \in \mathbb{N}$), or $\varphi_j(z) = e^{\langle z/a_j \rangle + b_j}$, for all $z \in \mathbb{C}^n$, with $a_j \in \mathbb{C}^n, b_j \in \mathbb{C}$.

We consider in this proof the case where

$$\begin{cases} f_j(z) = A_j(\langle z/\lambda_j \rangle + \mu_j) + \overline{B_j}(\langle z/\gamma_j \rangle + \delta_j)^{s_j} \\ g_j(z) = B_j(\langle z/\lambda_j \rangle + \mu_j) - \overline{A_j}(\langle z/\gamma_j \rangle + \delta_j)^{s_j} \end{cases}$$

for each $z \in \mathbb{C}^n$ and all $j \in \{1, \dots, n\}$ (the proof of the other cases are similar of this proof). Therefore,

$$\begin{aligned} u(z, w) = & (|A_1|^2 + |B_1|^2)[|w - \langle z/\lambda_1 \rangle - \mu_1|^2 + |\langle z/\gamma_1 \rangle + \delta_1|^{2s_1}] + \dots \\ & + (|A_q|^2 + |B_q|^2)[|w - \langle z/\lambda_q \rangle - \mu_q|^2 + |\langle z/\gamma_q \rangle + \delta_q|^{2s_q}], \end{aligned}$$

$(z, w) \in \mathbb{C}^n \times \mathbb{C}$.

Define

$$\begin{aligned} v(z, w) = & |w - \langle z/\lambda_1 \rangle - \mu_1|^2 + |\langle z/\gamma_1 \rangle + \delta_1|^{2s_1} + \dots \\ & + |w - \langle z/\lambda_q \rangle - \mu_q|^2 + |\langle z/\gamma_q \rangle + \delta_q|^{2s_q}, \end{aligned}$$

$(z, w) \in \mathbb{C}^n \times \mathbb{C}$. u and v are functions of class C^∞ on $\mathbb{C}^n \times \mathbb{C}$.

We have in fact u is strictly psh on $\mathbb{C}^n \times \mathbb{C}$ if and only if v is strictly psh on $\mathbb{C}^n \times \mathbb{C}$. Because this situation, we study the function v .

Let $T : \mathbb{C}^n \times \mathbb{C} \rightarrow \mathbb{C}^n \times \mathbb{C}$, $T(z, w) = (z, w + \langle z/\lambda_1 \rangle)$, for $(z, w) \in \mathbb{C}^n \times \mathbb{C}$. T is a \mathbb{C} -linear bijective transformation over $\mathbb{C}^n \times \mathbb{C}$. Put $v_1 = v \circ T$. Then v_1 is a function of class C^∞ on $\mathbb{C}^n \times \mathbb{C}$.

We have v is strictly psh on $\mathbb{C}^n \times \mathbb{C}$ if and only if v_1 is strictly psh on $\mathbb{C}^n \times \mathbb{C}$.

$$\begin{aligned} v_1(z, w) = & |w - \mu_1|^2 + |\langle z/\gamma_1 \rangle + \delta_1|^{2s_1} + |w - \langle z/\lambda_2 - \lambda_1 \rangle - \mu_2|^2 \\ & + |\langle z/\gamma_2 \rangle + \delta_2|^{2s_2} + \dots + |w - \langle z/\lambda_q - \lambda_1 \rangle - \mu_q|^2 \\ & + |\langle z/\gamma_q \rangle + \delta_q|^{2s_q}, \end{aligned}$$

for $(z, w) \in \mathbb{C}^n \times \mathbb{C}$.

The Levi hermitian form of v_1 is

$$\begin{aligned} L(v_1)(z, w)(\alpha, \beta) = & |\beta|^2 + s_1^2 |\langle \alpha/\gamma_1 \rangle|^2 |\langle z/\gamma_1 \rangle + \delta_1|^{2s_1-2} \\ & + |\beta - \langle \alpha/\lambda_2 - \lambda_1 \rangle|^2 + s_2^2 |\langle \alpha/\gamma_2 \rangle|^2 |\langle z/\gamma_2 \rangle + \delta_2|^{2s_2-2} + \dots \\ & + |\beta - \langle \alpha/\lambda_q - \lambda_1 \rangle|^2 + s_q^2 |\langle \alpha/\gamma_q \rangle|^2 |\langle z/\gamma_q \rangle + \delta_q|^{2s_q-2}, \end{aligned}$$

for $(z, w), (\alpha, \beta) \in \mathbb{C}^n \times \mathbb{C}$.

Fix now $(z_0, w_0) \in \mathbb{C}^n \times \mathbb{C}$. Let $(\alpha, \beta) \in \mathbb{C}^n \times \mathbb{C}$ with $L(v)(z_0, w_0)(\alpha, \beta) = 0$. Then

$$\left\{ \begin{array}{l} \beta = 0 \\ s_1^2 |\langle \alpha/\gamma_1 \rangle|^2 |\langle z/\gamma_1 \rangle| + \delta_1 |^{2s_1-2} = 0 \\ \langle \alpha/\lambda_2 - \lambda_1 \rangle = 0 \\ s_2^2 |\langle \alpha/\gamma_2 \rangle|^2 |\langle z/\gamma_2 \rangle| + \delta_2 |^{2s_2-2} = 0 \\ \cdot \\ \cdot \\ \cdot \\ \langle \alpha/\lambda_q - \lambda_1 \rangle = 0 \\ s_q^2 |\langle \alpha/\gamma_q \rangle|^2 |\langle z/\gamma_q \rangle| + \delta_q |^{2s_q-2} = 0. \end{array} \right.$$

Thus

$$\left\{ \begin{array}{l} \beta = 0 \\ \langle \alpha/\lambda_2 - \lambda_1 \rangle = 0 \\ \cdot \\ \cdot \\ \cdot \\ \langle \alpha/\lambda_q - \lambda_1 \rangle = 0 \\ s_1^2 |\langle \alpha/\gamma_1 \rangle|^2 |\langle z/\gamma_1 \rangle| + \delta_1 |^{2s_1-2} = 0 \\ \cdot \\ \cdot \\ \cdot \\ s_q^2 |\langle \alpha/\gamma_q \rangle|^2 |\langle z/\gamma_q \rangle| + \delta_q |^{2s_q-2} = 0. \end{array} \right.$$

Therefore this above system is equivalent with $\beta = 0$ and the system

$$\left\{ \begin{array}{l} \langle \alpha/\lambda_2 - \lambda_1 \rangle = 0 \\ \cdot \\ \cdot \\ \cdot \\ \langle \alpha/\lambda_q - \lambda_1 \rangle = 0 \\ s_1^2 |\langle \alpha/\gamma_1 \rangle|^2 |\langle z/\gamma_1 \rangle| + \delta_1 |^{2s_1-2} = 0 \\ \cdot \\ \cdot \\ \cdot \\ s_q^2 |\langle \alpha/\gamma_q \rangle|^2 |\langle z/\gamma_q \rangle| + \delta_q |^{2s_q-2} = 0. \end{array} \right.$$

Consequently, v_1 is strictly psh on $\mathbb{C}^n \times \mathbb{C}$ if and only if (for each $(\alpha, \beta) \in \mathbb{C}^n \times \mathbb{C}$ and every $(z, w) \in \mathbb{C}^n \times \mathbb{C}$, the condition $L(v_1)(z, w)(\alpha, \beta) = 0$ implies that $\alpha = 0$

and $\beta = 0$). Then the system

$$\begin{cases} < \alpha/\lambda_2 - \lambda_1 > = 0 \\ \cdot \\ \cdot \\ < \alpha/\lambda_q - \lambda_1 > = 0 \\ s_1^2 |< \alpha/\gamma_1 >|^2 |< z/\gamma_1 > + \delta_1 |^{2s_1-2} = 0 \\ \cdot \\ \cdot \\ s_q^2 |< \alpha/\gamma_q >|^2 |< z/\gamma_q > + \delta_q |^{2s_q-2} = 0 \end{cases}$$

implies that $\alpha = 0$.

Using algebraic methods, we have then $(\lambda_2 - \lambda_1, \dots, \lambda_q - \lambda_1, \gamma_1, \dots, \gamma_q)$ is a basis of $\mathbb{C}^n = \mathbb{C}^{2q-1}$ and $s_1 = \dots = s_q = 1$ (\mathbb{C}^n considered a complex vector space of dimension n). \square

Theorem 10. Let $n = 2q$, $n \in \mathbb{N}$, $n \geq 1$, $q \in \mathbb{N}$. Let $f_1, g_1, \dots, f_q, g_q : \mathbb{C}^n \rightarrow \mathbb{C}$ be $2q$ holomorphic functions and $A_1, B_1, \dots, A_q, B_q \in \mathbb{C} \setminus \{0\}$. Define

$$u_j(z, w) = |A_j w - \overline{f_j}(z)|^2 + |B_j w - \overline{g_j}(z)|^2, \quad u = (u_1 + \dots + u_q),$$

for $(z, w) \in \mathbb{C}^n \times \mathbb{C}$ and $j \in \{1, \dots, q\}$. The following conditions are equivalent

- (A) u_1, \dots, u_q are convex functions on $\mathbb{C}^n \times \mathbb{C}$ and u is strictly psh on $\mathbb{C}^n \times \mathbb{C}$;
- (B) For every $j \in \{1, \dots, q\}$,

$$\begin{cases} f_j(z) = \overline{A_j} (< z/\lambda_j > + \mu_j) + B_j \varphi_j(z) \\ g_j(z) = \overline{B_j} (< z/\lambda_j > + \mu_j) - A_j \varphi_j(z) \end{cases}$$

(for each $z \in \mathbb{C}^n$, with $\lambda_j \in \mathbb{C}^n$, $\mu_j \in \mathbb{C}$, $\varphi_j : \mathbb{C}^n \rightarrow \mathbb{C}$ be a holomorphic function and $|\varphi_j|^2$ is a convex function on \mathbb{C}^n) and

$$(\lambda_1, \dots, \lambda_q, (\overline{\frac{\partial \varphi_1}{\partial z_1}}(a), \dots, \overline{\frac{\partial \varphi_1}{\partial z_n}}(a)), \dots, (\overline{\frac{\partial \varphi_q}{\partial z_1}}(a), \dots, \overline{\frac{\partial \varphi_q}{\partial z_n}}(a)))$$

is a basis of \mathbb{C}^n for all $a \in \mathbb{C}^n$.

(We can also study the problem u is strictly psh on $\mathbb{C}^n \times \mathbb{C}$ and not strictly convex on all not empty Euclidean open ball of $\mathbb{C}^n \times \mathbb{C}$, ...).

4. Real Convexity and Complex Convexity

Question. An original question of complex analysis is now to find exactly the set of all continuous functions $f_1, \dots, f_N : D \rightarrow \mathbb{C}$ (D is a convex domain of \mathbb{C}^n , $n \geq 1$, $N \geq 1$) such that φ is psh on $D \times \mathbb{C}$, where $\varphi(z, w) = \log(|w - f_1(z)|^2 + \dots + |w - f_N(z)|^2)$, for $(z, w) \in D \times \mathbb{C}$.

Observe that for $N = 1$, this is exactly all the holomorphic functions over D . But for $N \geq 2$, the set of solution contains several classes of functions.

Example. $N = 2$ and $D = \mathbb{C}^n$. Put

$$k_1(z) = (\langle z/a \rangle + b) + (\overline{\langle z/c \rangle + d})^s,$$

$$k_2(z) = (\langle z/a \rangle + b) - (\overline{\langle z/c \rangle + d})^s,$$

$z \in \mathbb{C}^n$, $a, c \in \mathbb{C}^n \setminus \{0\}$, $b, d \in \mathbb{C}$, $s \in \mathbb{N} \setminus \{0\}$. $k_1, k_2, \overline{k_1}$ and $\overline{k_2}$ are not holomorphic functions over \mathbb{C}^n . The function ψ is psh on $\mathbb{C}^n \times \mathbb{C}$, where $\psi(z, w) = \log(|w - k_1(z)|^2 + |w - k_2(z)|^2)$, $(z, w) \in \mathbb{C}^n \times \mathbb{C}$.

Theorem 11. Let $g_1, g_2, k : \mathbb{C}^n \rightarrow \mathbb{C}$ be three analytic functions, $n \geq 1$. Let $(A_1, A_2) \in (\mathbb{C} \setminus \{0\})^2$. Put $u(z, w) = |A_1(w - \overline{k}(z)) - g_1(z)|^2 + |A_2(w - \overline{k}(z)) - g_2(z)|^2$, $v(z, w) = |\overline{A_1}w - \overline{g_1}(z)|^2 + |\overline{A_2}w - \overline{g_2}(z)|^2$, for $(z, w) \in \mathbb{C}^n \times \mathbb{C}$. The following conditions are equivalent

- (A) u is convex on $\mathbb{C}^n \times \mathbb{C}$;
- (B) k is an affine function and

$$\begin{cases} g_1(z) = A_1(\langle z/a \rangle + b) + \overline{A_2}(\langle z/c \rangle + d)^m \\ g_2(z) = A_2(\langle z/a \rangle + b) - \overline{A_1}(\langle z/c \rangle + d)^m \end{cases}$$

(for each $z \in \mathbb{C}^n$, where $a, c \in \mathbb{C}^n, b, d \in \mathbb{C}$), or

$$\begin{cases} g_1(z) = A_1(\langle z/\lambda \rangle + \mu) + \overline{A_2}e^{(\langle z/\gamma \rangle + \delta)} \\ g_2(z) = A_2(\langle z/\lambda \rangle + \mu) - \overline{A_1}e^{(\langle z/\gamma \rangle + \delta)} \end{cases}$$

(for each $z \in \mathbb{C}^n$, where $\lambda, \gamma \in \mathbb{C}^n, \mu, \delta \in \mathbb{C}$);

- (C) v is convex on $\mathbb{C}^n \times \mathbb{C}$ and k is an affine function on \mathbb{C}^n .

Theorem 12. Let $A_1, A_2 \in \mathbb{C} \setminus \{0\}$. Consider three holomorphic functions $g_1, g_2, k : \mathbb{C}^n \rightarrow \mathbb{C}$, $n \geq 1$. Put

$$v(z, w) = |A_1(w - \overline{k}(z)) - g_1(z)|^2 + |A_2(w - \overline{k}(z)) - g_2(z)|^2,$$

$$u(z, w) = |A_1w - g_1(z)|^2 + |A_2w - g_2(z)|^2,$$

$$u_1(z, w) = |A_1(w - \overline{k}(z))|^2 + |A_2(w - \overline{k}(z))|^2,$$

for $(z, w) \in \mathbb{C}^n \times \mathbb{C}$. The following conditions are equivalent

- (A) v is strictly psh and convex on $\mathbb{C}^n \times \mathbb{C}$;
- (B) $n \in \{1, 2\}$, k is an affine function and

$$\begin{cases} g_1(z) = A_1(\langle z/\lambda \rangle + \mu) + \overline{A_2}\varphi(z) \\ g_2(z) = A_2(\langle z/\lambda \rangle + \mu) - \overline{A_1}\varphi(z) \end{cases}$$

(for each $z \in \mathbb{C}^n$, where $\lambda \in \mathbb{C}^n$, $\mu \in \mathbb{C}$ and $\varphi : \mathbb{C}^n \rightarrow \mathbb{C}$ be analytic, $|\varphi|^2$ is convex on \mathbb{C}^n and $(|k|^2 + |\varphi|^2)$ is strictly psh on \mathbb{C}^n);

(C)

$$\begin{cases} |A_2g_1 - A_1g_2|^2 \text{ is convex on } \mathbb{C}^n, \\ (\overline{A_1}g_1 + \overline{A_2}g_2) \text{ is affine on } \mathbb{C}^n, \\ k \text{ is affine on } \mathbb{C}^n, \text{ and} \\ \text{the function } (|k|^2 + \frac{1}{(|A_1|^2 + |A_2|^2)^2} |A_2g_1 - A_1g_2|^2) \text{ is strictly psh on } \mathbb{C}^n; \end{cases}$$

- (D) u is convex on $\mathbb{C}^n \times \mathbb{C}$, u_1 is convex on $\mathbb{C}^n \times \mathbb{C}$ and the function $(u + u_1)$ is strictly psh on $\mathbb{C}^n \times \mathbb{C}$.

(If $n = 1$, we can study the strict plurisubharmonicity of v and u on a neighborhood of $\partial D(0, 1) \times D(0, 1)$).

Remark 2. Let $A_1, A_2 \in \mathbb{C} \setminus \{0\}$ with $(A_1\overline{A_2} \neq \overline{A_1}A_2)$ and $g_1, g_2 : \mathbb{C} \rightarrow \mathbb{C}$ be two analytic functions. Put $u(z, w) = |A_1w - g_1(z)|^2 + |A_2w - g_2(z)|^2$, $v(z, w) = |A_1w - \overline{g_1}(z)|^2 + |A_2w - \overline{g_2}(z)|^2$, for $(z, w) \in \mathbb{C}^2$. If u is strictly psh on \mathbb{C}^2 , then v is strictly psh on \mathbb{C}^2 (and the converse is false).

By a simple study of u and v , we prove that this property is not true for the class of convex functions (respectively strictly psh and convex, strictly convex, strictly psh convex and not strictly convex on all not empty Euclidean open ball of \mathbb{C}^2, \dots). This is one of the great differences between the above classes of functions.

A good comparison between the subject strictly convex and the concept (convex and strictly psh) can be follows by the following two theorems.

Theorem 13. Fix $g_1, g_2 : \mathbb{C}^n \rightarrow \mathbb{C}$ be two holomorphic functions, $n \in \mathbb{N} \setminus \{0\}$. Let $(A_1, A_2) \in \mathbb{C}^2$. Define

$$v(z, w) = |A_1w - \overline{g_1}(z)|^2 + |A_2w - \overline{g_2}(z)|^2, \quad (z, w) \in \mathbb{C}^n \times \mathbb{C}.$$

The following conditions are equivalent

- (A) v is strictly convex on $\mathbb{C}^n \times \mathbb{C}$;
- (B) $n = 1$, $(A_1, A_2) \in \mathbb{C}^2 \setminus \{(0, 0)\}$ and

$$\begin{cases} g_1(z) = \overline{A_1}(az + b) + A_2(cz + d) \\ g_2(z) = \overline{A_2}(az + b) - A_1(cz + d) \end{cases}$$

(for each $z \in \mathbb{C}$, where $a, b, c, d \in \mathbb{C}$, $c \neq 0$).

Theorem 14. Let $g_1, g_2 : \mathbb{C}^n \rightarrow \mathbb{C}$ be two holomorphic functions, $n \in \mathbb{N} \setminus \{0\}$. Let $(A_1, A_2) \in \mathbb{C}^2$. Define

$$u(z, w) = |A_1 w - g_1(z)|^2 + |A_2 w - g_2(z)|^2, \quad (z, w) \in \mathbb{C}^n \times \mathbb{C}.$$

The following conditions are equivalent

- (A) u is strictly psh and convex on $\mathbb{C}^n \times \mathbb{C}$, but u is not strictly convex in all not empty Euclidean open ball of $\mathbb{C}^n \times \mathbb{C}$;
- (B) $n = 1$, $(A_1, A_2) \in \mathbb{C}^2 \setminus \{(0, 0)\}$ and

$$\begin{cases} g_1(z) = A_1(az + b) + \overline{A_2}e^{(cz+d)} \\ g_2(z) = A_2(az + b) - \overline{A_1}e^{(cz+d)} \end{cases}$$

for each $z \in \mathbb{C}$, with $a, b, d \in \mathbb{C}$ and $c \in \mathbb{C} \setminus \{0\}$.

Now one can observe that there exists a great differences between the classes (convex and strictly psh) and strictly convex functions in all of the above two theorems.

The representation theorems for another cases

We begin by

Theorem 15. Let $k(w) = (aw + b)^m$, for all $w \in \mathbb{C}$, where $a \in \mathbb{C} \setminus \{0\}$ and $b \in \mathbb{C}$, $m \in \mathbb{N}$, $m \geq 2$. ($|k|^2$ is convex on \mathbb{C}). Let $(A_1, A_2) \in \mathbb{C}^2 \setminus \{0\}$ and consider two holomorphic functions $g_1, g_2 : \mathbb{C}^n \rightarrow \mathbb{C}$, $n \geq 1$. Define

$$u(z, w) = |A_1 k(w) - g_1(z)|^2 + |A_2 k(w) - g_2(z)|^2, \quad (z, w) \in \mathbb{C}^n \times \mathbb{C}.$$

We have

- (A) u is convex on $\mathbb{C}^n \times \mathbb{C}$ if and only if

$$\begin{cases} g_1(z) = \overline{A_2}\varphi(z) \\ g_2(z) = -\overline{A_1}\varphi(z) \end{cases}$$

for each $z \in \mathbb{C}^n$, where $\varphi : \mathbb{C}^n \rightarrow \mathbb{C}$, φ is holomorphic and $|\varphi|^2$ is convex on \mathbb{C}^n ;

- (B) u is convex on $\mathbb{C}^n \times \mathbb{C}$ and $u(\cdot, 0)$ is strictly psh on \mathbb{C}^n if and only if $n = 1$ and $|\varphi|^2$ is strictly sh on \mathbb{C} .

(The same case for $k(w) = e^{(a_1 w + b_1)}$, for all $w \in \mathbb{C}$, with $a_1 \in \mathbb{C} \setminus \{0\}$ and $b_1 \in \mathbb{C}$).

Observe that, in all not empty convex domain G subset of \mathbb{C}^n , ($n \geq 2$), there exists $K : G \rightarrow \mathbb{R}$ be a function of class C^2 such that K is strictly psh on G , but K is not convex in all not empty Euclidean open ball subset of G . For example $K_1(z, w) = |w - e^{\bar{z}}|^2$, $(z, w) \in \mathbb{C}^2$. K_1 is strictly psh on \mathbb{C}^2 , but K_1 is not convex in all Euclidean open ball of \mathbb{C}^2 (consider $K_1(\bar{z}, w)$).

The converse can be studied and investigated by the following.

Theorem 16. Let $(A_1, A_2) \in \mathbb{C}^2 \setminus \{0\}$ and $n \geq 1$.

Let $\varphi(w) = (aw + b)^m$, where $a \in \mathbb{C} \setminus \{0\}$, $b \in \mathbb{C}$, $m \in \mathbb{N}$, $m \geq 2$ (for all $w \in \mathbb{C}$) and $g_1, g_2 : \mathbb{C}^n \rightarrow \mathbb{C}$ be two holomorphic functions. Define

$$u(z, w) = |A_1\varphi(w) - g_1(z)|^2 + |A_2\varphi(w) - g_2(z)|^2,$$

for $(z, w) \in \mathbb{C}^n \times \mathbb{C}$. The following conditions are equivalent

- (A) u is convex and not strictly psh at all point of $\mathbb{C}^n \times \mathbb{C}$;
- (B) We have the following two cases

$$\begin{cases} g_1(z) = \overline{A_2}(\langle z/\lambda \rangle + \mu)^s \\ g_2(z) = -\overline{A_1}(\langle z/\lambda \rangle + \mu)^s \end{cases}$$

(for every $z \in \mathbb{C}^n$, where $\lambda \in \mathbb{C}^n$, $\mu \in \mathbb{C}$, $s \in \mathbb{N}$ such that $(s = 0)$, or $(n = 1, \lambda = 0)$, or $(n \geq 2)$), or

$$\begin{cases} g_1(z) = \overline{A_2}e^{\langle z/\lambda_1 \rangle + \mu_1} \\ g_2(z) = -\overline{A_1}e^{\langle z/\lambda_1 \rangle + \mu_1} \end{cases}$$

(for each $z \in \mathbb{C}^n$, where $\lambda_1 \in \mathbb{C}^n$, $\mu_1 \in \mathbb{C}$, such that $(n = 1, \lambda_1 = 0)$, or $(n \geq 2)$).
(The same situation if $\varphi(w) = e^{(aw+b)}$, for $w \in \mathbb{C}$, where $a \in \mathbb{C} \setminus \{0\}$, $b \in \mathbb{C}$).

In general observe that if k is an arbitrary holomorphic function on \mathbb{C} , there does not exists $(B_1, B_2) \in \mathbb{C}^2 \setminus \{0\}$, there does not exists $n \geq 1$ and $f_1, f_2 : \mathbb{C}^n \rightarrow \mathbb{C}$ be two holomorphic functions such that v is convex on $\mathbb{C}^n \times \mathbb{C}$; $v(z, w) = |B_1k(w) - f_1(z)|^2 + |B_2k(w) - f_2(z)|^2$, $(z, w) \in \mathbb{C}^n \times \mathbb{C}$. The example is given by the following theorem which is fundamental in mathematical analysis.

Theorem 17. Let $(A_1, A_2) \in (\mathbb{C} \setminus \{0\})^2$ and $n \in \mathbb{N} \setminus \{0\}$. Define $p_1(w) = w^3$, $p_2(w) = w^4 + w^2$ and $p_3(w) = w^3 + w$, for $w \in \mathbb{C}$ and p be an analytic polynomial over \mathbb{C} , $\deg(p) \leq 2$. Let $\varphi = (\varphi_1, \varphi_2)$, where $\varphi_1, \varphi_2 : \mathbb{C}^n \rightarrow \mathbb{C}$ be two analytic functions. Define

$$\begin{aligned} u_\varphi(z, w) &= |A_1p_1(w) - \varphi_1(z)|^2 + |A_2p_1(w) - \varphi_2(z)|^2, \\ v_\varphi(z, w) &= |A_1p_2(w) - \varphi_1(z)|^2 + |A_2p_2(w) - \varphi_2(z)|^2, \\ \psi_\varphi(z, w) &= |A_1p_3(w) - \varphi_1(z)|^2 + |A_2p_3(w) - \varphi_2(z)|^2 \quad \text{and} \\ \rho_\varphi(z, w) &= |A_1p(w) - \varphi_1(z)|^2 + |A_2p(w) - \varphi_2(z)|^2, \end{aligned}$$

for $(z, w) \in \mathbb{C}^n \times \mathbb{C}$. We have the following four assertions:

- (A) There exists an infinite number of holomorphic functions $g_1, g_2 : \mathbb{C}^n \rightarrow \mathbb{C}$, $g = (g_1, g_2)$ and u_g is convex on $\mathbb{C}^n \times \mathbb{C}$.
- (B) There does not exists an holomorphic function $f = (f_1, f_2)$, where $f_1, f_2 : \mathbb{C}^n \rightarrow \mathbb{C}$ such that v_f is convex on $\mathbb{C}^n \times \mathbb{C}$.

- (C) *There does not exist an holomorphic function $k = (k_1, k_2)$, where $k_1, k_2 : \mathbb{C}^n \rightarrow \mathbb{C}$ such that ψ_k is convex on $\mathbb{C}^n \times \mathbb{C}$.*
- (D) *For all polynomial p analytic on \mathbb{C} , $\deg(p) \leq 2$, there exists always an infinite number of holomorphic functions $\theta_1, \theta_2 : \mathbb{C}^n \rightarrow \mathbb{C}$, $\theta = (\theta_1, \theta_2)$ and ρ_θ is convex on $\mathbb{C}^n \times \mathbb{C}$.*

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DOI: 10.7862/rf.2018.10

Abidi Jamel

email: abidijamel1@gmail.com

ORCID: 0000-0002-2552-6848

Département de Mathématiques

Faculté des Sciences de Tunis

1060 Tunis

TUNISIA

Received 29.09.2017

Accepted 28.07.2018

FG-coupled Fixed Point Theorems for Contractive Type Mappings in Partially Ordered Metric Spaces

Deepa Karichery and Shaini Pulickakunnel

ABSTRACT: In this paper we prove FG-coupled fixed point theorems for Kannan, Reich and Chatterjea type mappings in partially ordered complete metric spaces using mixed monotone property.

AMS Subject Classification: 47H10, 54F05.

Keywords and Phrases: FG-coupled fixed point; Mixed monotone property; Contractive type mappings; Partially ordered space.

1. Introduction and Preliminaries

Banach contraction theorem is one of the fundamental theorems in metric fixed point theory. Banach proved existence of unique fixed point for a self contraction in complete metric space. Since the contractions are always continuous, Kannan introduced a new type of contractive map known as Kannan mapping [8] and proved analogues results of Banach contraction theorem. The importance of Kannan mapping is that it can be discontinuous and it characterizes completeness of the space [14, 15]. In [11] Reich introduced a new type of contraction which is a generalization of Banach contraction and Kannan mapping and proved existence of unique fixed point in complete metric spaces. Later Chatterjea defined a contraction similar to Kannan mapping known as Chatterjea mapping [4] and proved various fixed point results. Inspired by these contractions, several authors did research in this area using different spaces and by weakening the contraction conditions [2, 7, 9, 12].

The concept of coupled fixed point was introduced by Guo and Lakshmikantham [6]. They proved fixed point theorems using mixed monotone property in cone spaces.

In [3] Gnana Bhaskar and Lakshmikantham proved coupled fixed point theorems for contractions in partially ordered complete metric spaces using mixed monotone property. Kannan, Chatterjea and Reich type contractions are further explored in coupled fixed point theory and the results are reported in [1, 5, 13]. Recently the concept of FG-coupled fixed point was introduced in [10] and they proved FG-coupled fixed point theorems for various contractive type mappings.

In this paper we prove existence of FG-coupled fixed point theorems using Kannan, Chatterjea and Reich type contraction on partially ordered complete metric spaces.

Now we recall some basic concepts of coupled and FG-coupled fixed points.

Definition 1.1 ([3]). An element $(x, y) \in X \times X$ is said to be a coupled fixed point of the map $F : X \times X \rightarrow X$ if $F(x, y) = x$ and $F(y, x) = y$.

Definition 1.2 ([10]). Let (X, d_X, \leq_{P_1}) and (Y, d_Y, \leq_{P_2}) be two partially ordered metric spaces and $F : X \times Y \rightarrow X$ and $G : Y \times X \rightarrow Y$. We say that F and G have mixed monotone property if for any $x, y \in X$

$$x_1, x_2 \in X, \quad x_1 \leq_{P_1} x_2 \Rightarrow F(x_1, y) \leq_{P_1} F(x_2, y) \text{ and } G(y, x_1) \geq_{P_2} G(y, x_2)$$

$$y_1, y_2 \in Y, \quad y_1 \leq_{P_2} y_2 \Rightarrow F(x, y_1) \geq_{P_1} F(x, y_2) \text{ and } G(y_1, x) \leq_{P_2} G(y_2, x).$$

Definition 1.3 ([10]). An element $(x, y) \in X \times Y$ is said to be FG-coupled fixed point if $F(x, y) = x$ and $G(y, x) = y$.

If $(x, y) \in X \times Y$ is an FG-coupled fixed point then $(y, x) \in Y \times X$ is a GF-coupled fixed point. Partial order \leq on $X \times Y$ is defined as $(u, v) \leq (x, y) \Leftrightarrow x \geq_{P_1} u, y \leq_{P_2} v \quad \forall (x, y), (u, v) \in X \times Y$. Also the iteration is given by $F^{n+1}(x, y) = F(F^n(x, y), G^n(y, x))$ and $G^{n+1}(y, x) = G(G^n(y, x), F^n(x, y))$ for every $n \in \mathbb{N}$ and $(x, y) \in X \times Y$.

2. Main Results

Theorem 2.1. Let $(X, d_X, \leq_{P_1}), (Y, d_Y, \leq_{P_2})$ be two partially ordered complete metric spaces. Let $F : X \times Y \rightarrow X$ and $G : Y \times X \rightarrow Y$ be two continuous functions having the mixed monotone property. Assume that there exist $p, q, r, s \in \left[0, \frac{1}{2}\right)$ satisfying

$$d_X(F(x, y), F(u, v)) \leq p d_X(x, F(x, y)) + q d_X(u, F(u, v)); \forall x \geq_{P_1} u, y \leq_{P_2} v \quad (1)$$

$$d_Y(G(y, x), G(v, u)) \leq r d_Y(y, G(y, x)) + s d_Y(v, G(v, u)); \forall x \leq_{P_1} u, y \geq_{P_2} v. \quad (2)$$

If there exist $x_0 \in X, y_0 \in Y$ satisfying $x_0 \leq_{P_1} F(x_0, y_0)$ and $y_0 \geq_{P_2} G(y_0, x_0)$ then there exist $x \in X, y \in Y$ such that $x = F(x, y)$ and $y = G(y, x)$.

Proof. Given $x_0 \leq_{P_1} F(x_0, y_0) = x_1$ and $y_0 \geq_{P_2} G(y_0, x_0) = y_1$.

Define $x_{n+1} = F(x_n, y_n)$ and $y_{n+1} = G(y_n, x_n)$ for $n = 1, 2, 3..$

Then we can easily show that $\{x_n\}$ is increasing in X and $\{y_n\}$ is decreasing in Y .

Using inequalities (1) and (2) we get

$$\begin{aligned} d_X(x_{n+1}, x_n) &= d_X(F(x_n, y_n), F(x_{n-1}, y_{n-1})) \\ &\leq p d_X(x_n, F(x_n, y_n)) + q d_X(x_{n-1}, F(x_{n-1}, y_{n-1})) \\ &= p d_X(x_n, x_{n+1}) + q d_X(x_{n-1}, x_n) \end{aligned}$$

ie, $(1 - p) d_X(x_{n+1}, x_n) \leq q d_X(x_{n-1}, x_n)$

$$\begin{aligned} \text{ie, } d_X(x_n, x_{n+1}) &\leq \frac{q}{1 - p} d_X(x_{n-1}, x_n) \\ &= \delta_1 d_X(x_{n-1}, x_n) \text{ where } \delta_1 = \frac{q}{1 - p} < 1 \\ &\leq \delta_1^2 d_X(x_{n-2}, x_{n-1}) \\ &\vdots \\ &\leq \delta_1^n d_X(x_0, x_1). \end{aligned}$$

Similarly we get $d_Y(y_{n+1}, y_n) \leq \delta_2^n d_Y(y_1, y_0)$ where $\delta_2 = \frac{r}{1 - s} < 1$.

Consider $m > n$

$$\begin{aligned} d_X(x_m, x_n) &\leq d_X(x_m, x_{m-1}) + d_X(x_{m-1}, x_{m-2}) + \dots + d_X(x_{n+1}, x_n) \\ &\leq \delta_1^{m-1} d_X(x_1, x_0) + \delta_1^{m-2} d_X(x_1, x_0) + \dots + \delta_1^n d_X(x_1, x_0) \\ &= \delta_1^n \left(1 + \delta_1 + \dots + \delta_1^{m-n-1}\right) d_X(x_1, x_0) \\ &\leq \frac{\delta_1^n}{1 - \delta_1} d_X(x_1, x_0). \end{aligned}$$

Since $0 \leq \delta_1 < 1$, δ_1^n converges to 0 (as $n \rightarrow \infty$). Therefore $\{F^n(x_0, y_0)\}$ is a Cauchy sequence in X . Similarly we can prove that $\{G^n(y_0, x_0)\}$ is a Cauchy sequence in Y . Since by the completeness of X and Y , there exist $x \in X$ and $y \in Y$ such that $\lim_{n \rightarrow \infty} F^n(x_0, y_0) = x$ and $\lim_{n \rightarrow \infty} G^n(y_0, x_0) = y$.

Now we have to prove the existence of FG-coupled fixed point.

Consider,

$$\begin{aligned} d_X(F(x, y), x) &= \lim_{n \rightarrow \infty} d_X(F(F^n(x_0, y_0), G^n(y_0, x_0)), F^n(x_0, y_0)) \\ &= \lim_{n \rightarrow \infty} d_X(F^{n+1}(x_0, y_0), F^n(x_0, y_0)) \\ &= 0 \end{aligned}$$

ie, $F(x, y) = x$. Similarly we get $G(y, x) = y$. □

By replacing the continuity of F and G by other conditions we obtain the following existence theorems of FG-coupled fixed point.

Theorem 2.2. Let (X, d_X, \leq_{P_1}) and (Y, d_Y, \leq_{P_2}) be two partially ordered complete metric spaces and $F : X \times Y \rightarrow X$, $G : Y \times X \rightarrow Y$ be two mappings having the mixed monotone property. Assume that X and Y satisfy the following property

(i) If a non-decreasing sequence $\{x_n\} \rightarrow x$ then $x_n \leq_{P_1} x \forall n$.

(ii) If a non-increasing sequence $\{y_n\} \rightarrow y$ then $y \leq_{P_2} y_n \forall n$.

Also assume that there exist $p, q, r, s \in \left[0, \frac{1}{2}\right)$ satisfying

$$d_X(F(x, y), F(u, v)) \leq p d_X(x, F(x, y)) + q d_X(u, F(u, v)); \forall x \geq_{P_1} u, y \leq_{P_2} v \quad (3)$$

$$d_Y(G(y, x), G(v, u)) \leq r d_Y(y, G(y, x)) + s d_Y(v, G(v, u)); \forall x \leq_{P_1} u, y \geq_{P_2} v. \quad (4)$$

If there exist $x_0 \in X$, $y_0 \in Y$ satisfying $x_0 \leq_{P_1} F(x_0, y_0)$ and $y_0 \geq_{P_2} G(y_0, x_0)$ then there exist $x \in X$, $y \in Y$ such that $x = F(x, y)$ and $y = G(y, x)$.

Proof. Following as in the proof of Theorem 2.1 we get $\lim_{n \rightarrow \infty} F^n(x_0, y_0) = x$ and $\lim_{n \rightarrow \infty} G^n(y_0, x_0) = y$.

Now we have

$$\begin{aligned} d_X(F(x, y), x) &\leq d_X(F(x, y), F^{n+1}(x_0, y_0)) + d_X(F^{n+1}(x_0, y_0), x) \\ &= d_X(F(x, y), F(F^n(x_0, y_0), G^n(y_0, x_0))) + d_X(F^{n+1}(x_0, y_0), x) \\ &\leq p d_X(x, F(x, y)) + q d_X(F^n(x_0, y_0), F(F^n(x_0, y_0), G^n(y_0, x_0))) \\ &\quad + d_X(F^{n+1}(x_0, y_0), x) \quad (\text{using (3)}) \end{aligned}$$

ie, $d_X(F(x, y), x) \leq p d_X(x, F(x, y))$ as $n \rightarrow \infty$.

This holds only when $d_X(F(x, y), x) = 0$. Therefore we get $F(x, y) = x$.

Similarly using (4) and $\lim_{n \rightarrow \infty} G^n(y_0, x_0) = y$ we can prove $y = G(y, x)$. \square

Remark 2.1. If we put $k = m$ and $l = n$ in Theorems 2.1 and 2.2, we get Theorems 2.7 and 2.8 respectively of [10].

Theorem 2.3. Let (X, d_X, \leq_{P_1}) , (Y, d_Y, \leq_{P_2}) be two partially ordered complete metric spaces. Let $F : X \times Y \rightarrow X$ and $G : Y \times X \rightarrow Y$ be two continuous functions having the mixed monotone property. Assume that there exist $p, q, r, s \in \left[0, \frac{1}{2}\right)$ satisfying

$$d_X(F(x, y), F(u, v)) \leq p d_X(x, F(u, v)) + q d_X(u, F(x, y)); \forall x \geq_{P_1} u, y \leq_{P_2} v \quad (5)$$

$$d_Y(G(y, x), G(v, u)) \leq r d_Y(y, G(v, u)) + s d_Y(v, G(y, x)); \forall x \leq_{P_1} u, y \geq_{P_2} v. \quad (6)$$

If there exist $x_0 \in X$, $y_0 \in Y$ satisfying $x_0 \leq_{P_1} F(x_0, y_0)$ and $y_0 \geq_{P_2} G(y_0, x_0)$ then there exist $x \in X$, $y \in Y$ such that $x = F(x, y)$ and $y = G(y, x)$.

Proof. As in Theorem 2.1 we have $\{x_n\}$ increasing in X and $\{y_n\}$ decreasing in Y . We have

$$\begin{aligned} d_X(x_{n+1}, x_n) &= d_X(F(x_n, y_n), F(x_{n-1}, y_{n-1})) \\ &\leq p d_X(x_n, F(x_{n-1}, y_{n-1})) + q d_X(x_{n-1}, F(x_n, y_n)) \quad (\text{Using (5)}) \\ &= p d_X(x_n, x_n) + q d_X(x_{n-1}, x_{n+1}) \\ &\leq q [d_X(x_{n-1}, x_n) + d_X(x_n, x_{n+1})] \\ \text{ie, } d_X(x_n, x_{n+1}) &\leq \frac{q}{1-q} d_X(x_{n-1}, x_n) \\ &= \delta_1 d_X(x_{n-1}, x_n) \quad \text{where } \delta_1 = \frac{q}{1-q} < 1 \\ &\leq \delta_1^2 d_X(x_{n-2}, x_{n-1}) \\ &\vdots \\ &\leq \delta_1^n d_X(x_0, x_1). \end{aligned}$$

Similarly we get $d_Y(y_{n+1}, y_n) \leq \delta_2^n d_Y(y_1, y_0)$ where $\delta_2 = \frac{r}{1-r} < 1$

Now, we prove that $\{F^n(x_0, y_0)\}$ and $\{G^n(y_0, x_0)\}$ are Cauchy sequences in X and Y respectively.

For $m > n$,

$$\begin{aligned} d_X(x_m, x_n) &\leq d_X(x_m, x_{m-1}) + d_X(x_{m-1}, x_{m-2}) + \dots + d_X(x_{n+1}, x_n) \\ &\leq \delta_1^{m-1} d_X(x_1, x_0) + \delta_1^{m-2} d_X(x_1, x_0) + \dots + \delta_1^n d_X(x_1, x_0) \\ &\leq \frac{\delta_1^n}{1-\delta_1} d_X(x_1, x_0). \end{aligned}$$

Since $0 \leq \delta_1 < 1$, δ_1^n converges to 0 (as $n \rightarrow \infty$). Therefore $\{F^n(x_0, y_0)\}$ is a Cauchy sequence in X .

Similarly we can prove that $\{G^n(y_0, x_0)\}$ is a Cauchy sequence in Y .

By the completeness of X and Y , there exist $x \in X$ and $y \in Y$ such that $\lim_{n \rightarrow \infty} F^n(x_0, y_0) = x$ and $\lim_{n \rightarrow \infty} G^n(y_0, x_0) = y$.

As in the proof of Theorem 2.1 we can show that $x = F(x, y)$ and $y = G(y, x)$. \square

Theorem 2.4. Let (X, d_X, \leq_{P_1}) and (Y, d_Y, \leq_{P_2}) be two partially ordered complete metric spaces and $F : X \times Y \rightarrow X$, $G : Y \times X \rightarrow Y$ be two mappings having the mixed monotone property. Assume that X and Y satisfy the following property

(i) If a non-decreasing sequence $\{x_n\} \rightarrow x$ then $x_n \leq_{P_1} x \forall n$.

(ii) If a non-increasing sequence $\{y_n\} \rightarrow y$ then $y \leq_{P_2} y_n \forall n$.

Also assume that there exist $p, q, r, s \in [0, \frac{1}{2})$ satisfying

$$d_X(F(x, y), F(u, v)) \leq p d_X(x, F(u, v)) + q d_X(u, F(x, y)); \forall x \geq_{P_1} u, y \leq_{P_2} v \quad (7)$$

$$d_Y(G(y, x), G(v, u)) \leq r d_Y(y, G(v, u)) + s d_Y(v, G(y, x)); \quad \forall x \leq_{P_1} u, \quad y \geq_{P_2} v. \quad (8)$$

If there exist $x_0 \in X$, $y_0 \in Y$ satisfying $x_0 \leq_{P_1} F(x_0, y_0)$ and $y_0 \geq_{P_2} G(y_0, x_0)$ then there exist $x \in X$, $y \in Y$ such that $x = F(x, y)$ and $y = G(y, x)$.

Proof. Following as in the proof of Theorem 2.3 we get $\lim_{n \rightarrow \infty} F^n(x_0, y_0) = x$ and $\lim_{n \rightarrow \infty} G^n(y_0, x_0) = y$.

Consider

$$\begin{aligned} d_X(F(x, y), x) &\leq d_X(F(x, y), F^{n+1}(x_0, y_0)) + d_X(F^{n+1}(x_0, y_0), x) \\ &= d_X(F(x, y), F(F^n(x_0, y_0), G^n(y_0, x_0))) + d_X(F^{n+1}(x_0, y_0), x) \\ &\leq p d_X(x, F((F^n(x_0, y_0), G^n(y_0, x_0)))) + q d_X(F^n(x_0, y_0), F(x, y)) \\ &\quad + d_X(F^{n+1}(x_0, y_0), x) \\ &= p d_X(x, F^{n+1}(x_0, y_0)) + q d_X(F^n(x_0, y_0), F(x, y)) \\ &\quad + d_X(F^{n+1}(x_0, y_0), x) \end{aligned}$$

ie, $d_X(F(x, y), x) \leq q d_X(x, F(x, y))$ as $n \rightarrow \infty$, which implies that $d_X(F(x, y), x) = 0$. Therefore we get $F(x, y) = x$.

Similarly using (8) and $\lim_{n \rightarrow \infty} G^n(y_0, x_0) = y$, we get $y = G(y, x)$. \square

Remark 2.2. If we put $p = r$ and $q = s$ in Theorems 2.3 and 2.4, we get Theorems 2.9 and 2.10 respectively of [10].

The following example illustrates the above results.

Example 2.1. Let $X = [0, 1]$ and $Y = [-1, 1]$ with usual metric. Partial order on X is defined as $x \leq_{P_1} u$ if and only if $x = u$ and partial order on Y is defined as $y \leq_{P_2} v$ if and only if either $y = v$ or $(y, v) = (0, 1)$. The mapping $F : X \times Y \rightarrow X$ is defined by $F(x, y) = \frac{x+1}{2}$ and $G : Y \times X \rightarrow Y$ is defined as $G(y, x) = \frac{x-1}{2}$. Then F and G satisfies (1), (2), (5), (6) with $p, q, r, s \in [0, \frac{1}{2}]$. Also $(1, 0)$ is the FG-coupled fixed point.

Theorem 2.5. Let $(X, d_X, \leq_{P_1}), (Y, d_Y, \leq_{P_2})$ be two partially ordered complete metric spaces. Let $F : X \times Y \rightarrow X$ and $G : Y \times X \rightarrow Y$ be two continuous functions having the mixed monotone property. Assume that there exist a, b, c with $a + b + c < 1$ satisfying

$$d_X(F(x, y), F(u, v)) \leq a d_X(x, F(x, y)) + b d_X(u, F(u, v)) + c d_X(x, u); \quad (9)$$

$$\forall x \geq_{P_1} u, \quad y \leq_{P_2} v$$

$$d_Y(G(y, x), G(v, u)) \leq a d_Y(y, G(y, x)) + b d_Y(v, G(v, u)) + c d_Y(y, v); \quad (10)$$

$$\forall x \leq_{P_1} u, \quad y \geq_{P_2} v.$$

If there exist $x_0 \in X, y_0 \in Y$ satisfying $x_0 \leq_{P_1} F(x_0, y_0)$ and $y_0 \geq_{P_2} G(y_0, x_0)$ then there exist $x \in X, y \in Y$ such that $x = F(x, y)$ and $y = G(y, x)$.

Proof. Following as in Theorem 2.1 we have $\{x_n\}$ is increasing in X and $\{y_n\}$ is decreasing in Y .

Now we claim that

$$d_X(F^{n+1}(x_0, y_0), F^n(x_0, y_0)) \leq \left(\frac{b+c}{1-a}\right)^n d_X(x_0, x_1) \tag{11}$$

$$d_Y(G^{m+1}(y_0, x_0), G^m(y_0, x_0)) \leq \left(\frac{a+c}{1-b}\right)^m d_Y(y_0, y_1). \tag{12}$$

The proof is by mathematical induction with the help of (9) and (10). For $n = 1$, consider

$$\begin{aligned} d_X(F^2(x_0, y_0), F(x_0, y_0)) &= d_X(F(F(x_0, y_0), G(y_0, x_0)), F(x_0, y_0)) \\ &\leq a d_X(F(x_0, y_0), F^2(x_0, y_0)) + b d_X(x_0, F(x_0, y_0)) \\ &\quad + c d_X(F(x_0, y_0), x_0) \end{aligned}$$

ie, $d_X(F^2(x_0, y_0), F(x_0, y_0)) \leq \frac{b+c}{1-a} d_X(x_0, x_1)$.

Thus the inequality (11) is true for $n = 1$.

Now assume that (11) is true for $n \leq m$, and check for $n = m + 1$.

Consider,

$$\begin{aligned} &d_X(F^{m+2}(x_0, y_0), F^{m+1}(x_0, y_0)) \\ &= d_X(F(F^{m+1}(x_0, y_0), G^{m+1}(y_0, x_0)), F(F^m(x_0, y_0), G^m(y_0, x_0))) \\ &\leq a d_X(F^{m+1}(x_0, y_0), F^{m+2}(x_0, y_0)) + b d_X(F^m(x_0, y_0), F^{m+1}(x_0, y_0)) \\ &\quad + c d_X(F^{m+1}(x_0, y_0), F^m(x_0, y_0)) \end{aligned}$$

ie, $d_X(F^{m+2}(x_0, y_0), F^{m+1}(x_0, y_0)) \leq \frac{b+c}{1-a} d_X(F^m(x_0, y_0), F^{m+1}(x_0, y_0))$
 $\leq \left(\frac{b+c}{1-a}\right)^{m+1} d_X(x_0, x_1)$

ie, the inequality (11) is true for all $n \in \mathbb{N}$.

Similarly we can prove the inequality (12).

For $m > n$, consider

$$\begin{aligned} &d_X(F^n(x_0, y_0), F^m(x_0, y_0)) \\ &\leq d_X(F^n(x_0, y_0), F^{n+1}(x_0, y_0)) + d_X(F^{n+1}(x_0, y_0), F^{n+2}(x_0, y_0)) + \dots \\ &\quad + d_X(F^{m-1}(x_0, y_0), F^m(x_0, y_0)) \\ &\leq \left[\left(\frac{b+c}{1-a}\right)^n + \left(\frac{b+c}{1-a}\right)^{n+1} + \dots + \left(\frac{b+c}{1-a}\right)^{m-1} \right] d_X(x_0, x_1) \\ &\leq \frac{\delta_1^n}{1-\delta_1} d_X(x_0, x_1) \text{ where } \delta_1 = \frac{b+c}{1-a} < 1. \end{aligned}$$

Since $0 \leq \delta_1 < 1$, δ_1^n converges to 0 (as $n \rightarrow \infty$) ie, $\{F^n(x_0, y_0)\}$ is a Cauchy sequence in X. Similarly by using inequality (12) we can prove that $\{G^n(y_0, x_0)\}$ is a Cauchy sequence in Y.

By the completeness of X and Y, there exist $x \in X$ and $y \in Y$ such that $\lim_{n \rightarrow \infty} F^n(x_0, y_0) = x$ and $\lim_{n \rightarrow \infty} G^n(y_0, x_0) = y$.

As in the proof of Theorem 2.1, using continuity of F and G we can prove that $F(x, y) = x$ and $G(y, x) = y$. \square

If we take $X = Y$ and $F = G$ in the above theorem we get the following corollary.

Corollary 2.1. *Let (X, d, \leq) be a partially ordered complete metric space. Let $F : X \times X \rightarrow X$ be a continuous function having the mixed monotone property. Assume that there exist non-negative a, b, c such that $a + b + c < 1$ satisfying*

$$d(F(x, y), F(u, v)) \leq a d(x, F(x, y)) + b d(u, F(u, v)) + c d(x, u); \quad \forall x \geq u, y \leq v.$$

If there exist $x_0, y_0 \in X$ satisfying $x_0 \leq F(x_0, y_0)$ and $y_0 \geq F(y_0, x_0)$ then there exist $(x, y) \in X \times X$ such that $x = F(x, y)$ and $y = F(y, x)$.

Theorem 2.6. *Let (X, d_X, \leq_{P_1}) and (Y, d_Y, \leq_{P_2}) be two partially ordered complete metric spaces and $F : X \times Y \rightarrow X$, $G : Y \times X \rightarrow Y$ be two mappings having the mixed monotone property. Assume that X and Y satisfy the following property*

(i) *If a non-decreasing sequence $\{x_n\} \rightarrow x$ then $x_n \leq_{P_1} x \quad \forall n$.*

(ii) *If a non-increasing sequence $\{y_n\} \rightarrow y$ then $y \leq_{P_2} y_n \quad \forall n$.*

Also assume that there exist a, b, c with $a + b + c < 1$ satisfying

$$d_X(F(x, y), F(u, v)) \leq a d_X(x, F(x, y)) + b d_X(u, F(u, v)) + c d_X(x, u); \quad (13)$$

$$\forall x \geq_{P_1} u, y \leq_{P_2} v$$

$$d_Y(G(y, x), G(v, u)) \leq a d_Y(y, G(y, x)) + b d_Y(v, G(v, u)) + c d_Y(y, v); \quad (14)$$

$$\forall x \leq_{P_1} u, y \geq_{P_2} v.$$

If there exist $x_0 \in X, y_0 \in Y$ satisfying $x_0 \leq_{P_1} F(x_0, y_0)$ and $y_0 \geq_{P_2} G(y_0, x_0)$ then there exist $x \in X, y \in Y$ such that $x = F(x, y)$ and $y = G(y, x)$.

Proof. Following as in the proof of Theorem 2.5 we obtain $\lim_{n \rightarrow \infty} F^n(x_0, y_0) = x$ and $\lim_{n \rightarrow \infty} G^n(y_0, x_0) = y$.

We have

$$\begin{aligned} d_X(F(x, y), x) &\leq d_X(F(x, y), F^{n+1}(x_0, y_0)) + d_X(F^{n+1}(x_0, y_0), x) \\ &= d_X(F(x, y), F(F^n(x_0, y_0), G^n(y_0, x_0))) + d_X(F^{n+1}(x_0, y_0), x) \\ &\leq a d_X(x, F(x, y)) + b d_X(F^n(x_0, y_0), F(F^n(x_0, y_0), G^n(y_0, x_0))) \\ &\quad + c d_X(x, F^n(x_0, y_0)) + d_X(F^{n+1}(x_0, y_0), x) \\ &= a d_X(x, F(x, y)) + b d_X(F^n(x_0, y_0), F^{n+1}(x_0, y_0)) \\ &\quad + c d_X(x, F^n(x_0, y_0)) + d_X(F^{n+1}(x_0, y_0), x) \end{aligned}$$

ie, $d_X(F(x, y), x) \leq a d_X(x, F(x, y))$ as $n \rightarrow \infty$, which implies that $d_X(F(x, y), x) = 0$. Therefore $F(x, y) = x$.

Similarly using (14) and $\lim_{n \rightarrow \infty} G^n(y_0, x_0) = y$ we get $y = G(y, x)$. □

By assuming $X = Y$ and $F = G$ in the above theorem we will get the following corollary.

Corollary 2.2. *Let (X, d, \leq) be a partially ordered complete metric space and $F : X \times X \rightarrow X$ be a mapping having the mixed monotone property. Assume that X satisfy the following property*

(i) *If a non-decreasing sequence $\{x_n\} \rightarrow x$ then $x_n \leq x \forall n$.*

(ii) *If a non-increasing sequence $\{y_n\} \rightarrow y$ then $y \leq y_n \forall n$.*

Also assume that there exist non-negative a, b, c such that $a + b + c < 1$ satisfying

$$d(F(x, y), F(u, v)) \leq a d(x, F(x, y)) + b d(u, F(u, v)) + c d(x, u); \forall x \geq u, y \leq v.$$

If there exist $(x_0, y_0) \in X \times X$ satisfying $x_0 \leq F(x_0, y_0)$ and $y_0 \geq F(y_0, x_0)$ then there exist $x, y \in X$ such that $x = F(x, y)$ and $y = F(y, x)$.

Remark 2.3. If we take $c = 0$ in Theorems 2.5 and 2.6, we get Theorems 2.7 and 2.8 respectively of [10].

Theorem 2.7. *Let $(X, d_X, \leq_{P_1}), (Y, d_Y, \leq_{P_2})$ be two partially ordered complete metric spaces. Let $F : X \times Y \rightarrow X$ and $G : Y \times X \rightarrow Y$ be two continuous functions having the mixed monotone property. Assume that there exist non-negative a, b, c satisfying*

$$d_X(F(x, y), F(u, v)) \leq a d_X(x, F(u, v)) + b d_X(u, F(x, y)) + c d_X(x, u); \quad (15)$$

$$\forall x \geq_{P_1} u, y \leq_{P_2} v; 2b + c < 1$$

$$d_Y(G(y, x), G(v, u)) \leq a d_Y(y, G(v, u)) + b d_Y(v, G(y, x)) + c d_Y(y, v); \quad (16)$$

$$\forall x \leq_{P_1} u, y \geq_{P_2} v; 2a + c < 1.$$

If there exist $x_0 \in X, y_0 \in Y$ satisfying $x_0 \leq_{P_1} F(x_0, y_0)$ and $y_0 \geq_{P_2} G(y_0, x_0)$ then there exist $x \in X, y \in Y$ such that $x = F(x, y)$ and $y = G(y, x)$.

Proof. As in the proof of Theorem 2.1, it can be proved that $\{x_n\}$ is increasing in X and $\{y_n\}$ is decreasing in Y .

Now we claim that

$$d_X(F^{n+1}(x_0, y_0), F^n(x_0, y_0)) \leq \left(\frac{b+c}{1-b}\right)^n d_X(x_0, x_1) \quad (17)$$

$$d_Y(G^{n+1}(y_0, x_0), G^n(y_0, x_0)) \leq \left(\frac{a+c}{1-a}\right)^n d_Y(y_0, y_1). \quad (18)$$

We prove the claim by mathematical induction, using (15) and (16).

For $n = 1$, consider

$$\begin{aligned} d_X(F^2(x_0, y_0), F(x_0, y_0)) &= d_X(F(F(x_0, y_0), G(y_0, x_0)), F(x_0, y_0)) \\ &\leq a d_X(F(x_0, y_0), F(x_0, y_0)) + b d_X(x_0, F^2(x_0, y_0)) + c d_X(F(x_0, y_0), x_0) \\ &\leq b [d_X(x_0, F(x_0, y_0)) + d_X(F(x_0, y_0), F^2(x_0, y_0))] + c d_X(F(x_0, y_0), x_0) \end{aligned}$$

$$\text{ie, } d_X(F^2(x_0, y_0), F(x_0, y_0)) \leq \frac{b+c}{1-b} d_X(x_0, x_1).$$

Thus the inequality (17) is true for $n = 1$.

Now assume that (17) is true for $n \leq m$, then check for $n = m + 1$.

Consider,

$$\begin{aligned} d_X(F^{m+2}(x_0, y_0), F^{m+1}(x_0, y_0)) &= d_X(F(F^{m+1}(x_0, y_0), G^{m+1}(y_0, x_0)), F(F^m(x_0, y_0), G^m(y_0, x_0))) \\ &\leq a d_X(F^{m+1}(x_0, y_0), F^{m+1}(x_0, y_0)) + b d_X(F^m(x_0, y_0), F^{m+2}(x_0, y_0)) \\ &\quad + c d_X(F^{m+1}(x_0, y_0), F^m(x_0, y_0)) \\ &\leq b [d_X(F^m(x_0, y_0), F^{m+1}(x_0, y_0)) + d_X(F^{m+1}(x_0, y_0), F^{m+2}(x_0, y_0))] \\ &\quad + c d_X(F^{m+1}(x_0, y_0), F^m(x_0, y_0)) \end{aligned}$$

ie,

$$\begin{aligned} d_X(F^{m+2}(x_0, y_0), F^{m+1}(x_0, y_0)) &\leq \frac{b+c}{1-b} d_X(F^m(x_0, y_0), F^{m+1}(x_0, y_0)) \\ &\leq \left(\frac{b+c}{1-b}\right)^{m+1} d_X(x_0, x_1) \end{aligned}$$

ie, the inequality (17) is true for all $n \in \mathbb{N}$.

Similarly we can prove the inequality (18).

For $m > n$, consider

$$\begin{aligned} d_X(F^n(x_0, y_0), F^m(x_0, y_0)) &\leq d_X(F^n(x_0, y_0), F^{n+1}(x_0, y_0)) + d_X(F^{n+1}(x_0, y_0), F^{n+2}(x_0, y_0)) + \dots \\ &\quad + d_X(F^{m-1}(x_0, y_0), F^m(x_0, y_0)) \\ &\leq \left[\left(\frac{b+c}{1-b}\right)^n + \left(\frac{b+c}{1-b}\right)^{n+1} + \dots + \left(\frac{b+c}{1-b}\right)^{m-1} \right] d_X(x_0, x_1) \\ &\leq \frac{\delta_1^n}{1-\delta_1} d_X(x_0, x_1); \text{ where } \delta_1 = \frac{b+c}{1-b} < 1. \end{aligned}$$

Since $0 \leq \delta_1 < 1$, δ_1^n converges to 0 (as $n \rightarrow \infty$) ie, $\{F^n(x_0, y_0)\}$ is a Cauchy sequence in X . Similarly we can prove that $\{G^n(y_0, x_0)\}$ is a Cauchy sequence in Y .

Since X and Y are complete, there exist $x \in X$ and $y \in Y$ such that $\lim_{n \rightarrow \infty} F^n(x_0, y_0) = x$ and $\lim_{n \rightarrow \infty} G^n(y_0, x_0) = y$.

By continuity of F and G , as in the Theorem 2.1 we can show that $F(x, y) = x$ and $G(y, x) = y$. \square

If $X = Y$ and $F = G$ in the above theorem we get the following corollary.

Corollary 2.3. *Let (X, d, \leq) be a partially ordered complete metric space. Let $F : X \times X \rightarrow X$ be a continuous function having the mixed monotone property. Assume that there exist non-negative a, b, c such that $2a + c < 1$ and $2b + c < 1$ satisfying*

$$d(F(x, y), F(u, v)) \leq a d(x, F(u, v)) + b d(u, F(x, y)) + c d(x, u); \quad \forall x \geq u, y \leq v.$$

If there exist $(x_0, y_0) \in X \times Y$ satisfying $x_0 \leq F(x_0, y_0)$ and $y_0 \geq F(y_0, x_0)$ then there exist $x, y \in X$ such that $x = F(x, y)$ and $y = F(y, x)$.

In the following theorem we replace the continuity by other conditions to obtain FG-coupled fixed point.

Theorem 2.8. *Let (X, d_X, \leq_{P_1}) and (Y, d_Y, \leq_{P_2}) be two partially ordered complete metric spaces and $F : X \times Y \rightarrow X, G : Y \times X \rightarrow Y$ be two mappings having the mixed monotone property. Assume that X and Y satisfy the following property*

(i) *If a non-decreasing sequence $\{x_n\} \rightarrow x$ then $x_n \leq_{P_1} x \quad \forall n$.*

(ii) *If a non-increasing sequence $\{y_n\} \rightarrow y$ then $y \leq_{P_2} y_n \quad \forall n$.*

Also assume that there exist non-negative a, b, c satisfying

$$d_X(F(x, y), F(u, v)) \leq a d_X(x, F(u, v)) + b d_X(u, F(x, y)) + c d_X(x, u); \quad \forall x \geq_{P_1} u, y \leq_{P_2} v; \quad 2b + c < 1 \tag{19}$$

$$d_Y(G(y, x), G(v, u)) \leq a d_Y(y, G(v, u)) + b d_Y(v, G(y, x)) + c d_Y(y, v); \quad \forall x \leq_{P_1} u, y \geq_{P_2} v; \quad 2a + c < 1. \tag{20}$$

If there exist $x_0 \in X, y_0 \in Y$ satisfying $x_0 \leq_{P_1} F(x_0, y_0)$ and $y_0 \geq_{P_2} G(y_0, x_0)$ then there exist $x \in X, y \in Y$ such that $x = F(x, y)$ and $y = G(y, x)$.

Proof. Following as in the proof of Theorem 2.7 we get $\lim_{n \rightarrow \infty} F^n(x_0, y_0) = x$ and $\lim_{n \rightarrow \infty} G^n(y_0, x_0) = y$.

We have

$$\begin{aligned} d_X(F(x, y), x) &\leq d_X(F(x, y), F^{n+1}(x_0, y_0)) + d_X(F^{n+1}(x_0, y_0), x) \\ &= d_X(F(x, y), F(F^n(x_0, y_0), G^n(y_0, x_0))) + d_X(F^{n+1}(x_0, y_0), x) \\ &\leq a d_X(x, F(F^n(x_0, y_0), G^n(y_0, x_0))) + b d_X(F^n(x_0, y_0), F(x, y)) \\ &\quad + c d_X(x, F^n(x_0, y_0)) + d_X(F^{n+1}(x_0, y_0), x) \\ &= a d_X(x, F^{n+1}(x_0, y_0)) + b d_X(F^n(x_0, y_0), F(x, y)) \\ &\quad + c d_X(x, F^n(x_0, y_0)) + d_X(F^{n+1}(x_0, y_0), x) \end{aligned}$$

ie, $d_X(F(x, y), x) \leq b d_X(x, F(x, y))$ as $n \rightarrow \infty$, which implies that $d_X(F(x, y), x) = 0$. Therefore $F(x, y) = x$.

Also by using (20) and $\lim_{n \rightarrow \infty} G^n(y_0, x_0) = y$ we can show that $y = G(y, x)$. \square

Taking $X = Y$ and $F = G$ in the above corollary we get the corresponding coupled fixed point result.

Corollary 2.4. *Let (X, d, \leq) be a partially ordered complete metric spaces and $F : X \times Y \rightarrow X$ be a mapping having the mixed monotone property. Assume that X satisfy the following property*

(i) *If a non-decreasing sequence $\{x_n\} \rightarrow x$ then $x_n \leq x \forall n$.*

(ii) *If a non-increasing sequence $\{y_n\} \rightarrow y$ then $y \leq y_n \forall n$.*

Also assume that there exist non-negative a, b, c such that $2a + c < 1$ and $2b + c < 1$ satisfying

$$d(F(x, y), F(u, v)) \leq a d(x, F(u, v)) + b d(u, F(x, y)) + c d(x, u); \forall x \geq u, y \leq v.$$

If there exist $(x_0, y_0) \in X \times Y$ satisfying $x_0 \leq F(x_0, y_0)$ and $y_0 \geq F(y_0, x_0)$ then there exist $(x, y) \in X \times Y$ such that $x = F(x, y)$ and $y = F(y, x)$.

Remark 2.4. If we take $c = 0$ in Theorems 2.7 and 2.8, we get Theorems 2.9 and 2.10 respectively of [10].

Acknowledgment

The first author acknowledges financial support from Kerala State Council for Science, Technology and Environment (KSCSTE), in the form of fellowship. We also acknowledge the valuable suggestions made by the referee for improving this paper.

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DOI: 10.7862/rf.2018.11

Deepa Karichery

email: deepakarichery@gmail.com

ORCID: 0000-0002-4016-4814

Department of Mathematics

Central University of Kerala
INDIA - 671316

Shaini Pulickakunnel
email: shainipv@gmail.com
ORCID: 0000-0001-9958-9211
Department of Mathematics
Central University of Kerala
INDIA - 671316

Received 05.01.2018

Accepted 10.08.2018

Location of Zeros of Lacunary-type Polynomials

Idrees Qasim, Tawheeda Rasool and Abdul Liman

ABSTRACT: In this paper, we present some interesting results concerning the location of zeros of Lacunary-type of polynomial in the complex plane. By relaxing the hypothesis and putting less restrictive conditions on the coefficients of the polynomial, our results generalize and refines some classical results.

AMS Subject Classification: 30A01, 30C10, 30C15.

Keywords and Phrases: Lacunary polynomials; Zeros; Eneström-Kekeya Theorem.

1. Introduction

The study of the zeros of a polynomial dates from about the time when the geometric representation of complex numbers was introduced into mathematics. The first contributors of the subject were Gauss and Cauchy. Since the days of Gauss and Cauchy many mathematicians have contributed to the further growth of the subject. The classical results of Cauchy [3], concerning the bounds for the moduli of the zeros of a polynomial states

Theorem A. $P(z) := \sum_{j=0}^n a_j z^j$ is a polynomial of degree n and $M = \max_{0 \leq j \leq n-1} \left| \frac{a_j}{a_n} \right|$, then all the zeros of $P(z)$ lie in

$$|z| \leq 1 + M.$$

There exists several generalizations and improvements of this result (for reference see [12] and [13]). As an improvement of this result, Joyal, Labelle and Rahman [8]

proved that, if $B = \max_{0 \leq j \leq n-1} |a_j|$, then all the zeros of the polynomial $P(z) := z^n + \sum_{j=0}^{n-1} a_j z^j$ are contained in the circle

$$|z| \leq \frac{1}{2} \left\{ 1 + |a_{n-1}| + \{(1 - |a_{n-1}|)^2 + 4B\}^{\frac{1}{2}} \right\}. \quad (1.0)$$

Next we state the following elegant result which is commonly known as Eneström-Kakeya Theorem in the theory of distribution of zeros of polynomials.

Theorem B. Let $P(z) := \sum_{j=0}^n a_j z^j$ be a polynomial of degree n such that

$$a_n \geq a_{n-1} \geq \dots \geq a_1 \geq a_0 > 0,$$

then all the zeros of $P(z)$ lie in $|z| \leq 1$.

Theorem B was proved by Eneström [4], independently by Kakeya [9] and Hurwitz [7]. Applying this result to the polynomial $z^n P(\frac{1}{z})$, one gets equivalent form of Eneström-Kakeya Theorem which states that

Theorem C. If $P(z) := \sum_{j=0}^n a_j z^j$ is a polynomial of degree n such that

$$a_0 \geq a_1 \geq \dots \geq a_{n-1} \geq a_n > 0,$$

then all the zeros of $P(z)$ lie in $|z| \geq 1$.

Applying the above results to the polynomial $P(tz)$, the following more general result is immediate:

Theorem D. Let $P(z) := \sum_{j=0}^n a_j z^j$ be a polynomial of degree n such that

$$a_n t^n \geq a_{n-1} t^{n-1} \geq \dots \geq a_1 t \geq a_0 > 0,$$

then all the zeros of $P(z)$ lie in $|z| \leq t$ and in case

$$0 < a_n \leq a_{n-1} t^{n-1} \leq \dots \leq a_1 t^{n-1} \leq a_0 t^n,$$

then $P(z)$ has all zeros in $|z| \geq \frac{1}{t}$.

Now consider the class of polynomials

$$P(z) := a_0 + a_1 z + \dots + a_\mu z^\mu + a_{n_1} z^{n_1} + a_{n_2} z^{n_2} + \dots + a_{n_k} z^{n_k}, \quad (1.1)$$

$$0 < n_0 = \mu < n_1 < n_2 < \dots < n_k, \quad a_0 a_\mu a_{n_1} a_{n_2} \dots a_{n_k} \neq 0.$$

Here the coefficients a_j , $0 \leq j \leq \mu$, are fixed, the coefficients a_{n_j} , $j = 1, 2, \dots, k$ are arbitrary and the remaining coefficients a_j are zero.

Landau ([10] and [11]) Initiated the study of polynomials of this form in 1906-7.

He considered the case $\mu = 1$, $k = 1$ or 2 and proved that every trinomial

$$a_0 + a_1z + a_nz^n, \quad a_1a_n \neq 0, \quad n \geq 2$$

has at least one zero in the circle $|z| \leq 2 \left| \frac{a_0}{a_1} \right|$ and every quadrinomial

$$a_0 + a_1z + a_mz^m + a_nz^n, \quad a_1a_ma_n \neq 0, \quad 2 \leq m < n,$$

has at least one zero in the circle $|z| \leq \frac{17}{3} \left| \frac{a_0}{a_1} \right|$. Thus for these cases Landau proved the existence of a circle $|z| = R(a_0, a_1)$ containing at least one zero of $P(z)$. He also raised the question as to whether or not a circle with this same property existed in the case $\mu = 1$ and k arbitrary.

An affirmative reply was given in 1907 by Allardice [1] who proved that when $\mu = 1$, at least one zero of $P(z)$ lies in the circle

$$|z| \leq \left| \frac{a_0}{a_1} \right| \prod_{j=1}^k \frac{n_j}{n_j - 1}$$

and by Fejér ([6], [5]) who proved that, when $a_1 = a_2 = \dots = a_{\mu-1} = 0$, at least one zero of $P(z)$ lies in the circle

$$|z| \leq \left\{ \left| \frac{a_0}{a_\mu} \right| \prod_{j=1}^k \frac{n_j}{n_j - \mu} \right\}^{\frac{1}{\mu}}.$$

Another result which is instructive is the one due to Van Vleck [14], who proved that the polynomial

$$P(z) := 1 + a_rz^r + a_{r+1}z^{r+1} + \dots + a_nz^n, \quad r < n, \quad a_r \neq 0$$

has at least r zeros in the disk $|z| \leq \left[\frac{C(n,r)}{|a_r|} \right]^{\frac{1}{r}}$.

Recently Aziz and Rather [2] proved the following result for Lacunary-type of polynomials.

Theorem E. *For any given positive number t , all the zeros of the polynomial*

$$P(z) := a_nz^n + a_\mu z^\mu + a_{\mu-1}z^{\mu-1} + \dots + a_1z + a_0, \quad \mu < n, \quad a_0a_\mu a_n \neq 0$$

lie in the circle

$$|z| \leq \max \left\{ N_{p,t}^{\frac{1}{n-\mu}}, N_{p,t}^{\frac{1}{\mu}} \right\},$$

where

$$N_{p,t} = (\mu + 1)^{\frac{1}{q}} \left\{ \sum_{j=0}^{\mu} \left| \frac{a_j}{a_n t^{n-j}} \right|^p \right\}^{\frac{1}{p}},$$

$p > 1, q > 1$ with $p^{-1} + q^{-1} = 1$.

In this paper, we consider the following Lacunary-type of polynomials of type (1.1) with $k = 1$ and μ arbitrary

$$P(z) := a_n z^n + a_\mu z^\mu + a_{\mu-1} z^{\mu-1} + \dots + a_1 z + a_0, \quad \mu < n, \quad a_0 a_\mu a_n \neq 0, \quad (1.2)$$

and prove some results concerning the bounds for the zeros of polynomials of this form.

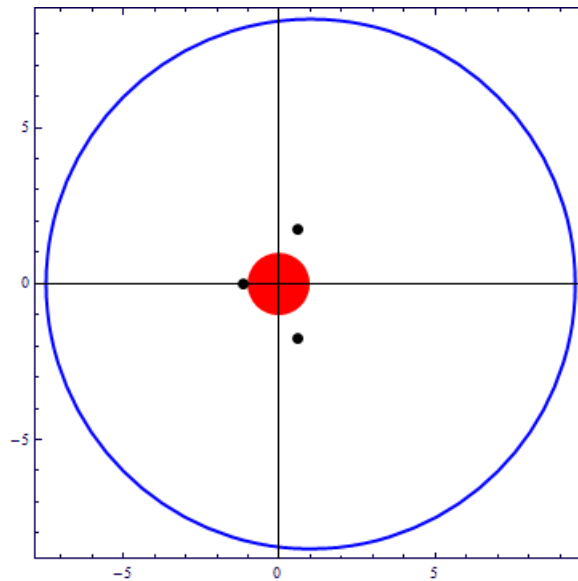
2. Main Results

Theorem 1. *Let $P(z)$ be a polynomial of type (1.2) which does not vanish in $|z| < t$, where $t > 0$, then for $p > 0, q > 0, p^{-1} + q^{-1} = 1$, all the zeros of $P(z)$ lie in*

$$|z - t| \leq A = (\mu + 2)^{\frac{1}{q}} \left\{ \sum_{j=0}^{\mu+1} \left| \frac{t a_j - a_{j-1}}{a_n t^{n-j}} \right|^p \right\}^{\frac{1}{p}}, \quad (2.1)$$

where $a_{-1} = a_{\mu+1} = 0$.

Theorem 1 states that if no zero of a polynomial $P(z)$ of type (1.2) lie in $|z| < t, t > 0$, then all its zeros will lie in the region between the circles $|z| < t$ and $|z - t| \leq A$, where A is defined above. As an example we take a polynomial $P(z) = z^3 + 2z + 3$. Here $\mu = 1$ and $n = 3$. We make use of *Wolfram Mathematica* to visualize the zeros of the above polynomial in a specific region (figure 1). The zeros of the polynomial are $(-1.17951, 0); (0.589755, -1.74454); (0.589755, 1.74454)$. Take $t = 1$, clearly $P(z)$ does not vanish in $|z| < 1$. Therefore it follows from Theorem 1 with $p = 2, q = 2$ that all the zeros of $P(z)$ lie in $|z - 1| \leq 8.48528$.



Thus it is clear from figure 1 that all the zeros of the above polynomial lie in the unshaded region between the circles $|z| < 1$ and $|z - 1| \leq 8.48528$.

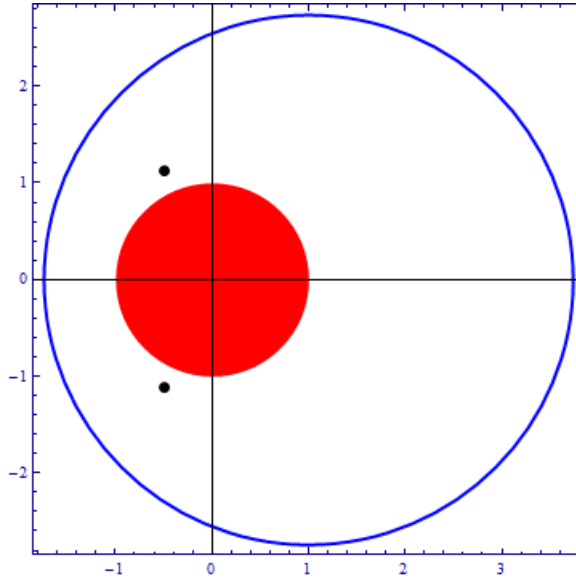
For $\mu = n - 1$, the Lacunary polynomial $P(z)$ in Theorem 1 reduces to a simple polynomial of degree n and yields the following result.

Corollary 1. Let $P(z) := \sum_{j=0}^n a_j z^j$, be a polynomial of degree n which does not vanish in $|z| < t$ where $t > 0$, then for $p > 0$, $q > 0$, $p^{-1} + q^{-1} = 1$, all the zeros of $P(z)$ lie in

$$|z - t| \leq (n + 1)^{\frac{1}{q}} \left\{ \sum_{j=0}^n \left| \frac{ta_j - a_{j-1}}{a_n t^{n-j}} \right|^p \right\}^{\frac{1}{p}}, \tag{2.2}$$

where $a_{-1} = 0$.

As an example, we take $P(z) = 2z^2 + 2z + 3$, having zeros $(-0.5, -1.11803)$; $(-0.5, 1.11803)$. Also take $t = 1$. Clearly these zeros does not lie in $|z| < 1$. Moreover from Corollary 1, it is clear that all the zeros lie in the region between the circles $|z| < 1$ and $|z - 1| < 2.738$ as is clear from figure 2.



For $\mu = 1$, we have the following result for trinomial of degree n .

Corollary 2. Let $P(z) = a_n z^n + a_1 z + a_0$, be a trinomial of degree n which does not vanish in $|z| < t$ where $t > 0$, then for $p > 0$, $q > 0$, $p^{-1} + q^{-1} = 1$, all the zeros of $P(z)$ lie in

$$|z - t| \leq 3^{\frac{1}{q}} \left\{ \sum_{j=0}^2 \left| \frac{ta_j - a_{j-1}}{a_n t^{n-j}} \right|^p \right\}^{\frac{1}{p}}. \tag{2.3}$$

Letting $q \rightarrow \infty$ so that $p = 1$ in Theorem 1, we get the following result.

Corollary 3. *Let $P(z)$ be a polynomial of type (1.2) which does not vanish in $|z| < t$, where $t > 0$, then all the zeros of $P(z)$ lie in the circle*

$$|z - t| \leq \sum_{j=0}^{\mu+1} \left| \frac{ta_j - a_{j-1}}{a_n t^{n-j}} \right|. \quad (2.4)$$

In particular for $t = 1$, we have the following Corollary by restricting the coefficients of the polynomial.

Corollary 4. *Let $P(z)$ be a polynomial of type (1.2) with real coefficients, which does not vanish in $|z| < 1$ and*

$$a_\mu \geq a_{\mu-1} \geq \cdots \geq a_1 \geq a_0 > 0, \quad a_n > 0,$$

then all the zeros of the polynomial lie in the circle

$$|z - 1| \leq 2 \frac{a_\mu}{a_n}. \quad (2.5)$$

In other words all the zeros of the polynomial $P(z)$ which does not vanish in $|z| < 1$ lie in the region

$$\left\{ z : 1 \leq |z| \cap |z - 1| \leq 2 \frac{a_\mu}{a_n} \right\}.$$

If we reverse the monotonicity of the coefficients of the polynomial, we get the following result.

Corollary 5. *Let $P(z)$ be a polynomial of type (1.2) with real coefficients, which does not vanish in $|z| < 1$ and*

$$a_0 \geq a_1 \geq \cdots \geq a_\mu > 0, \quad a_n > 0,$$

then all the zeros of $P(z)$ lie in the circle

$$|z - 1| \leq 2 \frac{a_0}{a_n}. \quad (2.6)$$

In other words if $P(z)$ does not vanish in $|z| < 1$, then all the zeros of $P(z)$ lie in the region

$$\left\{ z : 1 \leq |z| \cap |z - 1| \leq 2 \frac{a_0}{a_n} \right\}.$$

Again we make use of WOLFRAM MATHEMATICA to show that the bounds obtained in our results are sharper than the prior ones. For this we take the following

Lacunary polynomial of type (1.2)

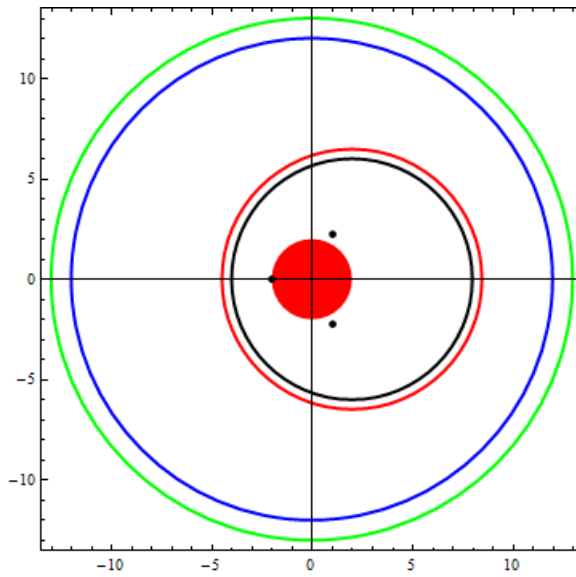
$$P(z) := z^3 + 2z + 12.$$

Its zeros are given below

$(-2), (1 - 2.23607i), (1 + 2.23607i)$. The bounds for the zeros of the above Lacunary polynomial $P(z)$ obtained by using different results are given in the following table.

Bounds for the zeros of $P(z) := 2z^8 + z + 5$ by using different results		
S.No.	Theorems	Bounds
1	Landau's Theorem for Trinomials	$ z \leq 12$
2	Theorem A	$ z \leq 13$
3	Theorem C	$ z \geq 1$
4	Corollary 2	$1 \leq z \cap z - 2 \leq 6.4807$
5	Corollary 3	$1 \leq z \cap z - 2 \leq 6$

In order to visualize the above bounds we draw the following figure in which circles of different colours represents different bounds obtained by various results.



3. Proof of Theorem

Proof of Theorem 1. Consider the polynomial

$$\begin{aligned} F(z) &= (t - z)P(z) = (t - z)(a_n z^n + a_\mu z^\mu + \dots + a_1 z + a_0) \\ &= -a_n z^{n+1} + ta_n z^n - a_\mu z^{\mu+1} + (ta_\mu - a_{\mu-1})z^\mu \\ &\quad + (ta_{\mu-1} - a_{\mu-2})z^{\mu-1} + \dots + (ta_1 - a_0)z + ta_0. \end{aligned}$$

This gives

$$|F(z)| = |-a_n z^{n+1} + ta_n z^n - a_\mu z^{\mu+1} + (ta_\mu - a_{\mu-1})z^\mu + (ta_{\mu-1} - a_{\mu-2})z^{\mu-1} \\ + \dots + (ta_1 - a_0)z + ta_0| \geq |a_n||z^n| \left[|z-t| - \sum_{j=0}^{\mu+1} \left| \frac{ta_j - a_{j-1}}{a_n} \right| \frac{1}{|z|^{n-j}} \right].$$

Since $p > 0$, $q > 0$ and $p^{-1} + q^{-1} = 1$, therefore we have by Hölder's inequality for $|z| \geq t$,

$$|F(z)| \geq |a_n||z^n| \left[|z-t| - (\mu+2)^{\frac{1}{q}} \left\{ \sum_{j=0}^{\mu+1} \left| \frac{ta_j - a_{j-1}}{a_n t^{n-j}} \right|^p \right\}^{\frac{1}{p}} \right] > 0$$

if

$$|z-t| > (\mu+2)^{\frac{1}{q}} \left\{ \sum_{j=0}^{\mu+1} \left| \frac{ta_j - a_{j-1}}{a_n t^{n-j}} \right|^p \right\}^{\frac{1}{p}}.$$

This shows that for $|z| \geq t$, $F(z)$ does not vanish in

$$|z-t| > (\mu+2)^{\frac{1}{q}} \left\{ \sum_{j=0}^{\mu+1} \left| \frac{ta_j - a_{j-1}}{a_n t^{n-j}} \right|^p \right\}^{\frac{1}{p}}.$$

Hence, we conclude that those zeros of $F(z)$ and therefore $P(z)$ whose modulus is greater than t lie in

$$|z-t| \leq (\mu+2)^{\frac{1}{q}} \left\{ \sum_{j=0}^{\mu+1} \left| \frac{ta_j - a_{j-1}}{a_n t^{n-j}} \right|^p \right\}^{\frac{1}{p}}.$$

This completes proof of Theorem 1. □

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DOI: 10.7862/rf.2018.12

Idrees Qasim

email: idreesf3@gmail.com
ORCID: 0000-0001-5838-6522
Department of Mathematics
National Institute of Technology Srinagar - 190006
INDIA

Tawheeda Rasool

email: tawheedrasool@gmail.com
Department of Mathematics
National Institute of Technology Srinagar - 190006
INDIA

Abdul Liman

email: abliman@rediffmail.com

Department of Mathematics

National Institute of Technology Srinagar - 190006

INDIA

Received 10.11.2017

Accepted 31.01.2018

Number of Zeros of a Polynomial (Lacunary-type) in a Disk

Idrees Qasim, Tawheeda Rasool and Abdul Liman

ABSTRACT: The problem of finding out the region which contains all or a prescribed number of zeros of a polynomial $P(z) := \sum_{j=0}^n a_j z^j$ has a long history and dates back to the earliest days when the geometrical representation of complex numbers was introduced. In this paper, we present certain results concerning the location of the zeros of Lacunary-type polynomials $P(z) := a_0 + \sum_{j=\mu}^n a_j z^j$ in a disc centered at the origin.

AMS Subject Classification: 30A01, 30C10, 30C15.

Keywords and Phrases: Zeros; Lacunary polynomial; Prescribed region.

1. Introduction and Statement of Results

The problem of locating some or all the zeros of a given polynomial as a function of its coefficients is of long standing interest in mathematics. This fact can be visualized by glancing at the references in the comprehensive books of Marden [9] and Milovanovic, Mitrinovic and Rassias [10], Rahman and Schmeisser [12] and by noting the abundance of recent publications on the subject [7, 8, 13].

Regarding the least number of zeros of polynomial $P(z) = \sum_{j=0}^n a_j z^j$ in a given circle Mohammad [11] proved the following:

Theorem A. Let $P(z) := \sum_{j=0}^n a_j z^j$ be a polynomial of degree n such that

$$a_n \geq a_{n-1} \geq a_{n-2} \geq \cdots \geq a_1 \geq a_0 > 0,$$

then the number of zeros of $P(z)$ in $|z| \leq \frac{1}{2}$ does not exceed

$$1 + \frac{1}{\log 2} \log \frac{a_n}{a_0}.$$

Dewan [3] generalized Theorem A to polynomials with complex coefficients and proved the following result:

Theorem B. Let $P(z) := \sum_{j=0}^n a_j z^j$ be a polynomial of degree n with complex coefficients. If $\operatorname{Re} a_j = \alpha_j$ and $\operatorname{Im} a_j = \beta_j$, $j = 0, 1, 2, \dots, n$ such that

$$\alpha_n \geq \alpha_{n-1} \geq \alpha_{n-2} \geq \dots \geq \alpha_1 \geq \alpha_0 > 0,$$

then the number of zeros of $P(z)$ in $|z| \leq \frac{1}{2}$ does not exceed

$$1 + \frac{1}{\log 2} \log \frac{\alpha_n + \sum_{j=0}^n |\beta_j|}{|a_0|}.$$

In this direction, recently Irshad et al [1] proved the following:

Theorem C. Let $P(z) := \sum_{j=0}^n a_j z^j$ be a polynomial of degree n with complex coefficients such that for some $\lambda \geq 1$, $0 \leq k \leq n$,

$$|a_n| \leq |a_{n-1}| \leq \dots \leq |a_{k+1}| \leq \lambda |a_k| \geq |a_{k-1}| \geq \dots \geq |a_1| \geq |a_0|,$$

and for some real β ,

$$|\arg a_j - \beta| \leq \alpha \leq \frac{\pi}{2}, \quad j = 0, 1, 2, \dots, n$$

then the number of zeros of $P(z)$ in $|z| \leq \frac{1}{2}$ does not exceed

$$\left. \begin{aligned} & \frac{1}{\log 2} \log \left\{ \frac{2\lambda |a_k| \cos \alpha + 2|\lambda - 1| |a_k| \sin \alpha}{|a_0|} \right. \\ & \left. + \frac{|a_n| (\sin \alpha - \cos \alpha + 1) + 2 \sin \alpha \sum_{j=0}^{n-1} |a_j| + 2|1 - \lambda| |a_k|}{|a_0|} \right\}. \end{aligned} \right\}$$

Chan and Malik [2] introduced the class of Lacunary polynomials of the form $P(z) := a_0 + \sum_{j=\mu}^n a_j z^j$, where $a_0 \neq 0$. Notice that when $\mu = 1$, we simply have the class of all polynomials of degree n . In [5] and [6] Landau proved that every trinomial

$$a_0 + a_1 z + a_n z^n, a_1 a_n \neq 0, n \geq 2$$

has at least one zero in the circle $|z| \leq 2 \frac{|a_0|}{|a_1|}$ and that of quadrinomial

$$a_0 + a_1 z + a_m z^m + a_n z^n, a_1 a_m a_n \neq 0, 2 \leq m \leq n$$

has at least one zero in the circle $|z| \leq \frac{17}{3} \left| \frac{a_0}{a_1} \right|$. These two polynomials are of the Lacunary-type $P(z) := a_0 + \sum_{j=\mu}^n a_j z^j$.

The aim of this paper is to study the number of zeros in a disc centered at the origin for such class of polynomials. We begin by proving the following result putting restrictions on the moduli of the coefficients. In fact we prove:

Theorem 1. *Let $P(z) := a_0 + \sum_{j=\mu}^n a_j z^j$, $1 \leq \mu \leq n - 1$, $a_0 \neq 0$, be a polynomial of degree n . If for some real α and β*

$$|\arg a_j - \beta| \leq \alpha \leq \frac{\pi}{2}, \quad \mu \leq j \leq n$$

and for some $t > 0$ and some k with $\mu \leq k \leq n$,

$$t^\mu |a_\mu| \leq \dots \leq t^{k-1} |a_{k-1}| \leq t^k |a_k| \geq t^{k+1} |a_{k+1}| \geq \dots \geq t^{n-1} |a_{n-1}| \geq t^n |a_n|$$

then the number of zeros of $P(z)$ in $|z| \leq \frac{t}{2}$ does not exceed

$$\frac{1}{\log 2} \log \frac{M}{|a_0|},$$

where

$$M = 2|a_0|t + |a_\mu|t^{\mu+1}(1 - \sin \alpha - \cos \alpha) + 2|a_k|t^{k+1} \cos \alpha + |a_n|t^{n+1}(1 - \sin \alpha - \cos \alpha) + 2 \sum_{j=\mu}^n |a_j|t^{j+1} \sin \alpha.$$

For $t = 1$, we get the following:

Corollary 1.1. *Let $P(z) := a_0 + \sum_{j=\mu}^n a_j z^j$, $1 \leq \mu \leq n - 1$, $a_0 \neq 0$ be a polynomial of degree n . If for some real α and β*

$$|\arg a_j - \beta| \leq \alpha \leq \frac{\pi}{2}, \quad \mu \leq j \leq n$$

and some k with

$$|a_\mu| \leq \dots \leq |a_{k-1}| \leq |a_k| \geq |a_{k+1}| \geq \dots \geq |a_{n-1}| \geq |a_n|$$

then the number of zeros of $P(z)$ in $|z| \leq \frac{1}{2}$ does not exceed

$$\frac{1}{\log 2} \log \frac{M}{|a_0|},$$

where

$$M = 2|a_0| + |a_\mu|(1 - \sin \alpha - \cos \alpha) + 2|a_k| \cos \alpha + |a_n|(1 - \sin \alpha - \cos \alpha) + 2 \sum_{j=\mu}^n |a_j| \sin \alpha.$$

With $k = n$ in Corollary 1.1, we get:

Corollary 1.2. Let $P(z) := a_0 + \sum_{j=\mu}^n a_j z^j$, $1 \leq \mu \leq n-1$, $a_0 \neq 0$ be a polynomial of degree n . If for some real α and β

$$|\arg a_j - \beta| \leq \alpha \leq \frac{\pi}{2}, \quad \mu \leq j \leq n$$

such that

$$|a_\mu| \leq \cdots \leq |a_{n-1}| \leq |a_n|$$

then the number of zeros of $P(z)$ in $|z| \leq \frac{1}{2}$ does not exceed

$$\frac{1}{\log 2} \log \frac{M}{|a_0|},$$

where

$$M = 2|a_0| + |a_\mu|(1 - \sin \alpha - \cos \alpha) + |a_n|(1 - \sin \alpha + \cos \alpha) + 2 \sum_{j=\mu}^n |a_j| \sin \alpha.$$

Choosing $k = \mu$ in Corollary 1.1, we get:

Corollary 1.3. Let $P(z) := a_0 + \sum_{j=\mu}^n a_j z^j$, $1 \leq \mu \leq n-1$, $a_0 \neq 0$ be a polynomial of degree n . If for some real α and β

$$|\arg a_j - \beta| \leq \alpha \leq \frac{\pi}{2}, \quad \mu \leq j \leq n$$

such that

$$|a_\mu| \geq \cdots \geq |a_{n-1}| \geq |a_n|$$

then the number of zeros of $P(z)$ in $|z| \leq \frac{1}{2}$ does not exceed

$$\frac{1}{\log 2} \log \frac{M}{|a_0|},$$

where

$$M = 2|a_0| + |a_\mu|(1 - \sin \alpha + \cos \alpha) + |a_n|(1 - \sin \alpha - \cos \alpha) + 2 \sum_{j=\mu}^n |a_j| \sin \alpha.$$

Taking $\mu = 1$ in Corollary 1.3, we have

Corollary 1.4. Let $P(z) := \sum_{j=0}^n a_j z^j$, $a_0 \neq 0$ be a polynomial of degree n . If for some real α and β

$$|\arg a_j - \beta| \leq \alpha \leq \frac{\pi}{2}, \quad \mu \leq j \leq n$$

such that

$$|a_1| \geq \dots \geq |a_{n-1}| \geq |a_n|$$

then the number of zeros of $P(z)$ in $|z| \leq \frac{1}{2}$ does not exceed

$$\frac{1}{\log 2} \log \frac{M}{|a_0|},$$

where

$$M = 2|a_0| + |a_1|(1 - \sin \alpha + \cos \alpha) + |a_n|(1 - \sin \alpha - \cos \alpha) + 2 \sum_{j=1}^n |a_j| \sin \alpha.$$

Next, we put restriction on the real part of coefficients of a polynomial and proved:

Theorem 2. Let $P(z) := a_0 + \sum_{j=\mu}^n a_j z^j$, $1 \leq \mu \leq n - 1$, $a_0 \neq 0$, be a polynomial of degree n with $Re a_j = \alpha_j$ and $Im a_j = \beta_j$ for $\mu \leq j \leq n$. Suppose that for some $t > 0$ and some k with $\mu \leq k \leq n$ we have

$$t^\mu \alpha_\mu \leq \dots \leq t^{k-1} \alpha_{k-1} \leq t^k \alpha_k \geq t^{k+1} \alpha_{k+1} \geq \dots \geq t^{n-1} \alpha_{n-1} \geq t^n \alpha_n$$

then the number of zeros of $P(z)$ in $|z| \leq \frac{t}{2}$ does not exceed

$$\frac{1}{\log 2} \log \frac{M}{|a_0|},$$

where

$$M = 2(|\alpha_0| + |\beta_0|)t + (|\alpha_\mu| - \alpha_\mu)t^{\mu+1} + 2\alpha_k t^{k+1} + (|\alpha_n| - \alpha_n)t^{n+1} + 2 \sum_{j=\mu}^n |\beta_j| t^{j+1}.$$

For $t = 1$ in Theorem 2, we obtain

Corollary 2.1. Let $P(z) := a_0 + \sum_{j=\mu}^n a_j z^j$, $1 \leq \mu \leq n - 1$, $a_0 \neq 0$, be a polynomial of degree n with $Re a_j = \alpha_j$ and $Im a_j = \beta_j$ for $\mu \leq j \leq n$. Suppose that for some k with $\mu \leq k \leq n$ we have

$$\alpha_\mu \leq \dots \leq \alpha_{k-1} \leq \alpha_k \geq \alpha_{k+1} \geq \dots \geq \alpha_{n-1} \geq \alpha_n$$

then the number of zeros of $P(z)$ in $|z| \leq \frac{1}{2}$ does not exceed

$$\frac{1}{\log 2} \log \frac{M}{|a_0|},$$

where

$$M = 2(|\alpha_0| + |\beta_0|) + (|\alpha_\mu| - \alpha_\mu) + 2\alpha_k + (|\alpha_n| - \alpha_n) + 2 \sum_{j=\mu}^n |\beta_j|.$$

For $k = n$ in Corollary 2.1, we get:

Corollary 2.2. Let $P(z) := a_0 + \sum_{j=\mu}^n a_j z^j$, $1 \leq \mu \leq n-1$, $a_0 \neq 0$, be a polynomial of degree n with $\operatorname{Re} a_j = \alpha_j$ and $\operatorname{Im} a_j = \beta_j$ for $\mu \leq j \leq n$ such that

$$\alpha_\mu \leq \cdots \leq \alpha_{n-1} \leq \alpha_n$$

then the number of zeros of $P(z)$ in $|z| \leq \frac{1}{2}$ does not exceed

$$\frac{1}{\log 2} \log \frac{M}{|a_0|},$$

where

$$M = 2(|\alpha_0| + |\beta_0|) + (|\alpha_\mu| - \alpha_\mu) + 2 \sum_{j=\mu}^n |\beta_j|.$$

For $k = \mu$, in Corollary 2.1, we get:

Corollary 2.3. Let $P(z) := a_0 + \sum_{j=\mu}^n a_j z^j$, $1 \leq \mu \leq n-1$, $a_0 \neq 0$, be a polynomial of degree n with $\operatorname{Re} a_j = \alpha_j$ and $\operatorname{Im} a_j = \beta_j$ for $\mu \leq j \leq n$ such that

$$\alpha_\mu \geq \cdots \geq \alpha_{n-1} \geq \alpha_n$$

then the number of zeros of $P(z)$ in $|z| \leq \frac{1}{2}$ does not exceed

$$\frac{1}{\log 2} \log \frac{M}{|a_0|},$$

where

$$M = 2(|\alpha_0| + |\beta_0|) + (|\alpha_\mu| + \alpha_\mu) + (|\alpha_\mu| - \alpha_\mu) + 2 \sum_{j=\mu}^n |\beta_j|.$$

For $\beta_j = 0$, $1 \leq j \leq n$ in Theorem 2, we have the following:

Corollary 2.4. Let $P(z) := a_0 + \sum_{j=\mu}^n a_j z^j$, $\mu \leq n-1$, where $a_0 \neq 0$. Suppose that for some $t > 0$ and some k we have

$$t^\mu a_\mu \leq \cdots \leq t^{k-1} a_{k-1} \leq t^k a_k \geq t^{k+1} a_{k+1} \geq \cdots \geq t^{n-1} a_{n-1} \geq t^n a_n$$

then the number of zeros of $P(z)$ in $|z| \leq \frac{t}{2}$ does not exceed

$$\frac{1}{\log 2} \log \frac{M}{|a_0|},$$

where

$$M = 2|a_0|t + (|a_\mu| - a_\mu)t^{\mu+1} + 2a_k t^{k+1} + (|a_n| - a_n)t^{n+1}.$$

Finally, we prove the following result:

Theorem 3. Let $P(z) := a_0 + \sum_{j=\mu}^n a_j z^j$, $1 \leq \mu \leq n-1$, where $a_0 \neq 0$, $Re a_j = \alpha_j$ and $Im a_j = \beta_j$ for $\mu \leq j \leq n$. Suppose that for some $t > 0$ and some k with $\mu \leq k \leq n$ we have

$$t^\mu \alpha_\mu \leq \dots \leq t^{k-1} \alpha_{k-1} \leq t^k \alpha_k \geq t^{k+1} \alpha_{k+1} \geq \dots \geq t^{n-1} \alpha_{n-1} \geq t^n \alpha_n$$

and for some $\mu \leq l \leq n$ we have

$$t^\mu \beta_\mu \leq \dots \leq t^{l-1} \beta_{l-1} \leq t^l \beta_l \geq t^{l+1} \beta_{l+1} \geq \dots \geq t^{n-1} \beta_{n-1} \geq t^n \beta_n$$

then the number of zeros of $P(z)$ in $|z| \leq \frac{t}{2}$ does not exceed

$$\frac{1}{\log 2} \log \frac{M}{|a_0|},$$

where

$$M = 2(|\alpha_0| + |\beta_0|)t + (|\alpha_\mu| - \alpha_\mu + |\beta_\mu| - \beta_\mu)t^{\mu+1} + 2(\alpha_k t^{k+1} + \beta_l t^{l+1})t^{n+1} + (|\alpha_n| - \alpha_n + |\beta_n| - \beta_n)t^{n+1}.$$

If we take $t = 1$, in Theorem 3 we obtain:

Corollary 3.1. Let $P(z) = a_0 + \sum_{j=\mu}^n a_j z^j$, $1 \leq \mu \leq n-1$, where $a_0 \neq 0$, $Re a_j = \alpha_j$ and $Im a_j = \beta_j$ for $\mu \leq j \leq n$. Suppose that for some k with $\mu \leq k \leq n$ we have

$$\alpha_\mu \leq \dots \leq \alpha_{k-1} \leq \alpha_k \geq \alpha_{k+1} \geq \dots \geq \alpha_{n-1} \geq \alpha_n$$

and for some $\mu \leq l \leq n$ we have

$$\beta_\mu \leq \dots \leq \beta_{l-1} \leq \beta_l \geq \beta_{l+1} \geq \dots \geq \beta_{n-1} \geq \beta_n.$$

Then the number of zeros of $P(z)$ in $|z| \leq \frac{1}{2}$ does not exceed

$$\frac{1}{\log 2} \log \frac{M}{|a_0|},$$

where

$$M = 2(|\alpha_0| + |\beta_0|) + (|\alpha_\mu| - \alpha_\mu + |\beta_\mu| - \beta_\mu) + 2(\alpha_k + \beta_l) + (|\alpha_n| - \alpha_n + |\beta_n| - \beta_n).$$

For $k = l = n$ in Corollary 3.1, we get the following:

Corollary 3.2. Let $P(z) = a_0 + \sum_{j=\mu}^n a_j z^j$, $1 \leq \mu \leq n-1$, where $a_0 \neq 0$, $Re a_j = \alpha_j$ and $Im a_j = \beta_j$ for $\mu \leq j \leq n$ such that

$$\alpha_\mu \leq \dots \leq \alpha_{n-1} \leq \alpha_n$$

and

$$\beta_\mu \leq \cdots \leq \beta_{n-1} \leq \beta_n$$

then the number of zeros of $P(z)$ in $|z| \leq \frac{1}{2}$ does not exceed

$$\frac{1}{\log 2} \log \frac{M}{|a_0|},$$

where

$$M = 2(|\alpha_0| + |\beta_0|) + (|\alpha_\mu| - \alpha_\mu + |\beta_\mu| - \beta_\mu) + (|\alpha_n| + \alpha_n + |\beta_n| + \beta_n).$$

In Corollary 3.1, if we choose $k = l = \mu$ we get:

Corollary 3.3. Let $P(z) = a_0 + \sum_{j=\mu}^n a_j z^j$, $1 \leq \mu \leq n-1$, where $a_0 \neq 0$, $\operatorname{Re} a_j = \alpha_j$ and $\operatorname{Im} a_j = \beta_j$ for $\mu \leq j \leq n$ such that

$$\alpha_\mu \geq \cdots \geq \alpha_{n-1} \leq \alpha_n$$

and

$$\beta_\mu \geq \cdots \geq \beta_{n-1} \geq \beta_n$$

then the number of zeros of $P(z)$ in $|z| \leq \frac{1}{2}$ does not exceed

$$\frac{1}{\log 2} \log \frac{M}{|a_0|},$$

where

$$M = 2(|\alpha_0| + |\beta_0|) + (|\alpha_\mu| + \alpha_\mu + |\beta_\mu| + \beta_\mu) + (|\alpha_n| - \alpha_n + |\beta_n| - \beta_n).$$

2. Lemma

For the proof of some these results we need the following lemma which is due to Govil and Rahman [4].

Lemma 2.1. For any two complex numbers b_0 and b_1 such that $|b_0| \geq |b_1|$ and

$$|\arg b_j - \beta| \leq \alpha \leq \frac{\pi}{2}, \quad j = 0, 1$$

for some β , then

$$|b_0 - b_1| \leq (|b_0| - |b_1|) \cos \alpha + (|b_0| + |b_1|) \sin \alpha.$$

An application of the Maximum modulus theorem shown in (p.171, [14]) we have the following interesting result:

Lemma 2.2. *Let $f(z)$ be regular and $|f(z)| \leq M$, in the circle $|z| \leq R$ and suppose that $f(0) \neq 0$, then the number of zeros of $f(z)$ in the circle $|z| \leq \frac{1}{2}R$ does not exceed $\frac{1}{\log 2} \log \left[\frac{M}{|f(0)|} \right]$.*

3. Proofs of Theorems

Proof of Theorem 1. Consider the polynomial

$$\begin{aligned} F(z) &= (t - z)P(z) \\ &= (t - z) \left(a_0 + \sum_{j=\mu}^n a_j z^j \right) \\ &= a_0 t + \sum_{j=\mu}^n a_j t z^j - a_0 z - \sum_{j=\mu}^n a_j z^{j+1} \\ &= a_0(t - z) + \sum_{j=\mu}^n a_j t z^j - \sum_{j=\mu+1}^{n+1} a_{j-1} z^j \\ &= a_0(t - z) + a_\mu t z^\mu + \sum_{j=\mu+1}^n (a_j t - a_{j-1}) z^j - a_n z^{n+1}. \end{aligned}$$

For $|z| = t$, we have

$$\begin{aligned} |F(z)| &\leq 2|a_0|t + |a_\mu|t^{\mu+1} + \sum_{j=\mu+1}^n |a_j t - a_{j-1}|t^j + |a_n|t^{n+1} \\ &= 2|a_0|t + |a_\mu|t^{\mu+1} + \sum_{j=\mu+1}^k |a_j t - a_{j-1}|t^j + \sum_{j=k+1}^n |a_{j-1} - a_j t|t^j + |a_n|t^{n+1}. \end{aligned}$$

Using Lemma 2.1 with $b_0 = a_j t$ and $b_1 = a_{j-1}$ when $1 \leq j \leq k$ and with $b_0 = a_{j-1}$ and $b_1 = a_j t$ when $k + 1 \leq j \leq n$,

$$\begin{aligned} |F(z)| &\leq 2|a_0|t + |a_\mu|t^{\mu+1} + \sum_{j=\mu+1}^k \{(|a_j|t - |a_{j-1}|) \cos \alpha + (|a_j|t + |a_{j-1}|) \sin \alpha\} t^j \\ &\quad + \sum_{j=k+1}^n \{(|a_{j-1}| - |a_j|t) \cos \alpha + (|a_j|t + |a_{j-1}|) \sin \alpha\} t^j + |a_n|t^{n+1} \end{aligned}$$

$$\begin{aligned}
&= 2|a_0|t + |a_\mu|t^{\mu+1} + \sum_{j=\mu+1}^k |a_j|t^{j+1} \cos \alpha - \sum_{j=\mu+1}^k |a_{j-1}|t^j \cos \alpha + \sum_{j=\mu+1}^k |a_j|t^{j+1} \sin \alpha \\
&\quad + \sum_{j=\mu+1}^k |a_{j-1}|t^j \sin \alpha + \sum_{j=k+1}^n |a_{j-1}|t^j \cos \alpha - \sum_{j=k+1}^n |a_j|t^{j+1} \cos \alpha \\
&\quad + \sum_{j=k+1}^n |a_{j-1}|t^j \sin \alpha + \sum_{j=k+1}^n |a_j|t^{j+1} \sin \alpha + |a_n|t^{n+1} \\
&= 2|a_0|t + |a_\mu|t^{\mu+1} - |a_\mu|t^{\mu+1} \cos \alpha + |a_k|t^{k+1} \cos \alpha + |a_\mu|t^{\mu+1} \sin \alpha + |a_k|t^{k+1} \sin \alpha \\
&\quad + 2 \sum_{j=\mu+1}^{k-1} |a_j|t^{j+1} \sin \alpha + |a_k|t^{k+1} \cos \alpha - |a_n|t^{n+1} \cos \alpha + |a_k|t^{k+1} \sin \alpha \\
&\quad + |a_n|t^{n+1} \sin \alpha + 2 \sum_{j=k+1}^{n-1} |a_j|t^{j+1} \sin \alpha + |a_n|t^{n+1} \\
&= 2|a_0|t + |a_\mu|t^{\mu+1} + |a_\mu|t^{\mu+1}(\sin \alpha - \cos \alpha) + 2 \sum_{j=\mu+1}^{n-1} |a_j|t^{j+1} \sin \alpha + 2|a_k|t^{k+1} \cos \alpha \\
&\quad + (\sin \alpha - \cos \alpha + 1)|a_n|t^{n+1} \\
&= 2|a_0|t + |a_\mu|t^{\mu+1}(1 - \sin \alpha - \cos \alpha) + 2|a_k|t^{k+1} \cos \alpha + |a_n|t^{n+1}(1 - \sin \alpha - \cos \alpha) \\
&\quad + 2 \sum_{j=\mu}^n |a_j|t^{j+1} \sin \alpha \\
&= M(\text{say}).
\end{aligned}$$

Now $F(z)$ is analytic in $|z| \leq t$ and $F(z) \leq M$ for $|z| = t$. Applying Lemma 2.2 to the polynomial $F(z)$, we get the number of zeros of $F(z)$ in $|z| \leq \frac{t}{2}$ does not exceed

$$\frac{1}{\log 2} \log \frac{M}{|f(0)|}.$$

Thus, the number of zeros of $F(z)$ in $|z| \leq \frac{t}{2}$ does not exceed

$$\frac{1}{\log 2} \log \left\{ \frac{2|a_0|t + |a_\mu|t^{\mu+1}(1 - \sin \alpha - \cos \alpha) + 2|a_k|t^{k+1} \cos \alpha}{|a_0|} + \frac{|a_n|t^{n+1}(1 - \sin \alpha - \cos \alpha) + 2 \sum_{j=\mu}^n |a_j|t^{j+1} \sin \alpha}{|a_0|} \right\}.$$

As the number of zeros of $P(z)$ in $|z| \leq \frac{1}{2}$ is also equal to the number of zeros $F(z)$ the theorem follows. \square

Proof of Theorem 2. Consider the polynomial

$$\begin{aligned}
 F(z) &= (t - z)P(z) \\
 &= (t - z) \left(a_0 + \sum_{j=\mu}^n a_j z^j \right) \\
 &= a_0 t + \sum_{j=\mu}^n a_j t z^j - a_0 z - \sum_{j=\mu}^n a_j z^{j+1} \\
 &= a_0(t - z) + \sum_{j=\mu}^n a_j t z^j - \sum_{j=\mu+1}^{n+1} a_{j-1} z^j
 \end{aligned}$$

and therefore

$$\begin{aligned}
 F(z) &= (\alpha_0 + i\beta_0)(t - z) + (\alpha_\mu + i\beta_\mu)t z^\mu + \sum_{j=\mu+1}^n (\alpha_j t - \alpha_{j-1})z^j \\
 &\quad + i \sum_{j=\mu+1}^n (\beta_j t - \beta_{j-1})z^j - (\alpha_n + i\beta_n)z^{n+1}.
 \end{aligned}$$

For $|z| = t$, we have

$$\begin{aligned}
 |F(z)| &\leq 2(|\alpha_0| + |\beta_0|)t + (|\alpha_\mu| + |\beta_\mu|)t^{\mu+1} + \sum_{j=\mu+1}^n |\alpha_j t - \alpha_{j-1}|t^j \\
 &\quad + \sum_{j=\mu+1}^n (|\beta_j|t + |\beta_{j-1}|)t^j + (|\alpha_n| + |\beta_n|)t^{n+1} \\
 &= 2(|\alpha_0| + |\beta_0|)t + (|\alpha_\mu| + |\beta_\mu|)t^{\mu+1} + \sum_{j=\mu+1}^k (\alpha_j t - \alpha_{j-1})t^j \\
 &\quad + \sum_{j=k+1}^n (\alpha_{j-1} - \alpha_j t)t^j + |\beta_\mu|t^{\mu+1} + 2 \sum_{j=\mu+1}^{n-1} |\beta_j|t^{j+1} + |\beta_n|t^{n+1} \\
 + (|\alpha_n| + |\beta_n|)t^{n+1} &= 2(|\alpha_0| + |\beta_0|)t + (|\alpha_\mu| + |\beta_\mu|)t^{\mu+1} - \alpha_\mu t^{\mu+1} + 2\alpha_k t^{k+1} \\
 - \alpha_n t^{n+1} + |\beta_\mu|t^{\mu+1} &+ 2 \sum_{j=\mu+1}^n |\beta_j|t^{j+1} + |\alpha_n|t^{n+1} = 2(|\alpha_0| + |\beta_0|)t \\
 + (|\alpha_\mu| - \alpha_\mu)t^{\mu+1} + 2\alpha_k t^{k+1} &+ (|\alpha_n| - \alpha_n)t^{n+1} + 2 \sum_{j=\mu}^n |\beta_j|t^{j+1} = M.
 \end{aligned}$$

Proceedings on the same lines of the proof of Theorem 1, the proof of this result follows.

Proof of Theorem 3. Consider the polynomial

$$F(z) = (t - z)P(z) = a_0(t - z) + a_\mu tz^\mu + \sum_{j=\mu+1}^n (a_j t - a_{j-1})z^j - a_n z^{n+1},$$

and so

$$\begin{aligned} F(z) &= (\alpha_0 + i\beta_0)(t - z) + (\alpha_\mu + i\beta_\mu)tz^\mu \\ &+ \sum_{j=\mu+1}^n ((\alpha_j + i\beta_j)t - (\alpha_{j-1} + i\beta_{j-1}))z^j - (\alpha_n + i\beta_n)z^{n+1} \\ &= (\alpha_0 + i\beta_0)(t - z) + (\alpha_\mu + i\beta_\mu)tz^\mu + \sum_{j=\mu+1}^n (\alpha_j t - \alpha_{j-1})z^j \\ &+ i \sum_{j=\mu+1}^n (\beta_j t - \beta_{j-1})z^j - (\alpha_n + i\beta_n)z^{n+1}. \end{aligned}$$

For $|z| = t$, we have

$$\begin{aligned} |F(z)| &\leq 2(|\alpha_0| + |\beta_0|)t + (|\alpha_\mu| + |\beta_\mu|)t^{\mu+1} + \sum_{j=\mu+1}^n |\alpha_j t - \alpha_{j-1}|t^j \\ &+ \sum_{j=\mu+1}^n |\beta_j t - \beta_{j-1}|t^j + (|\alpha_n| + |\beta_n|)t^{n+1} = 2(|\alpha_0| + |\beta_0|)t \\ &+ (|\alpha_\mu| + |\beta_\mu|)t^{\mu+1} + \sum_{j=\mu+1}^k (\alpha_j t - \alpha_{j-1})t^j + \sum_{j=k+1}^n (\alpha_{j-1} - \alpha_j t)t^j \\ &+ \sum_{j=\mu+1}^l (\beta_j t - \beta_{j-1})t^j + \sum_{j=l+1}^n (\beta_{j-1} - \beta_j t)t^j + (|\alpha_n| + |\beta_n|)t^{n+1} \\ &= 2(|\alpha_0| + |\beta_0|)t + (|\alpha_\mu| + |\beta_\mu|)t^{\mu+1} - \alpha_\mu t^{\mu+1} + 2\alpha_k t^{k+1} \\ &- \alpha_n t^{n+1} - \beta_\mu t^{\mu+1} + 2\beta_l t^{l+1} - \beta_n t^{n+1} + (|\alpha_n| + |\beta_n|)t^{n+1} \\ &= 2(|\alpha_0| + |\beta_0|)t + (|\alpha_\mu| - \alpha_\mu + |\beta_\mu| - \beta_\mu)t^{\mu+1} \\ &+ 2(\alpha_k t^{k+1} + \beta_l t^{l+1}) + (|\alpha_n| - \alpha_n + |\beta_n| - \beta_n)t^{n+1} = M. \end{aligned}$$

The result now follows as in the proof of Theorem 1. \square

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DOI: 10.7862/rf.2018.13

Idrees Qasim

email: idreesf3@gmail.com
ORCID: 0000-0001-5838-6522
Department of Mathematics
National Institute of Technology
Srinagar - 190006
INDIA

Tawheeda Rasool

email: tawheedrasool@gmail.com
Department of Mathematics
National Institute of Technology
Srinagar - 190006
INDIA

Abdul Liman

email: abliman22@rediffmail.com
Department of Mathematics
National Institute of Technology
Srinagar - 190006
INDIA

Received 12.08.2017

Accepted 14.08.2018

An Upper Bound for Third Hankel Determinant of Starlike Functions Related to Shell-like Curves Connected with Fibonacci Numbers

Janusz Sokół, Sedat İlhan and H. Özlem Güney

ABSTRACT: We investigate the third Hankel determinant problem for some starlike functions in the open unit disc, that are related to shell-like curves and connected with Fibonacci numbers. For this, firstly, we prove a conjecture, posed in [17], for sharp upper bound of second Hankel determinant. In the sequel, we obtain another sharp coefficient bound which we apply in solving the problem of the third Hankel determinant for these functions.

AMS Subject Classification: 30C45, 30C50.

Keywords and Phrases: Analytic functions; Convex function; Fibonacci numbers; Hankel determinant; Shell-like curve; Starlike function.

1. Introduction

Let \mathcal{A} denote the class of functions f which are *analytic* in the open unit disk $\mathbb{U} = \{z : z \in \mathbb{C} \text{ and } |z| < 1\}$ and let \mathcal{S} denote the class of functions in \mathcal{A} which are univalent in \mathbb{U} and normalized by the conditions $f(0) = f'(0) - 1 = 0$ and are of the form:

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n. \quad (1.1)$$

We say that f is subordinate to F in \mathbb{U} , written as $f \prec F$, if and only if $f(z) = F(w(z))$ for some analytic function w such that $|w(z)| \leq |z|$ for all $z \in \mathbb{U}$.

If $f \in \mathcal{A}$ and

$$\frac{zf'(z)}{f(z)} \prec p(z) \quad \text{or} \quad 1 + \frac{zf''(z)}{f'(z)} \prec p(z)$$

where $p(z) = \frac{1+z}{1-z}$, then we say that f is starlike or convex respectively. These functions form known classes denoted by \mathcal{S}^* or \mathcal{C} respectively. These classes are very important subclasses of the class \mathcal{S} in geometric function theory. In this paper we consider the following subclass of starlike functions.

Definition 1. The function $f \in \mathcal{A}$ belongs to the class \mathcal{SL} if it satisfies the condition that

$$\frac{zf'(z)}{f(z)} \prec \tilde{p}(z)$$

with

$$\tilde{p}(z) = \frac{1 + \tau^2 z^2}{1 - \tau z - \tau^2 z^2},$$

where $\tau = (1 - \sqrt{5})/2 \approx -0.618$. The class \mathcal{SL} was introduced in [16].

The function \tilde{p} is not univalent in \mathbb{U} , but it is univalent in the disc $|z| < \tau^2 \approx 0.38$. For example, $\tilde{p}(0) = \tilde{p}(-1/2\tau) = 1$ and $\tilde{p}(e^{\mp i \arccos(1/4)}) = \sqrt{5}/5$, and it may also be noticed that

$$\frac{1}{|\tau|} = \frac{|\tau|}{1 - |\tau|},$$

which shows that the number $|\tau|$ divides $[0, 1]$ such that it fulfils the golden section. The image of the unit circle $|z| = 1$ under \tilde{p} is a curve described by the equation given by

$$(10x - \sqrt{5})y^2 = (\sqrt{5} - 2x)(\sqrt{5}x - 1)^2,$$

which is translated and revolved trisectrix of Maclaurin. The curve $\tilde{p}(re^{it})$ is a closed curve without any loops for $0 < r \leq r_0 = (3 - \sqrt{5})/2 \approx 0.38$. For $r_0 < r < 1$, it has a loop, and for $r = 1$, it has a vertical asymptote. Since τ satisfies the equation $\tau^2 = 1 + \tau$, this expression can be used to obtain higher powers τ^n as a linear function of lower powers, which in turn can be decomposed all the way down to a linear combination of τ and 1. The resulting recurrence relationships yield Fibonacci numbers u_n :

$$\tau^n = u_n \tau + u_{n-1}. \tag{1.2}$$

In 1976, Noonan and Thomas [10] stated the s^{th} Hankel determinant for $s \geq 1$ and $k \geq 1$ as

$$H_s(k) = \begin{vmatrix} a_k & a_{k+1} & \cdots & a_{k+s-1} \\ a_{k+1} & a_{k+2} & \cdots & \vdots \\ \vdots & \vdots & \vdots & \vdots \\ a_{k+s-1} & \cdots & \cdots & a_{k+2(s-1)} \end{vmatrix} \tag{1.3}$$

where $a_1 = 1$.

This determinant has also been considered by several authors. For example, Noor [11] determined the rate of growth of $H_s(k)$ as $k \rightarrow \infty$ for functions f given by (1) with bounded boundary. Ehrenborg in [3] studied the Hankel determinant of exponential polynomials. The Hankel transform of an integer sequence and some of its properties were discussed by Layman in [8]. Also, several authors considered the case $s = 2$. Especially, $H_2(1) = a_3 - a_2^2$ is known as Fekete-Szegő functional and this functional is generalized to $a_3 - \mu a_2^2$ where μ is some real number [4]. Estimating for an upper bound of $|a_3 - \mu a_2^2|$ is known as the Fekete-Szegő problem. In [13], Raina and Sokół considered Fekete-Szegő problem for the class \mathcal{SL} . In 1969, Keogh and Merkes [7] solved this problem for the classes \mathcal{S}^* and \mathcal{C} . The second Hankel determinant is $H_2(2) = a_2 a_4 - a_3^2$. Janteng [5] found the sharp upper bound for $|H_2(2)|$ for univalent functions whose derivative has positive real part. Also, in [6] Janteng et al. obtained the bounds for $|H_2(2)|$ for the classes \mathcal{S}^* and \mathcal{C} . In [17], Sokół et al. considered second Hankel determinant problem for the class \mathcal{SL} and obtained sharp upper bounds for the functional $|a_2 a_4 - a_3^2|$ belonging to the class \mathcal{SL} . Also they gave a conjecture for sharp bound of $|a_2 a_4 - a_3^2|$ for functions in the class \mathcal{SL} . The third Hankel determinant is $H_3(1) = a_3(a_2 a_4 - a_3^2) - a_4(a_4 - a_2 a_3) + a_5(a_3 - a_2^2)$. Recently, Babalola [1], Raza and Malik [15] and Bansal et al. [2] have studied third Hankel determinant $H_3(1)$, for various classes of analytic and univalent functions.

In this paper, we investigate an upper bound on the modulus of $H_3(1)$ the functions belonging to the class \mathcal{SL} of analytic functions related to shell-like curves connected with Fibonacci numbers in the open unit disc defined by (1.1).

Now we recall the following lemmas which will be used in proving our main results. Let $\mathcal{P}(\beta)$, $0 \leq \beta < 1$, denote the class of analytic functions p in \mathbb{U} with $p(0) = 1$ and $\operatorname{Re}\{p(z)\} > \beta$. Especially, we will use \mathcal{P} instead of $\mathcal{P}(0)$.

Lemma 1.1. ([12]) *Let $p \in \mathcal{P}$ with $p(z) = 1 + c_1 z + c_2 z^2 + \dots$, then*

$$|c_n| \leq 2, \quad \text{for } n \geq 1. \tag{1.4}$$

If $|c_1| = 2$, then $p(z) \equiv p_1(z) \equiv (1 + xz)/(1 - xz)$ with $x = \frac{c_1}{2}$. Conversely, if $p(z) \equiv p_1(z)$ for some $|x| = 1$, then $c_1 = 2x$. Furthermore, we have

$$\left| c_2 - \frac{c_1^2}{2} \right| \leq 2 - \frac{|c_1|^2}{2}. \tag{1.5}$$

If $|c_1| < 2$, and $\left| c_2 - \frac{c_1^2}{2} \right| = 2 - \frac{|c_1|^2}{2}$, then $p(z) \equiv p_2(z)$, where

$$p_2(z) = \frac{1 + \bar{x}wz + z(wz + x)}{1 + \bar{x}wz - z(wz + x)}$$

and $x = \frac{c_1}{2}$, $w = \frac{2c_2 - c_1^2}{4 - |c_1|^2}$ and $|c_2 - \frac{c_1^2}{2}| = 2 - \frac{|c_1|^2}{2}$.

Lemma 1.2. ([14]) *Let $p \in \mathcal{P}$ with coefficients c_n as above, then*

$$|c_1 c_2 - c_3| \leq 2. \tag{1.6}$$

Lemma 1.3. ([9]) Let $p \in \mathcal{P}$ with coefficients c_n as above, then

$$|c_3 - 2c_1c_2 + c_1^3| \leq 2. \quad (1.7)$$

Lemma 1.4. ([16]) If $f(z) = z + \sum_{n=2}^{\infty} a_n z^n$ belongs to the class \mathcal{SL} , then

$$|a_n| \leq |\tau|^{n-1} u_n, \quad (1.8)$$

where u_n is the sequence of Fibonacci numbers and $\tau = \frac{1-\sqrt{5}}{2}$. Equality holds in (1.8) for the function $f_0(z) = \frac{z}{1-\alpha z - \alpha^2 z^2}$.

Lemma 1.5. ([13]) If $f(z) = z + \sum_{n=2}^{\infty} \lambda a_n z^n$ belongs to the class \mathcal{SL} , then

$$|a_3 - \lambda a_2^2| \leq \tau^2(2 + \lambda) \text{ for all } \lambda \in \mathbb{C}. \quad (1.9)$$

The above estimation is sharp. If $\lambda > 0$, then the equality in (1.9) is attained by the function $f_0(z) = \frac{z}{1-\alpha z - \alpha^2 z^2}$ while by the function $-f_0(-z)$, when $\lambda \leq 0$.

Especially, when $\lambda = 1$ in (1.9), we obtain $|a_3 - a_2^2| \leq 3\tau^2$.

In this study, we use ideas and techniques used in geometric function theory. The central problem considered here is the sharp upper bounds for the functionals $|H_2(2)|$ and $|a_2 a_3 - a_4|$ of functions in the class \mathcal{SL} depicted by the Fibonacci numbers, respectively. Also the third Hankel determinant $|H_3(1)|$ is considered using these functionals.

2. Main Results

In [17] it was proved that if $f(z) = z + a_2 z^2 + \dots$ belongs to \mathcal{SL} , then

$$|H_2(2)| = |a_2 a_4 - a_3^2| \leq \frac{11}{3} \tau^4.$$

And it was conjectured that $|H_2(2)| = |a_2 a_4 - a_3^2| \leq \tau^4$. Firstly, we present a proof of this.

Theorem 2.1. If $f(z) = z + a_2 z^2 + \dots$ belongs to \mathcal{SL} , then

$$|H_2(2)| = |a_2 a_4 - a_3^2| \leq \tau^4. \quad (2.1)$$

The bound is sharp.

Proof. For given $f \in \mathcal{SL}$, define $p(z) = 1 + p_1 z + p_2 z^2 + \dots$, by

$$\frac{z f'(z)}{f(z)} = p(z) = 1 + p_1 z + p_2 z^2 + \dots,$$

where $p \prec \tilde{p}$. If $p \prec \tilde{p}$, then there exists an analytic function w such that $|w(z)| \leq |z|$ in \mathbb{U} and $p(z) = \tilde{p}(w(z))$. Therefore, the function

$$h(z) = \frac{1 + w(z)}{1 - w(z)} = 1 + c_1z + c_2z^2 + \dots \tag{2.2}$$

is in the class \mathcal{P} . It follows that

$$w(z) = \frac{c_1z}{2} + \left(c_2 - \frac{c_1^2}{2}\right) \frac{z^2}{2} + \left(c_3 - c_1c_2 + \frac{c_1^3}{4}\right) \frac{z^3}{2} + \dots \tag{2.3}$$

and

$$\begin{aligned} \tilde{p}(w(z)) &= 1 + \tilde{p}_1 \left\{ \frac{c_1z}{2} + \left(c_2 - \frac{c_1^2}{2}\right) \frac{z^2}{2} + \left(c_3 - c_1c_2 + \frac{c_1^3}{4}\right) \frac{z^3}{2} + \dots \right\} \\ &\quad + \tilde{p}_2 \left\{ \frac{c_1z}{2} + \left(c_2 - \frac{c_1^2}{2}\right) \frac{z^2}{2} + \left(c_3 - c_1c_2 + \frac{c_1^3}{4}\right) \frac{z^3}{2} + \dots \right\}^2 \\ &\quad + \tilde{p}_3 \left\{ \frac{c_1z}{2} + \left(c_2 - \frac{c_1^2}{2}\right) \frac{z^2}{2} + \left(c_3 - c_1c_2 + \frac{c_1^3}{4}\right) \frac{z^3}{2} + \dots \right\}^3 + \dots \\ &= 1 + \frac{\tilde{p}_1 c_1 z}{2} + \left\{ \left(c_2 - \frac{c_1^2}{2}\right) \frac{\tilde{p}_1}{2} + \frac{c_1^2 \tilde{p}_2}{4} \right\} z^2 \\ &\quad + \left\{ \left(c_3 - c_1c_2 + \frac{c_1^3}{4}\right) \frac{\tilde{p}_1}{2} + \left(c_2 - \frac{c_1^2}{2}\right) \frac{c_1 \tilde{p}_2}{2} + \frac{c_1^3 \tilde{p}_3}{8} \right\} z^3 + \dots \end{aligned} \tag{2.4}$$

It is known that

$$\begin{aligned} \tilde{p}(z) &= 1 + \sum_{n=1}^{\infty} \tilde{p}_n z^n \\ &= 1 + (u_0 + u_2)\tau z + (u_1 + u_3)\tau^2 z^2 + \sum_{n=3}^{\infty} (u_{n-3} + u_{n-2} + u_{n-1} + u_n)\tau^n z^n \\ &= 1 + \tau z + 3\tau^2 z^2 + 4\tau^3 z^3 + 7\tau^4 z^4 + 11\tau^5 z^5 + \dots \end{aligned} \tag{2.5}$$

This shows that the relevant connection of \tilde{p} with the sequence of Fibonacci numbers u_n , such that $u_0 = 0, u_1 = 1, u_{n+2} = u_n + u_{n+1}$ for $n = 0, 1, 2, \dots$. Thus, $\tilde{p}_1 = \tau, \tilde{p}_2 = 3\tau^2$ and

$$\tilde{p}_n = (u_{n-1} + u_{n+1})\tau^n = (u_{n-3} + u_{n-2} + u_{n-1} + u_n)\tau^n = \tau \tilde{p}_{n-1} + \tau^2 \tilde{p}_{n-2} \quad (n = 3, 4, 5, \dots).$$

If $p(z) = 1 + p_1z + p_2z^2 + \dots$, then using (2.4) and (2.5), we have

$$p_1 = \frac{c_1}{2}\tau, \tag{2.6}$$

$$p_2 = \frac{1}{2} \left(c_2 - \frac{c_1^2}{2}\right) \tau + \frac{3}{4} c_1^2 \tau^2, \tag{2.7}$$

and

$$p_3 = \frac{1}{2} \left(c_3 - c_1 c_2 + \frac{c_1^3}{4} \right) \tau + \frac{3}{2} c_1 \left(c_2 - \frac{c_1^2}{2} \right) \tau^2 + \frac{1}{2} c_1^3 \tau^3. \quad (2.8)$$

Hence

$$\frac{zf'(z)}{f(z)} = 1 + a_2 z + (2a_3 - a_2^2) z^2 + (3a_4 - 3a_2 a_3 + a_2^3) z^3 + \dots = 1 + p_1 z + p_2 z^2 + \dots$$

and

$$a_2 = p_1, \quad a_3 = \frac{p_1^2 + p_2}{2}, \quad a_4 = \frac{p_1^3 + 3p_1 p_2 + 2p_3}{6}.$$

Therefore, we have

$$\begin{aligned} & |a_2 a_4 - a_3^2| \\ &= \frac{1}{12} |p_1^4 - 4p_1 p_3 + 3p_2^2| \\ &= \frac{1}{12} \left| \frac{c_1^4}{16} \tau^4 - 2c_1 \tau \left\{ \frac{\tau}{2} \left(c_3 - c_1 c_2 + \frac{c_1^3}{4} \right) + \frac{3c_1 \tau^2}{2} \left(c_2 - \frac{c_1^2}{2} \right) + \frac{c_1^3 \tau^3}{2} \right\} \right. \\ &\quad \left. + 3 \left\{ \frac{\tau}{2} \left(c_2 - \frac{c_1^2}{2} \right) + \frac{3c_1^2 \tau^2}{4} \right\}^2 \right| \\ &= \frac{\tau^2}{12} \left| \left(\frac{3c_1^4}{4} - \frac{3c_1^2}{4} \left(c_2 - \frac{c_1^2}{2} \right) \right) \tau + \frac{c_1^4}{2} - c_1 c_3 + c_1^2 c_2 + \frac{3}{4} \left(c_2 - \frac{c_1^2}{2} \right)^2 \right|. \quad (2.9) \end{aligned}$$

It is known (1.2), that

$$\forall n \in \mathbb{N}, \quad \tau = \frac{\tau^n}{u_n} - x_n, \quad x_n = \frac{u_{n-1}}{u_n}, \quad \lim_{n \rightarrow \infty} \frac{u_{n-1}}{u_n} = |\tau| \approx 0.618. \quad (2.10)$$

Applying (2.10) gives

$$\begin{aligned} |a_2 a_4 - a_3^2| &= \frac{\tau^2}{12} \left| \left(\frac{3c_1^4}{4} - \frac{3c_1^2}{4} \left(c_2 - \frac{c_1^2}{2} \right) \right) \frac{\tau^n}{u_n} + c_1 (c_1 c_2 - c_3) \right. \\ &\quad \left. + \frac{3}{4} c_2 \left(c_2 - \frac{c_1^2}{2} \right) + \frac{3}{8} (2x_n - 1) c_1^2 \left(c_2 - \frac{c_1^2}{2} \right) + \frac{2 - 3x_n}{4} c_1^4 \right|. \end{aligned}$$

Now, applying the triangle inequality, (1.4), (1.5) and (1.6) gives

$$\begin{aligned}
 |a_2a_4 - a_3^2| &\leq \frac{\tau^2}{12} \left| \frac{3c_1^4}{4} - \frac{3c_1^2}{4} \left(c_2 - \frac{c_1^2}{2} \right) \right| \frac{|\tau|^n}{u_n} \\
 &\quad + \frac{\tau^2}{12} \left\{ |c_1| |c_1c_2 - c_3| + \frac{3}{4} |c_2| \left| c_2 - \frac{c_1^2}{2} \right| \right. \\
 &\quad \left. + \frac{3}{8} (2x_n - 1) |c_1^2| \left| c_2 - \frac{c_1^2}{2} \right| + \frac{2 - 3x_n}{4} |c_1|^4 \right\} \\
 &\leq \frac{\tau^2}{12} \left| \frac{3c_1^4}{4} - \frac{3c_1^2}{4} \left(c_2 - \frac{c_1^2}{2} \right) \right| \frac{|\tau|^n}{u_n} \\
 &\quad + \frac{\tau^2}{12} \left\{ 2|c_1| + \frac{3}{2} \left(2 - \frac{|c_1|^2}{2} \right) \right. \\
 &\quad \left. + \frac{3}{8} (2x_n - 1) |c_1|^2 \left(2 - \frac{|c_1|^2}{2} \right) + \frac{2 - 3x_n}{4} |c_1|^4 \right\},
 \end{aligned}$$

because by (2.10), we have $x_n \rightarrow 0.618$ so $2x_n - 1 > 0$, $2 - 3x_n > 0$ for sufficiently large n . So, in above calculation, in the last line, we have got a function of variable $|c_1| =: y \in [0, 2]$ and after elementary calculations we can get that

$$\max_{y \in [0, 2]} \left\{ 2y + \frac{3}{2} \left(2 - \frac{y^2}{2} \right) + \frac{3}{8} (2x_n - 1) y^2 \left(2 - \frac{y^2}{2} \right) + \frac{2 - 3x_n}{4} y^4 \right\} = 12 - 12x_n \text{ at } y = 2. \tag{2.11}$$

Furthermore, it is clear that

$$\lim_{n \rightarrow \infty} \left| \frac{3c_1^4}{4} - \frac{3c_1^2}{4} \left(c_2 - \frac{c_1^2}{2} \right) \right| \frac{|\tau|^n}{u_n} = 0$$

and (2.10), (2.11) give

$$\begin{aligned}
 \lim_{n \rightarrow \infty} \left[\max_{y \in [0, 2]} \left\{ 2y + \frac{3}{2} \left(2 - \frac{y^2}{2} \right) + \frac{3}{8} (2x_n - 1) y^2 \left(2 - \frac{y^2}{2} \right) + \frac{2 - 3x_n}{4} y^4 \right\} \right] \\
 = 12 - 12|\tau| = 12\tau^2,
 \end{aligned}$$

so we have

$$|a_2a_4 - a_3^2| \leq 0 + \frac{\tau^2}{12} 12\tau^2 = \tau^4.$$

If we take in (2.2)

$$h(z) = \frac{1+z}{1-z} = 1 + 2z + 2z^2 + \dots,$$

then putting $c_1 = c_2 = c_3 = 2$ in (2.9) gives

$$|a_2a_4 - a_3^2| = \frac{\tau^2}{12} |12\tau + 12| = \frac{\tau^2}{12} |12\tau^2| = \tau^4.$$

and it shows that (2.1) is sharp. It completes the proof. □

Theorem 2.2. *If $f(z) = z + a_2z^2 + \dots$ belongs to \mathcal{SL} , then*

$$|a_2a_3 - a_4| \leq |\tau|^3. \quad (2.12)$$

The bound is sharp.

Proof. Let $f \in \mathcal{SL}$ and $p \in \mathcal{P}$ where $p(z) = 1 + p_1z + p_2z^2 + \dots$. From (2.6), (2.7), (2.8) and

$$\frac{zf'(z)}{f(z)} = 1 + a_2z + (2a_3 - a_2^2)z^2 + (3a_4 - 3a_2a_3 + a_2^3)z^3 + \dots = 1 + p_1z + p_2z^2 + \dots$$

we have

$$a_2a_3 - a_4 = \frac{1}{3}(p_1^3 - p_3).$$

So we obtain

$$\begin{aligned} |a_2a_3 - a_4| &= \frac{1}{3} |p_1^3 - p_3| \\ &= \frac{1}{3} \left| \frac{c_1^3}{8} \tau^3 - \frac{1}{2} \left(c_3 - c_1c_2 + \frac{c_1^3}{4} \right) \tau - \frac{3}{2} c_1 \left(c_2 - \frac{c_1^2}{2} \right) \tau^2 - \frac{1}{2} c_1^3 \tau^3 \right| \\ &= \frac{1}{3} \left| \left\{ \frac{1}{4} c_1 \left(c_2 - \frac{c_1^2}{2} \right) + \frac{1}{2} (c_1c_2 - c_3) - \frac{7}{4} c_1c_2 \right\} \tau + \frac{3}{8} c_1^3 - \frac{3}{2} c_1c_2 \right|. \end{aligned} \quad (2.13)$$

Applying (2.10), we have

$$\begin{aligned} |a_2a_3 - a_4| &= \frac{1}{3} \left| \left\{ \frac{1}{4} c_1 \left(c_2 - \frac{c_1^2}{2} \right) + \frac{1}{2} (c_1c_2 - c_3) - \frac{7}{4} c_1c_2 \right\} \frac{\tau^n}{u_n} \right. \\ &\quad \left. - \frac{1}{4} c_1 \left(c_2 - \frac{c_1^2}{2} \right) x_n - \frac{1}{2} (c_1c_2 - c_3) x_n + \frac{7}{4} c_1c_2x_n + \frac{3}{8} c_1^3 - \frac{3}{2} c_1c_2 \right| \\ &= \frac{1}{3} \left| \left\{ \frac{1}{4} c_1 \left(c_2 - \frac{c_1^2}{2} \right) + \frac{1}{2} (c_1c_2 - c_3) - \frac{7}{4} c_1c_2 \right\} \frac{\tau^n}{u_n} \right. \\ &\quad \left. + \frac{1}{2} (c_3 - 2c_1c_2 + c_1^3) x_n + \frac{3}{4} c_1 \left(c_2 - \frac{c_1^2}{2} \right) x_n \right. \\ &\quad \left. + \frac{5}{4} c_1c_2x_n - \frac{3}{4} c_1 \left(c_2 - \frac{c_1^2}{2} \right) - \frac{3}{4} c_1c_2 \right| \end{aligned} \quad (2.14)$$

Now, applying the triangle inequality, (1.4), (1.5),(1.6) and (1.7) gives

$$\begin{aligned}
 |a_2a_3 - a_4| &\leq \frac{1}{3} \left| \left\{ \frac{1}{4}c_1 \left(c_2 - \frac{c_1^2}{2} \right) + \frac{1}{2} (c_1c_2 - c_3) - \frac{7}{4}c_1c_2 \right\} \right| \frac{|\tau^n|}{u_n} \\
 &+ \frac{1}{2} |c_3 - 2c_1c_2 + c_1^3| x_n + \frac{|3x_n - 3|}{4} |c_1| \left| c_2 - \frac{c_1^2}{2} \right| + \frac{|5x_n - 3|}{4} |c_1||c_2| \\
 &\leq \frac{1}{3} \left| \left\{ \frac{1}{4}c_1 \left(c_2 - \frac{c_1^2}{2} \right) + \frac{1}{2} (c_1c_2 - c_3) - \frac{7}{4}c_1c_2 \right\} \right| \frac{|\tau^n|}{u_n} \\
 &+ \frac{1}{2} |c_3 - 2c_1c_2 + c_1^3| x_n + \frac{|3x_n - 3|}{4} |c_1| \left(2 - \frac{|c_1|^2}{2} \right) + \frac{|5x_n - 3|}{4} |c_1||c_2| \\
 &\leq \frac{1}{3} \left| \left\{ \frac{1}{4}c_1 \left(c_2 - \frac{c_1^2}{2} \right) + \frac{1}{2} (c_1c_2 - c_3) - \frac{7}{4}c_1c_2 \right\} \right| \frac{|\tau^n|}{u_n} \\
 &+ x_n + x_n|c_1| - \frac{3 - 3x_n}{8} |c_1|^3, \tag{2.15}
 \end{aligned}$$

because by (2.10), we have $x_n \rightarrow 0.618$ so $3x_n - 3 < 0$, $5x_n - 3 > 0$ for sufficiently large n . If we put $|c_1| =: y \in [0, 2]$ then and after elementary calculations we can get that $h(y) = x_n + x_ny - (3 - 3x_n)y^3/3$ increases in $[0, 2]$. Therefore,

$$\max_{y \in [0,2]} \{h(y)\} = \max_{y \in [0,2]} \left\{ x_n + x_ny - \frac{3 - 3x_n}{8} y^3 \right\} = 6x_n - 3 \text{ at } y = 2.$$

Because

$$\lim_{n \rightarrow \infty} \left| \frac{1}{4}c_1 \left(c_2 - \frac{c_1^2}{2} \right) + \frac{1}{2} (c_1c_2 - c_3) - \frac{7}{4}c_1c_2 \right| \frac{|\tau^n|}{u_n} = 0$$

and by (2.10)

$$\lim_{n \rightarrow \infty} \left[\max_{y \in [0,2]} \left\{ x_n + x_ny - \frac{3 - 3x_n}{8} y^3 \right\} \right] = 6|\tau| - 3 = -3(2\tau + 1) = -3\tau^3 = 3|\tau|^3,$$

we have

$$|a_2a_3 - a_4| \leq 0 + \frac{3|\tau|^3}{3} = |\tau|^3.$$

If we take in (2.2)

$$h(z) = \frac{1+z}{1-z} = 1 + 2z + 2z^2 + \dots,$$

then putting $c_1 = c_2 = c_3 = 2$ in (2.13) gives

$$|a_2a_3 - a_4| = |\tau|^3.$$

and it shows that (2.12) is sharp. It completes the proof. □

Now, we can obtain an upper bound for $|H_3(1)|$ in the class \mathcal{SL} as follows:

Theorem 2.3. *If $f(z) = z + a_2z^2 + \dots$ belongs to \mathcal{SL} , then*

$$|H_3(1)| \leq 20\tau^6. \tag{2.16}$$

Proof. Let $f \in \mathcal{SL}$. By the definition of third Hankel determinant,

$$H_3(1) = \begin{vmatrix} a_1 & a_2 & a_3 \\ a_2 & a_3 & a_4 \\ a_3 & a_4 & a_5 \end{vmatrix} = a_3(a_2a_4 - a_3^2) - a_4(a_4 - a_2a_3) + a_5(a_3 - a_2^2)$$

where $a_1 = 1$, we have

$$|H_3(1)| \leq |a_3||a_2a_4 - a_3^2| + |a_4||a_4 - a_2a_3| + |a_5||a_3 - a_2^2|. \quad (2.17)$$

Considering Lemma 1.4, Lemma 1.5, Theorem 2.1 and Theorem 2.2 in (2.17), we obtain

$$\begin{aligned} |H_3(1)| &\leq |a_3||a_2a_4 - a_3^2| + |a_4||a_4 - a_2a_3| + |a_5||a_3 - a_2^2| \\ &\leq 2\tau^2\tau^4 + 3|\tau|^3|\tau|^3 + 5\tau^23\tau^2 \\ &= 20\tau^6. \end{aligned}$$

□

3. Concluding, Remarks and Observations

In our present article, we have obtained sharp estimates of the third Hankel determinant for the class \mathcal{SL} of analytic functions related to shell-like curves connected with the Fibonacci numbers. Firstly, we have proved a conjecture given in [17] for sharp upper bound of second Hankel determinant. Secondly, we have obtained another sharp coefficient bound which will be used in the problem of finding the upper bound associated with the third Hankel determinant $H_3(1)$ for this class. Lastly, we have given an upper bound for functional $|H_3(1)|$ in the class \mathcal{SL} .

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DOI: 10.7862/rf.2018.14

Janusz Sokół

email: jsokol@ur.edu.pl

ORCID: 0000-0003-1204-2286

Faculty of Mathematics and Natural Sciences

University of Rzeszów

ul. Prof. Piłonia 1, 35-310 Rzeszów
POLAND

Sedat İlhan

email: sedati@dicle.edu.tr

ORCID: 0000-0002-6608-8848

Department of Mathematics, Faculty of Science

Dicle University

TR-21280 Diyarbakır

TURKEY

H. Özlem Güney

email: ozlemg@dicle.edu.tr

ORCID: 0000-0002-3010-7795

Department of Mathematics, Faculty of Science

Dicle University

TR-21280 Diyarbakır

TURKEY

Received 08.12.2017

Accepted 11.03.2018

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Manuscript completed in December 2018, Printed in December 2018.

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Order no. 113/18