

A Refinement of Schwarz's Lemma and its Applications

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ABSTRACT: By using the value of the second derivative of the function at 0, along with the values of the function and its first derivative at 0, we have obtained a refinement of well known Schwarz's lemma and have used this refinement to obtain refinements, of Aziz and Rather's inequalities [2004] for a polynomial of degree n having no zeros in $|z| < k$, ($k \geq 1$).

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1. Introduction and statement of results

Concerning the values of a function, analytic in the interior of a disc, we have the following well known result, known as Schwarz's lemma [2, p. 189-190].

Theorem A. *If $f(z)$ is analytic in $|z| < 1$, where it satisfies the inequality $|f(z)| \leq 1$, and if $f(0) = 0$, then the inequality*

$$|f(z)| \leq |z|$$

holds whenever $|z| < 1$. Moreover equality can occur only when $f(z) = ze^{i\alpha}$, where α is a real constant.

There is a generalization of Theorem A, known as generalization of Schwarz's lemma [6, p. 212], which can be stated as

Theorem B. *If $f(z)$ is analytic and $|f(z)| \leq 1$, in $|z| < 1$ then*

$$|f(z)| \leq \frac{|z| + |a|}{|a||z| + 1}, \quad |z| < 1,$$

where

$$a = f(0).$$

Govil et al. [4] obtained the following refinement of Theorem B, by using the value of the first derivative of the function at 0, along with the value of the function at 0.

Theorem C. *If $f(z)$ is analytic and $|f(z)| \leq 1$, in $|z| < 1$ then*

$$|f(z)| \leq \begin{cases} \frac{(1-|a|)|z|^2+|bz|+|a|(1-|a|)}{|a|(1-|a|)|z|^2+|bz|+(1-|a|)}, & |a| < 1, \\ 1 & , |a| = 1, \end{cases}, \quad |z| < 1, \quad (1.1)$$

where

$$a = f(0), \quad b = f'(0).$$

The example

$$f(z) = (a + \frac{b}{1+a}z - z^2)/(1 - \frac{b}{1+a}z - az^2)$$

shows that the estimate is sharp.

In this paper we have firstly obtained a refinement of Theorem C, thereby giving a refinement of Theorem A also, by using the value of the second derivative of the function at 0, along with the values of the function and its first derivative at 0. More precisely we have proved

Theorem 1. *Let $f(z)$ be analytic in $|z| < 1$, with*

$$|f(z)| \leq 1, \quad |z| < 1,$$

$$a = f(0), \quad b = f'(0), \quad c = f''(0), \quad (1.2)$$

$$\gamma = \begin{cases} \arg \bar{a}, & a \neq 0, \\ \text{any value}, & a = 0, \end{cases}, \quad (1.3)$$

$$A = 2(1 - |a|^2)(1 - |a|^2 - |b|), \quad (1.4)$$

$$B = 2|a||b|(1 - |a|^2 - |b|) + ||a|^2c - 2e^{i\gamma}|a|b^2 - c|, \quad (1.5)$$

$$C = 2|b|(1 - |a|^2 - |b|) + |a||a|^2c - 2e^{i\gamma}|a|b^2 - c| \quad (1.6)$$

and

$$D = 2|a|(1 - |a|^2)(1 - |a|^2 - |b|). \quad (1.7)$$

Then

$$|f(z)| \leq \begin{cases} \frac{A|z|^3+B|z|^2+C|z|+D}{D|z|^3+C|z|^2+B|z|+A}, & |a| < 1 \text{ and } |b| < 1 - |a|^2, \\ \frac{|z|+|a|}{|a||z|+1}, & |a| < 1 \text{ and } |b| = 1 - |a|^2, \\ 1 & , |a| = 1, \end{cases}, \quad |z| < 1. \quad (1.8)$$

Remark 1. By using the result [5, p. 172, exercise # 9] one can show that Theorem 1 is a refinement of Theorem C.

Further for a polynomial $p(z)$, let

$$\begin{aligned}\|p\|_q &= \left\{ \frac{1}{2\pi} \int_0^{2\pi} |p(e^{i\theta})|^q d\theta \right\}^{1/q}, \quad q > 0, \\ \|p\|_\infty &= \max_{|z|=1} |p(z)|.\end{aligned}$$

Then secondly we have used Theorem 1 to obtain a refinement of Aziz and Rather's result [1, Theorem 1 and Corollary 1]

Theorem D. Let $P(z) = \sum_{j=0}^n a_j z^j$ be a polynomial of degree n , having no zeros in $|z| < k, (k \geq 1)$, with

$$1 \leq s < n$$

and

$$\delta_{k,s} = \frac{C(n,s)|a_0|k^{s+1} + |a_s|k^{2s}}{C(n,s)|a_0| + |a_s|k^{s+1}}. \quad (1.9)$$

Then

$$\|P^{(s)}\|_q \leq \frac{n(n-1)\dots(n-s+1)}{\|\delta_{k,s} + z\|_q} \|P\|_q, \quad q > 0$$

and

$$\|P^{(s)}\|_\infty \leq n(n-1)\dots(n-s+1) \left\{ \frac{C(n,s)|a_0| + |a_s|k^{s+1}}{C(n,s)|a_0|(1+k^{s+1}) + k^{s+1}(k^{s-1}+1)|a_s|} \right\} \|P\|_\infty.$$

More precisely we have proved

Theorem 2. Let $P(z) = \sum_{j=0}^n a_j z^j$ be a polynomial of degree n , having no zeros in $|z| < k, (k \geq 1)$, with

$$1 \leq s < n,$$

$$E = \frac{C(n,s)|a_0| + |a_s|k^{s+1}}{C(n,s)|a_0|k + |a_s|k^s}, \quad (1.10)$$

$$\delta_{k,s} = k^s/E, \text{ (by (1.9))},$$

$$F = |n(s+1)a_0a_{s+1} - (n-s)a_1a_s|, \quad (1.11)$$

$$\begin{aligned}H &= n|a_0|^2 \{C(n,s)\}^2 + |a_0||a_s|nC(n,s)k^s(k^2-1) - n|a_s|^2k^{2s+2} \\ &\quad + k^{s+2}C(n,s)F, \\ J &= nk^2 \{C(n,s)\}^2 |a_0|^2 - nC(n,s)|a_0||a_s|k^s(k^2-1) - n|a_s|^2k^{2s} \\ &\quad + k^{s+2}C(n,s)F,\end{aligned}$$

$$d_{k,s} = \begin{cases} k^s J/H, \frac{k^s}{C(n,s)} \left| \frac{a_s}{a_0} \right| < 1, \\ k^s, \frac{k^s}{C(n,s)} \left| \frac{a_s}{a_0} \right| = 1, \end{cases}$$

$$a_{n+1} = 0,$$

$$\gamma = \begin{cases} \arg \left(\frac{k^s \overline{a_s}}{C(n,s)a_0} \right), \frac{k^s a_s}{C(n,s)a_0} \neq 0, \\ \text{any value}, \frac{k^s a_s}{C(n,s)a_0} = 0, \end{cases}$$

$$\begin{aligned} G = & |a_0 \{ n^2(n-1)a_0^2(s+2)(s+1)a_{s+2} - 2n(n-s)a_0((n-s-1)a_2a_s + \\ & (n-1)(s+1)a_1a_{s+1}) + 2(n-1)(n-s)^2a_1^2a_s \} \{ k^{2s}|a_s|^2 - \\ & (C(n,s))^2|a_0|^2 \} - 2e^{i\gamma}(n-1)|a_0||a_s|k^{2s} \{ n(s+1)a_0a_{s+1} - \\ & (n-s)a_1a_s \}^2|, \end{aligned} \quad (1.12)$$

$$\begin{aligned} L = & 2n^2(n-1) \{ (C(n,s))^5|a_0|^7 + (C(n,s))^4k^{s+3}|a_0|^6|a_s| - \\ & 2(C(n,s))^3k^{2s}|a_0|^5|a_s|^2 - 2(C(n,s))^2k^{3s+3}|a_0|^4|a_s|^3 \\ & + C(n,s)k^{4s}|a_0|^3|a_s|^4 + k^{5s+3}|a_0|^2|a_s|^5 \} + \\ & 2n(n-1)C(n,s)k^{s+1}|a_0|^2F(k^2-1) \{ |a_0|^3(C(n,s))^3 - \\ & (C(n,s))^2|a_0|^2|a_s|k^{s+1} - (C(n,s))|a_0||a_s|^2k^{2s} + \\ & |a_s|^3k^{3s+1} \} + C(n,s)k^{s+3}G \{ C(n,s)|a_0| + |a_s|k^{s+1} \} - \\ & 2(n-1)(C(n,s))^2k^{2s+3}F^2|a_0|^2 \{ kC(n,s)|a_0| + k^s|a_s| \}, \end{aligned} \quad (1.13)$$

$$\begin{aligned} S = & 2n^2(n-1)|a_0|^2 \{ (C(n,s))^5|a_0|^5k^3 + (C(n,s))^4|a_0|^4|a_s|k^s - \\ & 2(C(n,s))^3|a_0|^3|a_s|^2k^{2s+3} - 2(C(n,s))^2|a_0|^2|a_s|^3k^{3s} + \\ & C(n,s)|a_0||a_s|^4k^{4s+3} + |a_s|^5k^{5s} \} + 2n(n-1)|a_0|^2k^{s+1}C(n,s)(k^2-1) \\ & F \{ -(C(n,s))^3k|a_0|^3 + (C(n,s))^2k^s|a_0|^2|a_s| + C(n,s)k^{2s+1}|a_0||a_s|^2 \\ & - |a_s|^3k^{3s} \} - 2(n-1)(C(n,s))^2k^{2s+3}|a_0|^2F^2 \{ |a_0|C(n,s) + \\ & k^{s+1}|a_s| \} + Gk^{s+3}C(n,s) \{ |a_0|k + |a_s|k^s \}, \end{aligned} \quad (1.14)$$

$$\gamma_{k,s} = \begin{cases} \frac{k^s S}{L}, \frac{k^s}{C(n,s)} \left| \frac{a_s}{a_0} \right| < 1 \text{ and } \frac{k^{s+1}F}{n|a_0|^2C(n,s)} < 1 - \frac{k^{2s}}{(C(n,s))^2} \left| \frac{a_s}{a_0} \right|^2, & (1.15) \\ \frac{k^s}{E}, \frac{k^s}{C(n,s)} \left| \frac{a_s}{a_0} \right| < 1 \text{ and } \frac{k^{s+1}F}{n|a_0|^2C(n,s)} = 1 - \frac{k^{2s}}{(C(n,s))^2} \left| \frac{a_s}{a_0} \right|^2, & (1.16) \\ k^s, \frac{k^s}{C(n,s)} \left| \frac{a_s}{a_0} \right| = 1. & (1.17) \end{cases}$$

Then for $q > 0$

$$\begin{aligned}
 \|P^{(s)}\|_q &\leq \frac{n(n-1)\dots(n-s+1)}{\|\gamma_{k,s} + z\|_q} \|P\|_q, \\
 &\leq \frac{n(n-1)\dots(n-s+1)}{\|d_{k,s} + z\|_q} \|P\|_q, \\
 &\leq \frac{n(n-1)\dots(n-s+1)}{\|\delta_{k,s} + z\|_q} \|P\|_q, \\
 &\leq \frac{n(n-1)\dots(n-s+1)}{\|k^s + z\|_q} \|P\|_q.
 \end{aligned} \tag{1.18}$$

Remark 2. With the additional assumption

$$a_{n+2} = 0,$$

one can show that Theorem 2 is true for $s = n$ also.

By letting $q \rightarrow \infty$ in (1.18), we obtain

Corollary 1. Under the same hypotheses as in Theorem 2

$$\begin{aligned}
 \|P^{(s)}\|_\infty &\leq \frac{n(n-1)\dots(n-s+1)}{(\gamma_{k,s} + 1)} \|P\|_\infty, \\
 &\leq \frac{n(n-1)\dots(n-s+1)}{(d_{k,s} + 1)} \|P\|_\infty, \\
 &\leq \frac{n(n-1)\dots(n-s+1)}{(\delta_{k,s} + 1)} \|P\|_\infty, \\
 &\leq \frac{n(n-1)\dots(n-s+1)}{(k^s + 1)} \|P\|_\infty.
 \end{aligned}$$

Remark 3. With the additional assumption

$$a_{n+2} = 0,$$

one can show that Corollary 1 is true for $s = n$ also.

Remark 4. Corollary 1 is also a refinement of Govil and Rahman's result [3, Theorem 4].

2. Lemmas

For the proofs of the theorems we require the following lemmas.

Lemma 1. *If $f(z)$ is analytic and $|f(z)| \leq 1$, in $|z| \leq 1$ then*

$$|f(z)| \leq \frac{|z| + |a|}{1 + |a||z|}, \quad |z| \leq 1,$$

where

$$a = f(0).$$

Proof of Lemma 1. It easily follows from Theorem B.

Lemma 2. *If $f(z)$ is analytic and $|f(z)| \leq 1$, in $|z| \leq 1$ then*

$$|f(z)| \leq \left\{ \begin{array}{l} \frac{(1-|a|)|z|^2+|bz|+|a|(1-|a|)}{|a|(1-|a|)|z|^2+|bz|+(1-|a|)}, |a| < 1, \\ 1, |a| = 1, \end{array} \right\}, \quad |z| \leq 1,$$

where

$$a = f(0), \quad b = f'(0).$$

Proof of Lemma 2. It easily follows from Theorem C.

Remark 5. By using the result [5, p.172, exercise # 9] one can show that Lemma 2 is a refinement of Lemma 1.

Lemma 3. *Let $f(z)$ be analytic in $|z| \leq 1$, with*

$$|f(z)| \leq 1, \quad |z| \leq 1,$$

$$a = f(0), \quad b = f'(0), \quad c = f''(0),$$

$$\gamma = \left\{ \begin{array}{l} \arg \bar{a}, \quad a \neq 0, \\ \text{any value}, \quad a = 0, \end{array} \right\}.$$

Then

$$|f(z)| \leq \left\{ \begin{array}{l} \frac{A|z|^3+B|z|^2+C|z|+D}{D|z|^3+C|z|^2+B|z|+A}, |a| < 1 \text{ and } |b| < 1 - |a|^2, \\ \frac{|z|+|a|}{1+|a||z|}, |a| < 1 \text{ and } |b| = 1 - |a|^2, \\ 1, |a| = 1, \end{array} \right\}, \quad |z| \leq 1,$$

where A, B, C and D are, as in Theorem 1.

Proof of Lemma 3. It easily follows from Theorem 1.

Remark 6. By Remark 1 one can say that Lemma 3 is a refinement of Lemma 2.

Lemma 4. *Under the same hypotheses as in Theorem 2*

$$k^s |P^{(s)}(z)| \leq \delta_{k,s} |P^{(s)}(z)| \leq d_{k,s} |P^{(s)}(z)| \leq \gamma_{k,s} |P^{(s)}(z)| \leq |Q^{(s)}(z)|, \quad |z| = 1, \quad (2.1)$$

where

$$Q(z) = z^n \overline{P(1/\bar{z})}. \quad (2.2)$$

Proof of Lemma 4. Let

$$\begin{aligned} G(z) &= z^n \overline{P(k/\bar{z})} = k^n Q(z/k), \\ H(z) &= z^{n-s} \overline{G^{(s)}(1/\bar{z})} = (kz)^{n-s} \overline{Q^{(s)}(1/(k\bar{z}))}. \end{aligned} \quad (2.3)$$

Then by using the result [1, inequality 32] we get

$$k^s |P^{(s)}(kz)| \leq |H(z)|, \quad |z| = 1, \quad (2.4)$$

with $H(z)$ having all its zeros in $|z| \geq 1$. Further let

$$H(z) = \Phi(z)H_1(z), \quad (2.5)$$

with

$$H_1(z) \neq 0, \quad |z| = 1$$

and

$$\Phi(z) = \begin{cases} 1 & ; H(z) \neq 0 \text{ on } |z| = 1, \\ \prod_{\gamma=1}^m (z - z_\gamma), |z_\gamma| = 1 \forall \gamma; & H(z) \text{ has certain zeros on } |z| = 1. \end{cases} \quad (2.6)$$

Then by (2.4), (2.5) and (2.6) we have

$$k^s P^{(s)}(kz) = \Phi(z)P_1(z), \quad (2.7)$$

with

$$|P_1(z)| \leq |H_1(z)|, \quad |z| = 1. \quad (2.8)$$

Now

$$f(z) = \frac{P_1(z)}{H_1(z)} \quad (2.9)$$

is analytic in $|z| \leq 1$, with

$$|f(z)| \leq 1, \quad |z| = 1, \text{ (by (2.8))}$$

and therefore

$$|f(z)| \leq 1, \quad |z| \leq 1, \text{ (by maximum modulus principle),} \quad (2.10)$$

with

$$|f(z)| \leq \frac{|z| + |a|}{1 + |a||z|}, \quad |z| \leq 1, \text{ (a refinement of (2.10)),} \quad (2.11)$$

$$|f(z)| \leq \left\{ \begin{array}{l} \frac{(1-|a|)|z|^2 + |b||z| + |a|(1-|a|)}{|a|(1-|a|)|z|^2 + |b||z| + (1-|a|)}, |a| < 1, \\ 1, |a| = 1, \end{array} \right\}, \quad |z| \leq 1,$$

$$\text{(a refinement of (2.11)), (by Remark 5)} \quad (2.12)$$

and

$$|f(z)| \leq \left\{ \begin{array}{l} \frac{A|z|^3 + B|z|^2 + C|z| + D}{D|z|^3 + C|z|^2 + B|z| + A}, |a| < 1 \text{ and } |b| < 1 - |a|^2, \\ \frac{|z| + |a|}{1 + |a||z|}, |a| < 1 \text{ and } |b| = 1 - |a|^2, \\ 1, |a| = 1, \end{array} \right\}, \quad |z| \leq 1,$$

$$\text{(a refinement of (2.12)), (by Remark 6).} \quad (2.13)$$

Further if

$$g(z) = \frac{k^s P^{(s)}(kz)}{H(z)}$$

then

$$\begin{aligned} a &= f(0), \text{ (by (1.2))}, \\ &= g(0), \text{ (by (2.5), (2.7) and (2.9))}, \\ &= \frac{k^s a_s}{C(n, s) a_0}, \end{aligned} \quad (2.14)$$

$$\begin{aligned} b &= f'(0), \text{ (by (1.2))}, \\ &= g'(0), \text{ (by (2.5), (2.7) and (2.9))}, \\ &= \frac{k^{s+1}}{n a_0^2 C(n, s)} \{n(s+1) a_0 a_{s+1} - (n-s) a_1 a_s\}, \end{aligned} \quad (2.15)$$

$$\begin{aligned} c &= f''(0), \text{ (by (1.2))}, \\ &= g''(0), \text{ (by (2.5), (2.7) and (2.9))}, \\ &= \frac{k^{s+2}}{n^2 (n-1) a_0^3 C(n, s)} \{n^2 (n-1) a_0^2 (s+1) (s+2) a_{s+2} - \\ &\quad 2n(n-s) a_0 ((n-s-1) a_2 a_s + \\ &\quad (n-1)(s+1) a_1 a_{s+1}) + 2(n-1)(n-s)^2 a_1^2 a_s\} \end{aligned} \quad (2.16)$$

and on using (2.13), with

$$z = \frac{1}{k} e^{i\theta}, \quad 0 \leq \theta \leq 2\pi,$$

we get

$$\left| f\left(\frac{1}{k} e^{i\theta}\right) \right| \leq \left\{ \begin{array}{l} L/S, \frac{k^s}{C(n, s)} \left| \frac{a_s}{a_0} \right| < 1 \text{ and } \frac{k^{s+1} F}{n |a_0|^2 C(n, s)} < 1 - \frac{k^{2s}}{(C(n, s))^2} \left| \frac{a_s}{a_0} \right|^2, \\ E, \frac{k^s}{C(n, s)} \left| \frac{a_s}{a_0} \right| < 1 \text{ and } \frac{k^{s+1} F}{n |a_0|^2 C(n, s)} = 1 - \frac{k^{2s}}{(C(n, s))^2} \left| \frac{a_s}{a_0} \right|^2, \\ 1, \frac{k^s}{C(n, s)} \left| \frac{a_s}{a_0} \right| = 1, \end{array} \right\},$$

(by (2.14), (2.15), (2.16), (1.3), (1.4), (1.5), (1.6), (1.7), (1.11), (1.12), (1.13), (1.14) and (1.10)),

i.e.

$$\left| P_1\left(\frac{1}{k} e^{i\theta}\right) \right| \leq \left\{ \begin{array}{l} L/S, \frac{k^s}{C(n, s)} \left| \frac{a_s}{a_0} \right| < 1 \text{ and } \frac{k^{s+1} F}{n |a_0|^2 C(n, s)} < 1 - \frac{k^{2s}}{(C(n, s))^2} \left| \frac{a_s}{a_0} \right|^2, \\ E, \frac{k^s}{C(n, s)} \left| \frac{a_s}{a_0} \right| < 1 \text{ and } \frac{k^{s+1} F}{n |a_0|^2 C(n, s)} = 1 - \frac{k^{2s}}{(C(n, s))^2} \left| \frac{a_s}{a_0} \right|^2, \\ 1, \frac{k^s}{C(n, s)} \left| \frac{a_s}{a_0} \right| = 1, \end{array} \right\} \left| H_1\left(\frac{1}{k} e^{i\theta}\right) \right|, \quad (\text{by (2.9)}),$$

which, by (2.5) and (2.7), implies that

$$k^s |P^{(s)}(e^{i\theta})| \leq \left\{ \begin{array}{l} L/S, \frac{k^s}{C(n,s)} \left| \frac{a_s}{a_0} \right| < 1 \text{ and } \frac{k^{s+1}F}{n|a_0|^2 C(n,s)} < 1 - \frac{k^{2s}}{(C(n,s))^2} \left| \frac{a_s}{a_0} \right|^2, \\ E, \frac{k^s}{C(n,s)} \left| \frac{a_s}{a_0} \right| < 1 \text{ and } \frac{k^{s+1}F}{n|a_0|^2 C(n,s)} = 1 - \frac{k^{2s}}{(C(n,s))^2} \left| \frac{a_s}{a_0} \right|^2, \\ 1, \frac{k^s}{C(n,s)} \left| \frac{a_s}{a_0} \right| = 1, \end{array} \right\} \left| H\left(\frac{1}{k}e^{i\theta}\right) \right|. \quad (2.17)$$

And as

$$H(z/k) = z^{n-s} \overline{Q^{(s)}(1/\bar{z})}, \quad (\text{by (2.3)}),$$

we get, by using (1.15), (1.16) and (1.17) in (2.17), that

$$\gamma_{k,s} |P^{(s)}(e^{i\theta})| \leq |Q^{(s)}(e^{i\theta})|, \quad 0 \leq \theta \leq 2\pi. \quad (2.18)$$

Now as we have obtained (2.18) by using (2.13), we can similarly obtain

$$d_{k,s} |P^{(s)}(e^{i\theta})| \leq |Q^{(s)}(e^{i\theta})|, \quad 0 \leq \theta \leq 2\pi,$$

$$\delta_{k,s} |P^{(s)}(e^{i\theta})| \leq |Q^{(s)}(e^{i\theta})|, \quad 0 \leq \theta \leq 2\pi$$

and

$$k^s |P^{(s)}(e^{i\theta})| \leq |Q^{(s)}(e^{i\theta})|, \quad 0 \leq \theta \leq 2\pi,$$

by using (2.12), (2.11) and (2.10) respectively. Further

$$k^s \leq \delta_{k,s} \leq d_{k,s} \leq \gamma_{k,s} \quad (2.19)$$

follows from the fact

$$\left\{ \begin{array}{l} \frac{A|z|^3+B|z|^2+C|z|+D}{D|z|^3+C|z|^2+B|z|+A}, |a| < 1 \text{ and } |b| < 1 - |a|^2, \\ \frac{|z|+|a|}{1+|a||z|}, |a| < 1 \text{ and } |b| = 1 - |a|^2, \\ 1, |a| = 1, \end{array} \right\} \leq \left\{ \begin{array}{l} \frac{(1-|a|)|z|^2+|b||z|+|a|(1-|a|)}{|a|(1-|a|)|z|^2+|b||z|+(1-|a|)}, |a| < 1, \\ 1, |a| = 1, \end{array} \right\} \leq \frac{|z|+|a|}{1+|a||z|} \leq 1, \quad |z| \leq 1,$$

(by (2.13), (2.12), (2.11) and (2.10))

and the way, L. H. S. of inequality (2.17) was obtained from R. H. S. of inequality (2.18). This completes the proof of Lemma 4. \square

Remark 7. With the additional assumption

$$a_{n+2} = 0,$$

one can show that Lemma 4 is true for $s = n$ also.

3. Proofs of the theorems

Proof of Theorem 1. If

$$|f(0)| = 1$$

then result follows trivially. Therefore from now onwards we will assume that

$$|f(0)| < 1.$$

Now we consider the function

$$g(z) = \frac{e^{i\gamma}f(z) - |f(0)|}{|f(0)|e^{i\gamma}f(z) - 1}, \quad (3.1)$$

which is analytic in $|z| < 1$, with

$$\begin{aligned} |g(z)| &\leq 1, \quad |z| < 1, \\ g(0) &= 0, \\ |g(z)| &\leq |z|, \quad |z| < 1, \text{ (by Schwarz's lemma),} \\ g'(0) &= \frac{e^{i\gamma}f'(0)}{|f(0)|^2 - 1}, \end{aligned} \quad (3.2)$$

$$g''(0) = \frac{e^{i\gamma}}{(|f(0)|^2 - 1)^2} [f''(0)\{|f(0)|^2 - 1\} - 2e^{i\gamma}|f(0)|(f'(0))^2] \quad (3.3)$$

and the function

$$\Phi(z) = \begin{cases} \frac{g(z)}{z}, & 0 < |z| < 1, \\ g'(0), & z = 0, \end{cases} \quad (3.4)$$

$$(3.5)$$

which is analytic in $|z| < 1$, with

$$\begin{aligned} |\Phi(z)| &\leq 1, \quad |z| < 1, \\ \Phi'(0) &= \frac{1}{2}g''(0). \end{aligned} \quad (3.6)$$

We apply Theorem C to $\Phi(z)$. If

$$|\Phi(0)| = 1$$

then by (1.1) and (3.4)

$$\left| \frac{g(z)}{z} \right| \leq 1, \quad 0 < |z| < 1,$$

i.e.

$$\left| \frac{e^{i\gamma}f(z) - |f(0)|}{|f(0)|e^{i\gamma}f(z) - 1} \right| \leq |z|, \quad |z| < 1, \text{ (by (3.1)),}$$

which implies

$$|e^{i\gamma} f(z)| \leq \frac{|z| + |f(0)|}{1 + |f(0)||z|}, \quad |z| < 1,$$

thereby proving second part of (1.8). And if

$$|\Phi(0)| < 1$$

then by (1.1) and (3.4)

$$\left| \frac{g(z)}{z} \right| \leq \frac{(1 - |\Phi(0)|)|z|^2 + |\Phi'(0)||z| + |\Phi(0)|(1 - |\Phi(0)|)}{|\Phi(0)|(1 - |\Phi(0)|)|z|^2 + |\Phi'(0)||z| + (1 - |\Phi(0)|)}, \quad 0 < |z| < 1,$$

i.e.

$$\begin{aligned} \left| \frac{e^{i\gamma} f(z) - |f(0)|}{|f(0)|e^{i\gamma} f(z) - 1} \right| &\leq |z| \frac{(1 - |\Phi(0)|)|z|^2 + |\Phi'(0)||z| + |\Phi(0)|(1 - |\Phi(0)|)}{|\Phi(0)|(1 - |\Phi(0)|)|z|^2 + |\Phi'(0)||z| + (1 - |\Phi(0)|)}, \quad |z| < 1, \\ &= E_0, \text{ (say), } \quad |z| < 1, \end{aligned} \quad \text{(by (3.1)),}$$

which implies

$$\begin{aligned} |e^{i\gamma} f(z)| &\leq \frac{E_0 + |f(0)|}{1 + E_0|f(0)|}, \quad |z| < 1, \\ &= \frac{A|z|^3 + B|z|^2 + C|z| + D}{D|z|^3 + C|z|^2 + B|z| + A}, \quad |z| < 1, \\ &\text{(by (3.5), (3.6), (3.2), (3.3), (1.2), (1.4), (1.5), (1.6) and (1.7)),} \end{aligned}$$

thereby proving first part of (1.8). This completes the proof of Theorem 1. \square

Proof of Theorem 2. As Aziz and Rather [1, Theorem 1] had obtained

$$\|P^{(s)}\|_q \leq \frac{n(n-1)\dots(n-s+1)}{\|\delta_{k,s} + z\|_q} \|P\|_q,$$

by using

$$\delta_{k,s}|P^{(s)}(z)| \leq |Q^{(s)}(z)|, \quad |z| = 1, \text{ (by (2.2) and (2.1)),}$$

one can similarly obtain

$$\begin{aligned} \|P^{(s)}\|_q &\leq \frac{n(n-1)\dots(n-s+1)}{\|\gamma_{k,s} + z\|_q} \|P\|_q, \\ \|P^{(s)}\|_q &\leq \frac{n(n-1)\dots(n-s+1)}{\|d_{k,s} + z\|_q} \|P\|_q \end{aligned}$$

and

$$\|P^{(s)}\|_q \leq \frac{n(n-1)\dots(n-s+1)}{\|k^s + z\|_q} \|P\|_q,$$

by using

$$\gamma_{k,s}|P^{(s)}(z)| \leq |Q^{(s)}(z)|, \quad |z| = 1, \quad (\text{by (2.2) and (2.1)}),$$

$$d_{k,s}|P^{(s)}(z)| \leq |Q^{(s)}(z)|, \quad |z| = 1, \quad (\text{by (2.2) and (2.1)})$$

and

$$k^s|P^{(s)}(z)| \leq |Q^{(s)}(z)|, \quad |z| = 1, \quad (\text{by (2.2) and (2.1)})$$

respectively. Further

$$1 \leq k^s \leq \delta_{k,s} \leq d_{k,s} \leq \gamma_{k,s}, \quad (\text{by (2.19)})$$

implies

$$|k^s + e^{i\theta}| \leq |\delta_{k,s} + e^{i\theta}| \leq |d_{k,s} + e^{i\theta}| \leq |\gamma_{k,s} + e^{i\theta}|, \quad 0 \leq \theta \leq 2\pi,$$

thereby proving last three inequalities in (1.18). This completes the proof of Theorem 2. \square

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