ISSN 1733-6775

Journal of Mathematics and Applications

vol. 31 (2009)

Department of Mathematics
Rzeszów University of Technology
Rzeszów, Poland

Journal of Mathematics and Applications

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vol. 31 (2009)

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http://www.jma.prz.rzeszow.pl

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Text pepared to print in LATEX

p-ISSN 1733-6775

Publishing House of the Rzeszów University of Technology

Printed in July 2009
(55/09)

Journal of Mathematic and Applications

vol. 31 (2009)

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Journal of
Mathematics
and Applications
No 31, pp 7-18 (2009)

On generalization of some classes of Sălăgean-type multivalent harmonic functions

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Submitted by: *Jan Stankiewicz*

Abstract: In the present paper, we make generalization of the classes in [7] of Sălăgean-Type multivalent harmonic functions. We introduce sufficient coefficient condition for the class $\mathcal{H}_p^i(n;\lambda,\beta,m)$ and this condition be also necessary if certain restriction is imposed on the coefficients of these harmonic functions. Also we have obtained a representation theorem, inclusion relations and distortion bounds for these functions

AMS Subject Classification: *30C45*
Key Words and Phrases: *Multivalent harmonic functions, Sălăgean derivative operator, Inclusion relations, Distortion bounds, Representation theorem*

1. Introduction

A continuous function $f = u + iv$ is a complex valued harmonic function in a complex domain $\mathbb{C}$, if u and v are real harmonic. If Ω be any simply connected domain and $\Omega \subset \mathbb{C}$, then $f = h + \overline{g}$, where h and g are analytic in Ω, h is analytic part and g is co-analytic part of $f \cdot |g'(z)| < |h'(z)|$ if and only if f is locally univalent and sense preserving in Ω, see [3], [5]. Denote by

$\mathcal{H} = \{f : f = h + \overline{g}, f$ is harmonic univalent and sense-preserving in the open unit disk $U = \{z : |z| < 1\}\}$.

So $f = h + \overline{g} \in \mathcal{H}$ is normalized by $f(0) = h(0) = f_z(0) - 1 = 0$.

Ahuja and Jahangiri [1] defined the class $\mathcal{H}_p(n)$ $(p, n \in I\!N = \{1, 2, 3, \cdots\})$ consisting of all p-valent harmonic functions $f = h + \overline{g}$ that are sense-preserving U, and h, g are of the form

$$h(z) = z^p + \sum_{k=n+p}^{\infty} a_k z^k, \quad g(z) = \sum_{k=n+p-1}^{\infty} b_k z^k, \quad |b_{n+p-1}| < 1. \tag{1}$$

Let $f = h + \overline{g}$ given by (1), the modified Sălăgean operator of f is defined as:

$$D^i f(z) = D^i h(z) + (-1)^i \overline{D^i g(z)}, \quad p > i, \quad i \in I\!N_0 = \{0, 1, 2, \cdots\},$$

where $D^i h(z) = p^i z^p \sum\limits_{k=n+p}^{\infty} k^i a_k z^k$ and $D^i g(z) = \sum\limits_{k=n+p-1}^{\infty} k^i b_k z^k$ (see [4], [6]).

Let $\mathcal{H}_p^i(n)$ be a subclass consisting of harmonic functions $f_i = h + \overline{g}_i$, so that h and g_i are of the form:

$$h(z) = z^p - \sum_{k=n+p}^{\infty} a_k z^k, \quad g_i(z) = (-1)^i \sum_{k=n+p-1}^{\infty} b_k z^k, \quad \text{for} \quad a_k, b_k \geq 0, |b_{n+p-1}| < 1. \tag{2}$$

A function f in $\mathcal{H}_p(n)$ is said to be in the class $\mathcal{H}_p^i(n; \lambda, \beta, m)$ if

$$Re\left\{(1-\lambda)\frac{D^i f(z)}{\frac{\partial^i}{\partial\theta^i} z^p} + \lambda(1-m)\frac{D^{i+1} f(z)}{\frac{\partial^{i+1}}{\partial\theta^{i+1}} z^p} + \lambda m \frac{D^{i+2} f(z)}{\frac{\partial^{i+2}}{\partial\theta^{i+2}} z^p}\right\} > \frac{\beta}{p^{i+1}}, \tag{3}$$

where $0 \leq \beta < p, \lambda \geq 0, 0 \leq m \leq 1, p \geq i$ and $z = re^{\mathrm{i}\theta} \in U$.

As λ changes from 0 to 1, the family $\mathcal{H}_p^i(n; \lambda, \beta, m)$ provides a passage from the class of Sălăgean-type multivalent harmonic functions $\mathcal{H}_p^i R(n; \beta) \equiv \mathcal{H}_p^i(n; 0, \beta, m)$ consisting of functions f, where

$$Re\left\{\frac{D^i f(z)}{\frac{\partial^i}{\partial\theta^i} z^p}\right\} > \frac{\beta}{p^{i+1}}$$

and this class was studied in [7].

To the class of Sălăgean-type multivalent harmonic functions $\mathcal{H}_p^i S(n; \beta, m) \equiv \mathcal{H}_p^i(n; 1, \beta, m)$ consisting of functions f, where

$$Re\left\{(1-m)\frac{D^{i+1} f(z)}{\frac{\partial^{i+1}}{\partial\theta^{i+1}} z^p} + m\frac{D^{i+2} f(z)}{\frac{\partial^{i+2}}{\partial\theta^{i+2}} z^p}\right\} > \frac{\beta}{p^{i+1}},$$

to the class of Sălăgean-type multivalent harmonic functions (if $m = 0$) $\mathcal{H}_p^i T(n; \beta) \equiv \mathcal{H}_p^i(n; 1, \beta, 0)$ consisting of functions f satisfying

$$Re\left\{\frac{D^{i+1} f(z)}{\frac{\partial^{i+1}}{\partial\theta^{i+1}} z^p}\right\} > \frac{\beta}{p^{i+1}},$$

and this class was studied in [7].

If $m = 0$, then the class $\mathcal{H}_p^i(n; \lambda, \beta, m)$ reduces to the class $\mathcal{H}_p^i U(n; \lambda, \beta) \equiv \mathcal{H}_p^i(n; \lambda, \beta, 0)$ consisting of functions f such that

$$Re\left\{(1-\lambda)\frac{D^i f(z)}{\frac{\partial^i}{\partial\theta^i} z^p} + \lambda\frac{D^{i+1} f(z)}{\frac{\partial^{i+1}}{\partial\theta^{i+1}} z^p}\right\} > \frac{\beta}{p^{i+1}},$$

and this class was studied in [7].

Now, we define the subclass $\overline{\mathcal{H}}_p^i(n;\lambda,\beta,m) \equiv \mathcal{H}_p^i(n;\lambda,\beta,m) \cap \mathcal{H}_p^i(n)$. If $m=0$ and $i=0$, then the class $\mathcal{H}_p^i(n;\lambda,\beta,m)$ reduces to the class $\mathcal{H}_pV(n;\lambda,\beta) \equiv \mathcal{H}_p^0(n;\lambda,\beta,0)$ that was studied in [2].

2. Representation Theorem

In the following theorem, we find a coefficient bound for functions in $\mathcal{H}_p^i(n;\lambda,\beta,m)$.

Theorem 1. Let $f=h+\overline{g}$ be given by (1). Then $f \in \mathcal{H}_p^i(n;\lambda,\beta,m)$ if

$$\sum_{k=n+p}^{\infty} |p+(k-p)(\frac{mk}{p}+1)\lambda| k^i |a_k| + \sum_{k=n+p-1}^{\infty} |p+(k+p)(\frac{mk}{p}-1)\lambda| k^i |b_k| \leq p^{i+1}-\beta, \quad (4)$$

where $0 \leq \beta < p, \lambda \geq 0, 0 \leq m \leq 1, p \geq i$ and $z = re^{\mathrm{i}\theta} \in U$.

Proof. By using the fact $Re\ \alpha \geq 0$ if and only if $|1+\alpha| \geq |1-\alpha|$ in U, it suffices to show that

$$|p^{i+1}-\beta+p^{i+1}w| \geq |p^{i+1}+\beta-p^{i+1}w|,$$

where

$$w = (1-\lambda)\frac{D^i f(z)}{\frac{\partial^i}{\partial\theta^i} z^p} + \lambda(1-m)\frac{D^{i+1} f(z)}{\frac{\partial^{i+1}}{\partial\theta^{i+1}} z^p} + \lambda m \frac{D^{i+2} f(z)}{\frac{\partial^{i+2}}{\partial\theta^{i+2}} z^p}.$$

Substituting for h and g in w we obtain

$$\begin{aligned} w &= 1 + \sum_{k=n+p}^{\infty} [1+(\frac{k}{p}-1)(m\frac{k}{p}+1)\lambda]\frac{k^i}{p^i} a_k \frac{z^k}{z^p} \\ &\quad + \sum_{k=n+p-1}^{\infty} [1-(\frac{k}{p}+1)(1-m\frac{k}{p})\lambda](-1)^i \frac{k^i}{p^i} b_k \frac{\overline{z}^k}{z^p} \end{aligned}$$

and then we have

$$\begin{aligned}
&|p^{i+1}-\beta+p^{i+1}w|-|p^{i+1}+\beta-p^{i+1}w| \\
= \ & |2p^{i+1}-\beta+\sum_{k=n+p}^{\infty}[p+(k-p)(\frac{mk}{p}+1)\lambda]k^i a_k\frac{z^k}{z^p} \\
&+\sum_{k=n+p-1}^{\infty}[p-(k+p)(1-\frac{mk}{p})\lambda](-1)^i k^i b_k\frac{\overline{z}^k}{z^p}| \\
&-|\beta+\sum_{k=n+p}^{\infty}[p+(k-p)(\frac{mk}{p}+1)\lambda]k^i a_k\frac{z^k}{z^p} \\
&-\sum_{k=n+p-1}^{\infty}[p-(k+p)(1-\frac{mk}{p})\lambda](-1)^i k^i b_k\frac{\overline{z}^k}{z^p}| \\
\geq \ & 2p^{i+1}-\sum_{k=n+p}^{\infty}|p+(k-p)(\frac{mk}{p}+1)\lambda|k^i|a_k||z|^{k-p} \\
&-\sum_{k=n+p-1}^{\infty}|p+(k+p)(\frac{mk}{p}-1)\lambda|k^i|b_k||z|^{k-p} \\
&-\sum_{k=n+p}^{\infty}|p+(k-p)(\frac{mk}{p}+1)\lambda|k^i|a_k||z|^{k-p} \\
&-\sum_{k=n+p-1}^{\infty}|p+(k+p)(\frac{mk}{p}-1)\lambda|k^i|b_k||z|^{k-p} \\
\geq \ & 2[(p^{i+1}-\beta)-\sum_{k=n+p}^{\infty}|p+(k-p)(\frac{mk}{p}+1)\lambda|k^i|a_k| \\
&-\sum_{k=n+p-1}^{\infty}|p+(k+p)(\frac{mk}{p}-1)\lambda|k^i|b_k|]\geq 0.
\end{aligned}$$

The proof is complete.

The coefficient bound (4) given in Theorem 1 is sharp for the function

$$f(z)=z^p+\sum_{k=n+p}^{\infty}\frac{x_k}{|p+(k-p)(\frac{mk}{p}+1)\lambda|k^i}z^k+\sum_{k=n+p-1}^{\infty}\frac{\overline{y}_k}{|p+(k+p)(\frac{mk}{p}-1)\lambda|k^i}\overline{z}^k,$$

where $\sum\limits_{k=n+p}^{\infty}|x_k|+\sum\limits_{k=n+p-1}^{\infty}|y_k|=p^{i+1}-\beta.$

Theorem 2. Let $f_i=h+\overline{g}_i$ be given by (2). Then $f_i\in\overline{\mathcal{H}}_p^i(n;\lambda,\beta,m)$ if and only if

$$\sum_{k=n+p}^{\infty}|p+(k-p)(\frac{mk}{p}+1)\lambda|k^i a_k+\sum_{k=n+p-1}^{\infty}|p+(k+p)(\frac{mk}{p}-1)\lambda|k^i b_k\leq p^{i+1}-\beta. \quad (5)$$

Proof. From Theorem 1, we only want to prove the "only if" part of the theorem, since $\overline{\mathcal{H}}_p^i(n;\lambda,\beta,m) \subset \mathcal{H}_p^i(n;\lambda,\beta,m)$. If $f_i \in \overline{\mathcal{H}}_p^i(n;\lambda,\beta,m)$, then, for $z = re^{\mathrm{i}\theta}$ in U we get

$$
\begin{aligned}
& Re\left\{(1-\lambda)\frac{D^i f(z)}{\frac{\partial^i}{\partial\theta^i} z^p} + \lambda(1-m)\frac{D^{i+1} f(z)}{\frac{\partial^{i+1}}{\partial\theta^{i+1}} z^p} + \lambda m\frac{D^{i+2} f(z)}{\frac{\partial^{i+2}}{\partial\theta^{i+2}} z^p}\right\} \\
= \quad & Re\left\{\frac{(1-\lambda)}{p^i}\left(\frac{D^i h(z) + (-1)^i\overline{D^i g_i(z)}}{\mathrm{i}^i z^p}\right)\right. \\
& +\frac{\lambda(1-m)}{p^{i+1}}\left(\frac{D^{i+1} h(z) - (-1)^i\overline{D^{i+1} g_i(z)}}{\mathrm{i}^{i+1} z^p}\right) \\
& \left. +\frac{\lambda m}{p^{i+2}}\left(\frac{D^{i+2} h(z) + (-1)^i\overline{D^{i+2} g_i(z)}}{\mathrm{i}^{i+2} z^p}\right)\right\} \\
\geq \quad & 1 - \frac{1}{p^{i+1}}\sum_{k=n+p}^{\infty}|p+(k-p)(\frac{mk}{p}+1)\lambda|k^i a_k r^{k-p} \\
& -\frac{1}{p^{i+1}}\sum_{k=n+p-1}^{\infty}|p+(k+p)(\frac{mk}{p}-1)\lambda|k^i b_k r^{k-p} \geq \frac{\beta}{p^{i+1}}.
\end{aligned}
$$

This inequality must hold for all $z \in U$. In particular, letting $z = r \to 1$, it yields the required condition (5).

As special cases of Theorem 2, we obtain the following corollaries :

Corollary 1. **[7]** $f_i = h + \overline{g}_i \in \overline{\mathcal{H}}_p^i R(n;\beta) \equiv \mathcal{H}_p^i R(n;\beta) \cap \mathcal{H}_p^i(n)$ if and only if

$$\sum_{k=n+p}^{\infty}\frac{pk^i}{p^{i+1}-\beta}a_k + \sum_{k=n+p-1}^{\infty}\frac{pk^i}{p^{i+1}-\beta}b_k \leq 1.$$

Corollary 2. $f_i = h + \overline{g}_i \in \overline{\mathcal{H}}_p^i S(n;\beta,m) \equiv \mathcal{H}_p^i S(n;\beta,m) \cap \mathcal{H}_p^i(n)$ if and only if

$$\sum_{k=n+p}^{\infty}\frac{|p+(k-p)(\frac{mk}{p}+1)|k^i}{p^{i+1}-\beta}a_k + \sum_{k=n+p-1}^{\infty}\frac{|p+(k+p)(\frac{mk}{p}-1)|k^i}{p^{i+1}-\beta}b_k \leq 1.$$

Corollary 3. **[7]** $f_i = h + \overline{g}_i \in \overline{\mathcal{H}}_p^i T(n;\beta) \equiv \mathcal{H}_p^i T(n;\beta) \cap H_p^i(n)$ if and only if

$$\sum_{k=n+p}^{\infty}\frac{k^{i+1}}{p^{i+1}-\beta}a_k + \sum_{k=n+p-1}^{\infty}\frac{k^{i+1}}{p^{i+1}-\beta}b_k \leq 1.$$

Corollary 4. **[7]** $f_i = h + \overline{g}_i \in \overline{\mathcal{H}}_p^i U(n;\lambda,\beta) \equiv \mathcal{H}_p^i U(n;\lambda,\beta) \cap H_p^i(n)$ if and only if

$$\sum_{k=n+p}^{\infty}\frac{|\lambda k+(1-\lambda)p|k^i}{p^{i+1}-\beta}a_k + \sum_{k=n+p-1}^{\infty}\frac{|\lambda k-(1-\lambda)p|k^i}{p^{i+1}-\beta}b_k \leq 1.$$

In the following theorem, we determine a representation theorem for functions in $\overline{\mathcal{H}}_p^i(n;\lambda,\beta,m)$.

Theorem 3. $f_i = h + \overline{g}_i \in \overline{\mathcal{H}}_p^i(n;\lambda,\beta,m)$ if and only if f_i can be expressed as

$$f_i(z) = X_p h_p(z) + \sum_{k=n+p}^{\infty} X_k h_k(z) + \sum_{k=n+p-1}^{\infty} Y_k g_{k_i}(z),$$

where $h_p(z) = z^p, h_k(z) = \frac{p^{i+1}-\beta}{|p+(k-p)(\frac{mk}{p}+1)\lambda|k^i} z^k, (k = n+p, n+p+1, \cdots), g_{k_i}(z) = z^p + (-1)^i \frac{p^{i+1}-\beta}{|p+(k+p)(\frac{mk}{p}-1)\lambda|k^i} \overline{z}^k, (k = n+p-1, n+p, \cdots), X_p \geq 0, Y_{n+p-1} \geq 0, X_p + \sum_{k=n+p}^{\infty} X_k + \sum_{k=n+p-1}^{\infty} Y_k = 1$, and $X_k \geq 0, Y_k \geq 0$, for $k = n+p, n+p+1, \cdots$.

Proof. For functions f_i of the form (2), we have

$$\begin{aligned} f_i(z) &= X_p h_p(z) + \sum_{k=n+p}^{\infty} X_k h_k(z) + \sum_{k=n+p-1}^{\infty} Y_k g_{k_i}(z) \\ &= z^p - \sum_{k=n+p}^{\infty} \frac{p^{i+1}-\beta}{|p+(k-p)(\frac{mk}{p}+1)\lambda|k^i} X_k z^k \\ &\quad +(-1)^i \sum_{k=n+p-1}^{\infty} \frac{p^{i+1}-\beta}{|p+(k+p)(\frac{mk}{p}-1)\lambda|k^i} Y_k \overline{z}^k. \end{aligned}$$

Consequently, $f_i \in \overline{\mathcal{H}}_p^i(n;\lambda,\beta,m)$, since by (5), we have

$$\begin{aligned} &\sum_{k=n+p}^{\infty} |p+(k-p)(\frac{mk}{p}+1)\lambda|k^i a_k + \sum_{k=n+p-1}^{\infty} |p+(k+p)(\frac{mk}{p}-1)\lambda|k^i b_k \\ &= \sum_{k=n+p}^{\infty} |p+(k-p)(\frac{mk}{p}+1)\lambda|k^i \cdot \frac{p^{i+1}-\beta}{|p+(k-p)(\frac{mk}{p}+1)\lambda|k^i} |X_k| \\ &\quad + \sum_{k=n+p-1}^{\infty} |p+(k+p)(\frac{mk}{p}-1)\lambda|k^i \cdot \frac{p^{i+1}-\beta}{|p+(k+p)(\frac{mk}{p}-1)\lambda|k^i} |Y_k| \\ &= (p^{i+1}-\beta)\left(\sum_{k=n+p}^{\infty} |X_k| + \sum_{k=n+p-1}^{\infty} |Y_k|\right) = (p^{i+1}-\beta)(1-X_p) \leq p^{i+1}-\beta. \end{aligned}$$

Conversely, assume $f_i \in \overline{\mathcal{H}}_p^i(n;\lambda,\beta,m)$. Letting $X_p = 1 - \sum_{k=n+p}^{\infty} X_k - \sum_{k=n+p-1}^{\infty} Y_k$, where $X_k = \frac{|p+(k-p)(\frac{mk}{p}+1)\lambda|k^i}{p^{i+1}-\beta} a_k$ and $Y_k = \frac{|p+(k+p)(\frac{mk}{p}-1)\lambda|k^i}{p^{i+1}-\beta} b_k$, we obtain the required representation, since

$$\begin{aligned}
f_i(z) &= z^p - \sum_{k=n+p}^{\infty} a_k z^k + (-1)^i \sum_{k=n+p-1}^{\infty} b_k \overline{z}^k \\
&= z^p - \sum_{k=n+p}^{\infty} \frac{(p^{i+1}-\beta)X_k}{|p+(k-p)(\frac{mk}{p}+1)\lambda|k^i} z^k \\
&\quad +(-1)^i \sum_{k=n+p-1}^{\infty} \frac{(p^{i+1}-\beta)Y_k}{|p+(k+p)(\frac{mk}{p}-1)\lambda|k^i} \overline{z}^k \\
&= z^p - \sum_{k=n+p}^{\infty} (z^p - h_k(z))X_k - \sum_{k=n+p-1}^{\infty} (z^p - g_{k_i}(z))Y_k \\
&= \left(1 - \sum_{k=n+p}^{\infty} X_k - \sum_{k=n+p-1}^{\infty} Y_k\right) z^p + \sum_{k=n+p}^{\infty} h_k(z)X_k + \sum_{k=n+p-1}^{\infty} g_{k_i}(z)Y_k \\
&= X_p h_p(z) + \sum_{k=n+p}^{\infty} X_k h_k(z) + \sum_{k=n+p-1}^{\infty} Y_k g_{k_i}(z).
\end{aligned}$$

3. Inclusion Relations

In the following theorem, we discuss the inclusion relations between the above mentioned classes. The inclusion relations between the classes for the different values of λ are not so obvious.

Theorem 4. For $n \in I\!N$ and $0 \le \beta < p$, we have:

(1) $\overline{\mathcal{H}}_p^i S(n;\beta,m) \subset \overline{\mathcal{H}}_p^i(n;\lambda,\beta,m), 0 \le \lambda < 1$

(2) $\overline{\mathcal{H}}_p^i(n;\lambda,\beta,m) \subset \overline{\mathcal{H}}_p^i S(n;\beta,m), \lambda \ge 1$

(3) $\overline{\mathcal{H}}_p^i(n;\lambda,\beta,m) \subset \overline{\mathcal{H}}_p^i R(n;\beta), \lambda \ge 0$

(4) $\overline{\mathcal{H}}_p^i(n;\lambda,\beta,m) \subset \overline{\mathcal{H}}_p^i U(n;\lambda,\beta), \lambda \ge 0$

(5) $\overline{\mathcal{H}}_p^i S(n;\beta,m) \subset \overline{\mathcal{H}}_p^i R(n;\beta)$.

Proof (1) For $0 \le \lambda < 1$, we have

$$\begin{aligned}
&\sum_{k=n+p}^{\infty} |p+(k-p)(\frac{mk}{p}+1)\lambda|k^i a_k + \sum_{k=n+p-1}^{\infty} |p+(k+p)(\frac{mk}{p}-1)\lambda|k^i b_k \\
&\quad \le \sum_{k=n+p}^{\infty} |p+(k-p)(\frac{mk}{p}+1)|k^i a_k + \sum_{k=n+p-1}^{\infty} |p+(k+p)(\frac{mk}{p}-1)|k^i b_k \\
&\quad \le p^{i+1}-\beta. \text{ (by Corollary 2)}
\end{aligned}$$

Therefore (1) is obtained from Theorem 2.

(2) If $\lambda \geq 1$, then by Theorem 2

$$\begin{aligned}
&\sum_{k=n+p}^{\infty} |p+(k-p)(\frac{mk}{p}+1)|k^i a_k + \sum_{k=n+p-1}^{\infty} |p+(k+p)(\frac{mk}{p}-1)|k^i b_k \\
\leq & \sum_{k=n+p}^{\infty} |p+(k-p)(\frac{mk}{p}+1)\lambda|k^i a_k + \sum_{k=n+p-1}^{\infty} |p+(k+p)(\frac{mk}{p}-1)\lambda|k^i b_k \\
\leq & \ p^{i+1}-\beta.
\end{aligned}$$

Therefore, (2) is obtained from Corollary 2.

(3) If $\lambda \geq 0$, then by Theorem 2,

$$\begin{aligned}
&\sum_{k=n+p}^{\infty} pk^i a_k + \sum_{k=n+p-1}^{\infty} pk^i b_k \\
&\leq \sum_{k=n+p}^{\infty} |p+(k-p)(\frac{mk}{p}+1)\lambda|k^i a_k + \sum_{k=n+p-1}^{\infty} |p+(k+p)(\frac{mk}{p}-1)\lambda|k^i b_k \\
&\leq p^{i+1}-\beta.
\end{aligned}$$

Thus, (3) is obtained from Corollary 1.

(4) If $\lambda \geq 0$, then by Theorem 2,

$$\begin{aligned}
&\sum_{k=n+p}^{\infty} |\lambda k+(1-\lambda)p|k^i a_k + \sum_{k=n+p-1}^{\infty} |\lambda k-(1-\lambda)p|k^i b_k \\
&= \sum_{k=n+p}^{\infty} |p+(k-p)\lambda|k^i a_k + \sum_{k=n+p-1}^{\infty} |(k+p)\lambda-p|k^i b_k \\
&\leq \sum_{k=n+p}^{\infty} |p+(k-p)(\frac{mk}{p}+1)\lambda|k^i a_k + \sum_{k=n+p-1}^{\infty} |p+(k+p)(\frac{mk}{p}-1)\lambda|k^i b_k \\
&\leq p^{i+1}-\beta.
\end{aligned}$$

Thus, (4) is obtained from Corollary 4.

(5) In view of Corollaries 1 and 2, since

$$\begin{aligned}
&\sum_{k=n+p}^{\infty} pk^i a_k + \sum_{k=n+p-1}^{\infty} pk^i b_k \\
&\leq \sum_{k=n+p}^{\infty} |p+(k-p)(\frac{mk}{p}+1)|k^i a_k + \sum_{k=n+p-1}^{\infty} |p+(k+p)(\frac{mk}{p}-1)|k^i b_k.
\end{aligned}$$

The result follows.

4. Distortion Bounds

We introduce a distortion theorem for functions in $\overline{\mathcal{H}}_p^i(n;\beta,\lambda,m)$.

Theorem 5. If $f_i \in \overline{\mathcal{H}}_p^i(n;\lambda,\beta,m), \lambda \geq 1$ and $|z| = r < 1$, then

$$|f_i(z)| \leq (1 + b_{n+p-1}r^{n-1})r^p + \left(\frac{p^{i+1}-\beta}{(p+n(\frac{mn}{p}+m+1)\lambda)(n+p)^i}\right.$$
$$\left. - \frac{[(n+2p-1)(1-\frac{1}{p}(m(n+p-1)))\lambda - p](n+p-1)^i}{(p+n(\frac{mn}{p}+m+1)\lambda)(n+p)^i} b_{n+p-1}\right) r^{n+p}$$

and

$$|f_i(z)| \geq (1 - b_{n+p-1}r^{n-1})r^p - \left(\frac{p^{i+1}-\beta}{(p+n(\frac{mn}{p}+m+1)\lambda)(n+p)^i}\right.$$
$$\left. - \frac{[(n+2p-1)(1-\frac{1}{p}(m(n+p-1)))\lambda - p](n+p-1)^i}{(p+n(\frac{mn}{p}+m+1)\lambda)(n+p)^i} b_{n+p-1}\right) r^{n+p}.$$

Proof. We prove the left hand side inequality for $|f_i|$. Let $f_i \in \overline{\mathcal{H}}_p^i(n;\lambda,\beta,m)$, then by Theorem 2, we obtain:

$$\begin{aligned}
|f_i(z)| &= \left| z^p + (-1)^i b_{n+p-1}\overline{z}^{n+p-1} + \sum_{k=n+p}^{\infty}(a_k z^k + (-1)^i b_k \overline{z}^k)\right| \\
&\geq r^p - b_{n+p-1}r^{n+p-1} - \frac{p^{i+1}-\beta}{(p+n(\frac{mn}{p}+m+1)\lambda)(n+p)^i} \times \\
&\quad \sum_{k=n+p}^{\infty}\left(\frac{p+n(\frac{mn}{p}+m+1)\lambda}{p^{i+1}-\beta}a_k + \frac{p+n(\frac{mn}{p}+m+1)\lambda}{p^{i+1}-\beta}b_k\right)(n+p)^i r^k \\
&\geq r^p - b_{n+p-1}r^{n+p-1} - \frac{p^{i+1}-\beta}{(p+n(\frac{mn}{p}+m+1)\lambda)(n+p)^i} \times \\
&\quad \sum_{k=n+p}^{\infty}\left(\frac{p+(k-p)(\frac{mk}{p}+1)\lambda}{p^{i+1}-\beta}a_k + \frac{(k+p)(1-\frac{mk}{p})\lambda - p}{p^{i+1}-\beta}b_k\right)k^i r^k \\
&\geq (1 - b_{n+p-1}r^{n-1})r^p - \frac{p^{i+1}-\beta}{(p+n(\frac{mn}{p}+m+1)\lambda)(n+p)^i} \times \\
&\quad \left[1 - \frac{[(n+2p-1)(1-\frac{1}{p}(m(n+p-1)))\lambda - p](n+p-1)^i}{p^{i+1}-\beta}b_{n+p-1}\right] r^{n+p} \\
&\geq (1 - b_{n+p-1}r^{n-1})r^p - \left(\frac{p^{i+1}-\beta}{(p+n(\frac{mn}{p}+m+1)\lambda)(n+p)^i}\right. \\
&\quad \left. - \frac{[(n+2p-1)(1-\frac{1}{p}(m(n+p-1)))\lambda - p](n+p-1)^i}{(p+n(\frac{mn}{p}+m+1)\lambda)(n+p)^i}b_{n+p-1}\right) r^{n+p}.
\end{aligned}$$

The proof for the right hand side inequality can be done using similar arguments and this completes the proof of theorem.

The following result follows from the left hand side inequality in Theorem 5.

Corollary 5. If $f_i \in \overline{\mathcal{H}}_p^i(n;\lambda,\beta,m), \lambda \geq 1$, then the set

$$\begin{aligned}\{w \quad : \quad & |w| < [(p+n(\frac{mn}{p}+m+1)\lambda)(n+p)^i - p^{i+1} + \beta \\ & -[(p+n(\frac{mn}{p}+m+1)\lambda)(n+p)^i + [(n+2p-1)(1-\frac{1}{p}(m(n+p-1)))\lambda - p] \\ & \cdot(n+p-1)^i]b_{n+p-1}]/[(p+n(\frac{mn}{p}+m+1)\lambda)(n+p)^i]\}\end{aligned}$$

is included in $f_i(U)$.

By using arguments similar to those given in the proof of Theorem 5, we get the following corollaries.

Corollary 6. **[7]** If $f_i \in \overline{\mathcal{H}}_p^i R(n;\beta)$, then

$$|f_i(z)| \leq (1+b_{n+p-1}r^{n-1})r^p + \left(\frac{p^{i+1}-\beta}{p(n+p)^i} + \frac{(n+p-1)^i}{(n+p)^i}b_{n+p-1}\right)r^{n+p},$$

and

$$|f_i(z)| \geq (1-b_{n+p-1}r^{n-1})r^p - \left(\frac{p^{i+1}-\beta}{p(n+p)^i} + \frac{(n+p-1)^i}{(n+p)^i}b_{n+p-1}\right)r^{n+p}.$$

Corollary 7. **[7]** If $f_i \in \overline{\mathcal{H}}_p^i T(n;\beta)$, then

$$|f_i(z)| \leq (1+b_{n+p-1}r^{n-1})r^p + \left(\frac{p^{i+1}-\beta}{(n+p)^{i+1}} - \frac{(n+p-1)^{i+1}}{(n+p)^{i+1}}b_{n+p-1}\right)r^{n+p},$$

and

$$|f_i(z)| \geq (1-b_{n+p-1}r^{n-1})r^p - \left(\frac{p^{i+1}-\beta}{(n+p)^{i+1}} - \frac{(n+p-1)^{i+1}}{(n+p)^{i+1}}b_{n+p-1}\right)r^{n+p}.$$

Corollary 8. **[7]** If $f_i \in \overline{\mathcal{H}}_p^i U(n;\lambda,\beta)$, then

$$\begin{aligned}|f_i(z)| \quad \leq \quad & (1+b_{n+p-1}r^{n-1})r^p \\ & + \left(\frac{p^{i+1}-\beta}{(\lambda n+p)(n+p)^i} - \frac{[\lambda(n+2p-1)-p](n+p-1)^i}{(\lambda n+p)(n+p)^i}b_{n+p-1}\right)r^{n+p},\end{aligned}$$

and

$$\begin{aligned}|f_i(z)| \quad \geq \quad & (1-b_{n+p-1}r^{n-1})r^p \\ & - \left(\frac{p^{i+1}-\beta}{(\lambda n+p)(n+p)^i} - \frac{[\lambda(n+2p-1)-p](n+p-1)^i}{(\lambda n+p)(n+p)^i}b_{n+p-1}\right)r^{n+p}.\end{aligned}$$

Corollary 9. If $f_i \in \overline{\mathcal{H}}_p^i S(n;\beta,m)$, then

$$
\begin{aligned}
|f_i(z)| \quad &\leq \quad (1+b_{n+p-1}r^{n-1})r^p + \left(\frac{p^{i+1}-\beta}{(p+n(\frac{mn}{p}+m+1))(n+p)^i} \right. \\
&\left. - \frac{[(n+2p-1)(1-\frac{1}{p}(m(n+p-1)))-p](n+p-1)^i}{(p+n(\frac{mn}{p}+m+1))(n+p)^i} b_{n+p-1} \right) r^{n+p},
\end{aligned}
$$

and

$$
\begin{aligned}
|f_i(z)| \quad &\geq \quad (1-b_{n+p-1}r^{n-1})r^p - \left(\frac{p^{i+1}-\beta}{(p+n(\frac{mn}{p}+m+1))(n+p)^i} \right. \\
&\left. - \frac{[(n+2p-1)(1-\frac{1}{p}(m(n+p-1)))-p](n+p-1)^i}{(p+n(\frac{mn}{p}+m+1))(n+p)^i} b_{n+p-1} \right) r^{n+p}.
\end{aligned}
$$

Acknowledgement: the author waggas galib is thankful to his wife (Hnd Hekmat Abdulah) for her support for him to compelete this paper.

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Received 28.08.2008

Journal of
Mathematics
and Applications
No 31, pp 19-27 (2009)

On single-valued and multi-valued convergences

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ABSTRACT: We present a review of ideas of a general theory of convergence, developed independently of topology, with the stress on the duality of convergence and topology. Results and problems concerning sufficient and necessary conditions for a convergence to be topological, both in case of the single- and multi-valued cases, are recalled. We reconstruct, filling certain gaps, an example given in [7] to show that one of sufficient conditions in the theorems proved in [1] and [9] for multi-valued convergences to be topological is not necessary.

AMS Subject Classification: *40A05, 54Axx, 54Bxx, 54Dxx*
Key Words and Phrases: *$\mathcal{L}^*$-Fréchet space, single-valued and multi-valued convergence, topological convergence, Galois connection*

1. Introduction

Convergence of sequences is often defined at first directly, without using any topology. The corresponding topology, if it exists at all, is discussed usually later. This is because the description of convergence via topology is more complicated or even impossible at all (see e.g. [14]). Therefore it makes sense to develop, independently of topology, a general theory of convergence and it was initiated already by Fréchet and Urysohn. A convergence in an arbitrary set X can be defined, in general, by indicating convergent sequences of elements of X and their limits in X. Fréchet [8] and Urysohn [16] considered only single-valued convergences, i.e. convergences with unique limits. The so-called $\mathcal{L}^*$-Fréchet spaces satisfy three Fréchet's conditions which are fulfilled by every convergence defined by a topology. The study of single-valued and then multi-valued convergences was continued by many authors (see e.g. [13], [5], [15], [3], [1], [7], [9], [10], [12]).

An important result concerning single-valued convergences is Kisyński's theorem [13] which says that Fréchet's conditions are also sufficient for a single-valued convergence to be topological. The situation in the multi-valued case is more complicated. A characterization of topological multi-valued convergences given in [15] appeared to be incorrect: a mistake in the proof was found during the international conference on convergence held in Szczyrk in 1979. A respective counter-example was given by the second author who later gave in [10] (see also [11], [12]) a full characterization of topological multi-valued convergences by means of sequential closures.

Simpler sufficient conditions for a multi-valued convergence to be topological are given in [1] and [9], but one of the conditions is not necessary as the first of the two interesting examples sketched in [7] shows. However the examples are quite complicated and contain ambiguities, so the ideas of the constructions may be not so easy to follow for the reader.

The aim of this note is to reconstruct the first example shown in [7] with more carefulness, filling gaps found in the original text. We will present this example in section 3. The second example from [7], which brought the negative answer to the problem of V. Koutnik (posed during the mentioned conference), but originally also contained some gaps and inaccuracies, is reconstructed in [2]. In section 2, we recall all necessary definitions and main properties of the notions under consideration, using the formalism proposed in [3].

The present article is a result of discussions between the authors during the seminar conducted by the second author at the University of Rzeszów.

2. Basic definitions and statements

We will use the notation from [3] with certain modifications. By $\mathbb{N}$ we will denote as usual the set of all positive integers and by X a fixed nonempty set. The symbol $k_n \nearrow \infty$ with $k_i \in \mathbb{N}$ $(i \in \mathbb{N})$ means that the sequence $\{k_i\} = \{k_i\}_{i=1}^{\infty}$ is strictly increasing. In general, we use a shorter notation $\{\xi_i\}$ instead of $\{\xi_i\}_{i=1}^{\infty}$ for the sequence $\xi_1, \xi_2, \dots$ of elements of an arbitrary set. Thus the symbol $\{\xi_i\}$ may denote either the sequence $\xi_1, \xi_2, \dots$ or the one-element set consisting of ξ_i for a fixed $i \in \mathbb{N}$. This will not lead, however, to misunderstanding because of a clear context.

By capital letters A, B, … we denote subsets of X, i.e. elements of 2^X; by the scribed letter $\mathcal{F}$ (with or without indices) - an arbitrary family of subsets A of X, i.e. $\mathcal{F} \subset 2^X$; by the Gothic letter $\mathfrak{F}$ - the class 2^{2^X} of all families $\mathcal{F} \subset 2^X$ of subsets of X. By Greek letters $\xi, \eta, \dots$ (with or without indices) we denote elements of X; by the corresponding Latin letters $x, y, \dots$ - sequences $\{\xi_i\}, \{\eta_i\}, \dots$ of elements of X, respectively, i.e. elements of $X^{\mathbb{N}}$; by the scribed letter $\mathcal{G}$ (with or without indices) - an arbitrary mapping which assigns to each sequence $x = \{\xi_i\} \in X^{\mathbb{N}}$ a subset $A \in 2^X$; by the Gothic letter $\mathfrak{G}$ - the class $(2^X)^{(X^{\mathbb{N}})}$ of all such mappings.

We write $y \prec x$, if $y = \{\eta_i\}$ is a subsequence of $x = \{\xi_i\}$, i.e. if $\eta_i = \xi_{k_i}$ for certain $k_i \in \mathbb{N}$ $(i \in \mathbb{N})$ such that $k_i \nearrow \infty$. If $x = \{\xi_i\}$ with $\xi_i = \xi \in X$ for $i \in \mathbb{N}$, then we denote the constant sequence x by $\dot{\xi}$. For a given $x = \{\xi_i\} \in X^{\mathbb{N}}$ and $A \subset X$ we write $x \sqsubset A$ if there exists an index $i_0 \in \mathbb{N}$ such that $\xi_i \in A$ for $i \in \mathbb{N}$, $i > i_0$. For a given sequence $x = \{\xi_i\},\ \xi_i \in X$, we denote by (x) the set of all its elements, i.e.

$$(x) := \{\xi_i\colon\ i \in \mathbb{N}\}. \tag{1}$$

In particular, if $X = Y^{\mathbb{N}}$ is the set of all sequences of elements of a certain set Y and $x = \{\xi_i\},\ \xi_i \in X$, with $\xi_i = \{\eta_{i,j}\},\ \eta_{i,j} \in Y\ (i, j \in \mathbb{N})$, then by $((x))$ we denote the union of all the sets (ξ_i) in the sense of (1) for $\xi_i \in Y^{\mathbb{N}}(i \in \mathbb{N})$, i.e.

$$((x)) := \bigcup_{i\in\mathbb{N}} (\xi_i) = \{\eta_{i,j}\colon\ i, j \in \mathbb{N}\}. \tag{2}$$

Now we collect definitions we need further on.

Definition 1. *By a topology in X we mean an arbitrary family $\mathcal{F} \in \mathfrak{F}$ satisfying the conditions:*

$(T1)$ $\emptyset \in \mathcal{F}$ *and* $X \in \mathcal{F}$*;*

$(T2)$ *if* $A \in F$ *and* $B \in \mathcal{F}$*, then* $A \cap B \in \mathcal{F}$*;*

$(T3)$ *if* $A_\gamma \in \mathcal{F}\,(\gamma \in \Gamma)$*, then* $\bigcup_{\gamma\in\Gamma} A_\gamma \in \mathcal{F}$ *for any nonempty set* Γ *of indices.*

Definition 2. *By a convergence in X we mean an arbitrary mapping $\mathcal{G}\colon\ X^{\mathbb{N}} \to 2^X$, i.e. $\mathcal{G} \in \mathfrak{G}$.*

Remark 1. The interpretation of a convergence as a mapping $\mathcal{G}:\ X^{\mathbb{N}} \to 2^X$ is very natural. For each sequence $x = \{\xi_i\} \in X^{\mathbb{N}}$, the set $\mathcal{G}(x)$ is a subset of X and is interpreted as the set of all limits of the sequence x. If the set $\mathcal{G}(x)$ is empty, the sequence x is divergent. If the set $\mathcal{G}(x)$ is nonempty, the sequence x is convergent to all elements of $\mathcal{G}(x)$. If the set $\mathcal{G}(x)$ contains exactly one element, the limit of the sequence x is unique.

The following conditions on convergences expressed in terms of Definition 2. correspond to the three Fréchet's conditions $(L1)$, $(L2)$, $(L3)$ in the definition of $\mathcal{L}^*$-Fréchet spaces:

S. *For every $\xi \in X$, we have $\xi \in \mathcal{G}(\dot{\xi})$;*

F. *If $y \prec x$, i.e. y is a subsequence of a sequence x, then $\mathcal{G}(x) \subset \mathcal{G}(y)$;*

U. *If $\xi \notin \mathcal{G}(x)$ $(\xi \in X, x \in X^{\mathbb{N}})$, then there exists a $y \prec x$ such that $\xi \notin \mathcal{G}(z)$ for each $z \prec y$.*

We introduce in Definitions 3. and 4. below the two operators T and L such that $T\colon \mathfrak{G} \to \mathfrak{F}$ and $L\colon \mathfrak{F} \to \mathfrak{G}$.

Definition 3. *Let $\mathcal{G} \in \mathfrak{G}$. By $T\mathcal{G} \in \mathfrak{F}$ we mean the family of all $A \subset X$ satisfying the implication: $A \cap \mathcal{G}(x) \neq \emptyset \Rightarrow x \sqsubset A$. The family $T\mathcal{G}$ is called the topology induced by the convergence $\mathcal{G}$.*

Definition 4. *Let $\mathcal{F} \in \mathfrak{F}$. By $L\mathcal{F} \in \mathfrak{G}$ we mean the convergence such that, for every sequence $x \in X^{\mathbb{N}}$, the set $(L\mathcal{F})(x)$ consists of all points $\xi \in X$ satisfying the implication: $\xi \in A \subset \mathcal{F} \Rightarrow x \sqsubset A$. The mapping $L\mathcal{F}$ is called the convergence induced by the family $\mathcal{F}$.*

Remark 2. Notice that the convergence $L\mathcal{F}$ induced by $\mathcal{F} \in \mathfrak{F}$ is defined in Definition 4. exactly as in case $\mathcal{F}$ is a topology, but we do not impose conditions (T_1)-(T_3) on $\mathcal{F}$, in general. Analogously, to define $T\mathcal{G} \in \mathfrak{F}$ we do not need to impose on $\mathcal{G}$ conditions **S**, **F**, **U** in Definition 3..

It is natural to define the inclusion between convergences as follows:

Definition 5. *Let $\mathcal{G}_1$, $\mathcal{G}_2 \in \mathfrak{G}$. We write $\mathcal{G}_1 \subset \mathcal{G}_2$, whenever $\mathcal{G}_1(x) \subset \mathcal{G}_2(x)$ for every $x \in X^{\mathbb{N}}$.*

The notions of sequential topology and topological convergence considered in the literature can be described in the following way:

Definition 6. *A family $\mathcal{F} \in \mathfrak{F}$ is called a sequential topology if $\mathcal{F}$ is induced by some convergence, i.e. there is a $\mathcal{G} \in \mathfrak{G}$ such that $\mathcal{F} = T\mathcal{G}$ (see Statement 1.).*

Definition 7. *A convergence $\mathcal{G} \in \mathfrak{G}$ is called topological if $\mathcal{G}$ is induced by some family $\mathcal{F} \in \mathfrak{F}$, i.e. there is an $\mathcal{F} \in \mathfrak{F}$ such that $\mathcal{G} = L\mathcal{F}$.*

An important particular case of convergences are so-called *single-valued* convergences described by the following *Hausdorff* condition.

Definition 8. *If a convergence $\mathcal{G} \in \mathfrak{G}$ satisfies the condition:*

H. *For each $x \in X^{\mathbb{N}}$, if $\xi, \eta \in \mathcal{G}(x)$, then $\xi = \eta$,*

i.e. each sequence has at most one limit, we call $\mathcal{G}$ a single-valued convergence. If $\mathcal{G} \in \mathfrak{G}$ not necessarily satisfies **H**, *we call $\mathcal{G}$ a multi-valued convergence.*

It is easy to check the following properties of the operators T and L (see [3]):

Statement 1.

1° *For every $\mathcal{F} \in \mathfrak{F}$, the convergence $G = L\mathcal{F}$ satisfies conditions* **S**, **F**, **U**.

2° *For every $\mathcal{G} \in \mathfrak{G}$, the family $\mathcal{F} = T\mathcal{G}$ of subsets of X is a topology in X.*

Statement 2.

1° *If $\mathcal{F}_1 \subset \mathcal{F}_2$ $(\mathcal{F}_1, \mathcal{F}_2 \in \mathfrak{F})$, then $L\mathcal{F}_2 \subset L\mathcal{F}_1$.*

2° *If $\mathcal{G}_1 \subset \mathcal{G}_2$ $(\mathcal{G}_1, \mathcal{G}_2 \in \mathfrak{G})$, then $T\mathcal{G}_2 \subset T\mathcal{G}_1$.*

Statement 3.

1° *For every* $\mathcal{F} \in \mathfrak{F}$, *we have* $\mathcal{F} \subset TL\mathcal{F}$.

2° *For every* $\mathcal{G} \in \mathfrak{G}$, *we have* $\mathcal{G} \subset LT\mathcal{G}$.

Statement 4.

1° *If* $\mathcal{F}_1 \subset \mathcal{F}_2$ $(\mathcal{F}_1, \mathcal{F}_2 \in \mathfrak{F})$, *then* $TL\mathcal{F}_1 \subset TL\mathcal{F}_2$.

2° *If* $\mathcal{G}_1 \subset \mathcal{G}_2$ $(\mathcal{G}_1, \mathcal{G}_2 \in \mathfrak{G})$, *then* $LT\mathcal{G}_1 \subset LT\mathcal{G}_2$.

Statement 5.

1° *If* $\mathcal{F}_1, \mathcal{F}_2 \in \mathfrak{F}$, *then* $L(\mathcal{F}_1 \cup \mathcal{F}_2) = L\mathcal{F}_1 \cap L\mathcal{F}_2$.

2° *If* $\mathcal{G}_1, \mathcal{G}_2 \in \mathfrak{G}$, *then* $T(\mathcal{G}_1 \cup G_2) = T\mathcal{G}_1 \cap TG_2$.

Statement 6.

1° *A family* $\mathcal{F}$ *is a sequential topology iff* $TL\mathcal{F} = \mathcal{F}$.

2° *A convergence* $\mathcal{G}$ *is topological iff* $LT\mathcal{G} = \mathcal{G}$.

Statement 7.

1° *If* $\mathcal{F}_1$ *and* $\mathcal{F}_2$ *are sequential topologies, then so is* $\mathcal{F}_1 \cap \mathcal{F}_2$.

2° *If* $\mathcal{G}_1$ *and* $\mathcal{G}_2$ *are topological convergences, then so is* $\mathcal{G}_1 \cap \mathcal{G}_2$.

Remark 3. As an immediate consequence of Statements 2 and 3 we may conclude that the operators T and L define a Galois connection between the two complete lattices $\mathfrak{F}$ and $\mathfrak{G}$ (see e.g. [4], p. 56).

The following beautiful result is a complete characterization of topological convergences among all single-valued convergences:

Theorem 1. (Kisyński's theorem; see [13]) *A single-valued convergence* $\mathcal{G}$ *is topological iff* $\mathcal{G}$ *satisfies conditions* **S**, **F**, **U**.

Various additional conditions are usually imposed on multi-valued convergences (see e.g. [1], [9], [10], [7]). Before recalling some of them we introduce a definition.

Definition 9. *A set* $A \subset X$ *is called* $\mathcal{G}$-closed *if* $(x) \subset A$ *implies* $\mathcal{G}(x) \subset A$ *for every* $x \in X^{\mathbb{N}}$.

Consider the following conditions:

D. *If* $\eta_n \in G(\dot{\xi_n})$ *for* $n \in \mathbb{N}$, *then* $\mathcal{G}(\xi_n) \supset G(\eta_n)$;

C. *For each* $x \in X^{\mathbb{N}}$, *the set* $\mathcal{G}(x)$ *is* $\mathcal{G}$*-closed*;

C$'$. *For each* $x \in X^{\mathbb{N}}$ *there exists a subsequence* y *of* x *such that the set* $\bigcup_{z \prec y} \mathcal{G}(z)$ *is* $\mathcal{G}$*-closed*;

H$'$. *For every* $\xi \in X$, *if* $\eta_1, \eta_2 \in \mathcal{G}(\dot{\xi})$, *then* $\eta_1 = \eta_2$.

In [1], the following result was obtained:

Theorem 2. (see [1]) *If a convergence $\mathcal{G}$ satisfies conditions* **S***,* **F***,* **U***,* **D***,* **C** *and* $\mathbf{C}'$*, then $\mathcal{G}$ is topological.*

The above theorem was strengthened in [9] in the following way:

Theorem 3. (see [9]) *If convergence $\mathcal{G}$ satisfies conditions* **S***,* **F***,* **U***,* **D** *and* $\mathbf{C}'$*, then $\mathcal{G}$ is topological.*

Remark 3. Every topological convergence satisfies conditions **S**, **F**, **U**, **D**, but not condition **C**, in general (see [7]).

In the next section, we present an example of topological convergence which does not fulfil condition $\mathbf{C}'$ (cf. [7]). On the other hand, it can be shown that condition $\mathbf{C}'$ cannot be omitted in Theorem 3.. A respective example of a non-topological convergence which does not fulfil condition $\mathbf{C}'$, but satisfies conditions $\mathbf{S}, \mathbf{F}, \mathbf{U}, \mathbf{D}, \mathbf{C}$ and, in addition, $\mathbf{H}'$, is shown in [7] and [2]. This answers negatively the question posed by V. Koutnik.

3. Example of topological convergence

We will show that condition $\mathbf{C}'$ may not hold for topological convergences.

Example. Let $X = \mathbb{N} \cup \mathbb{N}_1 \cup \mathbb{N}_2$, where $\mathbb{N}$ is the set of all positive integers, $\mathbb{N}_1$ is the set of all increasing sequences $l = \{k_i\}$ with $k_i \in \mathbb{N}$ $(i \in \mathbb{N})$, and $\mathbb{N}_2$ is the set of all sequences $m = \{l_i\}$ with $l_i \in \mathbb{N}_1$ $(i \in \mathbb{N})$ such that, considering l_i as sequences of elements from $\mathbb{N}$, we have

$$(l_i) \cap (l_{i'}) = \emptyset, \qquad \text{whenever } \ i,\ i' \in \mathbb{N},\ \ i \neq i',$$

according to the notation in (1). Obviously, we have $\mathbb{N} \cap \mathbb{N}_1 = \mathbb{N} \cap \mathbb{N}_2 = \mathbb{N}_1 \cap \mathbb{N}_2 = \emptyset$.

Now, we construct the bases $\mathcal{B}(\xi)$ of neighbourhoods $U(\xi)$ at each $\xi \in X$ as follows:

1° if $\xi = k \in \mathbb{N}$, then we define the basis $\mathcal{B}(\xi)$ at ξ to consist of the single neighbourhood $U(k) := \{k\}$ of $\xi = k$, i.e. $\mathcal{B}(\xi) := \{U(k)\}$;

2° if $\xi = l = \{k_i\} \in \mathbb{N}_1$, then we define, for all $p \in \mathbb{N}$, the neighbourhoods $U_p(\xi)$ of $\xi = l$ by

$$U_p(l) := \{l\} \cup \{k_i\colon\ i \in \mathbb{N}, i \geq p\}$$

and the basis $\mathcal{B}(\xi)$ at $\xi = l$ by $\mathcal{B}(\xi) := \{U_p(l)\colon\ p \in \mathbb{N}\}$;

3° if $\xi = m = \{l_i\} \in \mathbb{N}_2$, where $l_i = \{k_{i,j}\}$ with $k_{i,j} \in \mathbb{N}$ $(i, j \in \mathbb{N})$, then we define, for every $r \in \mathbb{N}$ and an arbitrary sequence $\{q_s\}$ of positive integers, the following neighbourhoods of $\xi = m$:

$$U_{r,\{q_s\}}(m) := \{m\} \cup \{l_i\colon\ i \in \mathbb{N}, i \geq r\} \cup \{k_{i,j}\colon\ j \in \mathbb{N}, j \geq q_i, i \in \mathbb{N}, i \geq r\}$$

and the basis $\mathcal{B}(\xi)$ at $\xi = m$ by $\mathcal{B}(\xi) := \{U_{r,\{q_s\}}(m)\colon\ r \in \mathbb{N}, \{q_s\} \in \mathbb{N}^{\mathbb{N}}\}$.

We are going to show that the family $\mathcal{B} := \{\mathcal{B}(\xi)\colon\ \xi \in X\}$, where the bases $\mathcal{B}(\xi)$ for $\xi \in X$ are defined above, satisfies the conditions:

$(BP1)$ $\quad \forall_{x\in X}\ \ \mathcal{B}(x)\neq\emptyset;\ \ \forall_{U\in\mathcal{B}(x)}\ \ x\in U;$

$(BP2)$ $\quad \forall_{x\in X}\ \forall_{U\in\mathcal{B}(x)}\ \forall_{y\in U}\ \exists_{V\in\mathcal{B}(y)}\ \ V\subset U;$

$(BP3)$ $\quad x\in X\ \big[U_1,U_2\in\mathcal{B}(x)\ \Rightarrow\ \exists_{U\in\mathcal{B}(x)}\ \ U\subset U_1\cap U_2\big].$

Condition $(BP1)$ is obviously satisfied. To prove $(BP2)$ we consider the following three cases:

1° Let $x=k\in\mathbb{N}$. If $U\in\mathcal{B}(x)=\mathcal{B}(k)$ and $y\in U$, then $y=x=k$ and $V:=U\in\mathcal{B}(y)=\mathcal{B}(x)=\{\{k\}\}$.

2° Let $x=l=\{k_j\}\in\mathbb{N}_1$. If $U\in\mathcal{B}(x)$ and $y\in U$, then $y\in U_p(l)$ for some $p\in\mathbb{N}$. There are two possibilities: (a) $y=l$ or (b) $y=k_j$ for some $j\in\mathbb{N}$, $j\geq p$. We put $V:=U_p(l)$ in case (a) and $V:=\{k_j\}$ in case (b). Clearly, $V\in\mathcal{B}(y)$ and $y\in V\subset U$ in both cases (a) and (b).

3° Let $x=m=\{l_i\}\in\mathbb{N}_2$. If $U\in\mathcal{B}(x)=\mathcal{B}(m)$ and $y\in U$, then $y\in U_{r,\{q_s\}}(m)$ for some $r\in\mathbb{N}$, and $\{q_s\}\in\mathbb{N}^{\mathbb{N}}$. There are three possibilities: (a) $y=m$ or (b) $y=l_i\in\mathbb{N}_1$ for some $i\in\mathbb{N}$, $i\geq r$, or (c) $y=k_{i,j}\in\mathbb{N}$ for some $i\in\mathbb{N}$, $i\geq r$ and $j\in\mathbb{N}$, $j\geq q_i$. We put $V:=U_{r,\{q_s\}}(m)$ in case (a), $V:=U_p(l_i)$ with arbitrarily fixed $p\geq q_i$ in case (b), and $V:=\{k_{i,j}\}\in\mathcal{B}(y)$ in case (c). Obviously, $V\in\mathcal{B}(y)$ and $y\in V\subset U$ in all the cases (a), (b) and (c), so $(BP2)$ is satisfied in all the above cases $1^\circ,2^\circ$ and 3°.

We will verify condition $(BP3)$ only in case 3°, i.e. for $x=m\in\mathbb{N}_2$. If $U_1,U_2\in\mathcal{B}(m)$, then $U_1=U_{r_1,\{q_s\}}(m)$ and $U_2=U_{r_2,\{\bar{q}_s\}}(m)$ for some $r_1,r_2\in\mathbb{N}$ and $\{q_s\},\{\bar{q}_s\}\in\mathbb{N}^{\mathbb{N}}$. Let us define $U:=U_{\tilde{n},\{\tilde{q}_s\}}(m)$, where $\tilde{n}:=\max(r_1,r_2)$, $\tilde{q}_s:=\max(q_s,\bar{q}_s)$ for $s\in\mathbb{N}$. Obviously, $U\in\mathcal{B}(x)$ and $U\subset U_1\cap U_2$, as desired. Analogously one can check that $(BP3)$ is fulfilled in cases 1° and 2°. Thus the family $\mathcal{B}$ satisfies conditions $(BP1)$, $(BP2)$ and $(BP3)$.

Using well known topological arguments (see e.g. [6], p. 39, p. 58), we conclude that $\mathcal{B}$ uniquely defines a topology $\mathcal{F}$ in X such that $\mathcal{B}$ is a neighourhood system for $\mathcal{F}$ and $\mathcal{F}$ is a T_1-topology. Therefore the convergence $L\mathcal{F}$ satisfies condition $\mathbf{H}'$. Since $L\mathcal{F}$ is a topological convergence, it automatically satisfies conditions **S**, **F**, **U** (see [3]).

We will show that $L\mathcal{F}$ fulfils also conditions **D** and **C**. Let $\eta_n\in L\mathcal{F}(\dot{\xi_n})$ for $n\in\mathbb{N}$ and let $\eta\in L\mathcal{F}(\{\eta_n\})$, i.e. $\eta_n\in U$ for each $U\in\mathcal{F}$ such that $\eta\in U$ and for sufficiently large n, so $\xi_n\in U$ for sufficiently large n, which means that $\eta\in L\mathcal{F}(\{\xi_n\})$. Consequently, $L\mathcal{F}(\{\eta_n\})\subset L\mathcal{F}(\xi_n\})$ and thus LF satisfies condition **D**. Now, denoting $x=\{\xi_n\}\in X^{\mathbb{N}}$, assume that (a) $\eta_n\in L\mathcal{F}(x)$ for $n\in\mathbb{N}$ and (b) $\eta\in L\mathcal{F}(\{\eta_n\})$. By (b), for each $U\in\mathcal{F}$ with $\eta\in U$ there is an $n_0\in\mathbb{N}$ such that $\eta_n\in U$ for $n\geq n_0$. Hence, by (a), we can select a sequence $\{k_n\}$ of positive integers such that $k_n\nearrow\infty$ and $\xi_j\in U$ for $j\geq k_n\geq n\geq n_0$. Consequently, $\eta\in L\mathcal{F}(\{\xi_n\})$, so $L\mathcal{F}(x)$ is an $L\mathcal{F}$-closed set, i.e. LF satisfies condition **C**.

Before proving that $L\mathcal{F}$ does not satisfy condition $\mathbf{C}'$, we will prove the following implication:

$$l=\{k_j\}\in\mathbb{N}_1\ \Rightarrow\ l\in L\mathcal{F}(\{k_j\})\subset\mathbb{N}_1. \tag{3}$$

The relation $l \in L\mathcal{F}(l)$ in (3) is obvious, by definition of $L\mathcal{F}$. To prove that $L\mathcal{F}(l) \subset \mathbb{N}_1$, since the relation $k \notin L\mathcal{F}(l)$ for every $k \in \mathbb{N}$ is evident, it suffices to show that $m \notin L\mathcal{F}(l)$ for each $m \in \mathbb{N}_2$

Assume, on the contrary, that $m = \{l_i\} \in \mathbb{N}_2$ and $m \in L\mathcal{F}(l)$, where $l = \{k_j\}$. Of course, we have $k_j \in ((m))$, in the sense of the notation in (2), for sufficiently large j. Notice that, for each $i \in \mathbb{N}$, only a finite number of elements of $\{k_j\}$ belongs to (l_i). On the contrary, suppose that there exists an $i_0 \in \mathbb{N}$ such that $k_j \in (l_{i_0})$ for infinitely many $j \in \mathbb{N}$. Then there exists a sequence $\tilde{l}$ such that $\tilde{l} \prec l$ and $\tilde{l} \prec l_{i_0}$. Hence, by condition **F**, we have $m \in LF(\tilde{l})$. On the other hand, since $(\tilde{l}) \subset (l_{i_0})$ and $(l_{i_0}) \cap (l_i) = \emptyset$ for $i \neq i_0$, we have $\tilde{l} \not\subset U_{i_0+1,\{q_s\}}$ for arbitrary $\{q_s\} \in \mathbb{N}^{\mathbb{N}}$, which is impossible.

Let $\bar{k}_i := \max\{k_j : j \in \mathbb{N},\ k_j \in (l_i)\}$ and let $\bar{q}_i := \bar{k}_i + 1$ for $i \in \mathbb{N}$. Then for each $j \in \mathbb{N}$ we have $k_j \notin U_{1,\{\bar{q}_i\}}(m) \in \mathcal{B}(m)$, which contradicts the assumption that $m \in (LF)(\{k_j\})$ and completes the proof of (3).

Now, let x be an arbitrary increasing sequence of positive integers and let y be an arbitrary its subsequence. Of course, we can choose subsequences $y_1, y_2, \ldots$ of sequence y such that

$$(y_i) \cap (y_{i'}) = \emptyset \qquad \text{for} \quad i, i' \in \mathbb{N},\ i \neq i'.$$

By (3), we have

$$y_i \in \bigcup_{z \prec y} L\mathcal{F}(z) \subset \mathbb{N}_1 \qquad \text{for } i \in \mathbb{N}.$$

On the other hand, we have $m \in L\mathcal{F}(\{y_i\})$, where $m = \{y_i\} \in \mathbb{N}_2$, so the set $\bigcup_{z \prec y} L\mathcal{F}(z)$ is not $L\mathcal{F}$-closed.

Thus we have proved that convergence $L\mathcal{F}$ satisfies conditions **S**, **F**, **U**, **D** and **C**, but it does not satisfy condition **C**$'$.

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Received 12.02.2009

J o u r n a l of
Mathematics
and Applications
No 31, pp 29-36 (2009)

Differential superordinations defined by an integral operator

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ABSTRACT: By using the integral operator $I^m f$, we introduce a class of holomorphic functions denoted by $J_m(\alpha)$ and we obtain some superordinations results related to this class.

AMS Subject Classification: *30C45*
Key Words and Phrases: *differential subordinations, differential superordinations, integral operator*

1. Introduction and preliminaries

Let U_r, $0 < r \leq 1$, be the disc of center zero and radius r,

$$U_r = \{z \in \mathbb{C} : |z| < r\},$$

and let U be the unit disc of the complex plane

$$U = \{z \in \mathbb{C} : |z| < 1\}.$$

Also let

$$\dot{U} = U \setminus \{0\}.$$

Let $\mathcal{H}(U)$ be the space of holomorphic functions in U.
For $a \in \mathbb{C}$ and $n \in \mathbb{N}^*$ we let:

$$\mathcal{H}[a, n] = \{f \in \mathcal{H}(U), f(z) = a + a_n z^n + \dots, z \in U\}$$

and

$$A_n = \left\{f \in \mathcal{H}(U),\ f(z) = z + a_{n+1} z^{n+1} + \dots,\ z \in U\right\}$$

with

$$A = A_1.$$

Let f and F be members of $\mathcal{H}(U)$. The function f is said to be subordinate to F, or F is said to be superordinate to f, if there exists an analytic function w in U, with $w(0)=0$ and $|w(z)|<1$, such that $f(z)=F(w(z))$; in such a case we write $f(z)\prec F(z)$.

If F is univalent, then $f(z)\prec F(z)$ if and only if $f(0)=F(0)$ and $f(U)\subset F(U)$.

Let Ω be any set in the complex plane $\mathbb{C}$, let p be analytic in the unit disk U and let $\psi(\gamma,s,t;z):\mathbb{C}^3\times U\to\mathbb{C}$.

In a series of articles of S.S. Miller, P.T. Mocanu and D.J. Hallenbeck, S. Ruscheweyh have determined properties of functions p that satisfy the differential subordination

$$\{\psi(p(z),zp'(z),z^2p''(z);z)|\ z\in U\}\subset\Omega.$$

In this article we consider the dual problem of determining properties of functions p that satisfy the differential superordination

$$\Omega\subset\{\psi(p(z),zp'(z),z^2p''(z);z)|\ z\in U\}.$$

These results have been first presented in [3].

Definition 1.1. *Let $\varphi:\mathbb{C}^2\times U\to\mathbb{C}$ and let h be analytic in U. If p and $\varphi(p(z),zp'(z);z)$ are univalent in U and satisfy the (first-order) differential superordination*

$$h(z)\prec\varphi(p(z),zp'(z);z) \tag{1}$$

then p is called a solution of the differential superordination. An analytic function q is called a subordinant of the solutions of the differential superordination, or more simply a subordinant if $q\prec p$ for all p satisfying (1). A univalent subordinant $\widetilde{q}$ that satisfies $q\prec\widetilde{q}$ for all subordinants q of (1) is said to be the best subordinant.

Note that the best subordinant is unique up to a rotation of U.

For Ω a set in $\mathbb{C}$, with φ and p as given in Definition 1.1, suppose (1) is replaced by

$$\Omega\subset\{\varphi(p(z),zp'(z);z)|\ z\in U\}. \tag{2}$$

Although this more general situation is a "differential containment", the condition in (2) will also be referred to as a differential superordination, and the definitions of solution, subordinant and best dominant as given above can be extend to this generalization.

Before obtaining some of the main results we need to introduce a class of univalent functions defined on the unit disc that have some nice boundary properties.

Definition 1.2. [3] *We denote by Q the set of functions f that are analytic and injective on $\overline{U}\setminus E(f)$, where*

$$E(f)=\left\{\zeta\in\partial U:\lim_{z\to\zeta}f(z)=\infty\right\}$$

and are such that $f'(\zeta)\neq 0$ for $\zeta\in\partial U\setminus E(f)$.

The subclass of Q for which $f(0) = a$ is denoted by $Q(a)$.

Definition 1.3. [3] *Let Ω be a set in $\mathbb{C}$ and $q \in \mathcal{H}[a,n]$ with $q'(z) \neq 0$. The class of admissible functions $\Phi_n[\Omega, q]$, consist of those functions $\varphi : \mathbb{C}^2 \times \overline{U} \to \mathbb{C}$ that satisfy the admissibility condition*

$$\varphi\left(q(z), \frac{zq'(z)}{m}; \zeta\right) \in \Omega \tag{3}$$

where $z \in U$, $\zeta \in \partial U$ and $m \geq n \geq 1$.

In order to prove the new results we shall use the following lemmas:

Lemma 1.4. [3] *Let h be convex in U, with $h(0) = a$, $\gamma \neq 0$ and $\mathrm{Re}\gamma \geq 0$. If $p \in \mathcal{H}[a,1] \cap Q$ and $p(z) + \dfrac{zp'(z)}{\gamma}$ is univalent in U with*

$$h(z) \prec p(z) + \frac{zp'(z)}{\gamma}$$

then

$$q(z) \prec p(z)$$

where

$$q(z) = \frac{\gamma}{z^\gamma} \int_0^z h(t)\, t^{\gamma-1} dt, \ z \in U.$$

The function q is convex and is the best subordinant.

Lemma 1.5. [3] *Let q be convex in U and let h be defined by*

$$h(z) = q(z) + \frac{zq'(z)}{\gamma}, \ z \in U,$$

with $\mathrm{Re}\,\gamma \geq 0$. If $p \in \mathcal{H}[a,1] \cap Q$, $p(z) + \dfrac{zp'(z)}{\gamma}$ is univalent in U, and

$$q(z) + \frac{zq'(z)}{\gamma} \prec p(z) + \frac{zp'(z)}{\gamma}, \ z \in U$$

then

$$q(z) \prec p(z),$$

where

$$q(z) = \frac{\gamma}{z^\gamma} \int_0^z h(t) t^{\gamma-1} dt.$$

The function q is the best subordinant.

Definition 1.6. [6] *For $f \in A$ and $m \geq 0$, $m \in \mathbb{N}$, the operator $I^m f$ is defined by*

$$I^0 f(z) = f(z)$$

$$I^1 f(z) = \int_0^z f(t)\, t^{-1} dt$$

$$I^m f(z) = I\left[I^{m-1} f(z)\right],\ z \in U.$$

Remark 1.7. *If we denote by $l(z) = -\log(1-z)$, then*

$$I^m f(z) = [(\underbrace{l * l * l \cdots * l}_{n-times}) * f](z), f \in \mathcal{H}(U), f(0) = 0.$$

By "$*$" we denote the Hadamard product or convolution (i.e. if $f(z) = \sum_{j=0}^{\infty} a_j z^j$, $g(z) = \sum_{j=0}^{\infty} b_j z^j$ then $(f * g)(z) = \sum_{j=0}^{\infty} a_j b_j z^j$).

Remark 1.8. $I^m f(z) = \int_0^z \int_0^{t_m} \ldots \int_0^{t_2} \dfrac{f(t_1)}{t_1 t_2 \ldots t_m} dt_1 dt_2 \ldots dt_m$, $f \in \mathcal{H}(U)$, $f(0) = 0$.

Remark 1.9. $D^m I^m f(z) = I^m D^m f(z) = f(z)$, $f \in \mathcal{H}(U)$, $f(0) = 0$, *where D^m is the Sălăgean differential operator (see [6]).*

2. Main results

Definition 2.1. *For $0 \leq \alpha < 1$ and $m \in \mathbb{N}$, let $J_m(\alpha)$ denote the class of functions $f \in A$ which satisfy the inequality*

$$\mathrm{Re}\, [I^m f(z)]' > \alpha.$$

Remark 2.2. *If $m = 0$ then $J_0(\alpha)$ is the class of the functions which satisfy $\mathrm{Re} f'(z) > \alpha$, functions with bounded boundary rotation [see [4]].*

Theorem 2.3. *Let*

$$h(z) = \frac{1 + (2\alpha - 1)z}{1 + z},\ z \in U$$

a convex function in U, with $h(0) = 1$.

Let $f \in J_m(\alpha)$, and suppose that $[I^m f(z)]'$ is univalent and

$$\left[I^{m+1} f(z)\right]' \in \mathcal{H}[1,1] \cap Q.$$

If

$$h(z) \prec [I^m f(z)]',\ z \in U, \tag{4}$$

then

$$q(z) \prec [I^{m+1} f(z)]',\ z \in U,$$

where

$$q(z) = 2\alpha - 1 + 2(1-\alpha)\frac{\log(1+z)}{z}. \tag{5}$$

The function q is convex and is the best subordinant.

Proof. Let $f \in J_m(\alpha)$. By using the properties of the operator $I^m f(z)$ we have

$$I^m f(z) = z[I^{m+1} f(z)]', \; z \in U. \tag{6}$$

Differentiating (6), we obtain

$$[I^m f(z)]' = \left[I^{m+1} f(z)\right]' + z\left[I^{m+1} f(z)\right]'', \; z \in U. \tag{7}$$

If we denote by $p(z) = [I^{m+1} f(z)]'$ then (7) becomes

$$[I^m f(z)]' = p(z) + zp'(z), \; z \in U.$$

Then (4) becomes

$$h(z) \prec p(z) + zp'(z), \; z \in U.$$

By using Lemma 1.4 for $\gamma = 1$, we have

$$q(z) \prec p(z) = [I^{m+1} f(z)]', \; z \in U,$$

where

$$\begin{aligned} q(z) &= \frac{1}{z}\int_0^z \frac{1+(2\alpha-1)t}{1+t}dt = \\ &= 2\alpha - 1 + 2(1-\alpha)\frac{\log(1+z)}{z}. \end{aligned}$$

Moreover, the function q is the best subordinant. ∎

Theorem 2.4. *Let*

$$h(z) = \frac{1+(2\alpha-1)}{1+z}$$

be convex in U, with $h(0) = 1$.

Let $f \in J_m(\alpha)$, and suppose that $[I^m f(z)]'$ is univalent and

$$\frac{I^m f(z)}{z} \in \mathcal{H}[1,1] \cap Q.$$

If

$$h(z) \prec [I^m f(z)]', \; z \in U \tag{8}$$

then

$$q(z) \prec \frac{I^m f(z)}{z}, \; z \in \dot{U}$$

where

$$q(z) = 2\alpha - 1 + 2(1-\alpha)\frac{\log(1+z)}{z}.$$

The function q is convex and is the best subordinant.

Proof. We let

$$p(z) = \frac{I^m f(z)}{z}, \ z \in \dot{U}$$

and we obtain

$$I^m f(z) = zp(z), \ z \in U \tag{9}$$

By differentiating (9) we obtain

$$[I^m f(z)]' = p(z) + zp'(z), \ z \in U.$$

Then (8) becomes

$$h(z) \prec p(z) + zp'(z), \ z \in U.$$

By using Lemma 1.4 we have

$$q(z) \prec p(z) = \frac{I^m f(z)}{z}, \ z \in \dot{U}$$

where

$$\begin{aligned} q(z) &= \frac{1}{z}\int_0^z \frac{1+(2\alpha-1)t}{1+t}dt = \\ &= 2\alpha - 1 + 2(1-\alpha)\frac{\log(1+z)}{z}. \end{aligned}$$

The function q is convex and is the best subordinant. ∎

Theorem 2.5. *Let q be convex in U and let h be defined by*

$$h(z) = q(z) + zq'(z), \ z \in U.$$

Let $f \in J_m(\alpha)$ and suppose that $[I^m f(z)]'$ is univalent in U

$$[I^{m+1} f(z)]' \in \mathcal{H}[1,1] \cap Q$$

and

$$h(z) \prec [I^m f(z)]', \ z \in U. \tag{10}$$

Then

$$q(z) \prec [I^{m+1} f(z)]', \ z \in U$$

where

$$q(z) = \frac{1}{z}\int_0^z h(t)dt, \ z \in U.$$

The function q is the best subordinant.

Proof. Let $f \in J_m(\alpha)$. By using the properties of the operator $I^m f(z)$, we have

$$I^m f(z) = z[I^{m+1} f(z)]', \ z \in U. \tag{11}$$

Differentiating (11) we obtain

$$[I^m f(z)]' = \left[I^{m+1} f(z)\right]' + z\left[I^{m+1} f(z)\right]'', \ z \in U. \tag{12}$$

If we let $p(z) = \left[I^{m+1} f(z)\right]'$ then (12) becomes

$$[I^m f(z)]' = p(z) + zp'(z), \ z \in U.$$

By using Lemma 1.5 for $\gamma = 1$ we have

$$q(z) \prec p(z) = [I^{m+1} f(z)]', \ z \in U,$$

where

$$q(z) = \frac{1}{z}\int_0^z h(t)dt.$$

The function q is the best subordinant. ■

Theorem 2.6. *Let q be convex in U and let h be defined by*

$$h(z) = q(z) + zq'(z), \ z \in U.$$

Let $f \in J_m(\alpha)$ and suppose that $[I^m f(z)]'$ is univalent in U,

$$\frac{I^m f(z)}{z} \in \mathcal{H}[1,1] \cap Q$$

and

$$h(z) \prec [I^m f(z)]', \ z \in U. \tag{13}$$

Then

$$q(z) \prec \frac{I^m f(z)}{z}, \ z \in \dot{U},$$

where

$$q(z) = \frac{1}{z}\int_0^z h(t)dt.$$

The function q is the best subordinant.

Proof. We let

$$p(z) = \frac{I^m f(z)}{z}, \ z \in \dot{U}$$

and we obtain

$$I^m f(z) = zp(z), \ z \in U. \tag{14}$$

By differentiating (14), we obtain

$$[I^m f(z)]' = p(z) + zp'(z), \ z \in U,$$

Then (13) becomes

$$q(z) + zq'(z) \prec p(z) + zp'(z).$$

By using Lemma 1.5 for $\gamma = 1$ we have

$$q(z) \prec p(z) = \frac{I^m f(z)}{z}, \ z \in \dot{U}$$

where

$$q(z) = \frac{1}{z}\int_0^z h(t)dt.$$

The function q is the best subordinant. ■

Remark 2.7. *We remark that similar results to those in this paper, but for a differential operator were obtained by Gh. Oros and G. I. Oros in [5].*

Acknowledgement: This work is supported by Romanian Ministry of Education and Research, UEFISCSU Grant PN-II-ID 524/2007.

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Received 12.11.2008

Journal of
Mathematics
and Applications
No 31, pp 37-49 (2009)

Subordination results and integral means inequalities for generalized k-starlike functions

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Submitted by: *Leopold Koczan*

Abstract: In the paper, we introduce a generalized class of k-starlike functions associated with Wright generalized hypergeometric functions and obtain the sequential subordination results and integral means inequalities. Some interesting consequences of our results are also pointed out

AMS Subject Classification: *30C45, 30C80*
Key Words and Phrases: *starlike, convex, subordinating factor sequence, integral means, Hadamard product, generalized hypergeometric function, Wright function, varying arguments*

1 Introduction and Preliminaries

Let A denote the class of functions of the form

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n \tag{11}$$

which are analytic in the open disc $U = \{z : |z| < 1\}$. For functions $f \in A$ given by (1.1) and $g \in A$ given by

$$g(z) = z + \sum_{n=2}^{\infty} b_n z^n, \quad z \in U,$$

we define the Hadamard product (or Convolution) of f and g by

$$(f * g)(z) = z + \sum_{n=2}^{\infty} a_n b_n z^n, \quad z \in U. \tag{12}$$

For positive real parameters $\alpha_1, A_1 \ldots, \alpha_l, A_l$ and $\beta_1, B_1 \ldots, \beta_m, B_m$ $(l, m \in N = 1, 2, 3, ...)$ such that

$$1 + \sum_{n=1}^{m} B_m - \sum_{n=1}^{l} A_n \geq 0, \quad z \in U, \tag{13}$$

the Wright generalized hypergeometric function [32]

$$\begin{aligned} {}_l\Psi_m[(\alpha_1, A_1), \ldots, (\alpha_l, A_l); \quad & (\beta_1, B_1), \ldots, (\beta_m, B_m); z] \\ = \quad & {}_l\Psi_m[(\alpha_n, A_n)_{1,l}; (\beta_n, B_n)_{1,m}; z] \end{aligned}$$

is defined by

$$\begin{aligned} {}_l\Psi_m[(\alpha_t, A_t)_{1,l}; \quad & (\beta_t, B_t)_{1,m}; z] \\ = \quad & \sum_{n=0}^{\infty} \{\prod_{t=0}^{l} \Gamma(\alpha_t + tA_t\}\{\prod_{t=0}^{m} \Gamma(\beta_t + tB_t\}^{-1} \frac{z^n}{n!}, \quad z \in U. \end{aligned}$$

If $l \leq m+1$, $A_n = 1 (n = 1, ..., l)$ and $B_n = 1 (n = 1, ..., m)$, we have the relationship:

$$\Omega \, {}_l\Psi_m[(\alpha_n, 1)_{1,l}; (\beta_n, 1)_{1,m}; z] = {}_lF_m(\alpha_1, ..., \alpha_l; \beta_1, ..., \beta_m; z), \quad z \in U, \tag{14}$$

where ${}_lF_m(\alpha_1, ..., \alpha_l; \beta_1, ..., \beta_m; z)$ is the generalized hypergeometric function and

$$\Omega = \left(\prod_{t=0}^{l} \Gamma(\alpha_t) \right)^{-1} \left(\prod_{t=0}^{m} \Gamma(\beta_t) \right). \tag{15}$$

In [5] Dziok and Raina introduced the linear operator by using Wright generalized hypergeometric function. Let

$${}_l\phi_m[(\alpha_t, A_t)_{1,l}; (\beta_t, B_t)_{1,m}; z] = \Omega z \, {}_l\Psi_m[(\alpha_t, A_t)_{1,l} (\beta_t, B_t)_{1,m}; z], \quad z \in U,$$

and

$$\mathcal{H} = \mathcal{H}[(\alpha_n, A_n)_{1,l}; (\beta_n, B_n)_{1,m}] : A \to A$$

be a linear operator defined by

$$\mathcal{H}f(z) = z \, {}_l\phi_m[(\alpha_t, A_t)_{1,l}; (\beta_t, B_t)_{1,m}; z] * f(z).$$

We observe that, for f of the form (1.1), we have

$$\mathcal{H}f(z) = z + \sum_{n=2}^{\infty} \sigma_n \, a_n z^n, , \quad z \in U, \tag{16}$$

where

$$\sigma_n = \frac{\Omega \, \Gamma(\alpha_1 + A_1(n-1)) \ldots \Gamma(\alpha_p + A_p(n-1))}{(n-1)! \Gamma(\beta_1 + B_1(n-1)) \ldots \Gamma(\beta_q + B_q(n-1))}, \tag{17}$$

and Ω is given by (1.5).

In view of the relationship (1.4) the linear operator (1.6) includes the Dziok-Srivastava operator (see [7], [6] and [1]), so that it includes (as its special cases) various other linear operators introduced and studied by Bernardi [2], Carlson and Shaffer [4], Libera [13], Livingston [15], Ruscheweyh [23], Srivastava and Owa [29], and others.

For $0 \leq \gamma < 1$ and $k \geq 0$, we define the class $\mathcal{W}_m^l(\gamma, k)$ of functions f of the form (1.1) and satisfying the analytic criterion

$$\operatorname{Re}\left\{\frac{z(\mathcal{H}f(z))'}{\mathcal{H}f(z)} - \gamma\right\} > k\left|\frac{z(\mathcal{H}f(z))'}{\mathcal{H}f(z)} - 1\right|, \quad z \in U, \tag{18}$$

where $\mathcal{H}f(z)$ is given by (1.6).

Also we denote by $\mathcal{T}$, the class of analytic functions with varying arguments (introduced by Silverman [26]) consisting of functions f of the form (1.1) for which there exists a real number η such that

$$\theta_n + (n-1)\eta = \pi (\mathrm{mod}\ 2\pi), \text{ where } \arg(a_n) = \theta_n \text{ for all } \ n \geq 2. \tag{19}$$

Moreover let us put

$$\mathcal{T}\mathcal{W}_m^l(\gamma, k) := \mathcal{W}_m^l(\gamma, k) \cap \mathcal{T}.$$

If, $A_t = 1 (t = 1, ...l), B_t = 1 (t = 1, ...m)$ and by suitably specializing the values of l, m, $\alpha_1, \alpha_2, \ldots, \alpha_l$, $\beta_1, \beta_2, \ldots, \beta_m$, γ and k in the class $\mathcal{W}_m^l(\gamma, k)$, we obtain the various subclasses, we present some examples.

Example 11. *If $l = 2$ and $m = 1$ with $\alpha_1 = \alpha_2 = \beta_1 = 1$, then we obtain the class*

$$\begin{aligned} UST(\gamma, k) &:= \mathcal{W}_1^2(\gamma, k) \\ &= \left\{ f \in A : Re \left\{\frac{zf'(z)}{f(z)} - \gamma\right\} > k\left|\frac{zf'(z)}{f(z)} - 1\right|, \ z \in U \right\}. \end{aligned}$$

of $k-$starlike functions of order γ, $0 \leq \gamma < 1$, which was introduced in [3]. We observe that $S^*(\gamma) := UST(\gamma, 0)$ is well-known class of starlike functions of order γ.

Example 12. *If $l = 2$ and $m = 1$ with $\alpha_1 = \delta + 1\ (\delta > -1)$, $\alpha_2 = \beta_1 = 1$, then*

$$\begin{aligned} R_\delta(\gamma, k) = \mathcal{W}_1^2(\gamma, k) &= \left\{ f \in A : Re \left\{\frac{z(D^\delta f(z))'}{D^\delta f(z)} - \gamma\right\}\right. \\ &> \left. k\left|\frac{z(D^\delta f(z))'}{D^\delta f(z)} - 1\right|, \ z \in U \right\}, \end{aligned}$$

where D^δ is called Ruscheweyh derivative of order δ $(\delta > -1)$ defined by

$$D^\delta f(z) := \frac{z}{(1-z)^{\delta+1}} * f(z) \equiv \mathcal{H}_1^2(\delta + 1, 1; 1) f(z).$$

We observe that

$$K := R_1(0, 0)$$

is the well known class of convex functions. Also let

$$\mathcal{T}R_\delta(\gamma,k) := R_\delta(\gamma,k)\cap\mathcal{T},$$

The class $\mathcal{T}R_\delta(\gamma,0)$ was studied in [22].

Example 13. *If $l=2$ and $m=1$ with $\alpha_1=c+1(c>-1)$, $\alpha_2=1$, $\beta_1=c+2$, then*

$$\begin{aligned}\mathcal{W}_1^2(\gamma,k)=B_c(\gamma,k) &= \left\{f\in A : Re\left(\frac{z(J_cf(z))'}{J_cf(z)}-\gamma\right)\right.\\ &> \left. k\left|\frac{z(J_cf(z))'}{J_cf(z)}-1\right|,\ z\in U\right\},\end{aligned}$$

where J_c is a Bernardi operator [2] defined by

$$J_cf(z) := \frac{c+1}{z^c}\int_0^z t^{c-1}f(t)dt \equiv \mathcal{H}_1^2(c+1,1;c+2)f(z).$$

Note that the operator J_1 was studied earlier by Libera [13] and Livingston [15]. Further,

$$\mathcal{T}B_c(\gamma,k)=B_c(\gamma,k)\cap\mathcal{T}.$$

Example 14. *If $l=2$ and $m=1$ with $\alpha_1=a\,(a>0)$, $\alpha_2=1$, $\beta_1=c\,(c>0)$, then*

$$\begin{aligned}\mathcal{W}_1^2(\gamma,k)\equiv L_c^a(\gamma,k) &= \left\{f\in A : Re\left(\frac{z(L(a,c)f(z))'}{L(a,c)f(z)}-\gamma\right)\right.\\ &> \left. k\left|\frac{z(L(a,c)f(z))'}{L(a,c)f(z)}-1\right|,\ z\in U\right\},\end{aligned}$$

where $L(a,c)$ is a well-known Carlson-Shaffer linear operator [4] defined by

$$L(a,c)f(z) := \left(\sum_{k=0}^{\infty}\frac{(a)_k}{(c)_k}z^{k+1}\right)*f(z)\equiv\mathcal{H}_1^2(a,1;c)f(z).$$

The class $L_c^a(\gamma,k)$ was introduced in [17] and also

$$\mathcal{T}L_c^a(\gamma,k)=L_c^a(\gamma,k)\cap\mathcal{T}$$

was introduced and studied in [18, 19].

Remark 11. *Moreover specializing the parameters of the class $\mathcal{W}_m^l(\gamma,k)$, we can obtain classes introduced and studied by Goodman [9], Ma and Minda [16], Rønning [20, 21], Kanas et.al., [10, 11, 12] and others (see for example [30]).*

The object of the present paper is to investigate the coefficient estimates, extreme points. Further, we obtain the subordination results and integral means inequalities for the generalized class of k-starlike functions. Some interesting consequences of our results are also pointed out.

2 Coefficient Estimates

We first mention a sufficient condition for function f of the form (1.1) to belong to the class $\mathcal{W}_m^l(\gamma, k)$.

Theorem 21. *A function f of the form (1.1) belongs to the class $\mathcal{W}_m^l(\gamma, k)$ if*

$$\sum_{n=2}^{\infty}(kn+n-k-\gamma)\sigma_n\,|a_n| \leq 1-\gamma, \tag{21}$$

where σ_n is given by (1.7).

Proof. By definition of the class $\mathcal{W}_m^l(\gamma, k)$, it suffices to show that

$$k\left|\frac{z(\mathcal{H}f(z))'}{\mathcal{H}f(z)}-1\right| - \operatorname{Re}\left\{\frac{z(\mathcal{H}f(z))'}{\mathcal{H}f(z)}-1\right\} \leq 1-\gamma.$$

Simply calculations give

$$\begin{aligned} & k\left|\frac{z(\mathcal{H}f(z))'}{\mathcal{H}f(z)}-1\right| - \operatorname{Re}\left\{\frac{z(\mathcal{H}f(z))'}{\mathcal{H}f(z)}-\gamma\right\} \\ \leq \quad & (k+1)\left|\frac{z(\mathcal{H}f(z))'}{\mathcal{H}f(z)}-1\right| \leq (k+1)\,\frac{\sum\limits_{n=2}^{\infty}(n-1)\sigma_n|a_n||z|^{n-1}}{1-\sum\limits_{n=2}^{\infty}\sigma_n|a_n||z|^{n-1}}. \end{aligned}$$

Now the last expression is bounded above by $(1-\gamma)$ if (3.4) holds. ■

Our next theorem shows that the condition (2.1) is necessary as well for functions of the form (1.1) with (1.9) to belong to the class $\mathcal{TW}_m^l(\gamma, k)$.

Theorem 22. *Let f be given by (1.1) with (1.9). The function $f(z)$ belongs to the class $\mathcal{TW}_m^l(\gamma, k)$ if and only if (2.1) holds.*

Proof. In view of Theorem 2.1. we need only show that $f \in \mathcal{TW}_m^l(\gamma, k)$ satisfies the coefficient inequality (3.4). If $f \in \mathcal{TW}_m^l(\gamma, k)$, then by definition we have

$$k\left|\frac{z+\sum\limits_{n=2}^{\infty}n\sigma_n a_n z^n}{z+\sum\limits_{n=2}^{\infty}\sigma_n a_n z^n}-1\right| \leq \operatorname{Re}\left\{\frac{z+\sum\limits_{n=2}^{\infty}n\sigma_n a_n z^n}{z+\sum\limits_{n=2}^{\infty}\sigma_n a_n z^n}-\gamma\right\},$$

or

$$k\left|\frac{\sum\limits_{n=2}^{\infty}(n-1)\sigma_n a_n z^{n-1}}{1+\sum\limits_{n=2}^{\infty}\sigma_n a_n z^{n-1}}\right| \leq \operatorname{Re}\left\{\frac{(1-\gamma)+\sum\limits_{n=2}^{\infty}(n-\gamma)\sigma_n a_n z^{n-1}}{1+\sum\limits_{n=2}^{\infty}\sigma_n a_n z^{n-1}}\right\}.$$

In view of (1.9), we set $z = r^{i\eta}$ in the above inequality to obtain

$$\frac{\sum\limits_{n=2}^{\infty} k(n-1)\sigma_n|a_n|r^{n-1}}{1-\sum\limits_{n=2}^{\infty}\sigma_n|a_n|r^{n-1}} \le \frac{(1-\gamma)-\sum\limits_{n=2}^{\infty}(n-\gamma)\sigma_n|a_n|r^{n-1}}{1-\sum\limits_{n=2}^{\infty}\sigma_n|a_n|r^{n-1}}.$$

Thus

$$\sum_{n=2}^{\infty}(kn+n-k-\gamma)\sigma_n|a_n|r^{n-1} \le 1-\gamma, \tag{22}$$

and letting $r \to 1^-$ in (2.2) we obtain the desired inequality (2.1). ■

3 Subordination Results

Before stating and proving our subordination theorem for the class $\mathcal{TW}_m^l(\gamma, k)$, we need the following definitions and lemmas.

Definition 31. *For analytic functions g and h with $g(0) = h(0)$, g is said to be subordinate to h, denoted by $g \prec h$, if there exists an analytic function w such that $w(0) = 0$, $|w(z)| < 1$ and $g(z) = h(w(z))$, for all $z \in U$.*

Definition 32. *A sequence $\{b_n\}_{n=1}^{\infty}$ of complex numbers is said to be a subordinating sequence if, whenever $f(z) = \sum\limits_{n=1}^{\infty} a_n z^n$, $a_1 = 1$ is regular, univalent and convex in U, we have*

$$\sum_{n=1}^{\infty} b_n a_n z^n \prec f(z), \quad z \in U. \tag{31}$$

Lemma 31. [31]. *The sequence $\{b_n\}_{n=1}^{\infty}$ is a subordinating sequence if and only if*

$$Re\ \left\{1+2\sum_{n=1}^{\infty} b_n z^n\right\} > 0, \quad z \in U. \tag{32}$$

Theorem 33. *Let $g(z) \in K$, $f \in \mathcal{TW}_m^l(\gamma, k)$ and*

$$l > m,\ \alpha_{m+1} \ge 1,\ \alpha_j \ge \beta_j \text{ and } A_j \ge B_j\ \ (j = 2, \ldots, m). \tag{33}$$

Then

$$\frac{(2+k-\gamma)\sigma_2}{2\left[1-\gamma+(2+k-\gamma)\sigma_2\right]}(f * g)(z) \prec g(z) \tag{34}$$

and

$$Re\ \{f(z)\} > -\frac{\left[1-\gamma+(2+k-\gamma)\sigma_2\right]}{(2+k-\gamma)\sigma_2}, \quad z \in U. \tag{35}$$

The constant factor

$$\frac{(2+k-\gamma)\sigma_2}{2\left[1-\gamma+(2+k-\gamma)\sigma_2\right]} \tag{36}$$

in (3.4) cannot be replaced by a larger number.

Proof. Let $f \in \mathcal{TW}_m^l(\gamma,k)$ and suppose that

$$g(z)=z+\sum_{n=2}^{\infty}c_nz^n,\quad z\in U,$$

belongs to the class K. Then

$$\begin{aligned}&\frac{(2+k-\gamma)\sigma_2}{2[1-\gamma+(2+k-\gamma)\sigma_2]}(f*g)(z)\\ &\qquad=\frac{(2+k-\gamma)\sigma_2}{2[1-\gamma+(2+k-\gamma)\sigma_2]}\left(z+\sum_{n=2}^{\infty}c_na_nz^n\right).\end{aligned}$$

Thus, by Definition 3.2., the subordination result holds true if

$$\left\{\frac{(2+k-\gamma)\sigma_2}{2[1-\gamma+(2+k-\gamma)\sigma_2]}a_n\right\}_{n=1}^{\infty}$$

is a subordinating sequence, with $a_1=1$. In view of Lemma 3.1., this is equivalent to the following inequality

$$\operatorname{Re}\left\{1+\sum_{n=1}^{\infty}\frac{(2+k-\gamma)\sigma_2}{1-\gamma+(2+k-\gamma)\sigma_2}a_nz^n\right\}>0,\quad z\in U. \tag{37}$$

By (3.3) the sequence

$$d_n:=(kn+n-k-\gamma)\sigma_n,\quad n=2,3,\ldots$$

is increasing. In particular we obtain

$$(2+k-\gamma)\sigma_2\le(kn+n-k-\gamma)\sigma_n,\quad n\ge 2.$$

Thus, for $|z|=r<1$, we have

$$\begin{aligned}&\operatorname{Re}\left\{1+\frac{(2+k-\gamma)\sigma_2}{[1-\gamma+(2+k-\gamma)\sigma_2]}\sum_{n=1}^{\infty}a_nz^n\right\}\\
=\;&\operatorname{Re}\left\{1+\frac{(2+k-\gamma)\sigma_2}{1-\gamma+(2+k-\gamma)\sigma_2}z+\frac{\sum\limits_{n=2}^{\infty}(2+k-\gamma)\sigma_2a_nz^n}{1-\gamma+(2+k-\gamma)\sigma_2}\right\}\\
\ge\;&1-\frac{(2+k-\gamma)\sigma_2}{1-\gamma+(2+k-\gamma)\sigma_2}r-\frac{\sum\limits_{n=2}^{\infty}(kn+n-k-\gamma)\sigma_n\left|a_n\right|r^{n-1}}{1-\gamma+(2+k-\gamma)\sigma_2}r\\
\ge\;&1-\frac{(2+k-\gamma)\sigma_2}{1-\gamma+(2+k-\gamma)\sigma_2}r-\frac{1-\gamma}{1-\gamma+(2+k-\gamma)\sigma_2}r>0,\end{aligned}$$

where we have also made use the assertion (2.1) of Theorem 2.2.. This evidently proves the inequality (3.7) and hence the subordination result (3.4). The inequality (3.5) follows from (3.4) by taking

$$g(z) = \frac{z}{1-z} = z + \sum_{n=2}^{\infty} z^n, \quad z \in K.$$

Next we consider the function

$$F(z) = z - \frac{1-\gamma}{(2+k-\gamma)\sigma_2} z^2, \quad z \in K.$$

Clearly, $F \in \mathcal{TW}_m^l(\gamma, k)$. Thus by (3.4) we have

$$\frac{(2+k-\gamma)\sigma_2}{2[1-\gamma+(2+k-\gamma)\sigma_2]} F(z) \prec \frac{z}{1-z}.$$

It is easily verified that

$$\min\left\{\mathrm{Re}\ \left(\frac{(2+k-\gamma)\sigma_2}{2[1-\gamma+(2+k-\gamma)\sigma_2]} F(z)\right)\right\} = -\frac{1}{2}, \quad z \in U.$$

This shows that the constant (3.6) cannot be replaced by any larger one. ■

We observe that, if $A_t = 1 (t = 1, 2, ..., l)$ and $B_t = 1 (t = 1, 2, ..., m)$ specializing the parameters l, m, $\alpha_1, \alpha_2, \ldots, \alpha_p$, and $\beta_1, \beta_2, \ldots, \beta_q$, γ, and k in the above theorem and in view of Examples 1 to 4 in Section 1, we state the following corollaries for the subclasses defined in those examples.

Corollary 31. *If $f \in \mathcal{T}\mathbb{S}^*(\gamma, k)$, $g \in K$, then*

$$\frac{2+k-\gamma}{2[3+k-\gamma]} (f * g)(z) \prec g(z), \tag{38}$$

and

$$Re\{f(z)\} > -\frac{3+k-2\gamma}{2+k-\gamma}, \quad z \in U.$$

The constant factor

$$\frac{2+k-\gamma}{2[3+k-2\gamma]}$$

in (3.8) cannot be replaced by a larger one.

Remark 31. *Corollary 3.1., extend the result obtained by Singh [28] when $\gamma = k = 0$.*

Remark 32. *Corollary 3.1. extend the results obtained by Frasin [8] for the special values of γ and k.*

Corollary 32. *If $f \in \mathcal{T}R_\delta(\gamma, k)$, $\delta > 0$, $g \in K$, then*

$$\frac{(\delta+1)(2+k-\gamma)}{2[1-\gamma+(\delta+1)(2+k-\gamma)]}(f*g)(z) \prec g(z), \tag{39}$$

and

$$Re\{f(z)\} > -\frac{1-\gamma+(\delta+1)(2+k-\gamma)}{(\delta+1)(2+k-\gamma)}, \ \ z \in U.$$

The constant factor

$$\frac{(\delta+1)[(2+k-\gamma)]}{2[1-\gamma+(\delta+1)(2+k-\gamma)]}$$

in (3.9) cannot be replaced by a larger one.

Corollary 33. *If $f \in \mathcal{T}L_c^a(\gamma, k)$, $g \in K$, $a \geq c > 0$, then*

$$\frac{a(2+k-\gamma)}{2[c(1-\gamma)+a(2+k-\gamma)]}(f*g)(z) \prec g(z), \tag{310}$$

and

$$Re\{f(z)\} > -\frac{[c(1-\gamma)+a(2+k-\gamma)]}{a(2+k-\gamma)}, \ \ z \in U.$$

The constant factor

$$\frac{a(2+k-\gamma)}{2[c(1-\gamma)+a(2+k-\gamma)]}$$

in (3.10) cannot be replaced by a larger one.

4 Integral Means Inequalities

Due Littlewood [14] we obtain integral means inequalities for the functions in the family $\mathcal{TW}_m^l(\gamma, k)$.

Lemma 41. *[14]. If the functions f and g are analytic in U with $g \prec f$, then for $\eta > 0$, and $0 < r < 1$, we have*

$$\int_0^{2\pi} \left|g(re^{i\theta})\right|^\eta d\theta \leq \int_0^{2\pi} \left|f(re^{i\theta})\right|^\eta d\theta. \tag{41}$$

Silverman [27] found that the function

$$f_2(z) = z - \frac{z^2}{2}, \quad z \in U,$$

is often extremal over the family of functions with negative coefficients. He applied this function to resolve integral means inequality, conjectured in [24] and settled in [25], that

$$\int_0^{2\pi} \left|f(re^{i\theta})\right|^\eta d\theta \leq \int_0^{2\pi} \left|f_2(re^{i\theta})\right|^\eta d\theta,$$

for all functions f with negative coefficients, $\eta > 0$ and $0 < r < 1$. In [25], he also proved his conjecture for some subclasses of $\mathcal{T}$.

Applying Lemma 4.1. and Theorem 2.2., we prove the following result.

Theorem 41. *Let* $f \in \mathcal{TW}^l_m(\gamma, k)$, $\eta > 0$ *and* $f_2(z)$ *be defined by*

$$f_2(z) = z - \frac{1-\gamma}{(2+k-\gamma)\sigma_2} z^2, \quad z \in U.$$

Then for $z = re^{i\theta}$, $0 < r < 1$, *we have*

$$\int_0^{2\pi} |f(z)|^\eta \, d\theta \leq \int_0^{2\pi} |f_2(z)|^\eta \, d\theta. \tag{42}$$

Proof. For function $f(z)$ of the form (1.1) the inequality (4.2) is equivalent to the following:

$$\int_0^{2\pi} \left| 1 + \sum_{n=2}^{\infty} a_n z^{n-1} \right|^\eta d\theta \leq \int_0^{2\pi} \left| 1 - \frac{(1-\gamma)}{(2+k-\gamma)\sigma_2} z \right|^\eta d\theta.$$

By Lemma 4.1., it suffices to show that

$$1 + \sum_{n=2}^{\infty} a_n z^{n-1} \prec 1 - \frac{1-\gamma}{(2+k-\gamma)\sigma_2} z. \tag{43}$$

Setting

$$1 + \sum_{n=2}^{\infty} a_n z^{n-1} = 1 - \frac{1-\gamma}{(2+k-\gamma)\sigma_2} w(z), \quad z \in U,$$

and using (2.1), we obtain

$$\begin{aligned} |w(z)| &= \left| \sum_{n=2}^{\infty} \frac{[n(1+k) - (\gamma+k)]\sigma_n}{1-\gamma} a_n \, z^{n-1} \right| \\ &\leq |z| \sum_{n=2}^{\infty} \frac{[n(1+k) - (\gamma+k)]\sigma_n}{1-\gamma} |a_n| \leq |z|, \quad z \in U. \end{aligned}$$

Thus by definition od subordination we have (4.2) and this completes the proof. ∎

If $A_t = 1 (t = 1, 2, ..., l)$ and $B_t = 1 (t = 1, 2, ..., m)$ specializing the parameters l, m, $\alpha_1, \alpha_2, \ldots, \alpha_p$, and $\beta_1, \beta_2, \ldots, \beta_q$, γ, and k and in view of the Examples 1 to 4 in Section 1 and Theorem 4.1., we can state the following corollaries:

Corollary 41. *If* $f \in \mathcal{T}\mathbb{S}(\gamma, k)$ *and* $\eta > 0$, *then the assertion (4.2) holds true with*

$$f_2(z) = z - \frac{1-\gamma}{[2+k-\gamma)]} z^2, \quad z \in U.$$

Remark 41. *Fixing $k = 0$, Corollary 4.1. extend the integral means inequality obtained in [25].*

Corollary 42. *If $f \in \mathcal{T}R_\delta(\gamma, k)$ and $\eta > 0$, then the assertion (4.2) holds true with*

$$f_2(z) = z - \frac{(1-\gamma)}{(\delta+1)[2+k-\gamma]} z^2, \quad z \in U.$$

Corollary 43. *If $f \in \mathcal{T}B_c(\gamma, k)$ and $\eta > 0$, then the assertion (4.2) holds true with*

$$f_2(z) = z - \frac{(1-\gamma)(c+2)}{(c+1)[2+k-\gamma]} z^2, \quad z \in U.$$

Corollary 44. *If $f \in \mathcal{T}L_c^a(\gamma, k)$ and $\eta > 0$, then the assertion (4.2) holds true with*

$$f_2(z) = z - \frac{c(1-\gamma)}{a[2+k-\gamma]} z^2, \quad z \in U.$$

Concluding Remarks: Just as we pointed out the Wright generalized hypergeometric function contains the Hohlov operator, the Carlson-Shaffer operator, the Ruscheweyh derivative operator, the generalized Bernardi-Libera-Livingston operator, the fractional derivative operator, etc. The results presented here can provide interesting extensions and generalizations of those considered earlier for simpler analytic function classes. The details involved in the derivations of such specializations are fairly straight forward.

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Received 20.01.2009

J o u r n a l of
Mathematics
and Applications
No 31, pp 51-66 (2009)

The speed of convergence of random products of sums of independent random variables

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Abstract: Let $\{X_n, n \geq 1\}$ be a sequence of independent random variables such that $EX_n^2 < \infty$, $\sigma_n^2 = Var(X_n)$, $n \geq 1$. We set $S_n = \sum_{k=1}^n X_k, n \geq 1$. Let N denotes the standard normal random variable. In this paper we investigate the speed of convergence

$$\Big(\prod_{k=1}^{n} \frac{S_k - ES_k + a_k}{a_k}\Big)^{\gamma_n} \xrightarrow{D} e^N, \text{ as } n \to \infty,$$

in the Kolmogorov's metric for some sequences of positive reals $\{a_n, n \geq 1\}$ and $\{\gamma_n, n \geq 1\}$.

AMS Subject Classification: *Primary: 60F05; Secondary 60G50*
Key Words and Phrases: *Lognormal distribution, Product of sums, Central Limit Theorem, U - statitics, Law of Large Numbers, Speed of convergence*

1. Introduction

Let $\{X_n, n \geq 1\}$ be a sequence of independent random variables with mean $EX_n = \mu_n$ and variance $Var(X_n) = \sigma_n, n \geq 1$, and let $\{a_n, n \geq 1\}$ be a sequence of positive real numbers. In this paper we are interested in the limit behaviour of the products

$$\prod_{j=1}^{n} \frac{S_j - ES_j + a_j}{a_j}, \tag{1.1}$$

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[2]Research supported by the Deutsche Forschungsgemeinschaft through the German-Polish project 436 POL 125/0-1 and by TODEQ MTKD-CT-2005-030042.

as $n \to \infty$. This study begin Arnold and Villaseñor ([1]) in the case when $\{X_n, n \geq 1\}$ is a sequence of independent and exponentially distributed random variables and $a_n = ES_n, n \geq 1$. This result was generalized by Rempała and Wesołowski. In the paper [7] they omit the assumptions, that X_n, $n \geq 1$, are exponentially distributed. Furthermore the mentioned above result was generalized by Qi ([6]) and Lu and Qi ([3]) to the case of stable limit law.

In all these results there are computed the limit in weak sense of (1.1). Always there is considered the normalizing sequence $a_n = n\mu$ and always is considered the sequence $\{X_n, n \geq 1\}$ of independent and identically distributed random variables (i.i.d.). In [2] there was obtained the first result for independent only sequence $\{X_n, n \geq 1\}$ and for arbitrary normalizing sequence $\{a_n, n \geq 1\}$. This result is interesting even in i.i.d. case. If in the i.i.d. case we consider the sequence of reals $\{a_n, n \geq 1\}$ such that

$$X_n - EX_n + a_n - a_{n-1} > 0, \text{ a.s. }, n \geq 1,$$

$$\frac{n}{a_n^2} \to 0, \text{ and } \frac{\left(\sum_{k=1}^n k/a_k^2\right)^2}{\sum_{k=1}^n kA_{k+1}^n/a_k} \longrightarrow 0, \text{ as } n \to \infty,$$

then

$$\left(\prod_{j=1}^n \frac{S_j - j\mu + a_j}{a_j}\right)^{\gamma_n} \xrightarrow{D} e^N, \text{ as } n \to \infty, \tag{1.2}$$

where $\gamma_n^2 = \sum_{k=1}^n (A_k^n)^2\sigma^2, A_k^n = \sum_{k=1}^n 1/a_k, n \geq 1$. We consider this case separately (Theorem 2).

In this paper we investigate the speed of convergence in the mentioned above results. This problem is difficult due to a lot of results needed to establish the convergence type (1.2). It is the central limit theorem as well as the weak law of large numbers and the strong law of large numbers. In this paper we will give the estimation of the term

$$\Delta_n = \sup_x |P[(\prod_{j=1}^n \frac{S_j - ES_j + a_j}{a_j})^{\gamma_n} < x] - P[e^N < x]|.$$

This is the first paper concerning this problem. From now on C denotes the generic constants different in different places, maybe. For arbitrary $x, y \in \Re$ we write $x \wedge y = \min\{x, y\}$, $x \vee y = \max\{x, y\}$.

2. Main results

We begin with the result for a sequence of nonidentically distributed random variables $\{X_n, n \geq 1\}$.

Theorem 1. *Let $\{X_n,\ n \geq 1\}$ be a sequence of independent random variables, such that $EX_n = \mu_n$, $E(X_n - \mu_n)^2 = \sigma_n^2$. Moreover, let $\{a_n, n \geq 1\}$ be a nondecreasing and divergent to infinity sequence of positive real numbers (we put $a_o = 0$) such that $\frac{a_{n+1}}{a_n} = O(1)$, as $n \to \infty$, and for every $k \geq 1, \delta > 0$, denote*

$$\phi_k(\delta) = P[X_k - \mu_k + a_k - a_{k-1} < \delta].$$

Furthermore, let $\{\gamma_n, \quad n \geq 1\}$ *be a sequence of positive numbers such that* $\gamma_n \sum_{k=1}^{n} (A_k^n)^2 \sigma_k^2 \longrightarrow 1$*, as* $n \to \infty$*, where* $A_k^n = \sum_{i=k}^{n} \frac{1}{a_i}$*. If, for some* $r > 2, E|X_n|^r < \infty, n \geq 1$*, and*

$$\frac{a_n^r}{L_n + s_n^r} \sum_{j=n+1}^{\infty} \frac{E|\bar{X}_j|^r + s_{j+1}^r - s_j^r}{a_j^r} = O(1), \quad \text{as } n \to \infty,$$

where

$$L_n = \sum_{j=1}^{n} E|\bar{X}_j|^r, \quad s_n^2 = \sum_{j=1}^{n} \sigma_j^2, \quad n \geq 1,$$

then for every positive numbers ε, δ*, and* $m \in N$*, we have*

$$\begin{aligned}
\Delta_n \quad \leq \quad & C\{\frac{L_m + s_m^r}{a_m^r} + \gamma_n^{\frac{1}{2}} (\sum_{k=1}^{n} \frac{s_k^2}{a_k^2})^{\frac{1}{2}} + (\gamma_n \sum_{k=1}^{m} (A_k^m)^r E|\bar{X}_k|^r)^{\frac{1}{r+1}} m^{\frac{r-2}{2(r+1)}} \\
+ \quad & \sum_{k=1}^{m} \phi_k(\delta) + \frac{\gamma_n \ln(m) \sum_{k=1}^{m} E|\bar{X}_k|}{\varepsilon\delta} + \frac{\varepsilon}{\sqrt{2\pi}} \\
+ \quad & \frac{|\max\{\varrho_n, \varrho_n^{-1}\} - 1|}{\sqrt{2\pi e}} + \frac{\sum_{j=1}^{n} (A_j^n)^r E|\bar{X}_j|^r}{(\sum_{k=1}^{n} (A_k^n)^2 \sigma_k^2)^{\frac{r}{2}}}\},
\end{aligned}$$

where $\varrho_n = \gamma_n Var(\sum_{k=1}^{n} A_k^n X_k), \bar{X}_n = X_n - \mu_n, n \geq 1$.

In the case of i.i.d. sequence with arbitrary normalizing sequence $\{a_n, n \geq 1\}$ we have:

Theorem 2. *Let* $\{X_n, n \geq 1\}$ *be a sequence of i.i.d. random variables, such that* $EX_n = \mu$*,* $E(X_n - \mu)^2 = \sigma^2$*. Moreover, let* $\{a_n, n \geq 1\}$ $(a_0 = 0)$ *be a sequence of divergent to infinity positive real numbers such that* $\frac{a_{n+1}}{a_n} = O(1)$*. Furthermore let* γ_n *be a sequence of positive numbers such that*

$$\gamma_n \sum_{k=1}^{n} (A_k^n)^2 \sigma^2 \longrightarrow 1, \text{as } n \to \infty.$$

(i) If, for some $2 < r \leq 3, E|X_1|^r < \infty$ *and*

$$\exists_{k_o \in N} \ \exists_{c_o > 0} \ \sup_n \frac{n^{k_o - 1}}{a_n^{2k_o - r}} \leq c_o, \quad \text{and} \quad \lim_{n \to \infty} \frac{n}{a_n^2} = 0,$$

then for every positive number $\delta > 0$*, and* $m \in N$*, we have*

$$\begin{aligned}
\Delta_n \quad \leq \quad & C\{\frac{m}{a_m^r} + \gamma_n^{\frac{1}{2}} (\sum_{k=1}^{n} \frac{k}{a_k^2})^{\frac{1}{2}} + \gamma_n^{r/(r+1)} (\sum_{k=1}^{m} (A_k^m)^r)^{1/(r+1)} m^{\frac{r-2}{2(r+1)}} \\
+ \quad & (\frac{\gamma_n m}{\delta})^{\frac{r}{r+1}} + \frac{\sum_{j=1}^{n} (A_j^n)^r}{(\sum_{j=1}^{n} (A_j^n)^2)^{r/2}} + \frac{|\max\{\varrho_n, \varrho_n^{-1}\} - 1|}{\sqrt{2\pi e}} + \sum_{k=1}^{m} \phi_k(\delta)\}.
\end{aligned}$$

(ii) If, for some $T, c > 0,\ 0 < \delta' < 2,\ 0 < m < n,$ *and every* $t \in (0,T], Ee^{t\bar{X}_1} \leq e^{\frac{1}{2}t^2c^2}, \lim_{n\to\infty} \frac{n}{a_n^{\delta'}} = 0,$ *and* $\gamma_n \max\{\sum_{k=1}^m (A_k^m)^2, m\} \to 0,$ *as* $n \to \infty$, *then*

$$\begin{aligned}\Delta_n \quad & \leq C\{e^{-c_1(\frac{1}{2})a_m^{1\wedge(2-\delta')}} + \gamma_n|\ln(\gamma_n)|A_1^m + (\gamma_n \sum_{k=1}^n k/a_k^2)^{1/2} \\ + \quad & \frac{m\gamma_n|\ln(\gamma_n)|}{\delta} + \frac{\sum_{j=1}^n (A_j^n)^3}{(\sum_{j=1}^n (A_j^n)^2)^{3/2}} + \frac{|\max\{\varrho_n, \varrho_n^{-1}\} - 1|}{\sqrt{2\pi e}} + \sum_{k=1}^m \phi_k(\delta)\},\end{aligned}$$

where

$$c_1(x) = \begin{cases} \frac{xT}{2}, & \text{for } 0 < \delta' < 1 \\ \frac{c_o x^2}{2c^2} \wedge \frac{xT}{2}, & \text{for } \delta' = 1, \\ \frac{c_o x^2}{2c^2}, & \text{for } 1 < \delta' < 2. \end{cases} \tag{2.1}$$

The last result deals with the sequence $\{X_n, n \geq 1\}$ of i.i.d. with the normalizing constants $a_n = n\mu, n \geq 1$.

Theorem 3. *Let* $\{X_n, n \geq 1\}$ *be a sequence of i.i.d. random variables, such that* $EX_n = \mu > 0,\ E(X_n - \mu)^2 = \sigma^2$. *Moreover, let* $\gamma_n = \frac{\mu}{\sigma\sqrt{2n}}, n \geq 1$. *Then*

(i) If, for some $2 < r \leq 3,\ E|\bar{X}_1|^r < \infty$, *then for every positive integer* $m < n$ *and* $\delta > 0$, *we have*

$$\Delta_n \leq C\{m^{1-r} + \frac{\sqrt{\ln(n)}}{\sqrt[4]{n}} + n^{1-r/2} + (\frac{m}{n})^{\frac{r}{2(r+1)}} + (\frac{m}{\sqrt{n}\delta})^{\frac{r}{r+1}} + m\phi_1(\delta)\}.$$

(ii) If for some $c, T > 0$ *and every* $t \in (0,T], Ee^{t\bar{X}_1} \leq e^{\frac{1}{2}t^2c^2}$, *then for every positive* $0 < \delta < 1:$

$$\Delta_n \leq C\{\frac{\sqrt{\ln(n)}}{\sqrt[4]{n}} + \frac{(\ln(n))^2}{\delta\sqrt{n}} + \ln(n)\phi_1(\delta)\}.$$

Collorary 1. *Assume that there exists* $\delta_0 > 0$ *such that* $P[X_1 < \delta_0] = 0$.

(i) Let the assumption of Theorem 3 (i) holds. Then

$$\Delta_n \leq C\alpha_1^{n,r} n^{-\Theta_1(r)},$$

where

$$\Theta_1(r) = \begin{cases} \frac{r-2}{2}, & \text{for } r \in (2, r_o], \\ \frac{r(r-1)}{2(r^2+r-1)}, & \text{for } r \in (r_o, \frac{3+\sqrt{5}}{2}), \\ \frac{1}{4}, & \text{for } r \in [\frac{3+\sqrt{5}}{2}, 3], \end{cases}$$

$$\alpha_1^{n,r} = \begin{cases} \sqrt{\ln(n)}, & \text{for } r \in [\frac{3+\sqrt{5}}{2}, 3], \\ 1, & \text{otherwise,} \end{cases}$$

$$r_o = \frac{2}{3}(\sqrt{10}\cos(\frac{\pi}{3} - \frac{1}{3}\arccos(\frac{1}{10\sqrt{10}})) + 1) \approx 2,48.$$

(ii) Under the assumptions of Theorem 3 (ii), we have

$$\Delta_n \leq C\sqrt{\ln(n)}n^{-\frac{1}{4}}.$$

Collorary 2. *Assume that there exists $\delta_o, \lambda > 0$ such that for any $\delta_o > \delta > 0$ we have $P[X_1 < \delta] \leq C\delta^{\lambda}$.*

(i) If the assumptions of Theorem 3 (i) holds, then

$$\Delta_n \leq C\alpha_2^{n,r,\lambda} n^{-\Theta_2(\lambda,r)},$$

where

$$\Theta_2(\lambda, r) = \min\{\frac{r-2}{2}, \frac{(r-1)\lambda r}{2(r^2+\lambda r^2+\lambda r-\lambda)}, \frac{1}{4}\},$$

and

$$\alpha_2^{n,r,\lambda} = \begin{cases} \sqrt{\ln(n)}, & \text{if } \Theta_2(\lambda, r) = \frac{1}{4}, \\ 1, & \text{otherwise.} \end{cases}$$

(ii) Under the assumptions of Theorem 3 (ii), we have

$$\Delta_n \leq C\alpha_3^{n,\lambda} n^{-\Theta_3(\lambda)},$$

where

$$\Theta_3(\lambda) = \begin{cases} \frac{\lambda}{2(\lambda+1)}, & \text{for } \lambda \leq 1, \\ \frac{1}{4}, & \text{for } \lambda > 1, \end{cases}$$

$$\alpha_3^{n,\lambda} = \begin{cases} (\ln(n))^{\frac{2\lambda+1}{\lambda+1}}, & \text{for } \lambda \leq 1, \\ \sqrt{\ln(n)}, & \text{for } \lambda > 1. \end{cases}$$

3. Proofs

For the proof of our main result we need some auxiliary results.

Proposition 1. *Let $\{X_n, \ n \geq 1\}$ be a sequence of independent random variables, such that $EX_n = \mu_n$, $E(X_n - \mu_n)^2 = \sigma_n^2$. Moreover let $\{a_n, \ n \geq 1\}$ be a nondecreasing sequence of positive real numbers such that $\frac{a_{n+1}}{a_n} = O(1)$.*

(i) If, for some $r > 2, E|X_n|^r < \infty, n \geq 1$, then for every positive ε

$$P[|\frac{S_n}{a_n}| > \varepsilon] < C_1 \frac{L_n + s_n^r}{a_n^r \varepsilon^r}, \tag{3.1}$$

where $C_1 = 4(1+\frac{2}{r})^r + 2^{1-r/2}e^{r^2/2}(r+2)^r\Gamma(r/2+1)$.

(ii) If $\{X_n, n \geq 1\}$ is a sequence of independent identically distributed random variables with $E|X_n|^r < \infty, n \geq 1$, for some $r > 2$, then for every $\varepsilon > 0$, we have

$$P[|\frac{S_n}{a_n}| \geq \varepsilon] \leq C_1 \frac{n^{r/2}}{a_n^r \varepsilon^r}, \tag{3.2}$$

where C_1 *is such as in the point (i).*

If, additionally, there exist $k_0 \in N$, *and* c_o *such that*

$$\sup_n \frac{n^{k_o-1}}{a_n^{2k_o-r}} \leq c_o, \tag{3.3}$$

and

$$\lim_{n\to\infty} \frac{n}{a_n^2} = 0, \tag{3.4}$$

then

$$P[|\frac{S_n}{a_n}| \geq \varepsilon] \leq C_2 \frac{n}{a_n^r(\varepsilon^r \wedge \varepsilon^{k_o})}, \tag{3.5}$$

where $C_2 = 4(1+\frac{2}{r})^r E|X_1|^r + 2^{2-k_o} k_o!(r+2)^{2k_o} e^{rk_o} \sigma_1^{k_o} c_o$.

If $\{X_n, n \geq 1\}$ *is a sequence of independent and identically distributed random variables with* $Ee^{tX_n} \leq e^{\frac{1}{2}t^2c^2}$, *for some* $c > 0$ *and every* $t \in (0, T)$, *then*

$$P[|\frac{S_n}{a_n}| \geq \varepsilon] \leq 2e^{-\frac{\varepsilon a_n}{2}(\frac{\varepsilon a_n}{nc^2} \wedge T)}. \tag{3.6}$$

(iii) If $\{X_n, n \geq 1\}$ *is a sequence of independent identically distributed random variables with* $E|X_n|^r < \infty, n \geq 1$, *for some* $2 < r \leq 3$, *then for every* $\varepsilon > 0$, *we have*

$$P[|\frac{S_n}{n}| \geq \varepsilon] \leq C_3 \frac{1}{n^{r-1}(\varepsilon^r \wedge \varepsilon^2)}, \tag{3.7}$$

where $C_3 = 4(1+\frac{2}{r})^r E|X_1|^r + 4(r+2)^4 e^{2r} \sigma_1^2 \mu^{4-r}$.

If $Ee^{tX_n} \leq e^{\frac{1}{2}t^2c^2}$, *for some* $c > 0$ *and every* $t \in (0, T)$, *then*

$$P[|\frac{S_n}{n}| \geq \varepsilon] \leq 2e^{-n\varepsilon(T \wedge (\varepsilon/c^2))/2}. \tag{3.8}$$

Proof of Proposition 1. *From Fuk-Nagayev's inequality, we have*

$$P[|S_n| \geq x] \leq 2(1+\frac{2}{r})^r \frac{\sum_{i=1}^n E|X_i|^r}{x^r} + 2\exp\{-2(r+2)^{-2}e^{-r}\frac{x^2}{\sum_{j=1}^n \sigma_j^2}\}.$$

Let us put $x = a_n\varepsilon$, *then*

$$P[|\frac{S_n}{a_n}| \geq \varepsilon] \leq 4(1+\frac{2}{r})^r \frac{L_n}{\varepsilon^r a_n^r} + 2\exp\{-2(r+2)^{-2}e^{-r}\frac{a_n^2\varepsilon^2}{s_n^2}\}.$$

As for any $t \in R$

$$e^{-x} \leq \frac{\Gamma(t+1)}{x^t}, \qquad (x > 0),$$

putting $t = r/2$, we get (3.1) and (3.2). Putting $t = k_o$ from (3.3) and (3.4) we have

$$\exp\{-2^{-1}(r+2)^{-2}e^{-r}\sigma_1^{-2}\frac{a_n^2}{n}\} \le c_o(2(r+2)^2e^r)^{k_o}k_o!\frac{n}{a_n^r},$$

so we get (3.5). Because in the case $a_n = n$ (3.3) and (3.4) holds with $k_o = 2$, thus (3.7) follows from (3.5)

On the other hand (3.6) and (3.8) follows from Petrov ([5] chapter 4 theorem 16 p.81). ■

From Petrov [5] and Розовский [8] we cite the following result:

Proposition 2. *Let $\{\alpha_n, n \ge 1\}$ be an increasing sequence of positive numbers such that $\alpha_{n+1}/\alpha_n = O(1)$. Let $u(x)$ be a positive function such that, for some $\gamma > 0$, we have $u(x)x^{-\gamma} \downarrow 0$ as $x \uparrow \infty$ and*

$$\frac{1}{u(x)}\int_x^\infty \frac{u(y)}{y}dy = O(1), \quad as \quad x \to \infty. \tag{3.9}$$

Then, for every $\varepsilon > 0$, the following conditions are equivalent

$$P[|\frac{S_n}{\alpha_n}| > \varepsilon] = O(u(\alpha_n)),$$

and

$$P[\sup_{k\ge n}|\frac{S_k}{\alpha_k}| > \varepsilon] = O(u(\alpha_n)).$$

Proposition 3. *Let $\{X_n, \ \ n \ge 1\}$ be a sequence of independent random variables, such that $EX_n = \mu_n$, $E(X_n - \mu_n)^2 = \sigma_n^2$. Moreover let $\{a_n, \ \ n \ge 1\}$ be a nondecreasing sequence of positive real numbers such that $\frac{a_{n+1}}{a_n} = O(1)$.*

(i) Assume, for some $r > 2, E|X_n|^r < \infty, n \ge 1$. Furthermore, assume that $\frac{L_n + s_n^r}{a_n^r} \downarrow 0$ and

$$\frac{a_n^r}{L_n + s_n^r}\sum_{j=n+1}^\infty \frac{E|X_j - \mu_j|^r + s_{j+1}^r - s_j^r}{a_j^r} = O(1), \quad as \quad n \to \infty,$$

then for every $\varepsilon > 0$

$$P[\sup_{k\ge n}|\frac{S_k}{a_k}| > \varepsilon] < C\frac{L_n + s_n^r}{a_n^r}. \tag{3.10}$$

(ii) If $\{X_n, n \ge 1\}$ is a sequence of independent identically distributed random variables with $E|X_n|^r < \infty, n \ge 1$, for some $r > 2$, and the sequence $\{a_n, n \ge 1\}$ is such that $\frac{a_{n+1}}{a_n} = O(1)$ and $\frac{n}{a_n^2} \to 0$, as $n \to \infty$, and

$$\frac{a_{n+1}^r}{n^{r/2}}\sum_{j=n+1}^\infty \left(\frac{j}{a_j^2}\right)^{r/2}\frac{1}{j} = O(1), \quad as \quad n \to \infty,$$

then, for every $\varepsilon > 0$, we have

$$P[\sup_{k\geq n}|\frac{S_k}{a_k}| > \varepsilon] < C\frac{n^{r/2}}{a_n^r}. \tag{3.11}$$

If additionally there exist $k_0 \in N$, and c_0 such that

$$\sup_n \frac{n^{k_0-1}}{a_n^{2k_0-r}} \leq c_0,$$

then

$$P[\sup_{k\geq n}|\frac{S_k}{a_k}| \geq \varepsilon] \leq C\frac{n}{a_n^r}. \tag{3.12}$$

If $Ee^{tX_n} \leq e^{\frac{1}{2}t^2c^2}$, for some $c > 0$ and every $t \in (0,T)$, and for some $\delta > 0$, we have $\liminf \frac{a_n^{\delta'}}{n} \geq c_o$, then

$$P[\sup_{k\geq n}|\frac{S_k}{a_k}| \geq \varepsilon] \leq \begin{cases} Ce^{-c_1a_n}, & \text{for } 0 < \delta' \leq 1, \\ Ce^{-c_1a_n^{2-\delta}}, & \text{for } 1 < \delta' < 2, \end{cases} \tag{3.13}$$

where c_1 is defined in (2.1).

(iii) If $\{X_n, n \geq 1\}$ is a sequence of independent and identically distributed random variables with $E|X_n|^r < \infty, n \geq 1$, for some $r > 2$, then for every $\varepsilon > 0$, we have

$$P[\sup_{k\geq n}|\frac{S_k}{k}| \geq \varepsilon] \leq C\frac{1}{n^{r-1}}. \tag{3.14}$$

If $Ee^{tX_n} \leq e^{\frac{1}{2}t^2c^2}$, for some $c > 0$ and every $t \in (0,T)$, then

$$P[\sup_{k\geq n}|\frac{S_k}{k}| \geq \varepsilon] \leq Ce^{-c_1n}, \tag{3.15}$$

where c_1 is as in (2.1).

Proof of Proposition 3. *Proposition 3 follows from Propositions 1 and 2. For proof of (3.10), (3.11), (3.12), (3.13), (3.14) and (3.15) we consider*

$$\begin{aligned}
u(x) &= \frac{L_k + s_k^r + \frac{x-a_k}{a_{k+1}-a_k}(E|X_{k+1}|^r + s_{k+1}^r - s_k^r)}{x^r}, \quad \text{for } x \in (a_k, a_{k+1}], \\
u(x) &= \frac{k^{r/2} + \frac{x-a_k}{a_{k+1}-a_k}((k+1)^{r/2} - k^{r/2})}{x^r}, \quad \text{for } x \in (a_k, a_{k+1}], \\
u(x) &= \frac{k + \frac{x-a_k}{a_{k+1}-a_k}}{x^r}, \quad \text{for } x \in (a_k, a_{k+1}], \\
u(x) &= e^{-cx^{(2-\delta)\wedge 1}}, \\
u(x) &= \frac{1}{x^{r-1}}, \\
u(x) &= e^{-cx},
\end{aligned}$$

respectively. In all cases $u(x)$ is continuous monotonously tending to 0. *When we consider normalization by a_n we have $\frac{a_{n+1}}{a_n} = O(1)$ whereas for normalization by n we have $\frac{n+1}{n} = O(1)$. We must check only condition (3.9). In the first case we have, for $x \in (a_k, a_{k+1}]$:*

$$\begin{aligned}
\tfrac{1}{u(x)} & \int_x^\infty \frac{u(y)}{y} dy < \frac{a_{k+1}^r}{L_k + s_k^r} \sum_{j=k+1}^{\infty} \{ \int_{a_j}^{a_{j+1}} \frac{L_j + s_j^r}{y^{r+1}} dy \\
+ & \quad \frac{(a_{j+1} - a_j)(E|X_{j+1}|^r + s_{j+1}^r - s_j^r)}{a_j^{r+1}} \} \\
\leq & \quad \frac{a_{k+1}^r}{L_k + s_k^r} \sum_{j=k+1}^{\infty} \{ \frac{L_j + s_j^r}{r} (a_j^{-r} - a_{j+1}^{-r}) + C \frac{E|X_{j+1}|^r + s_{j+1}^r - s_j^r}{a_j^r} \} \\
\leq & \quad C \frac{a_{k+1}^r}{L_k + s_k^r} \sum_{j=k+1}^{\infty} \frac{E|X_{j+1}|^r + s_{j+1}^r - s_j^r}{a_j^r} = O(1),
\end{aligned}$$

and similarly in the case of proof of (3.11). For (3.12), we have

$$\begin{aligned}
\frac{1}{u(x)} \int_x^\infty \frac{u(y)}{y} dy &= \frac{x^r}{k} (\int_x^{a_{k+1}} \frac{k}{y^{r+1}} dy + \sum_{j=k+1}^{\infty} \{ \int_{a_j}^{a_{j+1}} \frac{j}{y^{r+1}} dy + \frac{C}{a_j^r} \}) \\
&\leq C \frac{a_{k+1}^r}{kr} (\frac{k}{a_k^r} + \sum_{j=k+1}^{\infty} a_j^{-r}) \\
&= (\frac{a_{k+1}}{a_k})^r \frac{1}{r} + \frac{a_{k+1}^r}{kr} \sum_{j=k+1}^{\infty} (\frac{j}{a_j^2})^{\frac{r}{2}} \frac{1}{j^{\frac{r}{2}}} \\
&\leq C (\frac{a_{k+1}}{a_k})^r \frac{1}{r} + \frac{a_{k+1}^r}{kr} \frac{(k+1)^{\frac{r}{2}}}{a_{k+1}^r} (k+1)^{1-\frac{r}{2}} \frac{1}{1-\frac{r}{2}} \\
&= C (\frac{a_{k+1}}{a_k})^r \frac{1}{r} + \frac{k+1}{k} \frac{2}{r(2-r)} = O(1).
\end{aligned}$$

To prove (3.13) we consider two cases. If $2 - \delta > 1$, then

$$\frac{1}{u(x)} \int_x^\infty \frac{1}{y} u(y) dy = C e^{c_o x} \int_x^\infty \frac{e^{-c_o y}}{y} dy \leq \frac{C e^{c_o x}}{x} \int_x^\infty e^{-c_o y} = \frac{C c_o}{x} = O(1)$$

otherwise

$$\begin{aligned}
& \frac{1}{u(x)} \int_x^\infty \frac{1}{y} u(y) dy = C e^{c_o x^{2-\delta}} \int_x^\infty \frac{e^{-c_o y^{2-\delta}}}{y} dy \leq \\
& \frac{C e^{c_o x^{2-\delta}}}{(2-\delta) x^{2-\delta}} \int_x^\infty (2-\delta) y^{1-\delta} e^{-c_o y^{2-\delta}} dy = \frac{C e^{c_o x^{2-\delta}}}{(2-\delta) x^{2-\delta}} \int_{x^{2-\delta}}^\infty e^{-c_o t} dt = \\
& \frac{C c_o}{(2-\delta) x^{2-\delta}} = O(1).
\end{aligned}$$

The last two cases of function $u(x)$ are obvious. ■

Let us consider the sequence of random variables $\{X_n, \quad n \geq 1\}$ defined on the probability space $(\Omega, \mathcal{A}, P)$.

Proof of Theorem 1. *Let us denote*

$$\begin{aligned}
\bar{X}_k &= X_k - \mu_k, \quad \bar{S}_k = S_k - \sum_{j=1}^{k} \mu_j, \quad k \geq 1, \\
\Delta_n &= \sup_x |P[(\prod_{k=1}^{n} \frac{\bar{S}_k + a_k}{a_k})^{\gamma_n} < x] - P[e^N < x]|, \\
\Delta(n) &= \sup_x |P[\frac{\sum_{k=1}^{n} A_k^n \bar{X}_k}{Var(\sum_{k=1}^{n} A_k^n \bar{X}_k)} < x] - \Phi(x)|.
\end{aligned}$$

Then

$$\Delta_n \leq \sup_x |P[\gamma_n \sum_{k=1}^{n} \ln(1 + \frac{\bar{S}_k}{a_k}) < x] - \Phi(x)|.$$

Let us note for arbitrary positive integer m and arbitrary positive $\varepsilon, \varepsilon', \varepsilon''$,

$$\begin{aligned}
A_1(m, \omega) = A_1 &= [\sup_{k>m} |\frac{\bar{S}_k}{a_k}| > \frac{1}{2}], \\
A_2(m, n, \omega) = A_2 &= [4\gamma_n \sum_{k=m+1}^{n} (\frac{\bar{S}_k}{a_k})^2 > \varepsilon], \\
A_3(m, n, \omega) = A_3 &= [\gamma_n |\sum_{k=1}^{m} \frac{\bar{S}_k}{a_k}| > \varepsilon'], \\
A_4(m, n, \omega) = A_4 &= [\gamma_n |\sum_{k=1}^{m} \ln(\frac{\bar{S}_k}{a_k} + 1)| > \varepsilon''], \qquad \omega \in \Omega.
\end{aligned}$$

Then, from the expansion of the logarithm function, we have for every $n \in N$

$$\begin{aligned}
\Delta_n \leq & \sup_x |P[\gamma_n \sum_{k=1}^{m} \ln(\frac{\bar{S}_k}{a_k} + 1) + \gamma_n \sum_{k=m+1}^{n} \frac{\bar{S}_k}{a_k} \\
& + 4\gamma_n \sum_{k=m+1}^{n} (\frac{\bar{S}_k}{a_k})^2 < x, A_1', A_2', A_3', A_4'] - \Phi(x)| \\
& + P[A_1] + P[A_2] + P[A_3] + P[A_4],
\end{aligned}$$

where $A' = \Omega \backslash A$. Because, for arbitrary events A and B we have $P[B] - P[A] \leq$

$P[B \cap A'] \leq P[B]$, *thus*

$$P[\gamma_n \sum_{k=1}^{n} \frac{\bar{S}_k}{a_k} < x - \varepsilon - \varepsilon' - \varepsilon''] - P[A_1] - P[A_2] - P[A_3] - P[A_4]$$
$$\leq P[\gamma_n \sum_{k=1}^{m} \ln(\frac{\bar{S}_k}{a_k} + 1) + \gamma_n \sum_{k=m+1}^{n} \frac{\bar{S}_k}{a_k} + 4\gamma_n \sum_{k=m+1}^{n} (\frac{\bar{S}_k}{a_k})^2 < x, A_1', A_2', A_3', A_4']$$
$$\leq \quad P[\gamma_n \sum_{k=1}^{n} \frac{\bar{S}_k}{a_k} < x + \varepsilon + \varepsilon' + \varepsilon''].$$

Therefore from [5] (p.161 (3.3) and (3.4)) we have

$$\begin{aligned} \Delta_n \quad &\leq \quad 2\sum_{i=1}^{4} P[A_i] + \max\{\sup_x |P[\gamma_n \sum_{k=1}^{n} \frac{\bar{S}_k}{a_k} < x - \varepsilon - \varepsilon' - \varepsilon''] - \Phi(x)|, \\ &\quad \sup_x |P[\gamma_n \sum_{k=1}^{n} \frac{\bar{S}_k}{a_k} < x + \varepsilon + \varepsilon' + \varepsilon''] - \Phi(x)|\} \\ &\leq \quad 2\sum_{i=1}^{4} P[A_i] + \Delta(n) + \frac{\varepsilon + \varepsilon' + \varepsilon''}{\sqrt{2\pi}} + \frac{|\max\{\varrho_n, \varrho_n^{-1}\} - 1|}{\sqrt{2\pi e}}. \end{aligned} \tag{3.16}$$

Now we sequentially evaluate all terms on the right hand side of (3.16). From Proposition 3 we have

$$I_1 = 2P[A_1] \leq C\frac{L_m + s_m^r}{a_m^r}. \tag{3.17}$$

Furthermore, from Markov's inequality we have

$$\begin{aligned} I_2 = \inf_{\varepsilon>0}(2P[A_2] + \frac{\varepsilon}{\sqrt{2\pi}}) \quad &\leq \quad \inf_{\varepsilon>0}\{8\gamma_n \sum_{k=m+1}^{n} \frac{E(\frac{\bar{S}_k}{a_k})^2}{\varepsilon} + \frac{\varepsilon}{\sqrt{2\pi}}\} \\ &\leq \quad 2\sqrt{2}(2\pi)^{-\frac{1}{4}}\gamma_n^{\frac{1}{2}}(\sum_{k=m+1}^{n} \frac{s_k^2}{a_k^2})^{\frac{1}{2}} \\ &\leq \quad C\gamma_n^{\frac{1}{2}}(\sum_{k=1}^{n} \frac{s_k^2}{a_k^2})^{\frac{1}{2}}. \end{aligned} \tag{3.18}$$

By a similar transformations like those in the proof of Proposition 1 and because

$$(\sum_{k=1}^{m} (A_k^m)^2 E\bar{X}_k{}^2)^{\frac{1}{2}} \leq m^{1/2-1/r}(\sum_{k=1}^{m} (A_k^m)^r E|\bar{X}_k|^r)^{\frac{1}{r}}$$

(the function $f(p) = (\frac{1}{m}\sum_{k=1}^{m} |t_k|)^p$ *is nondecreasing for every fixed sequence* $\{t_k, 1 \geq$

$k \geq m\}$) we achieve

$$\begin{aligned} I_3 &= \inf_{\varepsilon'>0}(2P[A_3] + \frac{\varepsilon'}{\sqrt{2\pi}}) \\ \leq \inf_{\varepsilon'>0} & \{C\gamma_n^r \frac{\sum_{k=1}^m (A_k^m)^r E|\bar{X}|^r + (\sum_{k=1}^m (A_k^m)^2 E|\bar{X}|^2)^{r/2}}{\varepsilon'^r} + \frac{\varepsilon'}{\sqrt{2\pi}}\} \\ \leq & \; Cm^{\frac{r-2}{2(r+1)}}(\sum_{k=1}^m (A_k^m)^r E|\bar{X}|^r)^{1/(r+1)}\gamma_n^{\frac{r}{r+1}}. \end{aligned} \tag{3.19}$$

On the other hand, for $x > -1$ it is true that $\frac{x}{x+1} < \ln(x+1) < x$, thus

$$\begin{aligned} I_4 &= \inf_{\varepsilon''>0}(2P[A_4] + \frac{\varepsilon''}{\sqrt{2\pi}}) \\ &\leq \inf_{\varepsilon''>0}\{2P[\gamma_n \sum_{k=1}^m \frac{\bar{S}_k}{a_k} > \varepsilon''] + 2P[\gamma_n \sum_{k=1}^m \frac{\frac{\bar{S}_k}{a_k}}{\frac{\bar{S}_k}{a_k}+1} < -\varepsilon] + \frac{\varepsilon''}{\sqrt{2\pi}}\} \\ \leq & \inf_{\varepsilon''>0}\{2P[\gamma_n \sum_{k=1}^m \frac{\bar{S}_k}{a_k} > \varepsilon''] + 2P[\gamma_n \sum_{k=1}^m \frac{\bar{S}_k}{\sum_{j=1}^k \widetilde{X}_j} < -\varepsilon'', \min_{1\leq j\leq m} \widetilde{X}_j \geq \delta] \\ & +2P[\min_{1\leq j\leq m} \widetilde{X}_j < \delta] + \frac{\varepsilon''}{\sqrt{2\pi}}\} \\ \leq & \inf_{\varepsilon''>0}\{\frac{\gamma_n^r m^{r/2-1}\sum_{k=1}^m (A_k^m)^r E|\bar{X}_k|^r}{\varepsilon''^r} + P[\gamma_n \sum_{k=1}^m |\frac{\bar{S}_k}{k}| > \varepsilon''\delta] \\ & +1 - \prod_{j=1}^m (1-\phi_j(\delta)) + \frac{\varepsilon''}{\sqrt{2\pi}}\}, \end{aligned}$$

where $\widetilde{X_n} = \bar{X}_n + a_n - a_{n-1}, n \geq 1$. Since

$$1 - \prod_{j=1}^m (1-\phi_j(\delta)) \leq \sum_{k=1}^m \phi_k(\delta)\prod_{j=1}^{k-1}(1-\phi_j(\delta)) \leq \sum_{k=1}^m \phi_j(\delta),$$

therefore

$$\begin{aligned} I_4 & \leq C\inf_{\varepsilon''>0}\{\frac{\gamma_n^r m^{r/2-1}(\sum_{k=1}^m (A_k^m)^r E|\bar{X}|^r}{\varepsilon''^r} \\ + & \quad \frac{\gamma_n \ln(m)\sum_{k=1}^m E|\bar{X}_k|}{\varepsilon''\delta} + \sum_{k=1}^m \phi_k(\delta) + \frac{\varepsilon''}{\sqrt{2\pi}}\}. \end{aligned} \tag{3.20}$$

Taking into account (3.17-3.20) and evaluation (cf. [5])

$$\Delta(n) \leq C\frac{\sum_{j=1}^n (A_j^n)^r E|X_j|^r}{(\sum_{k=1}^n (A_k^n)^2\sigma_k^2)^{\frac{r}{2}}}$$

we get the assertions. ■

Proof of Theorem 2. *Similarly as in the proof of Theorem 1 we evaluate terms in (3.16). From Proposition 1, Markov's inequality, Fuk's-Nagaev inequality and the evaluation*

$$P[\gamma_n \sum_{k=1}^{m} |\frac{\bar{S}_k}{k}| > \varepsilon''\delta] \leq \sum_{k=1}^{m} P[|\frac{\bar{S}_k}{k}| > \frac{\varepsilon''\delta}{\gamma_n m}] \leq C\frac{\gamma_n^r m^r}{\varepsilon''^r \delta^r},$$

we get

$$\begin{aligned}
I_1 &\leq C\frac{m}{a_m^r},\\
I_2 &\leq C\big(\gamma_n \sum_{k=m+1}^{n} \frac{k}{a_k^2}\big)^{\frac{1}{2}},\\
I_3 &\leq C\gamma_n^{\frac{r}{r+1}} m^{\frac{r-2}{2(r+1)}} (\sum_{k=1}^{m} (A_k^m)^r)^{\frac{1}{r+1}}\\
I_4 &\leq C\{\gamma_n^{r/(r+1)} m^{\frac{r-2}{2(r+1)}} (\sum_{k=1}^{m} (A_k^m)^r)^{1/(r+1)} + \big(\frac{\gamma_n m}{\delta}\big)^{\frac{r}{r+1}} + \sum_{k=1}^{m} \phi_k(\delta).
\end{aligned}$$

On the other hand, for the second part of Theorem 2 we use evaluations

$$\begin{aligned}
I_1 &\leq Ce^{-c_1 a_m^{1\wedge(2-\delta)}},\\
I_2 &\leq C\big(\gamma_n \sum_{k=1}^{n} \frac{k}{a_k^2}\big)^{\frac{1}{2}},\\
I_3 &\leq C\gamma_n |\ln(\gamma_n)| A_1^m\\
I_4 &\leq C\{\gamma_n |\ln(\gamma_n)| A_1^m + \frac{m\gamma_n |\ln(\gamma_n)|}{\delta} + \sum_{k=1}^{m} \phi_k(\delta),
\end{aligned}$$

thus the proof of Theorem 2 is ended. ■

Proof of Theorem 3. *At first we consider the asymptotic behaviour of* $\sum_{k=1}^{n}(\sum_{j=k}^{n} 1/j)^r$ *for arbitrary* r*. We have*

$$\sum_{k=1}^{n}(\sum_{j=k}^{n} \frac{1}{j})^r \sim \int_1^n \ln^r(\frac{n}{x})dx = n\int_0^n y^r e^{-y} dy,$$

on the other hand

$$\frac{1}{r+1} = \int_0^1 y^r e^{-y} dy \leq \int_0^n y^r e^{-y} dy \leq \Gamma(r+1),$$

where $\Gamma(x)$ *is the Euler's function. Thus*

$$C_1 n \leq \sum_{k=1}^{n}(\sum_{j=k}^{n} \frac{1}{j})^r \leq C_2 n. \tag{3.21}$$

The term's I_1 and I_2 we evaluate as in Theorem 2 whereas I_3 and I_4 follows from (3.21) and Proposition's 1 and 3:

$$\begin{aligned}
I_1 &\leq Cm^{1-r}, \\
I_2 &\leq C\sqrt{\ln(n)}/\sqrt[4]{n}, \\
I_3 &\leq C(\frac{m}{n})^{\frac{r}{2(r+1)}}, \\
I_4 &\leq C\{(\frac{m}{n})^{\frac{r}{2(r+1)}} + m\phi_1(\delta) + (\frac{m}{\delta\sqrt{n}})^{\frac{r}{r+1}}\}.
\end{aligned}$$

From (3.21), we have

$$\Delta(n) \leq Cn^{1-r/2},$$

and from the fact that $\varrho_n^2 = 1-\frac{H_n}{2n}$, where $H_n = \sum_{k=1}^n \frac{1}{k}$ is the n-th harmonic number, we get

$$\begin{aligned}
\frac{|\max\{\varrho_n, \varrho_n^{-1}\} - 1|}{\sqrt{2\pi e}} &\leq (\frac{H_n}{4\pi en(1 - H_n/(2n))})^{1/2} \\
&\leq C\sqrt{\frac{\ln(n)}{n}}
\end{aligned}$$

what ends the proof of (i). Part (ii) follows similarly from Theorem 2 (ii).

4. Examples and applications

Example 1. *Let $\{X_n, n \geq 1\}$ be a sequence of independent and identically distributed random variables with geometric distribution, i.e., $P[X_1 = k] = pq^{k-1}$, where $p = 1 - q$. Then*

$$EX_1 = \mu = \frac{1}{p}, \quad Var(X_1) = \sigma^2 = \frac{q}{p^2},$$

$$Ee^tX_1 \leq \frac{pe^t}{1 - qe^t} \leq \infty, \ \textit{for}\ t \in (0, -\ln(q)].$$

From Theorem 17 ([4]) and the last inequality above for $t < -\ln(q)$, there exists such g that

$$Ee^{t\bar{X}} \leq e^{gt^2}.$$

So, as $P[X < 1] = 0$ from Collorary 1 (ii), we have

$$\sup_x |P[(\prod_{j=1}^n \frac{pS_j}{j})^{\frac{1}{\sqrt{2qn}}} < x] - P[e^N < x]| \leq C\sqrt{\ln(n)}n^{-\frac{1}{4}}.$$

Example 2. *Let $\{X_n, n \geq 1\}$ be a sequence of independent identically distributed random variables with exponential distribution with parameter $\lambda > 0$, i.e.*

$$P[X_1 < x] = \begin{cases} 1 - e^{-\lambda x}, & \textit{for}\ \ x \in (0, \infty), \\ 0, & \textit{otherwise}. \end{cases}$$

Then

$$EX_1 = \mu = \frac{1}{\lambda}, \ \ Var(X_1) = \sigma^2 = \frac{1}{\lambda^2}.$$

Moreover

$$Ee^t\bar{X} = \frac{\lambda}{\lambda - t}e^{-\frac{t}{\lambda}} < e^{\frac{t}{\lambda-t}-\frac{t}{\lambda}},$$

thus

$$Ee^{t\bar{X}} < e^{\frac{c^2t^2}{2}},$$

for $T \in (0,\lambda)$, $t \in (0,T]$, *and* $c = \sqrt{\frac{2}{\lambda(\lambda-T)}}$. *As* $P[X_1 < \delta] = 1 - e^{-\lambda\delta} \le \lambda\delta$, *from Collorary 2 (ii), we have*

$$\sup_x |P[(\prod_{j=1}^{n} \frac{S_j\lambda}{j})^{\frac{1}{\sqrt{2n}}} < x] - P[e^N < x]| \le C(\ln(n))^{\frac{3}{2}} n^{-\frac{1}{4}}.$$

Example 3. *Let* $\{X_n.n \ge 1\}$ *be a sequence of independent and identically distributed random variables with Pareto's density function*

$$f(x) = \begin{cases} \frac{pa^p}{x^{p+1}}, & \text{for } x > a > 0, \\ 0, & \text{otherwise.} \end{cases}$$

Then, for $2 < p \le 3$, *we have*

$$EX_1 = \mu = \frac{pa}{p-1}, \ \ Var(X_1) = \sigma^2 = \frac{pa^2}{(p-2)(p-1)^2}.$$

As $P[X_1 < a] = 0$, *for* $a > 0$, *we have for* $p > \frac{3+\sqrt{5}}{2}$

$$\sup_x |P[(\prod_{j=1}^{n} \frac{S_j(p-1)}{jpa})^{\sqrt{\frac{p(p-2)}{2n}}} < x] - P[e^N < x]| \le C\sqrt{\ln(n)}n^{-\frac{1}{4}}.$$

The similar evaluation for another p may be obtained from Corollary 1, too.

Open problems.

(i) *As it seen from our examples the "worst" evaluated term in our Theorems 1-3 and Corollaries 1-2 is* I_2. *Is it possible to get the better evaluation of the term* $P[4\gamma_n \sum_{k=1}^{n}(\frac{\bar{S}_k}{a_k})^2 > \varepsilon]$*?*

(ii) *From [8] and from our Proposition 2 we know only asymptotic behaviour of* $P[\sup_{k\ge n} |\frac{\bar{S}_k}{a_k}| > \varepsilon]$. *Is it possible to evaluate the constants C in Theorems 1-3 and Corollaries 1-2?*

Acknowledgment

The authors would like to thank the referee for carefully reading the manuscript and for many valuable comments that improved the presentation of this paper.

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Received 19.03.2009

Journal of
Mathematics
and Applications
No 31, pp 67-76 (2009)

On a certain subclass of analytic functions defined by Sălăgean and Ruscheweyh operators

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ABSTRACT: In the present paper we define a new operator using the Sălăgean and Ruscheweyh operators. Denote by L_α^m the operator given by $L_\alpha^m : A_n \to A_n$, $L_\alpha^m f(z) = (1-\alpha)R^m f(z) + \alpha S^m f(z)$, $z \in U$, $n, m \in \mathbb{N}$, where $R^m f(z)$ denote the Ruscheweyh derivative, $S^m f(z)$ is the Sălăgean operator and $A_n = \{f \in \mathcal{H}(U),\ f(z) = z + a_{n+1}z^{n+1} + \dots,\ z \in U\}$ is the class of normalized analytic functions. A certain subclass, denoted by $S_m(\delta, \alpha)$, of analytic functions in the open unit disc is introduced by means of the new operator. By making use of the concept of differential subordination we will derive various properties and characteristics of the class $S_m(\delta, \alpha)$. Also, several differential subordinations are established regardind the operator L_α^m

AMS Subject Classification: *30C45, 30A20, 34A40*
Key Words and Phrases: *differential subordination, convex function, best dominant, differential operator*

1. Introduction

Denote by U the unit disc of the complex plane, $U = \{z \in \mathbb{C} : \ |z| < 1\}$ and $\mathcal{H}(U)$ the space of holomorphic functions in U.

Let $A_n = \{f \in \mathcal{H}(U),\ f(z) = z + a_{n+1}z^{n+1} + \dots,\ z \in U\}$ and $\mathcal{H}[a, n] = \{f \in \mathcal{H}(U),\ f(z) = a + a_n z^n + a_{n+1}z^{n+1} + \dots,\ z \in U\}$ for $a \in \mathbb{C}$ and $n \in \mathbb{N}$.

Denote by $K = \left\{f \in A_n,\ \mathrm{Re}\ \frac{zf''(z)}{f'(z)} + 1 > 0,\ z \in U\right\}$ the class of normalized convex functions in U.

If f and g are analytic functions in U, we say that f is subordinate to g, written $f \prec g$, if there is an analytic in U function w such that $w(0) = 0$, $|w(z)| < 1$ and

$f(z) = g(w(z))$ for all $z \in U$. If g is univalent, then $f \prec g$ if and only if $f(0) = g(0)$ and $f(U) \subseteq g(U)$.

Let $\psi : \mathbb{C}^3 \times U \to \mathbb{C}$ and h univalent in U. If p is analytic in U and satisfies the (second-order) differential subordination

$$\psi(p(z), zp'(z), z^2p''(z); z) \prec h(z), \qquad z \in U, \tag{1.1}$$

then p is called a solution of the differential subordination. The univalent function q is called a dominant of the solutions of the differential subordination, or more simply a dominant, if $p \prec q$ for all p satisfying (1.1).

A dominant $\widetilde{q}$ that satisfies $\widetilde{q} \prec q$ for all dominants q of (1.1) is said to be the best dominant of (1.1). The best dominant is unique up to a rotation of U.

1.0. (Sălăgean [8]) For $f \in A_n$, $m \in \mathbb{N}$, the operator S^m is defined by $S^m : A_n \to A_n$,

$$\begin{aligned} S^0 f(z) &= f(z), \\ S^1 f(z) &= zf'(z), ..., \\ S^{m+1} f(z) &= z\left(S^m f(z)\right)', \quad z \in U. \end{aligned}$$

1.0. If $f \in A_n$, $f(z) = z + \sum_{j=n+1}^{\infty} a_j z^j$, then $S^m f(z) = z + \sum_{j=n+1}^{\infty} j^m a_j z^j$, $z \in U$.

1.0. (Ruscheweyh [7]) For $f \in A_n$, $m \in \mathbb{N}$, the operator R^m is defined by $R^m : A_n \to A_n$,

$$\begin{aligned} R^0 f(z) &= f(z), \\ R^1 f(z) &= zf'(z), ..., \\ (m+1) R^{m+1} f(z) &= z\left(R^m f(z)\right)' + mR^m f(z), \quad z \in U. \end{aligned}$$

1.0. If $f \in A_n$, $f(z) = z + \sum\limits_{j=n+1}^{\infty} a_j z^j$, then $R^m f(z) = z + \sum\limits_{j=n+1}^{\infty} C_{m+j-1}^m a_j z^j$, $z \in U$.

Lemma 1.1. (Hallenbeck and Ruscheweyh [5]) *Let h be a convex function with $h(0) = a$, and let $\gamma \in C^*$ be a complex number with $Re\ \gamma \geq 0$. If $p \in \mathcal{H}[a,n]$ and*

$$p(z) + \frac{1}{\gamma} zp'(z) \prec h(z), \quad z \in U,$$

then

$$p(z) \prec g(z) \prec h(z), \quad z \in U,$$

where $g(z) = \frac{\gamma}{nz^{\gamma/n}} \int_0^z h(t) t^{\gamma/n-1} dt$, $z \in U$.

Lemma 1.2. *(Miller and Mocanu [6]) Let g be a convex function in U and let $h(z) = g(z) + n\alpha zg'(z)$, $z \in U$, where $\alpha > 0$ and n is a positive integer.*

If $p(z) = g(0) + p_n z^n + p_{n+1} z^{n+1} + \ldots$, $z \in U$, is holomorphic in U and

$$p(z) + \alpha zp'(z) \prec h(z), \quad z \in U,$$

then

$$p(z) \prec g(z)$$

and this result is sharp.

2. Main results

2.0. [1], [2] Let $\alpha \geq 0$, $m \in \mathbb{N}$. Denote by L_α^m the operator given by $L_\alpha^m : A_n \to A_n$,

$$L_\alpha^m f(z) = (1-\alpha) R^m f(z) + \alpha S^m f(z), \quad z \in U.$$

2.0. L_α^m is a linear operator and if $f \in A_n$, $f(z) = z + \sum_{j=n+1}^{\infty} a_j z^j$, then $L_\alpha^m f(z) = z + \sum_{j=n+1}^{\infty} \left(\alpha j^m + (1-\alpha) C_{m+j-1}^m\right) a_j z^j$, $z \in U$.

Theorem 2.1. *Let g be a convex function, $g(0) = 1$ and let h be the function $h(z) = g(z) + zg'(z)$, $z \in U$.*

If $\alpha \geq 0$, $m \in \mathbb{N}$, $f \in A_n$ and verifies the differential subordination

$$(L_\alpha^m f(z))' \prec h(z), \quad z \in U, \tag{2.1}$$

then

$$\frac{L_\alpha^m f(z)}{z} \prec g(z), \quad z \in U$$

and this result is sharp.

Proof. Consider

$$\begin{aligned} p(z) &= \frac{L_\alpha^m f(z)}{z} = \frac{z + \sum_{j=n+1}^{\infty} \left(\alpha j^m + (1-\alpha) C_{m+j-1}^m\right) a_j z^j}{z} \\ &= 1 + p_n z^n + p_{n+1} z^{n+1} + ..., \quad z \in U. \end{aligned}$$

We deduce that $p \in \mathcal{H}[1, n]$.

We have $L_\alpha^m f(z) = zp(z), z \in U$. Differentiating, we obtain $(L_\alpha^m f(z))' = p(z) + zp'(z), z \in U$.

Then (2.1) becomes

$$p(z) + zp'(z) \prec h(z) = g(z) + zg'(z), \quad z \in U.$$

By using Lemma 1.2., we have

$$p(z) \prec g(z), \quad z \in U, \quad i.e. \quad \frac{L_\alpha^m f(z)}{z} \prec g(z), \quad z \in U.$$

□

Theorem 2.2. *Let $h \in \mathcal{H}(U)$ with $h(0) = 1$ which verifies the inequality*

$$\operatorname{Re}\left(1 + \frac{zh''(z)}{h'(z)}\right) > -\frac{1}{2}, \quad z \in U.$$

If $\alpha \geq 0$, $m \in \mathbb{N}$, $f \in A_n$ and verifies the differential subordination

$$(L_\alpha^m f(z))' \prec h(z), \quad z \in U, \tag{2.2}$$

then

$$\frac{L_\alpha^m f(z)}{z} \prec q(z), \quad z \in U,$$

where $q(z) = \frac{1}{nz^{\frac{1}{n}}} \int_0^z h(t)t^{\frac{1}{n}-1}dt$. The function q is convex and it is the best dominant.

Proof. Let

$$p(z) = \frac{L_\alpha^m f(z)}{z} = 1 + \sum_{j=n+1}^{\infty} \left(\alpha j^m + (1-\alpha) C_{m+j-1}^m\right) a_j z^{j-1}$$

$$= 1 + \sum_{j=n+1}^{\infty} p_j z^{j-1}, \quad z \in U, p \in \mathcal{H}[1, n].$$

Differentiating, we obtain

$$(L_\alpha^m f(z))' = p(z) + zp'(z), \quad z \in U$$

and (2.2) becomes

$$p(z) + zp'(z) \prec h(z), \quad z \in U.$$

Using Lemma 1.1., we have

$$p(z) \prec q(z) = \frac{1}{nz^{\frac{1}{n}}} \int_0^z h(t)t^{\frac{1}{n}-1}dt, \quad z \in U,$$

i.e.

$$\frac{L_\alpha^m f(z)}{z} \prec q(z) = \frac{1}{nz^{\frac{1}{n}}} \int_0^z h(t)t^{\frac{1}{n}-1}dt, \quad z \in U$$

and q is the best dominant. $\square$

Theorem 2.3. *Let g be a convex function such that $g(0) = 1$ and let h be the function $h(z) = g(z) + zg'(z)$, $z \in U$.*

If $\alpha \geq 0$, $m \in \mathbb{N}$, $f \in A_n$ and the differential subordination

$$\left(\frac{zL_\alpha^{m+1} f(z)}{L_\alpha^m f(z)}\right)' \prec h(z), \quad z \in U \tag{2.3}$$

holds, then

$$\frac{L_\alpha^{m+1} f(z)}{L_\alpha^m f(z)} \prec g(z), \quad z \in U$$

and this result is sharp.

Proof. Consider

$$p(z)=\frac{L_\alpha^{m+1}f(z)}{L_\alpha^m f\,(z)}=\frac{z+\sum_{j=n+1}^{\infty}\left(\alpha j^{m+1}+(1-\alpha)\,C_{m+j}^{m+1}\right)a_j z^j}{z+\sum_{j=n+1}^{\infty}\left(\alpha j^{m}+(1-\alpha)\,C_{m+j-1}^{m}\right)a_j z^j}.$$

We have $p'\left(z\right)=\frac{\left(L_\alpha^{m+1}f(z)\right)'}{L_\alpha^m f(z)}-p\left(z\right)\cdot\frac{\left(L_\alpha^m f(z)\right)'}{L_\alpha^m f(z)}$ and we obtain $p\left(z\right)+z\cdot p'\left(z\right)=\left(\frac{zL_\alpha^{m+1}f(z)}{L_\alpha^m f(z)}\right)'$.
Relation (2.3) becomes

$$p(z)+zp'(z)\prec h(z)=g(z)+zg'(z),\quad z\in U.$$

By using Lemma 1.2., we have

$$p(z)\prec g(z),\quad z\in U,\quad i.e.\quad \frac{L_\alpha^{m+1}f(z)}{L_\alpha^m f\,(z)}\prec g(z),\quad z\in U.$$

□

Following the work done in [3] and [4], we introduce a new class of functions.

2.0. Let $\delta\in[0,1)$, $\alpha\geq 0$ and $m\in\mathbb{N}$. A function $f\in A_n$ is said to be in the class $S_m\left(\delta,\alpha\right)$ if it satisfies the inequality

$$\mathrm{Re}\ \left(L_\alpha^m f\left(z\right)\right)'>\delta,\qquad z\in U. \tag{2.4}$$

Theorem 2.4. *The set $S_m\left(\delta,\alpha\right)$ is convex.*

Proof. Let the functions

$$f_j\left(z\right)=z+\sum_{j=n+1}^{\infty}a_{jk}z^j,\quad k=1,2,\qquad z\in U$$

be in the class $S_m\left(\delta,\alpha\right)$. It is sufficient to show that the function

$$h\left(z\right)=\eta_1 f_1\left(z\right)+\eta_2 f_2\left(z\right)$$

is in the class $S_m\left(\delta,\alpha\right)$, with η_1 and η_2 nonnegative such that $\eta_1+\eta_2=1$.

Since $h\left(z\right)=z+\sum_{j=n+1}^{\infty}\left(\eta_1 a_{j1}+\eta_2 a_{j2}\right)z^j,\quad z\in U$, then

$$L_\alpha^m h\left(z\right)=z+\sum_{j=n+1}^{\infty}\left[\alpha j^m+(1-\alpha)\,C_{m+j-1}^m\right]\left(\eta_1 a_{j1}+\eta_2 a_{j2}\right)z^j,\quad z\in U. \tag{2.5}$$

Differentiating (2.5) we obtain
$\left(L_\alpha^m h\left(z\right)\right)'=1+\sum_{j=n+1}^{\infty}\left[\alpha j^m+(1-\alpha)\,C_{m+j-1}^m\right]\left(\eta_1 a_{j1}+\eta_2 a_{j2}\right)jz^{j-1}$,
$z\in U$.
Hence

$$\operatorname{Re}\left(L_\alpha^m h(z)\right)' = 1 + \operatorname{Re}\left(\eta_1 \sum_{j=n+1}^{\infty} j\left[\alpha j^m + (1-\alpha) C_{m+j-1}^m\right] a_{j1} z^{j-1}\right)$$
$$+\operatorname{Re}\left(\eta_2 \sum_{j=n+1}^{\infty} j\left[\alpha j^m + (1-\alpha) C_{m+j-1}^m\right] a_{j2} z^{j-1}\right). \tag{2.6}$$

Taking into account that $f_1, f_2 \in S_m(\delta, \alpha)$, we deduce

$$\operatorname{Re}\left(\eta_k \sum_{j=n+1}^{\infty} j\left[\alpha j^m + (1-\alpha) C_{m+j-1}^m\right] a_{jk} z^{j-1}\right) > \eta_k(\delta - 1), k = 1, 2. \tag{2.7}$$

Using (2.7) we get from (2.6)

$$\operatorname{Re}\left(L_a^m h(z)\right)' > 1 + \eta_1(\delta - 1) + \eta_2(\delta - 1), \qquad z \in U,$$

that is

$$\operatorname{Re}\left(L_a^m h(z)\right)' > \delta, \quad z \in U,$$

which is equivalent that $S_m(\delta, \alpha)$ is convex. □

Theorem 2.5. *Let g be a convex function in U and let $h(z) = g(z) + \frac{1}{c+2} z g'(z)$, $z \in U$, where $c > 0$.*

If $f \in S_m(\delta, \alpha)$ and $F(z) = I_c(f)(z) = \frac{c+2}{z^{c+1}} \int_0^z t^c f(t)\, dt$, $z \in U$, then

$$\left(L_\alpha^m f(z)\right)' \prec h(z), \qquad z \in U \tag{2.8}$$

implies

$$\left(L_\alpha^m F(z)\right)' \prec g(z), \qquad z \in U$$

and this result is sharp.

Proof. We have $z^{c+1} F(z) = (c+2) \int_0^z t^c f(t)\, dt$. Differentiating, with respect to z, we obtain

$$(c+1) F(z) + z F'(z) = (c+2) f(z) \tag{2.9}$$

and

$$(c+1) L_\alpha^m F(z) + z\left(L_\alpha^m F(z)\right)' = (c+2) L_\alpha^m f(z), \qquad z \in U. \tag{2.10}$$

Differentiating (2.10) we have

$$\left(L_\alpha^m F(z)\right)' + \frac{1}{c+2} z\left(L_\alpha^m F(z)\right)'' = \left(L_\alpha^m f(z)\right)', \qquad z \in U. \tag{2.11}$$

Using (2.11), the differential subordination (2.8) becomes

$$\left(L_\alpha^m F(z)\right)' + \frac{1}{c+2} z\left(L_\alpha^m F(z)\right)'' \prec g(z) + \frac{1}{c+2} z g'(z). \tag{2.12}$$

If we denote

$$p(z) = \left(L_\alpha^m F(z)\right)' \tag{2.13}$$

then $p \in \mathcal{H}[1, n]$.

Replacing (2.13) in (2.12) we obtain

$$p(z) + \frac{1}{c+2} z p'(z) \prec g(z) + \frac{1}{c+2} z g'(z), \qquad z \in U.$$

Using Lemma 1.2. we have

$$p(z) \prec g(z) \quad i.e. \quad (L_\alpha^m F(z))' \prec g(z), \qquad z \in U$$

and g is the best dominant. $\square$

Theorem 2.6. *Let* $h(z) = \frac{1+(2\delta-1)z}{1+z}$, $\delta \in [0,1)$ *and* $c > 0$.

If $\alpha \geq 0$, $m \in \mathbb{N}$ *and* I_c *is given by Theorem 2.5., then*

$$I_c[S_m(\delta, \alpha)] \subset S_m(\delta^*, \alpha), \tag{2.14}$$

where $\delta^* = 2\delta - 1 + \frac{(c+2)(2-2\delta)}{n} \beta\left(\frac{c+2}{n} - 2\right)$ *and* $\beta(x) = \int_0^1 \frac{t^{x+1}}{t+1} dt$.

Proof. The function h is convex and using the same steps as in the proof of Theorem 2.5. we get from the hypothesis of Theorem 2.6. that

$$p(z) + \frac{1}{c+2} z p'(z) \prec h(z),$$

where $p(z)$ is defined in (2.13).

Using Lemma 1.1. we deduce that

$$p(z) \prec g(z) \prec h(z),$$

that is

$$(L_\alpha^m F(z))' \prec g(z) \prec h(z),$$

where

$$\begin{aligned} g(z) &= \frac{c+2}{n z^{\frac{c+2}{n}}} \int_0^z t^{\frac{c+2}{n}-1} \frac{1+(2\delta-1)t}{1+t} dt = \\ & (2\delta - 1) + \frac{(c+2)(2-2\delta)}{n z^{\frac{c+2}{n}}} \int_0^z \frac{t^{\frac{c+2}{n}-1}}{1+t} dt. \end{aligned}$$

Since g is convex and $g(U)$ is symmetric with respect to the real axis, we deduce

$$\begin{aligned} \operatorname{Re}\ (L_\alpha^m F(z))' \geq \min_{|z|=1} \operatorname{Re}\ g(z) = \operatorname{Re}\ g(1) = \delta^* = \\ 2\delta - 1 + \frac{(c+2)(2-2\delta)}{n} \beta\left(\frac{c+2}{n} - 2\right). \end{aligned} \tag{2.15}$$

From (2.15) we deduce inclusion (2.14). $\square$

Theorem 2.7. *Let g be a convex function such that $g(0) = 1$, and let h be the function $h(z) = g(z) + zg'(z)$, $z \in U$.*

If $\alpha \geq 0$, $m \in \mathbb{N}$, $f \in A_n$ and the differential subordination

$$\left(L_\alpha^{m+1} f(z)\right)' + \frac{(1-\alpha)\, mz (R^m f(z))''}{m+1} \prec h(z), \quad z \in U \tag{2.16}$$

holds, then

$$[L_\alpha^m f(z)]' \prec g(z), \quad z \in U.$$

This result is sharp.

Proof. By using the properties of operator L_α^m, we obtain

$$L_\alpha^{m+1} f(z) = (1-\alpha) R^{m+1} f(z) + \alpha S^{m+1} f(z), \quad z \in U. \tag{2.17}$$

Then (2.16) becomes

$$\left((1-\alpha) R^{m+1} f(z) + \alpha S^{m+1} f(z)\right)' + \frac{(1-\alpha)\, mz \left(R^m f(z)\right)''}{m+1} \prec h(z),$$

with $z \in U$.

After a short calculation, we obtain $(1-\alpha)\left(R^m f(z)\right)' + \alpha \left(S^m f(z)\right)'$ $+z\left((1-\alpha)\left(R^m f(z)\right)'' + \alpha \left(S^m f(z)\right)''\right) \prec h(z), z \in U$.

Let

$$p(z) = (1-\alpha)\left(R^m f(z)\right)' + \alpha \left(S^m f(z)\right)' = (L_\alpha^m f(z))' = \tag{2.18}$$

$$1 + \sum_{j=n+1}^{\infty} \left(\alpha j^{m+1} + (1-\alpha)\, j C_{m+j-1}^m\right) a_j z^{j-1} = 1 + p_n z^n + p_{n+1} z^{n+1} + \ldots$$

We deduce that $p \in \mathcal{H}[1, n]$.

Using the notation in (2.18), the differential subordination becomes

$$p(z) + zp'(z) \prec h(z) = g(z) + zg'(z).$$

By using Lemma 1.2., we have

$$p(z) \prec g(z), \quad z \in U, \quad i.e. \quad (L_\alpha^m f(z))' \prec g(z), \quad z \in U$$

and this result is sharp. $\square$

Theorem 2.8. *Let $h(z) = \frac{1+(2\beta-1)z}{1+z}$, a convex function in U, $0 \leq \beta < 1$.*

If $\alpha \geq 0$, $m \in \mathbb{N}$, $f \in A_n$ and verifies the differential subordination

$$[L_\alpha^{m+1} f(z)]' + \frac{(1-\alpha)\, mz \left(R^m f(z)\right)''}{m+1} \prec h(z), \quad z \in U, \tag{2.19}$$

then

$$(L_\alpha^m f(z))' \prec q(z), \quad z \in U,$$

where q is given by $q(z) = 2\beta - 1 + \frac{2(1-\beta)}{nz^{\frac{1}{n}}} \int_0^z \frac{t^{\frac{1}{n}-1}}{1+t} dt$, $z \in U$. The function q is convex and it is the best dominant.

Proof. Following the same steps as in the proof of Theorem 2.7. and considering $p(z) = \left(L_\alpha^m f(z)\right)'$, the differential subordination (2.19) becomes

$$p(z) + zp'(z) \prec h(z) = \frac{1 + (2\beta - 1)z}{1 + z}, \quad z \in U.$$

By using Lemma 1.1. for $\gamma = 1$, we have $p(z) \prec q(z)$, i.e.,

$$\left(L_\alpha^m f(z)\right)' \prec q(z) = \frac{1}{nz^{\frac{1}{n}}} \int_0^z h(t)\, t^{\frac{1}{n}-1} dt = \frac{1}{nz^{\frac{1}{n}}} \int_0^z t^{\frac{1}{n}-1} \frac{1 + (2\beta - 1)\, t}{1 + t} dt$$

$$= 2\beta - 1 + \frac{2(1 - \beta)}{nz^{\frac{1}{n}}} \int_0^z \frac{t^{\frac{1}{n}-1}}{1 + t} dt, \quad z \in U.$$

□

Theorem 2.9. *Let $h \in \mathcal{H}(U)$ with $h(0) = 1$, which verifies the inequality*

$$\operatorname{Re}\left[1 + \frac{zh''(z)}{h'(z)}\right] > -\frac{1}{2}, \quad z \in U.$$

If $\alpha \geq 0$, $m \in \mathbb{N}$, $f \in A_n$ and satisfies the differential subordination

$$\left(L_\alpha^{m+1} f(z)\right)' + \frac{(1 - \alpha)\, mz \left(R^m f(z)\right)''}{m + 1} \prec h(z), \quad z \in U, \tag{2.20}$$

then

$$\left(L_\alpha^m f(z)\right)' \prec q(z), \quad z \in U,$$

where q is given by $q(z) = \frac{1}{nz^{\frac{1}{n}}} \int_0^z h(t) t^{\frac{1}{n}-1} dt$. The function q is convex and it is the best dominant.

Proof. Using the properties of operator L_α^m and considering $p(z) = \left(L_\alpha^m f(z)\right)'$, we obtain

$$\left(L_\alpha^{m+1} f(z)\right)' + \frac{(1 - \alpha)\, mz \left(R^m f(z)\right)''}{m + 1} = p(z) + zp'(z), \quad z \in U.$$

Then (2.20) becomes

$$p(z) + zp'(z) \prec h(z), \quad z \in U.$$

Since $p \in \mathcal{H}[1, n]$, using Lemma 1.1., we deduce

$$p(z) \prec q(z), \quad z \in U,$$

where

$$q(z) = \frac{1}{nz^{\frac{1}{n}}} \int_0^z h(t) t^{\frac{1}{n}-1} dt, \quad z \in U,$$

i.e.

$$\left(L_\alpha^m f(z)\right)' \prec q(z) = \frac{1}{nz^{\frac{1}{n}}} \int_0^z h(t) t^{\frac{1}{n}-1} dt, \quad z \in U$$

and q is the best dominant. □

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Received 22.01.2009

J o u r n a l of
Mathematics
and Applications
No 31, pp 77-90 (2009)

Coefficient conditions for univalency and starlikeness of analytic functions

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Abstract: In this paper, we find conditions on the coefficients $\{a_k\}$ such that the corresponding analytic function $f(z)$ and its partial sum $f_n(z)$ are close-to-convex with respect to some starlike function in the unit disc $\mathbb{D}$. We also find conditions on these coefficients so that the analytic function is starlike univalent in $\mathbb{D}$. As an application, we find conditions on the triplet (a, b, c) so that, the normalized Gaussian hypergeometric function and its particular cases, are in one of these classes.

AMS Subject Classification: *30C45, 30C15, 33C05*
Key Words and Phrases: *close-to-convex functions, starlike functions, Cesáro means, Gaussian hypergeometric functions*

1. Introduction

Let $\mathcal{A}$ be the class of analytic functions f in the unit disk $\mathbb{D} = \{z : |z| < 1\}$, normalized by the condition $f(0) = 0 = f'(0) - 1$ and $\mathcal{S} = \{f \in \mathcal{A} : \text{f is univalent in } \mathbb{D}\}$. A function $f \in \mathcal{S}$ is said to be starlike and convex if

$$\operatorname{Re}\left(\frac{zf'(z)}{f(z)}\right) > 0 \qquad \text{and} \quad \operatorname{Re}\left(1 + \frac{zf''(z)}{f'(z)}\right) > 0$$

respectively. The class of all starlike and convex functions are denoted as $\mathcal{S}^*$ and $\mathcal{C}$ respectively and $\mathcal{C} \subsetneq \mathcal{S}^* \subsetneq \mathcal{S}$. A function $f \in \mathcal{A}$ is known as close-to-convex with respect to a starlike function g if $\operatorname{Re}(\frac{zf'(z)}{g(z)}) > 0$. The class of all such function is denoted as $\mathcal{K}$. Every close-to-convex functions is univalent. Let $\mathcal{T}_{\mathcal{R}}$ be the subclass of $\mathcal{S}$, consist of all typically real functions, i.e, all $f \in \mathcal{S}$ such that $Im(f)Im(z) > 0$. For details regarding these classes, we refer to [3]. The following Lemma 1.1. gives another criteria for starlikeness.

Lemma 1.1. [12] Let $f \in \mathcal{A}$ be typically real in $\mathbb{D}$ and satisfies the condition that $\operatorname{Re} f'(z) > 0, z \in \mathbb{D}$. Then f is starlike univalent in $\mathbb{D}$.

Lemma 1.2. [10] *$f \in \mathcal{A}$ has real coefficients and is convex in the direction of imaginary axis if, and only if, $zf'(z)$ is typically real.*

A function $f \in \mathcal{A}$ is said to be convex in the direction of the imaginary axis if every line parallel to the imaginary axis either intersects $f(\mathbb{D})$ in an interval or does not intersect at all. The following result is well known.

Lemma 1.3. *Let $f \in \mathcal{A}$ has real coefficients, then f is convex in the direction of imaginary axis if, and only if,*

$$\operatorname{Re}\big((1-z^2)f'(z)\big) > 0 \qquad \forall z \in D$$

In this work, we are interested in the following problems:

Problem 1.1. *Find the condition on a_k, such that the functions defined by the series $f(z) = z + \displaystyle\sum_{k=2}^{\infty} a_k z^k$ and its partial sum $f(z) = z + \sum_{k=2}^{n} a_k z^k$ are starlike univalent.*

Problem 1.2. *Find the condition on a_k, such that the functions defined by the series $f(z) = z + \displaystyle\sum_{k=2}^{\infty} a_k z^k$ and its partial sum $f(z) = z + \sum_{k=2}^{n} a_k z^k$ are close-to-convex with respect to a particular starlike function.*

Non-negativity of cosine series, sine series and their partial sums play an important role in getting some partial answer of above problems. Many results regarding non-negativity of trigonometric series and their application to find the geometric nature such as starlikeness, convexity and univalency of various classes of analytic functions and polynomials, are available in the literature. For details we refer [1, 2, 4, 8, 15]. The following result given in [2] is very useful for our work.

Lemma 1.4. [2] *Let $(c_k)_{k=0}^{\infty}$ be non-increasing sequence of nonnegative real numbers such that $c_0 > 0$ and*

$$c_{2k} \leq \frac{2k}{2k+1} c_{2k-1}, \quad \textit{for} \quad k = 1, 2, 3, \cdots. \tag{1.1}$$

Then, for every positive integer N, M, we have

$$\begin{aligned} c_0 + c_1 \cos\theta + c_2 \cos 2\theta + c_3 \cos 3\theta + \cdots\cdots + c_N \cos N\theta &> 0, \\ c_1 \sin\theta + c_2 \sin 2\theta + c_3 \sin 3\theta + \cdots\cdots + c_{2M+1} \sin(2M+1)\theta &> 0. \end{aligned}$$

Following result is an immediate consequence of the Lemma 1.4., if we replace c_k by $r^k c_k, \forall r \in [0,1)$ and rewrite the hypothesis given in Lemma 1.4..

Lemma 1.5. *If $(c_k)_{k=0}^{\infty}$ satisfies the hypothesis of lemma(1.4.), then for $r \in [0,1)$ and every positive integer N, M, we have*

$$c_0 + c_1 r\cos\theta + c_2 r^2 \cos 2\theta + c_3 r^3 \cos 3\theta + \cdots\cdots + c_N r^N \cos N\theta > 0$$

and

$$c_1 r\sin\theta + c_2 r^2 \sin 2\theta + c_3 r^3 \sin 3\theta + \cdots\cdots + c_{2M+1} r^{2M+1} \sin(2M+1)\theta > 0.$$

For partial solution regarding the polynomial part of problems 1.1. and 1.2., we consider the Cesàro means of $f(z)$, which is defined as follows:
Let $f(z) = z + \sum_{k=2}^{\infty} a_k z^k$, then the n-th Cesàro means of order β of $f(z)$ is given as

$$\sigma_n^{\beta}(z,f) = \sum_{k=1}^{n} \frac{A_{n-k}^{\beta}}{A_n^{\beta}} a_k z^k, \tag{1.2}$$

for all $n \in \mathbb{N}$ and $\beta > -1$, where $A_n^{\beta} = \dfrac{(n+\beta)}{n} A_{n-1}^{\beta}$ and $A_0^{\beta} = 1$.

In particular,

$$\sigma_n^{\beta}(z) = \sum_{k=1}^{n} \frac{A_{n-k}^{\beta}}{A_n^{\beta}} z^k \tag{1.3}$$

Note that $\sigma_n^{\beta}(z,f) = \sigma_n^{\beta}(z) * f(z)$, where $*$ denotes the Hadamard product or convolution, which is defined as $(f*g)(z) = z + \sum_{k=2}^{\infty} a_k b_k z^k$ if $f(z) = z + \sum_{k=2}^{\infty} a_k z^k$ and $g(z) = z + \sum_{k=2}^{\infty} a_k z^k$. For details about the convolution and it's properties, we refer [3, 14].

There are many results available regarding the univalency of polynomials $\sigma_n^{\beta}(z)$, cf.[5, 7, 11]. The following is the most general one due to Lewis.

Lemma 1.6. [7] *For $\beta \geq 1$ and $n \in N$ we have $\sigma_n^{\beta}(z) \in \mathcal{K}$.*

By convolution property of convex functions and close-to-convex functions [14], we immediately have

Corollary 1.1. *For $\beta \geq 1$, $n \in N$ and $f \in \mathcal{C}$ we have $\sigma_n^{\beta}(z,f) \in \mathcal{K}$.*

In [13], Ruscheweyh established the following result which gives the geometric property of Cesàro means.

Lemma 1.7. [13] *Let $\beta \geq \alpha > 1$, $f \in \mathcal{C}_{(3-\alpha)/2}$. Then for all $n \in N$:*

$$\frac{n+\beta}{n} \sigma_n^{\beta}(z,f) \in \mathcal{C}_{(3-\alpha)/2}.$$

A corresponding result holds if $\mathcal{C}_{(3-\alpha)/2}$ is replaced by either ${\mathcal{S}^*}_{(3-\alpha)/2}$, or $\mathcal{K}_{(3-\alpha)/2}$.

2. Main Results

In this section, we state our main result which addresses problems 1.1. and 1.2. partially.

Theorem 2.1. *Let $(a_k)_{k=1}^{\infty}$ be sequence of non-negative real number such that $a_1 = 1$ and $(k+1)a_{k+1} \leq ka_k$. Then for all $k \geq 1$,*

(a): $(2k+1)^2(2n+2-2k)a_{2k+1} \leq (2k)^2(2n+2-2k+\beta)a_{2k}$
which implies $\dfrac{2n+2+\beta}{2n+2}\sigma^{\beta}_{2n+2}(z,f)$ is close-to-convex with respect to both the starlike functions z and $z/(1-z)$.
Further that, for the same condition $\dfrac{2n+2+\beta}{2n+2}\sigma^{\beta}_{2n+2}(z,f)$ is starlike univalent.

(b): $(2k+1)(2n+2-2k)a_{2k} \leq (2k-1)(2n+2-2k+\beta)a_{2k-1}$
which implies $\dfrac{2n+1+\beta}{2n+1}\sigma^{\beta}_{2n+1}(z,f)$ is close-to-convex with respect to the starlike function $z/(1-z^2)$.

Proof. (a): Consider

$$g_{2n+2}(z) := \frac{2n+2+\beta}{2n+2}\sigma^{\beta}_{2n+2}(z,f) = z + \frac{2n+2+\beta}{2n+2}\sum_{k=2}^{2n+2}\frac{A^{\beta}_{2n+2-k}}{A^{\beta}_{2n+2}}a_k z^k$$

and write

$$\mathrm{Re}\, g'_{2n+2}(z) = c_0 + \sum_{k=1}^{2n+1} c_k r^k cos k\theta \quad \text{and} \quad \mathrm{Im}\, g'_{2n+2}(z) = \sum_{k=1}^{2n+1} c_k r^k sin k\theta$$

where $c_0 = 1$,

$$c_k = \frac{2n+2+\beta}{2n+2}.\frac{A^{\beta}_{2n+1-k}}{A^{\beta}_{2n+2}}(k+1)a_{k+1}, \qquad \theta \in [0,\pi], \quad r \in [0,1).$$

Now we need to prove that, by hypothesis of this Theorem, c_k satisfy the conditions of Lemma 1.5.. By an easy compuation, for $k = 0,1,2,\ldots$, we have

$$\begin{aligned}
c_{k+1} - c_k &= \frac{(2n+2+\beta)}{(2n+2)A^{\beta}_{2n+2}}\left[A^{\beta}_{2n-k}(k+2)a_{k+2} - A^{\beta}_{2n+1-k}(k+1)a_{k+1}\right] \\
&= \frac{(2n+2+\beta)A^{\beta}_{2n-k}}{(2n+2)A^{\beta}_{2n+2}}\left[(k+2)a_{k+2} - \frac{2n+1-k+\beta}{2n+1-k}(k+1)a_{k+1}\right] \\
&= \frac{(2n+2+\beta)A^{\beta}_{2n-k}}{(2n+2)A^{\beta}_{2n+2}}\left[\big((k+2)a_{k+2} - (k+1)a_{k+1}\big) - \beta\frac{(k+1)a_{k+1}}{2n+1-k}\right] \\
&\leq 0.
\end{aligned}$$

Similarly, for $k = 1, 2, 3, \ldots$, by writing

$$M := \frac{(2n+2+\beta)A^{\beta}_{2n+1-2k}}{(2n+2)(2k+1)A^{\beta}_{2n+2}} > 0,$$

we have

$$c_{2k} - \frac{2k}{2k+1}c_{2k-1} = M\left[(2k+1)^2 a_{2k+1} - \frac{(2n+2-2k+\beta)}{2n+2-2k}(2k)^2 a_{2k}\right]$$

which is non-positive. Now c_k satisfies the hypothesis of Lemma 1.5.. Hence by using the minimum principle of harmonic functions, we have $\mathrm{Re}\, g'_{2n+2}(z) > 0$, and $\mathrm{Im}\, g'_{2n+2}(z) > 0$. Further, by reflection principle, $\mathrm{Im}\, g'_{2n+2}(z) > 0$ for $\mathrm{Im}\,(z) > 0$, implies $\mathrm{Im}\, g'_{2n+2}(z) < 0$ for $\mathrm{Im}\,(z) < 0$ and using $\cos k(2\pi - \theta) = \cos k\theta$, we get that, this result is also true for $\theta \in [\pi, 2\pi]$.

Combining all these observations, we have

$$\mathrm{Re}\,(1-z)g'_{2n+2}(z) = \mathrm{Re}\,(1-z)\mathrm{Re}\, g'_{2n+2}(z) + \mathrm{Im}\,(z)\mathrm{Im}\, g'_{2n+2}(z) > 0.$$

Hence $g_{2n+2}(z)$ is close-to-convex(univalent) with respect to z and $\frac{z}{1-z}$. Further using Lemma 1.1., we have that $g_{2n+2}(z)$ is starlike.

(b): Consider

$$g_{2n+1}(z) := \frac{2n+1+\beta}{2n+1}\sigma^{\beta}_{2n+1}(z, f) = z + \frac{2n+1+\beta}{2n+1}\sum_{k=2}^{2n+1}\frac{A^{\beta}_{2n+1-k}}{A^{\beta}_{2n+1}}a_k z^k$$

Then, we have, $\mathrm{Im}\, z g'_{2n+1}(z) = \displaystyle\sum_{k=1}^{2n+1} c_k r^k sink\theta$, where

$$c_k = \frac{2n+1+\beta}{2n+1}.\frac{A^{\beta}_{2n+1-k}}{A^{\beta}_{2n+1}}ka_k, \quad \theta \in [0, \pi], \quad r \in [0, 1).$$

Now, as in the previous part, to prove the result under the hypothesis of the Theorem, we need to verify the conditions given in Lemma 1.5.. Hence, for $k = 1, 2, 3, \ldots$, we have

$$\begin{aligned} c_{k+1} - c_k &= \frac{(2n+1+\beta)A^{\beta}_{2n-k}}{(2n+1)A^{\beta}_{2n+1}}\left[(k+1)a_{k+1} - \frac{2n+1+\beta}{2n+1}ka_k\right] \\ &= \frac{(2n+1+\beta)A^{\beta}_{2n-k}}{(2n+1)A^{\beta}_{2n+1}}\left[(k+1)a_{k+1} - ka_k - \frac{\beta}{2n+1}ka_k\right] \leq 0, \end{aligned}$$

and by writing $N := \dfrac{(2n+1+\beta)(2k)A^{\beta}_{2n+1-2k}}{(2n+1)(2k+1)A^{\beta}_{2n+1}} > 0$, we get

$$c_{2k} - \frac{2k}{2k+1}c_{2k-1} = N\left[(2k+1)a_{2k} - \frac{(2n+2-2k+\beta)}{2n+2-2k}(2k-1)a_{2k-1}\right]$$

which is non-positive. Hence we have $\operatorname{Im}(zg'_{2n+1}(z)) > 0$ in $\mathbb{D} \cap \{z : \operatorname{Im} z > 0\}$, using Lemma 1.5.. By reflection principle, we have $\operatorname{Im}(zg'(z)) < 0$ in $\mathbb{D} \cap \{z : \operatorname{Im} z < 0\}$, i.e, $zg'_{2n+1}(z)$ is typically real. Hence using the fact that $f(z)$ has real coefficients and is convex in the direction of imaginary axis implies that $f(z)$ is close-to-convex w.r.t the starlike function $\frac{z}{1-z^2}$ and Lemma 1.2., we get the required result and the proof is complete. ■

We observe that Theorem 2.1. is different from Lemma 1.7. by means of coefficient conditions. But in sense of order of close-to-convexity Lemma 1.7. is stronger than Theorem 2.1.. Now we will discuss some particular case of Theorem 2.1.. Since the class of all close-to-convex functions with respect to same starlike function forms a normal family, and for $\beta = 0$, $\frac{n+\beta}{n}\sigma_n^\beta(z,f) = f_n(z) = z + \sum_{k=2}^n a_k z^k$, we have the following result, partially supporting problem 1.1. and problem 1.2..

Theorem 2.2. *Let $(a_k)_{k=1}^\infty$ be sequence of non-negative real number with $a_1 = 1$ such that, for all $k \geq 1$,*

$$(k+1)a_{k+1} \leq k a_k \qquad \text{and} \quad (2k+1)^2 a_{2k+1} \leq (2k)^2 a_{2k}. \tag{2.1}$$

Then, $f(z) = \lim_{n\to\infty} f_{2n+2}(z) = z + \sum_{k=2}^\infty a_k z^k$ is close-to-convex with respect to both the starlike functions z and $z/(1-z)$. Further that, for the same condition, $f(z)$ is starlike univalent.

In [1], the results of [4] are improved and the following results are obtained, which seems to be the best available conditions on a_k so far, such that $f(z) = z + \sum_{k=2}^\infty a_k z^k$ are starlike univalent.

Lemma 2.1. [1] *Let $(a_k)_{k=1}^\infty$ be the sequence of real numbers such that $a_1 = 1$, and for that, $k \geq 3$, the quantities a_k, $\underline{\Delta} a_k$, $\underline{\Delta}^2 a_k$ are all non-negative. If*

1. $2a_1 - 4a_2 + 3a_3 \geq 0$
2. $2a_1 + a_2 - 12a_3 + 10a_4 \geq 0$
3. $2a_1 + 4a_2 + 6a_3 - 56a_4 + 45a_5 \geq 0$
4. $\underline{\Delta}^2 a_2 + \underline{\Delta}^2 a_4 + \frac{21}{16}\underline{\Delta}^2 a_6 \geq 0$,

then the function defined by $f(z) = z + \sum_{k=2}^\infty a_k z^k$, are starlike univalent.

Lemma 2.2. [1] *Let $(a_k)_{k=1}^\infty$ be the sequence of real numbers such that $a_1 = 1$, and for that, $k \geq 3$, the quantities a_k, $\underline{\Delta} a_k$, $\underline{\Delta}^2 a_k$ are all non-negative. If*

1. $2a_1 - 4a_2 + 3a_3 \geq 0$
2. $2a_1 + a_2 - 12a_3 + 10a_4 \geq 0$

3. $2a_1 + 4a_2 + 6a_3 - 56a_4 + 45a_5 \geq 0$
4. $\underline{\Delta}^2 a_2 + \underline{\Delta}^2 a_3 + \underline{\Delta}^2 a_5 + \frac{21}{16}\underline{\Delta}^2 a_6 \geq 0$
5. $\underline{\Delta}^2 a_2 + \underline{\Delta}^2 a_4 + \frac{21}{16}\underline{\Delta}^2 a_6 \geq 0$

then the function defined by $f(z) = z + \sum_{k=2}^{\infty} a_k z^k$, *are starlike univalent.*

Here by $\underline{\Delta}$ we mean $\underline{\Delta} a_k = k\, a_k - (k+1)a_{k+1}$ and $\underline{\Delta}^2 \equiv \underline{\Delta}(\underline{\Delta})$.

Now we provide some examples to support our results.

Example 2.1. *Consider the sequence* $(a_k)_{k=1}^{\infty}$ *such that* $a_1 = 1$, $a_2 = \frac{1}{2}$ *and*

$$(2k+1)a_{2k+1} = (2k+2)a_{2k+2} = \frac{(k!)^2 2^{2k}}{(2k+1)!}$$

Then by Theorem 2.2. *the function*

$$f(z) = z + \sum_{k=2}^{\infty} a_k z^k = z + \frac{1}{2}z^2 + \frac{2}{9}z^3 + \frac{1}{6}z^4 + \frac{8}{75}z^5 + \frac{4}{45}z^6 + \frac{16}{245}a^7 + \cdots$$

is starlike univalent. But Lemma 2.1. *and Lemma* 2.2. *fails to include this function as* $\underline{\Delta}^2 a_3 = 3a_3 - 8a_4 + 5a_5 = -\frac{2}{15} \ngeq 0$.

Hence Theorem 2.2. is better than the Lemma 2.1. and Lemma 2.2., in the sense that it covers some more cases.

The functions

$$z, \frac{z}{1-z}, \frac{z}{1-z^2}, \frac{z}{(1-z)^2} \quad \text{and } \frac{z}{1-z+z^2}$$

and their rotations are the only nine functions which are starlike univalent and have integer coefficients in $\mathbb{D}$, (see [6, 9] for details). Theorem 2.2. handles the close-to-convexity of $f(z)$ with respect to z and $z/(1-z)$. The following result, which can be obtained directly from Theorem 2.1., handles the close-to-convexity of $f(z)$ with respect to $z/(1-z^2)$.

Theorem 2.3. *Let* $(a_k)_{k=1}^{\infty}$ *be sequence of non-negative real number with* $a_1 = 1$ *such that, for all* $k \geq 1$,

$$(k+1)a_{k+1} \leq k a_k \qquad \text{and} \quad (2k+1)a_{2k} \leq (2k-1)a_{2k-1}. \tag{2.2}$$

Then $f(z) = \lim_{n\to\infty} f_{2n+1}(z) = z + \sum_{k=2}^{\infty} a_k z^k$ *is close-to-convex w.r.t* $\frac{z}{1-z^2}$.

Note that Theorem 2.1. refers to the starlike functions z, $z/(1-z)$ and $z/(1-z^2)$ upto partial sums. So far, we have no results for the remaining two cases of starlike functions, namely $z/(1-z)^2$ and $z/(1-z+z^2)$. Hence it will be interesting to see if one can find similar results with respect to these starlike functions. In particular, not many results related to the close-to-convexity of $f(z)$ with respect to the starlike function $z/(1-z+z^2)$ are available in the literature.

Using Vietori's lemma [15], some partial answer to the problem 1.2. is given in [1].

Lemma 2.3. [1] *Let* $(a_k)_{k=1}^{\infty}$ *be a sequence of nonnegative real numbers such that* $a_1 = 1$ *and if*

$$(k+1)a_{k+1} \leq ka_k, \quad (2k)^2 a_{2k} \leq (2k-1)^2 a_{2k-1}$$

Then the function defined by the series $\displaystyle\sum_{k=1}^{n} a_k z^k$, $\displaystyle\sum_{k=1}^{\infty} a_k z^k$ *are close-to-convex w.r.t* $\frac{z}{1-z^2}$.

Example 2.2. *Consider the sequence* $(a_k)_{k=1}^{\infty}$ *such that* $a_1 = 1$ *and*

$$(2k)a_{2k} = (2k+1)a_{2k+1} = \frac{(k!)^2 2^{2k}}{(2k+1)!}$$

Then by Theorem 2.3.*, the function*

$$f(z) = z + \frac{1}{3}z^2 + \frac{2}{9}z^3 + \frac{2}{15}z^4 + \frac{8}{75}z^5 + \cdots$$

is close-to-convex w.r.t $\frac{z}{1-z^2}$*, whereas Lemma* 2.3. *cannot be applied for this example.*

3. Application to Hypergeometric Functions

Consider the operator $\mathcal{H}_{a,b,c}(f)(z)$, is defined as

$$\mathcal{H}_{a,b,c}(f)(z) = zF(a,b;c;z) * f(z) \tag{3.1}$$

In particular, if a=1 in (3.1), the the operator $\mathcal{H}_{1,b,c}(f)(z)$ is known as Carlson and Shaffer operator and denoted as $\mathcal{L}_{b,c}$. Here by $F(a,b;c;z)$ we mean the Gaussian hypergeometric function $F(a,b;c;z)$, $z \in \mathbb{D}$, given by the series

$$F(a,b;c;z) = \sum_{n=0}^{\infty} \frac{(a,n)(b,n)}{(c,n)(1,n)} z^n \tag{3.2}$$

which is the solution of the homogenous hypergeometric differential equation

$$z(1-z)w''(z) + [c - (a+b+1)z]w'(z) - abw(z) = 0 \tag{3.3}$$

Here a, b, c are complex numbers such that $c \neq 0, -1, -2, -3, \cdots$, $(a,0) = 1$ for $a \neq 0$, and for each positive integer n, $(a,n) = a(a+1,n-1)$ is the Pochhammer symbol.

In this section, we find the condition on the triplet (a,b,c) and the coefficients $\{a_k\}$ such that $\mathcal{H}_{a,b,c}(f)$ and $\mathcal{L}_{b,c}(f)$ are (i) starlike univalent and (ii) close-to-convex with respect to a particular starlike function. These results are also expected to support Problem 1.1. and Problem 1.2. partially.

The following theorem on $\mathcal{H}_{a,b,c}(f)$ is obtained by using Theorem 2.2..

Theorem 3.1. *Let* $a_{k+1} \leq a_k, \forall k \geq 1$*, then* $\mathcal{H}_{a,b,c}(f)$ *is starlike univalent if one of the following conditions is satisfied.*

1. *$a<0, b<0$ be such that $c \geq \max(0, 2ab, a+b+1)$ and $(a,k)(b,k) \geq 0, \forall\, k \geq 1$.*
2. *$a>0, b>0$ and $c \geq T(a,b)$ where*

$$\begin{aligned} T(a,b) &= \max\Bigg\{0, 2ab, a+b+1, \frac{1}{6}(7a+7b+ab+4), \\ &\qquad \frac{1}{12}(15a+15b+6ab+4), \frac{1}{8}(9a+9b+9ab)\Bigg\}. \end{aligned}$$

Proof.

$$\mathcal{H}_{a,b,c}(f)(z) = zF(a,b;c;z) * f(z) = z + \sum_{k=2}^{\infty} b_k z^k \tag{3.4}$$

where, $b_k = \dfrac{(a,k-1)(b,k-1)}{(c,k-1)(1,k-1)} a_k, \forall k \geq 1$. Now it is enough to check if $\{b_k\}$ satisfies the hypothesis of Theorem 2.2.. For this, we write

$$(k+1)b_{k+1} - kb_k = \frac{(a,k-1)(b,k-1)}{(c,k)(1,k)} X(k), \tag{3.5}$$

where,

$$X(k) = (a+k-1)(b+k-1)(k+1)a_{k+1} - k^2(c+k-1)a_k. \tag{3.6}$$

We need to show that $X(k) \leq 0$. An easy computation, using the hypotheses given in the Theorem gives

$$\begin{aligned} X(k) &\leq \left[(a+k-1)(b+k-1)(k+1) - k^2(c+k-1)\right] a_{k+1} \\ &\leq [-c+2ab]a_{k+1}, \end{aligned}$$

proves $X(k) \leq 0$, since $c \geq 2ab$. The proof will be complete, if we prove the other inequality given in Theorem 2.2.. This implies that we need to prove that

$$(2k+1)^2 b_{2k+1} - (2k)^2 b_{2k} = \frac{(a,2k-1)(b,2k-1)}{(c,2k)(1,2k)} Y(k) \leq 0,$$

where,

$$Y(k) = (a+2k-1)(b+2k-1)(2k+1)^2 a_{2k+1} - (2k)^3(c+2k-1)a_{2k}.$$

Again an easy computation using the hypothesis of the Theorem leads to the fact that

$$\begin{aligned} Y(k) &\leq \left[(a+2k-1)(b+2k-1)(2k+1)^2 - (2k)^3(c+2k-1)\right] a_{2k+1} \\ &\leq [(28(a+b) - 24c + 4ab + 16)k^2 + (4ab - 26(a+b) + 24c - 24)k \\ &\quad +(7a+7b+ab+9-8c)]a_{2k+1} \\ &\leq \Big[(6(5a+5b+2ab-4c)+8)k - (21(a+b)+3ab-16c+8)\Big] a_{2k+1} \\ &\leq (9a+9b+9ab-8c)\, a_{2k+1} \leq 0, \end{aligned}$$

and the proof is complete. ■

Note that the convex function $f(z) = z/(1-z)$ satisfies the hypothesis given in Theorem 3.1.. Hence taking $a_k = 1, \forall k$, we get $\mathcal{H}_{a,b,c}(f)(z) = zF(a,b;c;z) * z/(1-z) = zF(a,b;c;z)$. This leaves us to have condition on triplet (a,b,c) such that $zF(a,b;c;z)$ is starlike univalent. Also the well-known Pólya-Schoenberg Theorem[14] ascertains that the convolution of close-to-convex(starlike) function with convex function is close-to-convex(starlike). Applying this result, we have

Corollary 3.1. *Let (a,b,c) satisfies the hypothesis of Theorem 3.1. with $a_k = 1$. Then, both $zF(a,b;c;z)$ and $\mathcal{H}_{a,b,c}(g)(z)$ are starlike univalent for any convex function $g(z) \in \mathcal{A}$.*

The following results can be proved in a similar way Theorem 3.1. is proved and hence we give only the statement of these results.

Theorem 3.2. *Let the triplet (a,b,c) and $\{a_k\}$ satisfy one of the following conditions.*

1. $(k+1)a_{k+1} \leq k a_k$ *with*
   - i. $a < 0,\ b < 0$ *such that* $c \geq ab$ *and* $(a,k)(b,k) \geq 0,\ \forall\, k \geq 1$, *or*
   - ii. $a > 0,\ b > 0$ *and* $c \geq \max\{0, ab, a+b, \frac{1}{8}(8a+8b+2ab-2), \frac{1}{4}(3a+3b+3ab-1)\}$.
2. $(k+1)^2 a_{k+1} \leq k^2 a_k$ *with*
   - i. $a < 0,\ b < 0$ *such that* $c \geq \frac{1}{2}ab$ *and* $(a,k)(b,k) \geq 0, \forall k \geq 1$, *or*
   - ii. $a > 0,\ b > 0$ *and* $c \geq \max\{0, \frac{1}{2}ab, (a+b-1), \frac{1}{2}(a+b+ab-1)\}$.
3. $(k+1)a_{k+1} \leq k a_k$, $(2k+1)^2 a_{2k+1} \leq (2k)^2 a_{2k}$, *with*
   - i. $a < 0,\ b < 0$ *such that* $c \geq ab$ *and* $(a,k)(b,k) \geq 0, \forall k \geq 1$, *or*
   - ii. $a > 0,\ b > 0$ *and* $c \geq \max\{0, ab, a+b-1\}$.
4. $(k+1)^2 a_{k+1} \leq k^2 a_k$, $(2k+1)^3 a_{2k+1} \leq (2k)^3 a_{2k}$, *with*
   - i. $a < 0,\ b < 0$ *such that* $c \geq \frac{1}{2}ab$ *and* $(a,k)(b,k) \geq 0, \forall k \geq 1$, *or*
   - ii. $a > 0,\ b > 0$ *and* $c \geq \max\{0, \frac{1}{2}ab, a+b-1, \frac{1}{3}(a+b+ab)\}$.

Then $\mathcal{H}_{a,b,c}(f)$ is starlike univalent.

Remark: Since $a_k = 1$ is not satisfied by any of the conditions given in Theorem 3.2., a result equivalent to Corollary 3.1. cannot be obtained for $zF(a,b;c;z)$. But when $a = 1$, then both results, namely Theorem 3.1. and Theorem 3.2. reduces to the following result, except for the cases, where $a < 0$.

Corollary 3.2. *If $b > 0$, then $\mathcal{L}_{b,c}(f)$ is starlike univalent if one of the following conditions is satisfied.*

(i) $a_{k+1} \leq a_k$,
$c \geq \max\{0, 2b, b+2, \frac{1}{6}(14b+11), \frac{1}{12}(21b+19), \frac{1}{8}(9b+18)\}$.

(ii) $(k+1)a_{k+1} \leq ka_k$, $c \geq \max\{0, b+1, \frac{1}{4}(5b+3), \frac{1}{2}(3b+1)\}$.

(iii) $(k+1)^2 a_{k+1} \leq k^2 a_k$, $c \geq b$.

(iv) $(k+1)a_{k+1} \leq ka_k$ *and* $(2k+1)^2 a_{2k+1} \leq (2k)^2 a_{2k}$, $c \geq b$.

(v) $(k+1)^2 a_{k+1} \leq k^2 a_k$ *and* $(2k+1)^3 a_{2k+1} \leq (2k)^3 a_{2k}$,
$c \geq \max\{0, b, \frac{1}{3}(2b+1)\}$.

The following result is immediate.

Corollary 3.3. *If* $b > 0$ *and* $c \geq \max\{0, 2b, b+2, \frac{1}{6}(14b+11), \frac{1}{12}(21b+19), \frac{1}{8}(9b+18)\}$, *then the incomplete beta function* $\phi(b, c; z)$ *is starlike univalent. Further, the Carlson-Shaffer operator* $\mathcal{L}_{b,c}(g)$ *is univalent for any convex function* $g \in \mathcal{A}$.

Instead of deducing the conditions for $\mathcal{L}_{b,c}$ from Theorems 3.1. and Theorem 3.2., we can apply Theorem 2.2., to get the following results for $\mathcal{L}_{b,c}(f)$.

Theorem 3.3. $\mathcal{L}_{b,c}(f)$ *is starlike univalent if* $b > 0$ *and one of the following conditions is satisfied*

(i) $c \geq \max\{0, 2b, b+2, \frac{1}{4}(6b+7), \frac{1}{4}(9b+5)\}$ *and* $a_{k+1} \leq a_k, \forall k \geq 1$.

(ii) $c \geq \frac{1}{2}(3b+1)$ *and* $(k+1)a_{k+1} \leq ka_k, \forall k \geq 1$.

(iii) $c \geq b$ *and* $(k+1)a_{k+1} \leq ka_k$, $(2k+1)^2 a_{2k+1} \leq (2k)^2 a_{2k}$.

The following result is similar to Corollary 3.1..

Corollary 3.4. *Let* $b > 0$ *and* $c \geq \max\{0, 2b, b+2, \frac{1}{4}(6b+7), \frac{1}{4}(9b+5)\}$. *Then, the incomplete beta function* $\phi(b, c; z)$ *is starlike univalent. Further, for every convex function* $g \in \mathcal{A}$, $\mathcal{L}_{b,c}(g)$ *univalent.*

We observe that Theorem 3.3. and Corollary 3.4. are stronger than Corollary 3.2. and Corollary 3.3.. But Corollary 3.2. and Corollary 3.3. covers more range for a, b, c than Theorem 3.3. and Corollary 3.4..

Now applying Theorem 2.3., we get following results for close-to-convexity of $\mathcal{H}_{a,b,c}(f)$ with respect to the starlike function $\frac{z}{1-z^2}$.

Theorem 3.4. *Let* (a, b, c) *and* $\{a_k\}$ *satisfy one of the following conditions.*

1. $a_{k+1} \leq a_k, \forall k \geq 1$, *together with*

   (i) $a < 0, b < 0$ *and* $c \geq \max\{0, 2ab, \frac{1}{6}(2a+2b+2ab+3), a+b+1\}$ *such that* $(a, k)(b, k) \geq 0$, $\forall\, k \geq 1$, *or*

   (ii) $a > 0, b > 0$, $c \geq \max\{0, 2ab, \frac{1}{2}(2a+2b+2ab-1), a+b+1, \frac{1}{6}(6a+6b+ab-2), \frac{1}{6}(2a+2b+2ab+3)\}$.

2. $(k+1)a_{k+1} \leq ka_k, (2k+1)a_{2k+1} \leq (2k-1)a_{2k-1}, \forall k \geq 1$, *together with*

   (i) $a < 0$, $b < 0$ *and* $c \geq ab$ *such that* $(a, k)(b, k) \geq 0$, $\forall\, k \geq 1$, *or*

(ii) $a > 0, b > 0,\ c \geq \max\{0, ab, a+b-1\}$.

3. $(k+1)^2 a_{k+1} \leq k^2 a_k$, $(2k+1)^2 a_{2k+1} \leq (2k-1)^2 a_{2k-1}, \forall k \geq 1$, *together with*

   (i) $a < 0,\ b < 0$ *and* $c \geq \frac{1}{2}ab$ *such that* $(a,k)(b,k) \geq 0,\ \forall k \geq 1$ *or*

   (ii) $a > 0, b > 0,\ c \geq \max\{0, \frac{1}{2}ab, a+b-2\}$.

4. $(k+2)a_{k+1} \leq k a_k, \forall k \geq 1$, *together with*

   (i) $a < 0,\ b < 0$ *and* $c \geq ab$ *such that* $(a,k)(b,k) \geq 0,\ \forall k \geq 1$, *or*

   (ii) $a > 0, b > 0,\ c \geq \max\{0, ab, a+b-1, \frac{1}{2}(2a+2b+ab-3)\}$.

Then, $\mathcal{H}_{a,b,c}(f)(z)$ *is close-to-convex w.r.t* $\frac{z}{1-z^2}$.

Proof. We prove the result for part 1 and the results for other cases can be obtained similarly. For proving part 1, we need to check that b_k given in equation (3.5) satisfy the requirements of Theorem 2.3.. Now writing

$$(k+1)b_{k+1} - kb_k = \frac{(a,k-1)(b,k-1)}{(c,k)(1,k)} X(k)$$

with $X(k)$ as in (3.6), we get that

$$\begin{aligned} X(k) &\leq [(a+k-1)(b+k-1) - k(c+k-1)]\,(k+1)a_{k+1} \\ &\leq [ab-c](k+1)a_{k+1} \leq 0, \end{aligned}$$

using the hypothesis of the Theorem. This proves first inequality in (2.2). It remains to prove the other inequality in (2.2). From equation (3.4), we have

$$(2k+1)b_{2k} - (2k-1)b_{2k-1} = \frac{(a,2k-2)(b,2k-2)}{(c,2k-1)(1,2k-1)} Z(k), \tag{3.7}$$

where,

$$\begin{aligned} Z(k) &= (a+2k-2)(b+2k-2)(2k+1)a_{2k} - (2k-1)^2(c+2k-2)a_{2k-1} \\ &\leq [4(a+b+1-c)k^2 - 2(a+b-ab-2c+5)k \\ &\qquad +(6-2(a+b)+ab-c)]a_{2k} \\ &\leq [2a+2b+2ab-6c+3]a_{2k} \leq 0, \end{aligned}$$

using the hypothesis of the Theorem and the proof is complete. ■

The following result can be obtained by applying $a_k = 1$, in first part of the hypothesis of Theorem 3.4.. Note that $a_k = 1$ can not be applied to the other case of Theorem 3.4..

Corollary 3.5. *Let* (a,b,c) *satisfies any one of the following conditions:*

(i) $a < 0, b < 0$ *and* $c \geq \max\{0, 2ab, \frac{1}{6}(2a+2b+2ab+3), a+b+1\}$ *such that* $(a,k)(b,k) \geq 0,\ \forall k \geq 1$.

(ii) $a > 0, b > 0$, $c \geq \max\{0, 2ab, \frac{1}{2}(2a+2b+2ab-1), a+b+1, \frac{1}{6}(6a+6b+ab-2), \frac{1}{6}(2a+2b+2ab+3)\}$.

Then, $zF(a,b;c;z)$ *is close-to convex w.r.t* $\frac{z}{1-z^2}$.

Corollary 3.6. *Under the hypothesis of Corollary* 3.5.*,* $\mathcal{H}_{a,b,c}(f)(z)$ *is univalent for every convex function* f.

Remark: In particular, taking $a = 1$ in Theorem 3.4., we have the close-to-convexity of $\mathcal{L}_{b,c}(f)$ w.r.t $\frac{z}{1-z^2}$, except for the conditions given in the Theorem with $a < 0$. We omit details of this result. But applying Theorem 2.3. on the coefficients of $\mathcal{L}_{b,c}(f)$, with $a_k = 1$, we get the following result immediately.

Corollary 3.7. *Let* $c \geq \begin{cases} b+2, & \text{if} \quad 0 < b \leq 1, \\ 3b, & \text{if} \quad b \geq 1. \end{cases}$
Then, the incomplete beta function $\phi(b,c;z)$ *is univalent.*

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Received 15.10.2008

J o u r n a l of
Mathematics
and Applications
No 31, pp 91-97 (2009)

Geometric result in the boundary behaviour of Blaschke products

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ABSTRACT: For a Blaschke product B with zeros in an angular domain having vertex on the unit circle we give a necessary and sufficient condition for the boundary behavior of B, in terms only of the distribution of the zeros. Moreover, we show, with a counterexample, the non-equivalence of two known results of Tanaka, concerning the specific boundary behavior of Blaschke products.

AMS Subject Classification: *30C55*
Key Words and Phrases: *Blaschke products, boundary behaviour*

1. Introduction

The boundary behavior of an analytic function in $D = \{z : \ |z| < 1\}$ is one of the fundamental subjects in the theory of analytic functions. One of the most important results in this direction is the following, known as Fatou's Theorem [3].

Theorem 1. *Let $f \in H^\infty$. There exists a unique function $f^* \in L^\infty(\partial D)$, which is defined almost everywhere as*

$$f^*(e^{i\theta}) = \lim_{r\to 1} \left| f(re^{i\theta}) \right| \tag{1}$$

Additionally, if for $\theta_0 \in \partial D$ the limit in (1) exists, then $f(z) \to f^*(e^{i\theta_0})$ when $z \to e^{i\theta_0}$ in a Stolz angle.

The limit in (1) is called radial limit of f in $e^{i\theta}$ and because of Fatou's Theorem, we call it also a not tangential limit. [4]

For a Blaschke product (see in [1])

$$B(z) = B(z, \{z_n\}) = \prod_{n=1}^{\infty} \frac{|z_n|}{z_n} \frac{z_n - z}{1 - \overline{z}_n z}, \ z \in D$$

we know that $|B(z)| < 1$ for $z \in D$ and $B(e^{i\theta}) = 1$ a.e.

Since $\lim_{r\to 1} \left|B(re^{i\theta})\right| < 1$ a.e., the study of the boundary behavior of a Blaschke product concerns the zero-measure set where the radial limit does not exist ([2]).

In general we have

$$0 \le \liminf_{r\to 1} \left|B(re^{i\theta})\right| \le \limsup_{r\to 1} \left|B(re^{i\theta})\right| \le 1. \tag{2}$$

Let us define

$$C(w\,,\,\{z_n\}) \;=\; \sum_{n=1}^{\infty} \frac{(1-|\,w\,|)(1-|\,z_n\,|)}{(1-|\,w\,|)^2+|\,w-z_n\,|^{\,2}}. \tag{3}$$

Using the definition of C, Tanaka in [5] proved some results, to determine when we have equalities or proper inequalities in the condition (2).

Let Γ be a Jordan curve ending in ∂D.

Theorem 2.
$\lim_{w\to 1,\ w\in\Gamma} B(w) = 0$ *if and only if* $\lim_{w\to 1,\ w\in\Gamma} C(w,\{z_n\}) = \infty$.

Theorem 3. $\limsup_{w\to 1,\ w\in\Gamma} C(w,\{z_n\}) = \infty$ *if and only if for every* $R > 0$ *, there is a sequence* $\{w_n\} \subset \Gamma$ *with* $w_n \to 1$ *, such that* $\lim_{w\to 1} B(w) = 0$ *when* w *belongs to the set* $\cup_{n\in N}\Gamma_n(R)$ *, where*

$$\Gamma_n(R) \;=\; \Gamma \cap \{z \in D \;/\; \rho(z,w_n) \le R\} \;\;,\;\; n \in N \;\;.$$

The above theorems are still valid if we substitute 1 with any point $e^{i\theta_0} \in \partial D$. But in this case, without loose of generality we will assume that $\theta_0 = 0$.

2. Main result

To formulate our results we need the notion of the asymptotically polynomial sequences. So, we give the following definition.

Definition. *Let* $\{\alpha_n\}$ *be a sequence with* $\alpha_n > 0,\ \alpha_n \ge \alpha_{n+1},\ n \in N$ *and* $\lim\limits_{n\to\infty} \alpha_n = 0$ *. We call* $\{\alpha_n\}$ **asymptotically polynomial** *if for every* $k \in N$ *there is an infinite sequence* $\{n_p\} \subset N$ *depending on* k *and such that* $\frac{a_{n_p}}{a_{n_p+k}} \to 1$ *as* $p \to \infty$.

If this condition is not satisfied, we say that $\{\alpha_n\}$ *is* **asymptotically exponential** .

In most cases it is relatively easy to see if a sequence $\{\alpha_n\}$ is asymptotically polynomial or not. In a plausible way we would say that $\{\alpha_n\}$ is asymptotically polynomial if it contains families of arbitrarily many asymptotically equal terms.

Our main theorem is

Theorem A. *Let* $\{z_n\} \subset D$ *be a Blaschke sequence. We suppose also that all* z_n *lie in an angular domain in* D *with vertex on* ∂D*, i.e. that there exist a* $\zeta \in \partial D$ *and* *a positive constant M so that $\frac{|\zeta - z_n|}{1-|z_n|} < M$ for all n (i.e. the zeros are inside a Stolz angle). Let $\alpha_n = 1 - |\, z_n\,|\ \ n \in N$.*

The sequence $\{\alpha_n\}$ is asymptotically polynomial, if and only if, for every $R > 0$ there exists a real number sequence $\{r_n\}$, $r_n \in (0,1)$, with $\lim_{n\to\infty} r_n = 1$, such that $\lim_{r\to 1} B(r) = 0$ when r is inside the set $\cup_{n\in N}\Gamma_n(R)$, where

$$\Gamma_n(R) \ = \ \{x \in R \ / \ \rho(x, r_n) \leq R\} \ \ n \in N \ \ .$$

Finally, we give a counterexample for which Theorem 2 is not valid. i.e. a specific Blaschke sequence for which we have $\limsup_{w\to 1,\ w\in\Gamma} C(w, \{z_n\}) = \infty$ but not $\lim_{w\to 1,\ w\in\Gamma} C(w, \{z_n\}) = \infty$.

Proof.

We prove first an auxiliary lemma.

Lemma. *Let $\{\alpha_n\}$ be a sequence with $\alpha_n > 0, \alpha_n \geq \alpha_{n+1}$ and $\lim\limits_{n\to\infty} \alpha_n = 0$. Then the sequences $c_n = \alpha_n \sum\limits_{k=1}^{n} \frac{1}{\alpha_k}$ and $d_n = \frac{1}{\alpha_n} \sum\limits_{k=n+1}^{\infty} \alpha_k$, $n \in N$, are both bounded if and only if $\{\alpha_n\}$ is asymptotically exponential .*

Proof. '$\Rightarrow$' : Let $\{\alpha_n\}$ be a positive and decreasing sequence and let

$$c_n = \alpha_n \sum_{k=1}^{n} \frac{1}{\alpha_k} < M < \infty \quad \forall\, n \in N. \tag{4}$$

Suppose that $\{\alpha_n\}$ is asymptotically polynomial and let $k > 2M$. Then we can find an infinite sequence $\{n_m\}$ such that $\frac{\alpha_{n_m}}{\alpha_{n_m+k}} \to 1$ as $m \to \infty$. It is very easy to see that for k succesive terms $\alpha_{n_m}, \alpha_{n_m+1}, \ldots, \alpha_{n_m+k-1}$ we have $\frac{\alpha_{n_m+l}}{\alpha_{n_m+l+1}} \to 1$ as $m \to \infty$ for every $l = 0, 1, \ldots, k-1$. For brevity we write m instead of n_m.

By induction we calculate easily that

$$c_{m+k-1} = \frac{\alpha_{m+k-1}}{\alpha_m} c_m + \sum_{i=0}^{k-2} \frac{\alpha_{m+k-1}}{\alpha_{m+k-1-i}}, \tag{5}$$

so

$$c_{m+k-1} > \sum_{i=1}^{k-1} \frac{\alpha_{m+k-1}}{\alpha_{m+i}}. \tag{6}$$

From $\lim_{m\to\infty} \frac{\alpha_{m+k-1}}{\alpha_{m+i}} = 1$ for every $i = 1, 2, ..., k-1$, it follows that

$$\lim_{m\to\infty} \sum_{i=1}^{k-1} \frac{\alpha_{m+k-1}}{\alpha_{m+i}} > 2M,$$

which implies, by (6), that $c_{m+k-1} > 2M$ for sufficiently large m. This stays in contradiction to (4).

The proof for the sequence $\{d_n\}$ is completely analogous , using instead of (5) the recursive relation

$$d_{n-k} = \frac{\alpha_n}{\alpha_{n-k}} d_n + \sum_{i=0}^{k-1} \frac{\alpha_{n-i}}{\alpha_{n-k}}.$$

'$\Leftarrow$' : Assume that the sequence $\{\alpha_n\}$ is asymptotically exponential. By definition this means that the families of asymptotically equal terms contain at most k_0 terms , where $k_0 \in N$. This implies that there exists infinite subsequence $\{n_i\} \subset N$ and $\lambda > 1$ such that $\forall\, i \in N$ $\frac{a_{n_i}}{a_{n_i+1}} \geq \lambda > 1$ and $n_{i+1} - n_i \leq k_0$, otherwise it would exists a family of more than k_0 asymptotically equal terms.

Now, let $n \in N$. Then there exist $i \in N$ such that $n_i \leq n < n_{i+1}$ or $a_{n_i} \geq \alpha_n > \alpha_{n_i+1}$. We have

$$c_n = \alpha_n \sum_{k=1}^{n} \frac{1}{\alpha_k} = \alpha_n \sum_{k=n_i+1}^{n} \frac{1}{\alpha_k} + \alpha_n \sum_{k=1}^{n_i} \frac{1}{\alpha_k} = \alpha_n \sum_{k=n_i+1}^{n} \frac{1}{\alpha_k} + \alpha_n \sum_{j=0}^{i-1} \sum_{k=n_j+1}^{n_{j+1}} \frac{1}{\alpha_k} \leq$$

$$\leq \alpha_n \sum_{k=n_i+1}^{n} \frac{1}{\alpha_n} + \alpha_{n_i} \sum_{j=0}^{i-1} \sum_{k=n_j+1}^{n_{j+1}} \frac{1}{\alpha_{n_j}} = n - n_i + \alpha_{n_i} \sum_{j=0}^{i-1} \frac{1}{\alpha_{n_j}} (n_{j+1} - n_j) \leq$$

$$k_0 + k_0 \alpha_{n_i} \sum_{j=0}^{i-1} \frac{1}{\alpha_{n_j}} = k_0 + k_0 \sum_{j=0}^{i-1} \frac{\alpha_{n_i}}{\alpha_{n_j}} \leq k_0 + k_0 \sum_{j=0}^{i-1} \frac{1}{\lambda^{i-j}} =$$

$$= k_0 + k_0 \sum_{j=1}^{i} \frac{1}{\lambda^j} \leq k_0 \sum_{j=0}^{\infty} \frac{1}{\lambda^j} = \frac{k_0 \lambda}{\lambda - 1} < \infty.$$

Again , the proof for the sequence $\{d_n\}$ is completely analogous , using instead of the equality

$$c_n = \alpha_n \sum_{k=1}^{n} \frac{1}{\alpha_k} = \alpha_n \sum_{k=n_i+1}^{n} \frac{1}{\alpha_k} + \alpha_n \sum_{j=0}^{i-1} \sum_{k=n_j+1}^{n_{j+1}} \frac{1}{\alpha_k},$$

the equality

$$d_n = \frac{1}{\alpha_n} \sum_{k=n+1}^{\infty} \alpha_k = \frac{1}{\alpha_n} \sum_{k=n+1}^{n_{i+1}-1} \alpha_k + \frac{1}{\alpha_n} \sum_{k=i+1}^{\infty} \sum_{j=n_k}^{n_{k+1}} \alpha_j \ .$$

■

Proof of the Theorem

$'' \Rightarrow''$ Let $\{z_n\}$ be a Blaschke sequence inside a Stolz angle and let $\{\alpha_n\}$ be asymptotically polynomial. According to the Theorem 3 it is enough to show that

$$\limsup_{r \to 1} C(r, \{z_n\}) = \infty \ . \tag{7}$$

We can easily see

$$1 - r \leq 1 - r\,|\,z_n\,| \leq |1 - rz_n\,| \tag{8}$$

and additionally

$$|\,z_n - r\,| = |\,(1 - r\,z_n) + (r\,z_n - r + z_n - 1)\,| \leq$$

$$\leq |\,1 - r\,z_n\,| + |\,(1+r)\,(1 - z_n)\,| \leq |\,1 - r\,z_n\,| + 2\,|\,1 - z_n\,| =$$

$$= |\,1 - r\,z_n\,| + 2\,|\,(1 - r\,z_n) + z_n(r-1)\,| \leq 5\,|\,1 - r\,z_n\,| \leq 5M\,(1 - r\,|z_n|) \tag{9}$$

The last inequality holds from Theorem hypothesis, that all z_n lie on an angular domain.

From (3), (8), (9) we have

$$C(r, \{z_n\}) \geq c \sum_{n=1}^{\infty} \frac{(1-r)(1-|z_n|)}{(1-r\,|z_n|)^2}$$

so, from (9) it suffices to show that

$$\limsup_{r\to 1} \sum_{n=1}^{\infty} \frac{(1-r)(1-|z_n|)}{(1-r\,|z_n|)^2} = \infty. \tag{10}$$

For $m \in N$ we set now $r_m = \frac{1}{1+a_m}$. It gives us

$$\limsup_{r\to 1} \sum_{n=1}^{\infty} \frac{(1-r)(1-|z_n|)}{(1-r\,|z_n|)^2} \geq \limsup_{m\to\infty} \sum_{n=1}^{\infty} \frac{(1-r_m)(1-|z_n|)}{(1-r_m\,|z_n|)^2} =$$

$$= \limsup_{m\to\infty} \sum_{n=1}^{\infty} \frac{\alpha_m \alpha_n (1+\alpha_m)}{(\alpha_m + \alpha_n)^2} \tag{11}$$

and since

$$\sum_{n=1}^{\infty} \frac{\alpha_m \alpha_n (1+\alpha_m)}{(\alpha_m+\alpha_n)^2} > \sum_{n=1}^{\infty} \frac{\alpha_m \alpha_n}{(\alpha_m+\alpha_n)^2} > \alpha_m \sum_{n=1}^{m} \frac{1}{\alpha_n} \frac{1}{(\frac{\alpha_m}{\alpha_n}+1)^2} > \frac{1}{4}\alpha_m \sum_{k=1}^{m} \frac{1}{\alpha_k} = \frac{c_m}{4}.$$

(10) follows immediately from (11) and our Lemma.

$'' \Leftarrow''$ From the Theorem 3 we get that

$$\sup_{r\in(0,1)} C(\,r\,,\,\{z_n\}\,) = \infty\ . \tag{12}$$

Let us define $\delta = 1 - r$. There exists a unique $m \in N$ such that $\alpha_m \geq \delta > \alpha_{m+1}$. Then we have

$$C(r, \{z_n\}) = \sum_{n=1}^{\infty} \frac{(1-r)(1-|z_n|)}{(1-r)^2 + |r - z_n|^2} \leq \sum_{n=1}^{\infty} \frac{(1-r)(1-|z_n|)}{(1-r)^2 + (r - |z_n|)^2} =$$

$$= \sum_{n=1}^{\infty} \frac{\delta\alpha_n}{\delta^2 + (\alpha_n - \delta)^2} = \sum_{n=1}^{m} \frac{\delta\alpha_n}{\delta^2 + (\alpha_n - \delta)^2} + \sum_{n=m+1}^{\infty} \frac{\delta\alpha_n}{\delta^2 + (\alpha_n - \delta)^2} \leq$$

$$\leq \alpha_m \sum_{n=1}^{m} \frac{1}{\alpha_n + 2\alpha_{m+1}(\frac{\alpha_{m+1}}{\alpha_n} - 1)} + \frac{1}{\alpha_{m+1}} \sum_{n=m+1}^{\infty} \frac{\alpha_n}{1 + (\frac{\alpha_n}{\delta} - 1)^2} \leq$$

$$\leq 2\alpha_m \sum_{n=1}^{m} \frac{1}{\alpha_n} + \frac{1}{\alpha_{m+1}} \sum_{n=m+1}^{\infty} \alpha_n = 2c_m + d_{m+1} + 1 , \tag{13}$$

where the sequences $\{c_n\}$ and $\{d_n\}$ are defined in Lemma. Inequality (13) and condition (12), show that at least one of $\{c_n\}$ and $\{d_n\}$ are not bounded. From our Lemma follows that $\{\alpha_n\}$ is asymptotically polynomial.

Counterexample

In the proof of Theorem A we show that if $\{\alpha_n\} = 1 - |z_n|$ is an asymptotically polynomial sequence where $\{z_n\}$ is lying inside a Stolz angle then

$$\limsup_{r \to 1} C(r, \{z_n\}) = \infty \tag{14}$$

Since Theorem 2 gives much better description for the specific boundary behavior, it is natural to ask the question if (14) can be reduced, at least for the specific cases of Theorem A, to

$$\lim_{r \to 1} C(r, \{z_n\}) = \infty$$

With the following counterexample we show that it is impossible.

Let $\rho_n = \frac{n\,(n+1)}{2}$ $n \in \mathbb{N}$, and $\alpha_n = \frac{1}{k!}$ when $\rho_{k-1} < n \leq \rho_k$. The sequence of zeros $z_n = 1 - \alpha_n$ is a Blashke sequence:

$$\sum_{n=1}^{\infty} \alpha_n = \sum_{i=1}^{\infty} \sum_{n=\rho_{i-1}+1}^{\rho_i} \frac{1}{i!} = \sum_{i=1}^{\infty} \frac{\rho_i - \rho_{i-1}}{i!} = \sum_{i=1}^{\infty} \frac{1}{(i-1)!} < \infty \ .$$

The number of succesive terms of the sequence $\{\alpha_n\}$ taking the same value is arbitrary large. Then we conclude that $\{\alpha_n\}$ is asymptotically polynomial and from Lemma we obtain that $\{c_n\}$ is not bounded. We have:

$$c_{\rho_n+1} = \alpha_{\rho_n+1} \sum_{k=1}^{\rho_n+1} \frac{1}{\alpha_k} = 1 + \frac{1}{(n+1)!} \sum_{\rho_0+1}^{\rho_n} \frac{1}{\alpha_k} =$$

$$= 1 + \frac{1}{(n+1)!} \sum_{i=1}^{n} \sum_{k=\rho_{i-1}+1}^{\rho_i} \frac{1}{\alpha_k} = 1 + \frac{1}{(n+1)!} \sum_{i=1}^{n} i!\, i =$$

$$= 2 - \frac{1}{(n+1)!} \to 2 \quad \text{when} \quad n \to \infty$$

Consequently, $\liminf_{n \to \infty} c_n < \infty$. And from (13) we get that

$$\liminf_{r \to 1} C(r, \{z_n\}) < \infty$$

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Received 12.12.2008

J o u r n a l of
Mathematics
and Applications

No 31, pp 99-106 (2009)

Subordinations and superordinations using the Dziok-Srivastava linear operator

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Abstract: By using the properties of the Dziok-Srivastava linear operator we obtain differential subordinations and superordinations by using functions from class $\mathcal{A}$. A sandwich-type result is also given. Theorem 1 from the paper gives sufficient conditions such that a function $f \in \mathcal{A}$ to be starlike, convex and α-convex.

AMS Subject Classification: *30C45, 30A10, 30C80*
Key Words and Phrases: *univalent functions, starlike functions, convex functions, differential subordination, differential superordination, Dziok-Srivastava linear operator*

1. Introduction and preliminaries

Let U denote the unit disc of the complex plane:

$$U = \{z \in \mathbb{C} : \ |z| < 1\}$$

and

$$\overline{U} = \{z \in \mathbb{C} : \ |z| \leq 1\}.$$

Let $\mathcal{H}$ denote the class of analytic functions defined on the open unit disk U.
Let $\mathcal{H}(U)$ denote the space of holomorphic functions in U and let

$$A_n = \{f \in \mathcal{H}(U), \ f(z) = z + a_{n+1}z^{n+1} + \dots, \ z \in U\}$$

with $A_1 = A$.
Let

$$\mathcal{H}[a, n] = \{f \in \mathcal{H}(U), \ f(z) = a + a_n z^n + a_{n+1}z^{n+1} + \dots, \ z \in U\},$$

$$S = \{f \in A; \ f \text{ is univalent in } U\}.$$

Let

$$K = \left\{ f \in A,\ \operatorname{Re} \frac{zf''(z)}{f'(z)} + 1 > 0,\ z \in U \right\},$$

denote the class of normalized convex function in U

$$S^* = \left\{ f \in A:\ \operatorname{Re} \frac{zf'(z)}{f(z)} > 0,\ z \in U \right\}$$

denote the class of starlike functions in U, and

$$M_\alpha = \left\{ f \in \mathcal{A}:\ \operatorname{Re} \left[(1-\alpha)\frac{zf'(z)}{f(z)} + \alpha \left(\frac{zf''(z)}{f'(z)} + 1 \right) \right] > 0 \right\}, \quad z \in U$$

denote the class of α-convex (Mocanu functions), with α real.

If f and g are analytic functions in U, then we say that f is subordinate to g, written $f \prec g$, if there is a function w analytic in U, with $w(0) = 0$, $|w(z)| < 1$, for all $z \in U$ such that $f(z) = g[w(z)]$, for $z \in U$.

If g is univalent, then $f \prec g$ if and only if $f(0) = g(0)$ and $f(U) \subset g(U)$.

The method of differential subordinations (also known as the admissible functions method) was introduced by P.T. Mocanu and S.S. Miller in 1978 [2] and 1981 [3] and developed in [4].

Let Ω and Δ be any sets in $\mathbb{C}$ and let p be an analytic function in the unit disk with $p(0) = a$ and let $\psi : \mathbb{C}^3 \times U \to \mathbb{C}$. The heart of this theory deals with generalizations of the following implication:

(i) $\{\psi(p(z), zp'(z), z^2p''(z); z) \mid z \in U\} \subset \Omega$ implies $p(U) \subset \Delta$.

Definition 1. [4, p.16] Let $\psi : \mathbb{C}^3 \times U \to \mathbb{C}$ and let h be univalent in U. If p is analytic in U and satisfies the (second-order) differential subordination

(ii) $\psi(p(z), zp'(z), z^2p''(z); z) \prec h(z)$, $z \in U$,

then p is called a solution of the differential subordination. The univalent function q is called a dominant of the solutions of the differential subordination, or more simply a dominant, if $p \prec q$ for all p satisfying (ii). A dominant $\widetilde{q}$ that satisfies $\widetilde{q} \prec q$ for all dominants q of (ii) is said to be the best dominant of (ii). (Note that the best dominant is unique up to a rotation of U).

In [5] the authors introduce the dual problem of the differential subordination which they call differential superordination.

Definition 2. [5] Let $f, F \in \mathcal{H}(U)$ and let F be univalent in U. The function F is said to be superordinate to f, or f is subordinate to F, written $f \prec F$, if $f(0) = F(0)$ and $f(U) \subset F(U)$.

Let Ω and Δ be any sets in $\mathbb{C}$ and let p be an analytic function in the unit disk and function $\varphi(r, s, t; z) : \mathbb{C}^3 \times U \to \mathbb{C}$. The heart of this theory deals with generalizations of the following implication:

(iii) $\Omega \subset \{\varphi(p(z), zp'(z), z^2p''(z); z) \mid z \in U\}$ implies $\Delta \subset p(U)$.

Definition 3. [5] Let $\varphi : \mathbb{C}^3 \times U \to \mathbb{C}$ and let h be analytic in U. If p and $\varphi(p(z), zp'(z), z^2p''(z); z)$ are univalent in U and satisfy the (second-order) differential superordination

(iv) $h(z) \prec \varphi(p(z), zp'(z), z^2p''(z); z)$
then p is called a solution of the differential superordination. An analytic function q is called a subordinant of the solutions of the differential superordination, or more simply a subordinant if $q \prec p$ for all p satisfying (iv). A univalent subordinant $\widetilde{q}$ that satisfies $q \prec \widetilde{q}$ for all subordinants q of (iv) is said to be the best subordinant. (Note that the best subordinant is unique up to a rotation of U).

Definition 4. [4, Definition 2.2b. p. 21] We denote by Q the set of functions f that are analytic and injective on $\overline{U} \setminus E(f)$, where

$$E(f) = \left\{\zeta \in \partial U;\ \lim_{z\to\zeta} f(z) = \infty\right\}$$

and are such that $f'(\zeta) \neq 0$ for $\zeta \in \partial U \setminus E(f)$.

The subclass of Q for which $f(0) = a$ is denoted by $Q(a)$.

In order to prove the new results we shall use the following lemmas:

Lemma A. [4, Theorem 3.4h, pp. 132] *Let q be univalent in U and let θ and ϕ be analytic in a domain D containing $q(U)$, with $\phi(w) \neq 0$, when $w \in q(U)$. Set*

$$Q(z) := zq'(z)\phi[q(z)], \quad h(z) := \theta[q(z)] + Q(z)$$

and suppose that either
(j) h is convex, or
(jj) Q is starlike.
In addition, assume that
(jjj) $\mathrm{Re}\,\dfrac{zh'(z)}{Q(z)} = \mathrm{Re}\left[\dfrac{\theta'[q(z)]}{\phi[q(z)]} + \dfrac{zQ'(z)}{Q(z)}\right] > 0.$
If p is analytic in U, with $p(0) = q(0)$, $p(U) \subset D$ and

$$\theta[p(z)] + zp'(z)\phi[p(z)] \prec \theta[q(z)] + zq'(z)\phi[q(z)] = h(z)$$

then $p \prec q$, and q is the best dominant.

Lemma B. [6, Th.3] *Let q be univalent in U, with $q(0) = a$ and θ and φ be analytic in a domain D containing $q(U)$. Define*

$$Q(z) = zq'(z)\varphi[q(z)], \quad h(z) = \theta[q(z)] + Q(z).$$

Suppose that:
(i) $\mathrm{Re}\left[\dfrac{\theta'[q(z)]}{\varphi[q(z)]}\right] > 0$ *and*
(ii) Q is starlike univalent in U.
If $p \in \mathcal{H}[a, 1] \cap Q$, with $p(U) \subset D$, and $\theta[p(z)] + zp'(z)\varphi[p(z)]$ is univalent in U, then

$$\theta[q(z)] + zq'(z)\varphi[q(z)] \prec \theta[p(z)] + zp'(z)\varphi[p(z)]$$

implies $q(z) \prec p(z)$ and $q(z)$ is the best subordinant.

In the paper [1] was defined the Dziok-Srivastava operator:

$$H_m^l(\alpha_1, \alpha_2, \dots, \alpha_l; \beta_1, \beta_2, \dots, \beta_m)f(z) \tag{1}$$
$$= z + \sum_{n=2}^{\infty} \frac{(\alpha_1)_{n-1}(\alpha_2)_{n-1}\dots(\alpha_l)_{n-1}}{(\beta_1)_{n-1}(\beta_2)_{n-1}\dots(\beta_l)_{n-1}} \cdot a_n \cdot \frac{z^n}{(n-1)!}$$

For simplicity, we write

$$H_m^l[\alpha_1]f(z) = H_m^l(\alpha_1, \alpha_2, \dots, \alpha_l; \beta_1, \beta_2, \dots, \beta_m)f(z).$$

For this operator we have the property

$$\alpha_1 H_m^l[\alpha_1+1]f(z) = z\{H_m^l[\alpha_1]f(z)\}' + (\alpha_1 - 1)H_m^l[\alpha_1]f(z) \tag{2}$$

2. Main results

Theorem 1. *Let* $0 \le \lambda \le 1$, $l, m \in \mathbb{N}_0 = \{0, 1, 2, \dots\}$, $l \le m+1$, $\alpha_i \in \mathbb{C}$, $i = 1, 2, \dots, l$, $\beta_j \in \mathbb{C} \setminus \{0, -1, -2, -3, \dots\}$, $j = 1, 2, \dots, m$, $\gamma \in \mathbb{C}^*$, *with* $\mathrm{Re}\,\frac{1}{\gamma} > 0$, $f \in \mathcal{A}$ *and* $H_m^l[\alpha_1]f(z)$ *the Dziok-Srivastava linear operator given by (1).*

We let

$$F(\lambda, l, m, \gamma, \alpha_1; z) = \tag{3}$$
$$(1-\lambda)\left[1 - \frac{1}{\alpha_1} + \frac{\gamma+1}{\alpha_1} z \frac{[H_m^l[\alpha_1]f(z)]'}{H_m^l[\alpha_1]f(z)} + \frac{\gamma}{\alpha_1} z^2 \left(\frac{[H_m^l[\alpha_1]f(z)]'}{H_m^l[\alpha_1]f(z)}\right)'\right]$$
$$+\lambda\left[1 + \frac{\gamma+1}{\alpha_1} z \frac{[H_m^l[\alpha_1]f(z)]''}{[H_m^l[\alpha_1]f(z)]'} + \frac{\gamma}{\alpha_1} z^2 \left(\frac{[H_m^l[\alpha_1]f(z)]''}{[H_m^l[\alpha_1]f(z)]'}\right)'\right]$$

and

$$h(z) = \frac{1-z}{1+z} - \frac{2\gamma z}{(1+z)^2}, \quad z \in U. \tag{4}$$

If $f \in \mathcal{A}$, $\alpha_1 > 0$, $\frac{H_m^l[\alpha_1]f(z)}{z} \neq 0$ *in* U, *and the differential subordination*

$$F(\lambda, l, m, \gamma, \alpha_1; z) \prec h(z), \tag{5}$$

holds, where $F(\lambda, l, m, \gamma, \alpha_1; z)$ *is given by (3), and* h *is given by (4), then*

$$(1-\lambda)\frac{H_m^l[\alpha_1+1]f(z)}{H_m^l[\alpha_1]f(z)} + \lambda\frac{[H_m^l[\alpha_1+1]f(z)]'}{[H_m^l[\alpha_1]f(z)]'} \prec q(z) = \frac{1-z}{1+z},$$

and g *is the best dominant.*

Proof. Let

$$p(z) = (1-\lambda)\frac{H_m^l[\alpha_1+1]f(z)}{H_m^l[\alpha_1]f(z)} + \lambda\frac{[H_m^l[\alpha_1+1]f(z)]'}{[H_m^l[\alpha_1]f(z)]'}, \quad z \in U. \tag{6}$$

Using (1) in (6), we obtain

$$p(z) = 1 + p_1 z + p_2 z^2 + \dots, \quad p(0) = 1, \quad p \in \mathcal{H}[1,1].$$

Differentiating (6), we have

$$\gamma z p'(z) = \gamma(1-\lambda) z \left(\frac{H_m^l[\alpha_1+1]f(z)}{H_m^l[\alpha_1]f(z)} \right)' + \gamma\lambda z \left(\frac{[H_m^l[\alpha_1+1]f(z)]'}{[H_m^l[\alpha_1]f(z)]'} \right)'. \tag{7}$$

From (6) and (7) and using the property (2), we obtain

$$p(z) + \gamma z p'(z) = F(\lambda, l, m, \gamma, \alpha_1; z) \tag{8}$$

where F is given by (3).

Using (8), the differential subordination (5) becomes

$$p(z) + \gamma z p'(z) \prec h(z), \quad z \in U, \tag{9}$$

where h is given by (4).

In order to prove the theorem, we use Lemma A. For that let

$$q(z) = \frac{1-z}{1+z}, \quad q(0) = 1, \quad q(U) = \{w \in \mathbb{C} : \ \mathrm{Re}\ w \geq 0\}.$$

Define the functions

$$\theta : D \supset q(U) \to \mathbb{C}, \quad \theta(w) = w,$$

and

$$\phi : D \supset q(U) \to \mathbb{C}, \quad \phi(w) = \gamma \in \mathbb{C}^*, \quad \mathrm{Re}\ \frac{1}{\gamma} \geq 0.$$

We calculate:

$$Q(z) = z q'(z) \cdot \phi[q(z)] = \frac{-2\gamma z}{(1+z)^2},$$

and we have

$$\mathrm{Re}\ \frac{zQ'(z)}{Q(z)} = \mathrm{Re}\ \frac{1-z}{1+z} \geq 0, \quad z \in U, \text{ i.e. } Q \in S^*.$$

Also,

$$h(z) = \theta[q(z)] + Q(z) = q(z) + \gamma z q'(z) = \frac{1-z}{1+z} - \frac{2\gamma z}{(1+z)^2},$$

and we have

$$\mathrm{Re}\ \frac{zh'(z)}{Q(z)} = \mathrm{Re}\ \frac{1}{\gamma} + \mathrm{Re}\ \frac{zQ'(z)}{Q(z)} > 0, \quad z \in U.$$

Using Lemma A, we obtain

$$p(z) \prec q(z),$$

i.e.

$$(1-\lambda)\frac{H_m^l[\alpha_1+1]f(z)}{H_m^l[\alpha_1]f(z)}+\lambda\frac{[H_m^l[\alpha_1+1]f(z)]'}{[H_m^l[\alpha_1]f(z)]'}\prec\frac{1-z}{1+z}=q(z),\quad z\in U,$$

and q is the best dominant.

Remark 1. For $0\le\lambda\le1$, $\gamma\in\mathbb{C}^*$, $\mathrm{Re}\,\dfrac{1}{\gamma}\ge0$, $l=1$, $m=0$, $\alpha_1=1$ we have

$$H_0^1[1]f(z)=f(z),\quad H_0^1[2]f(z)=zf'(z)$$

and Theorem 1 can be rewritten as the following

Corollary 1. *If $f\in\mathcal{A}$, with $\dfrac{f(z)}{z}\neq0$, $z\in U$, and the differential subordination*

$$(1-\lambda)\left\{(1+\gamma)\frac{zf'(z)}{f(z)}+\gamma\frac{z^2f''(z)}{f(z)}-\gamma\left(\frac{zf'(z)}{f(z)}\right)^2\right\}$$

$$+\lambda\left\{1+(1+\gamma)\frac{zf''(z)}{f'(z)}+\gamma\frac{z^2f'''(z)}{f'(z)}-\gamma\left(\frac{zf''(z)}{f'(z)}\right)^2\right\}$$

$$\prec h(z)=\frac{1-z}{1+z}-\frac{2\gamma z}{(1+z)^2},$$

holds, then

$$\mathrm{Re}\,\left\{(1-\lambda)\frac{zf'(z)}{f(z)}+\lambda\left(1+\frac{zf''(z)}{f'(z)}\right)\right\}>0,\quad z\in U,$$

i.e. $f\in M_\lambda$ (class of Mocanu functions).

Remark 2. For $l=1$, $m=0$, $\alpha_1=1$, $\lambda=0$, $\gamma\in\mathbb{C}^*$, $\mathrm{Re}\,\dfrac{1}{\gamma}>0$, from Theorem 1 we get the following corollary:

Corollary 2. *Let $f\in\mathcal{A}$, be so that $\dfrac{f(z)}{z}\neq0$, $z\in U$ and let the following differential subordination*

$$(1+\gamma)\frac{zf'(z)}{f(z)}+\gamma\frac{z^2f''(z)}{f(z)}-\gamma\left(\frac{zf'(z)}{f(z)}\right)^2\prec h(z)=\frac{1-z}{1+z}-\frac{2\gamma z}{(1+z)^2}.$$

Then

$$\mathrm{Re}\,\frac{zf'(z)}{f(z)}>0,\quad z\in U,\ \textit{i.e. } f\in S^*.$$

Remark 3. For $l=1$, $m=0$, $\alpha_1=1$, $\lambda=1$, $\gamma\in\mathbb{C}^*$, $\mathrm{Re}\,\dfrac{1}{\gamma}\ge0$, from Theorem 1 we get the following corollary:

Corollary 3. *Let $f \in \mathcal{A}$, be so that $\dfrac{f(z)}{z} \neq 0$, $z \in U$ and let the following differential subordination*

$$1 + (1+\gamma)\frac{zf''(z)}{f'(z)} + \gamma\frac{z^2 f'''(z)}{f'(z)} - \gamma\left(\frac{zf''(z)}{f'(z)}\right)^2 \prec h(z) = \frac{1-z}{1+z} - \frac{2\gamma z}{(1+z)^2}.$$

Then

$$\mathrm{Re}\,\left(1 + \frac{zf''(z)}{f'(z)}\right) \geq 0, \quad z \in U, \ \textit{i.e. } f \in K.$$

Theorem 2. *Let $0 \leq \lambda \leq 1$, $0 \leq \alpha < 1$, $l, m \in \mathbb{N}_0 = \{0, 1, 2, \dots\}$, $l \leq m+1$, $\alpha_i \in \mathbb{C}$, $i = 1, 2, 3, \dots, l$, $\beta_j \in \mathbb{C} \setminus \{0, -1, -2, \dots\}$, $j = 1, 2, \dots, m$, $\gamma \in \mathbb{C}^*$, $\mathrm{Re}\,\dfrac{1}{\gamma} \geq 0$, and $H_m^l[\alpha_1]f(z)$ the Dziok-Srivastava linear operator given by (1).*

If $f \in \mathcal{A}$, $\dfrac{H_m^l[\alpha_1]f(z)}{z} \neq 0$, $z \in U$, the function $F(\lambda, \gamma, m, l, \alpha_1; z)$ given by (3) is univalent in U, then

$$h(z) = \frac{1 + (1-2\alpha)z}{1-z} + \frac{2(1-\alpha)\gamma z}{(1-z)^2} \prec F(\lambda, \gamma, m, l, \alpha_1; z) \tag{10}$$

implies

$$q(z) = \frac{1 + (1-2\alpha)z}{1-z} \prec p(z), \quad z \in U,$$

where p is given by (6).

The function q is the best dominant.

Proof. Let

$$q(z) = \frac{1 + (1-2\alpha)z}{1-z},$$

then

$$h(z) = q(z) + \gamma z q'(z) = \frac{1 + (1-2\alpha)z}{1-z} + \frac{2(1-\alpha)\gamma z}{(1-z)^2}, \quad z \in U.$$

Using (8), (10) becomes

$$q(z) + \gamma z q'(z) \prec p(z) + \gamma z p'(z), \quad z \in U,$$

where p is given by (6), and $p(0) = 1$, $p \in \mathcal{H}[1, 1] \cap Q$.

Using Lemma B, we have

$$q(z) \prec p(z),$$

i.e.

$$\frac{1 + (1-2\alpha)^2}{1-z} \prec (1-\lambda)\frac{H_m^l[\alpha_1+1]f(z)}{H_m^l[\alpha_1]f(z)} + \lambda\frac{[H_m^l[\alpha_1+1]f(z)]'}{[H_m^l[\alpha_1]f(z)]'}, \quad z \in U,$$

and q is the best dominant.

Using the conditions from Theorem 1 and Theorem 2, we obtain the following sandwich-type result:

Corollary 4. *If $f \in \mathcal{A}$ and $\dfrac{H_m^l[\alpha_1]f(z)}{z} \neq 0$, then*

$$\frac{1+(1-2\alpha)z}{1-z} + \frac{2(1-\alpha)\gamma z}{(1-z)^2} \prec F(\lambda, \gamma, m, l, \alpha_1; z) \prec \frac{1-z}{1+z} - \frac{2\gamma z}{(1+z)^2}$$

implies

$$\frac{1+(1-2\alpha)z}{1-z} \prec p(z) \prec \frac{1-z}{1+z}, \quad z \in U, \quad \gamma \in \mathbb{C}^*, \quad \mathrm{Re}\,\frac{1}{\gamma} \geq 0,$$

where F is given by (3) and p is given by (6).

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Received 29.01.2009

J o u r n a l of
Mathematics
and Applications
No 31, pp 107-115 (2009)

The study of a class of univalent functions defined by Ruscheweyh differential operator

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Abstract: By using a certain operator D^n, we introduce a class of holomorphic functions $M_n(h)$, h convex function, and we obtain some subordination results. We also show that, for $h(z) \equiv \alpha$, $0 \leq \alpha < 1$ and $z \in U$, the set $M_n(\alpha)$ is convex and we obtain some new differential subordinations related to certain integral operators.

AMS Subject Classification: Primary *30C80,* Secondary *30C45, 30A20, 34A40*
Key Words and Phrases: *differential operators, differential subordination, dominant, best dominant*

1. Introduction and preliminaries

Denote by U the unit disc of the complex plane :

$$U = \{z \in \mathbb{C} : \ |z| < 1\}.$$

Let $\mathcal{H}(U)$ be the space of holomorphic functions in U.
Let

$$\mathcal{A}_n = \{f \in \mathcal{H}(U),\ f(z) = z + a_{n+1}z^{n+1} + \ldots,\ z \in U\}$$

with $\mathcal{A}_1 = \mathcal{A}$.
For $a \in \mathbb{C}$ and $n \in \mathbb{N}$, let

$$\mathcal{H}[a, n] = \{f \in \mathcal{H}(U),\ f(z) = a + a_n z^n + a_{n+1}z^{n+1} + \ldots,\ z \in U\}.$$

Let

$$K = \left\{ f \in \mathcal{A},\ \text{Re}\, \frac{zf''(z)}{f'(z)} + 1 > 0,\ z \in U \right\},$$

denote the class of convex functions in U.

A function f, analytic in U, is said to be convex if it is univalent and $f(U)$ is convex.

If f and g are analytic functions in U, then we say that f is subordinate to g, written $f \prec g$, if there is a function w analytic in U, with $\omega(0) = 0$, $|\omega(z)| < 1$, for all $z \in U$ such that $f(z) = g[\omega(z)]$ for $z \in U$. If g is univalent, then $f \prec g$ if and only if $f(0) = g(0)$ and $f(U) \subseteq g(U)$.

Let $\psi : \mathbb{C}^3 \times U \to \mathbb{C}$ and let h be univalent in U. If p is analytic in U and satisfies the (second-order) differential subordination

$$\psi(p(z), zp'(z), z^2p''(z); z) \prec h(z), \quad z \in U \tag{i}$$

then p is called a solution of the differential subordination.

The univalent function q is called a dominant of the solution of the differential subordination, or more simply a dominant, if $p \prec q$ for all p satisfying (i).

A dominant $\widetilde{q}$ that satisfies $\widetilde{q} \prec q$ for all dominants q of (i) is said to be the best dominant of (i). (Note that the best dominant is unique up to a rotation of U).

In order to prove the original results we use the following lemmas:

Lemma A. [1, Lemma 1.4] *Let q be convex function in U with $q(0) = 1$ and let* $\mathrm{Re}\ c > 0$. *Let*

$$h(z) = q(z) + \frac{n}{c} zq'(z).$$

If $p(z) = 1 + p_n z^n + p_{n+1} z^{n+1} + \dots$ is analytic in U and

$$p(z) + \frac{1}{c} zp'(z) \prec h(z),$$

then

$$p(z) \prec q(z)$$

and q is the best dominant.

Lemma B. [1, Lema 1.5] *Let* $\mathrm{Re}\ \gamma > 0$ *and let*

$$\omega = \frac{k^2 + |\gamma|^2 - |k^2 - \gamma^2|}{4k \mathrm{Re}\ \gamma}.$$

Let h be an analytic function in U with $h(0) = 1$ and suppose that

$$\mathrm{Re}\ \left(\frac{zh''(z)}{h'(z)} + 1\right) > -\omega.$$

If

$$p(z) = 1 + p_k z^k + p_{k+1} z^{k+1} + \dots$$

is analytic in U and

$$p(z) + \frac{1}{\gamma} zp'(z) \prec h(z),$$

then $p(z) \prec q(z)$, where q is solution of the differential equation

$$q(z) + \frac{k}{\gamma} z q'(z) = h(z), \quad q(0) = 1,$$

given by

$$q(z) = \frac{\gamma}{k z^{\gamma/k}} \int_0^z t^{\frac{\gamma}{k}-1} h(t) dt.$$

Moreover q is the best dominant.

Definition 1. (St. Ruscheweyh [3]) For $f \in \mathcal{A}$, $n \in \mathbb{N}^* \cup \{0\}$, the operator D^n is defined by $D^n : \mathcal{A} \to \mathcal{A}$

$$\begin{array}{l} D^0 f(z) = f(z) \\ D^1 f(z) = z f'(z) \\ \dots \\ (n+1) D^{n+1} f(z) = z[D^n f(z)]' + n D^n f(z),\ z \in U, \end{array}$$

this is Ruscheweyh differential operator.

Remark 1. [2] If $f \in \mathcal{A}$, $f(z) = z + \sum_{j=2}^{\infty} a_j z^j$ then

$$D^n f(z) = z + \sum_{j=2}^{\infty} C_n^{n+j-1} a_j z^j, \quad z \in U.$$

Definition 2. For $h \in \mathcal{K}$ and $n \in \mathbb{N}$, we let $M_n(h)$ denote the class of functions $f \in \mathcal{A}$ which satisfy the inequality:

$$\mathrm{Re}\ [D^n f(z)]' \prec h(z), \quad z \in U.$$

2. Main results

Theorem 1. *The set $M_n(\alpha)$ is convex , $0 \le \alpha < 1$.*

Proof. Let the functions

$$f_i(z) = z + \sum_{k=2}^{\infty} a_{ki} z^k, \quad i = 1, 2, \quad z \in U$$

be in the class $M_n(\alpha)$. It is sufficient to show that the function

$$h(z) = \mu_1 f_1(z) + \mu_2 f_2(z)$$

with μ_1 and μ_2 nonnegative and $\mu_1 + \mu_2 = 1$, is in $M_n(\alpha)$.

Since

$$h(z) = z + \sum_{k=2}^{\infty} (\mu_1 a_{k1} + \mu_2 a_{k2}) z^k, \quad z \in U$$

then

$$D^n h(z) = z + \sum_{k=2}^{\infty} C_n^{n+k-1} (\mu_1 a_{k1} + \mu_2 a_{k2}) z^k, \quad z \in U. \tag{1}$$

Differentiating (1), we have

$$[D^n h(z)]' = 1 + \sum_{k=2}^{\infty} k C_n^{n+k-1} (\mu_1 a_{k1} + \mu_2 a_{k2}) z^{k-1}, \tag{2}$$

hence

$$\text{Re } [D^n h(z)]' = \text{Re } \left[1 + \sum_{k=2}^{\infty} k C_n^{n+k-1} (\mu_1 a_{k1} + \mu_2 a_{k2}) z^{k-1}\right] \tag{3}$$

$$= \text{Re } \left[1 + \sum_{k=2}^{\infty} k C_n^{n+k-1} \mu_1 a_{k1} z^{k-1} + \sum_{k=2}^{\infty} k C_n^{n+k-1} \mu_2 a_{k2} z^{k-1}\right] \tag{4}$$

$$= 1 + \text{Re } \left[\mu_1 \sum_{k=2}^{\infty} k C_n^{n+k-1} a_{k1} z^{k-1}\right] + \text{Re } \left[\mu_2 \sum_{k=2}^{\infty} k C_n^{n+k-1} a_{k2} z^{k-1}\right]. \tag{5}$$

Since $f_1, f_2 \in M_n(\alpha)$, we obtain

$$\text{Re } \left[\mu_i \sum_{k=2}^{\infty} k C_n^{n+k-1} a_{ki} z^{k-1}\right] > \mu_i(\alpha - 1), \quad i = 1, 2. \tag{4}$$

Using (4) in (3), we obtain

$$\text{Re } [D^n h(z)]' > 1 + \mu_1(\alpha - 1) + \mu_2(\alpha - 1), \quad z \in U,$$

and since $\mu_1 + \mu_2 = 1$, we deduce

$$\text{Re } [D^n h(z)]' > \alpha$$

i.e. $M_n(\alpha)$ is convex.

Theorem 2. *Let q be a convex function in U, with $q(0) = 1$, and let*

$$h(z) = q(z) + \frac{1}{c+2} z q'(z), \quad z \in U$$

where c is a complex number, with Re $c > -2$.

If $f \in M_n(h)$ and $F = I_c(f)$, where

$$F(z) = I_c(f)(z) = \frac{c+2}{z^{c+1}} \int_0^z t^c f(t) dt, \quad \text{Re } c > -2, \tag{5}$$

then

$$[D^n f(z)]' \prec h(z), \quad z \in U, \tag{6}$$

implies

$$[D^n F(z)]' \prec q(z), \quad z \in U,$$

and this result is sharp.

Proof. From (5), we deduce

$$z^{c+1} F(z) = (c+2) \int_0^z t^c f(t) dt, \quad \mathrm{Re}\, c > -2, \quad z \in U. \tag{7}$$

Differentiating (7), with respect to z, we obtain

$$(c+1)F(z) + zF'(z) = (c+2)f(z), \quad z \in U. \tag{8}$$

We are studying the property of linearity of D^n operator.

Let

$$\alpha f(z) + \beta g(z) = (\alpha + \beta)z + \sum_{j=2}^{\infty} (\alpha a_j + \beta b_j) z^j, \quad z \in U,$$

applying D^n operator, we have

$$D^n[\alpha f(z) + \beta g(z)] = (\alpha + \beta)z + \sum_{j=2}^{\infty} \frac{n!}{(n+j-1)!(j-1)!} (\alpha a_j + \beta b_j) z^j,$$

and we obtain

$$D^n \alpha f(z) = \alpha z + \sum_{j=2}^{\infty} \frac{n!}{(n+j-1)!(j-1)!} \alpha a_j z^j = \alpha D^n f(z),$$

$$D^n \beta g(z) = \beta z + \sum_{j=2}^{\infty} \frac{n!}{(n+j-1)!(j-1)!} \beta b_j z^j = \beta D^n g(z),$$

hence

$$D^n[\alpha f(z) + \beta g(z)] = D^n \alpha f(z) + D^n \beta g(z) = \alpha D^n f(z) + \beta D^n g(z), \quad z \in U. \tag{9}$$

We show that

$$D^n(zF'(z)) = z[D^n F(z)]', \quad z \in U.$$

Let

$$F(z) = z + A_2 z^2 + \cdots = z + \sum_{j=2}^{\infty} A_j z^j.$$

Differentiating with respect to z, we have

$$F'(z) = 1 + \sum_{j=2}^{\infty} j A_j z^{j-1},$$

and

$$zF'(z) = z + \sum_{j=2}^{\infty} jA_j z^j.$$

Applying D^n differential operator,we obtain

$$D^n(zF'(z)) = z + \sum_{j=2}^{\infty} C_n^{n+j-1} jA_j z^j, \quad z \in U. \tag{10}$$

Using Remark 1, we have

$$D^n F(z) = z + \sum_{j=2}^{\infty} C_n^{n+j-1} A_j z^j, \quad z \in U.$$

Differentiating with respect to z, we have

$$[D^n F(z)]' = 1 + \sum_{j=2}^{\infty} jC_n^{n+j-1} A_j z^{j-1},$$

and we obtain

$$z[D^n F(z)]' = z + \sum_{j=2}^{\infty} jC_n^{n+j-1} A_j z^j, \quad z \in U. \tag{11}$$

From (10) and (11) , we obtain

$$D^n(zF'(z)) = z[D^n F(z)]', \quad z \in U. \tag{12}$$

Using D^n differential operator and properties (9),(10),(11),(12) in (8)we deduce

$$(c+1)D^n F(z) + z[D^n F(z)]' = (c+2)D^n f(z), \quad z \in U. \tag{13}$$

Differentiating (13), we have

$$[D^n F(z)]' + \frac{z}{c+2}[D^n F(z)]'' = [D^n f(z)]', \quad z \in U. \tag{14}$$

Using (14), the differential subordination (6) becomes

$$[D^n F(z)]' + \frac{1}{c+2} z[D^n F(z)]'' \prec h(z) = q(z) + \frac{1}{c+2} zq'(z). \tag{15}$$

Let

$$p(z) = [D^n F(z)]' = \left[z + \sum_{j=2}^{\infty} C_n^{n+j-1} b_k z^k \right]' \tag{16}$$

$$= 1 + p_1 z + p_2 z^2 + \dots, \quad p \in \mathcal{H}[1,1].$$

Using (16) in (15), we have

$$p(z) + \frac{1}{c+2} z p'(z) \prec h(z) = q(z) + \frac{1}{c+2} z q'(z), \quad z \in U. \tag{17}$$

Using Lemma A, we obtain $p(z) \prec q(z)$, i.e.

$$[D^n F(z)]' \prec q(z), \quad z \in U,$$

and q is the best dominant.

Example 1. If we let $c = 1 + i$ and $q(z) = \dfrac{1}{1-z}$, then

$$h(z) = \frac{3 + i - z(2+i)}{(3+i)(1-z)^2}$$

and from Theorem 2,we deduce that if $f \in M_n(h)$ and F is given by

$$F(z) = \frac{3+i}{z^{2+i}} \int_0^z t^{1+i} f(t) dt \tag{18}$$

then

$$[D^n f(z)]' \prec \frac{3 + i - z(2+i)}{(3+i)(1-z)^2}, \quad z \in U,$$

implies

$$[D^n F(z)]' \prec \frac{1}{1-z}, \quad z \in U,$$

where F is given by (18).

Theorem 3. *Let* $\mathrm{Re}\ c > -2$ *and let*

$$\omega = \frac{1 + |c+2|^2 - |c^2 + 4c + 3|}{4 \cdot \mathrm{Re}\ (c+2)} \tag{19}$$

Let h be an analytic function in U, with $h(0) = 1$ and suppose that

$$\mathrm{Re}\ \frac{z h''(z)}{h'(z)} + 1 > -\omega.$$

If $f \in M_n(h)$ and $F = I_c(f)$, where F is defined by (5), then

$$[D^n f(z)]' \prec h(z), \quad z \in U, \tag{20}$$

implies

$$[D^n F(z)]' \prec q(z), \quad z \in U,$$

where q is the solution of the differential equation

$$q(z) + \frac{1}{c+2} z q'(z) = h(z), \quad h(0) = 1,$$

given by

$$q(z) = \frac{c+2}{z^{c+2}} \int_0^z t^{c+1} h(t) dt, \quad z \in U.$$

Moreover q is the best dominant.

Proof. In order to prove Theorem 3 we will use Lemma B. The value of ω is given by (19). From (17) we have

$$p(z) = [D^n F(z)]' = 1 + p_1 z + p_2 z^2 + \cdots, \quad p \in \mathcal{H}[1,1] \quad (z \in U).$$

Using Lemma B, we deduce $k = 1$. Using (15) and (17), the differential subordination (16) becomes

$$p(z) + \frac{1}{c+2} z p'(z) \prec h(z) = q(z) + \frac{1}{c+2} z q'(z), \quad z \in U. \tag{21}$$

From the subordination (21), by using Lemma B, we deduce $r = c + 2$ and

$$p(z) \prec q(z), \quad z \in U,$$

where

$$q(z) = \frac{c+2}{z^{c+2}} \int_0^z t^{c+1} h(t) dt, \quad z \in U,$$

i.e.

$$[D^n F(z)]' \prec q(z) = \frac{c+2}{z^{c+2}} \int_0^z t^{c+1} h(t) dt, \quad z \in U. \tag{22}$$

Moreover q is the best dominant.

Remark 2. If we put

$$h(z) = \frac{1 + (2\alpha - 1)z}{1 + z}$$

in Theorem 3, we obtain the following interesting result:

Corollary 1. *If $0 \le \alpha < 1$, $n \in \mathbb{N}$, $\mathrm{Re}\ c > -2$ and I_c is defined by (5), then*

$$I_c[M_n(\alpha)] \subset M_n(\delta),$$

where

$$\delta = \min_{|z|=1} \mathrm{Re}\ q(z) = \delta(c, \alpha, z)$$

and this results is sharp. Moreover

$$\delta = \delta(c, \alpha, z) = 2\alpha - 1 + (c+2)(2 - 2\alpha)\sigma(c, z) \tag{23}$$

where

$$\sigma(c, z) = \int_0^z \frac{t^{c+1}}{1+t} dt. \tag{24}$$

Proof. If we let

$$h(z) = \frac{1 + (2\alpha - 1)z}{1 + z},$$

then h is convex. By using Theorem 3,

$$[D^n F(z)]' \prec h(z)$$

implies

$$\begin{aligned} p(z) &= [D^n F(z)]' \prec q(z) = \frac{c+2}{z^{c+2}} \int_0^z t^{c+1} \cdot \frac{1 + (2\alpha - 1)t}{1 + t} dt \qquad (25) \\ &= 2\alpha - 1 + \frac{(c+2)(2 - 2\alpha)}{z^{c+2}} \int_0^z \frac{t^{c+1}}{1+t} dt \\ &= 2\alpha - 1 + \frac{(c+2)(2 - 2\alpha)}{z^{c+2}} \sigma(c, z) \end{aligned}$$

where σ is given by (24).

If Re $c > -2$, then from the convexity of q and the fact that $q(U)$ is symmetric with respect to the real axis, we deduce

$$\begin{aligned} \mathrm{Re}\, [D^n F(z)]' &\geq \min_{|z|=1} \mathrm{Re}\, q(z) = \mathrm{Re}\, q(1) = \delta(c, \alpha, z) \\ &= 2\alpha - 1 + (c+2)(2 - 2\alpha)\sigma(c, 1) \end{aligned}$$

where σ is given by (24).

From (25), we deduce $I_c[M_n(h)] \subset M_n(\delta)$, where δ is given by (23).

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Received 29.01.2009

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