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On a class of meromorphic functions defined by the convolution

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ABSTRACT: In the present paper we define some classes of meromorphic functions with fixed argument of coefficients. Next we obtain coefficient estimates, distortion theorems, integral means inequalities, the radii of convexity and starlikeness and convolution properties for the defined class of functions.

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Dedicated to Professor Leon Mikołajczyk

1 Introduction

Let $\widetilde{\mathcal{M}}$ denote the class of functions which are *analytic* in $\mathcal{D} = \mathcal{D}(1)$, where

$$\mathcal{D}(r) = \{z \in \mathbb{C} : 0 < |z| < r\} \quad (r \in (0,1])$$

and let $\mathcal{M}^k$ $(k \in \mathbb{N}_0 := \{0,1,2,...\})$ denote the class of functions $f \in \widetilde{\mathcal{M}}$ of the form

$$f(z) = \frac{1}{z} + \sum_{n=k}^{\infty} a_n z^n \quad (z \in \mathcal{D}). \tag{1}$$

Moreover, let $\mathcal{M} := \mathcal{M}^0$. Also, by $\mathcal{T}_\theta$ $(\theta \in \mathbb{R})$ we denote the class of functions $f \in \mathcal{M}$ of the form

$$f(z) = \frac{1}{z} + e^{i\theta} \sum_{n=0}^{\infty} |a_n| z^n \quad (z \in \mathcal{D}). \tag{2}$$

The class $\mathcal{T}_\theta$ is called the class of meromorphic functions with fixed argument of coefficients. For $\theta = \pi$ we obtain the class $\mathcal{T}_\pi$ of meromorphic functions with negative coefficients. Classes of functions with fixed argument of coefficients were considered in [1, 2, 3, 4].

A function $f \in \mathcal{M}$ is said to be *convex* in $\mathcal{D}(r)$ if

$$\mathfrak{Re}\left(1+\frac{zf''(z)}{f'(z)}\right) < 0 \ \ (z \in \mathcal{D}(r)).$$

A function $f \in \mathcal{M}$ is said to be *starlike* in $\mathcal{D}(r)$ if

$$\mathfrak{Re}\frac{zf'(z)}{f(z)} < 0 \ \ (z \in \mathcal{D}(r)). \tag{3}$$

Let $\mathcal{B}$ be a subclass of the class $\mathcal{M}$. We define *the radius of starlikeness of order* α and *the radius of convexity of order* α for the class $\mathcal{B}$ by

$$R_\alpha^*(\mathcal{B}) = \inf_{f\in\mathcal{B}} \left\{\sup\left\{r \in (0,1] : f \text{ is starlike in } \mathcal{D}(r)\right\}\right\},$$
$$R_\alpha^c(\mathcal{B}) = \inf_{f\in\mathcal{B}} \left\{\sup\left\{r \in (0,1] : f \text{ is convex in } \mathcal{D}(r)\right\}\right\},$$

respectively.

Let functions f, F be analytic in $\mathcal{U} := \mathcal{D}\cup\{0\}$. We say that f is *subordinate* to F, and write $f(z) \prec F(z)$ (or simply $f \prec F$), if and only if there exists a function ω analytic in $\mathcal{U}$, $|\omega(z)| \le |z| \quad (z \in \mathcal{U})$, such that

$$f(z) = F(\omega(z)) \quad (z \in \mathcal{U}).$$

In particular, if F is univalent in $\mathcal{U}$, we have the following equivalence:

$$f(z) \prec F(z) \Longleftrightarrow f(0) = F(0) \ \text{ and } f(\mathcal{U}) \subset F(\mathcal{U}).$$

For functions $f, g \in \widetilde{\mathcal{M}}$ of the form

$$f(z) = \sum_{n=-1}^{\infty} a_n z^n \ \text{ and } g(z) = \sum_{n=-1}^{\infty} b_n z^n,$$

by $f * g$ we denote *the Hadamard product* (or *convolution*) of f and g, defined by

$$(f * g)(z) = \sum_{n=-1}^{\infty} a_n b_n z^n \quad (z \in \mathcal{D}).$$

Let $\varphi \in \mathcal{M}^k$ be a given function of the form

$$\varphi(z) = \frac{1}{z} + \sum_{n=k}^{\infty} \alpha_n z^n \quad (z \in \mathcal{D};\ \alpha_n > 0,\ n = k, k+1, ...). \tag{4}$$

Assume that A, B are real parameters, $-1 \le A < B \le 1$, $(\cos\theta < 0$ or $B \ne 1)$. By $\mathcal{M}^k(\varphi; A, B)$ we denote the class of functions $f \in \mathcal{M}^k$ such that

$$z(\varphi * f)(z) \prec \frac{1+Az}{1+Bz}. \tag{5}$$

Now, we define the classes of functions with fixed argument of coefficients related to the class $\mathcal{M}^k(\varphi; A, B)$. Let us denote

$$\mathcal{M}_\theta^k(\varphi; A, B) := \mathcal{T}_\theta \cap \mathcal{M}^k(\varphi; A, B), \ \mathcal{M}(\varphi; A, B) := \mathcal{M}^0(\varphi; A, B).$$

In the present paper we obtain coefficient estimates, distortion theorems, integral means inequalities, and the radii of convexity and starlikeness for the class $\mathcal{M}_\theta^k(\varphi; A, B)$. We also derive convolution properties for the class of functions.

2 Coefficient estimates

Before stating and proving coefficient estimates in the class $\mathcal{M}(\varphi; A, B)$ we need the following lemma.

Lemma 1 *[6] Let f be a function of the form*

$$f(z) = \sum_{n=0}^{\infty} a_n z^n,$$

which is analytic in $\mathcal{D}$. If $f \prec g$ and g is convex univalent in $\mathcal{U}$, then

$$|a_n| \le 1 \qquad (n \in \mathbb{N}).$$

Theorem 1 *If a function f of the form* (1) *belongs to the class $\mathcal{M}(\varphi; A, B)$, then*

$$|a_n| \le \frac{B-A}{\alpha_n} \quad (n = 0, 1, \ldots), \tag{6}$$

The result is sharp.

Proof. Let a function f of the form (1) belong to the class $\mathcal{M}(\varphi; A, B)$ and let us put

$$g(z) = \frac{z(\varphi * f)(z) - 1}{A-B} \qquad \text{and} \qquad h(z) = \frac{z}{1+Bz}.$$

Then, by (5), we have $g \prec h$. Since the function g is given by

$$g(z) = \sum_{n=0}^{\infty} \frac{\alpha_n}{A-B} a_n z^{n+1}$$

and the function h is convex univalent in $\mathcal{U}$, by Lemma 1 we obtain

$$\frac{\alpha_n}{B-A} |a_n| \le 1 \quad (n \in \mathbb{N}_0). \tag{7}$$

Thus we have (6). The Equality in (7) holds for the functions g_n of the form

$$g_n(z) = h(z^{n+1}) = z^{n+1} + \sum_{j=n+2}^{\infty} b_j z^j \quad (n = 0, 1, \ldots),$$

for some b_j $(j = n+2, n+3, \ldots)$. Consequently, the equality in (6) holds true for the functions f_n of the form

$$f_n(z) = \frac{1}{z} + \frac{A-B}{\alpha_n} z^n + \sum_{j=n+1}^{\infty} \frac{A-B}{\alpha_j} b_{j+1} z^j \quad (n = 0, 1, \ldots).$$

■

Theorem 2 *If a function f of the form (2) belongs to the class $\mathcal{M}_\theta^k(\varphi; A, B)$, then*

$$\sum_{n=k}^{\infty} \alpha_n |a_n| \le \delta(\theta; A, B), \tag{8}$$

where

$$\delta(\theta; A, B) := \frac{B-A}{\sqrt{1 - B^2 \sin^2\theta} - B\cos\theta}. \tag{9}$$

Proof. Let a function f belong to the class $\mathcal{M}_\theta^k(\varphi; A, B)$. Then, by (5) and the definition of subordination, we have

$$z(\varphi * f)(z) = \frac{1 + A\omega(z)}{1 + B\omega(z)},$$

where $\omega(0) = 0$ and $|\omega(z)| < 1$ for $z \in \mathcal{U}$. Thus we obtain

$$|z(\varphi * f)(z) - 1| < |Bz(\varphi * f)(z) - A| \quad (z \in \mathcal{D}).$$

Hence, by (2), we have

$$\left| \sum_{n=k}^{\infty} \alpha_n |a_n| z^{n+1} \right| < \left| B - A + Be^{i\theta} \sum_{n=k}^{\infty} \alpha_n |a_n| z^{n+1} \right| \quad (z \in \mathcal{D}). \tag{10}$$

Putting $z = r$ $(0 \le r < 1)$, we find that

$$|w| < |B - A + Bwe^{i\theta}|, \tag{11}$$

where, for convenience,

$$w = \sum_{n=k}^{\infty} \alpha_n |a_n| r^{n+1}.$$

Since w is a real number, by (11) we have

$$(1 - B^2)w^2 - [2B(B-A)\cos\theta]\, w - (B-A)^2 < 0.$$

Solving this inequality with respect to w, we obtain

$$\sum_{n=k}^{\infty} \alpha_n |a_n| r^{n+1} < \delta(\theta; A, B),$$

which, upon letting $r \to 1^-$, readily yields the assertion (8) of Theorem 1. ■

Theorem 3 *A function f of the form* (2) *belongs to the class $\mathcal{M}_{\pi}^{k}(\varphi; A, B)$ if and only if*

$$\sum_{n=k}^{\infty} \alpha_n |a_n| \leq \frac{B-A}{1+B}. \tag{12}$$

Proof. By virtue of Theorem 1, we only need to show that the condition (12) is the sufficient condition. Let a function f of the form (2) satisfy the condition (12). Then, in view of (10), it is sufficient to prove that

$$\left|\sum_{n=k}^{\infty} \alpha_n |a_n| z^{n+1}\right| - \left|B - A - B\sum_{n=k}^{\infty} \alpha_n |a_n| z^{n+1}\right| < 0 \quad (z \in \mathcal{D}).$$

Indeed, letting $|z| = r$ $(0 < r < 1)$, we have

$$\begin{aligned}
&\left|\sum_{n=k}^{\infty} \alpha_n |a_n| z^{n+1}\right| - \left|B - A - B\sum_{n=k}^{\infty} \alpha_n |a_n| z^{n+1}\right| \\
&\leq \left(\sum_{n=k}^{\infty} \alpha_n |a_n| r^{n+1}\right) - \left(B - A - B\sum_{n=k}^{\infty} \alpha_n |a_n| r^{n+1}\right) \\
&< (1+B)\sum_{n=k}^{\infty} \alpha_n |a_n| - (B-A) \leq 0,
\end{aligned}$$

which implies that $f \in \mathcal{M}_{\pi}^{k}(\varphi; A, B)$. ■

Theorem 2 readily yields

Corollary 1 *If a function f of the form* (2) *belongs to the class $\mathcal{M}_{\theta}^{k}(\varphi; A, B)$, then*

$$|a_n| \leq \frac{\delta(\theta; A, B)}{\alpha_n} \quad (n = k, k+1, \ldots), \tag{13}$$

where $\delta(\theta; A, B)$ is defined by (9). *The result is sharp for $\theta = \pi$. Then the functions f_n of the form*

$$f_n(z) = \frac{1}{z} - \frac{B-A}{(1+B)\,\alpha_n} z^n \quad (z \in \mathcal{D};\ n = k, k+1, \ldots) \tag{14}$$

are the extremal functions.

3 Distortion theorems

From Theorem 2 we have the following lemma.

Lemma 2 *Let a function f of the form* (2) *belong to the class $\mathcal{M}_{\theta}^{k}(\varphi; A, B)$. If the sequence $\{\alpha_n\}$ defined by* (4) *satisfies the inequality*

$$\alpha_k \leq \alpha_n \quad (n = k, k+1, \ldots), \tag{15}$$

then

$$\sum_{n=k}^{\infty}|a_n| \leq \frac{\delta(\theta;A,B)}{\alpha_k}.$$

Moreover, if

$$n\alpha_k \leq \alpha_n \quad (k \geq 1,\ n = k, k+1, \dots), \tag{16}$$

then

$$\sum_{n=k}^{\infty} n\,|a_n| \leq \frac{k\delta(\theta;A,B)}{\alpha_k}.$$

Theorem 4 *Let a function f belong to the class $\mathcal{M}_\theta^k(\varphi;A,B)$. If the sequence $\{\alpha_n\}$ defined by* (4) *satisfies* (15)*, then*

$$\frac{1}{r} - \frac{\delta(\theta;A,B)}{\alpha_k}r^k \leq |f(z)| \leq \frac{1}{r} + \frac{\delta(\theta;A,B)}{\alpha_k}r^k \quad (|z| = r < 1). \tag{17}$$

Moreover, if (16) *holds, then*

$$\frac{1}{r^2} - \frac{k\delta(\theta;A,B)}{\alpha_k}r^{k-1} \leq |f'(z)| \leq \frac{1}{r^2} + \frac{k\delta(\theta;A,B)}{\alpha_k}r^{k-1} \quad (|z| = r < 1). \tag{18}$$

The result is sharp for $\theta = \pi$, with the extremal function f_k of the form (14).

Proof. Let a function f of the form (2) belong to the class $\mathcal{M}_\theta^k(\varphi;A,B)$, $|z| = r < 1$. Since

$$|f(z)| = \left|\frac{1}{z} + e^{i\theta}\sum_{n=k}^{\infty} a_n z^n\right| \leq \frac{1}{r} + \sum_{n=k}^{\infty}|a_n|\,r^n \leq \frac{1}{r} + \sum_{n=k}^{\infty}|a_n|$$

and

$$|f(z)| = \left|\frac{1}{z} + e^{i\theta}\sum_{n=k}^{\infty} a_n z^n\right| \geq \frac{1}{r} - \sum_{n=k}^{\infty}|a_n|\,r^n \geq \frac{1}{r} - \sum_{n=k}^{\infty}|a_n|,$$

then by Lemma 2 we have (17). Analogously we prove (18). ■

4 Integral means inequalities

Due to Littlewood [7] we obtain integral means inequalities for the functions from the class $\mathcal{M}_\theta^k(\varphi;A,B)$.

Lemma 3 *[7]. Let function f, g be analytic in $\mathcal{U}$. If $f \prec g$, then*

$$\int_0^{2\pi}\left|f(re^{it})\right|^\lambda \mathrm{d}t \leq \int_0^{2\pi}\left|g(re^{it})\right|^\lambda \mathrm{d}t \quad (0 < r < 1,\ \lambda > 0). \tag{19}$$

Silverman [8] found that the function

$$g(z)=z-\frac{z^2}{2}\quad (z\in\mathcal{D}),$$

is often extremal over the family of functions with negative coefficients. He applied this function to resolve integral means inequality, conjectured in [9] and settled in [10], that (19) holds true for all functions f with negative coefficients. In [10] he also proved his conjecture for some subclasses of $\mathcal{T}_\pi$.

Applying Lemma 3 and Theorem 2 we prove the following result.

Theorem 5 *Let the sequence* $\{\alpha_n\}$ *defined by* (4) *satisfy the inequality* (15). *If* $f\in\mathcal{M}_\theta^0(\varphi;A,B)$, *then*

$$\int_0^{2\pi}\left|f(re^{it})\right|^\lambda \mathrm{d}t\le\int_0^{2\pi}\left|g(re^{it})\right|^\lambda \mathrm{d}t\quad (0<r<1,\ \lambda>0), \tag{20}$$

where

$$g(z)=\frac{1}{z}+e^{i\theta}\frac{\delta(\theta;A,B)}{\alpha_0}\quad (z\in\mathcal{D}).$$

Proof. For function f of the form (2), the inequality (20) is equivalent to the following:

$$\int_0^{2\pi}\left|1+e^{i\theta}\sum_{n=0}^{\infty}|a_n|z^{n+1}\right|^\lambda \mathrm{d}t\le\int_0^{2\pi}\left|1+e^{i\theta}\frac{\delta(\theta;A,B)}{\alpha_0}z\right|^\lambda \mathrm{d}t.$$

By Lemma 3, it suffices to show that

$$\sum_{n=0}^{\infty}|a_n|\,z^{n+1}\prec\frac{\delta(\theta;A,B)}{\alpha_0}z. \tag{21}$$

Setting

$$w(z)=\sum_{n=0}^{\infty}\frac{\alpha_0}{\delta(\theta;A,B)}a_nz^{n+1}\quad (z\in\mathcal{D})$$

and using (15) and Theorem 2 we obtain

$$|w(z)|=\left|\sum_{n=0}^{\infty}\frac{\alpha_0}{\delta(\theta;A,B)}a_n\ z^{n+1}\right|\le|z|\sum_{n=0}^{\infty}\frac{\alpha_n}{\delta(\theta;A,B)}\,|a_n|\le|z|\quad (z\in\mathcal{D}).$$

Since

$$\sum_{n=0}^{\infty}a_nz^{n+1}=\frac{\delta(\theta;A,B)}{\alpha_0}w(z)\quad (z\in\mathcal{D}),$$

by definition od subordination we have (21) and this completes the proof. ■

5 The radii of convexity and starlikeness

Theorem 6 *If a function f belongs to the class $\mathcal{M}_{\theta}^{k}(\varphi; A, B)$, $k \geq 1$, then f is starlike in the disk $\mathcal{D}(r^*)$, where*

$$r^* := \inf_{n \geq k} \left(\frac{\alpha_n}{n\delta(\theta, A, B)} \right)^{\frac{1}{n+1}} \tag{22}$$

and $\delta(\theta, A, B)$, $\{\alpha_n\}$ are defined by (9) and (4), respectively. For $\theta = \pi$, the result is sharp, that is

$$R^* \left(\mathcal{M}_{\pi}^{k}(\varphi; A, B) \right) = r^*.$$

Proof. A function $f \in \mathcal{M}^k$ of the form (2) is starlike in the disk $\mathcal{D}(r)$ if and only if it satisfies the condition (3) or if

$$\left| \frac{zf'(z) + f(z)}{zf'(z) - f(z)} \right| < 1 \quad (z \in \mathcal{D}(r)). \tag{23}$$

Since

$$\left| \frac{zf'(z) + f(z)}{zf'(z) - f(z)} \right| = \left| \frac{e^{i\theta} \sum\limits_{n=k}^{\infty} (n+1) |a_n| z^n}{\frac{2}{z} - e^{i\theta} \sum\limits_{n=k}^{\infty} (n-1) |a_n| z^n} \right| \leq \frac{\sum\limits_{n=k}^{\infty} (n+1) |a_n| |z|^{n+1}}{2 - \sum\limits_{n=k}^{\infty} (n-1) |a_n| |z|^{n+1}},$$

putting $|z| = r$ the condition (23) be true if

$$\sum_{n=k}^{\infty} n |a_n| r^{n+1} \leq 1. \tag{24}$$

By Theorem 2, we have

$$\sum_{n=k}^{\infty} \frac{\alpha_n}{\delta(\theta, A, B)} |a_n| \leq 1,$$

Thus, the condition (24) be true if

$$nr^{n+1} \leq \frac{\alpha_n}{\delta(\theta, A, B)} \qquad (n = k, k+1, ...),$$

that is, if

$$r \leq \left(\frac{\alpha_n}{n\delta(\theta, A, B)} \right)^{\frac{1}{n+1}} \qquad (n = k, k+1, ...).$$

It follows that each function $f \in \mathcal{M}_{\theta}^{k}(\varphi; A, B)$ is starlike in the disk $\mathcal{D}(r^*)$, where r^* is defined by (22). For $\theta = \pi$ the functions f_n of the form (14) are extremal functions. ■

Theorem 7 *If a function f belongs to the class $\mathcal{M}_\theta^k(\varphi; A, B)$, then f is convex in the disk $\mathcal{D}(r^c)$, where*

$$r^c := \inf_{n \geq k} \left(\frac{\alpha_n}{n^2 \delta(\theta, A, B)} \right)^{\frac{1}{n+1}}$$

and $\delta(\theta, A, B)$, $\{\alpha_n\}$ are defined by (9) and (4), respectively. For $\theta = \pi$, the result is sharp, that is,

$$R^c\left(\mathcal{M}_\pi^k(\varphi; A, B)\right) = r^c.$$

Proof. The proof is analogous to that of Theorem 4, and we omit the details. ■

6 Cnonvolution properties

Let

$$f(z) = \frac{1}{z} + e^{i\alpha} \sum_{n=k}^{\infty} |a_n| z^n, \quad g(z) = \frac{1}{z} + e^{i\beta} \sum_{n=k}^{\infty} |b_n| z^n \quad (z \in \mathcal{D}). \tag{25}$$

We define modified Hadamard product for the functions f, g as follows:

$$f \circledast g(z) = \frac{1}{z} - \sum_{n=k}^{\infty} |a_n||b_n| z^n \quad (z \in \mathcal{D}).$$

Theorem 8 *Let $f \in \mathcal{M}_\alpha^k(\varphi; A, B)$ and $g \in \mathcal{M}_\beta^k(\psi; C, D)$. Then $f \circledast g \in \mathcal{M}_\pi^k(\varphi * \psi; E, F)$, whenever*

$$\delta(\pi, E, F) \geq \delta(\alpha, A, B)\delta(\beta, C, D). \tag{26}$$

Proof. Let

$$\psi(z) = \frac{1}{z} + \sum_{n=k}^{\infty} \beta_n z^n \quad (z \in \mathcal{D};\ \beta_n > 0,\ n = k, k+1, ...)$$

and let functions f, g of the form (25) belong to the classes $\mathcal{M}_\alpha^k(\varphi; A, B)$ and $\mathcal{M}_\beta^k(\psi; C, D)$, respectively. From Theorem 2 we have

$$\sum_{n=k}^{\infty} \frac{\alpha_n}{\delta(\alpha; A, B)} |a_n| \leq 1, \quad \sum_{n=k}^{\infty} \frac{\beta_n}{\delta(\beta; C, D)} |b_n| \leq 1.$$

Thus, by (26) we obtain

$$\begin{aligned} \sum_{n=k}^{\infty} \frac{\alpha_n \beta_n}{\delta(\pi, E, F)} |a_n b_n| &\leq \sum_{n=k}^{\infty} \frac{\alpha_n \beta_n}{\delta(\alpha; A, B)\delta(\beta; C, D)} |a_n| \, |b_n| \\ &\leq \sum_{n=k}^{\infty} \frac{\alpha_n}{\delta(\alpha; A, B)} |a_n| \sum_{n=k}^{\infty} \frac{\beta_n}{\delta(\beta; C, D)} |b_n| \leq 1. \end{aligned}$$

Applying Theorem 3 we get $f \circledast g \in \mathcal{M}_\pi^k(\varphi * \psi; E, F)$. ■

Theorem 9 *Let the sequence* $\{\alpha_n\}$ *defined by* (4) *satisfy the inequalities* (15). *If* $f, g \in \mathcal{M}_\theta^k(\varphi; A, B)$, *then* $f \circledast g \in \mathcal{M}_\pi^k(\varphi; C, D)$, *whenever*

$$(D-C)\,\alpha_0 \geq (1+D)\,[\delta(\theta, A, B)]^2 . \tag{27}$$

Proof. Let a functions f, g of the form (25) belong to the class $\mathcal{M}_\alpha^k(\varphi; A, B)$. Then by Theorem 2 we have

$$\sum_{n=k}^{\infty} \frac{\alpha_n}{\delta(\alpha; A, B)} |a_n| \leq 1, \quad \sum_{n=k}^{\infty} \frac{\alpha_n}{\delta(\alpha; A, B)} |b_n| \leq 1.$$

Thus, by the Cauchy-Schwarz inequality we obtain

$$\sum_{n=k}^{\infty} \frac{\alpha_n}{\delta(\theta, A, B)} \sqrt{|a_n b_n|} \leq 1 . \tag{28}$$

We have to prove that

$$\sum_{k=2}^{\infty} \alpha_n \frac{1+D}{D-C} |a_n b_n| \leq 1 .$$

Therefore, by (28) it is sufficient to show that

$$\frac{1+D}{D-C} |a_n b_n| \leq \frac{1}{\delta(\theta, A, B)} \sqrt{|a_n b_n|} \quad (n \geq 2)$$

or equivalently

$$\sqrt{|a_n b_n|} \leq \frac{D-C}{(1+D)\,\delta(\theta, A, B)} \quad (n \geq 2).$$

From (28) we have

$$\sqrt{|a_n b_n|} \leq \frac{\delta(\theta, A, B)}{\alpha_n} \quad (n \geq 2).$$

Consequently, we need only to prove that

$$\frac{D-C}{(1+D)\,\delta(\theta, A, B)} \geq \frac{\delta(\theta, A, B)}{\alpha_n} \quad (n \geq 2),$$

and this inequality follows from (27) and (15). ■

We note that for functions $f \in \mathcal{M}_\alpha^k(\varphi; A, B)$ and $g \in \mathcal{M}_{\pi-\alpha}^k(\psi; C, D)$ we have $f * g = f \circledast g$. Thus from Theorem 8 obtain following corollary.

Corollary 2 *If* $f \in \mathcal{M}_\alpha^k(\varphi; A, B)$ *and* $g \in \mathcal{M}_{\pi-\alpha}^k(\psi; C, D)$, *then* $f * g \in \mathcal{M}_\pi^k(\varphi * \psi; E, F)$, *whenever*

$$\delta(\pi, E, F) \geq \delta(\alpha, A, B)\delta(\pi - \alpha, C, D).$$

Putting $\theta = \pi$ in Theorem 9 we obtain following corollary.

Corollary 3 *Let the sequence* $\{\alpha_n\}$ *defined by* (4) *satisfy* (15). *If* $f, g \in \mathcal{M}_{\pi}^{k}(\varphi; A, B)$, *then* $f \circledast g \in \mathcal{M}_{\pi}^{k}(\varphi; C, D)$, *whenever*

$$(D - C)(1 + B)^2 \alpha_0 \geq (1 + D)(B - A)^2 .$$

Putting $C = A$ and $D = B$ in Corollary 3 we obtain following corollary.

Corollary 4 *Let the sequence* $\{\alpha_n\}$ *defined by* (4) *satisfy* (15). *If* $f, g \in \mathcal{M}_{\pi}^{k}(\varphi; A, B)$, *then* $f \circledast g \in \mathcal{M}_{\pi}^{k}(\varphi; A, B)$, *whenever*

$$\alpha_0 \geq \frac{B - A}{1 + B}.$$

Since for $\alpha = \beta = \pi$, $E = A$ and $F = B$ the condition (26) is true, then from Theorem 8 we have following corollary.

Corollary 5 *If* $f \in \mathcal{M}_{\pi}^{k}(\varphi; A, B)$ *and* $g \in \mathcal{M}_{\pi}^{k}(\psi; C, D)$, *then*

$$f \circledast g \in \mathcal{M}_{\pi}^{k}(\varphi * \psi; A, B) \cap \mathcal{M}_{\pi}^{k}(\varphi * \psi; C, D).$$

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