

# **Journal of Mathematics and Applications**

(e-ISSN 2300-9926)

vol. 36 (2013)

Issued with the consent of the Rector

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**Publishing House of Rzeszow University of Technology**  
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Text prepared to print in L<sup>A</sup>T<sub>E</sub>X  
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The printed version of JMA is an original version.

p-ISSN 1733-6775  
e-ISSN 2300-9926

Publisher: Publishing House of Rzeszow University of Technology,  
12 Powstańców Warszawy Ave., 35-959 Rzeszow (e-mail: [oficina1@prz.edu.pl](mailto:oficina1@prz.edu.pl))  
<http://oficina.portal.prz.edu.pl/en/>

Editorial Office: Rzeszow University of Technology, Department of Mathematics, P.O. BOX 85  
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<http://jma.prz.edu.pl>

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**vol. 36 (2013)**

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# A Companion of the generalized trapezoid inequality and applications

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Submitted by: Jan Stankiewicz

ABSTRACT: A sharp companion of the generalized trapezoid inequality is introduced. Applications to quadrature formula are pointed out.

AMS Subject Classification: 26D15, 26D20, 41A55

Keywords and Phrases: Trapezoid inequality, Midpoint inequality, Ostrowski's inequality Bounded variation, Lipschitzian, Monotonic

## 1. Introduction

The following trapezoid type inequality for mappings of bounded variation was proved in [7] (see also [6]):

**Theorem 1.1** Let  $f : [a, b] \rightarrow \mathbb{R}$ , be a mapping of bounded variation on  $[a, b]$ , Then

$$\left| \int_a^b f(t) dt - (b-a) \frac{f(a) + f(b)}{2} \right| \leq \frac{1}{2} (b-a) \bigvee_a^b(f). \quad (1.1)$$

The constant  $\frac{1}{2}$  is the best possible.

A generalization (1.1) for mappings of bounded variation, was considered by Cerone et al. in [6], as follows:

$$\left| (b-x)f(b) + (x-a)f(a) - \int_a^b f(t) dt \right| \leq \left[ \frac{b-a}{2} + \left| x - \frac{a+b}{2} \right| \right] \cdot \bigvee_a^b(f) \quad (1.2)$$

for all  $x \in [a, b]$ . The constant  $\frac{1}{2}$  is the best possible.

In the same way, the following midpoint type inequality for mappings of bounded variation was proved in [8]:

**Theorem 1.2** *Let  $f : [a, b] \rightarrow \mathbb{R}$ , be a mapping of bounded variation on  $[a, b]$ , Then*

$$\left| (b-a) f\left(\frac{a+b}{2}\right) - \int_a^b f(t) dt \right| \leq \frac{1}{2} (b-a) \bigvee_a^b(f). \quad (1.3)$$

*The constant  $\frac{1}{2}$  is the best possible.*

A weighted generalization of trapezoid inequality for mappings of bounded variation, was considered by Tseng et. al. [12]. In order to combine the midpoint and the trapezoid inequalities together Guessab and Schmeisser [13] have proved an interesting companion of Ostrowski type inequality for  $r$ -Hölder continuous mappings. Motivated by [13], Dragomir in [14], has proved the Guessab–Schmeisser companion of Ostrowski inequality for mappings of bounded variation. Recently, in [15, 16] the authors proved a generalization of weighted Ostrowski type inequality for mappings of bounded variation and thus they deduced several trapezoid type inequalities. For recent new results regarding Ostrowski's and generalized trapezoid type inequalities see [1]–[5].

In this paper, we give a companion of (1.2) for mappings of bounded variation, Lipschitzian type and monotonic nondecreasing. Applications to quadrature formulae are given.

## 2. The Results

The following result holds:

**Theorem 2.3** *Let  $f : [a, b] \rightarrow \mathbb{R}$  be a mapping of bounded of variation on  $[a, b]$ . Then we have the inequality*

$$\begin{aligned} \left| (x-a)(f(a) + f(b)) + (a+b-2x) f\left(\frac{a+b}{2}\right) - \int_a^b f(t) dt \right| \\ \leq \left[ \frac{b-a}{4} + \left| x - \frac{3a+b}{4} \right| \right] \cdot \bigvee_a^b(f), \quad (2.4) \end{aligned}$$

*for all  $x \in [a, \frac{a+b}{2}]$ . Furthermore, the constant  $\frac{1}{4}$  is the best possible in the sense that it cannot be replaced by a smaller one.*

**Proof.** Integrating by parts

$$\int_a^b K(t, x) df(t) dt = (x-a)(f(a) + f(b)) + (a+b-2x) f\left(\frac{a+b}{2}\right) - \int_a^b f(t) dt$$

where,

$$K(t, x) := \begin{cases} t-x, & t \in [a, \frac{a+b}{2}] \\ t-(a+b-x), & t \in [\frac{a+b}{2}, b]. \end{cases}$$

Using the fact that, for a continuous mapping  $p : [a, b] \rightarrow \mathbb{R}$  and bounded variation mapping  $\nu : [a, b] \rightarrow \mathbb{R}$ , then one has the inequality

$$\left| \int_a^b p(t) d\nu(t) \right| \leq \sup_{t \in [a, b]} |p(t)| \bigvee_a^b(\nu).$$

Applying the above inequality, for  $p(t) := K(t, x)$  and  $\nu(t) := f(t)$ , we get

$$\begin{aligned} \left| \int_a^b K(t, x) df(t) \right| &\leq \sup_{t \in [a, b]} |K(t, x)| \cdot \bigvee_a^b(f) = \max \left\{ x - a, \frac{a+b}{2} - x \right\} \cdot \bigvee_a^b(f) \\ &= \left[ \frac{b-a}{4} + \left| x - \frac{3a+b}{4} \right| \right] \cdot \bigvee_a^b(f), \end{aligned}$$

for all  $x \in [a, \frac{a+b}{2}]$ , which proves (2.4). To prove the sharpness of (2.4), assume that (2.4) holds with constant  $C > 0$ , i.e.,

$$\begin{aligned} \left| (x-a)(f(a) + f(b)) + (a+b-2x)f\left(\frac{a+b}{2}\right) - \int_a^b f(t) dt \right| \\ \leq \left[ C(b-a) + \left| x - \frac{3a+b}{4} \right| \right] \cdot \bigvee_a^b(f). \quad (2.5) \end{aligned}$$

Consider the mapping  $f : [a, b] \rightarrow \mathbb{R}$ , given by

$$f(t) = \begin{cases} 0, & t \in (a, b) \\ \frac{1}{2}, & t = a, b. \end{cases}$$

Therefore,  $\int_a^b f(t) dt = 0$  and  $\bigvee_a^b(f) = 1$ . Making of use (2.5) with  $x = \frac{3a+b}{4}$ , we get

$$\left| \frac{b-a}{2} \left[ \frac{1}{2} + 0 \right] - 0 \right| \leq C(b-a) \cdot 1$$

which gives that,  $C \geq \frac{1}{4}$ , and the theorem is completely proved. ■

**Remark 2.1** In the inequality (2.4), choose

1.  $x = a$ , then we get

$$\left| (b-a)f\left(\frac{a+b}{2}\right) - \int_a^b f(t) dt \right| \leq \frac{1}{2}(b-a) \cdot \bigvee_a^b(f) \quad (2.6)$$

2.  $x = \frac{3a+b}{4}$ , then we get

$$\left| \frac{b-a}{2} \left[ f\left(\frac{a+b}{2}\right) + \frac{f(a) + f(b)}{2} \right] - \int_a^b f(t) dt \right| \leq \frac{1}{4}(b-a) \cdot \bigvee_a^b(f) \quad (2.7)$$

3.  $x = \frac{a+b}{2}$ , then we get

$$\left| (b-a) \frac{f(a)+f(b)}{2} - \int_a^b f(t) dt \right| \leq \frac{1}{2} (b-a) \cdot \bigvee_a^b(f) \quad (2.8)$$

**Corollary 2.1** *If  $f \in C^{(1)}[a, b]$ . Then we have the inequality*

$$\begin{aligned} & \left| (x-a)(f(a)+f(b)) + (a+b-2x)f\left(\frac{a+b}{2}\right) - \int_a^b f(t) dt \right| \\ & \leq \left[ \frac{b-a}{4} + \left| x - \frac{3a+b}{4} \right| \right] \cdot \|f'\|_{1,[a,b]}, \end{aligned} \quad (2.9)$$

where  $\|\cdot\|_1$  is the  $L_1$  norm, namely  $\|f'\|_{1,[a,b]} := \int_a^b |f'(t)| dt$ .

**Corollary 2.2** *If  $f$  is  $K$ -Lipschitzian on  $[a, b]$  with the constant  $K > 0$ . Then we have the inequality*

$$\begin{aligned} & \left| (x-a)(f(a)+f(b)) + (a+b-2x)f\left(\frac{a+b}{2}\right) - \int_a^b f(t) dt \right| \\ & \leq K(b-a) \left[ \frac{b-a}{4} + \left| x - \frac{3a+b}{4} \right| \right]. \end{aligned} \quad (2.10)$$

**Corollary 2.3** *If  $f$  is monotonic on  $[a, b]$ . Then we have the inequality*

$$\begin{aligned} & \left| (x-a)(f(a)+f(b)) + (a+b-2x)f\left(\frac{a+b}{2}\right) - \int_a^b f(t) dt \right| \\ & \leq \left[ \frac{b-a}{4} + \left| x - \frac{3a+b}{4} \right| \right] \cdot |f(b) - f(a)|. \end{aligned} \quad (2.11)$$

A refinement of (2.10), may be stated as follows:

**Theorem 2.4** *Let  $f : [a, b] \rightarrow \mathbb{R}$  be an  $L$ -Lipschitzian mapping on  $[a, b]$ . Then we have the inequality*

$$\begin{aligned} & \left| (x-a)(f(a)+f(b)) + (a+b-2x)f\left(\frac{a+b}{2}\right) - \int_a^b f(t) dt \right| \\ & \leq L \left[ \frac{(b-a)^2}{8} + 2 \left( x - \frac{3a+b}{4} \right)^2 \right], \end{aligned} \quad (2.12)$$

for all  $x \in [a, \frac{a+b}{2}]$ . Furthermore, the constant  $\frac{1}{8}$  is the best possible in the sense that it cannot be replaced by a smaller one.

**Proof.** Using the fact that, for a Riemann integrable function  $p : [a, b] \rightarrow \mathbb{R}$  and  $L$ -Lipschitzian function  $\nu : [a, b] \rightarrow \mathbb{R}$ , then one has the inequality

$$\left| \int_a^b p(t) d\nu(t) \right| \leq L \int_a^b |p(t)| dt.$$

Applying the above inequality, for  $p(t) := K(t, x)$  and  $\nu(t) := f(t)$ , we get

$$\begin{aligned} \left| \int_a^b K(t, x) df(t) \right| &\leq L \int_a^b |K(t, x)| = L \left[ (x-a)^2 + \left( \frac{a+b}{2} - x \right)^2 \right] \\ &= L \left[ \frac{(b-a)^2}{8} + 2 \left( x - \frac{3a+b}{4} \right)^2 \right], \end{aligned}$$

for all  $x \in [a, \frac{a+b}{2}]$ , which proves (2.12). To prove the sharpness of (2.12), assume that (2.12) holds with constant  $C > 0$ , i.e.,

$$\begin{aligned} \left| (x-a)(f(a) + f(b)) + (a+b-2x)f\left(\frac{a+b}{2}\right) - \int_a^b f(t) dt \right| \\ \leq \left[ C(b-a) + \left| x - \frac{3a+b}{4} \right| \right] \cdot \bigvee_a^b(f). \quad (2.13) \end{aligned}$$

Consider the mapping  $f : [a, b] \rightarrow \mathbb{R}$ , given by  $f(t) := t - \frac{3a+b}{4}$ . Therefore,  $f$  is Lipschitzian with  $L = 1$  and  $\int_a^b f(t) dt = \frac{(b-a)^2}{4}$ . Making of use (2.13) with  $x = \frac{3a+b}{4}$ , we get

$$\frac{(b-a)^2}{8} \leq C(b-a)^2$$

which gives that,  $C \geq \frac{1}{8}$ , and the theorem is completely proved. ■

**Remark 2.2** In the inequality (2.12), choose

1.  $x = a$ , then we get

$$\left| (b-a)f\left(\frac{a+b}{2}\right) - \int_a^b f(t) dt \right| \leq L \frac{(b-a)^2}{4} \quad (2.14)$$

2.  $x = \frac{3a+b}{4}$ , then we get

$$\left| \frac{b-a}{2} \left[ f\left(\frac{a+b}{2}\right) + \frac{f(a) + f(b)}{2} \right] - \int_a^b f(t) dt \right| \leq L \frac{(b-a)^2}{8} \quad (2.15)$$



3.  $x = \frac{a+b}{2}$ , then we get

$$\left| (b-a) \frac{f(a) + f(b)}{2} - \int_a^b f(t) dt \right| \leq L \frac{(b-a)^2}{4} \quad (2.16)$$

A refinement of (2.11), may be stated as follows:

**Theorem 2.5** *Let  $f : [a, b] \rightarrow \mathbb{R}$  be a monotonic non-decreasing on  $[a, \frac{a+b}{2}]$  and on  $[\frac{a+b}{2}, b]$ . Then we have the inequality*

$$\begin{aligned} & \left| (x-a)(f(a) + f(b)) + (a+b-2x)f\left(\frac{a+b}{2}\right) - \int_a^b f(t) dt \right| \\ & \leq (x-a)(f(b) - f(a)) + (2x-a-b)f\left(\frac{a+b}{2}\right) \\ & \quad + 2\left(\frac{3a+b}{4} - x\right)(f(a+b-x) - f(x)) \quad (2.17) \end{aligned}$$

for all  $x \in [a, \frac{a+b}{2}]$ .

**Proof.** Using the fact that, for a monotonic non-decreasing function  $\nu : [a, b] \rightarrow \mathbb{R}$  and continuous function  $p : [a, b] \rightarrow \mathbb{R}$ , then one has the inequality

$$\left| \int_a^b p(t) d\nu(t) \right| \leq \int_a^b |p(t)| d\nu(t).$$

Applying the above inequality, for  $p(t) := K(t, x)$  and  $\nu(t) := f(t)$ , we get

$$\left| \int_a^b K(t, x) df(t) \right| \leq \int_a^b |K(t, x)| df(t)$$

By the integration by parts formula for the Stieltjes integral we have

$$\begin{aligned}
 \int_a^b |K(t, x)| df(t) &= \int_a^{\frac{a+b}{2}} |t-x| df(t) + \int_{\frac{a+b}{2}}^b |t-(a+b-x)| df(t) \\
 &= \int_a^x (x-t) df(t) + \int_x^{\frac{a+b}{2}} (t-x) df(t) \\
 &\quad + \int_{\frac{a+b}{2}}^{a+b-x} (a+b-x-t) df(t) + \int_{a+b-x}^b (t+x-a-b) df(t) \\
 &= (x-t)f(t)|_a^x + \int_a^x f(t) dt + (x-t)f(t)|_x^{\frac{a+b}{2}} - \int_x^{\frac{a+b}{2}} f(t) dt \\
 &\quad + (a+b-x-t)f(t)|_{\frac{a+b}{2}}^{a+b-x} + \int_{\frac{a+b}{2}}^{a+b-x} f(t) dt \\
 &\quad + (t+x-a-b)f(t)|_{a+b-x}^b - \int_{a+b-x}^b f(t) dt \\
 &= (x-a)(f(b)-f(a)) + (2x-a-b)f\left(\frac{a+b}{2}\right) \\
 &\quad + \int_a^x f(t) dt - \int_x^{\frac{a+b}{2}} f(t) dt + \int_{\frac{a+b}{2}}^{a+b-x} f(t) dt - \int_{a+b-x}^b f(t) dt
 \end{aligned}$$

Now, by the monotonicity property of  $f$ , we have

$$\int_a^x f(t) dt \leq (x-a)f(x), \quad \int_x^{\frac{a+b}{2}} f(t) dt \geq \left(\frac{a+b}{2}-x\right)f(x),$$

and

$$\begin{aligned}
 \int_{\frac{a+b}{2}}^{a+b-x} f(t) dt &\leq \left(\frac{a+b}{2}-x\right)f(a+b-x), \\
 \int_{a+b-x}^b f(t) dt &\geq (x-a)f(a+b-x)
 \end{aligned}$$

giving that

$$\begin{aligned}
 \int_a^b |K(t, x)| df(t) &\leq (x-a)(f(b)-f(a)) + (2x-a-b)f\left(\frac{a+b}{2}\right) \\
 &\quad + \int_a^x f(t) dt - \int_x^{\frac{a+b}{2}} f(t) dt + \int_{\frac{a+b}{2}}^{a+b-x} f(t) dt - \int_{a+b-x}^b f(t) dt \\
 &\leq (x-a)(f(b)-f(a)) + (2x-a-b)f\left(\frac{a+b}{2}\right) \\
 &\quad + 2\left(\frac{3a+b}{4}-x\right)(f(a+b-x)-f(x)),
 \end{aligned}$$

which is required. ■

**Remark 2.3** In the inequality (2.17), choose

1.  $x = a$ , then we get

$$\left| (b-a) f\left(\frac{a+b}{2}\right) - \int_a^b f(t) dt \right| \leq \frac{(b-a)}{2} \left[ f(b) - 2f\left(\frac{a+b}{2}\right) - f(a) \right]. \quad (2.18)$$

2.  $x = \frac{3a+b}{4}$ , then we get

$$\left| \frac{b-a}{2} \left[ f\left(\frac{a+b}{2}\right) + \frac{f(a)+f(b)}{2} \right] - \int_a^b f(t) dt \right| \leq \frac{(b-a)}{4} \left[ f(b) - 2f\left(\frac{a+b}{2}\right) - f(a) \right]. \quad (2.19)$$

3.  $x = \frac{a+b}{2}$ , then we get

$$\left| (b-a) \frac{f(a)+f(b)}{2} - \int_a^b f(t) dt \right| \leq \frac{(b-a)}{2} \left[ f(b) - 2f\left(\frac{a+b}{2}\right) - f(a) \right]. \quad (2.20)$$

### 3. Applications to Quadrature Formulae

Let  $I_n : a = x_0 < x_1 < \dots < x_n = b$  be a division of the interval  $[a, b]$ ,  $\xi_i \in [x_i, x_{i+1}]$ ,  $h_i = x_{i+1} - x_i$ , ( $i = 0, 1, 2, \dots, n-1$ ) and  $\nu(h) := \max\{h_i | i = 0, 1, 2, \dots, n-1\}$ .

Define the quadrature

$$T_n(f, I_n, \xi) = \sum_{i=0}^{n-1} \left[ (\xi_i - x_i) (f(x_i) + f(x_{i+1})) + (x_i + x_{i+1} - 2\xi_i) f\left(\frac{x_i + x_{i+1}}{2}\right) \right].$$

In the following, we establish some upper bounds for the error approximation of  $\int_a^b f(t) dt$  by the quadrature  $T(f, I_n, \xi)$ .

**Theorem 4.1** Let  $f$  be as in Theorem 2.3. Then we have

$$\int_a^b f(t) dt = T_n(f, I_n, \xi_n) + R_n(f, I_n, \xi_n) \quad (4.1)$$

where,  $R_n(f, I_n, \xi_n)$  satisfies the estimation

$$\begin{aligned} |R_n(f, I_n, \xi_n)| &\leq \left[ \frac{1}{4} \nu(h) + \sup_{i=0, n-1} \left| \xi_i - \frac{3x_i + x_{i+1}}{4} \right| \right] \cdot \bigvee_a^b(f) \\ &\leq \frac{1}{2} \nu(h) \bigvee_a^b(f). \end{aligned} \quad (4.2)$$

**Proof.** Applying Theorem 2.3 on the intervals  $[x_i, x_{i+1}]$ ,  $i = 1, 2, \dots, n-1$ , we get

$$\begin{aligned} &\left| \left[ (\xi_i - x_i)(f(x_i) + f(x_{i+1})) + (x_i + x_{i+1} - 2\xi_i) f\left(\frac{x_i + x_{i+1}}{2}\right) \right] - \int_{x_i}^{x_{i+1}} f(t) dt \right| \\ &\leq \left[ \frac{1}{4} h_i + \left| \xi_i - \frac{3x_i + x_{i+1}}{4} \right| \right] \cdot \bigvee_{x_i}^{x_{i+1}}(f). \end{aligned}$$

Summing the above inequality over  $i$  from 0 to  $n-1$  and using the generalized triangle inequality, we deduce that

$$\begin{aligned} \left| T(f, \xi_n, I_n) - \int_a^b f(t) dt \right| &\leq \sum_{i=0}^{n-1} \left[ \frac{1}{4} h_i + \left| \xi_i - \frac{3x_i + x_{i+1}}{4} \right| \right] \cdot \bigvee_{x_i}^{x_{i+1}}(f) \\ &\leq \sup_{i=0, n-1} \left[ \frac{1}{4} h_i + \left| \xi_i - \frac{3x_i + x_{i+1}}{4} \right| \right] \cdot \sum_{i=0}^{n-1} \bigvee_{x_i}^{x_{i+1}}(f) \\ &\leq \left[ \frac{1}{4} \nu(h) + \sup_{i=0, n-1} \left| \xi_i - \frac{3x_i + x_{i+1}}{4} \right| \right] \cdot \bigvee_a^b(f) \end{aligned}$$

which completely proves the first inequality in (4.2).

For the second inequality, we observe that

$$\left| \xi_i - \frac{3x_i + x_{i+1}}{4} \right| \leq \frac{1}{4} h_i$$

it follows that

$$\sup_{i=0, n-1} \left| \xi_i - \frac{3x_i + x_{i+1}}{4} \right| \leq \frac{1}{4} \sup_{i=0, n-1} h_i = \frac{1}{4} \nu(h)$$

which proves the second inequality in (4.2). ■

**Theorem 4.2** Let  $f$  be as in Theorem 2.4. Then (4.1) holds where,  $R_n(f, I_n, \xi_n)$  satisfies the estimation

$$|R_n(f, I_n, \xi_n)| \leq L \sum_{i=0}^{n-1} \left[ \frac{1}{8} h_i^2 + 2 \left( \xi_i - \frac{3x_i + x_{i+1}}{4} \right)^2 \right] \quad (4.3)$$

**Proof.** Applying Theorem 2.4 on the intervals  $[x_i, x_{i+1}]$ ,  $i = 1, 2, \dots, n-1$ , we get

$$\left| \left[ (\xi_i - x_i)(f(x_i) + f(x_{i+1})) + (x_i + x_{i+1} - 2\xi_i) f\left(\frac{x_i + x_{i+1}}{2}\right) \right] - \int_{x_i}^{x_{i+1}} f(t) dt \right| \leq L \left[ \frac{1}{8}h_i^2 + 2 \left( \xi_i - \frac{3x_i + x_{i+1}}{4} \right)^2 \right].$$

Summing the above inequality over  $i$  from 0 to  $n-1$  and using the generalized triangle inequality, we deduce that

$$\left| T(f, \xi_n, I_n) - \int_a^b f(t) dt \right| \leq L \sum_{i=0}^{n-1} \left[ \frac{1}{8}h_i^2 + 2 \left( \xi_i - \frac{3x_i + x_{i+1}}{4} \right)^2 \right],$$

which completely proves the inequality in (4.3). ■

**Remark 4.1** One may state another result for monotonic mappings by applying Theorem 2.5. We shall left the details to the interested readers.

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**DOI: 10.7862/rf.2013.1**

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*Received 1.12.2011, Revised 30.06.2013, Accepted 25.10.2013*

# Preserving subordination and superordination results of generalized Srivastava-Attiya operator

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Submitted by: Jan Stankiewicz

ABSTRACT: In this paper, we obtain some subordination and superordination-preserving results of the generalized Srivastava-Attiya operator. Sandwich-type result is also obtained.

AMS Subject Classification: 30C45

Keywords and Phrases: Analytic function, Hadamard product, differential subordination, superordination

## 1. Introduction

Let  $H(U)$  be the class of functions analytic in  $U = \{z \in \mathbb{C} : |z| < 1\}$  and  $H[a, n]$  be the subclass of  $H(U)$  consisting of functions of the form  $f(z) = a + a_n z^n + a_{n+1} z^{n+1} + \dots$ , with  $H_0 = H[0, 1]$  and  $H = H[1, 1]$ . Denote  $A(p)$  by the class of all analytic functions of the form

$$f(z) = z^p + \sum_{n=1}^{\infty} a_{p+n} z^{p+n} \quad (p \in \mathbb{N} = \{1, 2, 3, \dots\}; z \in U) \quad (1.1)$$

and let  $A(1) = A$ . For  $f, F \in H(U)$ , the function  $f(z)$  is said to be subordinate to  $F(z)$ , or  $F(z)$  is superordinate to  $f(z)$ , if there exists a function  $\omega(z)$  analytic in  $U$  with  $\omega(0) = 0$  and  $|\omega(z)| < 1 (z \in U)$ , such that  $f(z) = F(\omega(z))$ . In such a case we write  $f(z) \prec F(z)$ . If  $F$  is univalent, then  $f(z) \prec F(z)$  if and only if  $f(0) = F(0)$  and  $f(U) \subset F(U)$  (see [14] and [15]).

Let  $\phi : \mathbb{C}^2 \times U \rightarrow \mathbb{C}$  and  $h(z)$  be univalent in  $U$ . If  $p(z)$  is analytic in  $U$  and satisfies the first order differential subordination:

$$\phi(p(z), zp'(z); z) \prec h(z), \quad (1.2)$$

then  $p(z)$  is a solution of the differential subordination (1.2). The univalent function  $q(z)$  is called a dominant of the solutions of the differential subordination (1.2) if  $p(z) \prec q(z)$  for all  $p(z)$  satisfying (1.2). A univalent dominant  $\tilde{q}$  that satisfies  $\tilde{q} \prec q$  for all dominants of (1.2) is called the best dominant. If  $p(z)$  and  $\phi(p(z), zp'(z); z)$  are univalent in  $U$  and if  $p(z)$  satisfies the first order differential superordination:

$$h(z) \prec \phi(p(z), zp'(z); z), \quad (1.3)$$

then  $p(z)$  is a solution of the differential superordination (1.3). An analytic function  $q(z)$  is called a subordinant of the solutions of the differential superordination (1.3) if  $q(z) \prec p(z)$  for all  $p(z)$  satisfying (1.3). A univalent subordinant  $\tilde{q}$  that satisfies  $q \prec \tilde{q}$  for all subordinants of (1.3) is called the best subordinant (see [14] and [15]).

The general Hurwitz-Lerch Zeta function  $\Phi(z, s, a)$  is defined by:

$$\Phi(z, s, a) = \sum_{n=0}^{\infty} \frac{z^n}{(n+a)^s}, \quad (1.4)$$

$$(a \in \mathbb{C} \setminus \mathbb{Z}_0^-; \mathbb{Z}_0^- = \{0, -1, -2, \dots\}; s \in \mathbb{C} \text{ when } |z| < 1; R\{s\} > 1 \text{ when } |z| = 1).$$

For interesting properties and characteristics of the Hurwitz-Lerch Zeta function  $\Phi(z, s, a)$  (see [3], [8], [9], [11] and [19]).

Recently, Srivastava and Attiya [18] introduced the linear operator  $L_{s,b} : A \rightarrow A$ , defined in terms of the Hadamard product by

$$L_{s,b}(f)(z) = G_{s,b}(z) * f(z) \quad (z \in U; b \in \mathbb{C} \setminus \mathbb{Z}_0^-; s \in \mathbb{C}), \quad (1.5)$$

where for convenience,

$$G_{s,b} = (1+b)^s [\Phi(z, s, b) - b^{-s}] \quad (z \in U). \quad (1.6)$$

The Srivastava-Attiya operator  $L_{s,b}$  contains among its special cases, the integral operators introduced and investigated by Alexander [1], Libera [7] and Jung et al. [6].

Analogous to  $L_{s,b}$ , Liu [10] defined the operator  $J_{p,s,b} : A(p) \rightarrow A(p)$  by

$$J_{p,s,b}(f)(z) = G_{p,s,b}(z) * f(z) \quad (z \in U; b \in \mathbb{C} \setminus \mathbb{Z}_0^-; s \in \mathbb{C}; p \in \mathbb{N}), \quad (1.7)$$

where

$$G_{p,s,b} = (1+b)^s [\Phi_p(z, s, b) - b^{-s}]$$

and

$$\Phi_p(z, s, b) = \frac{1}{b^s} + \sum_{n=0}^{\infty} \frac{z^{n+p}}{(n+1+b)^s}. \quad (1.8)$$

It is easy to observe from (1.7) and (1.8) that

$$J_{p,s,b}(f)(z) = z^p + \sum_{n=1}^{\infty} \left( \frac{1+b}{n+1+b} \right)^s a_{n+p} z^{n+p}. \quad (1.9)$$



We note that

- (i)  $J_{p,0,b}(f)(z) = f(z)$ ;
- (ii)  $J_{1,1,0}(f)(z) = Lf(z) = \int_0^z \frac{f(t)}{t} dt$ , where the operator  $L$  was introduced by Alexander [1];
- (iii)  $J_{1,s,b}(f)(z) = L_{s,b}f(z)$  ( $s \in \mathbb{C}, b \in \mathbb{C} \setminus \mathbb{Z}_0^-$ ), where the operator  $L_{s,b}$  was introduced by Srivastava and Attiya [18];
- (iv)  $J_{p,1,\nu+p-1}(f)(z) = F_{\nu,p}(f(z))$  ( $\nu > -p, p \in \mathbb{N}$ ), where the operator  $F_{\nu,p}$  was introduced by Choi et al. [4];
- (v)  $J_{p,\alpha,p}(f)(z) = I_p^\alpha f(z)$  ( $\alpha \geq 0, p \in \mathbb{N}$ ), where the operator  $I_p^\alpha$  was introduced by Shams et al. [17];
- (vi)  $J_{p,m,p-1}(f)(z) = J_p^m f(z)$  ( $m \in \mathbb{N}_0 = \mathbb{N} \cup \{0\}, p \in \mathbb{N}$ ), where the operator  $J_p^m$  was introduced by El-Ashwah and Aouf [5];
- (vii)  $J_{p,m,p+l-1}(f)(z) = J_p^m(l)f(z)$  ( $m \in \mathbb{N}_0, p \in \mathbb{N}, l \geq 0$ ), where the operator  $J_p^m(l)$  was introduced by El-Ashwah and Aouf [5].

It follows from (1.9) that:

$$z(J_{p,s+1,b}(f)(z))' = (b+1)J_{p,s,b}(f)(z) - (b+1-p)J_{p,s+1,b}(f)(z). \quad (1.10)$$

To prove our results, we need the following definitions and lemmas.

**Definition 1** [14]. Denote by  $F$  the set of all functions  $q(z)$  that are analytic and injective on  $\bar{U} \setminus E(q)$  where

$$E(q) = \left\{ \zeta \in \partial U : \lim_{z \rightarrow \zeta} q(z) = \infty \right\}$$

and are such that  $q'(\zeta) \neq 0$  for  $\zeta \in \partial U \setminus E(q)$ . Further let the subclass of  $F$  for which  $q(0) = a$  be denoted by  $F(a)$ ,  $F(0) \equiv F_0$  and  $F(1) \equiv F_1$ .

**Definition 2** [15]. A function  $L(z, t)$  ( $z \in U, t \geq 0$ ) is said to be a subordination chain if  $L(\cdot, t)$  is analytic and univalent in  $U$  for all  $t \geq 0$ ,  $L(z, \cdot)$  is continuously differentiable on  $[0; 1)$  for all  $z \in U$  and  $L(z, t_1) \prec L(z, t_2)$  for all  $0 \leq t_1 \leq t_2$ .

**Lemma 1** [16]. The function  $L(z, t) : U \times [0; 1) \rightarrow \mathbb{C}$  of the form

$$L(z, t) = a_1(t)z + a_2(t)z^2 + \dots \quad (a_1(t) \neq 0; t \geq 0)$$

and  $\lim_{t \rightarrow \infty} |a_1(t)| = \infty$  is a subordination chain if and only if

$$\operatorname{Re} \left\{ \frac{z \partial L(z, t) / \partial z}{\partial L(z, t) / \partial t} \right\} > 0 \quad (z \in U, t \geq 0).$$

**Lemma 2** [12]. Suppose that the function  $\mathcal{H} : \mathbb{C}^2 \rightarrow \mathbb{C}$  satisfies the condition

$$\operatorname{Re} \{ \mathcal{H}(is; t) \} \leq 0$$

for all real  $s$  and for all  $t \leq -n(1+s^2)/2$ ,  $n \in \mathbb{N}$ . If the function  $p(z) = 1 + p_n z^n + p_{n+1} z^{n+1} + \dots$  is analytic in  $U$  and

$$\operatorname{Re} \left\{ \mathcal{H}(p(z); zp'(z)) \right\} > 0 \quad (z \in U),$$

then  $\operatorname{Re}\{p(z)\} > 0$  for  $z \in U$ .

**Lemma 3** [13]. Let  $\kappa, \gamma \in \mathbb{C}$  with  $\kappa \neq 0$  and let  $h \in H(U)$  with  $h(0) = c$ . If  $\operatorname{Re}\{\kappa h(z) + \gamma\} > 0$  ( $z \in U$ ), then the solution of the following differential equation:

$$q(z) + \frac{zq'(z)}{\kappa q(z) + \gamma} = h(z) \quad (z \in U; q(0) = c)$$

is analytic in  $U$  and satisfies  $\operatorname{Re}\{\kappa q(z) + \gamma\} > 0$  for  $z \in U$ .

**Lemma 4** [14]. Let  $p \in F(a)$  and let  $q(z) = a + a_n z^n + a_{n+1} z^{n+1} + \dots$  be analytic in  $U$  with  $q(z) \neq a$  and  $n \geq 1$ . If  $q$  is not subordinate to  $p$ , then there exists two points  $z_0 = r_0 e^{i\theta} \in U$  and  $\zeta_0 \in \partial U \setminus E(q)$  such that

$$q(U_{r_0}) \subset p(U); \quad q(z_0) = p(\zeta_0) \quad \text{and} \quad z_0 p'(z_0) = m \zeta_0 p'(\zeta_0) \quad (m \geq n).$$

**Lemma 5** [15]. Let  $q \in H[a; 1]$  and  $\varphi : \mathbb{C}^2 \rightarrow \mathbb{C}$ . Also set  $\varphi(q(z), zq'(z)) = h(z)$ . If  $L(z, t) = \varphi(q(z), tzq'(z))$  is a subordination chain and  $p \in H[a; 1] \cap F(a)$ , then

$$h(z) \prec \varphi(p(z), zp'(z)),$$

implies that  $q(z) \prec p(z)$ . Furthermore, if  $\varphi(q(z), zq'(z)) = h(z)$  has a univalent solution  $q \in F(a)$ , then  $q$  is the best subordinant.

In the present paper, we aim to prove some subordination-preserving and superordination-preserving properties associated with the integral operator  $J_{p,s,b}$ . Sandwich-type result involving this operator is also derived.

## 2. Main results

Unless otherwise mentioned, we assume throughout this section that  $b \in \mathbb{C} \setminus \mathbb{Z}_0^-, s \in \mathbb{C}$ ,  $\operatorname{Re}(b) > 0$ ,  $p \in \mathbb{N}$  and  $z \in U$ .

**Theorem 1.** Let  $f, g \in A(p)$  and

$$\operatorname{Re} \left\{ 1 + \frac{z\phi''(z)}{\phi'(z)} \right\} > -\delta \quad \left( \phi(z) = \frac{J_{p,s-1,b}(g)(z)}{z^p}; z \in U \right), \quad (2.1)$$

where  $\delta$  is given by

$$\delta = \frac{1 + |b+1|^2 - |1 - (b+1)^2|}{4[1 + \operatorname{Re}(b)]} \quad (z \in U). \quad (2.2)$$

Then the subordination condition

$$\frac{J_{p,s-1,b}(f)(z)}{z^p} \prec \frac{J_{p,s-1,b}(g)(z)}{z^p} \quad (2.3)$$

implies that

$$\frac{J_{p,s,b}(f)(z)}{z^p} \prec \frac{J_{p,s,b}(g)(z)}{z^p} \quad (2.4)$$

and the function  $\frac{J_{p,s,b}(g)(z)}{z^p}$  is the best dominant.

**Proof.** Let us define the functions  $F(z)$  and  $G(z)$  in  $U$  by

$$F(z) = \frac{J_{p,s,b}(f)(z)}{z^p} \quad \text{and} \quad G(z) = \frac{J_{p,s,b}(g)(z)}{z^p} \quad (z \in U) \quad (2.5)$$

and without loss of generality we assume that  $G(z)$  is analytic, univalent on  $\bar{U}$  and

$$G'(\zeta) \neq 0 \quad (|\zeta| = 1).$$

If not, then we replace  $F(z)$  and  $G(z)$  by  $F(\rho z)$  and  $G(\rho z)$ , respectively, with  $0 < \rho < 1$ . These new functions have the desired properties on  $\bar{U}$ , so we can use them in the proof of our result and the results would follow by letting  $\rho \rightarrow 1$ .

We first show that, if

$$q(z) = 1 + \frac{zG''(z)}{G'(z)} \quad (z \in U), \quad (2.6)$$

then

$$\operatorname{Re}\{q(z)\} > 0 \quad (z \in U).$$

From (1.10) and the definition of the functions  $G, \phi$ , we obtain that

$$\phi(z) = G(z) + \frac{zG'(z)}{b+1}. \quad (2.7)$$

Differentiating both sides of (2.7) with respect to  $z$  yields

$$\phi'(z) = \left(1 + \frac{1}{b+1}\right)G'(z) + \frac{zG''(z)}{b+1}. \quad (2.8)$$

Combining (2.6) and (2.8), we easily get

$$1 + \frac{z\phi''(z)}{\phi'(z)} = q(z) + \frac{zq'(z)}{q(z) + b + 1} = h(z) \quad (z \in U). \quad (2.9)$$

It follows from (2.1) and (2.9) that

$$\operatorname{Re}\{h(z) + b + 1\} > 0 \quad (z \in U). \quad (2.10)$$

Moreover, by using Lemma 3, we conclude that the differential equation (2.9) has a solution  $q(z) \in H(U)$  with  $h(0) = q(0) = 1$ . Let

$$\mathcal{H}(u, v) = u + \frac{v}{u + b + 1} + \delta,$$

where  $\delta$  is given by (2.2). From (2.9) and (2.10), we obtain  $\operatorname{Re} \left\{ \mathcal{H} \left( q(z); zq'(z) \right) \right\} > 0$  ( $z \in U$ ).

To verify the condition

$$\operatorname{Re} \{ \mathcal{H}(i\vartheta; t) \} \leq 0 \quad \left( \vartheta \in \mathbb{R}; t \leq -\frac{1+\vartheta^2}{2} \right), \quad (2.11)$$

we proceed as follows:

$$\begin{aligned} \operatorname{Re} \{ \mathcal{H}(i\vartheta; t) \} &= \operatorname{Re} \left\{ i\vartheta + \frac{t}{b+1+i\vartheta} + \delta \right\} = \frac{t(1+\operatorname{Re}(b))}{|b+1+i\vartheta|^2} + \delta \\ &\leq -\frac{\Upsilon(b, \vartheta, \delta)}{2|b+1+i\vartheta|^2}, \end{aligned}$$

where

$$\Upsilon(b, \vartheta, \delta) = [1 + \operatorname{Re}(b) - 2\delta]\vartheta^2 - 4\delta \operatorname{Im}(b)\vartheta - 2\delta|b+1|^2 + 1 + \operatorname{Re}(b). \quad (2.12)$$

For  $\delta$  given by (2.2), the coefficient of  $\vartheta^2$  in the quadratic expression  $\Upsilon(b, \vartheta, \delta)$  given by (2.12) is positive or equal to zero. To check this, put  $b+1 = c$ , so that

$$1 + \operatorname{Re}(b) = c_1 \quad \text{and} \quad \operatorname{Im}(b) = c_2.$$

We thus have to verify that

$$c_1 - 2\delta \geq 0,$$

or

$$c_1 \geq 2\delta = \frac{1 + |c|^2 - |1 - c^2|}{2c_1}.$$

This inequality will hold true if

$$2c_1^2 + |1 - c^2| \geq 1 + |c|^2 = 1 + c_1^2 + c_2^2,$$

that is, if

$$|1 - c^2| \geq 1 - \operatorname{Re}(c^2),$$

which is obviously true. Moreover, the quadratic expression  $\Upsilon(b, \vartheta, \delta)$  by  $\vartheta$  in (2.12) is a perfect square for the assumed value of  $\delta$  given by (2.2). Hence we see that (2.11) holds. Thus, by using Lemma 2, we conclude that

$$\operatorname{Re} \{ q(z) \} > 0 \quad (z \in U),$$

that is, that  $G$  defined by (2.5) is convex (univalent) in  $U$ . Next, we prove that the subordination condition (2.3) implies that

$$F(z) \prec G(z),$$

for the functions  $F$  and  $G$  defined by (2.5). Consider the function  $L(z, t)$  given by

$$L(z, t) = G(z) + \frac{(1+t)zG'(z)}{b+1} \quad (0 \leq t < \infty; z \in U). \quad (2.13)$$

We note that

$$\left. \frac{\partial L(z, t)}{\partial z} \right|_{z=0} = G'(0) \left( 1 + \frac{1+t}{b+1} \right) \neq 0 \quad (0 \leq t < \infty; z \in U; \operatorname{Re}\{b+1\} > 0).$$

This show that the function

$$L(z, t) = a_1(t)z + \dots,$$

satisfies the condition  $a_1(t) \neq 0$  ( $0 \leq t < \infty$ ). Further, we have

$$\operatorname{Re} \left\{ \frac{z\partial L(z, t)/\partial z}{\partial L(z, t)/\partial t} \right\} = \operatorname{Re}\{b+1 + (1+t)q(z)\} > 0 \quad (0 \leq t < \infty; z \in U).$$

Since  $G(z)$  is convex and  $\operatorname{Re}\{b+1\} > 0$ . Therefore, by using Lemma 1, we deduce that  $L(z, t)$  is a subordination chain. It follows from the definition of subordination chain that

$$\phi(z) = G(z) + \frac{zG'(z)}{b+1} = L(z, 0)$$

and

$$L(z, 0) \prec L(z, t) \quad (0 \leq t < \infty),$$

which implies that

$$L(\zeta, t) \notin L(U, 0) = \phi(U) \quad (0 \leq t < \infty; \zeta \in \partial U). \quad (2.14)$$

If  $F$  is not subordinate to  $G$ , by using Lemma 4, we know that there exist two points  $z_0 \in U$  and  $\zeta_0 \in \partial U$  such that

$$F(z_0) = G(\zeta_0) \quad \text{and} \quad z_0 F'(z_0) = (1+t)\zeta_0 G'(\zeta_0) \quad (0 \leq t < \infty). \quad (2.15)$$

Hence, by using (2.5), (2.13), (2.15) and (2.3), we have

$$L(\zeta_0, t) = G(\zeta_0) + \frac{(1+t)\zeta_0 G'(\zeta_0)}{b+1} = F(z_0) + \frac{z_0 F'(z_0)}{b+1} = \frac{J_{p,s-1,b}(f)(z_0)}{z_0^p} \in \phi(U).$$

This contradicts (2.14). Thus, we deduce that  $F \prec G$ . Considering  $F = G$ , we see that the function  $G$  is the best dominant. This completes the proof of Theorem 1.

We now derive the following superordination result.

**Theorem 2.** Let  $f, g \in A(p)$  and

$$\operatorname{Re} \left\{ 1 + \frac{z\phi''(z)}{\phi'(z)} \right\} > -\delta \quad \left( \phi(z) = \frac{J_{p,s-1,b}(g)(z)}{z^p}; z \in U \right), \quad (2.16)$$

where  $\delta$  is given by (2.2). If the function  $\frac{J_{p,s-1,b}(f)(z)}{z^p}$  is univalent in  $U$  and  $\frac{J_{p,s,b}(f)(z)}{z^p} \in F$ , then the superordination condition

$$\frac{J_{p,s-1,b}(g)(z)}{z^p} \prec \frac{J_{p,s-1,b}(f)(z)}{z^p} \quad (2.17)$$

implies that

$$\frac{J_{p,s,b}(g)(z)}{z^p} \prec \frac{J_{p,s,b}(f)(z)}{z^p} \quad (2.18)$$

and the function  $\frac{J_{p,s,b}(g)(z)}{z^p}$  is the best subdominant.

**Proof.** Suppose that the functions  $F, G$  and  $q$  are defined by (2.5) and (2.6), respectively. By applying similar method as in the proof of Theorem 1, we get

$$\operatorname{Re}\{q(z)\} > 0 \quad (z \in U).$$

Next, to arrive at our desired result, we show that  $G \prec F$ . For this, we suppose that the function  $L(z, t)$  be defined by (2.13). Since  $G$  is convex, by applying a similar method as in Theorem 1, we deduce that  $L(z, t)$  is subordination chain. Therefore, by using Lemma 5, we conclude that  $G \prec F$ . Moreover, since the differential equation

$$\phi(z) = G(z) + \frac{zG'(z)}{b+1} = \varphi(G(z), zG'(z))$$

has a univalent solution  $G$ , it is the best subdominant. This completes the proof of Theorem 2.

Combining the above-mentioned subordination and superordination results involving the operator  $J_{p,s,b}$ , the following "sandwich-type result" is derived.

**Theorem 3.** Let  $f, g_j \in A(p)$  ( $j = 1, 2$ ) and

$$\operatorname{Re}\left\{1 + \frac{z\phi_j''(z)}{\phi_j'(z)}\right\} > -\delta \quad \left(\phi_j(z) = \frac{J_{p,s-1,b}(g_j)(z)}{z^p} \quad (j = 1, 2); z \in U\right),$$

where  $\delta$  is given by (2.2). If the function  $\frac{J_{p,s-1,b}(f)(z)}{z^p}$  is univalent in  $U$  and  $\frac{J_{p,s,b}(f)(z)}{z^p} \in F$ , then the condition

$$\frac{J_{p,s-1,b}(g_1)(z)}{z^p} \prec \frac{J_{p,s-1,b}(f)(z)}{z^p} \prec \frac{J_{p,s-1,b}(g_2)(z)}{z^p} \quad (2.19)$$

implies that

$$\frac{J_{p,s,b}(g_1)(z)}{z^p} \prec \frac{J_{p,s,b}(f)(z)}{z^p} \prec \frac{J_{p,s,b}(g_2)(z)}{z^p} \quad (2.20)$$

and the functions  $\frac{J_{p,s,b}(g_1)(z)}{z^p}$  and  $\frac{J_{p,s,b}(g_2)(z)}{z^p}$  are, respectively, the best subdominant and the best dominant.

**Remark.** (i) Putting  $b = p$  and  $s = \alpha$  ( $\alpha \geq 0, p \in \mathbb{N}$ ) in our results of this paper, we obtain the results obtained by Aouf and Seoudy [2];

(ii) Specializing the parameters  $s$  and  $b$  in our results of this paper, we obtain the results for the corresponding operators  $F_{\nu,p}$ ,  $J_p^m$  and  $J_p^m(l)$  which are defined in the introduction.

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**DOI: 10.7862/rf.2013.2**

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*Received 14.12.2011, Revised 20.06.2013, Accepted 25.10.2013*



# Influence of boundary conditions on 2D wave propagation in a rectangle

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ABSTRACT: Work is devoted to generalization of a differential method of spatial characteristics to case of the flat task about distribution of waves in rectangular area of the final sizes with gaps in boundary conditions. On the basis of the developed numerical technique are received the settlement certainly - differential ratios of dynamic tasks in special points of front border of rectangular area, where boundary conditions on coordinate aren't continuous. They suffer a rupture of the first sort in points in which action P - figurative dynamic loading begins. Results of research are brought to the numerical decision.

AMS Subject Classification: *isotropic environment, dynamic load, plane deformation, special point, tension, speed, wave progress, numerical solution, algorithm*

Keywords and Phrases: *65L10; 65L15; 65L60; 76E06*

## 1. Introduction

Modern engineering and technology widely employ massive elements of constructions, containing cracks, holes, inclusions and other inhomogeneities of various nature and purpose. Performance of these elements under dynamic loads puts a number of questions concerning with dynamic problems of solid mechanics. In particular, evaluation of dynamic stresses near cuts, holes, pores, inclusions and singular points of a boundary is of great practical importance for mechanical and civil engineering, rock mechanics, seismology and fault detection. Solving arising problems and studying unsteady wave fields discloses significant physical features and provides data on the strength and reliability of a construction. Meanwhile, the problem of finding unsteady wave fields is quite difficult. In many practically important cases, the problem is additionally complicated by discontinuous behaviour of a solution. Such are cases when a finite elastic region contains discontinuities in boundary conditions, holes or

inclusions with corner points and/or cuts with corners, which are sources of high stress concentration. It is impossible to solve such problems without developing efficient numerical methods. Accordingly, modern studies of unsteady waves in solids focus on the development and improvement of numerical techniques. For the dynamic problems, they include various modifications of finite differences, discrete steps, spatial characteristics, finite elements, Godunov's mesh-characteristic method, boundary integral equations, method of sources, etc. Among the methods, the finite difference methods, based on using characteristic surfaces and compatibility equations on them, have certain advantages. They provide utmost correspondence between the dependence regions of the starting differential equations and approximating difference equations, what notably increases the accuracy of results for smooth and discontinuous solutions; they also provide correct identification of boundaries and contacts. In 1960, an explicit scheme of second order was suggested for a system of partial differential equations of second order in three variables [1]. The scheme employed characteristics and it was used for studying plane waves [2]. Later on, the method of spatial characteristics has been developed for solving particular dynamic problems of solid mechanics [3], [4], [5], [6], [7], [8], [9] [10], [11], [12], [13].

## 2. Problem formulation.

Consider plane-strain deformation of an elastic rectangle  $0_1 \leq \ell, -L_2 \leq L$  The conventional dynamic equations of plane-strain elasticity (see [14]) are used in the form suggested in the paper [2]:

$$v_{1,t} - p_{,1} - q_{,1} - \tau_{,2} = 0; \quad v_{2,t} - p_{,2} + q_{,2} - \tau_{,1} = 0; \quad (2.1)$$

$$\gamma^2(\gamma^2 - 1)^{-1} p_{,t} - v_{1,1} - v_{2,2} = 0; \quad \gamma^2 q_{,t} - v_{1,1} + v_{2,2} = 0; \quad \gamma^2 \tau_{,t} - v_{1,2} - v_{2,1} = 0,$$

Herein, the dimensionless time  $\bar{t}$ , spatial coordinates  $\bar{x}_i$ , stresses  $p, q, \tau$  and velocities  $v_1, v_2$  are defined via the corresponding physical time  $t$ , coordinates  $x_i$ , stresses  $\sigma_{11}, \sigma_{22}, \sigma_{12}$  and displacements  $u_i$  in accordance with [2], as

$$\bar{t} = \frac{tc_1}{b}; \bar{x}_i = \frac{x_i}{b}; \quad v_i = \frac{1}{c_1} \frac{\partial u_i}{\partial t}, (i = 1, 2) \quad p = \frac{\sigma_{11} + \sigma_{22}}{2\rho c_1^2};$$

$$q = \frac{\sigma_{11} - \sigma_{22}}{2\rho c_1^2}; \quad \tau = \frac{\sigma_{12}}{\rho c_1^2}; \gamma = \frac{c_1}{c_2},$$

with  $b$  being a characteristic length. Further on, the overbar in the notation of the dimensionless time and coordinates is omitted.

We assume that before loading, the body does not move and it is stress-free. Therefore, the initial conditions are:

$$v_1(x_1, x_2, 0) = v_2(x_1, x_2, 0) = p(x_1, x_2, 0) = q(x_1, x_2, 0) = \tau(x_1, x_2, 0) = 0. \quad (2.2)$$

The boundary conditions (BC) for solving the system (2.1) are as follows. The boundary  $x_1 = 0$  of the rectangle is loaded by the normal traction  $p + q$ , prescribed

on the part  $L^* \leq x_2 \leq L^{**}$  as a step function, changing in time  $t$  with the amplitude  $A$  and the angular frequency  $T$ . The shear traction  $\tau$  is zero. Hence, at  $L^* \leq x_2 \leq L^{**}$ , the BC are:

$$p + q = f(x_2, t) = A \sin(\omega t), \tau = 0 \quad \text{for } 0 \leq t \leq t^*. \quad (2.3)$$

The load acts from the moment  $t = 0$  till  $t = t^*$  and then ceases to zero, so that

$$p + q = 0, \tau = 0 \quad \text{for } t \geq t^* \quad (2.4)$$

The remaining part of the upper boundary and the entire lower boundary ( $x_1 = l$ ) of the rectangle are traction-free:

$$p + q = 0, \tau = 0 \quad \text{for } t \geq 0. \quad (2.5)$$

The boundaries  $x_2 = \pm L$  are clamped. Hence at any time, the velocity at their points is zero:

$$v_1(x_1, t) = v_2(x_1, t) = 0 \quad \text{for } t \geq 0. \quad (2.6)$$

We are interested in finding fields of stresses and velocities caused by the fronts of incidental and diffracted elastic waves for  $t > 0$ . The problem consists in solving the system of partial differential equations (2.1) under the initial condition (2.2) and the boundary conditions (2.3) - (2.6). The solution is obtained by the method of spatial characteristics, presented in detail in [2]. Note, however, that the method, as it is suggested in [2], is applicable only to regions with continuous change of the input parameters. Thus we have developed an algorithm, presented below for finding the solution near the singular points  $x_2 = L^*$  and  $x_2 = L^{**}$  of the boundary  $x_1 = 0$ , where the load suffers the discontinuity of the first kind.

We represent the sides of the rectangle by  $n_1$  and  $n_2$  segments, respectively. Thus the division steps are  $h_1 = l/n_1$  and  $h_2 = L/n_2$ . The nodal points are  $(x_1^i, x_2^j)$  with  $x_1^i = ih_1$  ( $i = 0, 1, 2, \dots, n_1$ ) and  $x_2^j = jh_2$  ( $j = -n_2, -n_2 + 1, -n_2 + 2, \dots, -1, 0, 1, 2, \dots, n_2 - 1, n_2$ ). These points coincide with those, which appear at lines of boundary nodes of a rectangular mesh covering the considered rectangle.

Consider for certainty the point  $E_1(x_2 = L^{**})$  of the boundary  $x_1 = 0$  (fig. 1). In its vicinity, two corner points I and II are distinguished. For the corners I and II, we derive and employ finite difference approximations, obtained by integration along bi-characteristics and the axis of the characteristic cone. Note that for the corner I the equations are similar to those for the upper right corner R of the considered region:

$$\delta v_1^I - \delta v_2^I + \alpha_8 \delta p^I = A_1, \quad \delta v_1^I + \delta v_2^I + \alpha_2 \delta q^I = A_2, \quad (2.7)$$

while for the corner II they are similar to those for the upper left corner M:

$$\delta v_1^{II} + \delta v_2^{II} + \alpha_8 \delta p^{II} = A_3, \quad \delta v_1^{II} - \delta v_2^{II} + \alpha_2 \delta q^{II} = A_4, \quad (2.8)$$

The right-hand sides in (2.8) and (2.9) are defined by equations:

$$\begin{aligned}
A_1 &= k(v_{1,1} + p_{,1} + q_{,1} - \tau_{,1} + v_{2,2} - p_{,2} + q_{,2} + \tau_{,2}) \\
&\quad - \alpha_0(v_{1,2} + v_{2,1}) - \alpha_9(v_{1,12} - v_{2,12} - \alpha_5 p_{,12} + \alpha_3 \tau_{,12}); \\
A_2 &= k(v_{1,1} + p_{,1} + q_{,1} + \tau_{,1} - v_{2,2} + p_{,2} - q_{,2} + \tau_{,2}) \\
&\quad - \alpha_0(v_{1,2} - v_{2,1}) + \alpha_1(v_{1,2} - v_{2,1}) - \alpha_5 q_{,12} + \alpha_9(v_{1,12} + v_{2,12}); \\
A_3 &= k(v_{1,1} + p_{,1} + q_{,1} + \tau_{,1} + v_{2,2} + p_{,2} - q_{,2} + \tau_{,2}) \\
&\quad + \alpha_0(v_{1,2} + v_{2,1}) + \alpha_9(v_{1,12} + v_{2,12}) + \alpha_5 p_{,12} + \alpha_3 \tau_{,12}; \\
A_4 &= k(v_{1,1} + p_{,1} + q_{,1} - \tau_{,1} - v_{2,2} - p_{,2} + q_{,2} + \tau_{,2}) \\
&\quad + \alpha_0(v_{1,2} - v_{2,1}) - \alpha_1(v_{1,2} - v_{2,1}) + \alpha_5 q_{,12} - \alpha_9(v_{1,12} - v_{2,12}).
\end{aligned}$$

In accordance with (2.3), to the left of the point  $_1$  and at the point  $_1$  itself, we have prescribed the normal traction  $p + q$ . For its increment  $\delta p^I + \delta q^I$ , we may write:

$$\delta p^I + \delta q^I = A[\sin(\omega t) - \sin(\omega(t - k))]. \quad (2.9)$$

where  $k$  is the number of a time step. Besides, we need to meet the continuity conditions for the normal velocities and the normal and shear tractions at adjacent points of corners:

$$\delta v_1^I = \delta v_1^{II}, \quad \delta v_2^I = \delta v_2^{II},$$

$$\delta p^I - \delta q^I = \delta p^{II} - \delta q^{II}, \quad \delta \tau^I = \delta \tau^I = \delta \tau^{II}. \quad (2.10)$$

The system (2.8) - (2.11) uniquely defines the increments of the velocities  $\delta v_1^I$ ,  $\delta v_1^{II}$ ,  $\delta v_2^I$ ,  $\delta v_2^{II}$  and stresses  $\delta p^I$ ,  $\delta q^I$ ,  $\delta \tau^I$ ,  $\delta p^{II}$ ,  $\delta q^{II}$ ,  $\delta \tau^{II}$  at the point  $_1$ , where the BC is discontinuous:

$$\delta v_1 = \Delta_1/\Delta, \quad \delta v_2 = \Delta_2/\Delta, \quad \delta p^I = \Delta_3/\Delta, \quad (2.11)$$

$$\delta q^I = \Delta_4/\Delta, \quad \delta \tau = 0, \quad \delta p^{II} = \Delta_5/\Delta, \quad \delta q^{II} = \Delta_6/\Delta.$$

The determinants entering (2.10) are given by formulae:

$$\begin{aligned}
\Delta_1 &= - [\alpha_2 \alpha_8 (3(A_1 + A_2) - 2(\alpha_2 + \alpha_8)f(x_2, t) - A_3 \\
&\quad - A_4) + \alpha_8^2(A_2 + A_4) + \alpha_2^2(A_1 + A_3)], \\
\Delta_2 &= \alpha_2 \alpha_8 (A_1 - A_2 - A_3 + A_4) - \alpha_8^2(A_2 - A_4) + \alpha_2^2(A_1 - A_3), \\
\Delta_3 &= 2\alpha_2(A_2 - A_3 - \alpha_2 f(x_2, t)) + 2\alpha_8(A_4 - A_1 - \alpha_2 f(x_2, t)), \\
\Delta_4 &= \alpha_2(A_3 - A_2) - 2\alpha_8(A_4 - A_1) + (\alpha_2 + \alpha_8)f(x_2, t), \\
\Delta_5 &= 2\alpha_8(A_2 - A_3) + 2\alpha_2(2A_2 - 2A_3 - A_4 + A_1) - 2\alpha_2(\alpha_2 + \alpha_8)f(x_2, t), \\
\Delta_6 &= 2\alpha_2(A_1 - A_4) + 2\alpha_8(2A_1 + A_2 - A_3 - 2A_4) - 2\alpha_8(\alpha_2 + \alpha_8)f(x_2, t), \\
\Delta &= -2(\alpha_2 + \alpha_8)^2.
\end{aligned}$$

### 3. Main results

Equations (2.12) serve us for finding the solution at the right singular point  $E_1$ . Similar equations are used for the left singular point  $E_2$  (fig . 1). They present the basis of an algorithm for solving unsteady problems of dynamic elasticity involving discontinuities of the first kind at points of the boundary. The algorithm is employed in a subroutine, which is included into a general program for calculations on a conventional laptop.

As an illustration, we present results for the rectangular region  $0 \leq x_1 \leq 5$  and  $|x_2| \leq 5$ . The part, at which the external load acts, is:  $-4.85 \leq x_2 \leq 4.85$ ; hence only 3 percent of the upper boundary is free of the loads. The load, being symmetric with respect to the  $x_1$ -axis, we consider only the right half of the rectangle ( $x_2 \geq 0$ ). The spatial steps are taken equal:  $h_1 = h_2 = h = 0.05$ . The time step  $k$  is chosen to meet the stability condition for an explicit finite-difference scheme [2]:

$$\left(\frac{k}{h}\right)^2 \leq \min\left\{\frac{\gamma^2}{\gamma^2 + 1}, \frac{\gamma^2}{2(\gamma^2 - 1)}\right\}.$$

In calculations, we set  $k = 0.025$ . The amplitude  $A$  of the applied load entering (2.3) is taken unit, and the period is  $= 100k$ ; consequently, the angular frequency is  $\omega = \pi/(100k) = 0.4\pi$ . The duration of the external pulse is  $t^* = T = 2.5$ . Fig. 2 presents the velocity  $v_1$  at five 'observation' points on the boundary of  $x_1 = 0 : x_2 = 0$  (point 1),  $x_2 = 20h$  (2),  $x_2 = 40h$  (3),  $x_2 = 60h$  (4), and  $x_2 = 80h$  (5). At each of the points, for the first hundred time steps, the form of the curve is defined by the form of the applied sinusoidal pulse. Firstly, distortion arises at the point (5) with the coordinate  $x_2 = 80h$ , which is closest to the singular point  $E_1$  having the coordinate  $x_2 = 97h$ . It arises because of the influence of waves diffracted by the singular point and propagating with the normalized speed  $c_1 = 1$  of longitudinal waves. Then, in certain intervals, the influence of this point appears at points 4, 3, 2, and 1, successively. The intensity of the influence is relatively small. Notably more strong effect is caused by the longitudinal wave diffracted by the corner point R of the rectangle ( $x_1 = 0, x_2 = 100h$ ), which propagates with the speed  $c_1$ , as well. For the point 1 at the center of the rectangle, the waves, diffracted by the corners M and R, arrive at the moment  $t = 400k = 10$  simultaneously with the wave reflected from the lower boundary ( $x_1 = 100h$ ). Consequently, for the time exceeding 400 k, we may see the result of interference of various waves. Summarizing, we conclude that the suggested finite-difference equations provide a means for solving dynamic problems involving points of a boundary, where BC suffer discontinuity of the first kind. Numerical realization of the approach has shown its stability at a sufficiently long interval of time. The results correctly reproduce the general picture and specific features of wave processes. The approach may be used for studying dynamic stresses and strains in homogeneous and layered media.

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**DOI: 10.7862/rf.2013.3**

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*Received 22.08.2012, Revisted 12.10.2012, Accepted 25.10.2013*

# On duality between order and algebraic structures in Boolean systems

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ABSTRACT: We present an extension of the known one-to-one correspondence between Boolean algebras and Boolean rings with unit being two types of Boolean systems endowed with order and algebraic structures, respectively. Two equivalent generalizations of Boolean algebras are discussed. We show that there is a one-to-one correspondence between any of the two mentioned generalized Boolean algebras and Boolean rings without unit.

AMS Subject Classification: *06E05, 06E20, 06E75*

Keywords and Phrases: *lattice, 0-lattice, join-semilattice, 0-join-semilattice, distributive 0-lattice, B-ring,  $\Delta$ -join-semilattices, Boolean algebra, algebraic ring, Boolean ring*

## 1. Introduction

There exists a beautiful one-to-one correspondence between the following two notions defined in two distinct ways: (i) a *Boolean algebra*, introduced in terms of a partially ordered set (a poset for short) as a 0-1-lattice which is distributive and complemented; (ii) a *Boolean ring with unit*, i.e. an algebraic ring with idempotent multiplication which contains a unit. The correspondence between the Boolean systems (i) and (ii) can be described by suitable homomorphisms which transmit the order structure of posets to the algebraic structure of Boolean rings and vice versa, so one can say about some kind of duality between the two structures (cf. Introduction in [7]). This fact is well known and presented in numerous papers and monographs (see e.g. [41], [42], [43], [3], [40], [45], [44], [24], [2], [21]).

A similar duality holds in a more general but still classical situation, when both (i) and (ii) above are considered without units, roughly speaking. More exactly, in terms of the notions used in this paper, there is a bijection between (I) *B-rings*, a kind of generalized Boolean algebras, lattices in a given poset  $X$ , in which  $\sup X$  may not



exist (see Definition 6), and (II) *Boolean rings*, i.e. algebraic rings with idempotent multiplication in which a unit may not exist (see Definition 16).

Various types of further extensions of Boolean algebras and rings were investigated by many authors (see e.g. [23], [4], [10], [11], [5], [12], [6], [13], [14], [15], [7]). In particular, very interesting general duality results concerning generalized orthomodular lattices are given in [10] and [7].

In this paper, we discuss certain aspects of duality between order and algebraic structures of Boolean systems. To present mutual relations between  $B$ -rings and Boolean rings (Theorems 4 and 6 in section 5) we give a list of properties of the binary operation (denoted here by  $-$ ) of *difference* in  $B$ -rings, defined as an extension of the partial binary operation  $\ominus$  of *proper difference*. The latter is introduced under condition  $(R)$  postulated for  $B$ -rings in distributive 0-lattice (see Definitions 6 and 7). Some properties of difference in  $B$ -rings collected in section 3 were partly formulated and proved in our earlier paper [9]. We recall them in Proposition 12 and 14, changing slightly the notation and completing the omitted proofs.

On the other hand, we present also another approach in which the starting point is not a 0-lattice (*join* and *meet* of any two elements exist), but a special type of 0-join-semilattice (only *join* of any two elements exists), namely a  $\Delta$ -*join-semilattice* (see Definitions 2 and 13) which satisfy, in addition, conditions  $(I_m)$  and  $(\Delta)$ . The conditions allow one to define uniquely the binary operation  $\setminus$  of difference in  $\Delta$ -join-semilattices and guarantee that meets of any two of its elements exist. Moreover, the distributivity law and condition  $(R)$ , assumed for  $B$ -rings, are satisfied. Consequently,  $\Delta$ -join-semilattices appear to be an equivalent description of  $B$ -rings (see Theorem 3 in section 5). The mutual relations between  $\Delta$ -join-semilattices and Boolean rings (Theorems 5 and 7 in section 5) are consequences of Theorems 4 and 6, in view of Theorem 3.

In our approach to Boolean systems based on the partial order structure a special role is played by various types of binary operations (partial and full) of difference. In this context it should be noted that there exist much more general, categorical and fuzzy ( $MV$ -algebras) approaches, in which the notion of  $D$ -posets, introduced by F. Kôpka and F. Chovanec (see [30], [32], [8], [31]) on the base of the partial operation of difference, plays an important role as a model in quantum probability theory. The notion was intensively investigated by R. Frič and M. Papčo (see [16]-[20], [33]-[35]).

Our interest in generalizing the notion of Boolean algebras to  $B$ -rings or  $\Delta$ -join-semilattices is connected with the results of B. J. Pettis (see [36]) who extended to Boolean  $\sigma$ -rings certain theorems of W. Sierpiński (see [39]) on generated families of subsets of a given set (see also [28] and [22]). On the other hand,  $B$ -rings are natural objects for generalizing the theory of A. Rényi of conditional probability spaces (see [37], [38], [25], [26], [27], [1], [29]). We are going to discuss these issues in our forthcoming publications.

For completeness of the presentation we give full proofs of most of the results formulated in this paper.

## 2. Semilattices and lattices

We start from certain basic definitions and their consequences. In particular, we recall some of definitions and results from our article [9].

All material in this section is given without proofs, because the properties formulated here either follow immediately from the respective definitions or can be received in a similar way as in [9].

Let  $(X, \leq)$  be a *poset*, i.e. a partially ordered set, and let  $A$  be its non-empty subset. We say that  $\inf A$  [ $\sup A$ ] *exists* in  $X$  if there is an element  $x \in X$  such that  $1^\circ$   $x \leq y$  for all  $y \in A$  and  $2^\circ$   $X \ni x_1 \leq y$  for all  $y \in A$  implies  $x_1 \leq x$  [ $1^\circ$   $y \leq x$  for all  $y \in A$  and  $2^\circ$   $y \leq x_1 \in X$  for all  $y \in A$  implies  $x \leq x_1$ ]. Clearly,  $x$  satisfying  $1^\circ$  and  $2^\circ$  is unique and we denote  $x := \inf A$  [ $x := \sup A$ ]. In particular,  $\inf\{x\} = \sup\{x\} = x$  for each  $x \in X$ . If for given elements  $x$  and  $y$  of a poset  $(X, \leq)$  the symbol  $\inf\{x, y\}$  is used in any equality that we state is true, it means that  $\inf\{x, y\}$  exists in  $X$  (and the equality holds).

**Definition 1.** A poset  $(X, \leq)$  is called a *join-semilattice* if

$$(J) \quad \forall_{x, y \in X} \quad \sup\{x, y\} \text{ exists in } X.$$

The element  $\sup\{x, y\}$  is called the *join* of  $x$  and  $y$  (see [3], p. 6, Definition).

If a given poset  $(X, \leq)$  is a join-semilattice we denote

$$x \vee y := \sup\{x, y\} \tag{2.1}$$

for any  $x, y \in X$ . Formula (2.1) determines the binary operation  $\vee$  of join on  $X$ . We denote further a given join-semilattice  $(X, \leq)$  by  $(X, \vee)$  to mark explicitly the operation  $\vee$  of join defined by (2.1). Let us remark that  $\inf\{x, y\}$  for given elements  $x$  and  $y$  of a join-semilattice  $(X, \vee)$ , called the *meet* of  $x$  and  $y$  (see [3], p. 6, Definition), may exist or not, so the corresponding operation of meet may be treated, in general, as a partial binary operation on  $X$ .

In an analogous way we can define a *meet-semilattice* with a similar remark. We will not discuss further the notion of meet-semilattice which is dual to join-semilattice.

In Propositions 1 and 2 below we formulate a list of properties of join-semilattices corresponding to the respective properties of lattices given in [9] (see Statement 1), denoting them by  $(j)$  with the corresponding indices.

**Proposition 1.** *Let  $(X, \vee)$  be a join-semilattice. Then*

$$(j_1) \quad \sup\{x_1, \dots, x_n\} \text{ exists in } X$$

for arbitrary  $n \in \mathbb{N}$  and  $x_1, \dots, x_n \in X$ . Moreover,

$$\sup\{x_1, \dots, x_n\} = \sup\{\sup\{x_1, \dots, x_{n-1}\}, x_n\}$$

for each  $n \in \mathbb{N} \setminus \{1\}$  and  $x_1, \dots, x_n \in X$ .

**Proposition 2.** *Let  $(X, \vee)$  be a join-semilattice. Then*

- (j<sub>2</sub>)  $x \vee x = x,$
- (j<sub>3</sub>)  $x \vee y = y \vee x,$
- (j<sub>5</sub>)  $(x \vee y) \vee z = x \vee (y \vee z),$
- (j<sub>6</sub>)  $x \leq x \vee y, \quad y \leq x \vee y,$
- (j<sub>7</sub>)  $x \leq y \Leftrightarrow x \vee y = y,$
- (j<sub>8</sub>)  $x \leq y \Rightarrow z \vee x \leq z \vee y$

for arbitrary  $x, y, z \in X$ .

**Definition 2.** A join-semilattice  $(X, \vee)$  is called (a) 0-join-semilattice; (b) 1-join-semilattice; (c) 0-1-join-semilattice, whenever

- (J0)  $\inf X$  exists in  $X$ ;
- (J1)  $\sup X$  exists in  $X$ ;
- (J2)  $\inf X$  and  $\sup X$  exist in  $X$ ,

respectively.

The elements  $\inf X$  and  $\sup X$  are called the *zero* and the *unit* (more exactly: the *order zero* and the *order unit*) in a given join-semilattice  $(X, \vee)$  and denoted by 0 and 1, respectively. We denote a given (a) 0-join-semilattice; (b) 1-join-semilattice; (c) 0-1-join-semilattice  $(X, \vee)$  by (a)  $(X, \vee, 0)$ ; (b)  $(X, \vee, 1)$ ; (c)  $(X, \vee, 0, 1)$ , respectively, to mark explicitly the existence of the zero or/and the unit in the join-semilattice  $(X, \vee)$ .

**Proposition 3.** *If  $(X, \vee, 0)$  is a 0-join-semilattice, then*

$$(j_0) \quad 0 \leq x, \quad x \vee 0 = 0 \vee x = x, \quad \inf\{x, 0\} = 0$$

for each  $x \in X$ . If  $(X, \vee, 1)$  is a 1-join-semilattice, then

$$(j_1) \quad x \leq 1, \quad x \vee 1 = 1 \vee x = 1, \quad \inf\{x, 1\} = x$$

for each  $x \in X$ .

In the sequel, we will not discuss the notions of 1-join-semilattice or 0-1-join-semilattice.

**Definition 3.** A poset  $(X, \leq)$  is called a *lattice* if

$$(L) \quad \forall_{x, y \in X} \quad \inf\{x, y\} \text{ and } \sup\{x, y\} \text{ exist in } X.$$

In other words, a poset  $(X, \leq)$  is a lattice if it is a join-semilattice and a meet-semilattice.

If a given poset  $(X, \leq)$  is a lattice we denote

$$x \wedge y := \inf \{x, y\} \quad \text{and} \quad x \vee y := \sup \{x, y\} \quad (2.2)$$

for any  $x, y \in X$ . The formulae in (2.2) define the binary operations  $\wedge$  and  $\vee$  on  $X$ , respectively. Any lattice  $(X, \leq)$  with the operations  $\wedge$  of meet and  $\vee$  of join given by (2.2) will be denoted by  $(X, \wedge, \vee)$ .

Next, we recall some properties of an arbitrary lattice (see [9]).

**Proposition 4.** *Let  $(X, \wedge, \vee)$  be a lattice. Then*

$$(l_1) \quad \inf \{x_1, \dots, x_n\} \quad \text{and} \quad \sup \{x_1, \dots, x_n\} \quad \text{exist in } X$$

for arbitrary  $n \in \mathbb{N}$  and  $x_1, \dots, x_n \in X$ . Moreover,

$$\inf \{x_1, \dots, x_n\} = \inf \{\inf \{x_1, \dots, x_{n-1}\}, x_n\}$$

and

$$\sup \{x_1, \dots, x_n\} = \sup \{\sup \{x_1, \dots, x_{n-1}\}, x_n\}$$

for each  $n \in \mathbb{N} \setminus \{1\}$  and  $x_1, \dots, x_n \in X$ .

**Proposition 5.** *Let  $(X, \wedge, \vee)$  be a lattice. Then*

$$(l_2) \quad x \wedge x = x, \quad x \vee x = x,$$

$$(l_3) \quad x \wedge y = y \wedge x, \quad x \vee y = y \vee x,$$

$$(l_4) \quad (x \wedge y) \wedge z = x \wedge (y \wedge z),$$

$$(l_5) \quad (x \vee y) \vee z = x \vee (y \vee z),$$

$$(l_6) \quad x \wedge y \leq x \leq x \vee y, \quad x \wedge y \leq y \leq x \vee y,$$

$$(l_7) \quad x \leq y \Leftrightarrow x \wedge y = x \Leftrightarrow x \vee y = y,$$

$$(l_8) \quad x \leq y \Rightarrow z \wedge x \leq z \wedge y \quad \& \quad z \vee x \leq z \vee y,$$

$$(l_9) \quad (x \wedge z) \vee (y \wedge z) \leq (x \vee y) \wedge z,$$

$$(l_{10}) \quad (x \wedge y) \vee z \leq (x \vee z) \wedge (y \vee z)$$

for arbitrary  $x, y, z \in X$ .

**Definition 4.** A lattice  $(X, \wedge, \vee)$  is called *distributive* if

$$(D) \quad (x \vee y) \wedge z = (x \wedge z) \vee (y \wedge z), \quad x, y, z \in X,$$

or

$$(D') \quad (x \wedge y) \vee z = (x \vee z) \wedge (y \vee z), \quad x, y, z \in X.$$

**Remark 1.** Conditions  $(D)$  and  $(D')$  are equivalent (see e.g. [3], p. 11, Theorem 9).

**Definition 5.** A lattice  $(X, \wedge, \vee)$  is called (a) *0-lattice*; (b) *1-lattice*; (c) *0-1-lattice*, whenever

- (L0)  $\inf X$  exists in  $X$ ;
- (L1)  $\sup X$  exists in  $X$ ;
- (L2)  $\inf X$  and  $\sup X$  exist in  $X$ ,

respectively.

The elements  $\inf X$  and  $\sup X$  are called the *zero* and the *unit* (more exactly: the *order zero* and the *order unit*) in a given lattice  $(X, \wedge, \vee)$  and denoted by  $0$  and  $1$ , respectively. We denote a given (a) 0-lattice; (b) 1-lattice; (c) 0-1-lattice  $(X, \wedge, \vee)$  by (a)  $(X, \wedge, \vee, 0)$ , (b)  $(X, \wedge, \vee, 1)$ , (c)  $(X, \wedge, \vee, 0, 1)$ , respectively, to mark explicitly the existence of the zero or/and the unit in the lattice  $(X, \wedge, \vee)$ .

**Proposition 6.** *If  $(X, \wedge, \vee, 0)$  is a 0-lattice, then*

$$(l0) \quad 0 \leq x, \quad x \vee 0 = 0 \vee x = x, \quad x \wedge 0 = 0 \wedge x = 0$$

for any  $x \in X$ . *If  $(X, \wedge, \vee, 1)$  is a 1-lattice, then*

$$(l1) \quad x \leq 1, \quad x \vee 1 = 1 \vee x = 1, \quad x \wedge 1 = 1 \wedge x = x$$

for any  $x \in X$ .

### 3. Properties of $B$ -rings

**Definition 6.** A distributive 0-lattice  $(X, \wedge, \vee, 0)$  is called a *B-ring* if it satisfies the following condition:

$$(R) \quad \forall_{x,y \in X, y \leq x} \exists_{z \in X} \quad z \wedge y = 0 \quad \text{and} \quad z \vee y = x,$$

i.e. if the poset  $(X, \leq)$  satisfies conditions  $(L)$ ,  $(L0)$ ,  $(D)$  and  $(R)$ .

**Definition 7.** Let  $(X, \wedge, \vee, 0)$  be a  $B$ -ring and  $x, y$  be elements of  $X$  such that  $y \leq x$ . Then by a *proper difference* of  $x$  and  $y$  we mean an element  $z$  of  $X$  satisfying the identities in  $(R)$ .

Let us recall the following known uniqueness result (see e.g. [3], p. 12, Theorem 10):

**Theorem 1.** *Let  $(X, \wedge, \vee)$  be a distributive lattice and let  $y, z_1, z_2$  be its arbitrary elements. If  $z_1 \wedge y = z_2 \wedge y$  and  $z_1 \vee y = z_2 \vee y$ , then  $z_1 = z_2$ .*

**Remark 2.** In any distributive 0-lattice  $(X, \wedge, \vee, 0)$ , if for given  $x, y \in X$  with  $y \leq x$  there exists an element  $z \in X$  satisfying the two equalities in (R), then this element, the proper difference of  $x$  and  $y$ , is unique, in view of Theorem 1.

For any elements  $x$  and  $y$  such that  $y \leq x$  of any distributive 0-lattice, we denote the proper difference of  $x$  and  $y$  by  $x \ominus y$ , i.e.

$$x \ominus y = z \quad (y \leq x) \quad \text{if} \quad y \wedge z = 0 \quad \text{and} \quad y \vee z = x. \quad (3.3)$$

In an arbitrary  $B$ -ring  $(X, \wedge, \vee, 0)$ , formula (3.3) determines the partial binary operation  $\ominus$  of proper difference in  $X$  defined for all pairs of elements  $x, y \in X$  such that  $y \leq x$ . Though the operation  $\ominus$  of proper difference does not formally appear in Definition 6 it is uniquely determined, due to condition (R) and Theorem 1, via formula (3.3). Therefore we will use in the sequel the notation  $(X, \wedge, \vee, \ominus, 0)$  for  $B$ -rings, i.e. for those distributive 0-lattices  $(X, \wedge, \vee, 0)$  which satisfy condition (R).

We formulate below some properties of the operation  $\ominus$ . We start with the assertions, which are direct consequences of (R),  $(l_6)$  and the above notation.

**Proposition 7.** *If  $(X, \wedge, \vee, \ominus, 0)$  is a  $B$ -ring, then*

$$\forall_{x, y \in X, y \leq x} \quad x \ominus y \leq x, \quad (x \ominus y) \wedge y = 0, \quad (x \ominus y) \vee y = x.$$

**Proposition 8.** *If  $(X, \wedge, \vee, \ominus, 0)$  is a  $B$ -ring, then*

$$\forall_{x, y \in X, y \leq x} \quad x \ominus (x \ominus y) = y.$$

The following de Morgan properties of the operation  $\ominus$  are true:

**Proposition 9.** *If  $(X, \wedge, \vee, \ominus, 0)$  is a  $B$ -ring, then*

$$a \ominus (x \wedge y) = (a \ominus x) \vee (a \ominus y) \quad \text{and} \quad a \ominus (x \vee y) = (a \ominus x) \wedge (a \ominus y)$$

for arbitrary  $a, x, y \in X$  such that  $x \leq a$  and  $y \leq a$ .

**Proof.** Fix  $a, x, y \in X$  with  $x \leq a$  and  $y \leq a$ . Taking into account Proposition 7, we can collect the relations:

$$x \leq a, \quad y \leq a, \quad a \ominus x \leq a, \quad a \ominus y \leq a. \quad (3.4)$$

Moreover, according to (R) and Remark 2, we have

$$x \wedge (a \ominus x) = y \wedge (a \ominus y) = 0 \quad \text{and} \quad x \vee (a \ominus x) = y \vee (a \ominus y) = a. \quad (3.5)$$

Applying (D),  $(l_6)$ ,  $(l_8)$  and the first part of (3.5), we get

$$(x \wedge y) \wedge [(a \ominus x) \vee (a \ominus y)] \leq [x \wedge (a \ominus x)] \vee [y \wedge (a \ominus y)] = 0$$

and, consequently,

$$(x \wedge y) \wedge [(a \ominus x) \vee (a \ominus y)] = 0. \quad (3.6)$$

On the other hand, taking into account (3.4), using properties  $(l_2)$ ,  $(l_3)$ ,  $(l_5)$ - $(l_8)$ , the distributivity in the form of  $(D')$  and the second part of (3.5), we obtain the following chain of equalities:

$$\begin{aligned}
& (x \wedge y) \vee [(a \ominus x) \vee (a \ominus y)] \\
&= [(x \wedge y) \vee (a \ominus x)] \vee [(x \wedge y) \vee (a \ominus y)] \\
&= [(x \vee (a \ominus x)) \wedge (y \vee (a \ominus x))] \vee [(x \vee (a \ominus y)) \wedge (y \vee (a \ominus y))] \\
&= [a \wedge (y \vee (a \ominus x))] \vee [(x \vee (a \ominus y)) \wedge a] \\
&= [y \vee (a \ominus x)] \vee [x \vee (a \ominus y)] \\
&= [x \vee (a \ominus x)] \vee [y \vee (a \ominus y)],
\end{aligned}$$

i.e., applying again the second part of (3.5) and  $(l_2)$ , we get

$$(x \wedge y) \vee [(a \ominus x) \vee (a \ominus y)] = a. \quad (3.7)$$

According to condition  $(R)$  and Remark 2, the equalities (3.6) and (3.7) mean that the first identity in Proposition 9 holds.

The second identity can be shown in an analogous way.  $\square$

**Definition 8.** A  $B$ -ring  $(X, \wedge, \vee, \ominus, 0)$  is called a  $B$ -algebra if  $\sup X$  exists in  $X$ , i.e. if conditions  $(L)$ ,  $(L2)$ ,  $(D)$  and  $(R)$  are satisfied.

We denote  $\sup X$  in a  $B$ -algebra by 1 and, consequently, the given  $B$ -algebra by  $(X, \wedge, \vee, \ominus, 0, 1)$ .

The following assertion is straightforward:

**Proposition 10.** *If  $(X, \wedge, \vee, \ominus, 0)$  is a  $B$ -ring and  $a$  is a fixed element of  $X$ , then  $(X_a, \wedge, \vee, \ominus, 0)$ , where  $X_a := \{x \in X : x \leq a\}$ , is a  $B$ -algebra with the unit  $1_a := \sup X_a = a$ .*

In an arbitrary  $B$ -algebra  $(X, \wedge, \vee, \ominus, 0, 1)$ , for every  $x \in X$ , the following condition is satisfied:

$$(C_x) \quad \exists_{z \in X} \quad z \wedge x = 0 \quad \text{and} \quad z \vee x = 1.$$

The condition results directly from  $(R)$ . An element  $z$  satisfying the equalities in condition  $(C_x)$  is unique for any  $x \in X$  (see Remark 2).

**Definition 9.** Let  $(X, \wedge, \vee, 0, 1)$  be a 0-1-lattice. Fix  $x \in X$ . By a *complement* of  $x$  we mean an element  $z \in X$  satisfying condition  $(C_x)$ . If condition  $(C_x)$  is satisfied for every  $x \in X$ , then the 0-1-lattice  $(X, \wedge, \vee, 0, 1)$  is called *complemented* (see [3], p. 16).

**Remark 3.** In any distributive 0-1-lattice  $(X, \wedge, \vee, 0, 1)$ , if for a given  $x \in X$  condition  $(C_x)$  is satisfied by  $z \in X$ , then this  $z$ , the complement of  $x$ , is unique, in view of Theorem 1.

In a distributive 0-1-lattice  $(X, \wedge, \vee, 0, 1)$ , we denote, for any element  $x \in X$  for which condition  $(C_x)$  is satisfied, the complement of  $x$  by  $x'$ . In particular, if a given distributive 0-1-lattice  $(X, \wedge, \vee, 0, 1)$  is complemented we may consider the unary operation  $'$  of complement in  $X$ , denoting such a lattice by  $(X, \wedge, \vee, ', 0, 1)$ .

**Definition 10.** According to [3], pp. 17-18, by a *Boolean algebra* we mean a complemented distributive 0-1-lattice  $(X, \wedge, \vee, ', 0, 1)$  such that the unary operation  $'$  of complement satisfies the following conditions:

$$(A1) \quad x \wedge x' = 0, \quad x \vee x' = 1,$$

$$(A2) \quad (x')' = x,$$

$$(A3) \quad (x \wedge y)' = x' \vee y', \quad (x \vee y)' = x' \wedge y' \text{ for any } x, y \in X.$$

**Theorem 2.** *Every B-algebra  $(X, \wedge, \vee, \ominus, 0, 1)$  is a Boolean algebra  $(X, \wedge, \vee, ', 0, 1)$  with the unary operation  $'$  defined by*

$$x' := 1 \ominus x \tag{3.8}$$

for any  $x \in X$ . Conversely, every Boolean algebra  $(X, \wedge, \vee, ', 0, 1)$  is a B-algebra  $(X, \wedge, \vee, \ominus, 0, 1)$  with the binary operation  $\ominus$  defined by

$$x \ominus y := x \wedge y'$$

for  $x, y \in X$  such that  $y \leq x$ .

**Proof.** Let  $(X, \wedge, \vee, \ominus, 0, 1)$  be a B-algebra. Hence, in particular, the 0-1-lattice  $(X, \wedge, \vee, 0, 1)$  is distributive and complemented with the operation  $'$  of complement defined by formula (3.8). That the operation  $'$  satisfies conditions (A1) and (A2) follows from Propositions 7 and 8, respectively, applied for  $x := 1$  and  $y := x$ . That condition (A3) is satisfied follows from Proposition 9 for  $a := 1$ . This proves the first part of Theorem 2.

To prove the second part, since a given Boolean algebra  $(X, \wedge, \vee, ', 0, 1)$  is a distributive 0-1-lattice, it suffices to verify that condition (R) is fulfilled. Fix  $x, y \in X$  such that  $y \leq x$  and put  $z := x \wedge y'$ . We obtain

$$y \wedge z = y \wedge (x \wedge y') = (y \wedge y') \wedge x = 0$$

and

$$y \vee z = y \vee (x \wedge y') = (y \vee x) \wedge (y \vee y') = x,$$

applying the two equalities in (A1) as well as properties  $(l_3)$ ,  $(l_4)$ ,  $(l_7)$ ,  $(l_0)$ ,  $(l_1)$  and  $(D')$ . Hence  $z$  satisfies the equalities in (R) and, by Theorem 1 and formula (3.3),  $z = x \ominus y$ . This completes the proof of Theorem 2.  $\square$

In any B-ring  $(X, \wedge, \vee, \ominus, 0)$ , the partial binary operation  $\ominus$  can be extended to the binary operation (denoted here by  $-$ ) on  $X$  defined for an arbitrary pair of elements of  $X$  in the following way:



**Definition 11.** If  $(X, \wedge, \vee, \ominus, 0)$  is a  $B$ -ring, then we define the difference  $x - y$  of  $x$  and  $y$  by

$$x - y := x \ominus (x \wedge y) \quad (3.9)$$

for arbitrary  $x, y \in X$ .

Notice that definition in (3.9) makes sense, because  $x \wedge y \leq x$  for any  $x, y \in X$ , due to property  $(l_6)$ . The binary operation  $-$  of difference defined above is consistent with the partial binary operation  $\ominus$  of proper difference, introduced in Definition 7, according to the following obvious property:

**Proposition 11.** *Let  $(X, \wedge, \vee, \ominus, 0)$  be a  $B$ -ring. If  $x, y \in X$  and  $y \leq x$ , then  $x - y = x \ominus y$ .*

It should be noted that the above extension  $-$  of the operation  $\ominus$  of proper difference satisfies an extended form of condition  $(R)$  with the corresponding uniqueness property (cf. Theorem 1 and Remark 2). Namely, the following assertion is true:

**Proposition 12.** *Let  $(X, \wedge, \vee, \ominus, 0)$  be a  $B$ -ring. Then*

$$(\bar{R}) \quad \forall_{x, y \in X} \exists_{z \in X} \quad z \wedge y = 0 \quad \text{and} \quad z \vee y = x \vee y,$$

and an element  $z$  in  $(\bar{R})$  is unique, given by  $z := x - y$ .

Proposition 12 is a consequence of the following two facts. That for arbitrary  $x, y \in X$  in a  $B$ -ring the two identities required in condition  $(\bar{R})$  are satisfied by  $z := x - y$  follows from the two last equalities from part 2° of Proposition 14 proved below. That this  $z$  is unique follows directly from the known uniqueness result cited above as Theorem 1.

In addition to the uniqueness property concerning the operation of difference, formulated in Proposition 12, we will give below a series of properties of this operation in any  $B$ -ring. Let us start with the following easy consequence of Definition 11 and Proposition 12:

**Proposition 13.** *Let  $(X, \wedge, \vee, \ominus, 0)$  be a  $B$ -ring. We have*

$$x - 0 = x \ominus 0 = x, \quad x - x = x \ominus x = 0, \quad 0 - x = 0 \quad (3.10)$$

for every  $x \in X$ .

Properties 1-15 formulated in [9] contain the statement expressed in Proposition 12 and several other properties recalled in Proposition 14 below. Proofs of some of the properties were omitted in [9]. We complete below all the omitted proofs. The properties will be used in the proofs of certain characterizations given in section .

**Proposition 14.** *Let  $(X, \wedge, \vee, \ominus, 0)$  be a  $B$ -ring. The following properties of the binary operation  $-$  of difference on  $X$  hold true:*

$$1^\circ \quad x - y \leq x, \quad (x - y) \wedge (x \wedge y) = 0, \quad (x - y) \vee (x \wedge y) = x,$$

- 2°  $(x - y) \wedge (y - x) = 0, \quad (x - y) \wedge y = 0, \quad (x - y) \vee y = x \vee y,$   
3°  $(x - y) \vee (y - x) \vee (x \wedge y) = x \vee y,$   
4°  $x \leq y \Rightarrow (z - y) \wedge x = 0, \quad x \leq y \Rightarrow (z - y) - x = z - y,$   
5°  $x \leq y \Rightarrow z - y \leq z - x,$   
6°  $z - (x \vee y) = (z - x) \wedge (z - y), \quad z - (x \wedge y) = (z - x) \vee (z - y),$   
7°  $z - (x \vee y) = (z - x) - y, \quad (x \vee y) - z = (x - z) \vee (y - z),$   
8°  $(x - y) \wedge z = (x \wedge z) - (y \wedge z)$

for any  $x, y, z \in X$ .

**Proof.** Since  $x - y = x \ominus (x \wedge y)$ , by the definition in (3.9), the three properties in 1° follow directly from Proposition 7.

To prove the first property in 2° assume that

$$u \leq x - y = x \ominus (x \wedge y) \quad \text{and} \quad u \leq y - x = y \ominus (x \wedge y). \quad (3.11)$$

Therefore, by the first property in 1°, we have  $u \leq x$ ,  $u \leq y$  and, consequently,  $u \leq x \wedge y$ , due to  $(l_8)$ ,  $(l_3)$  and  $(l_2)$ . Hence  $u = 0$ , in view of (3.11) and the second assertion in Proposition 7. Thus

$$(x - y) \wedge (y - x) = \inf\{x - y, y - x\} = 0, \quad (3.12)$$

as required.

To show the two remaining properties in 2° together with property 3° we use the representations:

$$y = [y \ominus (x \wedge y)] \vee (x \wedge y) = (y - x) \vee (x \wedge y), \quad (3.13)$$

following from Proposition 7 and (3.9). Applying (3.13), the distributivity condition (D) as well as the properties  $(l_2)$ ,  $(l_3)$  and  $(l_5)$ , we obtain

$$\begin{aligned} (x - y) \wedge y &= (x - y) \wedge [(y - x) \vee (x \wedge y)] \\ &= [(x - y) \wedge (y - x)] \vee [(x - y) \wedge (x \wedge y)] = 0, \end{aligned}$$

and

$$\begin{aligned} (x - y) \vee y &= (x - y) \vee (y - x) \vee (x \wedge y) \\ &= [(x - y) \vee (x \wedge y)] \vee [(y - x) \vee (y \wedge x)] = x \vee y, \end{aligned}$$

in view of (3.12) and the two last identities in 1°. The proof of the third property in 2° was given also in [9] (see the proof of Property 5 there).

To prove the assertions in 4° and 5° assume that  $x, y, z \in X$  and  $x \leq y$ . We have

$$(z - y) \wedge x \leq (z - y) \wedge y = 0, \quad (3.14)$$

by  $(l_8)$  and the second identity in  $2^\circ$ , and hence

$$z - y = [(z - y) - x] \vee [(z - y) \wedge x] = (z - y) - x, \quad (3.15)$$

by the third identity in  $1^\circ$ . The assertions in  $4^\circ$  follow from (3.14) and (3.15).

Further,  $z \wedge x \leq z \wedge y \leq z$  and  $z - y \leq z$ , according to our assumption and due to  $(l_6)$ ,  $(l_8)$  and the first property in  $1^\circ$ , so we may write

$$z - y = (z - y) \wedge z = [z \ominus (z \wedge y)] \wedge [(z \ominus (z \wedge x)) \vee (z \wedge x)], \quad (3.16)$$

in view of  $(l_7)$  and Proposition 7. But, since  $z \wedge x \leq z \wedge y$ , we have

$$[z \ominus (z \wedge y)] \wedge (z \wedge x) = [z - (z \wedge y)] \wedge (z \wedge x) = 0,$$

by the first property in  $4^\circ$ , already proved. Therefore, in view of the distributivity condition  $(D)$ , the equations in (3.16) yield

$$z - y = [z \ominus (z \wedge y)] \wedge [(z \ominus (z \wedge x))] = (z - y) \wedge (z - x),$$

which means, due to  $(l_7)$ , that the assertion in  $5^\circ$  is proved (see also the proof of Property 10 in [9]).

The two assertions in  $6^\circ$  are consequences of Definition 11 and Proposition 9. Namely, we conclude from them the identity

$$\begin{aligned} z - (x \vee y) &= z \ominus [z \wedge (x \vee y)] = z \ominus [(z \wedge x) \vee (z \wedge y)] \\ &= [z \ominus (z \wedge x)] \wedge [z \ominus (z \wedge y)] = (z - x) \wedge (z - y), \end{aligned}$$

using additionally the distributivity property  $(D)$ , and the identity

$$\begin{aligned} z - (x \wedge y) &= z \ominus [z \wedge (x \wedge y)] = z \ominus [(z \wedge x) \wedge (z \wedge y)] \\ &= [z \ominus (z \wedge x)] \vee [z \ominus (z \wedge y)] = (z - x) \vee (z - y), \end{aligned}$$

using here, in addition, properties  $(l_2) - (l_4)$ .

Complete proofs of the three remaining properties formulated in  $7^\circ$  and  $8^\circ$  are given in [9] (see Properties 13-15).  $\square$

We will need also some additional properties of the operation of difference in  $B$ -rings.

**Proposition 15.** *Let  $(X, \wedge, \vee, \ominus, 0)$  be a  $B$ -ring. Then*

$$[(x - y) \vee (y - x)] \wedge (x \wedge y) = 0 \quad (3.17)$$

$$(x - y) \wedge [z - (x \vee y)] = 0, \quad (y - x) \wedge [z - (x \vee y)] = 0, \quad (3.18)$$

$$(x - y) \wedge (x \wedge y \wedge z) = 0, \quad (y - x) \wedge (x \wedge y \wedge z) = 0 \quad (3.19)$$

for any  $x, y, z \in X$ .

**Proof.** Applying (D), ( $l_6$ ), ( $l_8$ ) and the second property in part 2° of Proposition 14, we get

$$[(x - y) \vee (y - x)] \wedge (x \wedge y) \leq [(x - y) \wedge y] \vee [(y - x) \wedge x] = 0,$$

i.e. equality (3.17) is shown.

In view of property ( $l_3$ ), it remains to prove the first equalities in (3.18) and (3.19). We have  $x - y \leq x$  and  $z - (x \vee y) \leq z - x$ , by the first assertion in 1° and by 5° from Proposition 14. Therefore, using ( $l_6$ ) and ( $l_2$ ), we get

$$(x - y) \wedge [z - (x \vee y)] \leq x \wedge (z - x) = 0,$$

and, similarly,

$$(x - y) \wedge (x \wedge y \wedge z) \leq [x - (x \wedge y)] \wedge (x \wedge y) = 0,$$

due to the second assertion in part 2° of Proposition 14. The required equalities are proved and the proof of the proposition is completed.  $\square$

## 4. Properties of $\Delta$ -join-semilattices

**Definition 12.** We say that a 0-join-semilattice  $(X, \vee, 0)$  is *meet-invariant* if the following implication holds:

$$(I_m) \quad \text{if } \inf\{x, y\} = 0, \text{ then } \inf\{x \vee z, y \vee z\} = z \quad \text{for any } x, y, z \in X.$$

**Proposition 16.** *Each distributive 0-lattice  $(X, \wedge, \vee, 0)$  is meet-invariant.*

**Proof.** Suppose that  $x, y \in X$  and  $\inf\{x, y\} = 0$ . Then, from the distributivity law ( $D'$ ), we have

$$\inf\{x \vee z, y \vee z\} = (x \vee z) \wedge (y \vee z) = (x \wedge y) \vee z = z$$

for every  $z \in X$ , i.e. condition ( $I_m$ ) is fulfilled.  $\square$

**Definition 13.** We say that a 0-join-semilattice  $(X, \vee, 0)$  is a  $\Delta$ -join-semilattice if it is meet-invariant and satisfies the following condition:

$$(\Delta) \quad \forall_{x, y \in X} \exists_{z \in X} \quad \inf\{y, z\} = 0 \quad \text{and} \quad y \vee z = x \vee y.$$

In particular, condition ( $\Delta$ ) in any 0-join-semilattice  $(X, \vee, 0)$  yields:

$$(\Delta_0) \quad \forall_{x, y \in X, y \leq x} \exists_{z \in X} \quad \inf\{y, z\} = 0 \quad \text{and} \quad y \vee z = x.$$

**Proposition 17.** *If  $(X, \vee, 0)$  is a  $\Delta$ -join-semilattice, then for every pair of elements  $x, y \in X$  there exists a unique  $z$  such that the equalities in condition ( $\Delta$ ) are satisfied. In particular, a similar uniqueness holds in case of ( $\Delta_0$ ).*

**Proof.** Fix  $x, y \in X$ . Suppose that there exist elements  $z_1, z_2$  in  $X$  satisfying condition  $(\Delta)$ , that is:

$$\inf\{y, z_i\} = 0 \quad \text{and} \quad y \vee z_i = x \vee y \quad (i = 1, 2).$$

Hence, by condition  $(I_m)$  we have

$$z_1 = \inf\{y \vee z_1, z_2 \vee z_1\} = \inf\{y \vee z_2, z_1 \vee z_2\} = z_2,$$

which proves that the assertion is true.  $\square$

**Definition 14.** Let  $(X, \vee, 0)$  be a  $\Delta$ -join-semilattice and let  $x, y$  be arbitrary elements of  $X$ . Then by a *difference* of  $x$  and  $y$  we mean an element  $z$  of  $X$  satisfying the identities in  $(\Delta)$ .

A difference of any elements  $x$  and  $y$  in an arbitrary  $\Delta$ -join-semilattice  $(X, \vee, 0)$  exists and is unique (see condition  $(\Delta)$  and Proposition 17). We denote the difference of elements  $x$  and  $y$  by  $x \searrow y$ , i.e.

$$x \searrow y = z \quad \text{if} \quad \inf\{y, z\} = 0 \quad \text{and} \quad y \vee z = x \vee y. \quad (4.20)$$

If  $(X, \vee, 0)$  is a  $\Delta$ -join-semilattice, then formula (4.20) determines the binary operation  $\searrow$  of difference in  $X$ . In particular, if  $y \leq x$  we call  $x \searrow y$  the *proper difference* of  $x$  and  $y$ , i.e.

$$x \searrow y = z \quad (y \leq x) \quad \text{if} \quad \inf\{y, z\} = 0 \quad \text{and} \quad y \vee z = x. \quad (4.21)$$

**Remark 4.** Though the operation  $\searrow$  of difference does not formally appear in Definition 13 it is uniquely determined, due to conditions  $(\Delta)$ ,  $(I_m)$  and Proposition 17, via formula (4.20). Therefore we will use in the sequel the notation  $(X, \vee, \searrow, 0)$  for  $\Delta$ -join-semilattices, i.e. for those 0-join-semilattices  $(X, \vee, 0)$  which are meet-invariant and satisfy condition  $(\Delta)$ .

**Lemma 1.** *If  $(X, \vee, \searrow, 0)$  is a  $\Delta$ -join-semilattice, then*

$$x \searrow y \leq x \quad (4.22)$$

for any  $x, y \in X$ .

**Proof.** Let  $z := x \searrow y$ . In view of Definition 14 the element  $z$  satisfies condition  $(\Delta)$ , i.e. the following equalities hold:

$$\inf\{y, z\} = 0 \quad \text{and} \quad y \vee z = x \vee y. \quad (4.23)$$

Then, by  $(I_m)$  and (4.23), we have

$$x = \inf\{y \vee x, z \vee x\} = \inf\{y \vee z, x \vee z\}. \quad (4.24)$$

On the other hand, in view of  $(j_6)$ , we have the two relations:

$$z \leq y \vee z \quad \text{and} \quad z \leq x \vee z,$$

which imply, due to (4.24), that

$$z \leq \inf\{y \vee z, x \vee z\} = x.$$

The relation (4.22) is proved.  $\square$

**Lemma 2.** *If  $(X, \vee, \searrow, 0)$  is a  $\Delta$ -join-semilattice, then  $x \searrow (x \searrow y) \leq y$  for any  $x, y \in X$ .*

**Proof.** Let  $z := x \searrow y$  and  $u := x \searrow (x \searrow y) = x \searrow z$ . By Lemma 1, we have  $z \leq x$  and thus  $x \searrow z$  is a proper difference. Hence, by Definition 14 (see (4.20) and (4.21)), we obtain the equalities

$$\inf\{y, z\} = 0 \quad \text{and} \quad y \vee z = x \vee y \tag{4.25}$$

as well as the identities

$$\inf\{z, u\} = 0 \quad \text{and} \quad z \vee u = x. \tag{4.26}$$

Then, in view of (4.25), (4.26),  $(j_3)$  and  $(j_5)$  we have

$$y \vee z = x \vee y = (z \vee u) \vee y = (y \vee z) \vee u.$$

Hence, by  $(j_7)$ , we get  $u \leq y \vee z$ . On the other hand, we have  $u \leq u \vee y$ , by  $(j_6)$ . The last two relations, the first equality in (4.26) and condition  $(I_m)$  imply

$$u \leq \inf\{z \vee y, u \vee y\} = y$$

and the assertion is shown.  $\square$

**Lemma 3.** *Let  $(X, \vee, \searrow, 0)$  be a  $\Delta$ -join-semilattice and  $x, y, u \in X$ . If  $u \leq x \vee y$  and  $\inf\{x, u\} = 0$ , then  $u \leq y$ .*

**Proof.** Since  $u \leq x \vee y$  and  $u \leq u \vee y$ , by  $(j_6)$ , we have

$$u \leq \inf\{x \vee y, u \vee y\} = y$$

in view of condition  $(I_m)$  and the assumption that  $\inf\{x, u\} = 0$ .  $\square$

**Lemma 4.** *Let  $(X, \vee, \searrow, 0)$  be a  $\Delta$ -join-semilattice and  $x, y, u \in X$ . If  $u \leq x$  and  $u \leq y$ , then  $\inf\{u, x \searrow y\} = 0$ .*

**Proof.** Let  $z := x \searrow y$ . In view of Definition 14, the element  $z$  satisfies condition  $(\Delta)$ , i.e. the following equalities hold:

$$\inf\{y, z\} = 0 \quad \text{and} \quad y \vee z = x \vee y. \tag{4.27}$$

Let  $u \leq x$ ,  $u \leq y$  and let  $v \leq u$ ,  $v \leq x \searrow y = z$ . Since  $v \leq u \leq y$  and  $v \leq z$ , we have

$$v \leq \inf\{y, z\} = 0,$$

by (4.27). Hence  $v = 0$ . Consequently,  $\inf\{u, x \searrow y\} = 0$ .  $\square$

**Lemma 5.** *Let  $(X, \vee, \searrow, 0)$  be a  $\Delta$ -join-semilattice and  $x, y, u \in X$ . If  $u \leq x$  and  $u \leq y$ , then  $u \leq x \searrow (x \searrow y)$ .*

**Proof.** Let  $y_1 := x \searrow y$  and  $z_1 := x \searrow (x \searrow y)$ . By Lemma 1, we have  $y_1 \leq x$ , so  $x \searrow y_1$  is a proper difference. By Definition 14 (see (4.21)), we get

$$\inf\{y_1, z_1\} = 0 \quad \text{and} \quad y_1 \vee z_1 = x. \quad (4.28)$$

Since  $u \leq x$ , we have

$$u \leq y_1 \vee z_1, \quad (4.29)$$

due to the second equality in (4.28). Since  $u \leq x$  and  $u \leq y$ , we have

$$\inf\{u, y_1\} = \inf\{u, x \searrow y\} = 0 \quad (4.30)$$

in view of Lemma 4. Hence, by Lemma 3, conditions (4.29) and (4.30) imply

$$u \leq z_1 = x \searrow (x \searrow y). \quad \square$$

**Proposition 18.** *Let  $(X, \vee, \searrow, 0)$  be a  $\Delta$ -join-semilattice. Then*

$$\inf\{x, y\} = x \searrow (x \searrow y) \in X, \quad (4.31)$$

*i.e.  $\inf\{x, y\}$  exists in  $X$  for any  $x, y \in X$ . Moreover, if  $\wedge$  is a binary operation on  $X$  given by*

$$x \wedge y := \inf\{x, y\} \quad \text{for } x, y \in X, \quad (4.32)$$

*then  $(X, \wedge, \vee, 0)$  is a 0-lattice satisfying conditions (R) and  $(I_m)$ .*

**Proof.** Fix arbitrarily  $x, y \in X$ . In view of Lemma 1 and 2, we have

$$x \searrow (x \searrow y) \leq x \quad \text{and} \quad x \searrow (x \searrow y) \leq y,$$

so  $x \searrow (x \searrow y)$  is a lower bound of elements  $x, y$ . Due to Lemma 5, the element  $x \searrow (x \searrow y)$  is the greatest lower bound of elements  $x, y$ , which means that the equality in (4.31) holds and so  $\inf\{x, y\}$  exists in  $X$ .

Consequently, formula (4.32) well defines the binary operation  $\wedge$  on  $X$  and  $(X, \wedge, \vee, 0)$  is a 0-lattice which satisfies condition (R). The latter results from the fact that (R) is an equivalent form of condition  $(\Delta_0)$  which is a particular case of condition  $(\Delta)$ , assumed for the given  $\Delta$ -join-semilattice  $(X, \vee, \searrow, 0)$ .

**Remark 5.** Notice that condition  $(I_m)$  can be now expressed in the following equivalent form:

$$(I'_m) \quad (x \vee z) \wedge (y \vee z) = z, \quad \text{whenever } x \wedge y = 0 \quad \text{for any } x, y, z \in X.$$

**Proposition 19.** *Let  $(X, \vee, \searrow, 0)$  be a  $\Delta$ -join-semilattice. Then the binary operation  $\wedge$  given by (4.32) is distributive, i.e.*

$$(x \vee z) \wedge (y \vee z) = (x \wedge y) \vee z \quad (4.33)$$

for arbitrary  $x, y, z \in X$ . Consequently  $(X, \wedge, \vee, \ominus, 0)$  is a B-ring, where the partial binary operation  $\ominus$  is given by

$$x \ominus y := x \searrow y$$

for arbitrary  $x, y \in X$  such that  $y \leq x$ .

**Proof.** Let  $x, y, z$  be arbitrary elements of  $X$  and denote

$$u := x \wedge y. \quad (4.34)$$

By (4.31) and (4.32) in Proposition 18, it follows that  $u \in X$ . Clearly, we have

$$u \leq x, \quad u \leq y, \quad x \searrow u \leq x, \quad y \searrow u \leq y, \quad (4.35)$$

where  $x \searrow u$  and  $y \searrow u$  are proper differences, by (4.34) and Lemma 1. Moreover,

$$\inf\{u, x \searrow u\} = 0 = \inf\{u, y \searrow u\}, \quad (4.36)$$

by the first equality in  $(\Delta_0)$ .

To prove the identity

$$\inf\{x \searrow u, y \searrow u\} = 0 \quad (4.37)$$

suppose that  $v \leq x \searrow u$  and  $v \leq y \searrow u$ . Hence, by the last two relations in (4.35), we have  $v \leq x$  and  $v \leq y$ , so  $v \leq \inf\{x, y\} = u$ , in view of (4.34). But then (4.36) implies that  $v = 0$ , which proves equality (4.37).

On the other hand, by the second equality in  $(\Delta_0)$ , we have

$$x = u \vee (x \searrow u) \quad \text{and} \quad y = u \vee (y \searrow u). \quad (4.38)$$

Hence, by (4.38), (4.37) and  $(I_m)$ , we have

$$\begin{aligned} (x \vee z) \wedge (y \vee z) &= [(x \searrow u) \vee (u \vee z)] \wedge [(y \searrow u) \vee (u \vee z)] \\ &= u \vee z = (x \wedge y) \vee z, \end{aligned}$$

according to  $(j_3)$ ,  $(j_5)$  and the notation in (4.34), i.e. the distributivity law (4.33) is proved.  $\square$

## 5. Duality theorems

**Theorem 3.** *Assume that  $(X, \vee, \searrow, 0)$  is a  $\Delta$ -join-semilattice, where the binary operation  $\searrow$  is given by (4.20), according to conditions  $(I_m)$  and  $(\Delta)$ . Then  $(X, \wedge, \vee, \ominus, 0)$  is a B-ring with the binary lattice operation  $\wedge$ , given by*

$$x \wedge y := \inf\{x, y\} = x \searrow (x \searrow y) \quad \text{for } x, y \in X,$$



and the partial binary operation  $\ominus$  of proper difference, given by

$$x \ominus y := x \setminus y \quad \text{for } x, y \in X, y \leq x.$$

Conversely, assume that  $(X, \wedge, \vee, \ominus, 0)$  is a  $B$ -ring, where the partial binary operation  $\ominus$  of proper difference is given by (3.3), according to condition (R). Then  $(X, \vee, \setminus, 0)$  is a  $\Delta$ -join-semilattice with the binary operation  $\setminus$  given, according to  $(I_m)$ ,  $(\Delta)$  and (4.20), by

$$x \setminus y := x \ominus (x \wedge y) \quad \text{for } x, y \in X.$$

**Proof.** That a given  $\Delta$ -join-semilattice  $(X, \vee, \setminus, 0)$  is a  $B$ -ring follows directly from Propositions 18 and 19.

To show the second part of the assertion it suffices to notice that conditions  $(I_m)$  and  $(\Delta)$  in a given  $B$ -ring  $(X, \wedge, \vee, \ominus, 0)$  follow from Propositions 16 and 12, respectively, because the extension  $(\bar{R})$  of condition (R) is an equivalent form of condition  $(\Delta)$ .

The proof of Theorem 3 is completed.  $\square$

**Definition 15.** By an *algebraic ring*  $(X, +, \cdot)$  we mean a nonempty set  $X$  endowed with two binary operations:  $+$  (addition) and  $\cdot$  (multiplication) such that  $(X, +)$  is an Abelian group and the multiplication is associative and distributive with respect to the addition. By a *commutative ring* we mean an algebraic ring  $(X, +, \cdot)$  in which the operation  $\cdot$  is commutative.

**Definition 16.** By a *Boolean ring* we mean an algebraic ring  $(X, +, \cdot)$  in which the operation  $\cdot$  of the multiplication is idempotent, i.e.  $x^2 = x$  for every  $x \in X$ . By a *Boolean ring with unit* we mean a Boolean ring in which there is a unique neutral element of the multiplication.

**Remark 6.** It follows from Lemma 8 below that every Boolean ring is commutative.

Let  $(X, \wedge, \vee, \ominus, 0)$  be a  $B$ -ring. Define in  $(X, \wedge, \vee, \ominus, 0)$  the two binary operations  $+$  and  $\cdot$  as follows:

$$x + y := (x \vee y) \ominus (x \wedge y) = (x \vee y) - (x \wedge y) \quad (5.39)$$

and

$$x \cdot y := x \wedge y. \quad (5.40)$$

for arbitrary  $x, y \in X$ .

**Remark 7.** In any  $B$ -ring  $(X, \wedge, \vee, \ominus, 0)$ , the binary operation  $+$  on  $X$  can be defined in two ways by formula (5.39), because  $x \wedge y \leq x \vee y$  and thus the difference  $u - v$  and the proper difference  $u \ominus v$  of  $u := x \vee y$  and  $v := x \wedge y$  coincide, by Proposition 12, for any  $x, y \in X$ .

Due to our extension in Definition 11 of the partial binary operation  $\ominus$  of proper difference to the binary operation  $-$  of difference in  $B$ -rings, the above definition of the operation  $+$  can be expressed in an equivalent form, given by formula (5.41) below. In fact, it follows from Proposition 20 proved below that formulae (5.39) and (5.41) stand for two equivalent definition of the binary operation  $+$ .

Formulae (5.39) and (5.41) correspond to the known representations of symmetric difference of two sets.

**Proposition 20.** *Let  $(X, \wedge, \vee, \ominus, 0)$  be a  $B$ -ring. Then*

$$x + y = (x - y) \vee (y - x) \quad (5.41)$$

for arbitrary  $x, y \in X$ , where  $x + y$  is given by formula (5.39).

**Proof.** Fix  $x, y \in X$  and put  $z := (x - y) \vee (y - x)$ . We have to prove that  $z$  coincides with  $x + y$ , i.e. that  $z = a \ominus b$ , where  $a := x \vee y$  and  $b := x \wedge y$ . According to Definition 7 and Remark 2 it suffices to verify that the respective equations in condition  $(R)$  hold, i.e. we have the following identities:  $z \wedge b = 0$  and  $z \vee b = a$ . But they have been already proved in Proposition 15 in the form of (3.17) and in part 3° of Proposition 14.  $\square$

Now we will show some properties of a  $B$ -ring needed in the proof of Theorem 4.

**Lemma 6.** *Let  $(X, \wedge, \vee, \ominus, 0)$  be a  $B$ -ring. Then*

$$z - (x + y) = [z - (x \vee y)] \vee (x \wedge y \wedge z) \quad (5.42)$$

for arbitrary  $x, y, z \in X$ , where  $x + y$  is given by formula (5.39).

**Proof.** We will use in the proof condition  $(\bar{R})$ , a generalization of condition  $(R)$  proved in Proposition 12.

Fix  $x, y, z \in X$  and denote  $a := x - y$  and  $b := y - x$ , i.e.

$$x + y = a \vee b, \quad (5.43)$$

due to Proposition 20.

According to Proposition 12 it is enough to show the two equalities:

$$(x + y) \wedge c = 0 \quad \text{and} \quad (x + y) \vee c = (x + y) \vee z, \quad (5.44)$$

where

$$c := [z - (x \vee y)] \vee (x \wedge y \wedge z).$$

Using definition (5.39) of  $x + y$ , we get

$$(x + y) \wedge [z - (x \vee y)] = 0, \quad (5.45)$$

in view of the distributivity law  $(D)$  as well as both equalities in (3.18), proved in Proposition 15. In a similar way the equalities in (3.19) imply that

$$(x + y) \wedge (x \wedge y \wedge z) = 0. \quad (5.46)$$

From (5.45) and (5.46) again using (D), we obtain the first equality in (5.44).

To prove the second equality in (5.44), notice first that

$$(x + y) \vee c \leq (x + y) \vee z, \quad (5.47)$$

since  $z - (x \vee y) \leq z$  and  $x \wedge y \wedge z \leq z$ , by part 1° of Proposition 14 and (l<sub>6</sub>).

On the other hand, we have

$$x \vee y = a \vee b \vee (x \wedge y), \quad a \wedge z \leq a, \quad b \wedge z \leq b, \quad (5.48)$$

by property 3° in Proposition 14 and (l<sub>6</sub>). In view of (5.48) and (D), we have

$$\begin{aligned} (x \vee y) \wedge z &\leq [(a \vee b) \vee (x \wedge y)] \wedge z \leq (a \vee b) \vee (x \wedge y \wedge z) \\ &= (x + y) \vee (x \wedge y \wedge z), \end{aligned}$$

according to (l<sub>8</sub>) and our notation (5.43). Hence, by the third identity in part 1° in Proposition 14, we have

$$\begin{aligned} z &= [z \wedge (x \vee y)] \vee [z - (x \vee y)] \\ &\leq (x + y) \vee (x \wedge y \wedge z) \vee [z - (x \vee y)] = (x + y) \vee c \end{aligned}$$

and consequently,

$$(x + y) \vee z \leq (x + y) \vee c, \quad (5.49)$$

by (l<sub>2</sub>) and (l<sub>8</sub>). Now the second identity in (5.44) follows from relations (5.47) and (5.49).  $\square$

**Lemma 7.** *Let  $(X, \wedge, \vee, \ominus, 0)$  be a B-ring. Then*

$$(x + y) + z = x + (y + z) \quad (5.50)$$

for arbitrary  $x, y, z \in X$ .

**Proof.** By Proposition 20 and both identities in 7° in Proposition 14, we have

$$\begin{aligned} (x + y) - z &= [(x - y) \vee (y - x)] - z \\ &= [(x - y) - z] \vee [(y - x) - z] \\ &= [x - (y \vee z)] \vee [y - (x \vee z)] \end{aligned}$$

and thus, by (5.41) and (5.42), we obtain

$$\begin{aligned} (x + y) + z &= [(x + y) - z] \vee [z - (x + y)] \\ &= [x - (y \vee z)] \vee [y - (x \vee z)] \\ &\quad \vee [z - (x \vee y)] \vee (x \wedge y \wedge z) \end{aligned} \quad (5.51)$$

for arbitrary  $x, y, z \in X$ .

Replacing  $x$  by  $y$ ,  $y$  by  $z$  and  $z$  by  $x$  in (5.51), we get

$$(y + z) + x = [y - (z \vee x)] \vee [z - (y \vee x)] \vee [x - (y \vee z)] \vee (y \wedge z \wedge x). \quad (5.52)$$

We see that the right hand sides of (5.51) and (5.52) coincide due to  $(l_3)$ - $(l_5)$ , and hence (5.50) is true for all  $x, y, z \in X$ .  $\square$

We will prove the following theorem:

**Theorem 4.** *If  $(X, \wedge, \vee, \ominus, 0)$  is a  $B$ -ring, then  $(X, +, \cdot)$  is a Boolean ring with the operations  $+$  and  $\cdot$  given by (5.39) and (5.40), i.e.*

$$x + y := (x \vee y) \ominus (x \wedge y) = (x \vee y) - (x \wedge y)$$

and

$$x \cdot y := x \wedge y.$$

for all  $x, y \in X$  and the neutral element of the operation  $+$  in  $(X, +, \cdot)$  coincides with the above  $0$ , the order zero in  $(X, \wedge, \vee, \ominus, 0)$ .

**Proof.** Let  $(X, \wedge, \vee, \ominus, 0)$  be a given  $B$ -ring. By (5.41) and  $(l_3)$ , we have  $x + y = y + x$  for  $x, y \in X$ , i.e. the operation  $+$  is commutative and its associativity was proved in Lemma 7.

In view of (5.39) and  $(l_0)$ , we get

$$x + 0 = (x \vee 0) \ominus (x \wedge 0) = x \ominus 0 = x \quad \text{for } x \in X,$$

by the first equality in (3.10) in Proposition 13. Hence, the element  $0 = \inf X$  from the given  $B$ -ring is the neutral element (the unique  $0$  satisfying  $x + 0 = x$  for all  $x \in X$ ) of the operation  $+$  of addition. By equality (5.41) in Proposition 20 and by the second equality in (3.10) in Proposition 13, we have

$$x + x = (x - x) \vee (x - x) = 0 \vee 0 = 0 \quad \text{for } x \in X,$$

due to  $(l_0)$  or  $(l_2)$ , i.e. every element  $x$  is the inverse to itself, with respect to the operation  $+$  of addition in  $(X, +, \cdot)$ .

In view of  $(l_3)$ , we have

$$x \cdot y = x \wedge y = y \wedge x = y \cdot x \quad \text{for } x, y \in X,$$

i.e. the multiplication  $\cdot$  is commutative in  $(X, +, \cdot)$ . We have

$$x \cdot x = x \wedge x = x \quad \text{for } x \in X,$$

by  $(l_2)$ , so the multiplication  $\cdot$  is idempotent in  $(X, +, \cdot)$ . Moreover, by  $(l_4)$ , we get

$$(x \cdot y) \cdot z = (x \wedge y) \wedge z = x \wedge (y \wedge z) = x \cdot (y \cdot z) \quad \text{for } x, y, z \in X,$$

which proves that the multiplication  $\cdot$  is associative in  $(X, +, \cdot)$ .

In view of Proposition 20, part 8° of Proposition 14 and (D), we get

$$\begin{aligned} x \cdot z + y \cdot z &= [(x \wedge z) - (y \wedge z)] \vee [(y \wedge z) - (x \wedge z)] \\ &= [(x - y) \wedge z] \vee [(y - x) \wedge z] \\ &= [(x - y) \vee (y - x)] \wedge z = (x + y) \cdot z \end{aligned}$$

for arbitrary  $x, y, z \in X$ . The last equality implies that the operation  $+$  of addition is distributive with respect to the operation  $\cdot$  of multiplication.

The proof is completed.  $\square$

The following assertion follows directly from Theorems 3 and 4.

**Theorem 5.** *If  $(X, \vee, \searrow, 0)$  is a  $\Delta$ -join-semilattice, then  $(X, +, \cdot)$  is a Boolean ring with  $+$  and  $\cdot$  given by*

$$x + y := (x \vee y) \searrow [x \searrow (x \searrow y)] \quad \text{and} \quad x \cdot y := x \searrow (x \searrow y)$$

for all  $x, y \in X$  and the neutral element of the operation  $+$  in  $(X, +, \cdot)$  coincides with the above 0, the order zero in  $(X, \vee, \searrow, 0)$ .

Now assume that  $(X, +, \cdot)$  is a Boolean ring (with the neutral element 0 of the operation  $+$  of addition) and define in  $X$  the relation  $\leq$  in the following way:

$$x \leq y \quad :\Leftrightarrow \quad x \cdot y = x. \quad (5.53)$$

We will prove that  $(X, \leq)$  is a poset, satisfying conditions (L) and (L0), i.e. a 0-lattice with the lattice operations  $\wedge$  and  $\vee$  defined as in Definition 3:

$$x \wedge y := \inf\{x, y\} \quad \text{and} \quad x \vee y := \sup\{x, y\} \quad \text{for } x, y \in X, \quad (5.54)$$

with

$$0 = \inf X, \quad (5.55)$$

where 0 is the mentioned above neutral element of the operation  $+$  in the Boolean ring  $(X, +, \cdot)$ . Moreover, we will show that this 0-lattice is distributive and satisfies condition (R), i.e. it is a B-ring, with the partial binary operation  $\ominus$  of proper difference introduced according to Definition 7, and the identities

$$x \wedge y = x \cdot y \quad \text{and} \quad x \vee y = x + y + x \cdot y \quad (5.56)$$

are satisfied for all  $x, y \in X$ .

We recall certain algebraic properties of groups and Boolean rings that we need in the proof of Theorem 6.

**Lemma 8.** *If  $(X, +)$  is a group, then  $x + y = x$  implies  $y = 0$  for any  $x, y \in X$ . If  $(X, +, \cdot)$  is a Boolean ring, then it is commutative and  $x + x = 0$  for every  $x \in X$ .*

**Proof.** In a given group  $(X, +)$ , fix elements  $x$  and  $y$  such that  $x + y = x$ . Then

$$y = 0 + y = ((-x) + x) + y = (-x) + (x + y) = (-x) + x = 0,$$

which proves the first part of the lemma.

Since in a given Boolean ring  $(X, +, \cdot)$  the multiplication is idempotent, we have

$$x + y = (x + y)^2 = x^2 + y \cdot x + x \cdot y + y^2 = x + y + y \cdot x + x \cdot y$$

for arbitrary  $x, y \in X$  and hence, by the first part of the lemma,

$$y \cdot x + x \cdot y = 0 \quad \text{for } x, y \in X. \quad (5.57)$$

Putting  $y = x$  in (5.57), we get  $0 = x^2 + x^2 = x + x$  for each  $x \in X$ , which means that the last part of the lemma is true. Hence, by (5.57),

$$y \cdot x = -x \cdot y = x \cdot y \quad \text{for } x, y \in X,$$

which shows that the ring  $(X, +, \cdot)$  is commutative, as stated in the middle part of the lemma. Thus the proof is completed.  $\square$

**Theorem 6.** *If  $(X, +, \cdot)$  is a Boolean ring, then  $(X, \wedge, \vee, \ominus, 0)$  is a B-ring in a poset  $(X, \leq)$ , where  $\leq$  is defined by (5.53), the lattice operations  $\wedge$  and  $\vee$  are given by (5.54), the order zero  $0$  coincides with the neutral element of the addition in  $(X, +, \cdot)$  and the operation  $\ominus$ , determined by condition (R), is given by*

$$x \ominus y := x + y \quad \text{for } x, y \in X \quad \text{such that } y \leq x.$$

Moreover, the lattice operations  $\wedge$  and  $\vee$  defined in (5.54) are connected with the ring operations  $+$  and  $\cdot$  in the given  $(X, +, \cdot)$  by means of formulae in (5.56).

**Proof.** Let  $(X, +, \cdot)$  be a Boolean ring. We start with proving that the relation  $\leq$  defined in (5.53) is a partial order, i.e.  $(X, \leq)$  is a poset. In the proof of this fact as well as of properties of the operations  $+$  and  $\cdot$  defined by (5.54) we will often apply the definition of  $\leq$  given in (5.53).

By the assumption, we have  $x^2 = x$  and so  $x \leq x$  for any  $x \in X$ , in view of (5.53). If  $x \leq y$  and  $y \leq x$ , then  $x \cdot y = x$  and  $y \cdot x = y$ , according to (5.53), so  $x = y$  for any  $x, y \in X$ . Assume that  $x \leq y$  and  $y \leq z$  for given  $x, y, z \in X$ . By (5.53), we have  $x \cdot y = x$  and  $y \cdot z = y$ , so

$$x \cdot z = (x \cdot y) \cdot z = x \cdot (y \cdot z) = x \cdot y = x,$$

which means that  $x \leq z$ , in view of (5.53) again. Consequently,  $(X, \leq)$  is a poset.

If  $0$  denotes the neutral element of the addition  $+$  in the given Boolean ring, then  $0 \cdot x = 0$  and so, by (5.53), we have  $0 \leq x$  for any  $x \in X$ . On the other hand, fix  $a \in X$  such that  $a \leq x$  and so  $a \cdot x = a$  for all  $x \in X$ , due to (5.53). In particular, we have  $a = a \cdot 0 = 0$ . Hence  $0$  is the greatest lower bound of  $X$  and equality (5.55) holds.

Fix arbitrarily  $x, y \in X$  and denote

$$u := x + y + x \cdot y \quad \text{and} \quad v := x \cdot y. \quad (5.58)$$

By associativity, commutativity and idempotency of the operation  $\cdot$  of multiplication, we get

$$v \cdot x = (x \cdot y) \cdot x = x^2 \cdot y = x \cdot y = v \quad \text{and} \quad v \cdot y = x \cdot y^2 = x \cdot y = v,$$

i.e.  $v \leq x$  and  $v \leq y$ , according to (5.53). But if  $a \leq x$  and  $a \leq y$ , i.e.  $a \cdot x = a$  and  $a \cdot y = a$ , then we obtain

$$a \cdot v = a^2 \cdot (x \cdot y) = (a \cdot x) \cdot (a \cdot y) = a \cdot a = a,$$

due to the mentioned properties again, which means that  $a \leq v$ , according to (5.53). This shows that  $\inf\{x, y\} = v \in X$ .

Now, applying distributivity of  $+$  with respect to  $\cdot$  and other properties of these operations, we obtain

$$x \cdot u = x^2 + x \cdot y + x^2 \cdot y = x + x \cdot y + x \cdot y = x$$

and

$$y \cdot u = x \cdot y + y^2 + x \cdot y^2 = y + x \cdot y + x \cdot y = y,$$

in view of Lemma 8. Hence, by (5.53), we have

$$x \leq u \quad \text{and} \quad y \leq u. \quad (5.59)$$

On the other hand, if  $x \leq a$  and  $y \leq a$ , then  $x \cdot a = x$  and  $y \cdot a = y$ , so

$$a \cdot u = a \cdot x + a \cdot y + (a \cdot x) \cdot y = x + y + x \cdot y = u,$$

i.e.  $u \leq a$ , again due to (5.53). This and (5.59) mean that  $\sup\{x, y\} = u \in X$ .

We have thus shown that  $\inf\{x, y\}$  and  $\sup\{x, y\}$  exist in  $X$  and moreover, according to the notation introduced in (5.54) and (5.58), the equalities in (5.56) hold for all  $x, y \in X$ . Consequently, the poset  $(X, \leq)$  is a 0-lattice with the lattice operations  $\wedge$  and  $\vee$ , given by (5.54) and (5.56), and the order zero equal to the neutral element 0 of the operation  $+$  of additivity in  $(X, +, \cdot)$ . We may denote it now by  $(X, \wedge, \vee, 0)$ . In view of (5.56) and properties of the operations  $+$  and  $\cdot$  in  $(X, +, \cdot)$ , we have

$$\begin{aligned} x \wedge (y \vee z) &= x \cdot (y + z + y \cdot z) = x \cdot y + x \cdot z + x \cdot y \cdot z \\ &= x \cdot y + x \cdot z + (x \cdot y) \cdot (x \cdot z) = (x \wedge y) \vee (x \wedge z) \end{aligned}$$

for arbitrary  $x, y, z \in X$ , which proves that the 0-lattice  $(X, \wedge, \vee, 0)$  is distributive.

Fix now elements  $x, y \in X$  such that  $y \leq x$  and put  $z := x + y$ . By (5.56), Lemma 8 and properties of the operations  $+$  and  $\cdot$  in  $(X, +, \cdot)$ , we get

$$z \wedge y = z \cdot y = (x + y) \cdot y = x \cdot y + y^2 = y + y = 0,$$

and

$$\begin{aligned} z \vee y &= (x + y) + y + (x + y) \cdot y = x + (y + y) + x \cdot y + y^2 \\ &= x + (y + y) + (y + y) = x, \end{aligned}$$

which means that  $z$  satisfies the equalities in condition (R). Consequently the 0-lattice  $(X, \wedge, \vee, 0)$  is a  $B$ -ring in which the operation  $\ominus$  is defined by  $x \ominus y = z$ , where  $z := x + y$  for any  $x, y \in X$  such that  $y \leq x$ .  $\square$

Theorems 3 and 6 imply the following assertion:

**Theorem 7.** *If  $(X, +, \cdot)$  is a Boolean ring, then  $(X, \vee, \searrow, 0)$  is a  $\Delta$ -join-semilattice in a poset  $(X, \leq)$ , where  $\leq$  is defined by (5.53), the operation  $\vee$  is given by*

$$x \vee y := x + y + x \cdot y \quad \text{for } x, y \in X,$$

*the order zero 0 coincides with the neutral element of the addition in  $(X, +, \cdot)$  and the binary operation  $\searrow$ , determined by conditions  $(\Delta)$  and  $(I_m)$ , is given by*

$$x \searrow y := x + x \cdot y \quad \text{for } x, y \in X.$$

**Remark 8.** Notice that in Theorems 4 and 5 we do not assume that the order unit  $1 = \sup X$  exists in  $(X, \leq)$ ; consequently, the algebraic ring  $(X, +, \cdot)$  in the assertions may not contain a unit, a neutral element of the operation  $\cdot$  of multiplication. Similarly, in Theorems 6 and 7 we do not assume that the algebraic ring  $(X, +, \cdot)$  contains a neutral element of the multiplication  $\cdot$  and, consequently, the order unit  $\sup X$  may not exist in  $(X, \leq)$ . Adding the respective assumptions we get the respective particular cases of the above theorems.

**Acknowledgement.** We express our thanks to Professor A. Kamiński for his inspiration and advices and to the referees for very valuable suggestions which allowed us to improve the text of our paper.

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**DOI: 10.7862/rf.2013.4**

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*Received 12.03.2013, Revised 25.08.2013, Accepted 25.10.2013*

## New univalence criterions for special general integral operators

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ABSTRACT: In this work we consider some integral operators on the special subclasses of the set of analytic functions in the unit disc which are defined by the Hadamard product. Using the univalence criterions, we obtain new sufficient conditions for these operators to be univalent in the open unit disk. We give some applications of the main results.

AMS Subject Classification: *Primary 30C45; Secondary 30C80*

Keywords and Phrases: *Hadamard product, integral operator, univalent functions*

### 1. Introduction

Let  $\mathcal{H}$  denote the class of analytic functions in the unit disc  $\mathbb{D} = \{z : |z| < 1\}$  on the complex plane  $\mathbb{C}$ . Let  $\mathcal{A}$  denote the class of all functions  $f \in \mathcal{H}$  normalized by  $f(0) = 0$ ,  $f'(0) = 1$ . Let  $\mathcal{S}$  be the subclass of  $\mathcal{A}$  whose members are univalent in  $\mathbb{D}$ . Recall that a set  $E \subset \mathbb{C}$  is said to be starlike with respect to a point  $w_0 \in E$  if and only if the linear segment joining  $w_0$  to every other point  $w \in E$  lies entirely in  $E$ , while a set  $E$  is said to be convex if and only if it is starlike with respect to each of its points, that is if and only if the linear segment joining any two points of  $E$  lies entirely in  $E$ . The set of all functions  $f \in \mathcal{A}$  that are starlike univalent in  $\mathbb{D}$  will be denoted by  $\mathcal{S}^*$ . The set of all functions  $f \in \mathcal{A}$  that are convex univalent in  $\mathbb{D}$  by  $\mathcal{K}$ . Robertson introduced in [6] the classes  $\mathcal{S}^*(\alpha)$ ,  $\mathcal{K}(\alpha)$  of starlike and convex functions of order  $\alpha \leq 1$ , which are defined by

$$\mathcal{S}^*(\alpha) := \left\{ f \in \mathcal{A} : \Re \left\{ \frac{zf'(z)}{f(z)} \right\} > \alpha, \quad \text{for all } z \in \mathbb{D} \right\}, \quad (1.1)$$

$$\mathcal{K}(\alpha) := \left\{ f \in \mathcal{A} : \Re \left\{ 1 + \frac{zf''(z)}{f'(z)} \right\} > \alpha, \quad \text{for all } z \in \mathbb{D} \right\}. \quad (1.2)$$

If  $\alpha \in [0; 1)$ , then a function in either of these sets is univalent, if  $\alpha < 0$  it may fail to be univalent. In particular we denote  $\mathcal{S}^*(0) = \mathcal{S}^*$ ,  $\mathcal{K}(0) = \mathcal{K}$ . For functions  $f, g \in \mathcal{H}$  given by

$$f(z) = \sum_{n=1}^{\infty} a_n z^n, \quad g(z) = \sum_{n=1}^{\infty} b_n z^n; \quad (|z| < 1), \quad (1.3)$$

their Hadamard product (or convolution) is defined by:

$$(f * g)(z) = \sum_{n=1}^{\infty} a_n b_n z^n; \quad (|z| < 1). \quad (1.4)$$

For a function  $g \in \mathcal{H}$  we define the subclass  $\mathcal{S}^*(g; a, b, \lambda)$  of functions  $f \in \mathcal{H}$  satisfying the condition:

$$\left| \frac{z^\lambda (f * g)'(z)}{(f * g)^\lambda(z)} - a \right| < b; \quad (z \in \mathbb{D}, |a - 1| < b \leq a, \lambda \geq 1), \quad (1.5)$$

such that  $(f * g)(z) \neq 0$ . If  $f \in \mathcal{A}$ ,  $g(z) = z/(1 - z)$  ( $|z| < 1$ ) and  $\lambda = 1$  we have  $(f * g)(z) = f(z)$  and so:

$$\mathcal{S}^*(g; a, b, \lambda) = \mathcal{S}_0^*(a, b) = \left\{ f \in \mathcal{A} : \left| \frac{z f'(z)}{f(z)} - a \right| < b, z \in \mathbb{D}, |a - 1| < b \leq a \right\},$$

where the class  $\mathcal{S}_0^*(a, b)$  was introduced and studied by Jakubowski in 1972 (see [4]). Note that

$$\mathcal{S}_0^*(a, b) \subseteq \mathcal{S}_0^*(a - b) \subseteq \mathcal{S}^*(0) = \mathcal{S}^* \subseteq \mathcal{S}.$$

Recently Deniz, Raducanu and Orhan [2] defined the following general integral operator:

$$F_{\alpha, \beta}(z) = \left\{ \beta \int_0^z t^{\beta-1} \prod_{i=1}^n \left( \frac{(f_i * g_i)(t)}{h_i(t)} \right)^{\alpha_i} dt \right\}^{\frac{1}{\beta}}; \quad (\alpha_i, \beta \in \mathbb{C}, z \in \mathbb{D}), \quad (1.6)$$

where  $f_i, g_i, h_i \in \mathcal{A}$ ,  $\Re(\beta) > 0$ ,  $\alpha = (\alpha_1, \dots, \alpha_n)$  and  $(f_i * g_i)(z)/h_i(z) \neq 0$ . Note that all powers in (1.6) are principal ones.

Using the convolution, we introduce the following integral operator:

$$G_{\alpha, \beta}(z) = \int_0^z \prod_{i=1}^n ((f_i * g_i)'(t))^{\alpha_i} \left( \frac{(f_i * g_i)(t)}{h_i(t)} \right)^{\beta_i} dt; \quad (\alpha_i, \beta_i \in \mathbb{C}, z \in \mathbb{D}), \quad (1.7)$$

where  $\alpha = (\alpha_1, \dots, \alpha_n)$ ,  $\beta = (\beta_1, \dots, \beta_n)$  and  $(f_i * g_i)(z)/h_i(z) \neq 0$ .

**Remark 1.1** It is useful to see that the integral operators  $F_{\alpha, \beta}(z)$  and  $G_{\alpha, \beta}(z)$  extend some operators defined by many authors, for example:

1) If  $f_1 = \dots = f_n = f$ ,  $g_1 = \dots = g_n = z/(1-z)$ ,  $h_1 = \dots = h_n = z$ ,  $\alpha_1 = \dots = \alpha_n = \alpha$ ,  $\beta_1 = \dots = \beta_n = \beta$  and  $n = 1$ , then we obtain the following integral operators:

$$F_{\alpha,\beta}(z) = \left\{ \beta \int_0^z t^{\beta-1} \left( \frac{f(t)}{t} \right)^\alpha dt \right\}^{\frac{1}{\beta}}$$

and

$$G_{\alpha,\beta}(z) = \int_0^z (f'(t))^\alpha \left( \frac{f(t)}{t} \right)^\beta dt.$$

2) For  $g_1 = \dots = g_n = \frac{z}{(1-z)^2}$ ,  $h_1 = \dots = h_n = z$ , we obtain the integral operators:

$$F_{\alpha,\beta}(z) = \left\{ \beta \int_0^z t^{\beta-1} \prod_{i=1}^n (f'_i(t))^{\alpha_i} dt \right\}^{\frac{1}{\beta}}$$

and

$$G_{\alpha,\beta}(z) = \int_0^z \prod_{i=1}^n (f'_i(t) + t f''_i(t))^{\alpha_i} (f'_i(t))^{\beta_i} dt.$$

In this paper we give new sufficient conditions for the operators  $F_{\alpha,\beta}(z)$  and  $G_{\alpha,\beta}(z)$  to be univalent in  $\mathbb{D}$ , where the functions  $f_i$  belong to the class  $\mathcal{S}^*(g_i; a_i, b_i, \lambda_i)$  for all  $i = 1, \dots, n$ . In order to get our main results we will use the following lemmas, so we recall them here.

**Lemma 1.1** [5] *Let  $\beta \in \mathbb{C}$  with  $\Re(\beta) > 0$ . If  $f \in \mathcal{A}$  satisfies:*

$$\frac{1 - |z|^{2\Re(\beta)}}{\Re(\beta)} \left| \frac{z f''(z)}{f'(z)} \right| \leq 1, \quad (z \in \mathbb{D}),$$

then the function

$$F_\beta(z) = \left\{ \beta \int_0^z t^{\beta-1} f'(t) dt \right\}^{\frac{1}{\beta}} \tag{1.8}$$

is univalent in  $\mathbb{D}$ .

## 2. Main Results

Using the previous lemmas, we state and prove the following:

**Theorem 2.1** *Let  $f_i \in \mathcal{S}^*(g_i; a_i, b_i, \lambda_i)$ ,  $g_i, h_i \in \mathcal{A}$  for all  $i = 1, \dots, n$  and let  $|(f_i * g_i)(z)| < M$  for all  $z \in \mathbb{D}$  and  $M > 0$ . Assume also that  $|z h'_i(z)/h_i(z)| \leq 1$ . If  $c \in \mathbb{C} \setminus \{-1\}$  and  $\beta$  with  $\Re(\beta) > 0$*

$$\Re(\beta) \geq \left( \sum_{i=1}^n |\alpha_i| (1 + (a_i + b_i) M^{\lambda_i - 1}) \right), \quad (\alpha_i \in \mathbb{C}),$$

then the function defined by (1.6) is univalent in  $\mathbb{D}$ .

**Proof.** Define the function:

$$\phi(z) = \int_0^z \prod_{i=1}^n \left( \frac{(f_i * g_i)(t)}{h_i(t)} \right)^{\alpha_i} dt, \quad (\alpha_i \in \mathbb{C}, z \in \mathbb{D}).$$

Then we have  $\phi(0) = 0, \phi'(0) = 1$  and:

$$\phi'(z) = \prod_{i=1}^n \left( \frac{(f_i * g_i)(z)}{h_i(z)} \right)^{\alpha_i}, \quad (2.1)$$

also  $\phi(z)$  is analytic in  $\mathbb{D}$ . From (2.1) we obtain:

$$\frac{z\phi''(z)}{\phi'(z)} = \sum_{i=1}^n \alpha_i \left( \frac{z(f_i * g_i)'(z)}{(f_i * g_i)(z)} - \frac{zh_i'(z)}{h_i(z)} \right), \quad (z \in \mathbb{D}). \quad (2.2)$$

Since  $f_i \in \mathcal{S}^*(g_i; a_i, b_i, \lambda_i)$  and  $|(f_i * g_i)(z)| < M$ , by the well-known Schwarz lemma in complex analysis, we see that:

$$\begin{aligned} \left| \frac{z\phi''(z)}{\phi'(z)} \right| &= \left| \sum_{i=1}^n \alpha_i \left( \frac{z(f_i * g_i)'(z)}{(f_i * g_i)(z)} - \frac{zh_i'(z)}{h_i(z)} \right) \right| \\ &\leq \sum_{i=1}^n |\alpha_i| \left( \left| \frac{z^{\lambda_i} (f_i * g_i)'(z)}{(f_i * g_i)^{\lambda_i}(z)} \right| \left| \frac{(f_i * g_i)(z)}{z} \right|^{\lambda_i-1} + \left| \frac{zh_i'(z)}{h_i(z)} \right| \right) \\ &\leq \sum_{i=1}^n |\alpha_i| (1 + (a_i + b_i)M^{\lambda_i-1}). \end{aligned}$$

Now the last inequality shows that:

$$\begin{aligned} \frac{1 - |z|^{2\Re(\beta)}}{\Re(\beta)} \left| \frac{z\phi''(z)}{\phi'(z)} \right| &\leq \frac{1}{\Re(\beta)} \left| \frac{z\phi''(z)}{\phi'(z)} \right| \\ &\leq \frac{1}{\Re(\beta)} \left( \sum_{i=1}^n |\alpha_i| (1 + (a_i + b_i)M^{\lambda_i-1}) \right) \\ &\leq 1. \end{aligned}$$

Applying Lemma 1.1 for the function  $\phi(z)$ , we conclude that  $F_{\alpha,\beta}(z) \in \mathcal{S}$ . ■

Letting  $a_1 = b_1 = n = \lambda_1, f_1 = f, g_1 = z/(1-z)^2$  and  $h_1(z) = z$  in Theorem 2.1, we obtain the following result.

**Corollary 2.1** *Let  $f \in \mathcal{A}, |f(z)| < M$  for all  $z \in \mathbb{D}$  and  $|zf_i'(z)/f_i(z)| \leq 1$ . If  $\alpha, \beta \in \mathbb{C}$  with  $\Re(\beta) \geq 3|\alpha| > 0$ , then the function:*

$$F_{\alpha,\beta}(z) = \left\{ \beta \int_0^z t^{\beta-1} (f'(t))^\alpha dt \right\}^{\frac{1}{\beta}},$$

*is univalent in  $\mathbb{D}$ .*

We next give some sufficient conditions for the operator  $G_{\alpha,\beta}(z)$  to be univalent in  $\mathbb{D}$ .

**Theorem 2.2** *Let for  $i = 1, \dots, n, g_i, h_i \in \mathcal{A}, f_i \in \mathcal{A} \cap \mathcal{S}^*(g_i; a_i, b_i, \lambda_i)$  and:*

$$|(f_i * g_i)(z)| < M; \quad (z \in \mathbb{D}, M > 0).$$

*If  $f'_i \in \mathcal{S}^*(g_i(z)/z; a'_i, b'_i, 1)$  and  $|zh'_i(z)/h_i(z)| \leq 1$  for all  $z \in \mathbb{D}, i = 1, \dots, n$ , also:*

$$\sum_{i=1}^n |\alpha_i|(a'_i + b'_i) + |\beta_i| (1 + (a_i + b_i)M^{\lambda_i-1}) \leq 1,$$

*then the function  $G_{\alpha,\beta}(z)$  defined by (1.7) is univalent in  $\mathbb{D}$ .*

**Proof.** Define the function  $\phi(z)$  by:

$$\phi(z) = G_{\alpha,\beta}(z) = \int_0^z \prod_{i=1}^n ((f_i * g_i)'(t))^{\alpha_i} \left( \frac{(f_i * g_i)(t)}{h_i(t)} \right)^{\beta_i} dt; \quad (\alpha_i, \beta_i \in \mathbb{C}, z \in \mathbb{D}),$$

then we have:

$$\phi'(z) = \prod_{i=1}^n ((f_i * g_i)'(z))^{\alpha_i} \left( \frac{(f_i * g_i)(z)}{h_i(z)} \right)^{\beta_i},$$

$\phi(0) = 0, \phi'(0) = 1$  and  $\phi(z)$  is analytic in  $\mathbb{D}$ . Differentiating logarithmically from  $\phi'(z)$ , we obtain:

$$\frac{z\phi''(z)}{\phi'(z)} = \sum_{i=1}^n \alpha_i \frac{z(f_i * g_i)''(z)}{(f_i * g_i)'(z)} + \beta_i \left( \frac{z(f_i * g_i)'(z)}{(f_i * g_i)(z)} - \frac{zh'_i(z)}{h_i(z)} \right).$$

Because  $f_i, g_i \in \mathcal{A}$ , we see that:

$$\frac{(f_i * g_i)''(z)}{(f_i * g_i)'(z)} = \frac{(f'_i * (g_i/z))'(z)}{(f'_i * (g_i/z))(z)}.$$

Now, by suppositions of theorem, we find that:

$$\begin{aligned} \left| \frac{z\phi''(z)}{\phi'(z)} \right| &\leq \sum_{i=1}^n \left\{ |\alpha_i| \left| \frac{z(f'_i * (g_i/z))'(z)}{(f'_i * (g_i/z))(z)} \right| + \right. \\ &\quad \left. + |\beta_i| \left( \left| \frac{z^{\lambda_i}(f_i * g_i)'(z)}{(f_i * g_i)^{\lambda_i}(z)} \right| \cdot \left| \frac{(f_i * g_i)(z)}{z} \right|^{\lambda_i-1} + \left| \frac{zh'_i(z)}{h_i(z)} \right| \right) \right\} \\ &\leq \sum_{i=1}^n |\alpha_i|(a'_i + b'_i) + |\beta_i| (1 + (a_i + b_i)M^{\lambda_i-1}). \end{aligned}$$

Using Lemma 1.1 and the last inequality, we conclude that  $G_{\alpha,\beta}(z) \in \mathcal{S}$ . ■

Taking  $a_i = b_i = \lambda_i = a'_i = b'_i = 1, f_i = f, g_i = z/(1-z)^2$  and  $h_i(z) = z$  for  $i = 1, \dots, n$ , in Theorem 2.2, we obtain the following result.



**Corollary 2.2** Let  $f \in \mathcal{A}$ ,  $|zf''(z)/f'(z)| < 1$  and  $|zf'(z)| < M$  with  $M > 0$  for all  $z \in \mathbb{D}$ . If:

$$\left| \frac{z(2f'' + zf''')}{f' + zf''} - 1 \right| < 1 \quad (z \in \mathbb{D}),$$

and  $\sum_{i=1}^n 2|\alpha_i| + 3|\beta_i| \leq 1$ , for  $\alpha_i, \beta_i \in \mathbb{C}$ , then the function:

$$G_{\alpha, \beta}(z) = \int_0^z (f'(t) + tf''(t))^{n\alpha_i} (f'(t))^{n\beta_i} dt,$$

is univalent in  $\mathbb{D}$ .

**Proof.** Because  $g_i(z) = \frac{z}{(1-z)^2} = g(z)$ , it is easy to see that,  $(f * g)(z) = zf'(z)$ , also  $f' \in \mathcal{S}^*(g(z)/z; 1, 1, 1)$  if and only if:

$$\left| \frac{z(2f'' + zf''')}{f' + zf''} - 1 \right| < 1.$$

This completes the proof. ■

Using the proof of theorem (2.2), we obtain another result:

**Corollary 2.3** Suppose all conditions of Theorem 2.2 are satisfied, then the function  $G_{\alpha, \beta}(z)$  defined by (1.7) is in  $\mathcal{K}(\gamma)$ , where:

$$\gamma = 1 - \sum_{i=1}^n |\alpha_i|(a'_i + b'_i) + |\beta_i| (1 + (a_i + b_i)M^{\lambda_i - 1}).$$

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**DOI: 10.7862/rf.2013.5**

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*Received 12.03.2013, Revised 30.07.2013, Accepted 25.10.2013*

## Convolution properties of subclasses of analytic functions associated with the Dziok-Srivastava operator

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ABSTRACT: The aim of this paper is to introduce two new classes of analytic function by using principle of subordination and the Dziok-Srivastava operator. We further investigate convolution properties for these classes. We also find necessary and sufficient condition and coefficient estimate for them.

AMS Subject Classification: *30C45*

Keywords and Phrases: *analytic function; Hadmard product; starlike function; convex function; subordination and Dziok-Srivastava operator.*

### 1. Introduction

Let  $\mathcal{A}$  denote the class of analytic functions of the form

$$f(z) = z + \sum_{k=2}^{\infty} a_k z^k, \quad (1.1)$$

which are analytic in the open unit disk  $\mathbb{U} = \{z \in \mathbb{C} : |z| < 1\}$ . Let  $\mathcal{S}^*(\alpha)$  and  $\mathcal{K}(\alpha)$  ( $0 \leq \alpha < 1$ ) denote the subclasses of  $\mathcal{A}$  that consists, respectively, of starlike of order  $\alpha$  and convex of order  $\alpha$  in the disk  $\mathbb{U}$ . It is well known that  $\mathcal{S}^*(\alpha) \subset \mathcal{S}^*(0) = \mathcal{S}^*$  and  $\mathcal{K}(\alpha) = \mathcal{K}(0) = \mathcal{K}$ .

If  $f(z)$  and  $g(z)$  are analytic in  $\mathbb{U}$ , we say that  $f(z)$  is subordinate to  $g(z)$ , written  $f(z) \prec g(z)$  if there exists a Schwarz function  $\omega$ , which by definition is analytic in  $\mathbb{U}$  with  $\omega(0) = 0$  and  $|\omega(z)| < 1$ , such that  $f(z) = g(\omega(z))$ , for all  $z \in \mathbb{U}$ . Furthermore,

if the function  $g(z)$  is univalent in  $\mathbb{U}$ , then we have the following equivalence :

$$f(z) \prec g(z) \iff f(0) = g(0) \text{ and } f(\mathbb{U}) \subset g(\mathbb{U}).$$

For the function  $f(z)$  given by (1.1) and  $g(z)$  given by

$$g(z) = z + \sum_{k=2}^{\infty} b_k z^k \quad (1.2)$$

the Hadamard product or convolution of  $f(z)$  and  $g(z)$  is defined by

$$(f * g)(z) = z + \sum_{k=2}^{\infty} a_k b_k z^k = (g * f)(z) \quad (1.3)$$

Making use of principle of subordination between analytic functions. We introduce the subclasses  $\mathcal{S}^*[\lambda, \phi]$  and  $\mathcal{K}[\lambda, \phi]$  of the class  $\mathcal{A}$  for  $-1 \leq \lambda \leq 1$  which are defined by

$$\mathcal{S}^*[\lambda, \phi] = \left\{ f \in \mathcal{A} : \frac{zf'(z)}{[(1-\lambda)f(z) + \lambda zf'(z)]} \prec \phi(z) \quad (z \in \mathbb{U}) \right\} \quad (1.4)$$

and

$$\mathcal{K}[\lambda, \phi] = \left\{ f \in \mathcal{A} : \frac{zf''(z) + f'(z)}{[f'(z) + \lambda zf''(z)]} \prec \phi(z) \quad (z \in \mathbb{U}) \right\} \quad (1.5)$$

For complex parameters  $a_1, \dots, a_q; b_1, \dots, b_s$  ( $b_j \notin \mathbb{Z}_0^- = \{0, -1, -2, \dots\}; j = 1, \dots, s$ ), we define the generalized hypergeometric function  ${}_qF_s(a_1, \dots, a_i, \dots, a_q; b_1, \dots, b_s; z)$  by [12] the following infinite series:

$${}_qF_s(a_1, \dots, a_i, \dots, a_q; b_1, \dots, b_s; z) = \sum_{k=0}^{\infty} \frac{(a_1)_k \dots (a_q)_k}{(b_1)_k \dots (b_s)_k} \frac{z^k}{k!} \quad (1.6)$$

$$(q \leq s + 1; q, s \in \mathbb{N}_0 = \mathbb{N} \cup \{0\}; z \in \mathbb{U}),$$

where  $(\alpha)_k$  is Pochhammer symbol defined by

$$(\alpha)_k = \begin{cases} 1 & \text{for } k = 0 \\ \alpha(\alpha + 1) \dots (\alpha + k - 1) & \text{for } k \in \mathbb{N} \end{cases}$$

Dziok and Srivastava [4] considered a linear operator  $H(a_1, \dots, a_q; b_1, \dots, b_s) : \mathcal{A} \rightarrow \mathcal{A}$  defined by the following Hadamard product:

$$H(a_1, \dots, a_q; b_1, \dots, b_s)f(z) = h(a_1, \dots, a_q; b_1, \dots, b_s; z) * f(z) \quad (1.7)$$

where

$$h(a_1, \dots, a_i, \dots, a_q; b_1, \dots, b_s; z) = z {}_qF_s(a_1, \dots, a_q; b_1, \dots, b_s; z)$$

$$(q \leq s + 1; q, s \in \mathbb{N}_0; z \in \mathbb{U}).$$

If  $f(z) \in \mathcal{A}$  is given by (1.1), then we have

$$H(a_1, \dots, a_q; b_1, \dots, b_s)f(z) = z + \sum_{k=2}^{\infty} \Gamma_k[a_1; b_1] a_k z^k, \quad (1.8)$$

where

$$\Gamma_k[a_1; b_1] = \frac{(a_1)_{k-1} \dots (a_q)_{k-1}}{(b_1)_{k-1} \dots (b_s)_{k-1} (k-1)!} \quad (1.9)$$

The Dziok-Srivastava linear operator  $H_{q,s}[a_1; b_1]$  includes various other operators, which were considered in earlier works. We can quote here for example linear operators introduced by Carlson and Shaffer, Bernardi, Libera and Livingston, Choi, Saigo and Srivastava, Kim and Srivastava, Srivastava and Owa, Cho, Kwon and Srivastava, Ruscheweyh, Hohlov, Salagean, Noor, and others (see for details [8], [9] and []).

In recent years, many interesting subclasses of analytic functions associated with the Dziok-Srivastava operator  $H_{q,s}[a_1; b_1]$  and its many special cases were investigated by, for example, Murugusundaramoorthy and Magesh [7], Srivastava et al. ([13],[14]), Wang et al. [15] and others.

In this paper, we investigate convolution properties of the classes  $\mathcal{S}^*[a_1; \lambda, \phi]$  and  $\mathcal{K}[a_1; \lambda, \phi]$  associated with the operator  $H_{q,s}[a_1; b_1]$ . Using convolution properties, we find the necessary and sufficient condition and coefficient estimate for these classes.

## 2. Convolution properties

We assume that  $0 < \theta < 2\pi$ ,  $-1 \leq \lambda \leq 1$  throughout this section and  $\Gamma_k[a_1; b_1]$  is defined by (1.9)

**Theorem 1.** *The function  $f(z)$  defined by (1.1) is in the class  $\mathcal{S}^*[\lambda, \phi]$  if and only if.*

$$\frac{1}{z} \left[ f(z) * \frac{z - \frac{(\lambda-1)\phi(e^{i\theta})}{1-\phi(e^{i\theta})} z^2}{(1-z)^2} \right] \neq 0 \quad (z \in \mathbb{U}, 0 < \theta < 2\pi) \quad (2.1)$$

*Proof.* A function  $f(z)$  is in the class  $\mathcal{S}^*[\lambda, \phi]$  if and only if

$$\frac{zf'(z)}{[(1-\lambda)f(z) + \lambda zf'(z)]} \neq \phi(e^{i\theta}) \quad (z \in \mathbb{U}, 0 < \theta < 2\pi) \quad (2.2)$$

which is equivalent to

$$\begin{aligned} zf'(z) &\neq \phi(e^{i\theta}) [(1-\lambda)f(z) + \lambda zf'(z)], \\ \frac{1}{z} [zf'(z) [1 - \lambda\phi(e^{i\theta})] - (1-\lambda)\phi(e^{i\theta})f(z)] &\neq 0. \end{aligned} \quad (2.3)$$

Since

$$f(z) = f(z) * \frac{1}{(1-z)} \quad \text{and} \quad zf'(z) = f(z) * \frac{1}{(1-z)^2},$$

The equation (2.3) can be written as

$$\begin{aligned} \frac{1}{z} \left[ f(z) * \left( (1-\lambda\phi(e^{i\theta})) \frac{z}{(1-z)^2} - (1-\lambda)\phi(e^{i\theta}) \frac{z}{1-z} \right) \right] \\ = \frac{1-\phi(e^{i\theta})}{z} \left[ f(z) * \frac{z - ((\lambda-1)\phi(e^{i\theta})/(1-\phi(e^{i\theta})))z^2}{(1-z)^2} \right] \neq 0, \quad (0 < \theta < 2\pi). \end{aligned} \quad (2.4)$$

this completes the proof of Theorem 1.

**Theorem 2.** *The function  $f(z)$  defined by (1.1) is in the class  $\mathcal{K}[\lambda, \phi]$  if and only if.*

$$\frac{1}{z} \left[ f(z) * \frac{z - ((1 + (1 - 2\lambda)\phi(e^{i\theta}))/(\phi(e^{i\theta}) - 1))z^2}{(1 - z)^3} \right] \neq 0, (z \in \mathbb{U}). \quad (2.5)$$

*Proof.* Let us take

$$g(z) = \frac{z - ((\lambda - 1)\phi(e^{i\theta}))/(\phi(e^{i\theta}) - 1))z^2}{(1 - z)^2}, \quad (2.6)$$

from which we get

$$zg'(z) = \frac{z - ((1 + (1 - 2\lambda)\phi(e^{i\theta}))/(\phi(e^{i\theta}) - 1))z^2}{(1 - z)^3} \quad (0 < \theta < 2\pi). \quad (2.7)$$

Also from the identity  $zf'(z) * g(z) = f(z) * zg'(z)$ , ( $f, g \in \mathcal{A}$ ) and the fact that

$$f(z) \in \mathcal{K}[\lambda, \phi] \iff zf'(z) \in \mathcal{S}^*[\lambda, \phi].$$

the result (2.5) follows from Theorem 1.

**Theorem 3.** *A necessary and sufficient condition for the function  $f(z)$  defined by (1.1) to be in the class  $\mathcal{S}_{q,s}^*[a_1; \lambda, \phi]$  is that.*

$$1 + \sum_{k=2}^{\infty} \frac{(1 - \lambda)\phi(e^{i\theta}) + k(\lambda\phi(e^{i\theta}) - 1)}{\phi(e^{i\theta}) - 1} \Gamma_k[a_1, b_1] a_k z^{k-1} \neq 0 \quad (z \in \mathbb{U}, 0 < \theta < 2\pi) \quad (2.8)$$

*Proof.* From Theorem 1, we can say that  $f(z) \in \mathcal{S}_{q,s}^*[a_1, \lambda, \phi]$  if and only if

$$\frac{1}{z} \left[ H_{q,s}[a_1, b_1] f(z) * \frac{z - ((\lambda - 1)\phi(e^{i\theta}))/(\phi(e^{i\theta}) - 1))z^2}{(1 - z)^2} \right] \neq 0, \quad (z \in \mathbb{U}, 0 < \theta < 2\pi). \quad (2.9)$$

From (1.8), the left hand side of (2.9) can be written as

$$\frac{1}{z} \left[ H_{q,s}[a_1, b_1] f(z) * \left( \frac{z}{(1 - z^2)} - \frac{(1 - \lambda)\phi(e^{i\theta})}{\phi(e^{i\theta}) - 1} \frac{z^2}{(1 - z)^2} \right) \right], \quad (0 < \theta < 2\pi). \quad (2.10)$$

$$= \frac{1}{z} [z(H_{q,s}(a_1, b_1))f(z)' - \frac{(1 - \lambda)\phi(e^{i\theta})}{\phi(e^{i\theta}) - 1} \{z(H_{q,s}(a_1, b_1))f(z)' - (H_{q,s}(a_1, b_1))f(z)\}]. \quad (2.11)$$

$$= 1 + \sum_{k=2}^{\infty} \frac{(1 - \lambda)\phi(e^{i\theta}) + k(\lambda\phi(e^{i\theta}) - 1)}{\phi(e^{i\theta}) - 1} \Gamma_k[a_1, b_1] a_k z^{k-1}, \quad (0 < \theta < 2\pi). \quad (2.12)$$

Thus the proof is completed.

**Theorem 4.** A necessary and sufficient condition for the function  $f(z)$  defined by (1.1) to be in the class  $\mathcal{K}_{q,s}[a_1; \lambda, \phi]$  is that

$$1 + \sum_{k=2}^{\infty} k \frac{(1-\lambda)\phi(e^{i\theta}) + k(\lambda\phi(e^{i\theta}) - 1)}{\phi(e^{i\theta}) - 1} \Gamma_k[a_1, b_1] a_k z^{k-1} \neq 0, \quad (z \in \mathbb{U}, 0 < \theta < 2\pi) \quad (2.13)$$

*Proof.* From Theorem 1, we find that  $f(z) \in \mathcal{K}_{q,s}[a_1; \lambda, \phi]$  if and only if

$$\frac{1}{z} \left[ H_{q,s}[a_1, b_1] f(z) * \frac{z - ((1 + (1 - 2\lambda)\phi(e^{i\theta})) / (\phi(e^{i\theta}) - 1) z^2)}{(1 - z)^3} \right] \neq 0, \quad (z \in \mathbb{U}). \quad (2.14)$$

Using the definition (1.8), the above equation can be written as

$$\begin{aligned} & \frac{1}{z} \left[ H_{q,s}[a_1, b_1] f(z) * \left( \frac{z}{(1 - z)^3} - \frac{(1 + (1 - 2\lambda)\phi(e^{i\theta}))}{(\phi(e^{i\theta}) - 1)} \frac{z}{(1 - z)^3} \right) \right] \\ &= \frac{1}{z} \left[ \frac{z}{2} (z H_{q,s}[a_1, b_1] f(z))'' - \frac{(1 + (1 - 2\lambda)\phi(e^{i\theta}))}{2(\phi(e^{i\theta}) - 1)} z^2 (H_{q,s}[a_1, b_1] f(z))'' \right] \\ &= 1 + \sum_{k=2}^{\infty} k \frac{(1-\lambda)\phi(e^{i\theta}) + k(\lambda\phi(e^{i\theta}) - 1)}{\phi(e^{i\theta}) - 1} \Gamma_k[a_1, b_1] a_k z^{k-1} \end{aligned} \quad (2.15)$$

which proves the Theorem.

**Theorem 5.** If the function  $f(z)$  defined by (1.1) belongs to  $\mathcal{S}_{q,s}^*[a_1; \lambda, \phi]$  then

$$\sum_{k=2}^{\infty} (1 - \lambda) |\phi(e^{i\theta})| - |(\lambda\phi(e^{i\theta}) - 1)k| \Gamma_k[a_1, b_1] |a_k| \leq |1 - \phi(e^{i\theta})| \quad (2.16)$$

*Proof.* Since

$$\begin{aligned} & \left| 1 - \sum_{k=2}^{\infty} \frac{(1-\lambda)\phi(e^{i\theta}) + k(\lambda\phi(e^{i\theta}) - 1)}{1 - \phi(e^{i\theta})} \Gamma_k[a_1, b_1] a_k z^{k-1} \right| \quad (z \in \mathbb{U}) \\ & \geq 1 - \sum_{k=2}^{\infty} \left| \frac{(1-\lambda)\phi(e^{i\theta}) + k(\lambda\phi(e^{i\theta}) - 1)}{1 - \phi(e^{i\theta})} \right| \Gamma_k[a_1, b_1] |a_k| \\ & \Rightarrow \sum_{k=2}^{\infty} ((1 - \lambda) |\phi(e^{i\theta})| - |(\lambda\phi(e^{i\theta}) - 1)k| \Gamma_k[a_1, b_1] |a_k|) \leq |1 - \phi(e^{i\theta})| \end{aligned}$$

**Theorem 6.** If the function  $f(z)$  defined by (1.1) belongs to  $\mathcal{K}_{q,s}[a_1; \lambda, \phi]$  then

$$\sum_{k=2}^{\infty} ((1 - \lambda) |\phi(e^{i\theta})| - |(\lambda\phi(e^{i\theta}) - 1)k| \Gamma_k[a_1, b_1] |a_k|) \leq |1 - \phi(e^{i\theta})| \quad (2.17)$$

**Remark.** Putting  $\phi(e^{i\theta}) = \frac{1+Ae^{i\theta}}{1+Be^{i\theta}}$  and  $\lambda = 0$  in theorems 1 to 6, we get the results given recently by Aouf and Seoudy [2], Some of the results by Aouf and Seoudy also

contain the result due to Silverman ([10], [11]) and Ahuja [1].

**Acknowledgments:** The first author (S P G) is thankful to CSIR, New Delhi, India for awarding Emeritus Scientist under scheme No. 21(084)/10/EMR-II. The authors are also thankful to the anonymous reviewer for his/her useful comments.

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**DOI: 10.7862/rf.2013.6**

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*Received 22.06.2012, Revised 31.10.2013, Accepted 25.10.2013*

## Supra $b$ -compact and supra $b$ -Lindelöf spaces

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ABSTRACT: In this paper we introduce the notion of supra  $b$ -compact spaces and investigate its several properties and characterizations. Also we introduce and study the notion of supra  $b$ -Lindelöf spaces.

AMS Subject Classification: 54D20

Keywords and Phrases:  $b$ -open sets, supra  $b$ -open sets, supra  $b$ -compact spaces and supra  $b$ -Lindelöf spaces

### 1. Introduction and preliminaries

In 1983, A. S. Mashhour et al. [3] introduced the supra topological spaces. In 1996, D. Andrijevic [1] introduced and studied a class of generalized open sets in a topological space called  $b$ -open sets. This type of sets discussed by El-Atike [2] under the name of  $\gamma$ -open sets. In 2010, O. R. Sayed et al. [4] introduced and studied a class of sets and maps between topological spaces called supra  $b$ -open sets and supra  $b$ -continuous functions respectively. Now we introduce the concepts of supra  $b$ -compact and supra  $b$ -Lindelöf spaces and investigate several properties for these concepts.

Throughout this paper  $(X, \tau)$ ,  $(Y, \rho)$  and  $(Z, \sigma)$  (or simply  $X$ ,  $Y$  and  $Z$ ) denote topological spaces on which no separation axioms are assumed unless explicitly stated. For a subset  $A$  of  $(X, \tau)$ , the closure and the interior of  $A$  in  $X$  are denoted by  $Cl(A)$  and  $Int(A)$ , respectively. The complement of  $A$  is denoted by  $X - A$ . In the space  $(X, \tau)$ , a subset  $A$  is said to be  $b$ -open [1] if  $A \subseteq Cl(Int(A)) \cup Int(Cl(A))$ . The family of all  $b$ -open sets of  $(X, \tau)$  is denoted by  $BO(X)$ . A subcollection  $\mu \subseteq 2^X$  is called a supra topology [3] on  $X$  if  $X \in \mu$  and  $\mu$  is closed under arbitrary union.  $(X, \mu)$  is called a supra topological space. The elements of  $\mu$  are said to be supra open in  $(X, \mu)$  and the complement of a supra open set is called a supra closed set. The supra closure of a set  $A$ , denoted by  $Cl^\mu(A)$ , is the intersection of all supra closed sets including  $A$ . The supra interior of a set  $A$ , denoted by  $Int^\mu(A)$ , is the union of all supra open sets included in  $A$ . The supra topology  $\mu$  on  $X$  is associated with the topology  $\tau$  if  $\tau \subseteq \mu$ .

**Definition 1.1** [4] Let  $(X, \mu)$  be a supra topological space. A set  $A$  is called a supra  $b$ -open set if  $A \subseteq Cl^\mu(Int^\mu(A)) \cup Int^\mu(Cl^\mu(A))$ . The complement of a supra  $b$ -open set is called a supra  $b$ -closed set.

**Theorem 1.2** [4]. (i) Arbitrary union of supra  $b$ -open sets is always supra  $b$ -open.

(ii) Finite intersection of supra  $b$ -open sets may fail to be supra  $b$ -open.

**Definition 1.3** [4] The supra  $b$ -closure of a set  $A$ , denoted by  $Cl_b^\mu(A)$ , is the intersection of supra  $b$ -closed sets including  $A$ . The super  $b$ -interior of a set  $A$ , denoted by  $Int_b^\mu(A)$ , is the union of supra  $b$ -open sets included in  $A$ .

## 2. Supra $b$ -compact and supra $b$ -Lindelöf spaces

**Definition 2.1** A collection  $\{U_\alpha : \alpha \in \Delta\}$  of supra  $b$ -open sets in a supra topological space  $(X, \mu)$  is called a supra  $b$ -open cover of a subset  $B$  of  $X$  if  $B \subseteq \cup\{U_\alpha : \alpha \in \Delta\}$ .

**Definition 2.2** A supra topological space  $(X, \mu)$  is called supra  $b$ -compact (resp. supra  $b$ -Lindelöf) if every supra  $b$ -open cover of  $X$  has a finite (resp. countable) subcover.

The proof of the following theorem is straightforward and thus omitted.

**Theorem 2.3** If  $X$  is finite (resp. countable) then  $(X, \mu)$  is supra  $b$ -compact (resp. supra  $b$ -Lindelöf) for any supra topology  $\mu$  on  $X$ .

**Definition 2.4** A subset  $B$  of a supra topological space  $(X, \mu)$  is said to be supra  $b$ -compact (resp. supra  $b$ -Lindelöf) relative to  $X$  if, for every collection  $\{U_\alpha : \alpha \in \Delta\}$  of supra  $b$ -open subsets of  $X$  such that  $B \subseteq \cup\{U_\alpha : \alpha \in \Delta\}$ , there exists a finite (resp. countable) subset  $\Delta_0$  of  $\Delta$  such that  $B \subseteq \cup\{U_\alpha : \alpha \in \Delta_0\}$ .

Notice that if  $(X, \mu)$  is a supra topological space and  $A \subseteq X$  then  $\mu_A = \{U \cap A : U \in \mu\}$  is a supra topology on  $A$ .

$(A, \mu_A)$  is called a supra subspace of  $(X, \mu)$ .

**Definition 2.5** A subset  $B$  of a supra topological space  $(X, \mu)$  is said to be supra  $b$ -compact (resp. supra  $b$ -Lindelöf) if  $B$  is supra  $b$ -compact (resp. supra  $b$ -Lindelöf) as a supra subspace of  $X$ .

**Theorem 2.6** Every supra  $b$ -closed subset of a supra  $b$ -compact space  $X$  is supra  $b$ -compact relative to  $X$ .

**Prof:** Let  $A$  be a supra  $b$ -closed subset of  $X$  and  $\tilde{U}$  be a cover of  $A$  by supra  $b$ -open subsets of  $X$ . Then  $\tilde{U}^* = \tilde{U} \cup \{X - A\}$  is a supra  $b$ -open cover of  $X$ . Since  $X$  is supra  $b$ -compact,  $\tilde{U}^*$  has a finite subcover  $\tilde{U}^{**}$  for  $X$ . Now  $\tilde{U}^{**} - \{X - A\}$  is a finite subcover of  $\tilde{U}$  for  $A$ , so  $A$  is supra  $b$ -compact relative to  $X$ . ■

**Theorem 2.7** *Every supra  $b$ -closed subset of a supra  $b$ -Lindelöf space  $X$  is supra  $b$ -Lindelöf relative to  $X$ .*

**Prof:** Similar to the proof of the above theorem. ■

**Theorem 2.8** *Every supra subspace of a supra topological space  $(X, \mu)$  is supra  $b$ -compact relative to  $X$  if and only if every supra  $b$ -open subspace of  $X$  is supra  $b$ -compact relative to  $X$ .*

**Prof:**  $\Rightarrow$ ) Is clear.

$\Leftarrow$ ) Let  $Y$  be a supra subspace of  $X$  and let  $\tilde{U} = \{U_\alpha : \alpha \in \Delta\}$  be a cover of  $Y$  by supra  $b$ -open sets in  $X$ . Now let  $V = \cup \tilde{U}$ , then  $V$  is a supra  $b$ -open subset of  $X$ , so it is supra  $b$ -compact relative to  $X$ . But  $\tilde{U}$  is a cover of  $V$  so  $\tilde{U}$  has a finite subcover  $\tilde{U}^*$  for  $V$ . Then  $V \subseteq \cup \tilde{U}^*$  and therefore  $Y \subseteq V \subseteq \cup \tilde{U}^*$ . So  $\tilde{U}^*$  is a finite subcover of  $\tilde{U}$  for  $Y$ . Then  $Y$  is supra  $b$ -compact relative to  $X$ . ■

**Theorem 2.9** *Every supra subspace of a supra topological space  $(X, \mu)$  is supra  $b$ -Lindelöf relative to  $X$  if and only if every supra  $b$ -open subspace of  $X$  is supra  $b$ -Lindelöf relative to  $X$ .*

**Prof:** Similar to the proof of the above theorem. ■

For a family  $\tilde{A}$  of subsets of  $X$ , if all finite intersection of the elements of  $\tilde{A}$  are non-empty, we say that  $\tilde{A}$  has the finite intersection property.

**Theorem 2.10** *A supra topological space  $(X, \mu)$  is supra  $b$ -compact if and only if every supra  $b$ -closed family of subsets of  $X$  having the finite intersection property, has a non-empty intersection.*

**Prof:**  $\Rightarrow$ ) Let  $\tilde{A} = \{A_\alpha : \alpha \in \Delta\}$  be a supra  $b$ -closed family of subsets of  $X$  which has the finite intersection property. Suppose that  $\cap \{A_\alpha : \alpha \in \Delta\} = \phi$ . Let  $\tilde{U} = \{X - A_\alpha : \alpha \in \Delta\}$  then  $\tilde{U}$  is a supra  $b$ -open cover of  $X$ . Then  $\tilde{U}$  has a finite subcover  $\tilde{U}^* = \{X - A_{\alpha_1}, X - A_{\alpha_2}, \dots, X - A_{\alpha_n}\}$ . Now  $\tilde{A}^* = \{A_{\alpha_1}, A_{\alpha_2}, \dots, A_{\alpha_n}\}$  is a finite subfamily of  $\tilde{A}$  with  $\cap \{A_{\alpha_i} : i = 1, 2, \dots, n\} = \phi$  which is a contradiction.

$\Leftarrow$ ) Let  $\tilde{U} = \{U_\alpha : \alpha \in \Delta\}$  be a supra  $b$ -open cover of  $X$ . Suppose that  $\tilde{U}$  has no finite subcover. Now  $\tilde{A} = \{X - U_\alpha : \alpha \in \Delta\}$  is a supra  $b$ -closed family of subsets of  $X$  which has the finite intersection property. So by assumption we have  $\cap \{X - U_\alpha : \alpha \in \Delta\} \neq \phi$ . Then  $\cup \{U_\alpha : \alpha \in \Delta\} \neq X$  which is a contradiction. ■

The proof of the following theorem is straightforward and thus omitted.

**Theorem 2.11** *The finite (resp. countable) union of supra  $b$ -compact (resp. supra  $b$ -Lindelöf) sets relative to a supra topological space  $X$  is supra  $b$ -compact (resp. supra  $b$ -Lindelöf) relative to  $X$ .*

**Theorem 2.12** *Let  $A$  be a supra  $b$ -compact (resp. supra  $b$ -Lindelöf) set relative to a supra topological space  $X$  and  $B$  be a supra  $b$ -closed subset of  $X$ . Then  $A \cap B$  is supra  $b$ -compact (resp. supra  $b$ -Lindelöf) relative to  $X$ .*

**Prof:** We will show the case when  $A$  is supra  $b$ -compact relative to  $X$ , the other case is similar. Suppose that  $\tilde{U} = \{U_\alpha : \alpha \in \Delta\}$  is a cover of  $A \cap B$  by supra  $b$ -open sets in  $X$ . Then  $\tilde{O} = \{U_\alpha : \alpha \in \Delta\} \cup \{X - B\}$  is a cover of  $A$  by supra  $b$ -open sets in  $X$ , but  $A$  is supra  $b$ -compact relative to  $X$ , so there exist  $\alpha_1, \alpha_2, \dots, \alpha_n \in \Delta$  such that  $A \subseteq (\cup\{U_{\alpha_i} : i = 1, 2, \dots, n\}) \cup (X - B)$ . Then  $A \cap B \subseteq \cup\{(U_{\alpha_i} \cap B) : i = 1, 2, \dots, n\} \subseteq \cup\{U_{\alpha_i} : i = 1, 2, \dots, n\}$ . Hence,  $A \cap B$  is supra  $b$ -compact relative to  $X$ . ■

**Definition 2.13** [4] Let  $(X, \tau)$  and  $(Y, \rho)$  be two topological spaces and  $\mu$  be an associated supra topology with  $\tau$ . A function  $f : (X, \tau) \rightarrow (Y, \rho)$  is called a supra  $b$ -continuous function if the inverse image of each open set in  $Y$  is a supra  $b$ -open set in  $X$ .

**Theorem 2.14** A supra  $b$ -continuous image of a supra  $b$ -compact space is compact.

**Prof:** Let  $f : X \rightarrow Y$  be a supra  $b$ -continuous function from a supra  $b$ -compact space  $X$  onto a topological space  $Y$ . Let  $\tilde{O} = \{V_\alpha : \alpha \in \Delta\}$  be an open cover of  $Y$ . Then  $\tilde{U} = \{f^{-1}(V_\alpha) : \alpha \in \Delta\}$  is a supra  $b$ -open cover of  $X$ . Since  $X$  is supra  $b$ -compact,  $\tilde{U}$  has a finite subcover say  $\{f^{-1}(V_{\alpha_1}), f^{-1}(V_{\alpha_2}), \dots, f^{-1}(V_{\alpha_n})\}$ . Now  $\{V_{\alpha_1}, V_{\alpha_2}, \dots, V_{\alpha_n}\}$  is a finite subcover of  $\tilde{O}$  for  $Y$ . ■

**Theorem 2.15** A supra  $b$ -continuous image of a supra  $b$ -Lindelöf space is Lindelöf.

**Prof:** Similar to the proof of the above theorem. ■

**Definition 2.16** Let  $(X, \tau)$  and  $(Y, \rho)$  be two topological spaces and  $\mu, \eta$  be associated supra topologies with  $\tau$  and  $\rho$  respectively. A function  $f : (X, \tau) \rightarrow (Y, \rho)$  is called a supra  $b$ -irresolute function if the inverse image of each supra  $b$ -open set in  $Y$  is a supra  $b$ -open set in  $X$ .

**Theorem 2.17** If a function  $f : X \rightarrow Y$  is supra  $b$ -irresolute and a subset  $B$  of  $X$  is supra  $b$ -compact relative to  $X$ , then  $f(B)$  is supra  $b$ -compact relative to  $Y$ .

**Prof:** Let  $\tilde{O} = \{V_\alpha : \alpha \in \Delta\}$  be a cover of  $f(B)$  by supra  $b$ -open subsets of  $Y$ . Then  $\tilde{U} = \{f^{-1}(V_\alpha) : \alpha \in \Delta\}$  is a cover of  $B$  by supra  $b$ -open subsets of  $X$ . Since  $B$  is supra  $b$ -compact relative to  $X$ ,  $\tilde{U}$  has a finite subcover  $\tilde{U}^* = \{f^{-1}(V_{\alpha_1}), f^{-1}(V_{\alpha_2}), \dots, f^{-1}(V_{\alpha_n})\}$  for  $B$ . Now  $\{V_{\alpha_1}, V_{\alpha_2}, \dots, V_{\alpha_n}\}$  is a finite subcover of  $\tilde{O}$  for  $f(B)$ . So  $f(B)$  is supra  $b$ -compact relative to  $Y$ . ■

**Theorem 2.18** If a function  $f : X \rightarrow Y$  is supra  $b$ -irresolute and a subset  $B$  of  $X$  is supra  $b$ -Lindelöf relative to  $X$ , then  $f(B)$  is supra  $b$ -Lindelöf relative to  $Y$ .

**Prof:** Similar to the proof of the above theorem. ■

**Definition 2.19** [4]. A function  $f : (X, \tau) \rightarrow (Y, \rho)$  is called a supra  $b$ -open function if the image of each open set in  $X$  is a supra  $b$ -open set in  $(Y, \rho)$ .

The proof of the following theorem is straightforward and thus omitted.

**Theorem 2.20** Let  $f : (X, \tau) \rightarrow (Y, \rho)$  be a supra  $b$ -open surjection and  $\eta$  be a supra topology associated with  $\rho$ . If  $(Y, \eta)$  is supra  $b$ -compact (resp. supra  $b$ -Lindelöf) then  $(X, \tau)$  is compact (resp. Lindelöf).

**Definition 2.21** A subset  $F$  of a supra topological space  $(X, \mu)$  is called supra  $b$ - $F_\sigma$ -set if  $F = \cup\{F_i : i = 1, 2, \dots\}$  where  $F_i$  is a supra  $b$ -closed subset of  $X$  for each  $i = 1, 2, \dots$ .

**Theorem 2.22** A supra  $b$ - $F_\sigma$ -set  $F$  of a supra  $b$ -Lindelöf space  $X$  is supra  $b$ -Lindelöf relative to  $X$ .

**Prof:** Let  $F = \cup\{F_i : i = 1, 2, \dots\}$  where  $F_i$  is a supra  $b$ -closed subset of  $X$  for each  $i = 1, 2, \dots$ . Let  $\tilde{U}$  be a cover of  $F$  by supra  $b$ -open sets in  $X$ , then  $\tilde{U}$  is a cover of  $F_i$  for each  $i = 1, 2, \dots$  by supra  $b$ -open subsets of  $X$ . Since  $F_i$  is supra  $b$ -Lindelöf relative to  $X$ ,  $\tilde{U}$  has a countable subcover  $\tilde{U}_i = \{U_{i_1}, U_{i_2}, \dots\}$  for  $F_i$  for each  $i = 1, 2, \dots$ . Now  $\cup\{\tilde{U}_i : i = 1, 2, \dots\}$  is a countable subcover of  $\tilde{U}$  for  $F$ . So  $F$  is supra  $b$ -Lindelöf relative to  $X$ . ■

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DOI: 10.7862/rf.2013.7

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Received 16.11.2011, Revised 1.12.2012, Accepted 25.10.2013

# Some new generalized classes of difference sequences of fuzzy numbers defined by a sequence of Orlicz functions

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ABSTRACT: In the present paper we introduce some new generalized classes of difference sequence spaces of fuzzy numbers defined by a sequence of Orlicz functions. We also make an effort to study some topological properties and prove some inclusion relations between these spaces.

AMS Subject Classification: *40D05, 40A05, 46S40*

Keywords and Phrases: *fuzzy numbers, Orlicz function, difference sequence spaces*

## 1. Introduction and Preliminaries

Fuzzy set theory, compared to other mathematical theories, is perhaps the most easily adaptable theory to practice. The main reason is that a fuzzy set has the property of relativity, variability and inexactness in the definition of its elements. Instead of defining an entity in calculus by assuming that its role is exactly known, we can use fuzzy sets to define the same entity by allowing possible deviations and inexactness in its role. This representation suits well the uncertainties encountered in practical life, which make fuzzy sets a valuable mathematical tool. The concepts of fuzzy sets and fuzzy set operations were first introduced by Zadeh [11] and subsequently several authors have discussed various aspects of the theory and applications of fuzzy sets such as fuzzy topological spaces, similarity relations and fuzzy orderings, fuzzy measures of fuzzy events, fuzzy mathematical programming. Matloka [7] introduced bounded and convergent sequences of fuzzy numbers and studied some of their properties. For more details about sequence spaces of fuzzy numbers see ([1], [2], [3], [8], [10]) and references therein.

Let  $C(\mathbb{R}^n) = \{A \subset \mathbb{R}^n : A \text{ is compact and convex set}\}$ . The space  $C(\mathbb{R}^n)$  has a linear structure induced by the operations  $A + B = \{a + b : a \in A, b \in B\}$  and

$\lambda A = \{\lambda a : a \in A\}$  for  $A, B \in C(\mathbb{R}^n)$  and  $\lambda \in \mathbb{R}$ .

The Hausdorff distance between  $A$  and  $B$  in  $C(\mathbb{R}^n)$  is defined by

$$\delta_\infty(A, B) = \max \left\{ \sup_{a \in A} \inf_{b \in B} \|a - b\|, \sup_{b \in B} \inf_{a \in A} \|a - b\| \right\}.$$

It is well known that  $(C(\mathbb{R}^n), \delta_\infty)$  is a complete metric space.

A fuzzy number is a function  $X$  from  $\mathbb{R}^n$  to  $[0, 1]$  which is normal, fuzzy convex, upper semicontinuous and the closure of  $\{X \in \mathbb{R}^n : X(x) > 0\}$  is compact. These properties imply that for each  $0 < \alpha \leq 1$ , the  $\alpha$ -level set

$$X^\alpha = \{X \in \mathbb{R}^n : X(x) > \alpha\}$$

is non-empty compact, convex subset of  $\mathbb{R}^n$  with support  $X^0$ .

If  $\mathbb{R}^n$  is replaced by  $\mathbb{R}$ , then obviously the set  $C(\mathbb{R}^n)$  is reduced to the set of all closed bounded intervals  $A = [\underline{A}, \bar{A}]$  on  $\mathbb{R}$ , and also

$$\delta_\infty(A, B) = \max(|\underline{A} - \underline{B}|, |\bar{A} - \bar{B}|).$$

Let  $L(\mathbb{R})$  denote the set of all fuzzy numbers. The linear structure of  $L(\mathbb{R})$  induces the addition  $X + Y$  and the scalar multiplication  $\lambda X$  in terms of  $\alpha$ -level sets, by

$$[X + Y]^\alpha = [X]^\alpha + [Y]^\alpha$$

and

$$[\lambda X]^\alpha = \lambda[X]^\alpha$$

for each  $0 \leq \alpha \leq 1$ . The set  $\mathbb{R}$  of real numbers can be embedded in  $L(\mathbb{R})$  if we define  $\bar{r} \in L(\mathbb{R})$  by

$$\bar{r}(t) = \begin{cases} 1, & \text{if } t = r \\ 0, & \text{if } t \neq r. \end{cases}$$

The additive identity and multiplicative identity of  $L(\mathbb{R})$  are denoted by  $\bar{0}$  and  $\bar{1}$ , respectively. For  $r \in \mathbb{R}$  and  $X \in L(\mathbb{R})$ , the product  $rX$  is defined as follows :

$$rX(t) = \begin{cases} X(r^{-1}t), & \text{if } r \neq 0 \\ 0, & \text{if } r = 0. \end{cases}$$

Define a map  $d : L(\mathbb{R}) \times L(\mathbb{R}) \rightarrow \mathbb{R}$  by

$$d(X, Y) = \sup_{0 \leq \alpha \leq 1} \delta_\infty(X^\alpha, Y^\alpha).$$

For  $X, Y \in L(\mathbb{R})$  define  $X \leq Y$  if and only if  $X^\alpha \leq Y^\alpha$  for any  $\alpha \in [0, 1]$ . It is known that  $(L(\mathbb{R}), d)$  is complete metric space (see [7]).

A sequence  $X = (X_k)$  of fuzzy numbers is a function  $X$  from the set  $\mathbb{N}$  of natural numbers into  $L(\mathbb{R})$ . The fuzzy number  $X_k$  denotes the value of the function at  $k \in \mathbb{N}$ . By  $w(F)$  we denote the set of all sequences  $X = (X_k)$  of fuzzy numbers. A sequence



$X = (X_k)$  of fuzzy numbers is said to be bounded if the set  $\{X_k : k \in \mathbb{N}\}$  of fuzzy numbers is bounded.

By  $l_\infty(F)$  we denote the set of all bounded sequences  $X = (X_k)$  of fuzzy numbers. A sequence  $X = (X_k)$  of fuzzy numbers is said to be convergent to a fuzzy number  $X_0$  if for every  $\epsilon > 0$  there is a positive integer  $k_0$  such that  $d(X_k, X_0) < \epsilon$  for  $k > k_0$ .

We denote by  $c(F)$  the set of all convergent sequences  $X = (X_k)$  of fuzzy numbers. It is straightforward to see that  $c(F) \subset l_\infty \subset w(F)$ .

Nanda [9] studied the classes of bounded and convergent sequences of fuzzy numbers and showed that these are complete metric spaces.

The notion of difference sequence spaces was introduced by Kizmaz [5], who studied the difference sequence spaces  $l_\infty(\Delta)$ ,  $c(\Delta)$  and  $c_0(\Delta)$ . The notion was further generalized by Et and Çolak [4] by introducing the spaces  $l_\infty(\Delta^n)$ ,  $c(\Delta^n)$  and  $c_0(\Delta^n)$ . Let  $w$  be the space of all complex or real sequences  $x = (x_k)$  and let  $r$  be non-negative integer, then for  $Z = l_\infty, c, c_0$  we have sequence spaces

$$Z(\Delta^r) = \{x = (x_k) \in w : (\Delta^r x_k) \in Z\},$$

where  $\Delta^r x = (\Delta^r x_k) = (\Delta^{r-1} x_k - \Delta^{r-1} x_{k+1})$  and  $\Delta^0 x_k = x_k$  for all  $k \in \mathbb{N}$ , which is equivalent to the following binomial representation

$$\Delta^r x_k = \sum_{v=0}^r (-1)^v \binom{r}{v} x_{k+v}.$$

Taking  $r = 1$ , we get the spaces which were introduced and studied by Kizmaz [5].

An Orlicz function  $M : [0, \infty) \rightarrow [0, \infty)$  is a continuous, non-decreasing and convex function such that  $M(0) = 0$ ,  $M(x) > 0$  for  $x > 0$  and  $M(x) \rightarrow \infty$  as  $x \rightarrow \infty$ .

Lindenstrauss and Tzafriri [6] used the idea of Orlicz function to define the following sequence space,

$$\ell_M = \left\{ x \in w : \sum_{k=1}^{\infty} M\left(\frac{|x_k|}{\rho}\right) < \infty \right\}$$

which is called as an Orlicz sequence space. Also  $\ell_M$  is a Banach space with the norm

$$\|x\| = \inf \left\{ \rho > 0 : \sum_{k=1}^{\infty} M\left(\frac{|x_k|}{\rho}\right) \leq 1 \right\}.$$

Also, it was shown in [6] that every Orlicz sequence space  $\ell_M$  contains a subspace isomorphic to  $\ell_p$  ( $p \geq 1$ ). An Orlicz function  $M$  satisfies  $\Delta_2$ -condition if and only if for any constant  $L > 1$  there exists a constant  $K(L)$  such that  $M(Lu) \leq K(L)M(u)$  for all values of  $u \geq 0$ . An Orlicz function  $M$  can always be represented in the following integral form

$$M(x) = \int_0^x \eta(t) dt$$

where  $\eta$  is known as the kernel of  $M$ , is right differentiable for  $t \geq 0$ ,  $\eta(0) = 0$ ,  $\eta(t) > 0$ ,  $\eta$  is non-decreasing and  $\eta(t) \rightarrow \infty$  as  $t \rightarrow \infty$ .

The following inequality will be used throughout the paper. Let  $p = (p_k)$  be a sequence of positive real numbers with  $0 < p_k \leq \sup_k p_k = H$  and let  $K = \max\{1, 2^{H-1}\}$ . Then for sequences  $\{a_k\}$  and  $\{b_k\}$  in the complex plane, we have

$$|a_k + b_k|^{p_k} \leq K(|a_k|^{p_k} + |b_k|^{p_k}). \quad (1.1)$$

Let  $\sigma$  be a one-to-one mapping of the set of positive integers into itself such that  $\sigma^k(n) = \sigma(\sigma^{k-1}(n))$ ,  $k = 1, 2, 3, \dots$ . Let  $\mathcal{M} = (M_k)$  be a sequence of Orlicz function and  $p = (p_k)$  be a bounded sequence of positive real numbers. We define the following classes of sequences of fuzzy numbers :

$$c_0^F(\mathcal{M}, \Delta^r, p, \sigma, s) = \left\{ X = (X_k) \in w^F : \lim_k \frac{1}{k^s} \left[ M_k \left( \frac{d(\Delta^r X_{\sigma^k(n)}, \bar{0})}{\rho} \right) \right]^{p_k} = 0, \right. \\ \left. \text{uniformly in } n \text{ for some } \rho > 0, s \geq 0 \right\},$$

$$c^F(\mathcal{M}, \Delta^r, p, \sigma, s) = \left\{ X = (X_k) \in w^F : \lim_k \frac{1}{k^s} \left[ M_k \left( \frac{d(\Delta^r X_{\sigma^k(n)}, X_0)}{\rho} \right) \right]^{p_k} = 0, \right. \\ \left. \text{uniformly in } n \text{ for some } \rho > 0, s \geq 0 \right\}$$

and

$$l_\infty^F(\mathcal{M}, \Delta^r, p, \sigma, s) = \left\{ X = (X_k) \in w^F : \lim_k \frac{1}{k^s} \left[ M_k \left( \frac{d(\Delta^r X_{\sigma^k(n)}, \bar{0})}{\rho} \right) \right]^{p_k} = 0, \right. \\ \left. \text{for some } \rho > 0, s \geq 0 \right\}.$$

If we take  $\mathcal{M}(x) = x$ , we get the spaces as follows

$$c_0^F(\Delta^r, p, \sigma, s) = \left\{ X = (X_k) \in w^F : \lim_k \frac{1}{k^s} \left( \frac{d(\Delta^r X_{\sigma^k(n)}, \bar{0})}{\rho} \right)^{p_k} = 0, \right. \\ \left. \text{uniformly in } n \text{ for some } \rho > 0, s \geq 0 \right\},$$

$$c^F(\Delta^r, p, \sigma, s) = \left\{ X = (X_k) \in w^F : \lim_k \frac{1}{k^s} \left( \frac{d(\Delta^r X_{\sigma^k(n)}, X_0)}{\rho} \right)^{p_k} = 0, \right. \\ \left. \text{uniformly in } n \text{ for some } \rho > 0, s \geq 0 \right\}$$

and

$$l_\infty^F(\Delta^r, p, \sigma, s) = \left\{ X = (X_k) \in w^F : \lim_k \frac{1}{k^s} \left( \frac{d(\Delta^r X_{\sigma^k(n)}, \bar{0})}{\rho} \right)^{p_k} = 0, \right.$$

for some  $\rho > 0, s \geq 0$  }.

If  $p = (p_k) = 1, \forall k$ , we have

$$c_0^F(\mathcal{M}, \Delta^r, \sigma, s) = \left\{ X = (X_k) \in w^F : \lim_k \frac{1}{k^s} M_k \left( \frac{d(\Delta^r X_{\sigma^k(n)}, \bar{0})}{\rho} \right) = 0, \right. \\ \left. \text{uniformly in } n \text{ for some } \rho > 0, s \geq 0 \right\},$$

$$c^F(\mathcal{M}, \Delta^r, \sigma, s) = \left\{ X = (X_k) \in w^F : \lim_k \frac{1}{k^s} M_k \left( \frac{d(\Delta^r X_{\sigma^k(n)}, X_0)}{\rho} \right) = 0, \right. \\ \left. \text{uniformly in } n \text{ for some } \rho > 0, s \geq 0 \right\}$$

and

$$l_\infty^F(\mathcal{M}, \Delta^r, \sigma, s) = \left\{ X = (X_k) \in w^F : \lim_k \frac{1}{k^s} M_k \left( \frac{d(\Delta^r X_{\sigma^k(n)}, \bar{0})}{\rho} \right) = 0, \right. \\ \left. \text{for some } \rho > 0, s \geq 0 \right\}.$$

If we take  $r, s = 0, \sigma(n) = n + 1, \mathcal{M}(x) = x$  and  $p = (p_k) = 1$  then we obtain the classes  $c_0^F, c^F$  and  $l_\infty^F$  of ordinary null, convergent and bounded sequences of fuzzy numbers, respectively which were defined and studied by Matloka [7].

The main purpose of this paper is to study some new generalized difference sequence spaces of fuzzy numbers defined by a sequence of Orlicz functions. We also examine some properties of these sequence spaces.

## 2. Main results

**Proposition 2.1** *If  $d$  is a translation invariant metric on  $L(\mathbb{R})$  then*

$$(i) \quad d(X + Y, \bar{0}) \leq d(X, \bar{0}) + d(Y, \bar{0}),$$

$$(ii) \quad d(\lambda X, \bar{0}) \leq |\lambda| d(X, \bar{0}), |\lambda| > 1.$$

**Proof.** It is easy to prove so we omit the details. ■

**Theorem 2.2** *Let  $\mathcal{M} = (M_k)$  be a sequence of Orlicz functions, the spaces  $c_0^F(\mathcal{M}, \Delta^r, p, \sigma, s), c^F(\mathcal{M}, \Delta^r, p, \sigma, s)$  and  $l_\infty^F(\mathcal{M}, \Delta^r, p, \sigma, s)$  are closed under the operations of addition and scalar multiplication if  $d$  is a translation invariant metric.*

**Proof.** If  $d$  is translation metric, then

$$d(\Delta^r(X_{\sigma^k(n)} + Y_{\sigma^k(n)}), X_0 + Y_0) \leq d(\Delta^r X_{\sigma^k(n)}, X_0) + d(\Delta^r Y_{\sigma^k(n)}, Y_0) \quad (2.1)$$

and

$$d(\lambda \Delta^r X_{\sigma^k(n)}, \lambda X_0) \leq |\lambda| d(\Delta^r X_{\sigma^k(n)}, X_0) \quad (2.2)$$

where  $\lambda$  is a scalar with  $0 < \lambda \leq 1$ . It is easy to see that the spaces  $c_0^F(\mathcal{M}, \Delta^r, p, \sigma, s)$ ,  $c^F(\mathcal{M}, \Delta^r, p, \sigma, s)$  and  $l_\infty^F(\mathcal{M}, \Delta^r, p, \sigma, s)$  are closed under the operations of addition and scalar multiplication. ■

**Theorem 2.3** *If  $\mathcal{M} = (M_k)$  be a sequence of Orlicz functions, then*

$$c_0^F(\mathcal{M}, \Delta^r, p, \sigma, s) \subset c^F(\mathcal{M}, \Delta^r, p, \sigma, s) \subset l_\infty^F(\mathcal{M}, \Delta^r, p, \sigma, s).$$

**Proof.** The inclusion  $c_0^F(\mathcal{M}, \Delta^r, p, \sigma, s) \subset c^F(\mathcal{M}, \Delta^r, p, \sigma, s)$  is obvious. We have only to show that  $c^F(\mathcal{M}, \Delta^r, p, \sigma, s) \subset l_\infty^F(\mathcal{M}, \Delta^r, p, \sigma, s)$ . For this by using triangle inequality, we have

$$\begin{aligned} \frac{1}{k^s} \left[ M_k \left( \frac{d(\Delta^r X_{\sigma^k(n)}, \bar{0})}{\rho} \right) \right]^{p_k} &\leq \frac{1}{k^s} \left[ M_k \left( \frac{d(\Delta^r X_{\sigma^k(n)}, X_0)}{\rho} \right) \right]^{p_k} \\ &\quad + \frac{1}{k^s} \left[ M_k \left( \frac{d(X_0, \bar{0})}{\rho} \right) \right]^{p_k} \\ &\leq \frac{1}{k^s} \left[ M_k \left( \frac{d(\Delta^r X_{\sigma^k(n)}, X_0)}{\rho} \right) \right]^{p_k} \\ &\quad + \max \left( 1, \frac{1}{k^s} \left[ M_k \left( \frac{|X_0|}{\rho} \right) \right]^{p_k} \right). \end{aligned}$$

Thus  $X = (X_k) \in c^F(\mathcal{M}, \Delta^r, p, \sigma, s)$  implies that  $X = (X_k) \in l_\infty^F(\mathcal{M}, \Delta^r, p, \sigma, s)$ . This completes the proof. ■

**Theorem 2.4** *If  $\mathcal{M} = (M_k)$  be a sequence of Orlicz functions, then  $c^F(\mathcal{M}, \Delta^r, p, \sigma, s)$  is a complete metric space under the metric*

$$d(X, Y) = \inf \left\{ \rho > 0 : \sup_{n,k} \frac{1}{k^s} \left[ M_k \left( \frac{d(\Delta^r (X_{\sigma^k(n)} - Y_{\sigma^k(n)}))}{\rho} \right) \right]^{p_k} \leq 1 \right\}.$$

**Proof.** Let  $X = (X_k)$ ,  $Y = (Y_k) \in c^F(\mathcal{M}, \Delta^r, p, \sigma, s)$ . Let  $\{X^{(i)}\}$  be a Cauchy sequence in  $c^F(\mathcal{M}, \Delta^r, p, \sigma, s)$ . Then given any  $\epsilon > 0$  there exists a positive integer  $N$  depending on  $\epsilon$  such that  $d(X^{(i)}, X^{(j)}) < \epsilon$ , for all  $n, m \geq N$ . Hence

$$\sup_{n,k} \frac{1}{k^s} \left[ M_k \left( \frac{d(\Delta^r X_{\sigma^k(n)}^{(i)} - \Delta^r X_{\sigma^k(n)}^{(j)})}{\rho} \right) \right]^{p_k} < \epsilon \quad \forall i, j \geq N.$$

Consequently  $\{X_k^{(i)}\}$  is a Cauchy sequence in the metric space  $L(\mathbb{R})$ . But  $L(\mathbb{R})$  is complete. So,  $X_k^{(i)} \rightarrow X_k$  as  $i \rightarrow \infty$ . Hence there exists a positive integer  $n_0$  such that

$$\sup_{n,k} \frac{1}{k^s} \left[ M_k \left( \frac{d(\Delta^r X_{\sigma^k(n)}^{(n_0)} - \Delta^r X_{\sigma^k(n)})}{\rho} \right) \right]^{p_k} < \epsilon \quad \forall n_0 \geq N.$$

This implies that  $(X_k) \in c^F(\mathcal{M}, \Delta^r, p, \sigma, s)$ . Hence  $c^F(\mathcal{M}, \Delta^r, p, \sigma, s)$  is a complete metric space. This completes the proof. ■

**Theorem 2.5** *If  $\liminf \left(\frac{p_k}{q_k}\right) > 0$ , then  $c^F(\mathcal{M}, \Delta^r, q, \sigma, s) \subset c^F(\mathcal{M}, \Delta^r, p, \sigma, s)$ .*

**Proof.** Suppose that  $\liminf \left(\frac{p_k}{q_k}\right) > 0$  holds and  $X = (X_k) \in c^F(\mathcal{M}, \Delta^r, q, \sigma, s)$ . Then there is  $\beta > 0$  such that  $p_k > \beta q_k$  for large  $k \in N$ . Hence for large  $k$

$$\frac{1}{k^s} \left[ M_k \left( \frac{d(\Delta^r X_{\sigma^k(n)}, X_0)}{\rho} \right) \right]^{p_k} \leq \left( \frac{1}{k^s} \left[ M_k \left( \frac{d(\Delta^r X_{\sigma^k(n)}, X_0)}{\rho} \right) \right]^{q_k} \right)^\beta.$$

Since

$$\frac{1}{k^s} \left[ M_k \left( \frac{d(\Delta^r X_{\sigma^k(n)}, X_0)}{\rho} \right) \right]^{q_k} < 1$$

for each  $k, n$  and for some  $\rho > 0$ . Hence  $X = (X_k) \in c^F(\mathcal{M}, \Delta^r, p, \sigma, s)$ . ■

**Theorem 2.6** *If  $0 < p_k \leq q_k \leq 1$ , then  $l_\infty^F(\mathcal{M}, \Delta^r, q, \sigma, s)$  is closed subset of  $l_\infty^F(\mathcal{M}, \Delta^r, p, \sigma, s)$*

**Proof.** Suppose that  $0 < p_k \leq q_k \leq 1$  holds and  $X = (X_k) \in c^F(\mathcal{M}, \Delta^r, p, \sigma, s)$ . Then there is a constant  $L > 1$  such that

$$\frac{1}{k^s} \left[ M_k \left( \frac{d(\Delta^r X_{\sigma^k(n)}, \bar{0})}{\rho} \right) \right]^{q_k} \leq L$$

for each  $k, n$  and for some  $\rho > 0$ . This implies that

$$\frac{1}{k^s} \left[ M_k \left( \frac{d(\Delta^r X_{\sigma^k(n)}, \bar{0})}{\rho} \right) \right]^{p_k} \leq L$$

for each  $k$  and  $n$ . Hence  $X = (X_k) \in l_\infty^F(\mathcal{M}, \Delta^r, p, \sigma, s)$ . To show that  $l_\infty^F(\mathcal{M}, \Delta^r, q, \sigma, s)$  is closed, suppose that  $X^i = (X_k^i) \in l_\infty^F(\mathcal{M}, \Delta^r, q, \sigma, s)$ ,  $X^i \rightarrow X_0$  and  $X_0 \in l_\infty^F(\mathcal{M}, \Delta^r, p, \sigma, s)$ . Then for every  $\epsilon$ ,  $0 < \epsilon < 1$  there is  $i_0 \in N$  such that for all  $k, n$  and for some  $\rho > 0$

$$\frac{1}{k^s} \left[ M_k \left( \frac{d(\Delta^r (X_{\sigma^k(n)} - X_0), \bar{0})}{\rho} \right) \right]^{p_k} < \epsilon \text{ for } i > i_0.$$

Now

$$\begin{aligned} & \frac{1}{k^s} \left[ M_k \left( \frac{d(\Delta^r (X_{\sigma^k(n)} - X_0), \bar{0})}{\rho} \right) \right]^{q_k} \\ & < \frac{1}{k^s} \left[ M_k \left( \frac{d(\Delta^r (X_{\sigma^k(n)} - X_0), \bar{0})}{\rho} \right) \right]^{p_k} < \epsilon \text{ for } i > i_0. \end{aligned}$$

Therefore  $X = (X_k) \in l_\infty^F(\mathcal{M}, \Delta^r, q, \sigma, s)$  i.e.  $l_\infty^F(\mathcal{M}, \Delta^r, q, \sigma, s)$  is closed subset of  $l_\infty^F(\mathcal{M}, \Delta^r, p, \sigma, s)$ . ■

**Theorem 2.7** Let  $0 < h = \inf p_k \leq \sup p_k = H < \infty$ . For any sequence of Orlicz function  $\mathcal{M} = (M_k)$  which satisfies  $\Delta_2$ -condition, then

$$c^F(\Delta^r, p, \sigma, s) \subset c^F(\mathcal{M}, \Delta^r, p, \sigma, s).$$

**Proof.** Let  $X = (X_k) \in c^F(\Delta^r, p, \sigma, s)$ , so that  $\lim_k \frac{1}{k^s} [d(\Delta^r X_{\sigma^k(n)}, X_0)]^{p_k} = 0$ , uniformly in  $n$ . Let  $\epsilon > 0$  and choose  $\delta$  with  $0 < \delta < 1$  such that  $M_k(t) < \epsilon$  for  $0 \leq t \leq \delta$ . We can write

$$y_k = \frac{d(X_{\sigma^k(n)}, X_0)}{\rho}.$$

We consider

$$\sum_{\substack{y_k \leq \delta \\ k \in \mathbb{N}}} \frac{1}{k^s} [M_k(y_k)]^{p_k} + \sum_{\substack{y_k > \delta \\ k \in \mathbb{N}}} \frac{1}{k^s} [M_k(y_k)]^{p_k}.$$

For  $y_k \leq \delta$ , we have

$$\frac{1}{k^s} [M_k(y_k)]^{p_k} < \frac{1}{k^s} \max(\epsilon, \epsilon^h)$$

by using the continuity of  $(M_k)$ . For  $y_k > \delta$ , we have

$$y_k < \frac{y_k}{\delta} < 1 + \frac{y_k}{\delta}.$$

Since  $\mathcal{M} = (M_k)$  is non-decreasing and convex, it follows that

$$M_k(y_k) < M\left(1 + \frac{y_k}{\delta}\right) \leq \frac{1}{2}M_k(2) + \frac{1}{2}M_k\left(\frac{2y_k}{\delta}\right).$$

Since  $\mathcal{M} = (M_k)$  satisfies  $\Delta_2$ -condition, we have

$$M_k(y_k) \leq \frac{K}{2} \frac{y_k}{\delta} M_k(2) = K \frac{y_k}{\delta} M_k(2).$$

Thus we have

$$\frac{1}{k^s} [M_k(y_k)]^{p_k} \leq \frac{1}{k^s} \max(1, [KM_k(2)\delta^{-1}]^H [y_k]^{p_k}).$$

This implies that

$$\frac{1}{k^s} [M_k(y_k)]^{p_k} \leq \frac{1}{k^s} \max(\epsilon, \epsilon^h) \frac{1}{k^s} \max(1, [KM_k(2)\delta^{-1}]^H [y_k]^{p_k}).$$

Taking  $\epsilon \rightarrow 0$  and  $k \rightarrow \infty$ , it follows that  $X = (X_k) \in c^F(\mathcal{M}, \Delta^r, p, \sigma, s)$ . ■

**Theorem 2.8** Let  $\mathcal{M}$ ,  $\mathcal{M}'$  and  $\mathcal{M}''$  are sequences of Orlicz functions. Then

- (i)  $Z(\mathcal{M}', \Delta^r, p, \sigma, s) \subset Z(\mathcal{M} \cdot \mathcal{M}', \Delta^r, p, \sigma, s)$ ;
- (ii)  $Z(\mathcal{M}', \Delta^r, p, \sigma, s) \cap Z(\mathcal{M}'', \Delta^r, p, \sigma, s) \subset Z(\mathcal{M}' + \mathcal{M}'', \Delta^r, p, \sigma, s)$ , where  $Z = c_0^F, c^F, l_\infty^F$ .

**Proof.** It is easy to prove so we omit the details. ■

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DOI: 10.7862/rf.2013.8

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Received 07.05.2013, Revisted 20.10.2013, Accepted 25.10.2013

## On a study of double gai sequence space

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ABSTRACT: Let  $\chi^2$  denote the space of all prime sense double gai sequences and  $\Lambda^2$  the space of all prime sense double analytic sequences. This paper is devoted to the general properties of  $\chi^2$ .

AMS Subject Classification: *40A05, 40C05, 40D05*

Keywords and Phrases: *gai sequence, analytic sequence, double sequence, dual, monotone metric.*

### 1. Introduction

Throughout  $w, \chi$  and  $\Lambda$  denote the classes of all, gai and analytic scalar valued single sequences, respectively. We write  $w^2$  for the set of all complex sequences  $(x_{mn})$ , where  $m, n \in \mathbb{N}$ , the set of positive integers. Then,  $w^2$  is a linear space under the coordinate wise addition and scalar multiplication.

Some initial works on double sequence spaces is found in Bromwich[4]. Later on, they were investigated by Hardy[8], Moricz[12], Moricz and Rhoades[13], Basarir and Solankan[2], Tripathy[20], Colak and Turkmenoglu[6], Turkmenoglu[22], and many others.

Let us define the following sets of double sequences:

$$\begin{aligned}\mathcal{M}_u(t) &:= \left\{ (x_{mn}) \in w^2 : \sup_{m,n \in \mathbb{N}} |x_{mn}|^{t_{mn}} < \infty \right\}, \\ \mathcal{C}_p(t) &:= \left\{ (x_{mn}) \in w^2 : p\text{-}\lim_{m,n \rightarrow \infty} |x_{mn} - l|^{t_{mn}} = 1 \text{ for some } l \in \mathbb{C} \right\}, \\ \mathcal{C}_{0p}(t) &:= \left\{ (x_{mn}) \in w^2 : p\text{-}\lim_{m,n \rightarrow \infty} |x_{mn}|^{t_{mn}} = 1 \right\}, \\ \mathcal{L}_u(t) &:= \left\{ (x_{mn}) \in w^2 : \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} |x_{mn}|^{t_{mn}} < \infty \right\}, \\ \mathcal{C}_{bp}(t) &:= \mathcal{C}_p(t)\end{aligned}$$



where  $t = (t_{mn})$  is the sequence of strictly positive reals  $t_{mn}$  for all  $m, n \in \mathbb{N}$  and  $p - \lim_{m, n \rightarrow \infty}$  denotes the limit in the Pringsheim's sense. In the case  $t_{mn} = 1$  for all  $m, n \in \mathbb{N}$ ;  $\mathcal{M}_u(t), \mathcal{C}_p(t), \mathcal{C}_{0p}(t), \mathcal{L}_u(t), \mathcal{C}_{bp}(t)$  and  $\mathcal{C}_{0bp}(t)$  reduce to the sets  $\mathcal{M}_u, \mathcal{C}_p, \mathcal{C}_{0p}, \mathcal{L}_u, \mathcal{C}_{bp}$  and  $\mathcal{C}_{0bp}$ , respectively.

Now, we may summarize the knowledge given in some document related to the double sequence spaces. Gökhan and Colak [27,28] have proved that  $\mathcal{M}_u(t)$  and  $\mathcal{C}_p(t), \mathcal{C}_{bp}(t)$  are complete paranormed spaces of double sequences and gave the  $\alpha-, \beta-, \gamma-$  duals of the spaces  $\mathcal{M}_u(t)$  and  $\mathcal{C}_{bp}(t)$ . Quite recently, in her PhD thesis, Zelter [29] has essentially studied both the theory of topological double sequence spaces and the theory of summability of double sequences. Mursaleen and Edely [30] have recently introduced the statistical convergence and Cauchy for double sequences and given the relation between statistical convergent and strongly Cesàro summable double sequences. Nextly, Mursaleen [31] and Mursaleen and Edely [32] have defined the almost strong regularity of matrices for double sequences and applied these matrices to establish a core theorem and introduced the  $M$ -core for double sequences and determined those four dimensional matrices transforming every bounded double sequences  $x = (x_{jk})$  into one whose core is a subset of the  $M$ -core of  $x$ . More recently, Altay and Basar [33] have defined the spaces  $\mathcal{BS}, \mathcal{BS}(t), \mathcal{CS}_p, \mathcal{CS}_{bp}, \mathcal{CS}_r$  and  $\mathcal{BV}$  of double sequences consisting of all double series whose sequence of partial sums are in the spaces  $\mathcal{M}_u, \mathcal{M}_u(t), \mathcal{C}_p, \mathcal{C}_{bp}, \mathcal{C}_r$  and  $\mathcal{L}_u$ , respectively, and also examined some properties of those sequence spaces and determined the  $\alpha-$  duals of the spaces  $\mathcal{BS}, \mathcal{BV}, \mathcal{CS}_{bp}$  and the  $\beta(\vartheta) -$  duals of the spaces  $\mathcal{CS}_{bp}$  and  $\mathcal{CS}_r$  of double series. Quite recently Basar and Sever [34] have introduced the Banach space  $\mathcal{L}_q$  of double sequences corresponding to the well-known space  $\ell_q$  of single sequences and examined some properties of the space  $\mathcal{L}_q$ . Quite recently Subramanian and Misra [35] have studied the space  $\chi_M^2(p, q, u)$  of double sequences and gave some inclusion relations. We need the following inequality in the sequel of the paper. For  $a, b, \geq 0$  and  $0 < p < 1$ , we have

$$(a + b)^p \leq a^p + b^p \quad (1)$$

The double series  $\sum_{m, n=1}^{\infty} x_{mn}$  is called convergent if and only if the double sequence  $(s_{mn})$  is convergent, where  $s_{mn} = \sum_{i, j=1}^{m, n} x_{ij} (m, n \in \mathbb{N})$  (see[1]).

A sequence  $x = (x_{mn})$  is said to be double analytic if  $\sup_{m, n} |x_{mn}|^{1/m+n} < \infty$ . The vector space of all double analytic sequences will be denoted by  $\Lambda^2$ . A sequence  $x = (x_{mn})$  is called double gai sequence if  $((m+n)! |x_{mn}|)^{1/m+n} \rightarrow 0$  as  $m, n \rightarrow \infty$ . The double gai sequences will be denoted by  $\chi^2$ . Let  $\phi = \{\text{all finitesequences}\}$ .

Consider a double sequence  $x = (x_{ij})$ . The  $(m, n)^{th}$  section  $x^{[m, n]}$  of the sequence is defined by  $x^{[m, n]} = \sum_{i, j=0}^{m, n} x_{ij} \mathfrak{S}_{ij}$  for all  $m, n \in \mathbb{N}$ ; where  $\mathfrak{S}_{ij}$  denotes the double sequence whose only non zero term is a  $\frac{1}{(i+j)!}$  in the  $(i, j)^{th}$  place for each  $i, j \in \mathbb{N}$ .

An FK-space (or a metric space)  $X$  is said to have AK property if  $(\mathfrak{S}_{mn})$  is a Schauder basis for  $X$ . Or equivalently  $x^{[m, n]} \rightarrow x$ .

An FDK-space is a double sequence space endowed with a complete metrizable; locally convex topology under which the coordinate mappings  $x = (x_k) \rightarrow (x_{mn})(m, n \in$

$\mathbb{N}$ ) are also continuous.

If  $X$  is a sequence space, we give the following definitions:

- (i)  $X'$  = the continuous dual of  $X$ ;
- (ii)  $X^\alpha = \{a = (a_{mn}) : \sum_{m,n=1}^\infty |a_{mn}x_{mn}| < \infty, \text{ for each } x \in X\}$ ;
- (iii)  $X^\beta = \{a = (a_{mn}) : \sum_{m,n=1}^\infty a_{mn}x_{mn} \text{ is convergent, for each } x \in X\}$ ;
- (iv)  $X^\gamma = \{a = (a_{mn}) : \sup_{m,n} \geq 1 \left| \sum_{m,n=1}^{M,N} a_{mn}x_{mn} \right| < \infty, \text{ for each } x \in X\}$ ;
- (v) let  $X$  be an  $FK$  - space  $\supset \phi$ ; then  $X^f = \{f(\mathfrak{S}_{mn}) : f \in X'\}$ ;
- (vi) quad  $X^\delta = \{a = (a_{mn}) : \sup_{m,n} |a_{mn}x_{mn}|^{1/m+n} < \infty, \text{ for each } x \in X\}$ ;

$X^\alpha, X^\beta, X^\gamma$  are called  $\alpha$  - (or Köthe - Toeplitz) dual of  $X, \beta$  - (or generalized - Köthe - Toeplitz) dual of  $X, \gamma$  - dual of  $X, \delta$  - dual of  $X$  respectively.  $X^\alpha$  is defined by Gupta and Kamptan [24]. It is clear that  $X^\alpha \subset X^\beta$  and  $X^\alpha \subset X^\gamma$ , but  $X^\alpha \subset X^\gamma$  does not hold, since the sequence of partial sums of a double convergent series need not to be bounded.

The notion of difference sequence spaces (for single sequences) was introduced by Kizmaz [36] as follows

$$Z(\Delta) = \{x = (x_k) \in w : (\Delta x_k) \in Z\}$$

for  $Z = c, c_0$  and  $\ell_\infty$ , where  $\Delta x_k = x_k - x_{k+1}$  for all  $k \in \mathbb{N}$ . Here  $w, c, c_0$  and  $\ell_\infty$  denote the classes of all, convergent, null and bounded scalar valued single sequences respectively. The above spaces are Banach spaces normed by

$$\|x\| = |x_1| + \sup_{k \geq 1} |\Delta x_k|$$

Later on the notion was further investigated by many others. We now introduce the following difference double sequence spaces defined by

$$Z(\Delta) = \{x = (x_{mn}) \in w^2 : (\Delta x_{mn}) \in Z\}$$

where  $Z = \Lambda^2, \chi^2$  and  $\Delta x_{mn} = (x_{mn} - x_{mn+1}) - (x_{m+1n} - x_{m+1n+1}) = x_{mn} - x_{mn+1} - x_{m+1n} + x_{m+1n+1}$  for all  $m, n \in \mathbb{N}$

## 2. Definitions and Preliminaries

A sequence  $x = (x_{mn})$  is said to be double analytic if

$$\sup_{m,n} |x_{mn}|^{1/m+n} < \infty.$$

The vector space of all double analytic sequences is usually denoted by  $\Lambda^2$ . A sequence  $x = (x_{mn})$  is called double entire sequence if  $|x_{mn}|^{1/m+n} \rightarrow 0$  as  $m, n \rightarrow \infty$ . The vector space of double entire sequences is usually denoted by  $\Gamma^2$ . A sequence  $x = (x_{mn})$  is called double gai sequence if  $((m+n)! |x_{mn}|)^{1/m+n} \rightarrow 0$  as  $m, n \rightarrow \infty$ . The vector

space of double gai sequences is usually denoted by  $\chi^2$ . The space  $\chi^2$  is a metric space with the metric

$$d(x, y) = \sup_{m, n} \left\{ ((m+n)! |x_{mn} - y_{mn}|)^{1/m+n} : m, n : 1, 2, 3, \dots \right\} \quad (2)$$

for all  $x = \{x_{mn}\}$  and  $y = \{y_{mn}\}$  in  $\chi^2$ .

### 3. Main Results

**Proposition 3.1**  $\chi^2$  has monotone metric.

**Proof:** We know that

$$\begin{aligned} d(x, y) &= \sup_{m, n} \left\{ ((m+n)! |x_{mn} - y_{mn}|)^{1/m+n} : m, n : 1, 2, 3, \dots \right\} \\ d(x^n, y^n) &= \sup_{n, n} \left\{ ((2n)! |x_{nn} - y_{nn}|)^{1/2n} \right\} \end{aligned}$$

and

$$d(x^m, y^m) = \sup_{m, m} \left\{ ((2m)! |x_{mm} - y_{mm}|)^{1/2m} \right\}$$

Let  $m > n$ . Then

$$\begin{aligned} \sup_{m, m} \left\{ ((2m)! |x_{mm} - y_{mm}|)^{1/2m} \right\} &\geq \sup_{n, n} \left\{ ((2n)! |x_{nn} - y_{nn}|)^{1/2n} \right\} \\ d(x^m, y^m) &\geq d(x^n, y^n), \quad m > n \end{aligned} \quad (3)$$

Also  $\{d(x^n, y^n) : n = 1, 2, 3, \dots\}$  is monotonically increasing bounded by  $d(x, y)$ . For such a sequence

$$\sup_{n, n} \left\{ ((2n)! |x^{nn} - y^{nn}|)^{1/2n} \right\} = \lim_{n \rightarrow \infty} d(x^n, y^n) = d(x, y) \quad (4)$$

From(3) and (4) it follows that  $d(x, y) = \sup_{m, n} \left\{ ((m+n)! |x_{mn} - y_{mn}|)^{1/m+n} \right\}$  is a monotone metric for  $\chi^2$ . This completes the proof.

**Proposition 3.2** The dual space of  $\chi^2$  is  $\Lambda^2$ . In other words  $(\chi^2)^* = \Lambda^2$ .

**Proof:** We recall that

$$\mathfrak{S}_{mn} = \begin{pmatrix} 0, & 0, & \dots, & 0, & \dots \\ 0, & 0, & \dots, & 0, & \dots \\ \vdots & & & & \\ \vdots & & & & \\ \vdots & & & & \\ 0, & 0, & \dots, & \frac{1}{(m+n)!}, & 0, & \dots \\ 0, & 0, & \dots, & 0, & 0, & \dots \end{pmatrix}$$

with  $\frac{1}{(m+n)!}$  in the  $(m, n)th$  position and zero's else where. With

$$\begin{aligned}
 x &= \mathfrak{S}_{mn}, (|x_{mn}|)^{1/m+n} \\
 &= \begin{pmatrix} 0^{1/2}, \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & 0^{1/1+n} \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ 0^{1/m+1}, & \left(\frac{1}{(m+n)!}\right)^{1/m+n}, & \cdot & 0^{1/m+n+1} & \cdot & \cdot & \cdot \\ 0^{1/m+2}, & \cdot & \cdot & \cdot & \cdot & \cdot & 0^{1/m+n+2} \end{pmatrix} \\
 &= \begin{pmatrix} 0, \cdot & \cdot & \cdot & \cdot & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ 0, & \left(\frac{1}{(m+n)!}\right)^{1/m+n}, & \cdot & 0 \\ 0, & \cdot & \cdot & \cdot & 0 \end{pmatrix}
 \end{aligned}$$

which is a double gai sequence. Hence  $\mathfrak{S}_{mn} \in \chi^2$ . We have  $f(x) = \sum_{m,n=1}^{\infty} x_{mn}y_{mn}$ . With  $x \in \chi^2$  and  $f \in (\chi^2)^*$  the dual space of  $\chi^2$ . Take  $x = (x_{mn}) = \mathfrak{S}_{mn} \in \chi^2$ . Then

$$|y_{mn}| \leq \|f\| d(\mathfrak{S}_{mn}, 0) < \infty \quad \forall m, n \tag{5}$$

Thus  $(y_{mn})$  is a bounded sequence and hence an double analytic sequence. In other words  $y \in \Lambda^2$ . Therefore  $(\chi^2)^* = \Lambda^2$ . This completes the proof.

**Proposition 3.3**  $\chi^2$  is separable.

**Proof:**It is routine verification. Therefore omit the proof.

**Proposition 3.4**  $\Lambda^2$  is not separable.

**Proof:**Since  $|x_{mn}|^{1/m+n} \rightarrow 0$  as  $m, n \rightarrow \infty$ , so it may so happen that first row or column may not be convergent, even may not be bounded. Let  $S$  be the set that has double sequences such that the first row is built up of sequences of zeros and ones. Then  $S$  will be uncountable. Consider open balls of radius  $3^{-1}$  units. Then these open balls will not cover  $\Lambda^2$ . Hence  $\Lambda^2$  is not separable. This completes the proof.

**Proposition 3.5**  $\chi^2$  is not reflexive.

**Proof:** $\chi^2$  is separable by Proposition 3.3. But  $(\chi^2)^* = \Lambda^2$ , by Proposition 3.2. Since  $\Lambda^2$  is not separable, by Proposition 3.4. Therefore  $\chi^2$  is not reflexive. This completes the proof.

**Proposition 3.6**  $\chi^2$  is not an inner product space as such not a Hilbert space.

**Proof:** Let us take

$$x = x_{mn} = \begin{pmatrix} 1/2!, & 1/3!, & 0, & 0, & \dots \\ 0, & 0, & 0, & 0, & \dots \\ \vdots & & & & \\ \vdots & & & & \\ \vdots & & & & \end{pmatrix}$$

and

$$\begin{aligned} y = y_{mn} &= \begin{pmatrix} 1/2!, & -1/3!, & 0, & 0, & \dots \\ 0, & 0, & 0, & 0, & \dots \\ \vdots & \vdots & \vdots & & \end{pmatrix} \\ d(x, 0) &= \sup \begin{pmatrix} (2!|x_{11} - 0|)^{1/2}, & (3!|x_{12} - 0|)^{1/3}, & \dots \\ (3!|x_{21} - 0|)^{1/3}, & (4!|x_{22} - 0|)^{1/4}, & \dots \\ \vdots & \vdots & \vdots \end{pmatrix} \\ &= \sup \begin{pmatrix} (2!|1/2! - 0|)^{1/2}, & (3!|1/3! - 0|)^{1/3}, & \dots \\ 0, & 0, & \dots \\ \vdots & \vdots & \vdots \end{pmatrix} \\ &= \sup \begin{pmatrix} (1)^{1/2}, & (1)^{1/3}, & 0, & \dots \\ 0, & 0, & 0, & \dots \\ \vdots & \vdots & \vdots & \vdots \end{pmatrix} \\ d(x, 0) &= 1. \end{aligned}$$

Similarly  $d(x, 0) = 1$ . Hence  $d(x, 0) = d(y, 0) = 1$

$$\begin{aligned} x + y &= \begin{pmatrix} 1/2!, & 1/3!, & 0, & 0, & \dots \\ 0, & 0, & 0, & 0, & \dots \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0, & 0, & 0, & 0, & \dots \end{pmatrix} + \begin{pmatrix} 1/2!, & -1/3!, & 0, & 0, & \dots \\ 0, & 0, & 0, & 0, & \dots \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0, & 0, & 0, & 0, & \dots \end{pmatrix} \\ &= \begin{pmatrix} 1, & 0, & 0, & 0, & \dots \\ 0, & 0, & 0, & 0, & \dots \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0, & 0, & 0, & 0, & \dots \end{pmatrix} \end{aligned}$$

$$\begin{aligned} d(x + y, x + y) &= \sup \left\{ ((m+n)! (|x_{mn} + y_{mn}| - |x_{mn} - y_{mn}|))^{1/m+n} \right. \\ &\quad \left. : m, n = 1, 2, 3, \dots \right\} \end{aligned}$$

$$\begin{aligned}
 d(x_{mn} + y_{mn}, 0) &= \sup \begin{pmatrix} (2!|x_{11} + y_{11}|)^{1/2}, & (3!|x_{12} + y_{12}|)^{1/3}, & \dots \\ \vdots & \vdots & \end{pmatrix} \\
 &= \sup \begin{pmatrix} (2!|1/2! + 1/2!|)^{1/2}, & (3!|1/3! - 1/3!|)^{1/3}, & \dots \\ \vdots & \vdots & \end{pmatrix} \\
 &= \sup \begin{pmatrix} (2)^{1/2}, & 0, & \dots \\ 0, & 0, & \dots \\ \vdots & \vdots & \end{pmatrix} = \sup \begin{pmatrix} 1.414, & 0, & \dots \\ 0, & 0, & \dots \\ \vdots & \vdots & \end{pmatrix} = 1.414
 \end{aligned}$$

Therefore  $d(x + y, 0) = 1.414$ . Similarly  $d(x - y, 0) = 1.26$

By parellogram law,

$$\begin{aligned}
 [d(x + y, 0)]^2 + [d(x - y, 0)]^2 &= 2 [(d(x, 0))^2 + (d(0, y))^2] \implies \\
 (1.414)^2 + 1.26^2 &= 2 [1^2 + 1^2] \implies \\
 3.586996 &= 4.
 \end{aligned}$$

Hence it is not satisfied by the law. Therefore  $\chi^2$  is not an inner product space. Assume that  $\chi^2$  is a Hilbert space. But then  $\chi^2$  would satisfy reflexivity condition. [Theorem 4.6.6 [42]] . Proposition 3.5,  $\chi^2$  is not reflexive. Thus  $\chi^2$  is not a Hilbert space. This completes the proof.

**Proposition 3.7**  $\chi^2$  is rotund.

**Proof:** Let us take

$$x = x_{mn} = \begin{pmatrix} 1/2!, & 0, & 0, & 0 & \dots \\ 0, & 0, & 0, & 0, & \dots \\ \vdots & \vdots & \vdots & \vdots & \end{pmatrix} \quad \text{and} \quad y = y_{mn} = \begin{pmatrix} 1/2!, & 0, & 0, & 0, & \dots \\ 0, & 0, & 0, & 0, & \dots \\ \vdots & \vdots & \vdots & \vdots & \end{pmatrix}$$

Then  $x = (x_{mn})$  and  $y = (y_{mn})$  are in  $\chi^2$ . Also

$$d(x, y) = \sup \begin{pmatrix} (2!|x_{11} - y_{11}|)^{\frac{1}{2}}, & \dots & ((n + 1)!|x_{1n} - y_{1n}|)^{\frac{1}{1+n}}, & 0, & \dots \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ ((m + 1)!|x_{m1} - y_{m1}|)^{\frac{1}{m+1}}, & \dots & ((m + n)!|x_{mn} - y_{mn}|)^{\frac{1}{m+n}}, & 0, & \dots \\ 0, & \dots & 0, & \dots & \dots \end{pmatrix}$$

Therefore

$$d(x, 0) = \sup \begin{pmatrix} 1, & 0, & 0, & 0 & \dots \\ 0, & 0, & 0, & 0, & \dots \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 0, & 0, & 0, & 0, & \dots \end{pmatrix}, \quad d(0, y) = 1.$$

Obviously  $x = (x_{mn}) \neq y = (y_{mn})$ . But

$$\begin{aligned} (x_{mn}) + (y_{mn}) &= \begin{pmatrix} 1/2!, & 0, & 0, & 0, & \dots \\ 0, & 0, & 0, & 0, & \dots \\ \vdots & \vdots & \vdots & \vdots & \vdots \end{pmatrix} + \begin{pmatrix} 1/2!, & 0, & 0, & 0, & \dots \\ 0, & 0, & 0, & 0, & \dots \\ \vdots & \vdots & \vdots & \vdots & \vdots \end{pmatrix} \\ &= \begin{pmatrix} 1, & 0, & 0, & 0, & \dots \\ 0, & 0, & 0, & 0, & \dots \\ \vdots & \vdots & \vdots & \vdots & \vdots \end{pmatrix} \end{aligned}$$

$$\begin{aligned} d\left(\frac{x_{mn} + y_{mn}}{2}, 0\right) &= \sup \left( \begin{array}{cccc} \frac{(2!|x_{11}+y_{11}|)^{1/2}}{2}, & \dots & \frac{((1+n)!|x_{1n}+y_{1n}|)^{1/n+1}}{2}, & 0, \dots \\ \vdots & \ddots & \vdots & \vdots \\ \frac{((m+1)!|x_{m1}+y_{m1}|)^{1/m+1}}{2}, & \dots, & \frac{((m+n)!|x_{mn}+y_{mn}|)^{1/m+n}}{2}, & 0, \dots \end{array} \right) \\ d\left(\frac{x_{mn} + y_{mn}}{2}, 0\right) &= \sup \begin{pmatrix} (2^{1/2})/2, & 0, & 0, & 0, & \dots \\ 0, & 0, & 0, & 0, & \dots \\ \vdots & \vdots & \vdots & \vdots & \vdots \end{pmatrix} = 0.71. \end{aligned}$$

Therefore  $\chi^2$  is rotund. This completes the proof.

**Proposition 3.8** *Weak convergence and strong convergence are equivalent in  $\chi^2$ .*

**Proof: Step1:** Always strong convergence implies weak convergence.

**Step2:** So it is enough to show that weakly convergence implies strongly convergence in  $\chi^2$ .  $y^{(\eta)}$  tends to weakly in  $\chi^2$ , where  $(y_{mn}^{(\eta)}) = y^{(\eta)}$  and  $y = (y_{mn})$ . Take any  $x = (x_{mn}) \in \chi^2$  and

$$f(z) = \sum_{m,n=1}^{\infty} ((m+n)!|z_{mn}x_{mn}|)^{1/m+n} \text{ for each } z = (z_{mn}) \in \chi^2 \quad (6)$$

Then  $f \in (\chi^2)^*$  by Proposition 3.2. By hypothesis  $f(y^\eta) \rightarrow f(y)$  as  $\eta \rightarrow \infty$ .

$$f(y^{(\eta)} - y) \rightarrow 0 \text{ as } \eta \rightarrow \infty. \implies \quad (7)$$

$$\sum_{m,n=1}^{\infty} \left( |y_{mn}^{(\eta)} - y_{mn}|^{1/m+n} ((m+n)!|x_{mn}|)^{1/m+n} \right) \rightarrow 0 \text{ as } \eta \rightarrow \infty.$$

By using (6) and (7) we get since  $x = (x_{mn}) \in \Lambda^2$  we have

$$\sum_{m,n=1}^{\infty} |x_{mn}|^{1/m+n} < \infty \text{ for all } x \in \Lambda^2.$$

$$\begin{aligned}
 &\implies \sum_{m,n=1}^{\infty} \left( (m+n)! \left| y_{mn}^{(\eta)} - y_{mn} \right| \right)^{1/m+n} \rightarrow 0 \text{ as } \eta \rightarrow \infty. \\
 &\implies \sup_{m,n} \left( (m+n)! \left| (y_{mn}^{(\eta)} - y_{mn}), 0 \right| \right)^{1/m+n} \rightarrow 0 \text{ as } \eta \rightarrow \infty. \\
 &\implies \sup_{m,n} \left( (m+n)! \left| y_{mn}^{(\eta)} - y_{mn} \right| \right)^{1/m+n} \rightarrow 0 \text{ as } \eta \rightarrow \infty. \\
 &\implies d \left( (y^{(\eta)} - y), 0 \right) \rightarrow 0 \text{ as } \eta \rightarrow \infty. \\
 &\implies d \left( y^{(\eta)} - y \right) \rightarrow 0 \text{ as } \eta \rightarrow \infty.
 \end{aligned}$$

This completes the proof.

**Proposition 3.9** *There exists an infinite matrix A for which  $\chi_A^2 = \chi^2$ .*

**Proof:** Consider the matrix

$$\begin{pmatrix}
 2!y_{11}, & 3!y_{12}, & \dots, & (1+n)!y_{1n}, & 0, & 0 & \dots \\
 3!y_{21}, & 4!y_{22}, & \dots, & (2+n)!y_{2n}, & 0, & 0 & \dots \\
 \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
 (m+1)!y_{m1}, & (m+2)!y_{m2}, & \dots, & (m+n)!y_{mn}, & 0, & 0 & \dots \\
 0, & 0, & \dots, & 0, & 0, & 0 & \dots \\
 \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots
 \end{pmatrix}$$

$$= \begin{pmatrix}
 1, & 0, & 0, & \dots \\
 1, & 0, & 0, & \dots \\
 0, & 1, & 0, & \dots \\
 0, & 1, & 0, & \dots \\
 0, & 1, & 0, & \dots \\
 0, & 1, & 0, & \dots \\
 0, & 0, & 1, & 0, & \dots \\
 0, & 0, & 1, & 0, & \dots \\
 0, & 0, & 1, & 0, & \dots \\
 0, & 0, & 1, & 0, & \dots \\
 0, & 0, & 1, & 0, & \dots \\
 0, & 0, & 1, & 0, & \dots \\
 0, & 0, & 1, & 0, & \dots \\
 0, & 0, & 1, & 0, & \dots \\
 \vdots & & & & \\
 \vdots & & & & 
 \end{pmatrix}
 \begin{pmatrix}
 2!x_{11}, & 3!x_{12}, & \dots, & (1+n)!x_{1n}, & 0 & \dots \\
 3!x_{21}, & 4!x_{22}, & \dots, & (2+n)!x_{2n}, & 0 & \dots \\
 \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
 (m+1)!x_{m1}, & (m+2)!x_{m2}, & \dots, & (m+n)!x_{mn}, & 0, & \dots \\
 0, & 0, & \dots, & 0, & 0 & \dots \\
 \vdots & \vdots & \vdots & \vdots & \vdots & \vdots
 \end{pmatrix}$$



$$\begin{pmatrix}
2!y_{11}, & 3!y_{12}, & \dots, & (1+n)!y_{1n}, & 0, & 0 & \dots \\
3!y_{21}, & 4!y_{22}, & \dots, & (2+n)!y_{2n}, & 0, & 0 & \dots \\
\vdots & & & & & & \\
\vdots & & & & & & \\
(m+1)!y_{m1}, & (m+2)!y_{m2}, & \dots, & (m+n)!y_{mn}, & 0, & 0 & \dots \\
0, & 0, & \dots, & 0, & 0, & 0 & \dots \\
\vdots & & & & & & \\
\vdots & & & & & & \\
\vdots & & & & & & 
\end{pmatrix}
=
\begin{pmatrix}
2!x_{11}, \dots, & (1+n)!x_{1n}, & 0, \dots \\
2!x_{11}, \dots, & (1+n)!x_{1n}, & 0, \dots \\
3!x_{21}, \dots, & (2+n)!x_{2n}, & 0, \dots \\
3!x_{21}, \dots, & (2+n)!x_{2n}, & 0, \dots \\
3!x_{21}, \dots, & (2+n)!x_{2n}, & 0, \dots \\
3!x_{21}, \dots, & (2+n)!x_{2n}, & 0, \dots \\
4!x_{31}, \dots, & (3+n)!x_{3n}, & 0, \dots \\
4!x_{31}, \dots, & (3+n)!x_{3n}, & 0, \dots \\
4!x_{31}, \dots, & (3+n)!x_{3n}, & 0, \dots \\
4!x_{31}, \dots, & (3+n)!x_{3n}, & 0, \dots \\
4!x_{31}, \dots, & (3+n)!x_{3n}, & 0, \dots \\
4!x_{31}, \dots, & (3+n)!x_{3n}, & 0, \dots \\
4!x_{31}, \dots, & (3+n)!x_{3n}, & 0, \dots \\
4!x_{31}, \dots, & (3+n)!x_{3n}, & 0, \dots \\
4!x_{31}, \dots, & (3+n)!x_{3n}, & 0, \dots \\
\vdots & \vdots & \vdots
\end{pmatrix}$$

$$\begin{aligned}
2!y_{11}, \dots, (1+n)!y_{1n} &= 2!x_{11}, \dots, (1+n)!x_{1n} \\
3!y_{21}, \dots, (2+n)!y_{2n} &= 2!x_{11}, \dots, (1+n)!x_{1n} \\
4!y_{31}, \dots, (3+n)!y_{3n} &= 3!x_{21}, \dots, (2+n)!x_{2n} \\
5!y_{41}, \dots, (4+n)!y_{4n} &= 3!x_{21}, \dots, (2+n)!x_{2n} \\
6!y_{51}, \dots, (5+n)!y_{5n} &= 3!x_{21}, \dots, (2+n)!x_{2n} \\
7!y_{61}, \dots, (6+n)!y_{6n} &= 3!x_{21}, \dots, (2+n)!x_{2n} \\
&\vdots
\end{aligned}$$

and so on. For any  $x = (x_{mn}) \in \chi^2$ .

$$|(Ax)_{mn}| = \lim_{m, n \rightarrow \infty} ((m+n)! |\Sigma x_{mn}|)^{1/m+n} \leq d(x, 0)$$

where metric is taken  $\chi^2$ .

$$[d(x, 0)]_{\chi^2_A} \leq [d(x, 0)]_{\chi^2} \quad (8)$$

**Conversely,** Given  $x \in [d(x, 0)]_{\chi_A^2}$  fix any  $m, n$  then,

$$\begin{aligned} m, n \xrightarrow{\text{lim}} \infty ((m+n)! |x_{mn}|)^{1/m+n} &\leq (Ax)_{mn}. \\ \implies m, n \xrightarrow{\text{lim}} \infty ((m+n)! |x_{mn}|)^{1/m+n} &\leq [d(x, 0)]_{\chi_A^2} \\ &[d(x, 0)]_{\chi^2} \leq [d(x, 0)]_{\chi_A^2}. \end{aligned}$$

Therefore the matrix  $A = (x_{mn}^{lk})$  for which the summability field  $[d(x, 0)]_{\chi^2} = [d(x, 0)]_{\chi_A^2}$  is given by

$$A = \begin{pmatrix} 1, & 0, & 0, & \dots \\ 1, & 0, & 0, & \dots \\ 0, & 1, & 0, & \dots \\ 0, & 1, & 0, & \dots \\ 0, & 1, & 0, & \dots \\ 0, & 1, & 0, & \dots \\ 0, & 0, & 1, & 0, & \dots \\ 0, & 0, & 1, & 0, & \dots \\ 0, & 0, & 1, & 0, & \dots \\ 0, & 0, & 1, & 0, & \dots \\ 0, & 0, & 1, & 0, & \dots \\ 0, & 0, & 1, & 0, & \dots \\ 0, & 0, & 1, & 0, & \dots \\ 0, & 0, & 1, & 0, & \dots \\ \vdots & & & & \end{pmatrix}$$

**//Program for generalization:**

```
#include <iostream.h>
#include <conio.h>
#include <math.h>
#include <fstream.h>
void main()
{
clrscr();
int m,n,i,nn=0,j,count=1,k,lpp,abc;
ofstream fout,fout 1;
fout.open("aa1.txt");
fout1.open("aa2.txt");
cout << "enter the value of m:";
cin>> m;
for(i=1;i<=m;i++)
{
nn=nn+pow(2,i);
}
```

```

i=0
while(count<=nm)
{
cout<< " - ";
fout<< " - ";
for(abc=1;abc<=m+2;abc++)
{
cout<< " ";
fout<< " ";
}
cout<< " - \n";
fout<< " - \n";
for(j = 1; j <= m; j++)
{
for(k=1;k<=pow(2,j);k++)
{
for(pp=1;pp<=3;pp++)
{
fout1<< count + pp << "!Y" << count << ", " << pp << " ";
}
}
fout1<< "...(" << count << " + n)!Y" << count << ", n = ";
cout<< " | ";
fout<< " | ";
for(int q=1;q<=m+1;q++)
{
if(q==j)
{
cout<< "1";
fout<< "1";
}
else
{
cout<< "0";
fout<< "0";
}
}
for(int l=1;l<=3;l++)
{
fout1<< j + 1 << "!X" << "j" << ", " << l << " ";
}
}
fout1<< "...(" << j << " + n)!X" << j << "n";
cout<< " ... | \n";
fout<< " ... | \n";
fout1<< " ... | \n";
count++;

```

```

}
}
}
cout<< " \n \n \n \n";
fout<< " \n \n \n \n";
cout<< " | -";
fout<< " | -";
for(abc=1;abc<=m+1;abc++)
{
cout<< " ";
fout<< " ";
}
cout<< " - | ";
fout<< " - | ";
fout1<< " \n \n \n";
fout.close();
fout1.close();
getch();
}

```

**SAMPLE INPUT/OUTPUT:**

Enter the value of m=3

$$\begin{pmatrix}
 1, & 0, & 0, & \dots \\
 1, & 0, & 0, & \dots \\
 0, & 1, & 0, & \dots \\
 0, & 1, & 0, & \dots \\
 0, & 1, & 0, & \dots \\
 0, & 1, & 0, & \dots \\
 0, & 0, & 1, & 0, & \dots \\
 0, & 0, & 1, & 0, & \dots \\
 0, & 0, & 1, & 0, & \dots \\
 0, & 0, & 1, & 0, & \dots \\
 0, & 0, & 1, & 0, & \dots \\
 0, & 0, & 1, & 0, & \dots \\
 0, & 0, & 1, & 0, & \dots \\
 0, & 0, & 1, & 0, & \dots \\
 \vdots & & & & 
 \end{pmatrix}$$

$$\begin{aligned}
 2!Y_{1,1}, \dots, (1+n)!Y_{1,n} &= 2!X_{1,1}, \dots, (1+n)!X_{1,n} \\
 3!Y_{2,1}, \dots, (2+n)!Y_{2,n} &= 2!X_{1,1}, \dots, (1+n)!X_{1,n} \\
 4!Y_{3,1}, \dots, (3+n)!Y_{3,n} &= 3!X_{2,1}, \dots, (2+n)!X_{2,n} \\
 5!Y_{4,1}, \dots, (4+n)!Y_{4,n} &= 3!X_{2,1}, \dots, (2+n)!X_{2,n} \\
 6!Y_{5,1}, \dots, (5+n)!Y_{5,n} &= 3!X_{2,1}, \dots, (2+n)!X_{2,n} \\
 7!Y_{6,1}, \dots, (6+n)!Y_{6,n} &= 3!X_{2,1}, \dots, (2+n)!X_{2,n}
 \end{aligned}$$

$$\begin{aligned}
8!Y_{7,1}, \dots, (7+n)!Y_{7,n} &= 4!X_{3,1}, \dots, (3+n)!X_{3,n} \\
9!Y_{8,1}, \dots, (8+n)!Y_{8,n} &= 4!X_{3,1}, \dots, (3+n)!X_{3,n} \\
10!Y_{9,1}, \dots, (9+n)!Y_{9,n} &= 4!X_{3,1}, \dots, (3+n)!X_{3,n} \\
11!Y_{10,1}, \dots, (10+n)!Y_{10,n} &= 4!X_{3,1}, \dots, (3+n)!X_{3,n} \\
12!Y_{11,1}, \dots, (11+n)!Y_{11,n} &= 4!X_{3,1}, \dots, (3+n)!X_{3,n} \\
13!Y_{12,1}, \dots, (12+n)!Y_{12,n} &= 4!X_{3,1}, \dots, (3+n)!X_{3,n} \\
14!Y_{13,1}, \dots, (13+n)!Y_{13,n} &= 4!X_{3,1}, \dots, (3+n)!X_{3,n} \\
15!Y_{14,1}, \dots, (14+n)!Y_{14,n} &= 4!X_{3,1}, \dots, (3+n)!X_{3,n} \\
&\vdots
\end{aligned}$$

**Acknowledgement:** I wish to thank the referee's for their several remarks and valuable suggestions that improved the presentation of the paper.

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**DOI: 10.7862/rf.2013.9**

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*Received 05.03.2013, Revised 10.05.2013, Accepted 25.10.2013*



# Instability to differential equations of fourth order with a variable deviating argument

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ABSTRACT: The main purpose of this paper is to give two instability theorems to fourth order nonlinear differential equations with a variable deviating argument.

AMS Subject Classification: *34K20*

Keywords and Phrases: *Instability; Krasovskii criteria; differential equation; fourth order; deviating argument*

## 1. Introduction

In 2000, Ezeilo [5] proved two instability theorems for the fourth order nonlinear differential equations without delay

$$x^{(4)} + a_1x''' + g(x, x', x'', x''')x'' + h(x)x' + f(x, x', x'', x''') = 0 \quad (1.1)$$

and

$$x^{(4)} + p(x''', x'') + q(x, x', x'', x''')x'' + a_3x' + a_4x = 0. \quad (1.2)$$

In this paper, instead of Eq. (1.1) and Eq. (1.2), we consider the fourth order nonlinear differential equations with a variable deviating argument,  $\tau(t)$  :

$$\begin{aligned} x^{(4)}(t) &+ a_1x'''(t) + g(x(t-\tau(t)), \dots, x'''(t-\tau(t)))x'' \\ &+ h(x(t))x'(t) + f(x(t-\tau(t)), \dots, x'''(t-\tau(t)))x(t) = 0 \end{aligned} \quad (1.3)$$

and

$$\begin{aligned} x^{(4)}(t) &+ p(x'''(t), x''(t)) + q(x(t-\tau(t)), \dots, x'''(t-\tau(t)))x'' \\ &+ a_3x'(t) + a_4x(t) = 0. \end{aligned} \quad (1.4)$$

We write Eq. (1.3) and Eq. (1.4) in system form as

$$\begin{aligned}
 x' &= y, \\
 y' &= z, \\
 z' &= u, \\
 u' &= -a_1u - g(x(t - \tau(t)), \dots, u(t - \tau(t)))z - h(x)y \\
 &\quad - f(x(t - \tau(t)), \dots, u(t - \tau(t)))x
 \end{aligned} \tag{1.5}$$

and

$$\begin{aligned}
 x' &= y, \\
 y' &= z, \\
 z' &= u, \\
 u' &= -p(u, z) - q(x(t - \tau(t)), \dots, u(t - \tau(t)))z \\
 &\quad - a_3y - a_4x,
 \end{aligned} \tag{1.6}$$

respectively, where  $\tau(t)$  is fixed delay,  $t - \tau(t)$  is strictly increasing,  $\lim_{t \rightarrow \infty} (t - \tau(t)) = \infty$ ,  $t \in \mathbb{R}_+ = [0, \infty)$ ;  $a_1$ ,  $a_3$  and  $a_4$  are constants;  $g$ ,  $h$ ,  $f$ ,  $p$  and  $q$  are continuous functions in their respective arguments on  $\mathbb{R}^4$ ,  $\mathbb{R}$ ,  $\mathbb{R}^4$ ,  $\mathbb{R}^2$  and  $\mathbb{R}^4$ , respectively, with  $p(0, z) = 0$  and satisfy a Lipschitz condition in their respective arguments; the derivative  $\frac{\partial p}{\partial z}(u, z)$  exists and is also continuous. Hence, the existence and uniqueness of the solutions of Eq. (1.3) and Eq. (1.4) are guaranteed (see [[2], pp.14]). We assume in what follows that  $x(t)$ ,  $y(t)$ ,  $z(t)$  and  $u(t)$  are abbreviated as  $x, y, z$  and  $u$ , respectively.

So far, the instability of solutions to certain fourth order nonlinear scalar and vector differential equations without delay has been investigated by many authors (see Dong and Zhang [1], Ezeilo ([3]-[5]), Li and Duan [8], Li and Yu [9], Lu and Liao [10], Sadek [11], Skrapek [12], Sun and Hou [13], Tiryaki [14], Tunç ([15]-[18]), C. Tunç and E. Tunç [20] and the references cited thereof). However, by now, the instability of solutions to fourth order nonlinear differential equations with deviating arguments has only been studied by Tunç [19]. This paper is the second attempt on the topic in the literature. It is worth mentioning that throughout all of the papers, based on Krasovskii's properties (see Krasovskii [6]), the Lyapunov's second (or direct) method has been used as a basic tool to prove the results established therein. The motivation for this paper comes from the above mentioned papers. Our aim is to carry out the results established in Ezeilo [5] to nonlinear differential equations of fourth order, Eq. (1.3) and Eq. (1.4), with a deviating argument for the instability of zero solution of these equations.

Note that the instability criteria of Krasovskii [6] can be summarized as the following: According to these criteria, it is necessary to show here that there exists a Lyapunov function  $V(\cdot) \equiv V(x, y, z, u)$  which has Krasovskii properties, say  $(K_1)$ ,  $(K_2)$  and  $(K_3)$  :

$(K_1)$  In every neighborhood of  $(0, 0, 0, 0)$ , there exists a point  $(\xi, \eta, \zeta, \mu)$  such that  $V(\xi, \eta, \zeta, \mu) > 0$  ;

( $K_2$ ) the time derivative  $\dot{V} = \frac{d}{dt}V(x, y, z, u)$  along solution paths of the system (1.5) is positive semi-definite;

( $K_3$ ) the only solution  $(x, y, z, u) = (x(t), y(t), z(t), u(t))$  of the system (1.5) which satisfies  $\dot{V} = 0$ , ( $t \geq 0$ ), is the trivial solution  $(0, 0, 0, 0)$ .

Let  $r \geq 0$  be given, and let  $C = C([-r, 0], \mathfrak{R}^n)$  with

$$\|\phi\| = \max_{-r \leq s \leq 0} |\phi(s)|, \quad \phi \in C.$$

For  $H > 0$  define  $C_H \subset C$  by

$$C_H = \{\phi \in C : \|\phi\| < H\}.$$

If  $x : [-r, A) \rightarrow \mathfrak{R}^n$  is continuous,  $0 < A \leq \infty$ , then, for each  $t$  in  $[0, A)$ ,  $x_t$  in  $C$  is defined by

$$x_t(s) = x(t + s), \quad -r \leq s \leq 0, t \geq 0.$$

Let  $G$  be an open subset of  $C$  and consider the general autonomous delay differential system with finite delay

$$\dot{x} = F(x_t), \quad x_t = x(t + \theta), \quad -r \leq \theta \leq 0, t \geq 0,$$

where  $F(0) = 0$  and  $F : G \rightarrow \mathfrak{R}^n$  is continuous and maps closed and bounded sets into bounded sets. It follows from these conditions on  $F$  that each initial value problem

$$\dot{x} = F(x_t), \quad x_0 = \phi \in G$$

has a unique solution defined on some interval  $[0, A)$ ,  $0 < A \leq \infty$ . This solution will be denoted by  $x(\phi)(\cdot)$  so that  $x_0(\phi) = \phi$ .

**Definition 1.1.** Let  $F(0) = 0$ . The zero solution,  $x = 0$ , of  $\dot{x} = F(x_t)$  is stable if for each  $\varepsilon > 0$  there exists  $\delta = \delta(\varepsilon) > 0$  such that  $\|\phi\| < \delta$  implies that  $|x(\phi)(t)| < \varepsilon$  for all  $t \geq 0$ . The zero solution is said to be unstable if it is not stable.

**Theorem 1.1.** (*Instability Theorem of Cetaev's*). Let  $\Omega$  be a neighborhood of the origin. Let there be given a function  $V(x)$  and region  $\Omega_1$  in  $\Omega$  with the following properties:

- (i)  $V(x)$  has continuous first partial derivatives in  $\Omega_1$ .
- (ii)  $V(x)$  and  $\dot{V}(x)$  are positive in  $\Omega_1$ .
- (iii) At the boundary points of  $\Omega_1$  inside  $\Omega$ ,  $V(x) = 0$ .
- (iv) The origin is a boundary point of  $\Omega_1$ .

Under these conditions the origin is unstable (see LaSalle and Lefschetz [7]).

## 2. Main results

The first main result is the following theorem.

**Theorem 2.1.** *Suppose that*

$$f(x(t - \tau(t)), \dots, u(t - \tau(t))) - \frac{1}{4}g^2(x(t - \tau(t)), \dots, u(t - \tau(t))) > 0$$

for arbitrary  $x(t - \tau(t)), \dots, u(t - \tau(t))$ . Then the zero solution of Eq. (1.3) is unstable.

*Proof.* Consider the Lyapunov function  $V = V(x, y, z, u)$  defined by

$$V = yz + \frac{1}{2}a_1y^2 - xu - a_1xz - \int_0^x h(s)ds, \quad (\text{where } a_1 \text{ is a constant}),$$

so that

$$V(0, \varepsilon^2, \varepsilon, 0) = \varepsilon^3 + \frac{1}{2}a_1\varepsilon^4 > 0$$

for sufficiently small  $\varepsilon$ . In fact, if  $\varepsilon$  is an arbitrary positive constant, then

$$V(0, \varepsilon^2, \varepsilon, 0) > 0$$

for sufficiently small  $\varepsilon$ . Thus  $V$  satisfies the property  $(K_1)$ , (see [6]).

By an elementary differentiation the time derivative of  $V$  along the solutions of (1.5) can be estimated as follows

$$\begin{aligned} \dot{V} &= z^2 + xzg(x(t - \tau(t)), \dots, u(t - \tau(t))) + x^2f(x(t - \tau(t)), \dots, u(t - \tau(t))) \\ &= [z + 2^{-1}xg(x(t - \tau(t)), \dots, u(t - \tau(t)))]^2 \\ &\quad + \left[ f(x(t - \tau(t)), \dots, u(t - \tau(t))) - \frac{1}{4}g^2(x(t - \tau(t)), \dots, u(t - \tau(t))) \right] x^2 \\ &\geq \left[ f(x(t - \tau(t)), \dots, u(t - \tau(t))) - \frac{1}{4}g^2(x(t - \tau(t)), \dots, u(t - \tau(t))) \right] x^2 > 0. \end{aligned}$$

Thus  $V$  satisfies the property  $(K_2)$ , (see [6]).

Further, it follows that  $\dot{V} = 0 \Leftrightarrow x = 0$ . In turn, this implies that

$$x = y = z = u = 0.$$

Thus  $V$  satisfies the property  $(K_3)$ , (see [6]). This completes the proof of Theorem 2.1.  $\square$

**Example 2.1.** Consider nonlinear differential equation of fourth order with a variable deviating argument,  $\tau(t) = t/2$  :

$$\begin{aligned} x^{(4)} &+ x''' + \left\{ 2 + \frac{2}{1 + x^2(t/2) + \dots + x'''^2(t/2)} \right\} x'' \\ &+ 4xx' + (9 + x^2(t/2) + \dots + x'''^2(t/2))x = 0 \end{aligned}$$

so that

$$\begin{aligned}x' &= y, \\y' &= z, \\z' &= u, \\u' &= -u - \left\{2 + \frac{2}{1 + x^2(t/2) + \dots + u^2(t/2)}\right\}z - 4xy \\&\quad - \{9 + x^2(t/2) + \dots + u^2(t/2)\}x(t) = 0.\end{aligned}$$

We have the following estimates:

$$\begin{aligned}a_1 &= 1, \\ \tau(t) &= t/2, \\ g(x(t - \tau(t)), \dots, u(t - \tau(t))) &= 2 + \frac{2}{1 + x^2(t/2) + \dots + u^2(t/2)}, \\ h(x) &= 4x\end{aligned}$$

and

$$f(x(t - \tau(t)), \dots, u(t - \tau(t))) = 9 + x^2(t/2) + \dots + u^2(t/2)$$

so that

$$\begin{aligned}f(\cdot) - \frac{1}{4}g^2(\cdot) &= 9 + x^2(t/2) + \dots + u^2(t/2) \\ &\quad - \left[1 + \frac{1}{1 + x^2(t/2) + \dots + u^2(t/2)}\right]^2 > 0.\end{aligned}$$

This shows that the zero solution of the above equation is unstable.

The second main result is the following theorem.

**Theorem 2.2.** *Suppose that*

$$p(0, z) = 0, \quad a_4 > 0 \quad \text{and} \quad a_4 - \frac{1}{4}q^2(x(t - \tau(t)), \dots, u(t - \tau(t))) > 0$$

for arbitrary  $x(t - \tau(t)), \dots, u(t - \tau(t))$ , and  $\frac{\partial p}{\partial z}(u, z) \operatorname{sgn} u \leq 0$  for arbitrary  $u, z$ . Then the zero solution of Eq. (1.4) is unstable for arbitrary  $a_3$ .

*Proof.* Consider the Lyapunov function  $V_1 = V_1(x, y, z, u)$  defined by

$$V_1 = - \int_0^u p(s, z) ds - a_3 y u + \frac{1}{2} a_3 z^2 - a_4 x u + a_4 y z$$

so that

$$V_1(0, \varepsilon^2, \varepsilon, 0) = a_4 \varepsilon^3 + \frac{1}{2} a_3 \varepsilon^4 > 0, \quad (a_4 > 0), \quad (a_3 \in \mathfrak{R}),$$

for sufficiently small  $\varepsilon$ . Indeed, if  $\varepsilon$  is an arbitrary positive constant, then

$$V_1(0, \varepsilon^2, \varepsilon, 0) > 0$$

for sufficiently small  $\varepsilon$ . Thus  $V_1$  satisfies the property  $(K_1)$ , (see [6]).

The time derivative of  $V_1$  along the solutions of (1.6) can be calculated as follows:

$$\begin{aligned} \dot{V}_1 &= -u' \{p(u, z) + a_3y + a_4x\} + a_4z^2 - u \int_0^u \frac{\partial p}{\partial z}(s, z) ds \\ &= u' \{u' + q(x(t - \tau(t)), \dots, u(t - \tau(t)))z\} \\ &\quad + a_4z^2 - u \int_0^u \frac{\partial p}{\partial z}(s, z) ds. \end{aligned}$$

The last estimate leads

$$\begin{aligned} \dot{V}_1 &= \{u' + 2^{-1}q(x(t - \tau(t)), \dots, u(t - \tau(t)))z\}^2 \\ &\quad + \{a_4 - 4^{-1}q^2(x(t - \tau(t)), \dots, u(t - \tau(t)))\}z^2 \\ &\quad - u \int_0^u \frac{\partial p}{\partial z}(s, z) ds \end{aligned}$$

so that

$$\begin{aligned} \dot{V}_1 &\geq \{u' + 2^{-1}q(x(t - \tau(t)), \dots, u(t - \tau(t)))z\}^2 \\ &\quad + \{a_4 - 4^{-1}q^2(x(t - \tau(t)), \dots, u(t - \tau(t)))\}z^2 > 0. \end{aligned}$$

Thus  $V_1$  satisfies the property  $(K_2)$ , (see [6]).

On the other hand,  $\dot{V}_1 = 0 \Leftrightarrow z = 0$ , this implies that  $z = u = 0$ . System (1.6) and  $\dot{V}_1 = 0$  leads that

$$a_3y + a_4x = 0 \Rightarrow a_3x' + a_4x = 0.$$

Because of  $x'' = 0$ , it follows that  $x' = \text{constant}$  so that  $a_3x' + a_4x = 0 \Rightarrow x = \text{constant}$ . However, this implies  $x' = 0$  since  $a_4 \neq 0$ . Hence  $a_4 > 0$  implies  $x = 0$ . Thus  $V_1$  satisfies the property  $(K_3)$ , (see [6]). This completes the proof of Theorem 2.2.  $\square$

**Example 2.2.** Consider nonlinear differential equation of fourth order with a variable deviating argument,  $\tau(t) = t/2$ :

$$x^{(4)} - (\arctg x'')x''' + 2 \cos(x(t/2)) + \dots + x'''(t/2)x'' + 3x' + 4x = 0.$$

so that

$$\begin{aligned} x' &= y, \\ y' &= z, \\ z' &= u, \\ u' &= (\arctgz)u - 2 \cos(x(t/2)) + \dots + u(t/2)z - 3y - 4x. \end{aligned}$$

We have the following estimates:

$$\begin{aligned}\tau(t) &= t/2, & a_3 &= 3, & a_4 &= 4, \\ p(u, z) &= -(\operatorname{arctg} z)u, \\ \frac{\partial p}{\partial z}(u, z) \operatorname{sgn} u &= -\frac{u}{1+z^2} \operatorname{sgn} u \leq 0, \\ q(x(t-\tau(t)), \dots, u(t-\tau(t))) &= 2 \cos(x(t/2) + \dots + u(t/2)),\end{aligned}$$

so that

$$a_4 - \frac{1}{4}q^2(\cdot) = 4 - \cos^2(x(t/2) + \dots + u(t/2)) > 0.$$

This shows that the zero solution of the above equation is unstable.

#### Acknowledgement

The author would like to express his sincere appreciation to the reviewer for his/her helpful comments, corrections and suggestions which helped with improving the presentation and quality of this work.

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**DOI: 10.7862/rf.2013.10**

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*Received 1.08.2012,    Revisted 29.08.2012,    Accepted 25.10.2013*



# Instability to nonlinear vector differential equations of fifth order with constant delay

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Submitted by: Józef Banaś

ABSTRACT: We consider a certain vector differential equation of the fifth order with a constant delay. We give new certain sufficient conditions which guarantee the instability of the zero solution of that equation. An example is given to illustrate the theoretical analysis made in the paper.

AMS Subject Classification: 34K20

Keywords and Phrases: Vector differential equation, fifth order, instability, delay

## 1. Introduction

In 2003, Sadek [5] considered the nonlinear vector differential equation of the fifth order:

$$X^{(5)} + \Psi(\ddot{X})\ddot{X} + \Phi(\ddot{X}) + \Theta(\dot{X}) + F(X) = 0. \quad (1.1)$$

The author gave certain sufficient conditions, which guarantee the instability of the zero solution of Eq. (1.1).

In this paper, instead of Eq. (1.1), we consider its delay form as follows:

$$X^{(5)} + \Psi(\ddot{X})\ddot{X} + \Phi(X, \dot{X}, \ddot{X})\ddot{X} + H(\dot{X}(t - \tau)) + F(X(t - \tau)) = 0, \quad (1.2)$$

where  $X \in \mathbb{R}^n$ ,  $\tau > 0$  is the constant deviating argument,  $\Psi$  and  $\Phi$  are continuous  $n \times n$ -symmetric matrix functions for the arguments displayed explicitly,  $H : \mathbb{R}^n \rightarrow \mathbb{R}^n$  and  $F : \mathbb{R}^n \rightarrow \mathbb{R}^n$  with  $H(0) = F(0) = 0$ , and  $H$  and  $F$  are continuous functions for the arguments displayed explicitly. It is assumed the existence and the uniqueness of the solutions of Eq. (1.2).

Eq. (1.2) is the vector version for systems of real nonlinear differential equations of the fifth order:

$$\begin{aligned} x_i^{(5)} &+ \sum_{k=1}^n \psi_{ik}(x_1'', \dots, x_k'') x_k''' + \sum_{k=1}^n \phi_{ik}(x_1, \dots, x_k, \dots, x_1'', \dots, x_k'') x_k'' \\ &+ h_i(x_1'(t - \tau), \dots, x_n'(t - \tau)) + f_i(x_1(t - \tau), \dots, x_n(t - \tau)) = 0, \end{aligned}$$

for  $i = 1, 2, \dots, n$ .

Instead of Eq. (1.2), we consider its equivalent differential system

$$\begin{aligned}\dot{X} &= Y, \dot{Y} = Z, \dot{Z} = W, \dot{W} = U, \\ \dot{U} &= -\Psi(Z)W - \Phi(X, Y, Z)Z - H(Y) - F(X) \\ &\quad + \int_{t-\tau}^t J_H(Y(s))Z(s)ds + \int_{t-\tau}^t J_F(X(s))Y(s)ds,\end{aligned}\quad (1.3)$$

which was obtained by setting  $\dot{X} = Y$ ,  $\ddot{X} = Z$ ,  $\ddot{X} = W$  and  $X^{(4)} = U$  from Eq. (1.2).

$J_F(X)$  and  $J_H(Y)$  denote the Jacobian matrices corresponding to the functions  $F(X)$  and  $H(Y)$ , respectively. It is clear that

$$J_F(X) = \left( \frac{\partial f_i}{\partial x_j} \right)$$

and

$$J_H(Y) = \left( \frac{\partial h_i}{\partial y_j} \right), (i, j = 1, 2, \dots, n),$$

where  $(x_1, \dots, x_n)$ ,  $(y_1, \dots, y_n)$ ,  $(f_1, \dots, f_n)$  and  $(h_1, \dots, h_n)$  are components of  $X$ ,  $Y$ ,  $F$  and  $H$ , respectively. Throughout what follows, it is assumed that  $J_F(X)$  and  $J_H(Y)$  exist and are symmetric and continuous.

It should be noted that since 1978 till now the instability of the solutions of certain scalar differential equations of the fifth order without and with delay and vector differential equations of the fifth order without delay was discussed in the literature. For a comprehensive treatment of the subject, we refer the readers to the papers of Ezeilo [2], Sadek [5], Sun and Hou [6], Tunç ([7]-[14]), Tunç and Erdogan [15], Tunç and Karta [16], Tunç and Şevli [17] and the references cited in these sources. However, a review to date of the literature indicates that the instability of solutions of vector differential equations of the fifth order with delay has not been investigated. This paper is the first known publication regarding the instability of solutions for the nonlinear vector differential equations of the fifth order with a deviating argument. The motivation of this paper comes from the above papers done on scalar differential equations of the fifth order without and with delay and the vector differential equations of the fifth order without delay. Our aim is to achieve the results established in Sadek [[5], Theorem 3] to Eq. (1.2). By this work, we improve the results of Sadek [[5], Theorem 3] to a vector differential equation of the fifth order with delay. Based on Krasovskii's criterions [3], we prove our main result, and an example is also provided to illustrate the feasibility of the proposed result. The result to be obtained is new and different from that in the papers mentioned above.

Note that the instability criteria of Krasovskii [3] can be summarized as the following: According to these criteria, it is necessary to show here that there exists a Lyapunov- Krasovskii functional  $V(\cdot) \equiv V(X, Y, Z, W, U)$  which has Krasovskii properties, say  $(K_1)$ ,  $(K_2)$  and  $(K_3)$  :

( $K_1$ ) In every neighborhood of  $(0, 0, 0, 0, 0)$ , there exists a point  $(\xi_1, \dots, \xi_5)$  such that  $V(\xi_1, \dots, \xi_5) > 0$ ,

( $K_2$ ) the time derivative  $\frac{d}{dt}V(\cdot)$  along solution paths of (1.3) is positive semi-definite,

( $K_3$ ) the only solution  $(X, Y, Z, W, U) = (X(t), Y(t), Z(t), W(t), U(t))$  of (1.3) which satisfies  $\frac{d}{dt}V(\cdot) = 0$ , ( $t \geq 0$ ), is the trivial solution  $(0, 0, 0, 0, 0)$ .

The symbol  $\langle X, Y \rangle$  corresponding to any pair  $X, Y$  in  $\mathfrak{R}^n$  stands for the usual scalar product  $\sum_{i=1}^n x_i y_i$ , that is,  $\langle X, Y \rangle = \sum_{i=1}^n x_i y_i$ ; thus  $\langle X, X \rangle = \|X\|^2$ , and  $\lambda_i(\Omega)$ , ( $i = 1, 2, \dots, n$ ), are the eigenvalues of the real symmetric  $n \times n$  - matrix  $\Omega$ . The matrix  $\Omega$  is said to be negative-definite, when  $\langle \Omega X, X \rangle \leq 0$  for all nonzero  $X$  in  $\mathfrak{R}^n$ .

## 2. Main results

Before introduction of the main result, we need the following results.

**Lemma 2.1.** (Bellman [1]). Let  $A$  be a real symmetric  $n \times n$  -matrix and

$$a' \geq \lambda_i(A) \geq a > 0, (i = 1, 2, \dots, n),$$

where  $a'$  and  $a$  are constants.

Then

$$a' \langle X, X \rangle \geq \langle AX, X \rangle \geq a \langle X, X \rangle$$

and

$$a'^2 \langle X, X \rangle \geq \langle AX, AX \rangle \geq a^2 \langle X, X \rangle.$$

The following theorem, due to the Russian mathematician N. G. Četaev's (LaSalle and Lefschetz [4]).

**Theorem 2.1.** (Instability Theorem of Četaev's). Let  $\Omega$  be a neighborhood of the origin. Let there be given a function  $V(x)$  and region  $\Omega_1$  in  $\Omega$  with the following properties:

- (i)  $V(x)$  has continuous first partial derivatives in  $\Omega_1$ .
- (ii)  $V(x)$  and  $\dot{V}(x)$  are positive in  $\Omega_1$ .
- (iii) At the boundary points of  $\Omega_1$  inside  $\Omega$ ,  $V(x) = 0$ .
- (iv) The origin is a boundary point of  $\Omega_1$ .

Under these conditions the origin is unstable.

Let  $r \geq 0$  be given, and let  $C = C([-r, 0], \mathfrak{R}^n)$  with

$$\|\phi\| = \max_{-r \leq s \leq 0} |\phi(s)|, \phi \in C.$$

For  $H > 0$  define  $C_H \subset C$  by

$$C_H = \{\phi \in C : \|\phi\| < H\}.$$

If  $x : [-r, A) \rightarrow \mathfrak{R}^n$  is continuous,  $0 < A \leq \infty$ , then, for each  $t$  in  $[0, A)$ ,  $x_t$  in  $C$  is defined by

$$x_t(s) = x(t+s), -r \leq s \leq 0, t \geq 0.$$

Let  $G$  be an open subset of  $C$  and consider the general autonomous delay differential system with finite delay

$$\dot{x} = F(x_t), F(0) = 0, x_t = x(t+\theta), -r \leq \theta \leq 0, t \geq 0,$$

where  $F : G \rightarrow \mathfrak{R}^n$  is continuous and maps closed and bounded sets into bounded sets. It follows from these conditions on  $F$  that each initial value problem

$$\dot{x} = F(x_t), x_0 = \phi \in G$$

has a unique solution defined on some interval  $[0, A)$ ,  $0 < A \leq \infty$ . This solution will be denoted by  $x(\phi)(\cdot)$  so that  $x_0(\phi) = \phi$ .

**Definition 2.2.** The zero solution,  $x = 0$ , of  $\dot{x} = F(x_t)$  is stable if for each  $\varepsilon > 0$  there exists  $\delta = \delta(\varepsilon) > 0$  such that  $\|\phi\| < \delta$  implies that  $|x(\phi)(t)| < \varepsilon$  for all  $t \geq 0$ . The zero solution is said to be unstable if it is not stable.

The main result of this paper is the following theorem.

**Theorem 2.2.** *In addition to the basic assumptions imposed on  $\Psi$ ,  $\Phi$ ,  $H$  and  $F$  that appear in Eq. (1.2), we assume that there exist positive constants  $a_3$ ,  $a_4$  and  $a_5$  such that the following conditions hold:*

$$\Psi(Z), \Phi(X, Y, Z), J_H(Y) \text{ and } J_F(X) \text{ are symmetric,}$$

$$F(0) = 0, F(X) \neq 0, (X \neq 0), \lambda_i(J_F(X)) \leq -a_5$$

$$H(0) = 0, H(Y) \neq 0, (Y \neq 0), |\lambda_i(J_H(Y))| \leq a_4$$

and

$$\lambda_i(\Phi(X, Y, Z)) \geq a_3 \quad \text{for all } X \in \mathfrak{R}^n.$$

If

$$\tau < \min\left\{\frac{2}{\sqrt{n}}, \frac{2a_3}{2\sqrt{na_4} + \sqrt{na_5}}\right\},$$

then the zero solution of Eq. (1.2) is unstable.

*Remark 2.3.* It should be noted that there is no sign restriction on eigenvalues of  $\Psi$ , and it is clear that our assumptions have a very simple form and the applicability of them can be easily verified.

*Proof.* We define a Lyapunov-Krasovskii functional

$$\begin{aligned}
 V(\cdot) &= V(X(t), Y(t), Z(t), W(t), U(t)) : \\
 V(\cdot) &= -\langle Y, F(X) \rangle - \langle Z, U \rangle + \frac{1}{2} \langle W, W \rangle \\
 &\quad - \int_0^1 \langle H(\sigma Y), Y \rangle d\sigma - \int_0^1 \langle \sigma \Psi(\sigma Z) Z, Z \rangle d\sigma \\
 &\quad - \lambda \int_{-\tau}^0 \int_{t+s}^t \|Y(\theta)\|^2 d\theta ds - \mu \int_{-\tau}^0 \int_{t+s}^t \|Z(\theta)\|^2 d\theta ds,
 \end{aligned}$$

where  $\lambda$  and  $\mu$  are certain positive constants; the constants  $\lambda$  and  $\mu$  will be determined later in the proof.

It is clear that  $V(0, 0, 0, 0, 0) = 0$  and

$$V(0, 0, 0, \varepsilon, 0) = \frac{1}{2} \langle \varepsilon, \varepsilon \rangle = \frac{1}{2} \|\varepsilon\|^2 > 0$$

for all arbitrary  $\varepsilon \neq 0$ ,  $\varepsilon \in \mathfrak{R}^n$ , which verifies the property  $(P_1)$  of Krasovskii [3].

Using a basic calculation, the time derivative of  $V(\cdot)$  along solutions of (1.3) results in

$$\begin{aligned}
 \frac{d}{dt} V(\cdot) &= -\langle Y, J_F(X)Y \rangle + \langle \Psi(Z)W, Z \rangle + \langle Z, \Phi(X, Y, Z)Z \rangle \\
 &\quad + \langle H(Y), Z \rangle + \left\langle \int_{t-\tau}^t J_F(X(s))Y(s) ds, Z \right\rangle \\
 &\quad + \left\langle \int_{t-\tau}^t J_H(Y(s))Z(s) ds, Z \right\rangle \\
 &\quad - \frac{d}{dt} \int_0^1 \langle H(\sigma Y), Y \rangle d\sigma - \frac{d}{dt} \int_0^1 \langle \sigma \Psi(\sigma Z) Z, Z \rangle d\sigma \\
 &\quad - \lambda \frac{d}{dt} \int_{-\tau}^0 \int_{t+s}^t \|Y(\theta)\|^2 d\theta ds - \mu \frac{d}{dt} \int_{-\tau}^0 \int_{t+s}^t \|Z(\theta)\|^2 d\theta ds.
 \end{aligned}$$

It can be easily seen that

$$\begin{aligned}
\frac{d}{dt} \int_0^1 \langle H(\sigma Y), Y \rangle d\sigma &= \langle H(Y), Z \rangle, \\
\frac{d}{dt} \int_0^1 \langle \sigma \Psi(\sigma Z) Z, Z \rangle d\sigma &= \langle \Psi(Z) W, Z \rangle, \\
\frac{d}{dt} \int_{-\tau}^0 \int_{t+s}^t \|Y(\theta)\|^2 d\theta ds &= \|Y\|^2 \tau - \int_{t-\tau}^t \|Y(\theta)\|^2 d\theta, \\
\frac{d}{dt} \int_{-\tau}^0 \int_{t+s}^t \|Z(\theta)\|^2 d\theta ds &= \|Z\|^2 \tau - \int_{t-\tau}^t \|Z(\theta)\|^2 d\theta, \\
\langle \int_{t-\tau}^t J_F(X(s)) Y(s) ds, Z \rangle &\geq -\|Z\| \left\| \int_{t-\tau}^t J_F(X(s)) Y(s) ds \right\| \\
&\geq -\sqrt{na_5} \|Z\| \left\| \int_{t-\tau}^t Y(s) ds \right\| \\
&\geq -\sqrt{na_5} \|Z\| \int_{t-\tau}^t \|Y(s)\| ds \\
&\geq -\frac{1}{2} \sqrt{na_5} \tau \|Z\|^2 - \frac{1}{2} \sqrt{na_5} \int_{t-\tau}^t \|Y(s)\|^2 ds
\end{aligned}$$

and

$$\begin{aligned}
\langle \int_{t-\tau}^t J_H(Y(s)) Z(s) ds, Z \rangle &\geq -\|Z\| \left\| \int_{t-\tau}^t J_H(Y(s)) Z(s) ds \right\| \\
&\geq -\sqrt{na_4} \|Z\| \left\| \int_{t-\tau}^t Z(s) ds \right\| \\
&\geq -\sqrt{na_4} \|Z\| \int_{t-\tau}^t \|Z(s)\| ds \\
&\geq -\frac{1}{2} \sqrt{na_4} \tau \|Z\|^2 - \frac{1}{2} \sqrt{na_4} \int_{t-\tau}^t \|Z(s)\|^2 ds
\end{aligned}$$

so that

$$\begin{aligned}
\frac{d}{dt}V(\cdot) &\geq -\langle Y, J_F(X)Y \rangle + \langle Z, \Phi(X, Y, Z)Z \rangle \\
&\quad -\frac{1}{2}\sqrt{na_5}\tau \langle Z, Z \rangle - \frac{1}{2}\sqrt{na_5} \int_{t-\tau}^t \|Y(s)\|^2 ds \\
&\quad -\frac{1}{2}\sqrt{na_4}\tau \langle Z, Z \rangle - \frac{1}{2}\sqrt{na_4} \int_{t-\tau}^t \|Z(s)\|^2 ds \\
&\quad -\lambda\tau \langle Y, Y \rangle + \lambda \int_{t-\tau}^t \|Y(\theta)\|^2 d\theta \\
&\quad -\mu\tau \langle Z, Z \rangle + \mu \int_{t-\tau}^t \|Z(\theta)\|^2 d\theta \\
&\geq (a_5 - \lambda\tau) \|Y\|^2 \\
&\quad + \{a_3 - (\mu + \frac{1}{2}\sqrt{na_4} + \frac{1}{2}\sqrt{na_5})\tau\} \|Z\|^2 \\
&\quad + (\lambda - \frac{1}{2}\sqrt{na_5}) \int_{t-\tau}^t \|Y(s)\|^2 ds \\
&\quad + (\mu - \frac{1}{2}\sqrt{na_4}) \int_{t-\tau}^t \|Z(s)\|^2 ds.
\end{aligned}$$

Let

$$\lambda = \frac{1}{2}\sqrt{na_5}$$

and

$$\mu = \frac{1}{2}\sqrt{na_4}$$

so that

$$\frac{d}{dt}V(\cdot) \geq \{(a_5 - \frac{1}{2}\sqrt{na_5})\tau\} \|Y\|^2 + \{(a_3 - (\sqrt{na_4} + \frac{1}{2}\sqrt{na_5})\tau)\} \|Z\|^2.$$

If  $\tau < \min\{\frac{2}{\sqrt{n}}, \frac{2a_3}{2\sqrt{na_4} + \sqrt{na_5}}\}$ , then we have for some positive constants  $k_1$  and  $k_2$  that

$$\frac{d}{dt}V(\cdot) \geq k_1 \|Y\|^2 + k_2 \|Z\|^2 \geq 0,$$

which verifies the property  $(P_2)$  of Krasovskii [3].

On the other hand, it follows that

$$\frac{d}{dt}V(\cdot) = 0 \Leftrightarrow Y = \dot{X} = 0, Z = \dot{Y} = 0, W = \dot{Z} = 0, U = \dot{W} = 0 \quad \text{for all } t \geq 0.$$

Hence

$$X = \xi, Y = Z = W = U = 0,$$

where  $\xi$  is a constant vector.

Substituting foregoing estimates in the system (1.3), we get that  $F(\xi) = 0$ , which necessarily implies that  $\xi = 0$  since  $F(0) = 0$ . Thus, we have

$$X = Y = Z = W = U = 0 \quad \text{for all } t \geq 0.$$

Hence, the property ( $P_3$ ) of Krasovskii [3] holds

The proof of Theorem 2.2 is complete.  $\square$

**Example 2.4.** In a special case of Eq. (1.2), for  $n = 2$ , we choose

$$\begin{aligned} \Psi(Z) &= \begin{bmatrix} z_1 & 1 \\ 1 & z_2 \end{bmatrix}, \\ \Phi(X, Y, Z) &= \begin{bmatrix} 9 + \frac{1}{1+x_1^2+y_1^2+z_1^2} & 0 \\ 0 & 9 + \frac{1}{1+x_2^2+y_2^2+z_2^2} \end{bmatrix}, \\ H(Y(t-\tau)) &= \begin{bmatrix} 4y_1(t-\tau) \\ 4y_2(t-\tau) \end{bmatrix} \end{aligned}$$

and

$$F(X(t-\tau)) = \begin{bmatrix} -3x_1(t-\tau) \\ -3x_2(t-\tau) \end{bmatrix}.$$

Then, the matrix  $\Psi(Z)$  is symmetric, and, by an easy calculation, we obtain

$$\lambda_1(\Phi(X, Y, Z)) = 9 + \frac{1}{1+x_1^2+y_1^2+z_1^2},$$

$$\lambda_2(\Phi(X, Y, Z)) = 9 + \frac{1}{1+x_2^2+y_2^2+z_2^2},$$

$$J_H(Y) = \begin{bmatrix} 4 & 0 \\ 0 & 4 \end{bmatrix}$$

and

$$J_F(X) = \begin{bmatrix} -3 & 0 \\ 0 & -3 \end{bmatrix}$$

so that

$$\begin{aligned} \lambda_i(\Phi(X, Y, Z)) &\geq 9 = a_3 > 0, \\ |\lambda_i(J_H(Y))| &= 4 = a_4 \end{aligned}$$



and

$$\lambda_i(J_F(X)) \leq -3 = -a_5, \quad (i = 1, 2).$$

Thus, if

$$\tau < \min\left\{\frac{2}{\sqrt{2}}, \frac{8}{8\sqrt{2} + 3\sqrt{2}}\right\},$$

then all the assumptions of Theorem 2.2 hold.

### Acknowledgement

The author would like to express his sincere appreciation to the reviewer for his/her helpful comments, corrections and suggestions which helped with improving the presentation and quality of this work.

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**DOI: 10.7862/rf.2013.11**

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*Received 1.08.2012,    Revisted 29.08.2013,    Accepted 25.10.2013*

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Manuscript completed in December 2013, Printed in December 2013.

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