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The Fourier method
in three-dimensional dynamical problems
of the hemitropic theory of elasticity

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ABSTRACT: The basic three-dimensional dynamical problems for a hemitropic (non-centrally symmetric) micropolar elastic medium are considered. Using the Fourier method, the solvability of the formulated by us problems is proved in a classical sense.

1   Basic Notation and Equations

Let $D \subset R^3$ be a finite domain bounded by a closed surface S of the class $C^{2,\alpha}$; $0 < \alpha \leq 1$; $\overline{D} = D \cup S$, $L = (0, e)$, $\overline{L} = [0, e]$, $\Omega = D \times L$ be a cylinder in R^4, $\overline{\Omega} = \overline{D} \times \overline{L}$.

A system of differential equations of dynamics for the hemitropic micropolar elastic medium is of the form [1, 2]:

$$
\begin{cases}
(\mu + \alpha)\Delta u + (\lambda + \mu - \alpha)\operatorname{grad}\operatorname{div} u + (v + \eta)\Delta\omega \\
\quad + (\delta + v - \eta)\operatorname{grad}\operatorname{div}\omega + 2\alpha\operatorname{rot}\omega + X(x,t) = \rho\dfrac{\partial^2 u}{\partial t^2}, \\
(\nu + \beta)\Delta\omega - 4\alpha\omega + (\varepsilon + \nu - \beta)\operatorname{grad}\operatorname{div}\omega + (v + \eta)\Delta u \\
\quad + (\delta + v - \eta)\operatorname{grad}\operatorname{div} u + 2\alpha\operatorname{rot} u + 4\eta\operatorname{rot}\omega \\
\quad + Y(x,t) = \mathcal{J}\dfrac{\partial^2 \omega}{\partial t^2},
\end{cases}
\tag{1}
$$

where Δ is the three-dimensional Laplace operator; $u(x,t) = (u_1, u_2, u_3)$ is the displacement vector; $\omega(x,t) = (\omega_1, \omega_2, \omega_3)$ is the vector of rotation; $x = (x_1, x_2, x_3)$ is a point in R^3; t is time; $X(x,t)$ is the vector of body forces;

$Y(x,t)$ is the volumetrical moment; ρ is the medium density; $\mathcal{J}$ is a moment of inertia, and λ, μ, α, ε, ν, β, v, η, δ are the known elastic constants satisfying the following conditions: $\mu > 0$, $\alpha > 0$, $3\lambda + 2\mu > 0$, $\mu\nu - v^2 > 0$, $\alpha\beta - \eta^2 > 0$, $(3\lambda + 2\mu)(3\varepsilon + 2\nu) - (3\delta + 2v)^2 > 0$.

For the sake of brevity, the basic equations will be written in a matrix form. Towards this end, we adopt the following agreement: if we multiply the matrix $A = \|a_{ij}\|_{m \times n}$ of dimension n by the $u = (u_1, u_2, \ldots, u_n)$-dimensional vector n, then the vector is assumed to be a one-column matrix $u = \|u_i\|_{n \times 1}$, and the product Au is an m-dimensional vector.

The system (1) can be represented in a vector-matrix form as follows:

$$M(\partial x)V(x,t) + F(x,t) = r\frac{\partial^2 V}{\partial t^2}, \qquad (2)$$

where $M(\partial x) = \|M_{kj}(\partial x)\|_{6 \times 6}$, and also

$$M_{kj}(\partial x) = (\mu + \alpha)\delta_{kj}\Delta + (\lambda + \mu - \alpha)\frac{\partial^2}{\partial x_k \partial x_j} \quad \text{for} \quad k, j = 1, 2, 3,$$

$$M_{kj}(\partial x) = (v + \eta)\delta_{kj}\Delta + (\delta + v - \eta)\frac{\partial^2}{\partial x_k \partial x_j} - 2\alpha\sum_{e=1}^{3}\varepsilon_{kj}\frac{\partial}{\partial x_e},$$

$$\text{for} \quad k = 1, 2, 3, \quad j = 4, 5, 6 \quad \text{and} \quad k = 4, 5, 6, \quad j = 1, 2, 3;$$

$$M_{kj}(\partial x) = ((\nu + \beta)\Delta - 4\alpha)\delta_{kj} + (\varepsilon + \nu - \beta)\frac{\partial^2}{\partial x_k \partial x_j}$$

$$- 4\eta\sum_{e=1}^{3}\varepsilon_{kje}\frac{\partial}{\partial x_e} \quad \text{for} \quad k, j = 4, 5, 6;$$

δ_{kj} is the Kronecker symbol, $\varepsilon_k j e$ is the Levy-Civita symbol,

$$F(x,t) = (X(x,t), Y(x,t)) = (F_1, F_2, \ldots, F_6);$$

$$V(x,t) = (u(x,t), \omega(x,t)) = (u_1, u_2, u_3, \omega_1, \omega_2, \omega_3) = (v_1, v_2, \ldots, v_6);$$

r is the diagonal matrix of dimension 6×6; $r = \|r_{kj}\|_{6 \times 6}$, $r_{kj} = 0$, where $k \neq j$, $r_{ii} = \rho$ for $i = 1, 2, 3$; $r_{ii} = \mathcal{J}$ for $i = 4, 5, 6$.

The force stress $r^{(n)}(x,t) = (r_1^{(n)}(x,t), r_2^{(n)}(x,t), r_3^{(n)}(x,t))$ and the moment stress $\mu^{(n))}(x,t) = (\mu_1^{(n)}(x,t), \mu_2^{(n)}(x,t), \mu_3^{(n)}(x,t))$ at the point x and time t directed to $n = (n_1, n_2, n_3)$ are defined by the formulas [1, 2]:

$$r_i^{(n)} = \lambda n_i \operatorname{div} u + (\mu + \alpha)\sum_{j=1}^{3}\frac{\partial u_i}{\partial x_j}n_j + (\mu - \alpha)\sum_{j=1}^{3}\frac{\partial u_j}{\partial x_i}n_j +$$

$$+ \delta n_i \operatorname{div}\omega + (v + \eta)\sum_{j=1}^{3}\frac{\partial \omega_i}{\partial x_j}n_j + (v - \eta)\sum_{j=1}^{3}\frac{\partial \omega_j}{\partial x_i}n_j$$

$$+2\alpha \sum_{k,j=1}^{3} \varepsilon_{ijk}\omega_k n_j, \quad i = 1, 2, 3;$$

$$\mu_i^{(n)} = \delta n_i \operatorname{div} u + (v + \eta) \sum_{j=1}^{3} \frac{\partial u_i}{\partial x_j} n_j + (v - \eta) \sum_{j=1}^{3} \frac{\partial u_j}{\partial x_i} n_j$$

$$+ \varepsilon n_i \operatorname{div} \omega + (\nu + \beta) \sum_{j=1}^{3} \frac{\partial \omega_i}{\partial x_j} n_j + (\nu - \beta) \sum_{j=1}^{3} \frac{\partial \omega_j}{\partial x_i} n_j$$

$$+(v + \eta) \sum_{k,j=1}^{3} \varepsilon_{jki}\omega_k n_j + (v - \eta) \sum_{k,j=1}^{3} \varepsilon_{ijk}\omega_k n_j, \quad i = 1, 2, 3.$$

We introduce a matrix differential operator of dimension 6×6:

$$T(\partial x, n(x)) = \|T_{kj}(\partial x, n(x))\|_{6\times 6},$$

$$T_{kj}(\partial x, n(x)) = (\mu + \alpha)\delta_{kj}\frac{\partial}{\partial n(x)} + (\mu - \alpha)n_j\frac{\partial}{\partial x_k}$$

$$+ \lambda n_k \frac{\partial}{\partial x_j}, \quad k, j = 1, 2, 3;$$

$$T_{kj}(\partial x, n(x)) = (v + \eta)\delta_{kj}\frac{\partial}{\partial n(x)} + (v - \eta)n_j\frac{\partial}{\partial x_k} + \delta n_k\frac{\partial}{\partial x_j}$$

$$- 2\alpha \sum_{e=1}^{3} \varepsilon_{kje}n_e, \quad k = 1, 2, 3, \quad j = 4, 5, 6;$$

$$T_{kj}(\partial x, n(x)) = (v + \eta)\delta_{kj}\frac{\partial}{\partial n(x)} + (v - \eta)n_j\frac{\partial}{\partial x_k}$$

$$+ \delta n_k \frac{\partial}{\partial x_j}, \quad k = 4, 5, 6, \quad j = 1, 2, 3;$$

$$T_{kj}(\partial x, n(x)) = (\nu + \beta)\delta_{kj}\frac{\partial}{\partial n(x)} + (\nu - \beta)n_j\frac{\partial}{\partial x_k} + \beta n_k\frac{\partial}{\partial x_j}$$

$$- 2\nu \sum_{e=1}^{3} \varepsilon_{kje}n_e, \quad k, j = 4, 5, 6.$$

$T(\partial x, n(x))$ will be called a stress operator of the hemitropic theory of elasticity.

It is not difficult to check that

$$T(\partial x, n(x))V = (r^{(n)}(V), \mu^{(n)}(V)).$$

Note that $n(x)$ is an arbitrary unit vector at the point x. If $x \in S$, then $n(x)$ is the unit vector of the outer normal with respect to D.

2   Statement of the Basic Problems. The Conditions for the Given Vector-Functions

Let the hemitropic homogeneous elastic medium occupy the domain D. We consider the following two basic problems: find in a cylinder Ω a regular vector $V(x,t)$ ($x \in D$, $t \in L$, $V_i \in C^1(\overline{\Omega}) \cap C^2(\Omega)$, $i = \overline{1,6}$) satisfying

1) the equation

$$\forall(x,t) \in \Omega : M(\partial x)V(x,t) - r\frac{\partial^2 V(x,t)}{\partial t^2} = -F(x,t);$$

2) the initial conditions

$$\forall x \in \overline{D} : \lim_{t \to 0+} V(x,t) = \varphi(x), \quad \lim_{t \to 0+} \frac{\partial V(x,t)}{\partial t} = \psi(x);$$

3) one of the boundary conditions

$$\forall(z,t) \in S \times \overline{L} : \lim_{D \ni x \to z \in S} V(x,t) = 0, \ \ \text{for the first problem,}$$

$$\forall(z,t) \in S \times \overline{L} : \lim_{D \ni x \to z \in S} T(\partial x, n(x))V(x,t) = 0,$$

$$\text{for the second problem.}$$

The first problem we denote by $(I)_{F,\varphi,\psi}$, and the second one by $(II)_{F,\varphi,\psi}$.

The given vector-functions F, φ, ψ are assumed to satisfy the following conditions:

1) $F(\cdot,\cdot) \in C^2(\overline{\Omega})$, and the third order derivatives belong to the class $L_2(D)$. Moreover,

$$F|_S = MF|_S = 0, \quad t \in \overline{L} \quad \text{for the problem} \quad (I)_{F,\varphi,\psi}$$

and

$$TF|_S = 0, \quad t \in \overline{L} \quad \text{for the problem} \quad (II)_{F,\varphi,\psi};$$

2) $\varphi \in C^3(\overline{D})$, and the fourth order derivatives belong to the class $L_2(D)$. Moreover,

$$\varphi|_S = M\varphi|_S = 0, \quad \text{for the problem} \quad (I)_{F,\varphi,\psi}$$

and

$$T\varphi|_S = TM\varphi|_S = 0, \quad \text{for the problem} \quad (II)_{F,\varphi,\psi};$$

3) $\psi \in C^2(\overline{D})$, and the third order derivatives belong to the class $L_2(D)$. Moreover,

$$\psi|_S = M\psi|_S = 0, \quad \text{for the problem} \quad (I)_{F,\varphi,\psi}$$

and

$$T\psi|_S = 0, \quad \text{for the problem} \quad (II)_{F,\varphi,\psi}.$$

The symbol $\cdot|_S$ denotes the narrowing to KS.

3 Green's Formulas. The Uniqueness of Regular Solutions of the Problems Formulated Above

Let $V(x) = (u, \omega)$ and $V'(x) = (u', \omega')$ be arbitrary six-component vectors of the class $C^1(\overline{D})$ whose second derivatives belong to the class $L_2(D)$. Then the following Green's formulas are valid [3]:

$$\int_D (V'M(\partial x)V + E(V', V))dx = \int_S V'TV dS, \tag{1}$$

$$\int_D (V'M(\partial x)V - VM(\partial x)V')dx = \int_S (V'TV - VTV')dS, \tag{2}$$

where

$$
\begin{aligned}
E(V', V) = \sum_{k,j=1}^{3} \Big(& (\mu + \alpha)u'_{kj}u_{kj} + (\mu - \alpha)u'_{kj}u_{jk} \\
& + (v + \eta)u'_{kj}\omega_{kj} + (v - \eta)u'_{kj}\omega_{jk} + (\nu + \beta)\omega'_{kj}\omega_{kj} \\
& + (\nu - \beta)\omega'_{kj}\omega_{jk} + (v + \eta)\omega'_{kj}u_{kj} + (v - \eta)\omega'_{kj}u_{jk} \\
& + \delta(u'_{kk}\omega_{jj} + u_{kk}\omega'_{jj}) + \lambda u'_{kk}u_{jj} + \varepsilon\omega'_{kk}\omega_{jj} \Big),
\end{aligned}
\tag{3}
$$

where $u_{kj} = \frac{\partial u_j}{\partial x_k} - \sum_{e=1}^{3} \varepsilon_{kje}\omega_e$, $\omega_{kj} = \frac{\partial \omega_j}{\partial x_k}$. From (3) we have

$$
\begin{aligned}
E(V, V) = & \frac{3\lambda + 2\mu}{3}\Big(\operatorname{div} u + \frac{3\delta + 2v}{2\lambda + 2\mu}\operatorname{div}\omega\Big)^2 + \\
& + \frac{1}{3}\Big(3\varepsilon + 2\nu - \frac{(3\delta + 2v)^2}{3\lambda + 2\mu}\Big)(\operatorname{div}\omega)^2 + \\
& + \frac{\mu}{2}\sum_{k \neq j=1}\Big(\frac{\partial u_k}{\partial x_j} + \frac{\partial u_j}{\partial x_k} + \frac{v}{\mu}\Big(\frac{\partial \omega_k}{\partial x_j} + \frac{\partial \omega_j}{\partial x_k}\Big)\Big)^2 + \\
& + \frac{\mu}{3}\sum_{k,j=1}^{3}\Big(\frac{\partial u_k}{\partial x_k} - \frac{\partial u_j}{\partial x_j} + \frac{v}{\mu}\Big(\frac{\partial \omega_k}{\partial x_k} - \frac{\partial \omega_j}{\partial x_j}\Big)\Big)^2 + \\
& + \Big(\nu - \frac{v^2}{\mu}\Big)\Big(\frac{1}{2}\sum_{k \neq j=1}\Big(\frac{\partial \omega_k}{\partial x_j} + \frac{\partial \omega_j}{\partial x_k}\Big)^2 + \frac{1}{3}\sum_{k,j=1}^{3}\Big(\frac{\partial \omega_k}{\partial x_k} - \frac{\partial \omega_j}{\partial x_j}\Big)^2\Big) + \\
& + \Big(\beta - \frac{\eta^2}{\alpha}\Big)(\operatorname{rot}\omega)^2 + \alpha\Big(\operatorname{rot} u + \frac{\eta}{\alpha}\operatorname{rot}\omega - 2\omega\Big)^2.
\end{aligned}
\tag{4}
$$

From (3) and (4), on the basis of the conditions satisfying the elastic constants, we can conclude that

$$E(V', V) = E(V, V'), \quad E(V, V) \geq 0.$$

Let now prove the following

Theorem 1 *Problems $(I)_{F,\varphi,\psi}$ and $(II)_{F,\varphi,\psi}$ have no more than one solution.*

Proof.   We have to prove that the problems $(I)_{0,0,0}$ and $(II)_{0,0,0}$ have only zero solutions. To this end, we apply formula (1) to the vectors $V = V(x,t)$ and $V' = \frac{\partial V(x,t)}{\partial t}$, where $V(x,t) = (u(x,t), \omega(x,t))$ is a regular solution of the problem $(I)_{0,0,0}$ or of the problem $(II)_{0,0,0}$, and make use of the identities

$$V'MV = \frac{\partial V}{\partial t}MV = \Big(\frac{\partial u}{\partial t}, \frac{\partial \omega}{\partial t}\Big) \cdot \Big(\rho\frac{\partial^2 u}{\partial t^2}, \mathcal{J}\frac{\partial^2 \omega}{\partial t^2}\Big)$$
$$= \frac{\rho}{2}\frac{\partial}{\partial t}\Big|\frac{\partial u}{\partial t}\Big|^2 + \frac{\mathcal{J}}{2}\frac{\partial}{\partial t}\Big|\frac{\partial \omega}{\partial t}\Big|^2;$$
$$E(V',V) = E\Big(\frac{\partial V}{\partial t}, V\Big) = \frac{1}{2}\frac{\partial}{\partial t}E(V,V).$$

Then (1) takes the form

$$\frac{\partial}{\partial t}\int_D \Big(\frac{\rho}{2}\Big|\frac{\partial u}{\partial t}\Big|^2 + \frac{\mathcal{J}}{2}\Big|\frac{\partial \omega}{\partial t}\Big|^2 + \frac{1}{2}E(V,V)\Big)dx = \int_S \frac{\partial V}{\partial t}TV dS. \tag{5}$$

By virtue of the boundary conditions, the right-hand side of (5) for the both problems is equal to zero and, consequently,

$$\int_D \Big(\frac{\rho}{2}\Big|\frac{\partial u}{\partial t}\Big|^2 + \frac{\mathcal{J}}{2}\Big|\frac{\partial \omega}{\partial t}\Big|^2 + \frac{1}{2}E(V,V)\Big)dx = \text{const}. \tag{6}$$

Since this constant at the starting moment is equal to zero, it will remain unchanged in a due course, and $\Big|\frac{\partial u}{\partial t}\Big|^2 = 0$, $\Big|\frac{\partial \omega}{\partial t}\Big|^2 = 0$; $E(V,V) = 0$ thus $V = (u, \omega) \equiv 0$. ∎

4   Green's Tensors and the Problems for Eigenvalues

As we will see below, in investigating the above-formulated dynamical problems, of great importance are the solutions of some special corresponding problems of statics which are called the Green's tensors.

The first Green's tensor or the Green's tensor of the first basic problem of statics corresponding to the problem $(I)_{F,\varphi,\psi}$ is called the matrix $\overset{1}{G}(x,y)$ of dimension 6×6, depending on two points x and y and satisfying the following conditions:

1) $\forall x, y \in D$, $x \neq y$: $M(\partial x)\overset{1}{G}(x,y) = 0$,

2) $\forall z \in S$, $\forall y \in D$: $\overset{1}{G}(z,y) = 0$

3) $\overset{1}{G}(x,y) = \Gamma(x-y) - \overset{1}{g}(x,y)$, $x, y \in D$,

where $\Gamma(x-y)$ is the known matrix of fundamental solutions of the equation [3]

$$M(\partial x)V(x) = 0, \tag{7}$$

and $\overset{1}{g}(x,y)$ is a regular solution (including $x = y$) of equation (7) in the domain D.

It is clear that the proof of the existence of $\overset{1}{G}(x,y)$ is reduced to the solvability of the following first basic problem of statics for D: find a regular in D solution $\overset{1}{g}(x,y)$ of equation (7), satisfying the boundary condition

$$\forall z \in S, \ \ \forall y \in D: \ \ \overset{1}{g}(z,y) = \Gamma(z-y).$$

The solvability of such a problem has been proved in [3].

The second Green's tensor, or the Green's tensor of the second basic problem of statics in the domain D, corresponding to the problem $(II)_{F,\varphi,\psi}$, cannot, generally speaking, be defined analogously to the first tensor. Since the second basic problem of statics is not always solvable [3], we, following to H. Weyl [4, 5], represent $\overset{2}{G}(x,y)$ in the domain D in the form

$$\overset{2}{G}(x,y) = \Pi(x,y) - \overset{2}{g}(x,y),$$

where $\Pi(x,y)$ is the known matrix involving $\Gamma(x-y)$ and the vectors of rigid displacement, and $\overset{2}{g}(x,y)$ is a regular matrix in D, satisfying equation (7) and the boundary condition

$$\forall z \in S, \ \ \forall y \in D: \ \lim_{D \ni x \to z \in S} T(\partial x, n(x)) \overset{2}{g}(x,y) = T(\partial z, n(z)) \Pi(z,y).$$

It is shown that the necessary and sufficient condition for the obtained second problem of statics is fulfilled and hence the existence of $\overset{2}{G}(x,y)$ is proved.

Relying on the results of [3, 6], we can show that the Green's tensors possess the following properties:

1) $\overset{k}{G}(x,y) = \overset{k}{G}{}^{\top}(y,x), \quad$ where "$\top$" denotes transposition $k = 1,2$;

2) $\forall (x,y) \in D \times D: \ \overset{k}{G}_{mn}(x,y) = O(|x-y|^{-1}),$

$$\frac{\partial}{\partial x_j} \overset{k}{G}_{mn}(x,y) = O(|x-y|^{-2}), \tag{8}$$

$$k = 1,2; \ \ j = 1,2,3; \ \ m,n = \overline{1,6};$$

3) $\forall (x,y) \in D \times D: \ \ \frac{\partial}{\partial x_j} \overset{k}{G}{}^{(2)}_{mn}(x,y) = O(\ln|x-y|), \tag{9}$

$$j = 1,2,3; \ \ k = 1,2; \ \ m,n = \overline{1,6};$$

4) $\forall (x,y) \in \overline{D}' \times D: \ \frac{\partial^2}{\partial x_i \partial x_j} \overset{k}{G}{}^{(2)}_{mn}(x,y) = O(|x-y|^{-1}), \tag{10}$

$$k = 1,2; \ \ j = 1,2,3; \ \ m,n = \overline{1,6},$$

where $|x - y|$ is the distance between the points x and y, $\overline{D}' \subset D$ is an arbitrary closed domain, lying strictly in D, and $\overset{k}{G}{}^{(2)}(x, y)$ is an iterated kernel for $\overset{k}{G}(x, y)$:

$$\overset{k}{G}{}^{(2)}(x, y) = \int_D \overset{k}{G}(x, z)\overset{k}{G}(z, y)dz, \quad x \neq y, \quad k = 1, 2.$$

We rewrite equation (2) as follows:

$$\widetilde{M}(\partial x)\widetilde{V}(x, t) - \frac{\partial^2 \widetilde{V}(x, t)}{\partial t^2} = -\widetilde{F}(x, t),$$

where $\widetilde{M} = \varkappa^{-1} M \varkappa^{-1}$, $\widetilde{V} = \varkappa V$, $\widetilde{F} = \varkappa^{-1} F$, $\varkappa = \|\sqrt{r_{kj}}\|_{6 \times 6}$.

Consider now two problems for eigen-values:

$$\forall x \in D: \quad \widetilde{M}(\partial x)\overset{k}{W}(x) + \gamma \overset{k}{W}(x) = 0, \quad k = 1, 2;$$

$$\forall z \in S: \quad \lim_{D \ni x \to z \in S} \overset{1}{W}(x) = 0, \quad \lim_{D \ni x \to z \in S} T(\partial x, n(x))\overset{2}{W}(x) = 0.$$

The first problem we denote by $(I)_\gamma$ and the second one by $(II)_\gamma$. The eigen vector-function $\overset{k}{W}(x) = (\overset{k}{W}_1, \overset{k}{W}_2, \ldots, \overset{k}{W}_6)$, $k = 1, 2$ (not equal identically to zero) is said to be regular, if $\overset{k}{W}_i \in C^1(\overline{D}) \cap C^2(D)$, $i = \overline{1, 6}$; $k = 1, 2$.

It is not difficult to show [5] that the problems $(I)_\gamma$ and $(II)_\gamma$ are equivalent to the following system of integral equations:

$$\overset{k}{W}(x) = \gamma \int_D \overset{k}{K}(x, y)\overset{k}{W}(y)dy, \quad x \in D, \tag{11}$$

where $\overset{k}{K}(x, y) = \varkappa \overset{k}{G}(x, y)\varkappa$, $k = 1, 2$.

It follows from the above-mentioned properties of $\overset{k}{G}(x, y)$ that (11) is the integral equation with symmetric kernel of the class $L_2(D)$. Consequently, there exist a countable system of eigen-numbers $(\overset{k}{\gamma}_n)_{n=1}^\infty$, $|\overset{k}{\gamma}_n| \to \infty$ as $n \to \infty$, and the corresponding orthonormalized in D system of eigen-vectors $(\overset{k}{W}{}^n(x))_{n=1}^\infty$, $x \in D$, $k = 1, 2$, of equation (11) or of the problems $(I)_\gamma$ and $(II)_\gamma$, respectively. It is easy to state that all $\overset{1}{\gamma}_n > 0$, while $\overset{2}{\gamma}_n \geq 0$, where $\overset{2}{\gamma} = 0$ is the eigen sixth rank number, and the corresponding vectors are those of the rigid displacement $(\chi^{(n)}(x))_{n=1}^6$. In what follows, it will be assumed that $\overset{2}{\gamma}_n = 0$, $\overset{2}{W}{}^{(n)} = \chi^{(n)}$, $n = \overline{1, 6}$, $\overset{2}{\gamma}_n > 0$ for $n > 6$. It can be shown that the vectors $\overset{k}{W}{}^{(n)}(x)$ are regular in the domain D.

5   Lemmas on the Order of Fourier Coefficients

Lemma 1 *For any six-component vector $\Phi(x)$ satisfying the conditions $\Phi \in C^0(\overline{D})$, $\frac{\partial \Phi}{\partial x_i} \in L_2(D)$, $i = 1, 2, 3$; $\Phi|_S = 0$ the inequality*

$$\sum_{n=1}^{\infty} \Phi_n^2 \overset{1}{\gamma}_n \leq \int_D E(\varkappa^{-1}\Phi, \varkappa^{-1}\Phi)dx, \tag{12}$$

where

$$\Phi_n = \int_D \Phi(x) \overset{1}{W}^{(n)}(x)dx,$$

is valid. In particular, it follows from the above lemma that the numerical series in the left-hand side of (12) *converges.*

Proof.   Applying Green's formula (11) to the vectors $V' = \varkappa^{-1}\Phi(x)$ and $V = \varkappa^{-1}\overset{1}{W}^{(n)}(x)$ and taking into account the condition $\Phi|_S = 0$, we obtain

$$\int_D E(\varkappa^{-1}\Phi, \varkappa^{-1}\overset{1}{W}^{(n)})dx = \overset{1}{\gamma}_n \Phi_n. \tag{13}$$

In particular, assuming in (13) that $\Phi = \overset{1}{W}{}^m$, we obtain

$$\int_D E(\varkappa^{-1}\overset{1}{W}^{(m)}, \varkappa^{-1}\overset{1}{W}^{(n)})dx = \begin{cases} \overset{1}{\gamma}_n & \text{for} \quad m = n, \\ 0 & \text{for} \quad m \neq n. \end{cases} \tag{14}$$

Consider now a nonnegative value

$$\mathcal{J} = \int_D E(\varkappa^{-1}V, \varkappa^{-1}V)dx$$

and assume $V(x) = \Phi(x) - \sum_{n=1}^{n_0} \Phi_n \overset{1}{W}^{(n)}(x)$, then simple calculations show that

$$\mathcal{J} = \int_{\mathcal{J}} E(\varkappa^{-1}\Phi, \varkappa^{-1}\Phi)dx + \sum_{m,n=1}^{n_0} \int_D E(\varkappa^{-1}\overset{1}{W}^{(m)}, \varkappa^{-1}\overset{1}{W}^{(n)})dx$$

$$-2\sum_{n=1}^{n_0} \Phi_n \int_D E(\varkappa^{-1}\Phi, \varkappa^{-1}\overset{1}{W}^{(n)})dx. \tag{15}$$

Taking into account (13) and (14), from (15) we find that

$$\sum_{n=1}^{n_0} \Phi_n^2 \overset{1}{\gamma}_n \leq \int_D E(\varkappa^{-1}\Phi, \varkappa^{-1}\Phi)dx,$$

which proves our lemma. ∎

Lemma 2 *For any six-component vector* $\Phi(x)$ *satisfying the conditions* $\Phi \in C^0(\overline{D})$, $\frac{\partial \Phi}{\partial x_i} \in L_2(D)$, $i = 1, 2, 3$, *the inequality*

$$\sum_{n=1}^{\infty} \Phi_n^2 \gamma_n^2 \leq \int_D E(\varkappa^{-1}\Phi, \varkappa^{-1}\Phi)dx, \tag{16}$$

where

$$\Phi_n = \int_D \Phi(x)\overset{2}{W}{}^{(n)}(x)dx,$$

is valid.

Proof.   Applying Green's formula to the vectors $V' = \varkappa^{-1}\Phi(x)$ and $V = \varkappa^{-1}\overset{2}{W}{}^{(n)}(x)$ and taking into account the condition $T\overset{2}{W}{}^{(n)}|_S = 0$, we obtain

$$\int_D E(\varkappa^{-1}\Phi, \varkappa^{-1}\overset{2}{W}{}^{(n)})dx = \overset{2}{\gamma}_n \Phi.$$

Repeating further word for word the proof of Lemma 1, we will get inequality (16). ∎

Lemma 3 *For any six-component vector* $\Phi(x)$ *satisfying the conditions* 1) $\Phi \in C^1(\overline{D})$, 2) $\frac{\partial^2 \Phi}{\partial x_i \partial x_j} \in L_2(D)$ *and* 3) $\Phi|_S = 0$ *for the problem* $(I)_{F,\varphi,\psi}$ *and the condition* $T\Phi|_S = 0$ *for the problem* $(II)_{F,\varphi,\psi}$, *the inequality*

$$\sum_{n=1}^{\infty} \Phi_n^2 \overset{k}{\gamma}{}^2 \leq \int_D |\widetilde{M}\Phi|^2 dx, \quad k = 1, 2, \tag{17}$$

is valid.

In particular, it follows from the above theorem that the numerical series in the left-hand side (17) converges.

Proof.   Applying Green's formula (1) to the vectors $\varkappa^{-1}\Phi(x)$ and $\varkappa^{-1}\overset{k}{W}{}^{(n)}(x)$ and taking in each of the cases the boundary conditions $\Phi(x)$ and $\overset{k}{W}{}^{(n)}(x)$ on S, we obtain

$$\int_D \widetilde{M}\Phi\overset{k}{W}{}^{(n)}dx = -\overset{k}{\gamma}_n \Phi_n, \quad k = 1, 2,$$

whence

$$(\widetilde{M}\Phi)_n = -\overset{k}{\gamma}_n \Phi_n, \quad k = 1, 2. \tag{18}$$

For $\widetilde{M}\Phi(x)$, we write the Bessel's inequality

$$\sum_{n=1}^{\infty} (\widetilde{M}\Phi)_n^2 \leq \int_D |\widetilde{M}\Phi|^2 dx. \tag{19}$$

Taking into account (18), from (19) we get (17). ∎

6 Formal Scheme of the Fourier Method

We denote $\widetilde{\varphi}(x) = \varkappa\varphi(x)$, $\widetilde{\psi}(x) = \varkappa\psi(x)$. Let $\widetilde{V}^I(x,t)$ be a solution of the problem $(I)_{\widetilde{F},\widetilde{\varphi},\widetilde{\psi}}$. We write the representation $\widetilde{V}^I(x,t) = \widetilde{V}^1(x,t) + \widetilde{V}^2(x,t)$, where $\widetilde{V}^1(x,t)$ is a solution of the problem $(I)_{0,\widetilde{\varphi},\widetilde{\psi}}$, and $\widetilde{V}^2(x,t)$ is that of the problem $(I)_{\widetilde{F},0,0}$.

Applying to the problem $(I)_{0,\widetilde{\varphi},\widetilde{\psi}}$ a formal scheme of the Fourier method, we obtain

$$\widetilde{V}^1(x,y) = \sum_{n=1}^{\infty} \overset{1}{W}{}^{(n)}(x)\left(\widetilde{\varphi}_n \cos\sqrt{\overset{1}{\gamma}_n}t + \frac{\widetilde{\psi}_n}{\sqrt{\overset{1}{\gamma}_n}}\sin\sqrt{\overset{1}{\gamma}_n}t\right),$$

where

$$\widetilde{\varphi}_n = \int_D \widetilde{\varphi}(x)\overset{1}{W}{}^{(n)}(x)dx, \quad \widetilde{\psi}_n = \int_D \widetilde{\psi}(x)\overset{1}{W}{}^{(n)}(x)dx.$$

Formally, we decompose $\widetilde{V}^2(x,t)$ and $\widetilde{F}(x,t)$ into a series by the system $(\overset{1}{W}{}^{(n)}(x))_{n=1}^{\infty}$:

$$\widetilde{V}^2(x,t) = \sum_{n=1}^{\infty} \widetilde{V}_n^2(t)\overset{1}{W}{}^{(n)}(x), \quad \widetilde{F}(x,t) = \sum_{n=1}^{\infty} \widetilde{F}_n(t)\overset{1}{W}{}^{(n)}(x),$$

We get

$$\widetilde{V}^2(x,t) = \sum_{n=1}^{\infty} \overset{1}{W}{}^{(n)}(x)\frac{1}{\sqrt{\overset{1}{\gamma}_n}}\int_0^t \widetilde{F}_n(\tau)\sin\sqrt{\overset{1}{\gamma}_n}(t-\tau)d\tau.$$

Consequently, a solution of the problem $(I)_{\widetilde{F},\widetilde{\varphi},\widetilde{\psi}}$ is formally looks as follows:

$$\widetilde{V}^I(x,t) = \sum_{n=1}^{\infty} \overset{1}{W}{}^{(n)}(x)\left(\widetilde{\varphi}_n \cos\sqrt{\overset{1}{\gamma}_n}t + \frac{\widetilde{\psi}_n}{\sqrt{\overset{1}{\gamma}_n}}\sin\sqrt{\overset{1}{\gamma}_n}t\right)$$
$$+ \sum_{n=1}^{\infty} \overset{1}{W}{}^{(n)}(x)\frac{1}{\sqrt{\overset{1}{\gamma}_n}}\int_0^t \widetilde{F}_n(\tau)\sin\sqrt{\overset{1}{\gamma}_n}(t-\tau)d\tau. \tag{20}$$

Applying Green's formula (2) to the vectors $\varkappa^{-1}\widetilde{\varphi}(x)$ and $\varkappa^{-1}\overset{1}{W}{}^{(n)}(x)$ and taking into account that $\widetilde{\varphi}|_S = \overset{1}{W}{}^{(n)}|_S = 0$, we obtain

$$\int_D \widetilde{M\widetilde{\varphi}}(x)\overset{1}{W}{}^{(n)}(x) = -\overset{1}{\gamma}_n\widetilde{\varphi}_n,$$

that is,

$$(\widetilde{M\widetilde{\varphi}})_n = -\overset{1}{\gamma}_n\widetilde{\varphi}_n. \tag{21}$$

We apply now formula (2) to the vectors $\varkappa^{-1}\widetilde{M}\widetilde{\varphi}(x)$ and $\varkappa^{-1}\overset{1}{W}{}^{(n)}(x)$ and take into account that $\widetilde{M}\widetilde{\varphi}|_S = \overset{1}{W}{}^{(n)}|_S = 0$. Thus we get

$$(\widetilde{M}^2\widetilde{\varphi})_n = -\overset{1}{\gamma}_n(\widetilde{M}\widetilde{\varphi})_n,$$

whence by virtue of (21), we obtain

$$\widetilde{\varphi}_n = \frac{(\widetilde{M}^2\widetilde{\varphi})_n}{\overset{1}{\gamma}_n^2}. \tag{22}$$

Analogously, we find that

$$\widetilde{\psi}_n = -\frac{(\widetilde{M}\widetilde{\psi})_n}{\overset{1}{\gamma}_n}, \quad \widetilde{F}_n(t) = -\frac{(\widetilde{M}\widetilde{F})_n(t)}{\overset{1}{\gamma}_n}. \tag{23}$$

In view of (22) and (23), (20) takes the form

$$\widetilde{V}^I(x,t) = \sum_{n=1}^{\infty} \frac{\overset{1}{W}{}^{(n)}(x)}{\overset{1}{\gamma}_n^2}(\widetilde{M}^2\widetilde{\varphi})_n \cos\sqrt{\overset{1}{\gamma}_n}\,t$$

$$- \sum_{n=1}^{\infty} \frac{\overset{1}{W}{}^{(n)}(x)}{\overset{1}{\gamma}_n^{3/2}}(\widetilde{M}\widetilde{\varphi})_n \sin\sqrt{\overset{1}{\gamma}_n}\,t$$

$$- \sum_{n=1}^{\infty} \frac{\overset{1}{W}{}^{(n)}}{\overset{1}{\gamma}_n^{3/2}} \int_0^t (\widetilde{M}\widetilde{F})_n(\tau)\sin\sqrt{\overset{1}{\gamma}_n}(t-\tau)d\tau. \tag{24}$$

Let $\widetilde{V}^{II}(x,t)$ be a solution of the problem $(II)_{\widetilde{F},\widetilde{\varphi},\widetilde{\varphi}}$. Then, analogously, we get

$$\widetilde{V}^{II}(x,t) = \sum_{n=1}^{6} \chi^{(n)}(x)(\widetilde{\varphi}_n + t\widetilde{\psi}_n) + \sum_{n=7}^{\infty} \frac{\overset{2}{W}{}^{(n)}(x)}{\overset{2}{\gamma}_n^2}(\widetilde{M}^2\widetilde{\varphi})_n \cos\sqrt{\overset{2}{\gamma}_n}\,t$$

$$- \sum_{n=7}^{\infty} \frac{\overset{2}{W}{}^{(n)}(x)}{\overset{2}{\gamma}_n^{3/2}}(\widetilde{M}\widetilde{\psi})_n \sin\sqrt{\overset{2}{\gamma}_n}\,t + \sum_{n=1}^{6} \chi^{(n)}(x)\int_0^t \left(\int_0^t \widetilde{F}_n(\tau)d\tau\right)dt$$

$$- \sum_{n=7}^{\infty} \frac{\overset{2}{W}{}^{(n)}(x)}{\overset{2}{\gamma}_n^{3/2}} \int_0^t (\widetilde{M}\widetilde{F})_n(\tau)\sin\sqrt{\overset{2}{\gamma}_n}(t-\tau)d\tau. \tag{25}$$

7 Justification of the Fourier Method

To justify the Fourier method, we have to prove that the series appearing in (24) and (25) and those obtained by means of a single termwise differentiation of these

series converge uniformly in the closed cylinder $\overline{\Omega}$, while the series obtained by a double termwise differentiation of these series converge uniformly in the cylinder Ω.

We investigate here only the series appearing in (24) because those appearing in (25) are investigated analogously. First, we investigate the series

$$\sum_{n=1}^{\infty} \frac{W^{(n)}(x)}{\gamma_n^2} (\widetilde{M}^2 \widetilde{\varphi})_n \cos \sqrt{\gamma_n} t, \tag{26}$$

where for the sake of simplicity we adopt $\overset{1}{W}{}^{(n)} = W^{(n)}$ and $\overset{1}{\gamma} = \gamma_n$.

Simultaneously, we investigate the series obtained by a single and double termwise differentiation with respect to t of the series (26),

$$-\sum_{n=1}^{\infty} \frac{W^{(n)}(x)}{\gamma_n^{3/2}} (\widetilde{M}^2 \widetilde{\varphi})_n \sin \sqrt{\gamma_n} t, \tag{27}$$

$$\sum_{n=1}^{\infty} \frac{W^{(n)}(x)}{\gamma_n} (\widetilde{M}^2 \widetilde{\varphi})_n \cos \sqrt{\gamma_n} t. \tag{28}$$

Estimate a residual of the series (26) by using the Cauchy-Bunyakowski's inequality and majorizing simultaneously the cosine by unity. We have

$$\left| \sum_{n=m}^{m+p} \frac{W^{(n)}(x)}{\gamma_n^2} (\widetilde{M}^2 \widetilde{\varphi})_n \cos \sqrt{\gamma_n} t \right| \leq \left[\sum_{n=m}^{m+p} \frac{|W^{(n)}(x)|^2}{\gamma_n^4} \sum_{n=m}^{m+p} (\widetilde{M}^2 \widetilde{\varphi})_n^2 \right]^{1/2}. \tag{29}$$

Since $\widetilde{M}^2 \widetilde{\varphi} \in L_2(D)$, by virtue of Bessel's inequality we obtain

$$\sum_{n=1}^{\infty} (\widetilde{M}^2 \widetilde{\varphi})_n^2 \leq \int_D |\widetilde{M}^2 \widetilde{\varphi}|^2 dx. \tag{30}$$

In view of (30), it follows from (29) that to prove that the series (26) converges uniformly in $\overline{\Omega}$, it suffices to state that the sum of the series

$$\sum_{n=1}^{\infty} \frac{|W^{(n)}(x)|^2}{\gamma_n^4} \tag{31}$$

exists and is uniformly bounded in $\overline{D}$.

The Bessel's inequality provides us with

$$\sum_{n=1}^{\infty} \frac{|W^{(n)}(x)|^2}{\gamma_n^2} \leq \int_D |K(x,y)|^2 dy, \tag{32}$$

where $K(x,y) = \overset{1}{K}(x,y) = \varkappa \overset{1}{G}(x,y) \varkappa$. By virtue of (8), it follows from (32) that the sum of the series

$$\sum_{n=1}^{\infty} \frac{|W^{(n)}(x)|^2}{\gamma_n^2} \tag{33}$$

exists and is uniformly bounded in $\overline{D}$.

The same conclusion is especially valid for the series (31). The series (27) and (28) are investigated analogously.

Consider now the series obtained by a single and double termwise differentiation with respect to x of the series (26),

$$\sum_{n=1}^{\infty} \frac{\frac{\partial W^{(n)}(x)}{\partial x_i}}{\gamma_n^2}(\widetilde{M}^2\widetilde{\varphi})_n \cos\sqrt{\gamma_n}t, \quad i = 1, 2, 3; \tag{34}$$

$$\sum_{n=1}^{\infty} \frac{\frac{\partial^2 W^{(n)}(x)}{\partial x_i \partial x_j}}{\gamma_n^2}(\widetilde{M}^2\widetilde{\varphi})_n \cos\sqrt{\gamma_n}t, \quad i = 1, 2, 3. \tag{35}$$

Consider the iterated kernel $K^{(2)}(x, y)$ for the kernel $K(x, y)$. Then

$$W^{(n)}(x) = \gamma_n^2 \int_D K^{(2)}(x, y)W^{(n)}(y)dy.$$

The Bessel's inequality results in

$$\sum_{n=1}^{\infty} \frac{|\frac{\partial W^{(n)}(x)}{\partial x_i}|^2}{\gamma_n^4} \leq \int_D \left|\frac{\partial K^{(2)}(x, y)}{\partial x_i}\right|^2 dy, \quad i = 1, 2, 3; \tag{36}$$

$$\sum_{n=1}^{\infty} \frac{|\frac{\partial^2 W^{(n)}(x)}{\partial x_i \partial x_j}|^2}{\gamma_n^4} \leq \int_D \left|\frac{\partial^2 K^{(2)}(x, y)}{\partial x_i \partial x_j}\right|^2 dy, \quad i, j = 1, 2, 3. \tag{37}$$

By virtue of (9), it follows from (36) that the sum of the series appearing in the left-hand side of (36) exists and is uniformly bounded in $\overline{D}$.

On the strength of (10), it follows from (37) that the sum of the series appearing in the left-hand side of (37) exists and is uniformly bounded in $\overline{D}'$, where $\overline{D}' \subset D$ is an arbitrary closed domain lying strictly in D.

Repeating the above reasoning, we can prove that the series (34) converges uniformly in $\overline{\Omega}$, and the series (35) converges uniformly in Ω.

Let us now pass to the investigation of the second series (24). We rewrite it in the form

$$\sum_{n=1}^{\infty} \frac{W^{(n)}(x)}{\gamma_n^2}((\widetilde{M}\widetilde{\psi})_n\sqrt{\gamma_n})\sin\sqrt{\gamma_n}t. \tag{38}$$

Comparing the series (38) and (26), it is not difficult to notice that they are of the same structure, the only difference is that the cosine is replaced by the sine and $(\widetilde{M}^2\widetilde{\varphi})_n$ is replaced by $(\widetilde{M}\widetilde{\varphi})_n\sqrt{\gamma_n}$. In investigating the series (26) we have used the fact that the series

$$\sum_{n=1}^{\infty} (\widetilde{M}^2\widetilde{\varphi})_n^2$$

converges. As for the convergence of the series

$$\sum_{n=1}^{\infty}(\widetilde{M}\widetilde{\psi})_n^2\gamma_n,$$

it follows directly from the above-proven Lemma 1. Consequently, we can apply the above scheme to the series (38).

Finally, we investigate the third series appearing in (24) and rewrite it in the form

$$\sum_{n=1}^{\infty}\frac{W^{(n)}(x)}{\gamma_n^2}\int_0^t(\widetilde{M}\widetilde{F})_n(\tau)\sqrt{\gamma_n}\sin\sqrt{\gamma_n}(t-\tau)d\tau. \qquad (39)$$

It is clear from the above reasoning that we have to prove the convergence of the series

$$\sum_{n=1}^{\infty}\int_0^t((\widetilde{M}\widetilde{F})_n(\tau))^2\gamma_n d\tau.$$

The last statement follows directly from Lemma 1 and the well-known theorem on the limiting passage under the Lebesgue integral sign.

It remains for us to prove that the Fourier series

$$\sum_{n=1}^{\infty}\widetilde{F}_n(t)W^{(n)}(x) \qquad (40)$$

of the vector-function $\widetilde{F}(x,t)$ converges uniformly in the closed cylinder $\overline{\Omega}$. Consider the series

$$\sum_{n=1}^{\infty}W^{(n)}(x)\int_0^t\frac{d\widetilde{F}_n(\tau)}{d\tau}\,d\tau. \qquad (41)$$

and estimate the residual of the series (43) by means of the Cauchy-Bunyakowski's inequality

$$\left|\sum_{n=m}^{m+p}W^{(n)}(x)\int_0^t\frac{d\widetilde{F}_n(\tau)}{d\tau}d\tau\right|$$
$$\leq\left[\sum_{n=m}^{m+p}\frac{|W^{(n)}(x)|^2}{\gamma_n^2}\sum_{n=m}^{m+p}\int_0^e\left|\frac{d\widetilde{F}_n(\tau)}{d\tau}\right|^2\gamma_n^2 d\tau\right]^{1/2} \qquad (42)$$

Using Lemma 3 for the vector-function $\frac{\partial\widetilde{F}(x,t)}{\partial t}$ and the theorem on the limiting passage under the integral sign, we can state that the series

$$\sum_{n=1}^{\infty}\int_0^e\left|\frac{d\widetilde{F}_n(\tau)}{d\tau}\right|^2\gamma_n^2 d\tau.$$

converges.

Thus taking into account the fact that the sum of the series (31) is uniformly bounded, it follows from (7) that the series (41) converges uniformly in $\overline{\Omega}$.

From (41), we have

$$\sum_{n=1}^{\infty} W^{(n)}(x) \int_0^t \frac{d\widetilde{F}_n(\tau)}{d\tau} d\tau \;\;=\;\; \sum_{n=1}^{\infty} \widetilde{F}_n(t) W^{(n)}(x) \;\;-\;\; \sum_{n=1}^{\infty} \widetilde{F}_n(0) W^{(n)}(x). \quad (43)$$

Thus it is clear from (43) that to prove that the series (40) converges uniformly in $\overline{\Omega}$, it suffices to prove that the series

$$\sum_{n=1}^{\infty} \widetilde{F}_n(0) W^{(n)}(x) \tag{44}$$

converges uniformly in $\overline{D}$. We estimate the residual of the series (44),

$$\left| \sum_{n=m}^{m+p} \widetilde{F}_n(0) W^{(n)}(x) \right| \leq \left[\sum_{n=m}^{m+p} \frac{|W^{(n)}(x)|^2}{\gamma_n^2} \sum_{n=m}^{m+p} \widetilde{F}_n{}^2(0)\gamma_n^2 \right]^{1/2}. \tag{45}$$

Using Lemma 3 for $\widetilde{F}(x,0)$, we immediately find that the series

$$\sum_{n=1}^{\infty} \widetilde{F}_n{}^2(0)\gamma_n{}^2.$$

converges uniformly. Taking now into account that the sum of the series (31) is uniformly bounded, we can conclude from (45) that the series (44) converges uniformly in $\overline{D}$. Consequently, a full justification of the Fourier method for the problems under consideration is complete.

Thus we have proved the following

Theorem 2 *If F, φ and ψ are the given vector-functions satisfying the conditions mentioned in item 2, then the series (24) and (25) are the regular (classical) solutions of the problems $(I)_{F,\varphi,\psi}$ and $(II)_{F,\varphi,\psi}$, respectively.*

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The reticulation of residuated lattices induced by fuzzy prime spectrum

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ABSTRACT: In this paper, we use the fuzzy prime spectrum to define the reticulation $(L(A), \lambda)$ of a residuated lattice A. We obtain some related results. In particular, we show that the lattices of fuzzy filters of a residuated lattice A and $L(A)$ are isomorphic and the fuzzy prime spectrum of A and $L(A)$ are homomorphic topological space.

1 Introduction

M. Ward and R.P. Dilworth [12] introduced the concept of residuated lattices as generalization of ideal lattices of rings. These algebras have been widely studied (See [1], [2] and [6]).

The reticulation was first defined by simmons([10]) for commutative ring and L. Leustean made this construction for BL-algebras ([7]). C. Mureson defined the reticulations for residuated lattices ([8]). The reticulation of an algebra A is a pair $(L(A), \lambda)$ consisting of a bounded distributive lattice $L(A)$ and a surjective $\lambda : A \to L(A)$([8]). Hence we can transfer many properties between A and $L(A)$.

The concept of fuzzy sets were introduced by Zadeh in 1965 ([13]). This concept was applied to residuated lattices and proposed the notions of fuzzy filters and prime fuzzy filters in a residuated lattice ([3], [4] and [14]). We defined and studied fuzzy prime spectrum of a residuated lattice in ([5]).

In this paper, we use fuzzy prime spectrum to define the congruence relation $\cong$ on a residuated lattice A. Then we will show that $A/\cong$ is a bounded distributive lattice and $(A/\cong, \pi)$ is a reticulation of A. We will investigate some related results. Also, we obtain the relation between the reticulation of a residuated lattice induced by fuzzy prime spectrum and the reticulation of a residuated lattice which is defined in ([8]).

2   Preliminaries

We recall some definitions and theorems which will be needed in this paper.

Definition 2.1. ([1], [11]) A *residuated lattice* is an algebraic structure $(A, \wedge, \vee, \rightarrow, *, 0, 1)$ such that
(1) $(A, \wedge, \vee, 0, 1)$ is a bounded lattice with the least element 0 and the greatest element 1,
(2) $(A, *, 1)$ is a commutative monoid where 1 is a unit element,
(3) $x * y \leq z$ iff $x \leq y \rightarrow z$, for all $x, y, z \in A$.

In the rest of this paper, we denote the residuated lattice $(A, \wedge, \vee, *, \rightarrow, 0, 1)$ by A.

Proposition 2.2. ([6], [11]) Let A be a residuated lattice. Then we have the following properties: for all $x, y, z \in A$,
(1) $x \leq y$ if and only if $x \rightarrow y = 1$,
(2) $x * y \leq x \wedge y \leq x, y$,
(3) $x * (y \vee z) = (x * y) \vee (x * z)$.

Definition 2.3. ([6], [11]) Let F be a non-empty subset of a residuated lattice A. F is called a *filter* if
(1) $1 \in F$,
(2) if $x, x \rightarrow y \in F$, then $y \in F$, for all $x, y \in A$.
F is called *proper*, if $F \neq A$.

Theorem 2.4. ([6], [11]) A non-empty subset F of a residuated lattice A is a filter if and only if
(1) $x, y \in F$ implies $x * y \in F$,
(2) if $x \leq y$ and $x \in F$, then $y \in F$.

Definition 2.5. ([6]) Let X be a subset of a residuated lattice A. The smallest filter of A which contains X is said to be the *filter generated* by X and will be denoted by $< X >$.

Proposition 2.6. ([6]) Let X be a non-empty subset of a residuated lattice A. Then $< X >= \{a \in A : a \geq x_1 * \ldots * x_n$ for some $x_1, \ldots, x_n \in X\}$.

Definition 2.7. ([6], [11]) A proper filter F of a residuated lattice A is called *prime filter*, if for all $x, y \in A$, $x \vee y \in A$, implies $x \in A$ or $y \in A$.

Proposition 2.8. ([9]) (The prime filter theorem) Let A be a residuated lattice, F be a filter of A and $a \in A \backslash F$. Then there exists a prime filter of A that includes F and does not contain a.

Definition 2.9. ([13]) Let X be a non-empty subset. A *fuzzy set* in X is a mapping

$\mu : X \longrightarrow [0,1]$. For $t \in [0,1]$, the set $\mu_t = \{x \in X : \mu(x) \geq t\}$ is called a *level subset* of μ. We call that μ is *proper*, if it has more two distinct values.

Definition 2.10. Let X, Y be non-empty sets and $f : X \to Y$ be a function. Let μ be a fuzzy set in X and ν be a fuzzy set in Y. Then $f(\mu)$ is a fuzzy set in Y defined by

$$f(\mu)(y) = \begin{cases} \sup\{\mu(x) : x \in f^{-1}(y)\} & if \quad f^{-1}(y) \neq \emptyset \\ 0 & if \quad f^{-1}(y) = \emptyset \end{cases}$$

for all $y \in Y$ and $f^{-1}(\nu)$ is a fuzzy set in X defined by $f^{-1}(\nu)(x) = \nu(f(x))$ for all $x \in X$.

Definition 2.11. Let X be a lattice. A fuzzy set μ is called a *fuzzy lattice filter* in X if it satisfies: for all $x, y \in X$,
(1) $\mu(x) \leq \mu(1)$,
(2) $min\{\mu(x), \mu(y)\} \leq \mu(x \wedge y)$.
The set of all fuzzy lattice filter in X is denoted by $\mathcal{FL}(X)$.

Definition 2.12. ([3], [14]) Let A be a residuated lattice. A fuzzy set μ is called a *fuzzy filter* in A if it satisfies: for all $x, y \in A$,
$(fF1)$ $\mu(x) \leq \mu(1)$,
$(fF4)$ $min\{\mu(x), \mu(x \to y)\} \leq \mu(y)$.
The set of all fuzzy filter in A is denoted by $\mathcal{F}(A)$.

Theorem 2.13. ([3], [14]) Let A be a residuated lattice. A fuzzy set μ in A is a fuzzy filter if and only if it satisfies: for all $x, y \in A$,
$(fF1)$ $x \leq y$ imply $\mu(x) \leq \mu(y)$,
$(fF2)$ $min\{\mu(x), \mu(y)\} \leq \mu(x * y)$.

Proposition 2.14. ([3]) Let μ be a fuzzy filter of A. If $\mu(x \to y) = \mu(1)$, then $\mu(x) \leq \mu(y)$, for any $x, y \in A$.

Definition 2.15. Let μ be a fuzzy set in a residuated lattice A. The smallest fuzzy filter in A which contains μ is said to be the *fuzzy filter generated* by μ and will be denoted by $< \mu >$.

Proposition 2.16. Let μ be a fuzzy set of a residuated lattice A. Then $< \mu > (x) = \sup\{\min\{\mu(a_1), \dots, \mu(a_n)\} : x \geq a_1 * \dots * a_n$ for some $a_1, \dots, a_n \in X\}$, for all $x \in A$.

Definition 2.17. ([5]) Let μ be a proper fuzzy filter in a residuated lattice A. μ is called a *fuzzy prime filter* if $\mu(x \vee y) \leq \max\{\mu(x), \mu(y)\}$ for all $x, y \in A$.

Theorem 2.18. ([5]) A proper subset P of a residuated lattice A is a prime filter of A if and only if χ_P is a fuzzy prime filter in A.

Theorem 2.19. ([5]) Let A and A' be residuated lattices and $f : A \to A'$ be an epimorphism. If μ is a fuzzy prime filter in A which is constant on $\ker(f)$, then $f(\mu)$ is a fuzzy prime filter in A'.

Theorem 2.20. ([5]) Let A and A' be residuated lattices and $f : A \to A'$ be a homomorphism. If ν is a fuzzy prime filter in A', then $f^{-1}(\nu)$ is a fuzzy prime filter in A.

Notation: ([5]) We shall denote the set of all fuzzy prime filter μ in a residuated lattice A such that $\mu(1) = 1$ by $Fspec(A)$. For each fuzzy set ν in A, define $\mathcal{C}(\nu) = \{\mu \in Fspec(A) : \nu \leq \mu\}$. Let $\mu = \chi_{\{a\}}$ for $a \in A$. We shall denote $\mathcal{C}(\mu)$ by $\mathcal{C}(a)$ for all $a \in A$. Thus $\mathcal{C}(a) = \{\mu \in Fspec(A) : \mu(a) = 1\}$.

Proposition 2.21 ([5]) Let μ, ν be fuzzy sets in a residuated lattice A and $a, b \in A$. Then
(1) $\mu \leq \nu$ imply $\mathcal{C}(\nu) \subseteq \mathcal{C}(\mu) \subseteq Fspec(A)$.
(2) $\mathcal{C}(\bigcup_{i \in I} \nu_i) = \bigcap_{i \in I} \mathcal{C}(\nu_i)$.
(3) $\mathcal{C}(\mu) \cup \mathcal{C}(\nu) \subseteq \mathcal{C}(< \mu > \cap < \nu >)$.
(4) $\mathcal{C}(a \wedge b) = \mathcal{C}(a) \cup \mathcal{C}(b)$,
(5) $\mathcal{C}(\chi_A) = \bigcap_{a \in A} \mathcal{C}(a)$.

Theorem 2.22.([5]) Let $\mathcal{V}(a) = Fspec(A) \backslash \mathcal{C}(a)$ and $\mathcal{B} = \{\mathcal{V}(a) : a \in A\}$. Then $\mathcal{B}$ is a base for a topology on $Fspec(A)$. The topological space $Fspec(A)$ is called *fuzzy spectrum* of A.

3   The reticulation of residuated lattices

Definition 3.1. Let A be a residuated lattice. Define

$$a \cong b \qquad \text{if and only if} \qquad \mathcal{C}(a) = \mathcal{C}(b),$$

for all $a, b \in A$. Hence $a \cong b$ iff for any $\mu \in Fspec(A)$, $(\mu(a) = 1$ iff $\mu(b) = 1)$.

Theorem 3.2. The relation $\cong$ is a congruence relation on a residuated lattice A with respect to $*$, $\wedge$ and $\vee$.

Proof: It is clear that $\cong$ is an equivalence relation on A. Suppose that $a \cong b$ and $c \cong d$ where $a, b, c, d \in A$. We will show that $a * c \cong b * d$, $a \wedge c \cong b \wedge d$ and $a \vee c \cong b \vee d$.
(1) Let $\mu \in \mathcal{C}(a * c)$. So $\mu(a * c) = 1$. By Proposition 2.2 part (2) and Theorem 2.13, we have $1 = \mu(a * c) \leq \mu(a), \mu(c)$. We get that $\mu(a) = \mu(c) = 1$. By assumption, $\mu(b) = \mu(d) = 1$. Since $b * d \leq b * d$, then $d \leq b \to (b * d)$ by Definition 2.1 part (3). We obtain that $1 = \mu(d) \leq \mu(b \to b * d)$ by Theorem 2.13. Since μ is a fuzzy filter in A, we have $1 = \min\{\mu(b), \mu(b \to b * d)\} \leq \mu(b * d)$. Then $\mu(b * d) = 1$, that is $\mu \in \mathcal{C}(b * d)$. Hence $\mathcal{C}(a * c) \subseteq \mathcal{C}(b * d)$. Similarly, we can show that $\mathcal{C}(b * d) \subseteq \mathcal{C}(a * c)$. Therefore $a * c \cong b * d$.

(2) Let $a \wedge c \cong b \wedge d$ and $\mu \in \mathcal{C}(a \wedge c)$. Thus $\mu(a \wedge c) = 1$. Since $a \wedge c \leq a, c$, then $1 = \mu(a \wedge c) \leq \mu(a), \mu(c)$ by Theorem 2.13. By assumption $\mu(b) = \mu(d) = 1$. Since μ is a fuzzy filter in A and $b * d \leq b \wedge d$, then $1 = \min\{\mu(b), \mu(d)\} \leq \mu(b * d) \leq \mu(b \wedge d)$ by Theorem 2.13. Hence $\mu(b \wedge d) = 1$ and then $\mathcal{C}(a \wedge c) \subseteq \mathcal{C}(b \wedge d)$. Similarly, we can show that $\mathcal{C}(b \wedge d) \subseteq \mathcal{C}(a \wedge c)$. Therefor $a \wedge c \cong b \wedge d$.

(3) Let $a \vee c \cong b \vee d$ and $\mu \in \mathcal{C}(a \vee b)$. Then $\mu(a \vee b) = 1$. Since μ is a fuzzy prime filter in A, we have $\mu(a) = 1$ or $\mu(b) = 1$. By assumption $\mu(c) = 1$ or $\mu(d) = 1$. Hence $\mu(c \vee d) = \max\{\mu(c), \mu(d)\} = 1$. We obtain that $\mu \in \mathcal{C}(c \vee d)$ and then $\mathcal{C}(a \vee b) \subseteq \mathcal{C}(c \vee d)$. Similarly, we can prove that $\mathcal{C}(c \vee d) \subseteq \mathcal{C}(a \vee b)$. Hence $a \vee c \cong b \vee d$. ∎

Notation: Let $\cong$ be a the congruence relation on residuated lattice A which is defined in Definition 3.1. For all $a \in A$, the equivalence class of a is denoted by $[a]$, that is $[a] = \{b \in A : a \cong b\}$. The set of all equivalence classes is denoted by $A/\cong$.

Theorem 3.3. The algebra $(A/\cong, \wedge, \vee, [0], [1])$ is a bounded lattice, where

$$[a] \vee [b] = [a \vee b] \text{ and } [a] \wedge [b] = [a \wedge b]$$

for all $a, b \in A$.

Proof: By Theorem 3.2, the operation $\wedge$ and $\vee$ are well defined. The rest of the proof is routine. ∎

Example 3.4. Consider the residuated lattice A with the universe $\{0, a, b, c, d, 1\}$. Lattice ordering is such that $0 < a, b < c < 1$, $0 < b < d < 1$ but $\{a, b\}$ and $\{c, d\}$ are incomparable. The operations of $*$ and $\rightarrow$ are given by the tables below :

$*$	0	a	b	c	d	1
0	0	0	0	0	0	0
a	0	a	0	a	0	a
b	0	0	0	0	b	b
c	0	a	0	a	b	c
d	0	0	b	b	d	d
1	0	a	b	c	d	1

$\rightarrow$	0	a	b	c	d	1
0	1	1	1	1	1	1
a	d	1	d	1	d	1
b	c	c	1	1	1	1
c	b	c	d	1	d	1
d	a	a	c	c	1	1
1	0	a	b	c	d	1

Consider $0 \leq \nu_1(0) = \nu_1(a) = \nu_1(b) = \nu_1(c) < \nu_1(d) = \nu_1(1) = 1$ and $0 \leq \nu_2(0) = \nu_2(b) = \nu_2(d) < \nu_2(c) = \nu_2(c) = \nu_2(1) = 1$. Then $Fspec(A) = \{\nu_1, \nu_2\}$. We obtain that $[0] = [b]$, $[a] = [c]$. Therefore $A/\cong = \{[0], [a], [d], [1]\}$ where $[0] < [a], [d] < 1$ but $\{[a], [d]\}$ are incomparable.

Lemma 3.5. Let A be a residuated lattice and $a, b \in A$. Then
(i) $[a] \leq [b]$ if and only if $\mathcal{C}(b) \subseteq \mathcal{C}(a)$,
(ii) if $a \leq b$, then $[a] \leq [b]$,
(iii) $[a \wedge b] = [a * b]$.

Proof: (i) By Theorem 3.3 and Proposition 2.21 parts (1) and (4), we have $[a] \le [b]$ iff $[a] \wedge [b] = [a]$ iff $[a \wedge b] = [a]$ iff $\mathcal{C}(a) = \mathcal{C}(a \wedge b) = \mathcal{C}(a) \cup \mathcal{C}(b)$ iff $\mathcal{C}(b) \subseteq \mathcal{C}(a)$.

(ii) If $a \le b$, then $\mathcal{C}(a) \subseteq \mathcal{C}(b)$. We obtain that $[a] \le [b]$ by (i).

(iii) We will show that $\mathcal{C}(a * b) = \mathcal{C}(a \wedge b)$. Let $\mu \in \mathcal{C}(a * b)$. Then $\mu(a * b) = 1$. By Proposition 2.2 part (2) and Theorem 2.13, $\mu(a * b) \le \mu(a \wedge b)$. We get that $\mu(a \wedge b) = 1$ and then $\mu \in \mathcal{C}(a \wedge b)$. Hence $\mathcal{C}(a * b) \subseteq \mathcal{C}(a \wedge b)$.

Conversely, let $\mu \in \mathcal{C}(a \wedge b)$. Then $\mu(a \wedge b) = 1$. Since $a \wedge b \le a, b$, then $\mu(a) = \mu(b) = 1$ by Theorem 2.13. Since $b \le a \to (a * b)$ and μ is a fuzzy filter in A,

$$1 = \min\{\mu(a), \mu(b)\} \le \min\{\mu(a), \mu(a \to (a * b))\} \le \mu(a * b).$$

Hence $\mu(a * b) = 1$ and then $\mu \in \mathcal{C}(a * b)$. We get that $\mathcal{C}(a \wedge b) \subseteq \mathcal{C}(a * b)$. Therefor $[a \wedge b] = [a * b]$. ∎

Theorem 3.6. The bounded lattice $(A/\cong, \wedge, \vee, [0], [1])$ is distributive.

Proof: Let $a, b, c \in A$. By Lemma 3.5 and Proposition 2.2 part (3),

$$\begin{aligned}
[a] \wedge ([b] \vee [c]) &= [a \wedge (b \vee c)] = [a * (b \vee c)] \\
&= [(a * b) \vee (a * c)] = [a * b] \vee [a * c] \\
&= [a \wedge b] \vee [a \wedge c] = ([a] \wedge [b]) \vee ([a] \wedge [c]). \blacksquare
\end{aligned}$$

Definition 3.7. Let A be a residuated lattice and $\pi : A \to A/\cong$ be that canonical surjective map defined by $\pi(a) = [a]$. Then $(A/\cong, \pi)$ is called *the reticulation* of residuated lattice induced by fuzzy filters.

Lemma 3.8. Let A_1 and A_2 be residuated lattices and $f : A_1 \to A_2$ be a homomorphism of residuated lattices. Then $\mathcal{C}(a) = \mathcal{C}(b)$ implies $\mathcal{C}(f(a)) = \mathcal{C}(f(b))$, for any $a, b \in A_1$.

Proof: Suppose that $\mathcal{C}(a) = \mathcal{C}(b)$ where $a, b \in A_1$ and $\nu \in \mathcal{C}(f(a))$. Then $\nu \in Fspec(A_2)$ and $\nu(f(a)) = 1$. By Theorem 2.20, we have $f^{-1}(\nu) \in Fspec(A_1)$ and $f^{-1}(\nu)(a) = \nu(f(a)) = 1$. Thus $f^{-1}(\nu) \in \mathcal{C}(a) = \mathcal{C}(b)$. We get that $\nu(f(b)) = f^{-1}(\nu)(b) = 1$ and then $\nu \in \mathcal{C}(f(b))$. Hence $\mathcal{C}(f(a)) \subseteq \mathcal{C}(f(b))$. Similarly, we can show that $\mathcal{C}(f(b)) \subseteq \mathcal{C}(f(a))$. ∎

In the following theorem, we will define a functor from the category of residuated lattices to the category of bounded distributive lattices.

Theorem 3.9. Let A_1 and A_2 be residuated lattices and $f : A_1 \to A_2$ be a homomorphism of residuated lattices. Then $\bar{f} : A_1/\cong \to A_2/\cong$ is defined by $\bar{f}([a]) = [f(a)]$ is a homomorphism of lattices.

Proof: Let $[a] = [b]$. By Lemma 3.5 part (i), we obtain that $\mathcal{C}(a) = \mathcal{C}(b)$. By Lemma 3.8, $\mathcal{C}(f(a)) = \mathcal{C}(f(b))$. We have $[f(a)] = [f(b)]$ by Lemma 3.5 part (i). So $\bar{f}$ is well defined. Now, Let $a, b \in A_1$. Since f is a homomorphism of residuated lattices, then

$$\bar{f}([a] \wedge [b]) = \bar{f}([a \wedge b]) = [f(a \wedge b)] = [f(a)] \wedge [f(b)] = \bar{f}([a]) \wedge \bar{f}([b]).$$

Similarly, we can show that $\bar{f}([a] \vee [b]) = \bar{f}([a]) \vee \bar{f}([b])$. Also, $\bar{f}([0]) = [f(0)] = [0]$ and $\bar{f}([1]) = [f(1)] = [1]$. Hence $\bar{f}$ is a homomorphism of lattices.■

Lemma 3.10. Let μ be a fuzzy filter in a residuated lattice A and $a, b \in A$ such that $[a] = [b]$. Then $\mu(a) = \mu(b)$.

Proof: Suppose that μ is a fuzzy filter in A such that $\mu(a) \neq \mu(b)$. Then $\mu(a) < \mu(b)$ or $\mu(b) < \mu(a)$. Let $\mu(a) < \mu(b)$. Put $F = \{x \in A : \mu(x) \geq \mu(b)\}$, i.e. $F = \mu_{\mu(b)}$. Hence F is a filter of A such that $a \notin F$. Define $J = < F \cup \{b\} >$. Then J is a filter of A. We shall show that $a \notin J$. Suppose that $a \in J$. By Proposition 2.6, there exist $y_1, ..., y_n \in F \cup \{b\}$ such that $y_1 * ... * y_n \leq a$. If $y_i = b$ for some $1 \leq i \leq n$, then $y_1 * ... * y_{i-1} * y_{i+1} ... * y_n * b \leq a$. Hence $y_1 * ... * y_{i-1} * y_{i+1} ... * y_n \leq b \to a$. Since F is a filter, we have $b \to a \in F$, that is $\mu(b \to a) \geq \mu(b)$. So $\mu(b) = \min\{\mu(b), \mu(b \to a)\} \leq \mu(a)$ which is a contradiction. Now, suppose that $y_i \in F$ for all $1 \leq i \leq n$. Thus $y_1 * ... * y_n \in F$. We get that $a \in F$ which is a contradiction. Hence $a \notin J$ and J is a proper filter. By Proposition 2.8, there exists a prime filter P such that $J \subseteq P$ and $a \notin P$. By Theorem 2.18, χ_P is a fuzzy prime filter in A such that $\chi_P(b) = 1$ and $\chi_P(a) \neq 1$. We obtain that $\chi_P \in \mathcal{C}(b)$ but $\chi_P \notin \mathcal{C}(a)$ which is a contradiction. Hence $\mu(a) = \mu(b)$. ■

Theorem 3.11. Let μ be a fuzzy filter in a residuated lattice L. Then $\pi(\mu)$ is a fuzzy lattice filter in $A/ \cong$ and $\pi^{-1}(\pi(\mu)) = \mu$.

Proof: Let $[a], [b] \in A/ \cong$. Then $\pi(a) = [a]$ and $\pi(b) = [b]$. Since π is a homomorphism, we have $[a] \wedge [b] = [a \wedge b] = \pi(a \wedge b)$. We get that $a \wedge b = \pi^{-1}(x \wedge y)$. We have

$$
\begin{aligned}
\pi(\mu)([a] \wedge [b]) &= \sup\{\mu(z) : z \in \pi^{-1}[a \wedge b]\} \\
&\geq \sup\{\mu(x \wedge y) : x \in \pi^{-1}([a]), y \in \pi^{-1}([b])\} \\
&= \sup\{\min\{\mu(x), \mu(y)\} : x \in \pi^{-1}([a]), y \in \pi^{-1}([b])\} \\
&= \min\{\sup\{\mu(x) : x \in \pi^{-1}([a])\}, \sup\{\mu(b) : y \in \pi^{-1}([b])\}\} \\
&= \min\{\pi(\mu)(a), \pi(\mu)(b)\}.
\end{aligned}
$$

Let $[a] \leq [b]$. Then $\pi(a) \leq \pi(b)$. We shall show that $\pi(\mu)([a]) \leq \pi(\mu)([b])$. Suppose that $\pi(\mu)[a] > \pi(\mu)[b]$. Then there exists $x_0 \in \pi^{-1}([a])$ such that $\pi(x_0) = a$ and $\mu(x_0) > \sup\{\mu(y) : y \in \pi^{-1}(b)\}$. We have $\mu(y) \leq \mu(x_0)$ for all $y \in \pi^{-1}(b)$. Let $y \in \pi^{-1}(b)$ be arbitrary. Since π is a lattice homomorphism, then $[b] = [a] \vee [b] = \pi(x_0) \vee \pi(y) = \pi(x_0 \vee y)$. Hence $x_0 \vee y \in \pi^{-1}(b)$. Therefore $\mu(x_0 \vee y) < \mu(x_0)$. By Definition 2.17, $\mu(x_0 \vee y) \geq \max\{\mu(x_0), \mu(y)\} = \mu(x_0)$ which is a contradiction. Hence $\pi(\mu)$ is a fuzzy lattice filter in $A/ \cong$. By Lemma 3. 10, we have $\pi^{-1}(\pi(\mu))(a) = \pi(\mu)(\pi(a)) = \pi(\mu)[a] = \sup\{\mu(x) : x \in \pi^{-1}([a])\} = \sup\{\mu(x) : \pi(x) = [a]\} = \sup\{\mu(x) : [x] = [a]\} = \mu(a)$.■

Theorem 3.12. Let ν be a fuzzy lattice filter in a lattice $A/\cong$. Then $\pi^{-1}(\nu)$ is a fuzzy filter in A and $\pi(\pi^{-1}(\nu)) = \nu$.

Proof: Let $x, y \in A$. By Lemma 3.5 part (iii), we have $\pi^{-1}(\nu)(x * y) = \nu(\pi(x * y)) = \nu([x * y]) = \nu([x \wedge y]) = \nu([x] \wedge [y]) \geq \min\{\nu([x]), \nu([y])\} = \min\{\pi^{-1}(\nu)(x), \pi^{-1}(\nu)(y)\}$. Suppose that $x \leq y$. By Lemma 3.5 part (ii), we have $[x] \leq [y]$. Since ν is a fuzzy lattice filter in $A/\cong$, we have $\nu([x]) \leq \nu([y])$, that is $\pi^{-1}(\nu)(x) \leq \pi^{-1}(\nu)(y)$. By Lemma 3.10, we obtain that $\pi(\pi^{-1}(\nu))[x] = \sup\{\pi^{-1}(\nu)(y) : y \in \pi^{-1}([x])\} = \sup\{\pi^{-1}(\nu)(y) : \pi(y) = [x]\} = \sup\{\pi^{-1}(\nu)(y) : [y] = [x]\} = \nu([x])$.∎

Proposition 3.13. Let μ and ν be fuzzy filters in a residuated lattice A. Then $\nu \leq \mu$ if and only if $\pi(\nu) \leq \pi(\mu)$.

Proof: Suppose that $\nu \leq \mu$. Then $\pi(\nu)([x]) = \sup\{\nu(y) : y \in \pi^{-1}([x])\} \leq \sup\{\mu(y) : y \in \pi^{-1}([x])\} = \pi(\mu)([x])$. Conversely, let $\pi(\nu) \leq \pi(\mu)$. Then $\nu(a) = \pi^{-1}(\pi(\nu))(a) = \pi(\nu)(\pi(a)) \leq \pi(\mu)(\pi(a)) = \pi^{-1}(\pi(\mu))(a) = \mu(a)$.∎

Theorem 3.14. There is a lattice isomorphism between the lattices $\mathcal{F}(A)$ and $\mathcal{FL}(A/\cong)$.

Proof: Define $\varphi : \mathcal{F}(A) \to \mathcal{FL}(A/\cong)$ by $\varphi(\mu) = \pi(\mu)$ and $\psi : \mathcal{F}(L/\equiv) \to \mathcal{F}(L)$ by $\psi(\nu) = \pi^{-1}(\nu)$. By Theorems 3.11 and 3.12 φ and ψ are well defined and bijection. By the above Proposition φ is a lattice homomorphism. Hence ϕ is an isomorphism of lattices.∎

Theorem 3.15. Let μ be a fuzzy prime filter in a residuated lattice A. Then $\pi(\mu)$ is a fuzzy prime filter in $A/\cong$.

Proof: Since μ is a fuzzy prime filter in A, then μ is proper. So $\mu(0) \neq \mu(1)$. By Lemma 3.10, $\pi(\mu)(0) = \sup\{\mu(x) : x \in \pi^{-1}([0])\} = \sup\{\mu(x) : [x] = [0]\} = \mu(0) = 0$ and $\pi(\mu)(1) = \sup\{\mu(x) : x \in \pi^{-1}([1])\} = \sup\{\mu(x) : [x] = [1]\} = \mu(1) = 1$. Hence $\pi(\mu)$ is proper. We have $\pi(\mu)([x \vee y]) = \sup\{\mu(z) : z \in \pi^{-1}([x \vee y])\} = \sup\{\mu(x) : [z] = [x \vee y]\} = \mu(x \vee y) = \max\{\mu(x), \mu(y)\}$ Also, we have $\pi(\mu)[x] = \sup\{\mu(a) : a \in \pi^{-1}([x])\}. = \sup\{\mu(a) : [a] = [x]\} = \mu(x)$. Similarly, we can show that $\pi(\mu)[y] = \mu(y)$. We obtain that $\pi(\mu)(x \vee y) = \mu(x \vee y) = \max\{\mu(x), \mu(y)\} = \max\{\pi(\mu)(x), \pi(\mu)(y)\}$ and then $\pi(\mu)$ is a fuzzy prime filter in $A/\cong$.∎

Theorem 3.16. Let ν be a fuzzy prime filter in a lattice $A/\cong$. Then $\pi^{-1}(\nu)$ is a fuzzy prime filter in A.

Proof: By assumption ν is proper. Hence $\nu([0]) \neq \nu([1])$. We have $\pi^{-1}(\nu)(0) = \nu(\pi(0)) = \nu([0])$ and $\pi^{-1}(\nu)(1) = \nu(\pi(1)) = \nu([1])$. Hence $\pi^{-1}(\nu)(0) \neq \pi^{-1}(\nu)(1)$. That is $\pi^{-1}(\nu)$ is proper. Also, we have
$\pi^{-1}(\nu)(x \vee y) = \nu(\pi(x \vee y)) = \nu([x \vee y]) = \nu([x] \vee [y]) = \max\{\nu([x]), \nu([y])\} = \max\{\pi^{-1}(\nu)(x), \pi^{-1}(\nu)(y)\}$. ∎

Theorem 3.17. There exists a homomorphism between topological Space $Fspec(A)$ and $Fspec(A/\cong)$.

Proof: Consider φ in Theorem 3.14. The restriction φ to $Fspec(A)$ is denoted by $\bar{\varphi}$. By Theorems 3.15 and 3.16, $\bar{\varphi} : Fspec(A) \to Fspec(A/\equiv)$ is a bijective. We will show that $\bar{\varphi}$ is continuous and closed. Let $\mathcal{C}([a])$ be an arbitrary closed base set. Then

$$
\begin{aligned}
\bar{\varphi}^{-1}(\mathcal{C}([a])) &= \{\mu \in Fspec(A) : \bar{\varphi}(\mu) \in \mathcal{C}([a])\} \\
&= \{\mu \in Fspec(A) : \pi(\mu) \in \mathcal{C}([a])\} \\
&= \{\mu \in Fspec(A) : \pi(\mu)[a] = 1\} \\
&= \{\mu \in Fspec(A) : \mu(a) = 1\} = \mathcal{C}(a).
\end{aligned}
$$

Hence φ is continuous. Also, we have

$$
\begin{aligned}
\bar{\varphi}(\mathcal{C}(a)) &= \{\varphi(\mu) : \mu \in Fspec(A), \mu \in \mathcal{C}(a)\} \\
&= \{\pi(\mu) : \mu \in Fspec(A), \mu \in \mathcal{C}(a)\} \\
&= \{\pi(\mu) : \mu \in Fspec(A), \mu(a) = 1\} \\
&= \{\nu \in Fspec(A/\cong) : \nu([a]) = 1\} = \mathcal{C}([a]).
\end{aligned}
$$

Hence φ is closed.■

Let A be a residuated lattice. For any $a, b \in A$ define $a \equiv b$ iff for any $P \in Spec(A)$, $(a \in P$ iff $b \in P)$. Then $\equiv$ is a congruence relation on A respect to $*$, $\wedge$ and $\vee$. Let us denot by $\bar{a}$ the equivalence class of $a \in A$ and let $A/\equiv$ be the quotient set. We denote $\lambda : A \to A/\equiv$ the canonical surjective defined by $\lambda(a) = \bar{a}$. Then $(A/\equiv, \vee, \wedge, 0, 1)$ is a bounded distributive lattice and $(A/\equiv, \lambda)$ is a reticulation of A (See [8]).

Theorem 3.18. Let A be a residuated lattice. Then the congruence relation $\cong$ is equal to the congruence relation $\equiv$ on A.

Proof: Let $a, b \in A$ such that $a \cong b$. We have $(\mu(a) = 1$ iff $\mu(b) = 1)$ for any $\mu \in Fspec(A)$. Suppose that $P \in Spec(A)$. By Theorem 2.18, χ_P is a fuzzy prime filter. Hence $\chi_P(a) = 1$ iff $\chi_P(b) = 1$. We get that $a \in P$ iff $b \in P$. Hence $a \equiv b$ and then $\cong \subseteq \equiv$.
Conversely, let $a \equiv b$ and $\mu \in Fspec(A)$ such that $\mu(a) = 1$. We get that $a \in \mu_1$ and μ_1 is a proper filter of A. Hence $\mu_1 \in Spec(A)$. Since $a \equiv b$, then we have $b \in \mu_1$. We obtain that $\mu(b) = 1$. Similarly, we can prove that if $\mu(b) = 1$, then $\mu(b) = 1$. So $a \cong b$. Therefor $\equiv \subseteq \cong$.■

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Remarks concerning the pexiderized Gołąb–Schinzel functional equation

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ABSTRACT: This paper is devoted to proof of theorem concerning solutions of the pexiderized Gołąb–Schinzel functional equation. We provide explicite formulas expressing solutions of the equation. Our considerations refer to the paper [6].

In the paper we consider the pexiderized Gołąb–Schinzel functional equation, i.e. the equation

$$f(x + g(x)y) = h(x)k(y) \tag{1}$$

in the class of unknown functions $f, g, h, k : X \to \mathbb{K}$, where X is a linear space over a commutative field $\mathbb{K}$. This equation generalizes one of the Pexider equations, i.e.

$$f(x + y) = g(x)h(y),$$

which is very well-known for over hundred years (see [7]), as well as the Gołąb–Schinzel equation

$$f(x + f(x)y) = f(x)f(y),$$

which appeared in 1959 in [4] and has been extensively studied by many authors (for more information see a survey paper [1]).

In 1966 E. Vincze introduced equation (1) in [8]. Next papers concerning it have been published over forty years later (see [2], [6]).

The principal aim of the paper is to prove the theorem, which characterizes general solutions of the equation (1) combined with a partially pexiderized Gołąb–Schinzel equation, i.e. the equation

$$f(x + g(x)y) = f(x)f(y). \tag{2}$$

Our main result is:

Theorem 1. (cf. [6, Theorem 1]) *Let X be a linear space over a commutative field $\mathbb{K}$. Functions $f, g, h, k : X \to \mathbb{K}$ satisfy* (1) *iff they have one of the following forms:*

(i) $\begin{cases} f = 0, \\ h = 0, \\ g, k \text{ are arbitrary} \end{cases}$ *or* $\begin{cases} f = 0, \\ k = 0, \\ g, h \text{ are arbitrary}; \end{cases}$

(ii) *there are $a, b \in \mathbb{K} \setminus \{0\}$ such that* $\begin{cases} f = ab, \\ g \text{ is arbitrary}, \\ h = a, \\ k = b; \end{cases}$

(iii) *there is a $b \in \mathbb{K} \setminus \{0\}$ such that* $\begin{cases} f = bh, \\ g = 0, \\ h \text{ is arbitrary nonconstant}, \\ k = b; \end{cases}$

(iv) *there are $a, b, c \in \mathbb{K} \setminus \{0\}$ and functions $F, G : X \to \mathbb{K}$ with $F \neq 1$ and $F(0) = G(0) = 1$, such that F and G satisfy the equation* (2) *and*
$\begin{cases} f = abF, \\ g = cG, \\ h = aF, \\ k(x) = bF(cx) \ \text{for } x \in X; \end{cases}$

(v) *there are $x_0 \in X \setminus \{0\}$, $a, b \in \mathbb{K} \setminus \{0\}$ and functions $F, G : X \to \mathbb{K}$ with $F(0) = G(0) = 1$, $F(-x_0) = G(-x_0) = 0$, such that F and G satisfy the equation* (2) *and* $\begin{cases} f(x) = abF(x - x_0) & \text{for } x \in X, \\ g(x) = g(x_0)G(x - x_0) & \text{for } x \in X, \\ h(x) = aF(x - x_0) & \text{for } x \in X, \\ k(x) = bF(g(x_0)x) & \text{for } x \in X. \end{cases}$

Proof. By [6, Theorem 1 (i)–(iv)] conditions (i)–(iv) of the theorem holds. Now we have to prove (v). According to [6, Theorem 1(v)] there are $x_0 \in X \setminus \{0\}$, $a, b \in \mathbb{K} \setminus \{0\}$ and a function $f_0 : X \to \mathbb{K}$ with

$$f_0(x_0) = 1, \ \ f_0(0) = g(0) = 0, \tag{3}$$

such that f_0 and g satisfy the equation

$$f_0(x + g(x)y) = f_0(x)f_0(x_0 + g(x_0)y) \ \text{ for every } \ x, y \in X \tag{4}$$

and

$$\begin{cases} f = abf_0, \\ h = af_0, \\ k(x) = bf_0(x_0 + g(x_0)x) \ \text{ for } \ x \in X. \end{cases} \tag{5}$$

First consider the case, when $g(x_0) = 0$. Then equation (4) has the following form:

$$f_0(x + g(x)y) = f_0(x).$$

Suppose that $g(y_0) \neq 0$ for some $y_0 \in X$. Then, for every $z \in X$, there exists a $y \in X$ such that $z = y_0 + g(y_0)y$ and hence, by (4),

$$f_0(z) = f_0(y_0 + g(y_0)y) = f_0(y_0) \text{ for every } z \in X.$$

It means that f_0 is constant, what contradicts (3). So, $g = 0$ and f_0 is arbitrary. Hence, by (5), $f = abf_0$, $g = 0$, $h = af_0$ and $k = b$ with an arbitrary function f_0. Thus $f = bh$, $g = 0$, h is arbitrary and $k = b$ and consequently functions f, g, h, k have the same form as in condition (iii).

Now we consider the case, when $g(x_0) \neq 0$. Define functions $F, G : X \to \mathbb{K}$ as follows:

$$F(x) = f_0(x + x_0) \text{ for } x \in X,$$
$$G(x) = \frac{g(x+x_0)}{g(x_0)} \text{ for } x \in X.$$

Clearly $F(0) = G(0) = 1$ and $F(-x_0) = G(-x_0) = 0$. Moreover, by (4), for every $x, y \in X$ we have:

$$F(x + G(x)y) = F\left(x + \frac{g(x+x_0)}{g(x_0)}y\right) = f_0\left(x + x_0 + g(x + x_0)\frac{y}{g(x_0)}\right)$$
$$= f_0(x + x_0)f_0\left(x_0 + g(x_0)\frac{y}{g(x_0)}\right) = F(x)F(y).$$

Hence functions F, G satisfy (2), what ends the proof of condition (v). $\qquad \square$

Theorem 1 shows that the pexiderized Gołąb–Schinzel equation is tightly connected with the equation (2). The equation (2) has been considered by J. Chudziak [3] in the class of real functions f, g, where g is continuous, or by the author of [5] in the class of continuous on rays functions $f, g : X \to \mathbb{R}$ (where X is a real linear space).

Using Theorem 1 and the result of J. Chudziak [3, Theorem 1], we obtain the following corollary.

Corollary 1. *Functions $f, g, h, k : \mathbb{R} \to \mathbb{R}$ satisfy* (1) *and g is continuous if and only if they have one of the following forms:*

(i) $\begin{cases} f = 0, \\ g \text{ is arbitrary continuous,} \\ h = 0, \\ k \text{ is arbitrary,} \end{cases}$ *or* $\begin{cases} f = 0, \\ g \text{ is arbitrary continuous,} \\ h \text{ is arbitrary,} \\ k = 0; \end{cases}$

(ii) *there are $a, b \in \mathbb{R} \setminus \{0\}$ such that* $\begin{cases} f = ab, \\ g \text{ is arbitrary continuous,} \\ h = a, \\ k = b; \end{cases}$

(iii) *there is a $b \in \mathbb{R} \setminus \{0\}$ such that* $\begin{cases} f = bh, \\ g = 0, \\ h \text{ is arbitrary nonconstant,} \\ k = b; \end{cases}$

(iii) *there is a* $b \in \mathbb{R} \setminus \{0\}$ *such that* $\begin{cases} f = bh, \\ g = 0, \\ h \text{ is arbitrary nonconstant continuous on rays,} \\ k = b; \end{cases}$

(iv) *there are a nontrivial linear functional* $L : X \to \mathbb{R}$, $a, b, c \in \mathbb{R} \setminus \{0\}$ *and* $r > 0$

    such that $\begin{cases} f = abF, \\ g = cG, \\ h = aF, \\ k(x) = bF(cx) \text{ for } x \in X, \end{cases}$

    where F *and* G *are defined by one of the following four formulas:*

    $-$ $G = 1$ *and* $F = \exp L$;

    $-$ $\begin{cases} G(x) = L(x) + 1 & \text{for } x \in X, \\ F(x) = |L(x) + 1|^r & \text{for } x \in X; \end{cases}$

    $-$ $\begin{cases} G(x) = L(x) + 1 & \text{for } x \in X, \\ F(x) = |L(x) + 1|^r \mathrm{sgn}\,(L(x) + 1) & \text{for } x \in X; \end{cases}$

    $-$ $\begin{cases} G(x) = \max\{L(x) + 1, 0\} & \text{for } x \in X, \\ F(x) = (\max\{L(x) + 1, 0\})^r & \text{for } x \in X; \end{cases}$

(v) *there are a nontrivial linear functional* $L : X \to \mathbb{R}$, $a, b, c \in \mathbb{R} \setminus \{0\}$ *and* $r > 0$

    such that either $\begin{cases} f = ab(\phi \circ L), \\ g = cL, \\ h = a(\phi \circ L), \\ k(x) = b\phi(1 + cL(x)) \text{ for } x \in X, \end{cases}$

    where $\phi : \mathbb{R} \to \mathbb{R}$ *has one of the following two forms:*

$$\phi(\alpha) = |\alpha|^r \text{ for } \alpha \in \mathbb{R} \quad or \quad \phi(\alpha) = |\alpha|^r \mathrm{sgn}\,\alpha \text{ for } \alpha \in \mathbb{R},$$

    or $\begin{cases} f(x) = ab(\max\{L(x), 0\})^r & \text{for } x \in X, \\ g(x) = c\max\{L(x), 0\} & \text{for } x \in X, \\ h(x) = a(\max\{L(x), 0\})^r & \text{for } x \in X, \\ k(x) = b(\max\{cL(x) + 1, 0\})^r & \text{for } x \in X. \end{cases}$

At the end of the paper let us mention that equation (1) has been treated in [6] in the class of real continuous functions f, g, h, k (see [6, Corollary 1]), but the proof given there is not correct, because [6, Proposition 1] does not hold (to see this it is enough to choose functions $f(x) = g(x) = \max\{x, 0\}$ for $x \in \mathbb{R}$). Consequently, [6, Theorem 2] and [6, Corollary 1] were not stated thoroughly, because their proofs base on [6, Proposition 1].

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Controllability of the semilinear Benjamin-Bona-Mahony equation

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ABSTRACT: In this paper we prove the interior approximate controllability of the following Generalized Semilinear Benjamin-Bona-Mahony type equation (BBM) with homogeneous Dirichlet boundary conditions

$$
\begin{cases}
z_t - a\Delta z_t - b\Delta z = 1_\omega u(t,x) + f(t,z,u(t,x)), & t \in (0,\tau], \quad x \in \Omega, \\
z(t,x) = 0, \quad t \geq 0, \quad x \in \partial\Omega,
\end{cases}
$$

where $a \geq 0$ and $b > 0$ are constants, Ω is a domain in $I\!\!R^N$, ω is an open nonempty subset of Ω, 1_ω denotes the characteristic function of the set ω, the distributed control u belongs to $L^2(0,\tau; L^2(\Omega))$ and the nonlinear function $f : [0,\tau] \times I\!\!R \times I\!\!R \to I\!\!R$ is smooth enough and there are $c, d, e \in I\!\!R$, with $c \neq -1$, $ea + b > 0$, such that

$$
\sup_{(t,z,u) \in Q_\tau} |f(t,z,u) - ez - cu - d| < \infty,
$$

where $Q_\tau = [0,\tau] \times I\!\!R \times I\!\!R$. We prove that for all $\tau > 0$ and any nonempty open subset ω of Ω the system is approximately controllable on $[0,\tau]$. Moreover, we exhibit a sequence of controls steering the system from an initial state z_0 to an ϵ-neighborhood of the final state z_1 on time $\tau > 0$. As a consequence of this result we obtain the interior approximate controllability of the semilinear heat equation by putting $a = 0$ and $b = 1$.

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Key Words and Phrases: *interior controllability, semilinear BBM equation, strongly continuous semigroups*

1   Introduction.

As we pointed out in [11], the original Benjamin-Bona-Mohany Equation is a nonlinear one; even so, in this reference we proved the interior controllability of the linear BBM equation, which is essential for a subsequent study of the nonlinear BBM equation. So, in this paper we shall prove the interior controllability of the following Generalized Semilinear Benjamin-Bona-Mahony type equation (BBM) with homogeneous Dirichlet boundary conditions

$$\begin{cases} z_t - a\Delta z_t - b\Delta z = 1_\omega u(t,x) + f(t,z,u(t,x)), & t \in (0,\tau], \quad x \in \Omega, \\ z(t,x) = 0, & t \geq 0, \quad x \in \partial\Omega, \end{cases} \quad (1.1)$$

where $a \geq 0$ and $b > 0$ are constants, Ω is a domain in $I\!\!R^N$, ω is an open nonempty subset of Ω, 1_ω denotes the characteristic function of the set ω, the distributed control $u \in L^2(0,\tau; L^2(\Omega))$ and the nonlinear function $f : [0,\tau] \times I\!\!R \times I\!\!R \to I\!\!R$ is smooth enough and there are $c, d, e \in I\!\!R$, with $c \neq -1$, $ea + b > 0$, such that

$$\sup_{(t,z,u) \in Q_\tau} |f(t,z,u) - ez - cu - d| < \infty, \quad (1.2)$$

where $Q_\tau = [0,\tau] \times I\!\!R \times I\!\!R$. Under these conditions we prove the following statement: For all $\tau > 0$ and any nonempty open subset ω of Ω the system is approximately controllable on $[0,\tau]$. Moreover, we exhibit a sequence of controls steering the system from an initial state z_0 to an ϵ-neighborhood of the final state z_1 on time $\tau > 0$. As a consequence of this result we obtain the interior approximate controllability of the semilinear heat equation by putting $a = 0$ and $b = 1$.

We note that, the interior approximate controllability of the linear heat equation

$$\begin{cases} z_t(t,x) = \Delta z(t,x) + 1_\omega u(t,x) & \text{in} \quad (0,\tau] \times \Omega, \\ z = 0, & \text{on} \quad (0,\tau] \times \partial\Omega, \\ z(0,x) = z_0(x), & x \in \Omega, \end{cases} \quad (1.3)$$

has been study by several authors, particularly by [15],[16],[17]; and in a general fashion in [14].

The approximate controllability of the heat equation under nonlinear perturbation $f(z)$ independents of t and u variables

$$\begin{cases} z_t(t,x) = \Delta z(t,x) + 1_\omega u(t,x) + f(z) & \text{in} \quad (0,\tau] \times \Omega, \\ z = 0, & \text{on} \quad (0,\tau] \times \partial\Omega, \\ z(0,x) = z_0(x), & x \in \Omega, \end{cases} \quad (1.4)$$

has been studied by several authors, particularly in [6], [7] and [8], depending on conditions impose to the nonlinear term $f(z)$. For instance, in [7] and [8] the approximate controllability of the system (1.4) is proved if $f(z)$ is sublinear at infinity, i.e.,

$$|f(z)| \leq E|z| + D. \quad (1.5)$$

Also, in the above references, the authors mentioned that when f is superlinear at the infinity, the approximate controllability of the system (1.4) fails.

In this paper we use different technique for the linear part (see [14], [11]) and Schauder fixed point Theorem for the semilinear system.

Now, we shall describe the strategy of this work:

First, we observe that the hypothesis (1.2) is equivalent to the existence of $e, c \in \mathbb{R}$, with $c \neq -1$, $ea + b > 0$, such that

$$\sup_{(t,z,u) \in Q_\tau} |f(t,z,u) - ez - cu| < \infty, \tag{1.6}$$

where $Q_\tau = [0, \tau] \times \mathbb{R} \times \mathbb{R}$.

Second, we prove that the auxiliary linear system

$$\begin{cases} z_t - a\Delta z_t - b\Delta z = 1_\omega u(t,x) + ez + cu(t,x), & t \in (0,\tau], \quad x \in \Omega, \\ z(t,x) = 0, \quad t \geq 0, \qquad x \in \partial\Omega, \end{cases} \tag{1.7}$$

is approximately controllable.

After that, we write the system(1.1) as follows

$$\begin{cases} z_t - a\Delta z_t - b\Delta z = 1_\omega u(t,x) + ez + cu(t,x) + g(t,z,u(t,x)), & t \in (0,\tau], \ x \in \Omega, \\ z(t,x) = 0, \quad t \geq 0, \qquad x \in \partial\Omega, \end{cases}$$
$$\tag{1.8}$$

where $g(t,z,u) = f(t,z,u) - ez - cu$ is an smooth and bounded function.

The technique we use here to prove the controllability of the linear equation (1.7) is based in the following results:

Theorem 1.1. *(see Theorem 1.23 from [2], p. 20) Suppose $\Omega \subset \mathbb{R}^n$ is open, nonempty and connected set, and f is real analytic function in Ω with $f = 0$ on a non-empty open subset ω of Ω. Then, $f = 0$ in Ω.*

Lemma 1.1. *(see Lemma 3.14 from [4], p. 62)Let $\{\alpha_j\}_{j\geq 1}$ and $\{\beta_{i,j} : i = 1, 2, \ldots, m\}_{j\geq 1}$ be sequences of real numbers such that: $\alpha_1 > \alpha_2 > \alpha_3 \cdots$. Then*

$$\sum_{j=1}^\infty e^{\alpha_j t} \beta_{i,j} = 0, \quad \forall t \in [0, t_1], \quad i = 1, 2, \cdots, m$$

if and only if

$$\beta_{i,j} = 0, \quad i = 1, 2, \cdots, m; j = 1, 2, \cdots, \infty.$$

Finally, the approximate controllability of the system (1.8) follows from the controllability of (1.7) and Schauder fixed point Theorem.

2   Abstract Formulation of the Problem.

In this section we describe the space in which this problem will be situated as an abstract ordinary differential equation.

Let $Z = L^2(\Omega) = L^2(\Omega, I\!R)$ and consider the linear unbounded operator $A :$ $D(A) \subset Z \to Z$ defined by $A\phi = -\Delta\phi$, where

$$D(A) = H^2(\Omega, I\!R) \cap H_0^1(\Omega, I\!R).$$

The operator A has the following very well known properties: the spectrum of A consists of eigenvalues

$$0 < \lambda_1 < \lambda_2 < \cdots < \lambda_j < \cdots \qquad \text{with} \qquad \lambda_j \to \infty, \tag{2.1}$$

each one with finite multiplicity γ_j equal to the dimension of the corresponding eigenspace. Therefore:
a) There exists a complete orthonormal set $\{\phi_{j,k}\}$ of eigenvectors of A.
b) For all $z \in D(A)$ we have

$$Az = \sum_{j=1}^{\infty} \lambda_j \sum_{k=1}^{\gamma_j} < z, \phi_{j,k} > \phi_{j,k} = \sum_{j=1}^{\infty} \lambda_j E_j z, \tag{2.2}$$

where $< \cdot, \cdot >$ is the inner product in Z and

$$E_j z = \sum_{k=1}^{\gamma_j} < z, \phi_{j,k} > \phi_{j,k}. \tag{2.3}$$

So, $\{E_j\}$ is a family of complete orthogonal projections in Z and

$$z = \sum_{j=1}^{\infty} E_j z, \quad z \in Z. \tag{2.4}$$

c) $-A$ generates the analytic semigroup $\{e^{-At}\}$ given by

$$e^{-At}z = \sum_{j=1}^{\infty} e^{-\lambda_j t} E_j z. \tag{2.5}$$

Consequently, systems (1.1), (1.7) and (1.8) can be written respectively as abstract differential equations in Z:

$$z' + aAz' + bAz = 1_\omega u(t) + f^e(t, z, u), \quad z \in Z \quad t \in (0, \tau], \tag{2.6}$$
$$z' + aAz' + bAz = 1_\omega u(t) + ez + cu, \quad z \in Z \quad t \in (0, \tau], \tag{2.7}$$
$$z' + aAz' + bAz = 1_\omega u(t) + ez + cu + g^e(t, z, u), \quad z \in Z \quad t \in (0, \tau], \tag{2.8}$$

where $u \in L^2([0, \tau]; U)$, $U = Z$, $B_\omega : U \longrightarrow Z$, $B_\omega u = 1_\omega u$ is a bounded linear operator, $f^e : [0, \tau] \times Z \times U \to Z$ is defined by $f^e(t, z, u)(x) = f(t, z(x), u(x))$, $\forall x \in \Omega$ and $g^e(t, z, u) = f^e(t, z, u) - ez - cu$. On the other hand, the hypothesis (1.2) implies that

$$\sup_{(t,z,u)\in Z_\tau} \|f^e(t, z, u) - ez - cu\|_Z < \infty, \tag{2.9}$$

where $Z_\tau = [0, \tau] \times Z \times U$. Therefore, $g^e : [0, \tau] \times Z \times U \to Z$ is bounded and smooth enough.

Since $(I + aA) = a(A - (-\frac{1}{a})I)$ and $-\frac{1}{a} \in \rho(A)$ ($\rho(A)$ is the resolvent set of A), then the operator:

$$I + aA : D(A) \to Z$$

is invertible with bounded inverse

$$(I + aA)^{-1} : Z \to D(A).$$

Therefore, equations (2.6),(2.7) and (2.8) also can be written as follows

$$
\begin{aligned}
z' + b(I + aA)^{-1}Az &= (I + aA)^{-1}1_\omega u(t) && \text{(2.10)} \\
&\quad + (I + aA)^{-1}f^e(t, z, u), \quad z \in Z, \quad t \in (0, \tau].
\end{aligned}
$$

$$
\begin{aligned}
z' + b(I + aA)^{-1}Az &= (I + aA)^{-1}1_\omega u(t) + e(I + aA)^{-1}z && \text{(2.11)} \\
&\quad + c(I + aA)^{-1}u, \quad z \in Z, \quad t \in (0, \tau].
\end{aligned}
$$

$$
\begin{aligned}
z' + b(I + aA)^{-1}Az &= (I + aA)^{-1}1_\omega u(t) + e(I + aA)^{-1}z && \text{(2.12)} \\
&\quad + c(I + aA)^{-1}u + (I + aA)^{-1}g^e(t, z, u), \quad z \in Z, \quad t \in (0, \tau].
\end{aligned}
$$

Moreover, $(I + aA)$ and $(I + aA)^{-1}$ can be written in terms of the eigenvalues of A:

$$(I + aA)z = \sum_{j=1}^{\infty}(1 + a\lambda_j)E_j z$$

$$(I + aA)^{-1}z = \sum_{j=1}^{\infty}\frac{1}{1 + a\lambda_j}E_j z. \tag{2.13}$$

Therefore, if we put $B = (I + aA)^{-1}$ and $F(t, z, u) = (I + aA)^{-1}f^e(t, z, u)$, equations (2.10),(2.11) and (2.12) can be written in the form:

$$z' + bBAz = BB_\omega u(t) + F(t, z, u), \quad t \in (0, \tau], \tag{2.14}$$

$$z' + bBAz = BB_\omega u(t) + eBz + cBu, \quad t \in (0, \tau], \tag{2.15}$$

$$z' + bBAz = BB_\omega u(t) + eBz + cBu + G(t, z, u), \quad t \in (0, \tau], \tag{2.16}$$

where $B_\omega f = 1_\omega f$ is a linear a bounded operator from Z to Z and $u \in L^2(0, \tau; L^2(\Omega))$ $= L^2(0, \tau; Z)$ and $G(t, z, u) = F(t, z, u) - eBz - cBu$ is smooth enough and bounded.

Now, we formulate two simple propositions.

Proposition 2.1. *([11]) The operators bBA and $T(t) = e^{-bBAt}$ are given by the following expressions*

$$bBAz = \sum_{j=1}^{\infty}\frac{b\lambda_j}{1 + a\lambda_j}E_j z \tag{2.17}$$

$$T_b(t)z = e^{-bBAt}z = \sum_{j=1}^{\infty} e^{\frac{-b\lambda_j}{1+a\lambda_j}t} E_j z. \tag{2.18}$$

Moreover, the following estimate holds

$$\| T(t) \| \le e^{-\beta t}, \quad t \ge 0, \tag{2.19}$$

where

$$\beta = \inf_{j \ge 1} \left\{ \frac{b\lambda_j}{1+a\lambda_j} \right\} = \frac{b\lambda_1}{1+a\lambda_1}. \tag{2.20}$$

Observe that, due to the above notation, the system (2.14) can be written as follows

$$z' = -\mathcal{A}z + BB_\omega u(t) + F(t,z,u), \quad t \in (0,\tau], \tag{2.21}$$

where $\mathcal{A} = bBA$.

Proposition 2.2. *The operators* $eB - \mathcal{A}$ *and* $T_e(t) = e^{(eB-\mathcal{A})t}$ *are given by the following expressions*

$$(eB - \mathcal{A})z = \sum_{j=1}^{\infty} \frac{e - b\lambda_j}{1+a\lambda_j} E_j z \tag{2.22}$$

$$T_e(t)z = e^{(eB-\mathcal{A})t}z = \sum_{j=1}^{\infty} e^{\frac{e-b\lambda_j}{1+a\lambda_j}t} E_j z, \tag{2.23}$$

and

$$\| T_e(t) \| \le e^{\rho t}, \quad t \ge 0, \tag{2.24}$$

where

$$\rho = \frac{e - b\lambda_1}{1+a\lambda_1} \tag{2.25}$$

provided that $b + ea > 0$.

Notice that systems (2.15) and (2.16) can be written in the form:

$$z' = (eB - \mathcal{A})z + BB_\omega u(t) + cBu, \quad t \in (0,\tau], \tag{2.26}$$

$$z' = (eB - \mathcal{A})z + BB_\omega u(t) + cBu + G(t,z,u), \quad t \in (0,\tau]. \tag{2.27}$$

3 Controllability of the Auxiliary Linear Equation (1.7)

In this section we prove the interior controllability of the linear system (2.26). But, at the beginning we give the definition of approximate controllability for this system. To this end, notice that for an arbitrary $z_0 \in Z$ and $u \in L^2(0,\tau;U)$ the initial value problem

$$\begin{cases} z' = (eB - \mathcal{A})z + BB_\omega u(t) + cBu, & t \in (0,\tau], \\ z(0) = z_0, \end{cases} \tag{3.1}$$

where the control function u belong to $L^2(0, \tau; U)$, admits only one mild solution given by

$$z(t) = T_e(t)z_0 + \int_0^t T_e(t-s)(BB_\omega + cBI)u(s)ds, \quad t \in [0, \tau]. \quad (3.2)$$

Definition 3.1. (*Approximate Controllability*) *The system (2.26) is said to be approximately controllable on $[0, \tau]$ if for every z_0, $z_1 \in Z$, $\varepsilon > 0$ there exists $u \in L^2(0, \tau; U)$ such that the solution $z(t)$ of (3.2) corresponding to u verifies:*

$$\|z(\tau) - z_1\| < \varepsilon.$$

Definition 3.2. *For the system (2.26) we define the following concept: The controllability map (for $\tau > 0$) $G_{\mathbf{e}} : L^2(0, \tau; U) \longrightarrow Z$ is given by*

$$G_{\mathbf{e}}u = \int_0^\tau T_e(s)(BB_\omega + cBI)u(s)ds, \quad (3.3)$$

whose adjoint operator $G_{\mathbf{e}}^* : Z \longrightarrow L^2(0, \tau; Z)$ is given by

$$(G_{\mathbf{e}}^*z)(s) = (B_\omega^* + cI)B^*T_e^*(s)z, \quad \forall s \in [0, \tau], \quad \forall z \in Z. \quad (3.4)$$

The following lemma holds in general for a linear bounded operator $G : W \to Z$ between Hilbert spaces W and Z.

Lemma 3.1. *(see [4], [5], [1] and [14]) The equation (2.26) is approximately controllable on $[0, \tau]$ if and only if one of the following statements holds:*

a) $\overline{\mathrm{Rang}(G_{\mathbf{e}})} = Z$.

b) $Ker(G_{\mathbf{e}}^*) = \{0\}$.

c) $\langle G_{\mathbf{e}}G_{\mathbf{e}}^*z, z \rangle > 0$, $z \neq 0$ in Z.

d) $\lim_{\alpha \to 0^+} \alpha(\alpha I + G_{\mathbf{e}}G_{\mathbf{e}}^*)^{-1}z = 0$.

e) $\sup_{\alpha > 0} \|\alpha(\alpha I + G_{\mathbf{e}}G_{\mathbf{e}}^*)^{-1}\| \leq 1$.

f) $(B_\omega^* + eI)B^*T_e^*(t)z = 0, \quad \forall t \in [0, \tau], \quad \Rightarrow z = 0$.

g) *For all $z \in Z$ we have $G_e u_\alpha = z - \alpha(\alpha I + G_{\mathbf{e}}G_{\mathbf{e}}^*)^{-1}z$, where*

$$u_\alpha = G_{\mathbf{e}}^*(\alpha I + G_{\mathbf{e}}G_{\mathbf{e}}^*)^{-1}z, \quad \alpha \in (0, 1].$$

So, $\lim_{\alpha \to 0} G_{\mathbf{e}}u_\alpha = z$ and the error $E_\alpha z$ of this approximation is given by

$$E_\alpha z = \alpha(\alpha I + G_{\mathbf{e}}G_{\mathbf{e}}^*)^{-1}z, \quad \alpha \in (0, 1].$$

Remark 3.1. *The Lemma 3.1 implies that the family of linear operators* $\Gamma_\alpha : Z \rightarrow L^2(0, \tau; U)$*, defined for* $0 < \alpha \le 1$ *by*

$$\Gamma_\alpha z = (B_\omega^* + eI)B^*T_e^*(\cdot)(\alpha I + G_\mathbf{e}G_\mathbf{e}^*)^{-1}z = G_\mathbf{e}^*(\alpha I + G_\mathbf{a}G_\mathbf{e}^*)^{-1}z, \qquad (3.5)$$

is an approximate inverse for the right of the operator $G_\mathbf{a}$ *in the sense that*

$$\lim_{\alpha \to 0} G_\mathbf{e}\Gamma_\alpha = I. \qquad (3.6)$$

Theorem 3.1. *The system (2.7) is approximately controllable on* $[0, \tau]$*. Moreover, a sequence of controls steering the system (2.7) from initial state* z_0 *to an* ϵ *neighborhood of the final state* z_1 *at time* $\tau > 0$ *is given by the formula*

$$u_\alpha(t) = (B_\omega^* + eI)B^*T_e^*(t)(\alpha I + G_\mathbf{e}G_\mathbf{e}^*)^{-1}(z_1 - T_e(\tau)z_0),$$

and the error of this approximation E_α *is given by the expresion*

$$E_\alpha = \alpha(\alpha I + G_\mathbf{e}G_\mathbf{e}^*)^{-1}(z_1 - T(\tau)z_0).$$

Proof . It is enough to show that the restriction $G_{\mathbf{e},\omega} = G_\mathbf{e}|_{L^2(0,\tau;L^2(\omega))}$ of $G_\mathbf{e}$ to the space $L^2(0, \tau; L^2(\omega))$ has range dense, i.e., $\overline{\text{Rang}(G_{\mathbf{e},\omega})} = Z$ or $Ker(G_{\mathbf{e},\omega}^*) = \{0\}$. Consequently, $G_{\mathbf{a},\omega} : L^2(0, \tau; L^2(\omega)) \rightarrow Z$ takes the following form

$$G_{\mathbf{e},\omega}u = \int_0^\tau T_e(s)B(1+c)u(s)ds.$$

whose adjoint operator $G_{\mathbf{e},\omega}^* : Z \longrightarrow L^2(0, \tau; L^2(\omega))$ is given by

$$(G_{\mathbf{e},\omega}z)(s) = (1+c)B^*T_e^*(s)z, \quad \forall s \in [0, \tau], \quad \forall z \in Z.$$

Since B is given by the formula

$$Bz = \sum_{j=1}^\infty \frac{1}{1+a\lambda_j}E_jz,$$

and T_e by (2.23), we get that $B = B^*$ and $T_e^*(t) = T_e$.

Suppose that

$$(1+c)B^*T_e^*(t)z = 0, \quad \forall t \in [0, \tau].$$

Since $1 + c \ne 0$, this is equivalents to the equality

$$B^*T_e^*(t)z = 0, \quad \forall t \in [0, \tau].$$

On the other hand, we have

$$B^*T_e^*(t)z = \sum_{j=1}^\infty \frac{e^{-\gamma_j t}}{1+a\lambda_j}E_jz = 0,$$

where $\gamma_j = \frac{b\lambda_j - e}{1 + a\lambda_j}$, which satisfies the conditions:

$$0 < \gamma_1 < \gamma_2 < \cdots < \gamma_j < \cdots . \tag{3.7}$$

Hence, following the proof of Lemma 1.1, we obtain that

$$E_j z(x) = \sum_{k=1}^{\gamma_j} < z, \phi_{j,k} > \phi_{j,k}(x) = 0, \quad \forall x \in \omega, \quad j = 1, 2, 3, \ldots .$$

Since $\phi_{j,k}$ are analytic functions on Ω, from Theorem 1.1 we obtain that

$$E_j z(x) = \sum_{k=1}^{\gamma_j} < z, \phi_{j,k} > \phi_{j,k}(x) = 0, \quad \forall x \in \Omega, \quad j = 1, 2, 3, \ldots .$$

Therefore, $E_j z = 0, \quad j = 1, 2, 3, \ldots$, which implies that $z = 0$. So, $\overline{\text{Rang}(G_{\mathbf{e},\omega})} = Z$, and consequently $\overline{\text{Rang}(G_{\mathbf{e}})} = Z$. Hence, the system (2.26) is approximately controllable on $[0, \tau]$, and the remainder of the proof follows from Lemma 3.1.

$\Box$

4 Controllability of the Semilinear BBM Equation

In this section we prove the main result of this paper, the interior controllability of the semilinear BBM Equation given by (1.1), which is equivalent to prove the approximate controllability of the system (2.27). To this end, observe that for all $z_0 \in Z$ and $u \in L^2(0, \tau; U)$ the initial value problem

$$\begin{cases} z' = (eB - \mathcal{A})z + BB_\omega u(t) + cBu + G(t, z, u), & t \in (0, \tau], \\ z(0) = z_0 \end{cases} \tag{4.1}$$

where the control function u belongs to $L^2(0, \tau; U)$, admits only one mild solution given by the formula

$$z_u(t) = T_e(t)z_0 + \int_0^t T_e(t - s)(BB_\omega + cBI)u(s)ds \tag{4.2}$$

$$+ \int_0^t T_e(t - s)G(s, z_u(s), (s))ds, \quad t \in [0, \tau].$$

Definition 4.1. *(Approximate Controllability)* *The system (2.27) is said to be approximately controllable on $[0, \tau]$ if for every $z_0, z_1 \in Z$, $\varepsilon > 0$ there exists $u \in L^2(0, \tau; U)$ such that the solution $z(t)$ of (4.2) corresponding to u verifies*

$$\|z(\tau) - z_1\| < \varepsilon.$$

Definition 4.2. *For the system (2.27) we define the following concept: The nonlinear controllability map (for $\tau > 0$) $G_g : L^2(0, \tau; U) \longrightarrow Z$ is given by the formula*

$$G_g u = \int_0^\tau T_e(s)(BB_\omega + cBI)u(s)ds + \int_0^\tau T_e(s)G(s, z_u(s), u(s))ds = G_e(u) + H(u),$$

(4.3)

where $H : L^2(0, \tau; U) \longrightarrow Z$ is the nonlinear operator given by

$$H(u) = \int_0^\tau T_e(s)G(s, z_u(s), u(s))ds, \quad u \in L^2(0, \tau; U)$$

(4.4)

The following lemma is trivial.

Lemma 4.1. *The equation (2.27) is approximately controllable on $[0, \tau]$ if and only if $\overline{\mathrm{Rang}(G_g)} = Z$.*

Definition 4.3. *The following equation*

$$u_\alpha = \Gamma_\alpha(z - H(u_\alpha)) = G_e^*(\alpha I + G_e G_e^*)^{-1}(z - H(u_\alpha)), \quad (0 < \alpha \le 1),$$

(4.5)

will be called the controllability equations associated to the non linear equation (2.27).

Now, we are ready to present and prove the main result of this paper, which is the interior approximate controllability of the semilinear BBM equation (1.1), and for the proof we will use some ideas from Propositions 4.2 from [1].

Theorem 4.1. *If the operator H define by (4.4) is compact and $\overline{\mathrm{Rang}(H)}$ is compact set, then the system (2.27) is approximately controllable on $[0, \tau]$. Moreover, a sequence of controls steering the system (2.27) from initial state z_0 to an ϵ neighborhood of the final state z_1 at time $\tau > 0$ is given by the formula*

$$u_\alpha(t) = (B_\omega^* + eI)B^*T_e^*(t)(\alpha I + G_e G_e^*)^{-1}(z_1 - T(\tau)z_0 - H(u_\alpha)),$$

and the error of this approximation E_α is given by the

$$E_\alpha = \alpha(\alpha I + G_e G_e^*)^{-1}(z_1 - T(\tau)z_0 - H(u_\alpha)).$$

Proof For each fixed $z \in Z$ we consider the following family of nonlinear operators $K_\alpha : L^2(0, \tau; U) \to L^2(0, \tau; U)$, given by the formula

$$K_\alpha(u) = \Gamma_\alpha(z - H(u)) = G_e^*(\alpha I + G_e G_e^*)^{-1}(z - H(u)), \quad (0 < \alpha \le 1).$$

(4.6)

First, we prove that, for all $\alpha \in (0, 1]$ the operator K_α has a fixed point u_α. In fact, since the operator H is a compact operator, then the operator K_α is compact. On the other hand, since $G(t, z, u)$ is bounded and $\|T_e(t)\| \le Re^{Wt}, \quad t \ge 0$, there exists a constant $M > 0$ such that

$$\|H(u)\| \le M, \quad \forall u \in L^2(0, \tau; U).$$

Then,

$$\|K_\alpha(u)\| \le \|\Gamma_\alpha\|(\|z\| + M), \quad \forall u \in L^2(0, \tau; U).$$

Therefore, the operator K_α maps the ball $B_r(0) \subset L^2(0, \tau; U)$ of center zero and radio $r \ge \|\Gamma_\alpha\|(\|z\| + M)$ into itself. Hence, applying the Schauder fixed point Theorem we get that the operator K_α has a fixed point $u_\alpha \in B_r(0) \subset L^2(0, \tau; U)$.

Since $\overline{\text{Rang}(H)}$ is compact, without loss of generality, we can assume that the sequence $H(u_\alpha)$ converges to $y \in Z$. So,

$$u_\alpha = \Gamma_\alpha(z - H(u_\alpha)) = G_e^*(\alpha I + G_e G_e^*)^{-1}(z - H(u_\alpha)).$$

Then, we get

$$
\begin{aligned}
G_e u_\alpha &= G_e \Gamma_\alpha(z - H(u_\alpha)) = G_e G_e^*(\alpha I + G_e G_e^*)^{-1}(z - H(u_\alpha)) \\
&= (\alpha I + G_e G_e^* - \alpha I)(\alpha I + G_e G_e^*)^{-1}(z - H(u_\alpha)) \\
&= z - H(u_\alpha) - \alpha(\alpha I + G_e G_e^*)^{-1}(z - H(u_\alpha))
\end{aligned}
$$

Hence, we deduce the following equality

$$G_e u_\alpha + H(u_\alpha) = z - \alpha(\alpha I + G_e G_e^*)^{-1}(z - H(u_\alpha)).$$

To conclude the proof, it enough to prove that

$$\lim_{\alpha \to 0} \{-\alpha(\alpha I + G_e G_e^*)^{-1}(z - H(u_\alpha))\} = 0$$

From Lemma 3.1 d) we get that

$$
\begin{aligned}
\lim_{\alpha \to 0} \{-\alpha(\alpha I + G_e G_e^*)^{-1}(z - H(u_\alpha))\} &= -\lim_{\alpha \to 0} \{-\alpha(\alpha I + G_e G_e^*)^{-1} H(u_\alpha)\} \\
= -\lim_{\alpha \to 0} -\alpha(\alpha I + G_e G_e^*)^{-1} y &- \lim_{\alpha \to 0} -\alpha(\alpha I + G_e G_e^*)^{-1}(H(u_\alpha) - y) \\
&= \lim_{\alpha \to 0} -\alpha(\alpha I + G_e G_e^*)^{-1}(H(u_\alpha) - y).
\end{aligned}
$$

On the other hand, from Lemma 3.1 e, we obtain that

$$\|\alpha(\alpha I + GG^*)^{-1}(H(u_\alpha) - y)\| \le \|(H(u_\alpha) - y)\|.$$

Therefore, keeping in mind that $H(u_\alpha)$ converges to y, we conclude that

$$\lim_{\alpha \to 0} \{-\alpha(\alpha I + GG^*)^{-1}(z - H(u_\alpha))\} = 0$$

So, putting $z = z_1 - T_e(\tau)z_0$ and using (4.2), we obtain the desired result

$$
\begin{aligned}
z_1 &= \lim_{\alpha \to 0^+} \{T_e(\tau)z_0 + \int_0^\tau T_e(\tau - s)(BB_\omega + cBI)u_\alpha(s)ds \\
&\quad + \int_0^\tau T_e(\tau - s)G(s, z_{u_\alpha}(s), u_\alpha(s))ds\}
\end{aligned}
$$

$\square$

Remark 4.1. *In the particular case that $a = 0$ and $b = 1$ the operator H define by (4.4) is compact and $\overline{\text{Rang}(H)}$ is compact set (see [3]) , and as a consequence we obtain the interior approximate controllability of the semilinear heat equation (see [12]).*

5   Final Remark

Our technique is simple and can be applied to those system involving diffusion process like some control system governed by heat equations. For example, the strongly damped wave equations, beam equations and so on.

Let us provide these two examples where this technique may be used.

Example 5.1. *Notice that this technique can be applied to prove the interior controllability of the strongly damped wave equation with Dirichlet boundary conditions*

$$\begin{cases} w_{tt} + \eta(-\Delta)^{1/2}w_t + \gamma(-\Delta)w = 1_\omega u(t,x) + f(t,z,u(t)), & in \quad (0,\tau] \times \Omega, \\ w = 0, & in \quad (0,\tau] \times \partial\Omega, \\ w(0,x) = w_0(x), \quad w_t(0,x) = w_1(x), & in \quad \Omega, \end{cases}$$

in the space $Z_{1/2} = D((-\Delta)^{1/2}) \times L^2(\Omega)$, where Ω is a bounded domain in $I\!\!R^n$, ω is an open nonempty subset of Ω, 1_ω denotes the characteristic function of the set ω, the distributed control $u \in L^2(0,\tau;L^2(\Omega))$ and η, γ are positive numbers.

Example 5.2. *Another example, where this technique may be applied, is the partial differential equations modeling the structural damped vibrations of a string or a beam having the form*

$$\begin{cases} y_{tt} - 2\beta\Delta y_t + \Delta^2 y = 1_\omega u(t,x) + f(t,z,u(t)), & on \quad (0,\tau] \times \Omega, \\ y = \Delta y = 0, & on \quad (0,\tau] \times \partial\Omega, \\ y(0,x) = y_0(x), \quad y_t(0,x) = y_1(x), & in \quad \Omega, \end{cases} \qquad (5.1)$$

where Ω is a bounded domain in $I\!\!R^n$, ω is an open nonempty subset of Ω, 1_ω denotes the characteristic function of the set ω, the distributed control $u \in L^2(0,\tau;L^2(\Omega))$ and $y_0, y_1 \in L^2(\Omega)$.

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Multiplier sequence spaces of fuzzy numbers defined by a Musielak-Orlicz function

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ABSTRACT: In this paper we introduce some multiplier sequence spaces of fuzzy numbers by using a Musielak-Orlicz function $\mathcal{M} = (M_k)$ and multiplier function $u = (u_k)$ and prove some inclusion relations between the resulting sequence spaces.

1 Introduction and Preliminaries

Fuzzy set theory, compared to other mathematical theories, is perhaps the most easily adaptable theory to practice. The main reason is that a fuzzy set has the property of relativity, variability and inexactness in the definition of its elements. Instead of defining an entity in calculus by assuming that its role is exactly known, we can use fuzzy sets to define the same entity by allowing possible deviations and inexactness in its role. This representation suits well the uncertainties encountered in practical life, which make fuzzy sets a valuable mathematical tool. The concepts of fuzzy sets and fuzzy set operations were first introduced by Zadeh [20] and subsequently several authors have discussed various aspects of the theory and applications of fuzzy sets such as fuzzy topological spaces, similarity relations and fuzzy orderings, fuzzy measures of fuzzy events, fuzzy mathematical programming. Matloka [12] introduced bounded and convergent sequences of fuzzy numbers and studied some of their properties. For more details about sequence spaces see ([1], [2], [14], [17]) and refrences therein.
The study of Orlicz sequence spaces was initiated with a certain specific purpose in Banach space theory. Indeed, Lindberg [9] got interested in Orlicz spaces in connection with finding Banach spaces with symmetric Schauder bases having complementary

subspaces isomorphic to c_0. Parashar and Choudhary [16] have introduced and discussed some properties of the sequence spaces defined by using a Orlicz function M which generalized the well-known Orlicz sequence space l_M and strongly summable sequence spaces $[\mathcal{C}, 1, p]$, $[\mathcal{C}, 1, p]_0$ and $[\mathcal{C}, 1, p]_\infty$. Later on, Basarir and Mursaleen [2], Tripathy and Mahanta [19] used the idea of an Orlicz function to construct some spaces of complex sequences. The concept of statistical convergence was introduced by Fast [6] and also independently by Buck [3] and Schoenberg [18] for real and complex sequences. Further this concept was studied by Fridy [7, Connor [4]] and many others. Statistical convergence is closely related to the concept of convergence in Probability.

A fuzzy number is a fuzzy set on the real axis, i.e., a mapping $u : \mathbb{R}^n \to [0,1]$ which satisfies the following four conditions:

1. u is normal, i.e., there exist an $x_0 \in \mathbb{R}^n$ such that $u(x_0) = 1$;

2. u is fuzzy convex, i.e., for $x, y \in \mathbb{R}^n$ and $0 \le \lambda \le 1, u(\lambda x + (1 - \lambda)y) \ge \min[u(x), u(y)]$;

3. u is upper semi-continuous;

4. the closure of $\{x \in \mathbb{R}^n : u(x) > 0\}$, denoted by $[u]^0$, is compact.

Denote $L(\mathbb{R}^n) = \{u : \mathbb{R}^n \to [0,1] \setminus u \text{ satisfies } (1) - (4) \text{ above}\}$. If $u \in L(\mathbb{R}^n)$, then u is called a fuzzy number and $L(\mathbb{R}^n)$ is a fuzzy number space.

Let $C(\mathbb{R}^n)$ denote the family of all non empty, compact, convex subsets of $\mathbb{R}^n$. If $\alpha, \beta \in \mathbb{R}$ and $A, B \in C(\mathbb{R}^n)$, then

$$\alpha(A + B) = \alpha A + \alpha B, \quad (\alpha\beta)A = \alpha(\beta A), \quad 1A = A$$

and if $\alpha, \beta \ge 0$, then $(\alpha + \beta)A = \alpha A + \beta A$. The distance between A and B is defined by the Housdorff metric

$$\delta_\infty(A, B) = \max\{\sup_{a \in A} \inf_{b \in B} \|a - b\|, \sup_{b \in B} \inf_{a \in A} \|a - b\|\},$$

where $\|.\|$ denoted the usual Euclidean norm in $\mathbb{R}^n$. It is well known that $(C(\mathbb{R}^n), \delta_\infty)$ is a complete metric space. For $0 < \alpha \le 1$, the α-level set $[u]^\alpha$ is defined by $[u]^\alpha = \{x \in \mathbb{R}^n : u(x) \ge \alpha\}$. Then from (1)-(4), it follows that $[u]^\alpha \in (C(\mathbb{R}^n))$. For the addition and scalar multiplication in $L(\mathbb{R}^n)$, we have

$$[u + v]^\alpha = [u]^\alpha + [v]^\alpha, \quad [ku]^\alpha = k[u]^\alpha,$$

where $u, v \in L(\mathbb{R}^n), k \in \mathbb{R}$. Define, for each $1 \le q < \infty$,

$$d_q(u, v) = \left(\int_0^1 [\delta_\infty([u]^\alpha, [v]^\alpha)]^q \right)^{\frac{1}{q}}$$

and $d_\infty(u,v) = \sup_{0 \leq \alpha \leq 1} \delta_\infty([u]^\alpha, [v]^\alpha)$, where δ_∞ is the Housdorff metric.

The idea of statistical convergence depends on the density of subsets of the set $\mathbb{N}$ of natural numbers. A subset E of $\mathbb{N}$ is said to have density $\delta(E)$ if

$$\delta(E) = \lim_{n \to \infty} \frac{1}{n} \sum_{k=1}^{n} \chi_E(k) \text{ exists,}$$

where χ_E is the characteristic function of E. It is clear that any finite subset of $\mathbb{N}$ has zero natural density and $\delta(E^c) = 1 - \delta(E)$.

A sequence $x = (x_k)$ is said to be statistically convergent to the number L if for every $\epsilon > 0$, $\delta(\{k \in \mathbb{N} : |x_k - L| \geq \epsilon\}) = 0$. In this case, we write $S - \lim x_k = L$. A sequence $X = (X_k)$ of fuzzy numbers is said to be bounded if the set $\{X_k : k \in \mathbb{N}\}$ of fuzzy numbers is bounded and convergent to the fuzzy number X_0, written as $\lim_k X_k = X_0$, i.e if for every $\epsilon > 0$ there exists a positive integer k_0 such that $d(X_k, X_0) < \epsilon$, for $k > k_0$. By w^F, l^F_∞ and c^F denote the set of all, bounded and convergent sequences of fuzzy numbers, respectively see [12].

An Orlicz function is a function $M : [0, \infty) \to [0, \infty)$ which is continuous, non decreasing and convex with $M(0) = 0$, $M(x) > 0$ for $x > 0$ and $M(x) \to \infty$ as $x \to \infty$. Lindenstrauss and Tzafriri [10] used the idea of Orlicz function to define the following sequence space. Let w be the space of all real or complex sequences $x = (x_k)$, then

$$l_M = \left\{ x \in w : \sum_{k=1}^{\infty} M\left(\frac{|x_k|}{\rho}\right) < \infty \right\},$$

which is called as an Orlicz sequence space. Also l_M is a Banach space with the norm

$$\|x\| = \inf \left\{ \rho > 0 : \sum_{k=1}^{\infty} M\left(\frac{|x_k|}{\rho}\right) \leq 1 \right\}.$$

Also, it was shown in [10] that every Orlicz sequence space l_M contains a subspace isomorphic to $l_p(p \geq 1)$. The Δ_2-condition is equivalent to $M(Lx) \leq LM(x)$, for all L with $0 < L < 1$. An Orlicz function M can always be represented in the following integral form,

$$M(x) = \int_0^x \eta(t)dt$$

where η is known as the kernel of M, is right differentiable for $t \geq 0, \eta(0) = 0, \eta(t) > 0$, η is non-decreasing and $\eta(t) \to \infty$ as $t \to \infty$.

A sequence $\mathcal{M} = (M_k)$ of Orlicz functions is called a Musielak-Orlicz function see ([13],[14]). A sequence $\mathcal{N} = (N_k)$ of Orlicz functions defined by

$$N_k(v) = \sup\{|v|u - M_k : u \geq 0\}, \; k = 1, 2, ...$$

is called the complementary function of the Musielak-Orlicz function $\mathcal{M}$. For a given Musielak-Orlicz function $\mathcal{M}$, the Musielak-Orlicz sequence space $t_\mathcal{M}$ and its subspace $h_\mathcal{M}$ are defined as follows

$$t_\mathcal{M} = \left\{ x \in w : I_\mathcal{M}(cx) < \infty, \text{ for some } c > 0 \right\},$$

$$h_{\mathcal{M}} = \left\{ x \in w : I_{\mathcal{M}}(cx) < \infty, \text{ for all } c > 0 \right\},$$

where $I_{\mathcal{M}}$ is a convex modular defined by

$$I_{\mathcal{M}}(x) = \sum_{k=1}^{\infty} M_k(x_k), x = (x_k) \in t_{\mathcal{M}}.$$

We consider $t_{\mathcal{M}}$ equipped with the Luxemburg norm

$$\|x\| = \inf \left\{ k > 0 : I_{\mathcal{M}}\left(\frac{x}{k}\right) \leq 1 \right\},$$

or equipped with the Orlicz norm

$$\|x\|^0 = \inf \left\{ \frac{1}{k}\left(1 + I_{\mathcal{M}}(kx)\right) : k > 0 \right\}.$$

Let $\lambda = (\lambda_n)$ be a non-decreasing sequence of positive numbers tending to ∞ and $\lambda_{n+1} \leq \lambda_n + 1, \lambda_1 = 1$. The generalized De la Vallee-Poussin mean is defined by

$$t_n(x) = \frac{1}{\lambda_n} \sum_{k \in I_n} x_k,$$

where $I_n = [n - \lambda_n + 1, n]$.

The space $\hat{c}$ of all almost convergent sequences was introduced by Maddox [12] has defined $x = (x_k)$ to be strongly almost convergent to a number l if

$$\lim_n \frac{1}{n} \sum_{k=1}^{n} |x_{k+m} - l| = 0, \text{ uniformly in } m.$$

The following inequality will be used throughout this paper. Let $p = (p_k)$ be a sequence of positive real numbers with $0 < p_k \leq \sup p_k = H$, and let $D = \max(1, 2^{H-1})$. Then for $a_k, b_k \in \mathbb{C}$, the set of complex numbers for all $k \in \mathbb{N}$, we have

$$|a_k + b_k|^{p_k} \leq D\{|a_k|^{p_k} + |b_k|^{p_k}\} \tag{1}$$

Let Λ denote the set of all non-decreasing sequences $\lambda = (\lambda_n)$ of positive numbers tending to ∞ such that $\lambda_{n+1} \leq \lambda_n + 1, \lambda_1 = 1, \mathcal{M} = (M_k)$ be a Musielak-Orlicz function and $p = (p_k)$ be a bounded sequence of positive real numbers. A sequence $X = (X_k)$ of fuzzy numbers is said to be almost λ-statistically convergent to the fuzzy number X_0, with respect to the Musielak-Orlicz function, if for every $\epsilon > 0$

$$\lim_{n \to \infty} \frac{1}{\lambda_n}\left|\left\{ k \in I_n : \left[M_k\left(\frac{d(t_{km}(X), X_0)}{\rho}\right)\right]^{p_k} \geq \epsilon \right\}\right| = 0,$$

uniformly in m for some $\rho > 0$,

where the vertical bars indicate the number of elements in the enclosed set and

$$t_{km}(X) = \frac{X_m + X_{m+1} + \cdots + X_{m+k}}{k+1} = \frac{1}{k+1} \sum_{i=0}^{k} X_{m+i}.$$

The set of all almost λ-statistically convergent sequences of fuzzy numbers is denoted by $\hat{S}^F(\lambda, M_k, u, p)$. In this case, we write $X_k \to X_0(\hat{S}^F(\lambda, M_k, u, p))$. In the special cases $\lambda_n = n$ for all $n \in \mathbb{N}$ and $M_k(X) = X, p_k = 1, u_k = 1$ for all $k \in \mathbb{N}$, we shall write $\hat{S}^F(M_k, u, p)$ and $\hat{S}^F(\lambda)$ instead of $\hat{S}^F(\lambda, M_k, u, p)$, respectively. Furthermore, the set of all almost statistically convergent sequences of fuzzy numbers is denoted by $\hat{S}^F$

Let $\lambda \in \Lambda, \mathcal{M} = (M_k)$ be a Musielak-Orlicz function, $p = (p_k)$ be a bounded sequence of positive real numbers and $u = (u_k)$ be a sequence of strictly positive real numbers. Then we define the following classes of sequences in this paper:

$$\hat{w}^F(\lambda, \mathcal{M}, u, p) = \left\{X = (X_k) : \lim_{n \to \infty} \frac{1}{\lambda_n} \sum_{k \in I_n} u_k \left[M_k\left(\frac{d(t_{km}(X), X_0)}{\rho}\right)\right]^{p_k} = 0\right.$$

$$\left. \text{uniformly in } m, \text{ for some } \rho > 0\right\},$$

$$\hat{w}_0^F(\lambda, \mathcal{M}, u, p) = \left\{X = (X_k) : \lim_{n \to \infty} \frac{1}{\lambda_n} \sum_{k \in I_n} u_k \left[M_k\left(\frac{d(t_{km}(X), \bar{0})}{\rho}\right)\right]^{p_k} = 0\right.$$

$$\left. \text{uniformly in } m, \text{ for some } \rho > 0\right\}$$

and

$$\hat{w}_\infty^F(\lambda, \mathcal{M}, u, p) = \left\{X = (X_k) : \sup_{m,n} \frac{1}{\lambda_n} \sum_{k \in I_n} u_k \left[M_k\left(\frac{d(t_{km}(X), \bar{0})}{\rho}\right)\right]^{p_k} < \infty\right.$$

$$\left. \text{uniformly in } m, \text{ for some } \rho > 0\right\},$$

where

$$\bar{0}(t) = \begin{cases} 1, & t = (0, 0, 0, \cdots, 0) \\ 0, & \text{otherwise} \end{cases}.$$

If $X \in \hat{w}^F(\lambda, \mathcal{M}, u, p)$, we say that X is strongly almost λ-convergent with respect to the Musielak-Orlicz function $\mathcal{M} = (M_k)$. In this case we write $X_k \to X_0(\hat{w}^F(\lambda, \mathcal{M}, u, p))$. The following sequence spaces are defined by giving particular values to $\mathcal{M}, u, p$.

(i) For $\lambda_n = n$
$\hat{w}^F(\lambda, \mathcal{M}, u, p) = \hat{w}^F(\mathcal{M}, u, p), \hat{w}_0^F(\lambda, \mathcal{M}, u, p) = \hat{w}_0^F(\mathcal{M}, u, p)$, and $\hat{w}_\infty^F(\lambda, \mathcal{M}, u, p) = \hat{w}_\infty^F(\mathcal{M}, u, p)$,

(ii) If $\mathcal{M} = M_k(x) = x$ for all k, we get

$\hat{w}^F(\lambda, \mathcal{M}, u, p) = \hat{w}^F(u, p, \lambda), \hat{w}_0^F(\lambda, \mathcal{M}, u, p) = \hat{w}_0^F(u, p, \lambda),$ and $\hat{w}_\infty^F(\lambda, \mathcal{M}, u, p) = \hat{w}_\infty^F(u, p, \lambda),$

(iii) If $p_k = 1$ for all $k \in \mathbb{N}$, then
$\hat{w}^F(\lambda, \mathcal{M}, u, p) = \hat{w}^F(\mathcal{M}, u, \lambda), \hat{w}_0^F(\lambda, \mathcal{M}, u, p) = \hat{w}_0^F(\mathcal{M}, u, \lambda),$ and $\hat{w}_\infty^F(\lambda, \mathcal{M}, u, p) = \hat{w}_\infty^F(\mathcal{M}, u, \lambda),$

(iv) If $\mathcal{M} = M_k(x) = x$ for all k, and $p_k = 1$ for all $k \in \mathbb{N}$, then
$\hat{w}^F(\lambda, \mathcal{M}, u, p) = \hat{w}^F(u, \lambda), \hat{w}_0^F(\lambda, \mathcal{M}, u, p) = \hat{w}_0^F(u, \lambda),$ and $\hat{w}_\infty^F(\lambda, \mathcal{M}, u, p) = \hat{w}_\infty^F(u, \lambda),$

(v) If $p_k = 1$ for all $k \in \mathbb{N}$, and $u_k = 1$ for all k, then
$\hat{w}^F(\lambda, \mathcal{M}, u, p) = \hat{w}^F(\mathcal{M}, \lambda), \hat{w}_0^F(\lambda, \mathcal{M}, u, p) = \hat{w}_0^F(\mathcal{M}, \lambda),$ and $\hat{w}_\infty^F(\lambda, \mathcal{M}, u, p) = \hat{w}_\infty^F(\mathcal{M}, \lambda),$

(vi) If $\mathcal{M} = M_k(x) = x, p_k = 1$ and $u_k = 1$ for all k, then
$\hat{w}^F(\lambda, \mathcal{M}, u, p) = \hat{w}^F(\lambda), \hat{w}_0^F(\lambda, \mathcal{M}, u, p) = \hat{w}_0^F(\lambda),$ and $\hat{w}_\infty^F(\lambda, \mathcal{M}, u, p) = \hat{w}_\infty^F(\lambda).$
In this paper we shall prove properties of linearity and some inclusion relations between the classes of sequences $\hat{w}^F(\lambda, \mathcal{M}, u, p), \hat{w}_0^F(\lambda, \mathcal{M}, u, p), \hat{w}_\infty^F(\lambda, \mathcal{M}, u, p)$ and $\hat{S}^F(\lambda, \mathcal{M}, u, p)$.

2. Main Results

Theorem 2.1. *For any Musielak-Orlicz function* $\mathcal{M} = (M_k), p = (p_k)$ *be a bounded sequence of strictly positive real numbers and* $u = (u_k)$ *be a sequence of positive real numbers, we have*

$$\hat{w}_0^F(\lambda, \mathcal{M}, u, p) \subset \hat{w}^F(\lambda, \mathcal{M}, u, p) \subset \hat{w}_\infty^F(\lambda, \mathcal{M}, u, p).$$

Proof. The inclusion $\hat{w}_0^F(\lambda, \mathcal{M}, u, p) \subset \hat{w}^F(\lambda, \mathcal{M}, u, p)$ is obvious.
Let $X \in \hat{w}^F(\lambda, \mathcal{M}, u, p)$, then

$$
\begin{aligned}
\frac{1}{\lambda_n} \sum_{k \in I_n} u_k \Big[M_k \Big(\frac{d(t_{km}(X), \bar{0})}{2\rho} \Big) \Big]^{p_k} \quad &\leq \quad \frac{D}{\lambda_n} \sum_{k \in I_n} \frac{1}{2^{p_k}} u_k \Big[M_k \Big(\frac{d(t_{km}(X), X_0)}{\rho} \Big) \Big]^{p_k} \\
&+ \quad \frac{D}{\lambda_n} \sum_{k \in I_n} \frac{1}{2^{p_k}} u_k \Big[M_k \Big(\frac{d(X_0, \bar{0})}{\rho} \Big) \Big]^{p_k} \\
&\leq \quad \frac{D}{\lambda_n} \sum_{k \in I_n} u_k \Big[M_k \Big(\frac{d(t_{km}(X), X_0)}{\rho} \Big) \Big]^{p_k} \\
&+ \quad D \max_{k \in I_n} \Big\{ \max \Big\{ 1, \sup_k u_k \Big[M_k \Big(\frac{d(X_0, \bar{0})}{\rho} \Big) \Big]^H \Big\} \Big\},
\end{aligned}
$$

where $\sup_k p_k = H$ and $D = \max(1, 2^{H-1})$. Thus we get $X \in \hat{w}_\infty^F(\lambda, \mathcal{M}, u, p)$.

Theorem 2.2. *If $\mathcal{M} = (M_k)$ be a Musielak-Orlicz function, $p = (p_k)$ be a bounded sequence of positive real numbers and $u = (u_k)$ be a sequence of strictly positive real numbers, then $\hat{w}^F(\lambda, \mathcal{M}, u, p), \hat{w}_0^F(\lambda, \mathcal{M}, u, p)$ and $\hat{w}_\infty^F(\lambda, \mathcal{M}, u, p)$ are closed under the operations of addition and scalar multiplication.*

Proof. Let $X = (X_k), Y = (Y_k) \in \hat{w}_\infty^F(\lambda, \mathcal{M}, u, p)$ and $\alpha, \beta \in \mathbb{C}$. Then there exist positive numbers ρ_1, ρ_2 such that

$$\sup_{m,n} \frac{1}{\lambda_n} \sum_{k \in I_n} u_k \left[M_k \left(\frac{d(t_{km}(X), \bar{0})}{\rho_1} \right) \right]^{p_k} < \infty, \text{ uniformly in } m$$

and

$$\sup_{m,n} \frac{1}{\lambda_n} \sum_{k \in I_n} u_k \left[M_k \left(\frac{d(t_{km}(Y), \bar{0})}{\rho_2} \right) \right]^{p_k} < \infty, \text{ uniformly in } m.$$

Define $\rho_3 = \max(2|\alpha|\rho_1, 2|\beta|\rho_2)$. Since $\mathcal{M} = (M_k)$ is non-decreasing and convex, we have

$$\sup_{m,n} \frac{1}{\lambda_n} \sum_{k \in I_n} u_k \left[M_k \left(\frac{\alpha d(t_{km}(X), \bar{0}) + \beta d(t_{km}(Y), \bar{0})}{\rho_3} \right) \right]^{p_k}$$

$$\leq \quad \frac{1}{2} \sup_{m,n} \frac{1}{\lambda_n} \sum_{k \in I_n} u_k \left[M_k \left(\frac{d(t_{km}(X), \bar{0})}{\rho_1} \right) \right]^{p_k}$$

$$+ \quad \frac{1}{2} \sup_{m,n} \frac{1}{\lambda_n} \sum_{k \in I_n} u_k \left[M_k \left(\frac{d(t_{km}(X), \bar{0})}{\rho_2} \right) \right]^{p_k}$$

$$< \quad \infty.$$

This proves that $\hat{w}_\infty^F(\lambda, \mathcal{M}, u, p)$ is a linear space. Similarly we can prove for other cases.

Theorem 2.3. *If $0 < p_k \leq r_k < \infty$ for all $k \in \mathbb{N}$ and $\left(\frac{r_k}{p_k} \right)$ be bounded, then we have*

$$\hat{w}_\infty^F(\lambda, \mathcal{M}, u, r) \subseteq \hat{w}_\infty^F(\lambda, \mathcal{M}, u, p).$$

Proof. Let $X = (X_k) \in \hat{w}_\infty^F(\lambda, \mathcal{M}, u, r)$. Thus, we have

$$\sup_{m,n} \frac{1}{\lambda_n} \sum_{k \in I_n} u_k \left[M_k \left(\frac{d(t_{km}(X), \bar{0})}{\rho} \right) \right]^{r_k} < \infty, \text{ uniformly in } m.$$

Let $s_k = \sup_{m,n} \frac{1}{\lambda_n} \sum_{k \in I_n} u_k \left[M_k \left(\frac{d(t_{km}(X), \bar{0})}{\rho} \right) \right]^{r_k}$ and $\lambda_k = \frac{p_k}{r_k}$. Since $p_k \leq r_k$, we have $0 \leq \lambda_k \leq 1$. Take $0 < \lambda < \lambda_k$. Now define

$$u_k = \begin{cases} s_k & \text{if} \quad s_k \geq 1 \\ \\ 0 & \text{if} \quad s_k < 1 \end{cases}$$

and

$$v_k = \begin{cases} 0 & \text{if} \quad s_k \geq 1 \\ s_k & \text{if} \quad s_k < 1 \end{cases}$$

$s_k = u_k + v_k, \quad s_k^{\lambda_k} = u_k^{\lambda_k} + v_k^{\lambda_k}$. It follows that $u_k^{\lambda_k} \leq u_k \leq s_k, \quad v_k^{\lambda_k} \leq v_k^{\lambda}$. since $s_k^{\lambda_k} = u_k^{\lambda_k} + v_k^{\lambda_k}$, then $s_k^{\lambda_k} \leq s_k + v_k^{\lambda}$

$$\sup_{m,n} \frac{1}{\lambda_n} \sum_{k \in I_n} u_k \left[\left(M_k \left(\frac{d(t_{km}(X), \bar{0})}{\rho_1} \right) \right)^{r_k} \right]^{\lambda_k} \leq \sup_{m,n} \frac{1}{\lambda_n} \sum_{k \in I_n} u_k \left[M_k \left(\frac{d(t_{km}(X), \bar{0})}{\rho_1} \right) \right]^{r_k}$$

$$\Longrightarrow \sup_{m,n} \frac{1}{\lambda_n} \sum_{k \in I_n} u_k \left[\left(M_k \left(\frac{d(t_{km}(X), \bar{0})}{\rho_1} \right) \right)^{r_k} \right]^{p_k/r_k} \leq \sup_{m,n} \frac{1}{\lambda_n} \sum_{k \in I_n} u_k \left[M_k \left(\frac{d(t_{km}(X), \bar{0})}{\rho_1} \right) \right]^{r_k}$$

$$\Longrightarrow \sup_{n,m} \frac{1}{\lambda_n} \sum_{k \in I_n} u_k \left[\left(M_k \left(\frac{d(t_{km}(X), \bar{0})}{\rho_1} \right) \right) \right]^{p_k} \leq \sup_{m,n} \frac{1}{\lambda_n} \sum_{k \in I_n} u_k \left[M_k \left(\frac{d(t_{km}(X), \bar{0})}{\rho} \right) \right]^{r_k}$$

But

$$\sup_{m,n} \frac{1}{\lambda_n} \sum_{k \in I_n} u_k \left[M_k \left(\frac{d(t_{km}(X), \bar{0})}{\rho} \right) \right]^{r_k} < \infty, \text{ uniformly in } m.$$

Therefore

$$\sup_{m,n} \frac{1}{\lambda_n} \sum_{k \in I_n} u_k \left[M_k \left(\frac{d(t_{km}(X), \bar{0})}{\rho} \right) \right]^{p_k} < \infty, \text{ uniformly in } m.$$

Hence $x \in \hat{w}_\infty^F(\lambda, \mathcal{M}, u, p)$. Thus we get $\hat{w}_\infty^F(\lambda, \mathcal{M}, u, r) \subseteq \hat{w}_\infty^F(\lambda, \mathcal{M}, u, p)$.

Theorem 2.4. *Suppose $\mathcal{M} = (M_k)$ be a Musielak-Orlicz function, $p = (p_k)$ be a bounded sequence of positive real numbers and $u = (u_k)$ be a sequence of strictly positive real numbers. If $\sup_k (M_k(t))^{p_k} < \infty$ for all fixed $t > 0$, then*

$$\hat{w}^F(\lambda, \mathcal{M}, u, p) \subset \hat{w}_\infty^F(\lambda, \mathcal{M}, u, p).$$

 Proof. Let $X \in \hat{w}^F(\lambda, \mathcal{M}, u, p)$, then there exists a positive number $\rho_1 > 0$ such that

$$\lim_{n \to \infty} \frac{1}{\lambda_n} \sum_{k \in I_n} u_k \left[M_k \left(\frac{d(t_{km}(X), X_0)}{\rho_1} \right) \right]^{p_k} = 0, \text{ uniformly in } m.$$

Define $\rho = 2\rho_1$. Since $\mathcal{M} = (M_k)$ is non-decreasing and convex, for each k. So by using (1), we have

$$\sup_{m,n} \frac{1}{\lambda_n} \sum_{k \in I_n} u_k \Big[M_k \Big(\frac{d(t_{km}(X), \bar{0})}{\rho} \Big) \Big]^{p_k}$$

$$\leq \quad \sup_{m,n} \frac{1}{\lambda_n} \sum_{k \in I_n} u_k \Big[M_k \Big(\frac{d(t_{km}(X), X_0) + d(X_0, \bar{0})}{\rho} \Big) \Big]^{p_k}$$

$$\leq \quad D\Big\{ \sup_{m,n} \frac{1}{\lambda_n} \sum_{k \in I_n} \frac{1}{2^{p_k}} u_k \Big[M_k \Big(\frac{d(t_{km}(X), X_0)}{\rho_1} \Big) \Big]^{p_k}$$

$$+ \quad \sup_{m,n} \frac{1}{\lambda_n} \sum_{k \in I_n} \frac{1}{2^{p_k}} u_k \Big[M_k \Big(\frac{d(t_{km}(X), \bar{0})}{\rho_1} \Big) \Big]^{p_k} \Big\}$$

$$< \quad \infty.$$

Thus $X \in \hat{w}_\infty^F(\lambda, \mathcal{M}, u, p)$, which completes the proof.

Theorem 2.5. *Let $0 < h = \inf p_k \leq p_k \leq \sup p_k = H < \infty$. Then for a Musielak-Orlicz function $\mathcal{M} = (M_k)$ which satisfies the Δ_2-condition, we have $\hat{w}_0^F(\lambda, u, p) \subset \hat{w}_0^F(\lambda, \mathcal{M}, u, p), \hat{w}^F(\lambda, u, p) \subset \hat{w}^F(\lambda, \mathcal{M}, u, p)$ and $\hat{w}_\infty^F(\lambda, u, p) \subset \hat{w}_\infty^F(\lambda, \mathcal{M}, u, p)$.*

Proof. Let $X \in \hat{w}^F(\lambda, u, p)$, then we have

$$\frac{1}{\lambda_n} \sum_{k \in I_n} u_k \Big[\Big(\frac{d(t_{km}(X), X_0)}{\rho} \Big) \Big]^{p_k} \to 0 \ \text{ as } \ n \to \infty, \text{uniformly in } m.$$

Let $\epsilon > 0$ and choose δ with $0 < \delta < 1$ such that $M_k(t) < \epsilon$ for $0 \leq t \leq \delta$. Then

$$\frac{1}{\lambda_n} \sum_{k \in I_n} u_k \Big[M_k \Big(\frac{d(t_{km}(X), X_0)}{\rho} \Big) \Big]^{p_k} \quad = \quad \frac{1}{\lambda_n} \sum_{k \in I_n, d(t_{km}(X), X_0) \leq \delta} u_k \Big[M_k \Big(\frac{d(t_{km}(X), X_0)}{\rho} \Big) \Big]^{p_k}$$

$$+ \quad \frac{1}{\lambda_n} \sum_{k \in I_n, d(t_{km}(X), X_0) > \delta} u_k \Big[M_k \Big(\frac{d(t_{km}(X), X_0)}{\rho} \Big) \Big]^{p_k}$$

$$= \quad \sum_1 + \sum_2 .$$

where

$$\sum_1 = \frac{1}{\lambda_n} \sum_{k \in I_n, d(t_{km}(X), X_0) \leq \delta} u_k \Big[M_k \Big(\frac{d(t_{km}(X), X_0)}{\rho} \Big) \Big]^{p_k} < \max(\epsilon, \epsilon^H)$$

by using continuity of (M_k). For the second summation, we will make the following procedure. Thus we have

$$\frac{d(t_{km}(X), X_0)}{\rho} < 1 + \frac{d(t_{km}(X), X_0)/\rho}{\delta}.$$

Since $\mathcal{M} = (M_k)$ is non-decreasing and convex, so we have

$$
\begin{aligned}
u_k\Big[M_k\Big(\frac{d(t_{km}(X), X_0)}{\rho}\Big)\Big] \quad &< \quad u_k\Big[M_k\Big\{1 + \frac{d(t_{km}(X), X_0)/\rho}{\delta}\Big\}\Big] \\
&\leq \quad \frac{1}{2}u_k\Big[M_k(2)\Big] + \frac{1}{2}u_k\Big[M_k\Big\{2\frac{d(t_{km}(X), X_0)/\rho}{\delta}\Big\}\Big].
\end{aligned}
$$

Again, since $\mathcal{M} = (M_k)$ satisfies the Δ_2-condition, it follows that

$$
\begin{aligned}
u_k\Big[M_k\Big(\frac{d(t_{km}(X), X_0)}{\rho}\Big)\Big] \quad &\leq \quad \frac{1}{2}L\Big\{\frac{d(t_{km}(X), X_0)/\rho}{\delta}\Big\}u_k\Big[M_k(2)\Big] \\
&+ \quad \frac{1}{2}L\Big\{\frac{d(t_{km}(X), X_0)/\rho}{\delta}\Big\}u_k\Big[M_k(2)\Big] \\
&= \quad L\Big\{\frac{d(t_{km}(X), X_0)/\rho}{\delta}\Big\}u_k\Big[M_k(2)\Big].
\end{aligned}
$$

Thus, it follows that

$$
\sum_2 = \max_{k \in I_n}\Big\{1, \Big[\frac{Lu_k[M_k(2)]}{\delta}\Big]^H\Big\}\frac{1}{\lambda_n}\sum_{k \in I_n}\Big[\Big(\frac{d(t_{km}(X), X_0)}{\rho}\Big)\Big]^{p_k}.
$$

Taking the limit as $\epsilon \to 0$ and $n \to \infty$, it follows that $X \in \hat{w}^F(\lambda, \mathcal{M}, u, p)$. Similarly, we can prove that $\hat{w}_0^F(\lambda, u, p) \subset \hat{w}_0^F(\lambda, \mathcal{M}, u, p)$ and $\hat{w}_\infty^F(\lambda, u, p) \subset \hat{w}_\infty^F(\lambda, \mathcal{M}, u, p)$.

Theorem 2.6. *If $\mathcal{M} = (M_k)$ be a Musielak-Orlicz function, $p = (p_k)$ be a bounded sequence of positive real numbers and $u = (u_k)$ be a sequence of strictly positive real numbers, then*
(i) If $0 < \inf p_k \leq p_k \leq 1$ for all k, then $\hat{w}^F(\lambda, \mathcal{M}, u) \subseteq \hat{w}^F(\lambda, \mathcal{M}, u, p)$,
(ii) If $1 \leq p_k \leq \sup p_k = H < \infty$ then $\hat{w}^F(\lambda, \mathcal{M}, u, p) \subseteq \hat{w}^F(\lambda, \mathcal{M}, u)$.

Proof. (i) Let $X \in \hat{w}^F(\lambda, \mathcal{M}, u)$. Since $0 < \inf p_k \leq p_k \leq 1$, we get

$$
\frac{1}{\lambda_n}\sum_{k \in I_n}u_k\Big[M_k\Big(\frac{d(t_{km}(X), X_0)}{\rho}\Big)\Big]^{p_k} \leq \frac{1}{\lambda_n}\sum_{k \in I_n}u_k\Big[M_k\Big(\frac{d(t_{km}(X), X_0)}{\rho}\Big)\Big]
$$

and hence $X \in \hat{w}^F(\lambda, \mathcal{M}, u, p)$.
(ii) Let $X \in \hat{w}^F(\lambda, \mathcal{M}, u, p)$ and $1 \leq p_k \leq \sup p_k = H < \infty$. Then for every $0 < \epsilon < 1$, there exists a positive integer n_0 such that

$$
\frac{1}{\lambda_n}\sum_{k \in I_n}u_k\Big[M_k\Big(\frac{d(t_{km}(X), X_0)}{\rho}\Big)\Big]^{p_k} \leq \epsilon < 1
$$

for all $n \geq n_0$. This follows that

$$
\frac{1}{\lambda_n}\sum_{k \in I_n}u_k\Big[M_k\Big(\frac{d(t_{km}(X), X_0)}{\rho}\Big)\Big] \leq \frac{1}{\lambda_n}\sum_{k \in I_n}u_k\Big[M_k\Big(\frac{d(t_{km}(X), X_0)}{\rho}\Big)\Big]^{p_k}.
$$

and hence $X \in \hat{w}^F(\lambda, \mathcal{M}, u)$.

Theorem 2.7. *If $\mathcal{M} = (M_k)$ be a Musielak-Orlicz function, $p = (p_k)$ be a bounded sequence of positive real numbers, $u = (u_k)$ be a sequence of strictly positive real numbers and $0 < h = \inf p_k \leq p_k \leq \sup p_k = H < \infty$. Then $\hat{w}^F(\lambda, \mathcal{M}, u, p) \subset \hat{S}^F(\lambda)$.*

Proof. The proof of the theorem follows from the following inequality:

$$\frac{1}{\lambda_n} \sum_{k \in I_n} u_k \Big[M_k \Big(\frac{d(t_{km}(X), X_0)}{\rho} \Big) \Big]^{p_k}$$

$$\geq \frac{1}{\lambda_n} \sum_{k \in I_n, d(t_{km}(X), X_0) \geq \epsilon} u_k \Big[M_k \Big(\frac{d(t_{km}(X), X_0)}{\rho} \Big) \Big]^{p_k}$$

$$\geq \frac{1}{\lambda_n} \sum_{k \in I_n, d(t_{km}(X), X_0) \geq \epsilon} \min \Big\{ u_k [M_k(\epsilon_1)]^h, u_k [M_k(\epsilon_1)]^H \Big\}$$

$$\geq \frac{1}{\lambda_n} \Big| \Big\{ k \in I_n : d(t_{km}(X), X_0) \geq \epsilon \Big\} \Big| \min_{k \in I_n} \Big\{ u_k [M_k(\epsilon_1)]^h, u_k [M_k(\epsilon_1)]^H \Big\},$$

where $\epsilon_1 = \frac{\epsilon}{\rho}$.

Theorem 2.8. *Let $\mathcal{M} = (M_k)$ be a Musielak-Orlicz function, $X = (X_k)$ be a bounded sequence of fuzzy numbers and $0 < h = \inf p_k \leq p_k \leq \sup p_k = H < \infty$. Then $\hat{S}^F(\lambda) \subset \hat{w}^F(\lambda, \mathcal{M}, u, p)$.*

Proof. Suppose that $X \in l_\infty^F$ and $X_k \to X_0(\hat{S}^F(\lambda))$. Since $X \in l_\infty^F$, there exists a constant $K > 0$ such that $d(t_{km}(X), X_0) \leq K$ for all k, m. Given $\epsilon > 0$, we have

$$\frac{1}{\lambda_n} \sum_{k \in I_n} u_k \Big[M_k \Big(\frac{d(t_{km}(X), X_0)}{\rho} \Big) \Big]^{p_k}$$

$$= \frac{1}{\lambda_n} \sum_{k \in I_n, d(t_{km}(X), X_0) \geq \epsilon} u_k \Big[M_k \Big(\frac{d(t_{km}(X), X_0)}{\rho} \Big) \Big]^{p_k}$$

$$+ \frac{1}{\lambda_n} \sum_{k \in I_n, d(t_{km}(X), X_0) < \epsilon} u_k \Big[M_k \Big(\frac{d(t_{km}(X), X_0)}{\rho} \Big) \Big]^{p_k}$$

$$\leq \frac{1}{\lambda_n} \sum_{k \in I_n, d(t_{km}(X), X_0) \geq \epsilon} \max \Big\{ u_k [M_k(\frac{K}{\rho})]^h, u_k [M_k(\frac{K}{\rho})]^H \Big\}$$

$$+ \frac{1}{\lambda_n} \sum_{k \in I_n, d(t_{km}(X), X_0) < \epsilon} u_k [M_k(\frac{\epsilon}{\rho})]^{p_k}$$

$$\leq \max_{k \in I_n} \Big\{ u_k [M_k(T)]^h, u_k [M_k(T)]^H \Big\} \frac{1}{\lambda_n} \Big| \Big\{ k \in I_n : d(t_{km}(X), X_0) \geq \epsilon \Big\} \Big|$$

$$+ \max_{k \in I_n} \Big\{ u_k [M_k(\epsilon_1)]^h, u_k [M_k(\epsilon_1)]^H \Big\}.$$

where $T = \frac{K}{\rho}, \frac{\epsilon}{\rho} = \epsilon_1$. Hence $X \in \hat{w}^F(\lambda, \mathcal{M}, u, p)$.

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Close-to-convexity properties of basic hypergeometric functions using their Taylor coefficients

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ABSTRACT: In this paper, we find the conditions on parameters a, b, c and q such that the basic hypergeometric function $z\phi(a, b; c; q, z)$ and its q-Alexander transform are close-to-convex (and hence univalent) in the unit disc $\mathbb{D} := \{z : |z| < 1\}$.

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1 Introduction and Notation

Most of the mathematical functions which are encounted in numerous contexts are of hypergeometric type. The ordinary or Gaussian hypergeometric function $_2F_1(a, b; c; z)$ is defined by the series

$$_2F_1(a, b; c; z) = \sum_{n=0}^{\infty} \frac{(a, n)(b, n)}{(c, n)n!} z^n, \quad |z| < 1,$$

where a, b, c are complex numbers such that $c \neq 0, -1, -2, -3, ...,$ $(a, 0) = 1$ for $a \neq 0$ and

$$(a, n + 1) = (a + n)(a, n), \quad n = 0, 1, 2, \cdots.$$

In the exceptional case $c = -p$, $p = 0, 1, 2, \cdots$, $F(a, b; c; z)$ is defined if $a = -m$ or $b = -m$, where $m = 0, 1, 2, \cdots$ and $m \leq p$. Heine (see [1, 5]) defined 'q-analogue' or 'basic analogue' of Gaussian hypergeometric function in the following way

$$_2\Phi_1(a, b; c; q; z) = 1 + \frac{(1 - q^a)(1 - q^b)}{(1 - q^c)(1 - q)} z + \frac{(1 - q^a)(1 - q^{a+1})(1 - q^b)(1 - q^{b+1})}{(1 - q^c)(1 - q^{c+1})(1 - q)(1 - q^2)} z^2 + \cdots,$$

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where $|q| < 1$. For the base q, $0 < q < 1$, define

$$(a; q) = \frac{1 - q^a}{1 - q}, \quad (0; q) = 1.$$

Clearly, by L'-Hospitals' rule, we have

$$(a; q) \to a \text{ as } q \to 1,$$

and the basic factorial notation is

$$(a; q)! = (1; q)(2; q) \cdots (n; q).$$

Now we write

$$(1 - q^a)(1 - q^{a+1}) \cdots (1 - q^{a+n-1}) = (a; q)(a + 1; q) \cdots (a + n - 1; q)(1 - q)^n$$

$$= (a; q)_n (1 - q)^n, \text{ say,}$$

and thus in the limiting case $q \to 1$, we have

$$\lim_{q \to 1} (a; q)_n = \lim_{q \to 1} \prod_{j=1}^{n} \frac{1 - q^{a+j-1}}{(1 - q)^n} = (a, n).$$

With this observation, the Heine's series or the q-analogue of Gauss function defined above takes the following form [17]:

$$_2\Phi_1(a, b; c; q; z) = \sum_{n=0}^{\infty} \frac{(a; q)_n (b; q)_n}{(c; q)_n (q; q)_n} z^n, \quad |z| < 1.$$

We remark that in the limiting case $q \to 1$, the function $_2\Phi_1(a, b; c; q; z)$ reduces to $_2F_1(a, b; c; z)$.

The geometric properties of $_2F_1(a, b; c; z)$ for various values of a, b and c are well known. For details, we refer to [7, 9, 10, 13, 14] and references therein. Similar study about $_2\phi_1(a, b; c; q; z)$ is not available in the literature, except [6, 15, 16]. Hence the main objective of this work is to find the geometric properties of $_2\phi_1(a, b; c; q; z)$ from the parameters a, b and c for $0 < q < 1$. For this purpose the q-Gamma function $\Gamma_q(x)$ [2], which is the q-generalization of the Gamma function and defined by

$$\Gamma_q(x)_: = \frac{(q; q)_\infty}{(q^x; q)_\infty} (1 - q)^{1-x}, \qquad 0 < q < 1,$$

is used.

Throughout the sequel, we always asuume that $z \in \mathbb{D}$ where $\mathbb{D}$ is the unit disc given by $\{z : |z| < 1\}$. The class of normalized analytic functions

$$\mathcal{A} = \left\{ f : \mathbb{D} \to \mathbb{C} \mid f(z) = z + \sum_{n=2}^{\infty} A_n z^n \right\} \tag{1.1}$$

has been studied extensively, together with its subclass of univalent (schlicht) functions

$$\mathcal{S} = \{f \in \mathcal{A} \mid f \text{ is one-to-one in } \mathbb{D}\}. \tag{1.2}$$

For $f \in \mathcal{A}$, the q-difference operator of the basic differentiation is given by

$$(D_q f)(z) = \begin{cases} \dfrac{f(z) - f(qz)}{z(1-q)}, & z \neq 0 \\ f'(0), & z = 0. \end{cases}$$

Clearly $D_q \to \dfrac{d}{dq}$ as $q \to 1$. A function $f \in \mathcal{A}$ is called starlike ($f \in \mathcal{S}^*$) if

$$Re\left(\frac{zf'(z)}{f(z)}\right) > 0, \quad z \in \mathbb{D} \tag{1.3}$$

and $f \in \mathcal{A}$ is called close-to-convex ($f \in \mathcal{K}$) if there exists $g \in \mathcal{S}^*$ such that

$$Re\left(\frac{zf'(z)}{g(z)}\right) > 0, \quad z \in \mathbb{D}. \tag{1.4}$$

Using q-difference operator the authors in [6] generalize the family $\mathcal{S}^*$ as follows:

Definition 1.1. *A function $f \in \mathcal{A}$ is said to belong to the class $\mathcal{PS}_q$ if*

$$\left| \frac{z(D_q f)(z)}{f(z)} - \frac{1}{1-q} \right| \leq \frac{1}{1-q}, \quad z \in \mathbb{D}. \tag{1.5}$$

Clearly $\mathcal{PS}_q$ reduces to $\mathcal{S}^*$ as $q \to 1^-$. Not much is known about the class $\mathcal{PS}_q$, except what is discussed in [6] for the inclusion of the functions $_2\Phi_1(a, b; c; q; z)$ and for the study of certain continued fraction expansions given by [11]. Recently the second author, among other results, studied [15, 16] certain continued fraction expansion for $_2\Phi_1(a, b; c; q; z)$ and used it to improve the results given in [6]. We now generalize the class $\mathcal{K}$ in the spirit as the Definition 1.1 generalizes $\mathcal{S}^*$.

Definition 1.2. *A function $f \in \mathcal{A}$ is said to belong to the class $\mathcal{PK}_q$ if there exists $g \in \mathcal{S}^*$ such that*

$$\left| \frac{z(D_q f)(z)}{g(z)} - \frac{1}{1-q} \right| \leq \frac{1}{1-q}, \quad z \in \mathbb{D}. \tag{1.6}$$

We observe that (1.6) reduces to (1.4) as $q \to 1^-$ and hence in the limiting case $\mathcal{PK}_q$ reduces to $\mathcal{K}$. Particular choice of the function g used in the study of $\mathcal{PK}_q$ are interesting. According to Frideman [4], there exists only nine functions of the class $\mathcal{S}$ whose coefficients are rational integers. They are

$$\frac{z}{1 \pm z}, \quad \frac{z}{1 \pm z^2}, \quad \frac{z}{(1 \pm z)^2}, \quad \frac{z}{1 \pm z + z^2} \tag{1.7}$$

together with the identity function. It is easy to see that each of these functions maps the disc $\mathbb{D}$ onto a starlike domain and the last fact is easy to see from the analytic characterization given by (1.3). We remark that each of these functions plays an important role in function theory since these together with its rotation are extremal for interesting subfamilies of $\mathcal{S}$. We first state few useful criteria for a normalized power series $(A_0 = 0, \ A_1 = 1,)$ defined by

$$f(z) = \sum_{n=0}^{\infty} A_n z^n, \ \ B_n = \frac{(1 - q^n)A_n}{1 - q}, \tag{1.8}$$

to belong to $\mathcal{PK}_q$.

Lemma 1.1. *Let f be defined by (1.8) and $B_n = \dfrac{(1 - q^n)A_n}{1 - q}$. Then we have the following:*

(1) $\displaystyle\sum_{n=1}^{\infty} |B_{n+1} - B_n| \leq 1$ *implies $f \in \mathcal{PK}_q$ with $g(z) = z/(1 - z)$.*

(2) $\displaystyle\sum_{n=1}^{\infty} |B_{n-1} - B_n + B_{n+1}| \leq 1$ *implies $f \in \mathcal{PK}_q$ with $g(z) = z/(1 - z + z^2)$.*

(3) $\displaystyle\sum_{n=1}^{\infty} |B_{n-1} - 2B_n + B_{n+1}| \leq 1$ *implies $f \in \mathcal{PK}_q$ with $g(z) = z/(1 - z)^2$.*

(4) $\displaystyle\sum_{n=1}^{\infty} |B_{n-1} - B_{n+1}| \leq 1$ *implies $f \in \mathcal{PK}_q$ with $g(z) = z/(1 - z^2)$.*

Proof. (1) Suppose that $\displaystyle\sum_{n=1}^{\infty} |B_{n+1} - B_n| \leq 1$.

The power series converges for $|z| < 1$. Since

$$|B_n| = \left| \sum_{k=1}^{n-1} (B_k - B_{k+1}) - 1 \right| \leq \sum_{k=1}^{n-1} |B_k - B_{k+1}| + 1$$

$$\leq \sum_{k=1}^{\infty} |B_k - B_{k+1}| + 1 \leq 2$$

and

$$|B_n| = \left| \frac{(1 - q^n)A_n}{1 - q} \right| \leq 2, \tag{1.9}$$

we have

$$|A_n| \leq \frac{2}{1 + q + \cdots + q^{n-1}}.$$

Thus, by applying Root test, the radius of convergence of $f(z)$ is seen to be unity and so f is analytic in $\mathbb{D}$. Next we show that f belongs to $\mathcal{P}K_q$ with respect to the starlike function $g(z) = z/(1-z)$. For this we need to show that f satisfies the condition

$$\left|(1-z)(D_q f)(z) - \frac{1}{1-q}\right| \le \frac{1}{1-q}, \quad z \in \mathbb{D}. \tag{1.10}$$

By (1.8) and the defintion of q-difference operator, the above inequality can be rewritten in the equivalent form

$$T_q := \frac{1}{1-q} - \left|1 + \sum_{n=1}^{\infty}(B_{n+1} - B_n)z^n - \frac{1}{1-q}\right| \ge 0.$$

Applying triangle inequality, we find that

$$T_q \ge \frac{1}{1-q} - \left|1 - \frac{1}{1-q}\right| - \left|\sum_{n=1}^{\infty}(B_{n+1} - B_n)z^n\right|$$

$$\ge \frac{1}{1-q} - \frac{q}{1-q} - \sum_{n=1}^{\infty}|B_{n+1} - B_n||z|^n$$

$$\ge 1 - \sum_{n=1}^{\infty}|B_{n+1} - B_n| \ge 0, \qquad \text{by hypothesis.}$$

Thus (1.10) holds for all $z \in \mathbb{D}$. Hence $f \in \mathcal{P}K_q$ with $g(z) = z/(1-z)$.

Next we prove (2) and the rest follows similarly. Assume that

$$\sum_{n=1}^{\infty}|B_{n-1} - B_n + B_{n+1}| \le 1.$$

This implies that the power series converges for $|z| < 1$. Since

$$|B_n| = \left|\sum_{k=2}^{n-2} B_k - \sum_{k=1}^{n-1}[B_{k-1} - B_k + (k+1)B_{k+1}]\right|$$

$$\le \sum_{k=2}^{n-2}|B_k| + \sum_{k=1}^{n-1}|B_{k-1} - B_k + B_{k+1}| \le 1 + \sum_{k=2}^{\infty}|B_k|$$

which is less than or equal to a finite quantity and hence the radius of convergence, by Root test, is unity. Taking $g(z) = z/(1-z+z^2)$, to prove $f \in \mathcal{P}K_q$ with respect to $g(z)$, we need to show that f satisfies the condition

$$\left|(1-z+z^2)(D_q f)(z) - \frac{1}{1-q}\right| \le \frac{1}{1-q}, \qquad z \in \mathbb{D},$$

which after some computation is seen to be equivalent to

$$S_q := \frac{1}{1-q} - \left|1 - \sum_{n=1}^{\infty}[-B_{n-1} + nB_n - B_{n+1}]z^n - \frac{1}{1-q}\right| \ge 0.$$

As in the first case, we use triangle inequality and obtain that for $|z| < 1$, $S_q \geq 0$. Hence $f \in \mathcal{PK}_q$ with $g(z) = z/(1 - z + z^2)$. This completes the proof. $\qquad\square$

The following lemma is immediate from the proof of Lemma 1.1.

Lemma 1.2. *Let B_n be as in Lemma* 1.1 *and*

$$f(z) = \sum_{n=0}^{\infty}(-1)^{n-1}A_n z^n \quad (A_0 = 0, \ A_1 = 1). \tag{1.11}$$

Then we have the following:

(1) $\displaystyle\sum_{n=1}^{\infty}|B_{n+1} - B_n| \leq 1$ *implies* $f \in \mathcal{PK}_q$ *with* $g(z) = z/(1 + z)$.

(2) $\displaystyle\sum_{n=1}^{\infty}|B_{n-1} - B_n + B_{n+1}| \leq 1$ *implies* $f \in \mathcal{PK}_q$ *with* $g(z) = z/(1 + z + z^2)$.

(3) $\displaystyle\sum_{n=1}^{\infty}|B_{n-1} - 2B_n + B_{n+1}| \leq 1$ *implies* $f \in \mathcal{PK}_q$ *with* $g(z) = z/(1 + z)^2$.

(4) $\displaystyle\sum_{n=1}^{\infty}|B_{n-1} - B_{n+1}| \leq 1$ *implies* $f \in \mathcal{PK}_q$ *with* $g(z) = z/(1 + z^2)$.

Note that as $q \to 1$, Lemma 1.1 and Lemma 1.2 give criteria for close-to-convexity with reference to the eight different starlike functions defined by (1.7). In the special case when $q \to 1$, Lemma 1.1 gives results of Ozaki [12] (see also [8]) and for which applications have been obtained related to the univalency question of the Gaussian and the confluent hypergeometric functions by various authors. For example, we refer to [13, 14] and references therein.

Theorem 1.1. *If a and b are related by any one of the following conditions*

1. *(a)* $(1 - q^a)(1 - q^b) > (1 - q)$,

   (b) $\dfrac{\Gamma_q(a + b)}{\Gamma_q(a)\Gamma_q(b)} \leq \dfrac{2}{q}$.

2. *(a)* $(1 - q^{a-1})(1 - q^{b-1}) < -(1 - q)$ *and* $a + b > 2$,

   (b) $\dfrac{\Gamma_q(a + b)}{\Gamma_q(a)\Gamma_q(b)} \geq 0$.

Then the function $z\phi(a, b; a + b; q, z)$ belongs to $\mathcal{PK}_q$ with respect to $\dfrac{z}{1 - z}$.

Proof. Consider $\phi(a, b; a + b; q, z) = \sum\limits_{n=0}^{\infty} \dfrac{(a; q)_n (b; q)_n}{(a + b; q)_n (q; q)_n} z^n$, then

$$z\phi(a, b; a + b; q, z) = z + \sum_{n=2}^{\infty} A_n z^n$$

$$= z + \sum_{n=2}^{\infty} \frac{(a; q)_{n-1} (b; q)_{n-1}}{(a + b; q)_{n-1} (q; q)_{n-1}} z^n, \qquad (1.12)$$

therefore $B_n = \dfrac{1 - q^n}{1 - q} \dfrac{(a; q)_{n-1} (b; q)_{n-1}}{(a + b; q)_{n-1} (q; q)_{n-1}}$. We further write

$$B_{n+1} - B_n = \frac{1}{1 - q} \frac{(a; q)_{n-1} (b; q)_{n-1}}{(a + b; q)_n (q; q)_n} f(q, n),$$

where

$$f(q, n) = (1 - q^{n+1})(1 - q^{a+n-1})(1 - q^{b+n-1}) - (1 - q^n)^2 (1 - q^{a+b+n-1}).$$

If we take $S := \sum\limits_{n \geq 1} \left| B_{n+1} - B_n \right|$, then from Lemma (1.1), it is sufficient to show that $S \leq 1$.

We assume that the first hypothesis of the theorem is true. Now writing

$$f(q, n) > (1 - q^n)(1 - q^{a+n-1})(1 - q^{b+n-1}) - (1 - q^n)^2 (1 - q^{a+b+n-1})$$

$$= (1 - q^n) \left(((1 - q^{a+n-1})(1 - q^{b+n-1}) - (1 - q^n)(1 - q^{a+b+n-1})) \right),$$

to show that $f(q, n) > 0$, it is enough to show that

$$(1 - q^{a+n-1})(1 - q^{b+n-1}) - (1 - q^n)(1 - q^{a+b+n-1}) > 0.$$

Rewriting

$$(1 - q^{a+n-1})(1 - q^{b+n-1}) - (1 - q^n)(1 - q^{a+b+n-1})$$
$$= q^{n-1}((1 - q^a)(1 - q^b) - (1 - q)) + q^{a+b+2n-2}(1 - q)$$

we can easily see that the first term is positive from the given hypothesis 1(a), and the term $q^{a+b+2n-2}(1 - q)$, which is positive for $n \geq 1$, hence $f(q, n)$ is positive. Now,

$$S = \frac{1}{1 - q} \sum_{n=1}^{\infty} \frac{(a; q)_{n-1} (b; q)_{n-1}}{(a + b; q)_n (q; q)_n} f(q, n)$$

$$= -1 + \frac{q(1 - q^a)(1 - q^b)}{1 - q} \sum_{n=1}^{\infty} \frac{(a; q)_{n-1} (b; q)_{n-1}}{(a + b; q)_n (q; q)_n}$$

$$= -1 + q \frac{\Gamma_q(a + b)}{\Gamma_q(a)\Gamma_q(b)} \leq 1,$$

from the hypothesis $1(b)$. Now considering the second hypothesis and observing that $f(q, n)$ is q^n multiple of

$$2 - q - q^{a-1} - q^{b-1} + q^{a+b-1}$$
$$- q^n((1 - q^a)(1 - q^b) - q^{a+b} - q^{a+b-2} + 2q^{a+b-1})$$

and considering

$$q^n((1 - q^a)(1 - q^b) - q^{a+b} - q^{a+b-2} + 2q^{a+b-1})$$
$$= q^n(1 - q^a - q^b - q^{a+b-2} + 2q^{a+b-1})$$
$$= q^n(1 - q^a)(1 - q^b) - q^{a+b-1+n}(1 - q) + q^{a+b+n-2}(1 - q))$$

we see that the first two terms of the above expression are positive for all $a, b > 0$ and $n \geq 1$. Hence to show that $f(q, n)$ is negative it is enough to show that

$$2 - q - q^{a-1} - q^{b-1} + q^{a+b-1} + q^{a+b+n-2}(1 - q) < 0$$

which is clearly true from hypothesis $2(a)$ by taking $a + b > 2$ and using the inequality $q^{a+b+n-2} < q^{a+b-2}$. Now it is easy to see that

$$S = \frac{1}{1-q} \sum_{n=1}^{\infty} \frac{(a; q)_{n-1}(b; q)_{n-1}}{(a+b; q)_n(q; q)_n} f(q, n) = 1 - q\frac{\Gamma_q(a+b)}{\Gamma_q(a)\Gamma_q(b)} \leq 1$$

is true from hypothesis $2(b)$ and the proof is complete. $\qquad\square$

The following corollary is immediate.

Corollary 1.1. *Let* $f(z) = z + \sum\limits_{n=2}^{\infty} b_n z^n$ *and*

$$2 \geq B_2 \geq \ldots B_n - (n-2) \geq B_{n+1} - (n-1) \ldots \geq 0$$

or

$$0 \leq B_2 \leq B_3 + 1 \leq B_4 + 2 \ldots B_n + (n-2) \leq B_{n+1} + (n-1) \leq 2.$$

then $f \in \mathcal{PK}_q$ *with* $g(z) = z/(1-z)$.

Before proceeding for the next result, we give a list of functions.

$$g_1(q, a, b) = ((1 + q^a)(1 + q^b) - (1 + q^b))q - (1 + q),$$
$$g_2(q, a, b) = (1 - q^{a-1})(1 - q^{b-1}) \qquad \text{and}$$
$$g_3(q, a, b) = g_1(q, a, b)(1 - q) - 1 + q)(1 - q^a)(1 - q^b) - g_2(q, a, b)(1 - q)^2.$$

Theorem 1.2. *If a and b are related by any one of the following conditions*

1. *(a)* $(1 - q^{a-1})(1 - q^{b-1}) < -(1 - q)(1 - q^{a+b-2})$

(b) $(1 - q^a)(1 - q^b) > (1 - q)^2 q^{a+b-2}$ *and*

(c) $q \dfrac{(1 - q^a)(1 - q^b)\Gamma_q(a + b)}{\Gamma_q(a + 2)\Gamma_q(b + 2)(1 - q)^4} \, g_3(q, a, b) < 0.$

2. (a) $(1 - q^{a-1})(1 - q^{b-1}) > -(1 - q)(1 - q^{a+b-2})$

   (b) $(1 - q^a)(1 - q^b) < (1 - q)^2 q^{a+b-2},$ *and*

(c) $q \dfrac{(1 - q^a)(1 - q^b)\Gamma_q(a + b)}{\Gamma_q(a + 2)\Gamma_q(b + 2)(1 - q)^4} \, g_3(q, a, b) > -2.$

Then the function $z\phi(a, b; a + b; q, z)$ *belongs to* $\mathcal{PK}_q$ *with respect to* $\dfrac{z}{1 - z^2}.$

Proof. Consider $\phi(a, b; c; q, z) = \displaystyle\sum_{n=0}^{\infty} \dfrac{(a; q)_n (b; q)_n}{(c; q)_n (q; q)_n} z^n$, then

$$z\phi(a, b; c; q, z) = z + \sum_{n=2}^{\infty} A_n z^n z + \sum_{n=2}^{\infty} \dfrac{(a; q)_{n-1}(b; q)_{n-1}}{(c; q)_{n-1}(q; q)_{n-1}} z^n,$$

therefore $B_n = \dfrac{1 - q^n}{1 - q} \dfrac{(a; q)_{n-1}(b; q)_{n-1}}{(c; q)_{n-1}(q; q)_{n-1}}$, which gives

$$B_{n-1} - B_{n+1} = \dfrac{1}{1 - q} \dfrac{(a; q)_{n-2}(b; q)_{n-2}}{(c; q)_n (q; q)_n} g(q, n),$$

where

$$g(q, n) = (1 - q^n)(1 - q^{n-1})^2 (1 - q^{a+b+n-2})(1 - q^{a+b+n-1})$$
$$- (1 - q^{n+1})(1 - q^{a+n-1})(1 - q^{a+n-2})(1 - q^{b+n-1})(1 - q^{b+n-2}).$$

Now we take

$$S := \sum_{n \geq 1} \left| B_{n-1} - B_{n+1} \right|,$$

from Lemma 1.1 it is sufficient to show that $S \leq 1$.

For the first part, writing

$$B_{n-1} - B_{n+1} = (B_{n-1} - B_n) + (B_n - B_{n+1}),$$

we have

$$B_{n-1} - B_n = \dfrac{(a; q)_{n-2}(b; q)_{n-2}}{(1 - q)(a + b; q)_{n-2}(q; q)_{n-2}} M(f, q, n)$$

where

$$M(f, q, n) = (1 - q^{n-1})(1 - q^{a+b+n-2})(1 - q^{n-1}) - (1 - q^n)(1 - q^{a+n-2})(1 - q^{b+n-2})$$

$$= -q^{n-1}(A) + q^{2n-2}(B)$$

with

$$A = (1-q)(1-q^{a+b-2}) + (1-q^{a-1})(1-q^{b-1}) \qquad (1.13)$$

and

$$B = (1-q^a)(1-q^b) - (1-q)^2 q^{a+b-2}. \qquad (1.14)$$

This gives $B_{n-1} - B_n$ positive, since $A < 0$ from $1(a)$ and $B > 0$ from $1(b)$ of the hypotheses of the theorem. Similarly

$$B_n - B_{n+1} = \frac{(a;q)_{n-2}(b;q)_{n-2}}{(1-q)(a+b;q)_{n-2}(q;q)_{n-2}} M_1(f,q,n),$$

where

$$M_1(f,q,n) = (1-q^n)(1-q^{a+b+n-1})(1-q^n) - (1-q^{n+1})(1-q^{a+n-1})(1-q^{b+n-1})$$
$$= -q^n(A) + q^{2n}(B).$$

Hence

$$B_{n-1} - B_{n+1} = (B_{n-1} - B_n) + (B_n + B_{n+1})$$
$$= -(1+q)q^{n-1}A + (1+q^2)q^{2n-2}B = g(q,n)$$

which is positive from the hypothesis, since $A < 0$ from $1(a)$, $B > 0$ from $1(b)$. Combining these we have $B_{n-1} - B_{n+1}$ is positive. Further

$$S = \frac{1}{1-q} \sum_{n=0}^{\infty} \frac{(a;q)_{n-2}(b;q)_{n-2}}{(c;q)_n(q;q)_n} g(q,n)$$

$$= 1 + \frac{q(1-q^a)(1-q^b)}{1-q} \left(g_1(q,a,b) \sum_{n=1}^{\infty} \frac{(a;q)_{n-2}(b;q)_{n-2}}{(a+b;q)_n(q;q)_{n-2}} q^{2n-4} \right.$$

$$- (1+q) \sum_{n=1}^{\infty} \frac{(a;q)_{n-2}(b;q)_{n-2}}{(a+b;q)_n(q;q)_{n-3}} q^{n-2}$$

$$\left. - g_2(q,a,b) \sum_{n=1}^{\infty} \frac{(a;q)_{n-2}(b;q)_{n-2}}{(a+b;q)_n(q;q)_{n-1}} q^{3n-3} \right)$$

$$= 1 + q \frac{(1-q^a)(1-q^b)\Gamma_q(a+b)}{\Gamma_q(a+2)\Gamma_q(b+2)(1-q)^4} g_3(q,a,b) \le 1,$$

from $1(c)$ of the hypothesis.

Proceeding similar to the first part, we can easily see that $g(q,n)$ is negative from the hypothesis $2(a)$ and $2(b)$. Hence

$$S = \frac{1}{1-q} \sum_{n=0}^{\infty} \frac{(a;q)_{n-2}(b;q)_{n-2}}{(c;q)_n(q;q)_n} g(q,n)$$

$$= -1 - q \frac{(1 - q^a)(1 - q^b)\Gamma_q(a + b)}{\Gamma_q(a + 2)\Gamma_q(b + 2)(1 - q)^4} g_f(q, n) \leq 1$$

from $2(c)$ of the hypothesis and the proof is complete. $\qquad\qquad\square$

Theorem 1.3. *The function $z\phi(a, b; a + b; q, z)$ belongs to $\mathcal{PK}_q$ with respect to the starlike function $\dfrac{z}{(1 - z)^2}$, whenever*

$$\left((1 - q^{a-1})(1 - q^{b-1}) + (1 - q)(1 - q^{a+b-2}) \right) \times$$
$$\left((1 - q^a)(1 - q^b) - (1 - q)^2 q^{a+b-2} \right) > 0. \qquad (1.15)$$

Proof. Consider $\phi(a, b; c; q, z) = \displaystyle\sum_{n=0}^{\infty} \frac{(a; q)_n (b; q)_n}{(c; q)_n (q; q)_n} z^n$, then

$$z\phi(a, b; c; q, z) = z + \sum_{n=2}^{\infty} A_n z^n z + \sum_{n=2}^{\infty} \frac{(a; q)_{n-1}(b; q)_{n-1}}{(c; q)_{n-1}(q; q)_{n-1}} z^n,$$

therefore $B_n = \dfrac{1 - q^n}{1 - q} \dfrac{(a; q)_{n-1}(b; q)_{n-1}}{(c; q)_{n-1}(q; q)_{n-1}}$. Let $S := \displaystyle\sum_{n \geq 1} \left| B_{n-1} - 2B_n + B_{n+1} \right|$. Then, from Lemma 1.1, it is sufficient to show that $S \leq 1$. Infact, we show that $|S| = 1$.

Now

$$B_{n-1} - 2B_n + B_{n+1} = \frac{1}{1 - q} \frac{(a; q)_{n-2}(b; q)_{n-2}}{(c; q)_n (q; q)_n} h(q, n)$$

where

$$\begin{aligned}
h(q, n) &= (1 - q^{n-1})(1 - q^{a+b+n-1})(1 - q^{a+b+n-2})(1 - q^n)(1 - q^{n-1}) \\
&\quad - 2(1 - q^n)(1 - q^{a+n-2})(1 - q^{b+n-2})(1 - q^{a+b+n-1})(1 - q^n) \\
&\quad + (1 - q^{n+1})(1 - q^{a+n-1})(1 - q^{a+n-2})(1 - q^{b+n-1})(1 - q^{b+n-2}), \\
&= (1 - q)(q^{n-1} A + (1 + q)q^{2n-2} B),
\end{aligned}$$

where A and B are respectively, as in (1.13) and (1.14). Thus, taking

$$(1 - q^{a-1})(1 - q^{b-1}) > -(1 - q)(1 - q^{a+b-2}) \qquad \text{and}$$
$$(1 - q^a)(1 - q^b) > (1 - q)^2 q^{a+b-2}$$

satisfies the hypothesis (1.15) of the theorem, which means we get $B_{n-1} - 2B_n + B_{n+1}$ is positive. On the other hand, if we take

$$(1 - q^{a-1})(1 - q^{b-1}) < -(1 - q)(1 - q^{a+b-2}) \qquad \text{and}$$
$$(1 - q^a)(1 - q^b) < (1 - q)^2 q^{a+b-2}$$

we again see that the hypothesis (1.15) of the theorem is satisfied to yield $B_{n-1} - 2B_n + B_{n+1}$ negative. This means $B_{n-1} - 2B_n + B_{n+1} \neq 0$.

Now

$$
|S| = \left| \frac{1}{1-q} \sum_{n=0}^{\infty} \frac{(a;q)_{n-2}(b;q)_{n-2}}{(c;q)_n(q;q)_n} h(q,n) \right|
$$

$$
= \left| -1 + q(1-q^a)(1-q^b) \left(\frac{(1-q^a)(1-q^b)}{1-q} \sum_{n=1}^{\infty} \frac{(a;q)_{n-2}(b;q)_{n-2}}{(a+b;q)_n(q;q)_{n-1}} q^{2n-3} \right. \right.
$$

$$
\left. \left. - \sum_{n=1}^{\infty} \frac{(a;q)_{n-2}(b;q)_{n-2}}{(a+b;q)_n(q;q)_{n-1}} q^{n-2} + \sum_{n=1}^{\infty} \frac{(a;q)_{n-2}(b;q)_{n-2}}{(a+b;q)_n(q;q)_{n-1}} q^{a+b+3n-5} \right) \right|
$$

$$
= 1,
$$

which satisfies Lemma 1.1 and the proof is complete.   □

We define the q-Alexander transform, analogous to the Alexander transform [3] in the following way. Given $f \in \mathcal{A}$, the q-Alexander transform is given by

$$
\Lambda_{f,q}(z) = \int_0^z \frac{f(t)}{t} d_q(t), \qquad f \in \mathcal{A}, \quad z \in \mathbb{D}. \tag{1.16}
$$

Hence, for $f(z) = z + \sum_{n=2}^{\infty} A_n z^n$, we see that

$$
\Lambda_{f,q}(z) = z + \sum_{n=2}^{\infty} A_n \frac{1-q}{1-q^n} z^n.
$$

With this, we give our next result.

Theorem 1.4. *Let a, b and c satisfy any one of the following properties.*

1. $a, b \in (1, \infty)$ and $\Gamma_q(a+b-1) \leq 2\Gamma_q(a)\Gamma_q(b)$,

2. $a \in (0,1)$, $b \in (1-a, 1)$ and $\Gamma_q(a+b-1) \leq 2\Gamma_q(a)\Gamma_q(b)$,   and

3. $a \in (0,1)$, $b \in (1, \infty)$ and $\dfrac{\Gamma_q(a+b-1)}{\Gamma_q(a)\Gamma_q(b)} \geq 0$.

Then the q-Alexander transform (1.16) of the function $z\phi(a,b; a+b-1; q, z)$ is in PK_q with $g(z) = \dfrac{z}{1-z}$.

Proof. Given $f(z) = z\phi(a,b; a+b-1; q, z)$, $\Lambda_{f,q}(z)$ is given by

$$
z + \sum_{n=2}^{\infty} \frac{(a;q)_n(b;q)_n}{(a+b-1;q)_n(q;q)_n} \frac{1-q}{1-q^n} z^n, \qquad z \in \mathbb{D}.
$$

Then, in both the cases, viz., $a, b \in (1, \infty)$ and $a \in (0, 1)$, $b \in (1 - a, 1)$,

$$\left| B_{n+1} - B_n \right| = \left| \frac{(a; q)_{n-1}(b; q)_{n-1}}{(a + b - 1; q)_n (q; q)_n} \times \right.$$

$$\left. \left[(1 - q^{a+n-1})(1 - q^{b+n-1}) - (1 - q^{a+b+n-2})(1 - q^n) \right] \right|,$$

so that

$$S := \sum_{n=1}^{\infty} \left| B_{n+1} - B_n \right| = \sum_{n=1}^{\infty} \frac{(a - 1; q)_n (b - 1; q)_n}{(a + b - 1; q)_n (q; q)_n} q^n$$

$$= \frac{\Gamma_q(a + b - 1)}{\Gamma_q(a) \Gamma_q(b)} - 1 \leq 1,$$

since $\Gamma_q(a + b - 1) \leq 2 \Gamma_q(a) \Gamma_q(b)$. In the case $a \in (0, 1)$ and $b \in (1, \infty)$,

$$S := \sum_{n=1}^{\infty} \left| B_{n+1} - B_n \right| = -\sum_{n=1}^{\infty} \frac{(a - 1; q)_n (b - 1; q)_n}{(a + b - 1; q)_n (q; q)_n} q^n$$

$$= 1 - \frac{\Gamma_q(a + b - 1)}{\Gamma_q(a) \Gamma_q(b)} \leq 1,$$

using $\dfrac{\Gamma_q(a + b - 1)}{\Gamma_q(a) \Gamma_q(b)} \geq 0$ and the proof is complete. $\qquad \square$

Corollary 1.2. *Let a and b satisfy any one of the following conditions*

1. $a, b \in (0, \infty)$ and $\Gamma_q(a + b - 1) \leq 2 \Gamma_q(a) \Gamma_q(b)$,

2. $a \in (-1, 0)$, $b \in (-1 - a, 0)$ and $\Gamma_q(a + b - 1) \leq 2 \Gamma_q(a) \Gamma_q(b)$,

3. $a \in (-1, 0)$, $b \in (0, \infty)$ and $\dfrac{\Gamma_q(a + b - 1)}{\Gamma_q(a) \Gamma_q(b)} \geq 0$.

Then the function $\dfrac{(1 - q^{a+b})(1 - q)}{(1 - q^a)(1 - q^b)} \left[z \phi(a, b; a + b; q, z) \right]$ belongs to PK_q with respect to $g(z) = \dfrac{z}{1 - z}$.

Proof. The q-Alexander transform of $g(z) = z \phi(a + 1, b + 1; c + 1; q, z)$ is

$$\Lambda_{g,f}(z) = \int_0^z \frac{g(t)}{t} d_q t = \frac{(1 - q)(1 - q^c)}{(1 - q^a)(1 - q^b)} \left[\phi(a, b; c; q, z) - 1 \right]$$

and the results follow from Theorem 1.4 by replacing $a = a + 1$, $b = b + 1$ and $c = a + b + 1$. $\qquad \square$

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Maximal ideal space of certain
α-Lipschitz operator algebras

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ABSTRACT: In a recent paper by A. Ebadian and A.A. Shokri [1], a α-Lipschitz operator from a compact metric space X into a unital bounded commutative Banach algebra B is defined. Let (X, d) be a nonempty compact metric space, $0 < \alpha \leq 1$ and $(B, \| \, . \, \|)$ be a unital bounded commutative Banach algebra. Let $\mathrm{Lip}_\alpha(X, B)$ be the algebra of all bounded continuous operators $f : X \to B$ such that

$$p_\alpha(f) := \sup \left\{ \frac{\| f(x) - f(y) \|}{d^\alpha(x, y)} : x, y \in X, x \neq y \right\} < \infty \ .$$

In this work, we characterize the maximal ideal space of $\mathrm{Lip}_\alpha(X, B)$.

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1   Introduction

Let (X, d) be a compact metric space with at least two elements and $(B, \| \, . \, \|)$ be a Banach space over the scaler field $\mathbb{F}$ $(= \mathbb{R}$ or $\mathbb{C})$. For a constant $0 < \alpha \leq 1$ and an operator $f : X \to B$, set

$$p_\alpha(f) := \sup_{s \neq t} \frac{\| f(t) - f(s) \|}{d^\alpha(s, t)} \quad ; \quad (s, t \in X) \ ,$$

which is called the Lipschitz constant of f. Define

$$Lip_\alpha(X, B) := \{ f : X \to B \quad : \quad p_\alpha(f) < \infty \}$$

and

$$lip_\alpha(X, B) := \{f : X \to B : \quad \frac{\| f(t) - f(s) \|}{d^\alpha(s, t)} \to 0 \quad as \quad d(s, t) \to 0, \ s, t \in X, \ s \neq t\}.$$

The elements of $\text{Lip}_\alpha(X, B)$ and $\text{lip}_\alpha(X, B)$ are called big and little α-Lipschitz operators, respectively [1]. Let $C(X, B)$ be the set of all continuous operators from X into B and for each $f \in C(X, B)$, define

$$\| f \|_\infty := \sup_{x \in X} \| f(x) \| .$$

For f, g in $C(X, B)$ and λ in $\mathbb{F}$, define

$$(f + g)(x) := f(x) + g(x), \quad (\lambda f)(x) := \lambda f(x), \quad (x \in X).$$

It is easy to see that $(C(X, B), \| . \|_\infty)$ becomes a Banach space over $\mathbb{F}$ and $\text{Lip}_\alpha(X, B)$ is a linear subspace of $C(X, B)$. For each element f of $\text{Lip}_\alpha(X, B)$, define

$$\| f \|_\alpha := \| f \|_\infty + p_\alpha(f).$$

When $(B, \| . \|)$ is a Banach space, Cao, Zhang and Xu [6] proved that $\left(\text{Lip}_\alpha(X, B), \| . \|_\alpha\right)$ is a Banach space over $\mathbb{F}$ and $\text{lip}_\alpha(X, B)$ is a closed linear subspace of $\left(\text{Lip}_\alpha(X, B), \| . \|_\alpha\right)$, and when $(B, \| . \|)$ is a unital commutative Banach algebra, A. Ebadian and A.A. Shokri [1] proved that $(\text{Lip}_\alpha(X, B), \| . \|_\alpha)$ is a Banach algebra over $\mathbb{F}$ under pointwise multiplication and $\text{lip}_\alpha(X, B)$ is a closed linear subalgebra of $(\text{Lip}_\alpha(X, B), \| . \|_\alpha)$. Furthermore, Sherbert [4,5], Weaver [7,8], Honary and Mahyar [9], Johnson [3], Cao, Zhang and Xu [6], Ebadian [2], Bade, Curtis and Dales [11], and etc studied some properties of Lipschitz algebras.

Finally, in this paper, we will study the Maxima Ideal space of $\text{Lip}_\alpha(X, B)$.

2   Maximal Ideal Space of $\text{Lip}_\alpha(X, B)$

In this section, let us use (X, d) to denote a compact metric space in $\mathbb{C}$ which has at least two elements, $(B, \| . \|)$ to denote a unital bounded commutative Banach algebra with unit $\mathbf{e}$ over the scalar field $\mathbb{F}(= \mathbb{R}$ or $\mathbb{C})$, $\text{Lip}_\alpha(X) = \text{Lip}_\alpha(X, \mathbb{C})$ and $0 < \alpha < 1$. Let E_1 and E_2 be Banach spaces with dual spaces E_1^* and E_2^*. Then we define for $X \in E_1 \otimes E_2$

$$\| X \|_\varepsilon = \sup \left\{ \ | \langle X , \ \phi_1 \otimes \phi_2 \rangle \ |: \ \phi_j \in B_1[0, E_j^*] \ for \ j = 1, 2\right\},$$

where

$$X = \sum_{k=1}^m x_1^{(k)} \otimes x_2^{(k)} , \ (m \in \mathbb{N} , \ x_1^{(k)} \in E_1 , \ x_2^{(k)} \in E_2 , \ 1 \leq k \leq m) ,$$

and

$$\langle X, \phi_1 \otimes \phi_2 \rangle = \langle \sum_{k=1}^m x_1^{(k)} \otimes x_2^{(k)}, \phi_1 \otimes \phi_2 \rangle = \sum_{k=1}^m \phi_1(x_1^{(k)}) \phi_2(x_2^{(k)}),$$

and $B_1[0, E_j^*]$ is called ball in $E*_j$ with radius 1 centered at 0 for $j = 1, 2$. We call $\| \, . \, \|_\varepsilon$ the injective norm on $E_1 \otimes E_2$. The injective tensor product $E_1 \check{\otimes} E_2$ is the completion of $E_1 \otimes E_2$ with respect to $\| \, . \, \|_\varepsilon$ [10].

Theorem 2.1. $\left(Lip_\alpha(X, B), \| \, . \, \|_\alpha \right)$ *is isometrically isomorphic to* $\left(Lip_\alpha(X) \check{\otimes} B, \| \, . \, \|_\varepsilon \right)$.

Proof. See [1]. □

Lemma 2.2. *Let* $f \in Lip_\alpha(X, B)$ *and*

$$\varphi(x) := \| f(x) \|^{1/2}, \quad (x \in X).$$

Then $\varphi \in Lip_\alpha(X)$.

Proof. Firstly, we show that $\varphi \in C(X)$. For this purpose, suppose that $x \in X$ and $\{x_n\} \subset X$ is a sequence such that $x_n \longrightarrow x$ (in X). Let $f \in Lip_\alpha(X, B)$. Then $f \in C(X, B)$, and so $f(x_n) \longrightarrow f(x)$ (with $\| \, . \, \|$). Thus for every $\varepsilon > 0$, there is $N \in \mathbb{N}$ such that for every $n \geq N$,

$$\| f(x_n) - f(x) \| < 2 \| f(x) \|^{1/2} \varepsilon.$$

Now for every $n \geq N$ we have

$$
\begin{aligned}
| \varphi(x_n) - \varphi(x) | &= \left| \| f(x_n) \|^{1/2} - \| f(x) \|^{1/2} \right| \\
&= \left| \frac{\| f(x_n) \| - \| f(x) \|}{\| f(x_n) \|^{1/2} + \| f(x) \|^{1/2}} \right| \\
&\leq \frac{\| f(x_n) - f(x) \|}{\| f(x_n) \|^{1/2} + \| f(x) \|^{1/2}} \\
&\leq \frac{2 \| f(x) \|^{1/2} \varepsilon}{2 \| f(x) \|^{1/2}} \\
&= \varepsilon \quad , \qquad \left(f(x) \neq 0 \right).
\end{aligned}
$$

Also this holds for $f(x) = 0$.

This implies that $\varphi(x_n) \longrightarrow \varphi(x)$, so $\varphi \in C(X)$. Now, we show that $p_\alpha(\varphi) < \infty$. For every $x, y \in X$ such that $x \neq y$, we have

$$p_\alpha(\varphi) = \sup_{x \neq y} \frac{| \varphi(x) - \varphi(y) |}{d^\alpha(x, y)} \, .$$

Since $f \in Lip_\alpha(X, B)$, $p_\alpha(f) < \infty$. So

$$\sup_{x \neq y} \frac{\| f(x) - f(y) \|}{d^\alpha(x, y)} < \infty \, ,$$

and then

$$\sup_{x \neq y} \frac{| \, \| f(x) \| - \| f(y) \| \, |}{d^\alpha(x, y)} < \infty \, .$$

So

$$\sup_{x \neq y} \frac{\left| \left(\| f(x) \|^{1/2} + \| f(y) \|^{1/2} \right) \left(\| f(x) \|^{1/2} - \| f(y) \|^{1/2} \right) \right|}{d^{\alpha}(x,y)} < \infty$$

Since B is bounded, $\| f \| < \infty$, $(x \in X)$. Thus

$$\sup_{x \neq y} \frac{\left| \| f(x) \|^{1/2} - \| f(y) \|^{1/2} \right|}{d^{\alpha}(x,y)} < \infty$$

and so

$$\sup_{x \neq y} \frac{\left| \varphi(x) - \varphi(y) \right|}{d^{\alpha}(x,y)} < \infty.$$

Therefore $p_{\alpha}(\varphi) < \infty$. Hence $\varphi \in \mathrm{Lip}_{\alpha}(X)$. □

Remark 2.3. *Note that, in lemma 2.2., we suppose that $0 < \alpha < 1$. Because for $\alpha = 1$, the function $f(x) = x^{1}/2$ on $[0,1]$ is not Lipschitz, where $B = \mathbb{C}$ and $d(x,y) = |x - y|$, $(x, y \in X)$.*

Lemma 2.4. *Let $f \in \mathrm{Lip}_{\alpha}(X, B)$ and*

$$g(x) := \begin{cases} \| f(x) \|^{-\frac{1}{2}} f(x) \,, & f(x) \neq 0 \; ; \\ 0 \,, & f(x) = 0 \; . \end{cases} \quad (x \in X).$$

Then $g \in \mathrm{Lip}_{\alpha}(X, B)$.

Proof. Case 1: $f(x) \neq 0$, $(x \in X)$. Let

$$\varphi(x) := \| f(x) \|^{1/2}, \quad (x \in X).$$

Then by lemma 2.2., $\varphi \in \mathrm{Lip}_{\alpha}(X)$. Let $x \in X$ and $\{x_n\} \subset X$ be a sequence such that $x_n \longrightarrow x$ in X. Since $f \in C(X, B)$, $f(x_n) \longrightarrow f(x)$ with $\| . \|$. So

$$\| f(x_n) \|^{-1/2} \longrightarrow \| f(x) \|^{-1/2} .$$

For every $\varepsilon > 0$, we have

$$\begin{aligned} \| g(x_n) - g(x) \| &= \left\| \| f(x_n) \|^{-1/2} f(x_n) - \| f(x) \|^{-1/2} f(x) \right\| \\ &\leq \| f(x_n) \|^{-1/2} \| f(x_n) - f(x) \| \\ &+ \| f(x) \| \; \left| \| f(x_n) \|^{-1/2} - \| f(x) \|^{-1/2} \right| \\ &< \varepsilon. \end{aligned}$$

So $g \in C(X, B)$. Now we have

$$f(x) = \| f(x) \|^{1/2} g(x), \quad (x \in X) .$$

Since $f \in \text{Lip}_\alpha(X, B)$, $\| f \|_\alpha < \infty$. So $\| f \|_\infty < \infty$. Then $\| g \|_\infty < \infty$. Also $p_\alpha(f) < \infty$, thus

$$\sup_{x \neq y} \frac{\| f(x) - f(y) \|}{d^\alpha(x, y)} < \infty \ ,$$

$$\sup_{x \neq y} \frac{\| \| f(x) \|^{1/2} g(x) - \| f(y) \|^{1/2} g(y) \|}{d^\alpha(x, y)} < \infty \ .$$

Then

$$\sup_{x \neq y} \frac{\| \| f(x) \|^{1/2} g(x) - \| f(x) \|^{1/2} g(y) + \| f(x) \|^{1/2} g(y) - \| f(y) \|^{1/2} g(y) \|}{d^\alpha(x, y)} < \infty \ .$$

So

$$\sup_{x \neq y} \frac{\| \varphi(x) \big(g(x) - g(y) \big) + g(y) \big(\varphi(x) - \varphi(y) \big) \|}{d^\alpha(x, y)} < \infty \ ,$$

$$\left(\sup_{x \neq y} \varphi(x) \times \frac{\| g(x) - g(y) \|}{d^\alpha(x, y)} \right) - \left(\sup_{x \neq y} \| g(y) \| \times \frac{\| \varphi(x) - \varphi(y) \|}{d^\alpha(x, y)} \right) < \infty \ .$$

Hence

$$\| \varphi \|_\infty \, p_\alpha(g) - \| g \|_\infty \, p_\alpha(\varphi) < \infty \ .$$

Since $\| g \|_\infty < \infty$ and $\| \varphi \|_\infty < \infty$ and $p_\alpha(\varphi) < \infty$, $p_\alpha(g) < \infty$. So $g \in \text{Lip}_\alpha(X, B)$.

Case 2: $f(x) = 0$, $(x \in X)$. Firstly, we show that g is continuous. Let $x \in X$ with $f(x) = 0$ be fixed. Let $\varepsilon > 0$ and $n \in \mathbb{N}$ with $\frac{2}{n} < \varepsilon$. Then V defined by

$$V := \{ t \in X \ : \ \| f(t) \| < \frac{1}{n^2} \}$$

is a neighborhood of x satisfying $\| g(t) \| < \infty$ for each $t \in V$. Indeed, $f(t) = 0$ implies that

$$\| g(t) \| = \| 0 \| = 0 < \varepsilon.$$

If $t \in V$ satisfies $f(t) \neq 0$, then there is $k \geq n$ with

$$\frac{1}{(k+1)^2} < \| f(t) \| \leq \frac{1}{k^2}.$$

Since $\frac{1}{(k+1)^2} < \| f(t) \|$, $\frac{1}{k+1} < \| f(t) \|^{1/2}$. So

$$\frac{1}{k+1} \| f(t) \|^{-1/2} < 1.$$

Thus we get

$$\begin{aligned}
\| g(t) \| &= \| \| f(t) \|^{-1/2} f(t) \| \\
&= \| \frac{1}{k+1} \| f(t) \|^{-1/2} (k+1) f(t) \| \\
&< (k+1) \| f(t) \| \\
&\leq \frac{k+1}{k^2} \leq \frac{2k}{k^2} = \frac{2}{k} \leq \frac{2}{n} < \varepsilon \ .
\end{aligned}$$

Which proves the continuity of g. Now for every $x, y \in X$, $x \neq y$ we have

$$
\begin{aligned}
p_\alpha(g) &= \sup_{x \neq y} \frac{\| g(x) - g(y) \|}{d^\alpha(x, y)} \\
&= \sup_{x \neq y} \frac{\| 0 - 0 \|}{d^\alpha(x, y)} = 0 < \varepsilon \ ,
\end{aligned}
$$

so $g \in \text{Lip}_\alpha(X, B)$. $\square$

Theorem 2.5. *Every character χ on $Lip_\alpha(X, B)$ is of form $\chi = \psi o \delta_z$ for some character ψ on B and some $z \in X$.*

Proof. Let

$$
j : Lip_\alpha(X) \to Lip_\alpha(X, B)
$$

$$
h \mapsto h \otimes \mathbf{e} \ ,
$$

be the canonical embedding. Since $\big(\text{Lip}_\alpha(X, B), \| \ . \ \|_\alpha \big)$ is isometrically isomorphic to $\big(\text{Lip}_\alpha(X) \check{\otimes} B, \| \ . \ \|_\varepsilon \big)$ by theorem 2.1., j is a well define map. Then there is $z \in X$ such that $\chi o j$ is the evaluation in z. Consider the ideal

$$
I := \big\{ f \in Lip_\alpha(X, B) : f(z) = 0 \big\}.
$$

We will show that I is contained in the kernel of χ. Given $f \in I$ we define

$$
\varphi(x) := \| f(x) \|^{1/2} \ (x \in X).
$$

By lemma 2.2., $\varphi \in \text{Lip}_\alpha(X)$ and has the same zeros as f. The function $g : X \to B$ defined by

$$
g(x) := \begin{cases} \| f(x) \|^{-1/2} f(x) & \text{if } f(x) \neq 0 \\ 0 & \text{if } f(x) = 0 \ , \end{cases}
$$

is in $\text{Lip}_\alpha(X, B)$, by lemma 2.4. Now for every $x \in X$ with $f(x) \neq 0$ we have

$$
\begin{aligned}
f(x) &= \| f(x) \|^{1/2} g(x) = \varphi(x)g(x) \\
&= \varphi(x) \, \mathbf{e} \, g(x) = (\varphi \otimes \mathbf{e})(x)g(x) \\
&= \Big((\varphi \otimes \mathbf{e})g\Big)(x) = \Big(j(\varphi)g\Big)(x).
\end{aligned}
$$

So $f = j(\varphi)g$. Since φ has the same zeros as f, we conclude

$$
\chi(f) = \chi\big(j(\varphi)g\big) = \big(\chi o j\big)(\varphi)\chi(g) = \delta_z(\varphi)\chi(g) = \varphi(z)\chi(g) = 0.
$$

The evaluation δ_z is an epimorphism and since $ker\delta_z = I \subset ker\chi$, we obtain the desired factorization $\chi = \psi o \delta_z$ for some character ψ on B. $\square$

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Complete convergence
under special hypotheses

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ABSTRACT: We prove Baum-Katz type theorems along subsequences
of random variables under Komlós-Saks and Mazur-Orlicz type bounded-
ness hypotheses

1 Introduction and main results

Throughout the paper we shall work with real valued random variables on a complete
probability space $(\Omega, \mathcal{F}, P)$. The following Baum-Katz type result (cf. [5]) quantifies
the rate of convergence in the strong law of large numbers for general sequences of
random variables in the form of a complete convergent series:

Theorem 0. *If $(X_n)_{n \geq 1}$ is an L^p-norm bounded sequence for some $0 < p < 2$,
i.e., $\sup_{n \geq 1} \|X_n\|_p \leq C$ for some $C > 0$, then there exists a subsequence $(Y_n)_{n \geq 1}$ of
$(X_n)_{n \geq 1}$ such that, for all $0 < r \leq p$, we have*

$$\sum_{n=1}^{\infty} n^{p/r-2} P\left(\left\{ \omega \in \Omega : \left| \sum_{j=1}^{n} Y_j(\omega) \right| > \varepsilon n^{1/r} \right\} \right) < \infty \text{ for } \varepsilon > 0. \tag{1}$$

*In particular the strong law of large numbers holds along the subsequence $(Y_n)_{n \geq 1}$,
i.e., $Y_n / n^{1/p} \to 0$ a.s.*

The examples in [6], [4] and [3] show that (1) may fail if one drops the L^p-norm
boundedness hypothesis. Inspired by the celebrated Komlós-Saks and Mazur-Orlicz
extensions of the law of large numbers, in this note we shall prove two versions of

the Baum-Katz theorem under special boundedness hypotheses, more general than L^p−norm boundedness condition required in Theorem 0.

Theorem 1. *Let $0 < p < 2$ and $(X_n)_{n\geq 1}$ a sequence such that $\limsup_n |X_n(\omega)|^p < \infty$ for all $\omega \in \Omega$. Then there exists a subsequence $(Y_n)_{n\geq 1}$ of $(X_n)_{n\geq 1}$ such that (1) holds for all $0 < r \leq p$.*

Theorem 2. *Let $0 < p < 2$ and $(X_n)_{n\geq 1}$ a sequence satisfying the following condition: for every subsequence $(\tilde{X}_n)_{n\geq 1}$ of $(X_n)_{n\geq 1}$ and $n \geq 1$, there exists a convex combination Z_n of $\left\{ |\tilde{X}_n|^p, |\tilde{X}_{n+1}|^p, \ldots \right\}$, such that $\limsup_n |Z_n(\omega)| < \infty$ for all $\omega \in \Omega$. Then there exists a subsequence $(Y_n)_{n\geq 1}$ of $(X_n)_{n\geq 1}$ such that (1) holds for all $0 < r \leq p$.*

Remarks. (i) Both Theorems 1 and 2 hold for uniformly bounded sequences $(X_n)_{n\geq 1}$ in $L^p, 0 < p < 2$. On $[0,1]$ endowed with the Lebesgue measure, the sequence $X_n(\omega) = n^2$ if $0 \leq \omega \leq 1/n$ and 0 otherwise, satisfies Theorem 2 because $X_n \to 0$ Lebesgue-a.s., yet it does not satisfy Theorem 1 with $p = 1$ because it is not bounded in $L^1[0,1]$. As a matter of fact, both Theorems 1 and 2 may fail for unbounded sequences, e.g., $X_n = n$.

(ii) The idea beneath Theorems 1 and 2 is to construct a rich family of uniformly integrable subsequences of $(X_n)_{n\geq 1}$ as in [2], for which condition (1) holds; note that the hypotheses in [6] and [3] cannot produce Baum-Katz type theorems, as the families of subsequences therein are no longer uniformly integrable.

2   Proofs of the results

Proof of Theorem 1. Note that $\limsup_n |X_n(\omega)|^p < \infty$ is equivalent to

$$\sup_{n\geq 1} |X_n(\omega)|^p < \infty$$

for all $\omega \in \Omega$. For any natural number $m \geq 1$, let us define

$$A_m = \left\{ \omega \in \Omega : \sup_{n\geq 1} |X_n(\omega)|^p \leq m \right\}.$$

Assume that $r < p$ and fix $a > p/r - 1$. As $P(A_m) \to 1$ as $m \to \infty$, we can choose $m_1 \geq 1$ such that $P(A_{m_1}) > 1 - 2^{-a}$. Integrating and applying Fatou's lemma, we obtain

$$\sup_{n\geq 1} \int_{A_{m_1}} |X_n(\omega)|^p dP(\omega) \leq m_1. \tag{2}$$

We now apply the Biting Lemma (cf. [1]) to the sequence $(X_n)_{n\geq 1}$ and obtain: an increasing sequence of sets $(B_k^1)_{k\geq 1}$ in $\mathcal{F}$ with $P(B_k^1) \to 1$ as $k \to \infty$, and a subsequence $(X_n^1)_{n\geq 1}$ of $(X_n)_{n\geq 1}$ such that $(X_n^1)_{n\geq 1}$ is uniformly integrable on each

set $A_{m_1} \cap B_k^1$, $k \geq 1$. The latter fact together with estimate (2) show that Theorem 0 applies to the sequence $(X_n^1)_{n \geq 1}$ and gives

$$\sum_{n=1}^{\infty} n^{p/r-2} P\left(\left\{\omega \in A_{m_1} \cap B_k^1 : \left|\sum_{j=1}^{n} X_j^1(\omega)\right| > \varepsilon n^{1/r}\right\}\right) < \infty \text{ for } \varepsilon > 0 \text{ and } k \geq 1.$$

Another application of the Biting Lemma to $(X_n^1)_{n \geq 1}$, instead of $(X_n)_{n \geq 1}$, produces: a measurable set A_{m_2} with $P(A_{m_2}) > 1 - 3^{-a}$, an increasing sequence of sets $(B_k^2)_{k \geq 1}$ in $\mathcal{F}$ with $P(B_k^2) \to 1$ as $k \to \infty$, and a subsequence $(X_n^2)_{n \geq 1}$ of $(X_n^1)_{n \geq 1}$ such that $(X_n^2)_{n \geq 1}$ is uniformly integrable on each set $A_{m_2} \cap B_k^2$, $k \geq 1$, such that

$$\sum_{n=1}^{\infty} n^{p/r-2} P\left(\left\{\omega \in A_{m_2} \cap B_k^2 : \left|\sum_{j=1}^{n} X_j^2(\omega)\right| > \varepsilon n^{1/r}\right\}\right) < \infty \text{ for } \varepsilon > 0 \text{ and } k \geq 1.$$

By induction, we construct for each $i \geq 1$: a measurable set A_{m_i} with $P(A_{m_i}) > 1 - (i+1)^{-a}$, an increasing sequence of sets $(B_k^i)_{k \geq 1}$ in $\mathcal{F}$ with $P(B_k^i) \to 1$ as $k \to \infty$, and a subsequence $(X_n^i)_{n \geq 1}$ of $(X_n^{i-1})_{n \geq 1}$, with the convention that $(X_n^0)_{n \geq 1}$ is precisely $(X_n)_{n \geq 1}$, such that $(X_n^i)_{n \geq 1}$ is uniformly integrable on each set $A_{m_i} \cap B_k^i$, $k \geq 1$, and

$$\sum_{n=1}^{\infty} n^{p/r-2} P\left(\left\{\omega \in A_{m_i} \cap B_k^i : \left|\sum_{j=1}^{n} X_j^i(\omega)\right| > \varepsilon n^{1/r}\right\}\right) < \infty \text{ for } \varepsilon > 0 \text{ and } k, i \geq 1.$$

Now define $Y_n := X_n^n$ and, using a diagonal argument in the above formula, we obtain that

$$\sum_{n=1}^{\infty} n^{p/r-2} P\left(\left\{\omega \in A_{m_n} \cap B_k^n : \left|\sum_{j=1}^{n} Y_j(\omega)\right| > \varepsilon n^{1/r}\right\}\right) < \infty \text{ for } \varepsilon > 0 \text{ and } k \geq 1. \quad (3)$$

As $P(B_k^n) \to 1$ as $k \to \infty$ for all $n \geq 1$, formula (3) and the dominated convergence theorem imply that

$$\sum_{n=1}^{\infty} n^{p/r-2} P\left(\left\{\omega \in A_{m_n} : \left|\sum_{j=1}^{n} Y_j(\omega)\right| > \varepsilon n^{1/r}\right\}\right) < \infty \text{ for } \varepsilon > 0. \quad (4)$$

Therefore, to prove that series (1) converges for our subsequence $(Y_n)_{n \geq 1}$ and $r < p$, it suffices to prove (4) with A_{m_n} replaced by its complement, i.e.,

$$\sum_{n=1}^{\infty} n^{p/r-2} P\left(\left\{\omega \in \Omega \setminus A_{m_n} : \left|\sum_{j=1}^{n} Y_j(\omega)\right| > \varepsilon n^{1/r}\right\}\right) < \infty \text{ for } \varepsilon > 0. \quad (5)$$

Indeed, the latter series is

$$\leq \sum_{n=1}^{\infty} n^{p/r-2} P\left(\left\{\omega \in \Omega \setminus A_{m_n}\right\}\right) \leq \sum_{n=1}^{\infty} n^{p/r-2-1} < \infty \quad (6)$$

as $P(A_{m_n}) > 1 - (n+1)^{-a} > 1 - n^{-a}$ and $a > p/r - 1$. The proof is achieved in the case $r < p$.

If $r = p$, then we modify the induction process as follows: choose measurable sets A_{m_i} with $P(A_{m_i}) > i/(i+1)$ for all $i \geq 1$; as such, the diagonal argument above gives the following replacement of (4):

$$\sum_{n=1}^{\infty} \frac{1}{n}\left(\left\{\omega \in A_{m_n} : \left|\sum_{j=1}^{n} Y_j(\omega)\right| > \varepsilon n^{1/r}\right\}\right) < \infty \text{ for } \varepsilon > 0.$$

To show that series (1) converges for our subsequence $(Y_n)_{n \geq 1}$ and $r = p$, it suffices to prove the following replacement of (5):

$$\sum_{n=1}^{\infty} \frac{1}{n} P\left(\left\{\omega \in \Omega \setminus A_{m_n} : \left|\sum_{j=1}^{n} Y_j(\omega)\right| > \varepsilon n^{1/r}\right\}\right) < \infty \text{ for } \varepsilon > 0.$$

Indeed, the latter series is

$$\leq \sum_{n=1}^{\infty} \frac{1}{n} P\left(\left\{\omega \in \Omega \setminus A_{m_n}\right\}\right) \leq \sum_{n=1}^{\infty} \frac{1}{n(n+1)} < \infty$$

by the choice of $P(A_{m_n}), n \geq 1$. The latter is the substitute of (6) in the case $r = p$, and the proof is now complete.

Proof of Theorem 2. By hypothesis we can write

$$Z_n = \sum_{i \in I_n} \lambda_i^n |\tilde{X}_{n+i}|^p \text{ for some } \lambda_i^n \geq 0 \text{ with } \sum_{i \in I_n} \lambda_i^n = 1,$$

and where I_n are finite subsets of $\{0, 1, 2, \ldots\}$. In addition, the sequence $(Z_n)_{n \geq 1}$ satisfies the condition $\sup_{n \geq 1} |Z_n(\omega)|^p < \infty$ for all $\omega \in \Omega$. For any natural number $m \geq 1$, let us define $A_m = \left\{\omega \in \Omega : \sup_{n \geq 1} |Z_n(\omega)| \leq m\right\}$. As $P(A_m) \to 1$ as $m \to \infty$, we can choose $m_1 \geq 1$ such that $P(A_{m_1}) > 1 - 2^{-a}$ or $1/2$, according to $p > r$ or $p = r$, and where $a > p/r - 1$ is fixed. Integrating and applying Fatou's lemma, we obtain

$$\sup_{n \geq 1} \sum_{i \in I_n} \lambda_i^n \int_{A_{m_1}} |\tilde{X}_{n+i}(\omega)|^p dP(\omega) \leq m_1.$$

Hence there is a subsequence $(\bar{X}_n)_{n \geq 1}$ of $(\tilde{X}_n)_{n \geq 1}$ (therefore of $(X_n)_{n \geq 1}$ as well), such that

$$\sup_{n \geq 1} \int_{A_{m_1}} |\bar{X}_n(\omega)|^p dP(\omega) \leq m_1,$$

which is precisely eq. (2) along a subsequence. The remainder of the proof goes exactly as in the proof of Theorem 1.

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On parameters of independence, domination and irredundance in edge-coloured graphs and their products

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ABSTRACT: In this paper we study some parameters of domination, independence and irredundance in some edge-coloured graphs and their products. We present several general properties of independent, dominating and irredundance sets in edge-coloured graphs and we give relationships between the independence, domination and irredundant numbers of an edge-coloured graph. We generalize some classical results concerning independence, domination and irredundance in graphs. Moreover we study G-join of edge-coloured graphs which preserves considered parameters with respect to related parameters in product factors.

1   Introduction

Consider a finite connected graph G with a vertex set $V(G)$ and an edge set $E(G)$. A *path* from a vertex x_1 to a vertex x_n, $n \geq 2$, in G is a sequence of distinct vertices $x_1, ..., x_n$ such that $x_i x_{i+1} \in E(G)$, for $i = 1, ..., n-1$; we denote it simply by $x_1...x_n$. If $x_1 = x_n$ the path form a *cycle*. An *edge-m-colouring* of G is a mapping $c : E(G) \rightarrow \{1, ..., m\}$. We then say that G is *edge-m-coloured* by c. An m-coloured graph G is *monochromatic* if $c(e) = c(f)$ for any $e, f \in E(G)$. We abuse the notation slightly and call $c(G) = c(e)$, for any $e \in E(G)$. A path (cycle) is m-coloured if its edges are coloured using m-colours. A *path* is called *monochromatic* if its edges are coloured alike. The set of all vertices y for which there is a monochromatic path $y...x$ is called *the chromatic neighborhood of x* and is denoted by $N_G^{mp}(x)$. We write $N_G^{mp}[x]$

instead of $N_G^{mp}(x) \cup \{x\}$. For a subset X of $V(G)$ we write $N_G^{mp}(X)$ and $N_G^{mp}[X]$ instead of $\bigcup\limits_{x \in X} N_G^{mp}(x)$ and $\bigcup\limits_{x \in X} N_G^{mp}[X]$, respectively.

A subset $S \subset V(G)$ is said to be *independent by monochromatic paths* of the edge-coloured graph G if for any two different vertices $x, y \in S$ there is no monochromatic path between them. In addition a subset containing only one vertex and the empty set are independent by monochromatic paths. For convenience we will write an *imp-set* of G instead of an independent by monochromatic paths set of G. For any proper edge-colouring of the graph G an imp-set of G is an independent set in the classical sense. Moreover every imp-set of G is independent. The *lower* and *upper independence by monochromatic paths numbers* $i_{mp}(G)$ *and* $\alpha_{mp}(G)$ of G are respectively the minimum and maximum cardinalities of maximal imp-set of vertices of G.

A subset $Q \subseteq V(G)$ is *dominating by monochromatic paths*, shortly dmp-set of the edge-coloured graph G if for each $x \in V(G) \setminus Q$ there exists a monochromatic path $x...y$, for some $y \in Q$. We will write a *dmp-set* of G instead of dominating by monochromatic paths set of G. For proper edge-colouring of the graph G a dmp-set of G is a dominating set of G in the classical sense. Moreover every dominating set of G is a dmp-set. The *lower* and *upper by monochromatic paths numbers* $\gamma_{mp}(G)$ *and* $\Gamma_{mp}(G)$ of G are respectively the minimum and maximum cardinalities of minimal dmp-set of vertices of G.

Parameters $\gamma_{mp}(G)$ and $\alpha_{mp}(G)$ will be named as the *domination by monochromatic paths* and *independence by monochromatic paths numbers*, respectively.

Let G be an edge-coloured graph and $X \subset V(G)$. For every $x \in X$, define $I_G^{mp}(x, X) = N_G^{mp}[x] - N_G^{mp}[X - \{x\}]$ the set of *private chromatic neighbours of the vertex relative to the set* X. If $I_G^{mp}(x, X) = \emptyset$, then x is said to be *redundant by monochromatic path in* X. A set X of vertices containing no redundant by monochromatic paths vertex is called *irredundant by monochromatic paths*. The *lower and upper irreduance by monochromatic paths number* $ir_{mp}(G)$ and $IR_{mp}(G)$ of a graph G are respectively the minimum and maximum cardinalities of maximal irredundant by monochromatic paths set of vertices of G. The parameter $ir_{mp}(G)$ is the *irredundance by monochromatic paths number* of an edge-coloured graph G. In this paper we will write an *irmp-set* of G instead of an irredundant by monochromatic paths set of G. For the proper edge-colouring of the graph G an irmp-set of G is an irredundant set in the classical sense.

Note that for the proper edge-colouring of the graph G we have the following equalities: $\alpha_{mp}(G) = \alpha(G)$, $\gamma_{mp}(G) = \gamma(G)$, $\Gamma_{mp}(G) = \Gamma(G)$, $i_{mp}(G) = i(G)$, $ir_{mp}(G) = ir(G)$ and $IR_{mp}(G) = IR(G)$.

The concepts of independence, domination and irredundance have existed in literature for a long time, see [14]. There are several generalizations of these concepts, for instance generalization in distance sense see [10, 13].

Concept of independence and domination by monochromatic path in graphs were studied in [1-8] and [15-21]. More generalized concept was considered recently in [9].

In this paper we study parameters of independence, domination and irredundance by monochromatic paths in an edge-coloured graphs and their products. We give some general properties of imp-sets, dmp-sets and irmp-sets in an edge-coloured graph and

we give some relationships between studied parameters which generalize results for independent sets, dominating sets and irredundance sets in the classical sense.

2 General properties of imp-sets and dmp-sets in graphs

In this section we give some relations between imp-sets, dmp-sets and irmp-sets in graphs.

Theorem 2.1. *[20] For an arbitrary edge-coloured graph G and a subset $S \subset V(G)$ the following conditions are equivalent:*

1. *S is a maximal imp-set od G.*

2. *S is an imp-set of G and a dmp-set of G.*

3. *S is both a maximal imp-set and a minimal dmp-set of G.*

Theorem 2.2. *Let X be an irmp-set of an edge-coloured graph G. If there exists $x \in X$ such that $x \in I_G^{mp}(x, X)$ then $I_G^{mp}(x, X) \nsubseteq N_G^{mp}[v]$, for any $v \in V(G) - N_G^{mp}[X]$.*

Proof. Assume that there exists $x \in X$ such that $I_G^{mp}(x, X) \subseteq N_G^{mp}[v]$ for some $v \in V(G) - N_G^{mp}[X]$. Then $x \in I_G^{mp}(x, X) \subseteq N_G^{mp}[v]$, that is, $v \in N_G^{mp}[x]$, which contradict the choice of the vertex $v \in V(G) - N_G^{mp}[X]$. $\square$

Theorem 2.3. *Let G be an edge-coloured graph and let Q be a dmp-set in G. Then Q is a minimal dmp-set in G if and only if $I_G^{mp}(x, Q) \neq \emptyset$, for each $x \in Q$.*

Proof. If Q is a minimal dmp-set in G, then for each $x \in Q$ we have that $N_G^{mp}[x] \cup N_G^{mp}[Q - \{x\}] = N_G^{mp}[Q] = V(G)$. Since $N_G^{mp}[Q - \{x\}] \subset V(G)$, so $I_G^{mp}(x, Q) \neq \emptyset$. Assume now that Q is a dmp-set in G and $I_G^{mp}(x, Q) \neq \emptyset$, for each $x \in Q$. Suppose on contrary that Q is not minimal. This means that for some $x \in Q$, $Q - \{x\}$ is a dmp-set in G. Therefore $N_G^{mp}[Q - \{x\}] = V(G)$ and since $N_G^{mp}[x] \subseteq V(G)$, so $I_G^{mp}(x, Q) = \emptyset$, contrary to the hypothesis. $\square$

From the definition of an irmp-set and Theorem 2.3 it follows the following relationships between minimal dmp-sets and maximal irmp-sets:

Corollary 1. *Let Q be a dmp-set of an edge-coloured graph G. Then Q is a minimal dmp-set of G if and only if Q is a maximal irmp-set of G.*

In view of the facts that every maximal imp-set of a graph G is a minimal dmp-set and every minimal dmp-set is a maximal irmp-set it follows the following string of inequalities:

Proposition 2.4. *For any edge-coloured graph G,*

$$ir_{mp}(G) \leq \gamma_{mp}(G) \leq i_{mp}(G) \leq \alpha_{mp}(G) \leq \Gamma_{mp}(G) \leq IR_{mp}(G).$$

Theorem 2.5. *If X is a smallest maximal irmp-set in an edge-coloured graph G and X is an imp-set, then*

$$ir_{mp}(G) = \gamma_{mp}(G) = i_{mp}(G).$$

Proof. By Proposition 2.4 we obtain that $ir_{mp}(G) \leq \gamma_{mp}(G) \leq i_{mp}(G)$, so it suffices to prove that $ir_{mp}(G) = i_{mp}(G)$. Suppose on the contrary that $ir_{mp}(G) \neq i_{mp}(G)$. Then $|X| = ir_{mp}(G) < i_{mp}(G)$ and this implies that X is not a maximal imp-set in G. Consequently $V(G) - N_G^{mp}[X] \neq \emptyset$ and for any $x \in V(G) - N_G^{mp}[X]$ the set $X \cup \{x\}$ is an imp-set of G. Therefore by previous considerations $X \cup \{x\}$ is an irmp-set in G, contrary to the maximality of X. $\square$

Theorem 2.6. *Let $G_1, ..., G_n$ be the connected components of an edge-coloured graph G and X be a maximal irmp-set of G and $X_i = X \cap V(G_i)$. Then $X_i \neq \emptyset$ and X_i is a maximal irmp-set of G_i, for each $i = 1, ..., n$.*

Proof. If $X_i = \emptyset$ for some i, $1 \leq i \leq n$ then we can observe that $X \cup \{y\}$ is an irmp-set of G for any $y \in V(G_i)$ which contradicts the maximality of X. This implies that $X_i \neq \emptyset$ for each $i = 1, ..., n$. Because X is a maximal irmp-set of G hence $I_G^{mp}(x, X) \neq \emptyset$, for each $x \in X$ and since $X_i \subseteq X$, so X_i is also an irmp-set of G_i. Suppose that there exists $1 \leq i \leq n$ such that X_i is not a maximal irmp-set of G_i. Then there exists at least one vertex $y \in V(G_i)$ such that $X_i \cup \{y\}$ is also an irmp-set, and consequently we have that $X \cup \{y\}$ is also an irmp-set of G, a contradiction to the maximality of X. $\square$

Theorem 2.7. *Let G be an edge-coloured graph. If X is a maximal irmp-set of G then for any $u \in V(G) - N_G^{mp}[X]$ there exists some $x \in X$ such that*
(1) $I_G^{mp}(x, X) \subseteq N_G^{mp}(u)$,
(2) for $x_1, x_2 \in I_G^{mp}(x, X)$ such that $x_1 \neq x_2$ either there is a monochromatic path $x_1...x_2$ in G or there exist $y_1, y_2 \in X - \{x\}$ such that there is a monochromatic path from x_1 to each vertex of $I_G^{mp}(y_1, X)$ and there is a monochromatic path from x_2 to each vertex of $I_G^{mp}(y_2, X)$.

Proof. (1). From the assumption about maximality of X we obtain that the set $X \cup \{u\}$ is not an irmp-set in G. Consequently $I_G^{mp}(x, X \cup \{u\}) = \emptyset$ for some $x \in X \cup \{x\}$. Since $u \in V(G) - N_G^{mp}[X]$, hence there is no monochromatic path $u...y$, for every $y \in X$, so $u \in I_G^{mp}(u, X \cup \{u\})$ and therefore $x \neq u$. Because $I_G^{mp}(x, X \cup \{u\}) = N_G^{mp}[x] - N_G^{mp}[X \cup \{u\} - \{x\}] = N_G^{mp}[x] - N_G^{mp}[X - \{x\}] - N_G^{mp}[u] = \emptyset$, then $I_G^{mp}(x, X) = N_G^{mp}[x] - N_G^{mp}[X - \{x\}] \subseteq N_G^{mp}[u]$ and this gives $I_G^{mp}(x, X) \subseteq N_G^{mp}(u)$ as $u \notin I_G^{mp}(x, X)$.
(2). Let x_1, x_2 be two distinct vertices of $I_G^{mp}(x, X)$ such that there is no monochromatic path $x_1...x_2$ in G and suppose on the contrary that for x_1 or x_2, say for x_1 and for all $y_i \in X - \{x\}$ there is $z_i \in I_G^{mp}(y_i, X)$ that there are no monochromatic paths $z_i...x_1$ in G. Then $x_2 \in I_G^{mp}(x, X \cup \{x_1\})$, $u \in I_G^{mp}(x_1, X \cup \{x_1\})$, $z_i \in I_G^{mp}(y_i, X \cup \{x_1\})$ for each $y_i \in X - \{x\}$ and therefore $X \cup \{x_1\}$ is an irmp-set in G, which contradicts the maximality of X. $\square$

Theorem 2.8. *Let X be a smallest maximal irmp-set in G. Let $X_0 \subset X$ be a subset such that $X_0 \cup \{x\}$ is an imp-set in G, for each $x \in X$ and $|X_0| = k < |X|$. Then $\gamma_{mp}(G) \leq 2ir_{mp}(G) - k - 1$*

Proof. Because $|X_0| = k < |X|$ so $X - X_0 \neq \emptyset$. Let $X - X_0 = \{x_1, ..., x_n\}$. Clearly $n \geq 2$. For each $x_i \in X - X_0$ we choose any $x_i' \in I_G^{mp}(x_i, X)$ and we define the set $X' = X \cup \{x_1', ..., x_n'\}$. From the assumption of X_0 we obtain that $x_i \notin I_G^{mp}(x_i, X)$, so $x_i' \neq x_i$ for $i = 1, ..., n$ and therefore $|X'| \leq 2ir_{mp}(G) - k$. Moreover the assumption of the set X_0 implies that for every $x_p \in X - X_0$ there is $x_q \in X - X_0$ and a monochromatic path $x_p...x_q$ in G. We shall show that X' is a dmp-set in G. Assume on the contrary that X' is not a dmp-set and let $u \in V(G) - N_G^{mp}[X']$. Consequently X is not a dmp-set in G. Thus for every $y \in X$ there is no monochromatic path $u...y$ in G. Then Theorem 2.7 (1) gives that $I_G^{mp}(x, X) \subseteq N_G^{mp}(u)$ for some $x \in X$. If $x \in X_0$, then $x \in I_G^{mp}(x, X)$ and there is a monochromatic path $x...u$, which contradicts the assumption. If $x \in X - X_0$, then $x = x_i$ (for some $i \in \{1, ..., n\}$) and by previous considerations $x_i' \in I_G^{mp}(x_i, X)$. Because $I_G^{mp}(x_i, X) \subseteq N_G(u)$ so there is a monochromatic path $x_i'...u$ in G, a contradiction. Therefore X' is a dmp-set. Since $X \subset X'$, hence Corollary 1 implies that X' is not a minimal dmp-set. Consequently $\gamma_{mp}(G) < |X'| \leq 2ir_{mp}(G) - k$ and $\gamma_{mp} \leq 2ir_{mp}(G) - k - 1$. $\qquad \square$

Corollary 2. *For any edge-coloured graph G, $\gamma_{mp}(G) \leq 2ir_{mp}(G) - 1$.*

Proof. Let X be a smallest maximal irmp-set in G. If X is an imp-set then by Proposition 2.4 we have that $\gamma_{mp}(G) \leq ir_{mp}(G)$ and therefore $\gamma_{mp}(G) \leq 2ir_{mp}(G) - 1$. If X is not an imp-set and $G[X]$ has a subset X_0 on k vertices such that $X_0 \cup \{x\}$ is an imp-set in G, for each $x \in X$, then by Theorem 2.8 it follows that $\gamma_{mp}(G) \leq 2ir_{mp}(G) - k - 1 \leq 2ir_{mp}(G) - 1$. $\qquad \square$

A vertex x of an edge-coloured graph is called *a monochromatic vertex* if it belongs to exactly one maximal (with respect to set inclusion) connected monochromatic subgraph of G. A connected monochromatic subgraph of a graph G containing at least one monochromatic vertex is called *a monochromatic simplex of G*. Note that if x is a monochromatic vertex of G then $G[N_G^{mp}[x]]$ contains a subgraph being the unique monochromatic simplex of G containing x. A graph G is *monochromatic simplical* if every vertex of G is a monochromatic vertex or belong to the monochromatic simplex. Certainly, if G is a monochromatic simplical graph and $M_1, ..., M_n$ are the monochromatic simplices in G, then $V(G) = \bigcup\limits_{i=1}^{n} V(M_i)$.

The following theorem was proved in [20].

Theorem 2.9. *[20] If an edge-coloured graph G has n monochromatic simplices and every vertex of G belongs to exactly one monochromatic simplex of G, then $\gamma_{mp}(G) = i_{mp}(G) = \alpha_{mp}(G) = \Gamma_{mp}(G) = n$*

The *monochromatic covering number* $\theta_{mp}(G)$ of an edge-coloured graph G is the smallest integer n for which there exists a partition $V_1, ..., V_n$ of the vertex set $V(G)$

such that each V_i induces a connected monochromatic subgraph of G. It is easy to observe that $\alpha_{mp}(G) \leq \theta_{mp}(G)$.

To prove the next theorem we need the following lemma:

Lemma 2.10. *Let G be an edge-coloured graph without p-coloured cycles, $2 \leq p \leq 4$ and let S and T be disjoint sets of vertices of G. If $G[S]$ and $G[T]$ are connected and monochromatic then there is a vertex $s_0 \in S$ such that $N_G^{mp}(s_0) \cap T = N_G^{mp}(S) \cap T$.*

Proof. The proof is by induction on $m = |S \cap N_G^{mp}(T)|$. If $m \leq 1$ then the result is obvious. Assume that $m > 1$ and that the result is true for all $m' < m$. Let $s \in S \cap N_G^{mp}(T)$. By the induction hypothesis there is $s' \in S - \{s\}$ such that $N_G^{mp}(s') \cap T = N_G^{mp}(S - \{s\}) \cap T$. Evidently if $N_G^{mp}(s) \cap T \subseteq N_G^{mp}(s') \cap T$ or $N_G^{mp}(s') \cap T \subseteq N_G^{mp}(s) \cap T$, then s' or s, respectively is the desired vertex. To complete the proof it suffices to show that at least one of two sets $N_G^{mp}(s) \cap T$ and $N_G^{mp}(s') \cap T$ contains the other one. Suppose to the contrary that neither $N_G^{mp}(s) \cap T \subseteq N_G^{mp}(s') \cap T$ nor $N_G^{mp}(s') \cap T \subseteq N_G^{mp}(s) \cap T$. Then for every $t \in (N_G^{mp}(s) - N_G^{mp}(s')) \cap T$ and every $t' \in (N_G^{mp}(s') - N_G^{mp}(s)) \cap T$ vertices s, s', t, t' belong to a p-coloured cycle, $2 \leq p \leq 4$, a contradiction.

This completes the proof of this Lemma. $\qquad\square$

Theorem 2.11. *If G is an edge-coloured graph without p-coloured cycles, $2 \leq p \leq 4$, then the following statements are equivalent:*
(1) every vertex of G belongs to exactly one monochromatic simplex
(2) $i_{mp}(G) = \alpha_{mp}(G) = \theta_{mp}(G)$.

Proof. Let $M_1, ..., M_n$ be the monochromatic simplices of G. If every vertex of G belongs to exactly one of them, then by Theorem 2.9 we have that $i_{mp}(G) = \alpha_{mp}(G) = n$ and consequently $\theta_{mp}(G) \leq n$. From this fact and by $\alpha_{mp}(G) \leq \theta_{mp}(G)$ we have that $\alpha_{mp}(G) = \theta_{mp}(G)$. This proves the first implication. To prove the converse implication assume that $M_1, ..., M_n$ are monochromatic subgraphs covering G, where $n = \theta_{mp}(G) = \alpha_{mp}(G) = i_{mp}(G)$. Firstly we shall show that $M_1, ..., M_n$ are mutually disjoint

Suppose on contrary that $v \in M_i \cap M_j$ where $(i \neq j)$ and assume that S is any maximal imp-set of G such that $v \in S$. Because $|S \cap (V(M_i) \cup V(M_j))| = 1$ and $|S \cap V(M_k)| \leq 1$ for $k = 1, ..., n$ we have that $|M| \leq n - 1 < \alpha_{mp}(G)$, a contradiction with the maximality of S. Next we prove that $M_1, ..., M_n$ are monochromatic simplices of the graph G.

Assume on the contrary that at least one of the monochromatic subgraphs is not a monochromatic simplex of G. Without loos of generality we can assume that M_n is not a monochromatic simplex of G. Clearly $n \geq 2$ and for every vertex $x \in V(M_n)$ there is a monochromatic path $x...y$ to some vertex y of $V(G) - V(M_n)$ and $c(xy) \neq c(M_n)$. Let S be any minimal subset of $V(G) - V(M_n)$ such that $V(M_n) \subseteq N_G^{mp}(S)$, say $|S| = k$. We shall show that the set S is an imp-set in G. Suppose on the contrary that S is not an imp-set. Hence there exist $u, v \in S$ and a monochromatic path $u...v$ in G. Applying Lemma 2.10 to sets $\{u, ..., v\}$ and $V(M_n)$ we have that there is a vertex $s_0 \in \{u, ..., v\}$ such that $N_G^{mp}(\{u, ..., v\}) \cap V(M_n) = N_G^{mp}(s_0) \cap V(M_n)$. Clearly $V(M_n) \subseteq N_G^{mp}((S - \{u, v\}) \cup s_0)$. We consider the following casses:

1. $(S - \{u, v\}) \cup \{s_0\}$ is an imp-set of G.

   Then we obtain the contradiction with the minimality of S.
2. $(S - \{u, v\}) \cup \{s_0\}$ is not an imp-set of G.

   Then there is $z \in S \setminus \{u, v\}$ and a monochromatic path $z...s_0$ in G. Applying Lemma 2.10 to sets $\{z, ..., s_0\}$ and $V(M_n)$ and proving analogously as above we obtain either case 1 or case 2. Using at most $k - 1$ steps we obtain the contradiction with the maximality of S.

   Thus S is an imp-set and this gives that $|S \cap V(M_i)| \leq 1$ for $i = 1, ..., n - 1$. Consequently $k = |S| \leq n - 1$ and we can assume that $|S \cap V(M_i)| = 1$ for $i = 1, ..., k$. Let $J \subseteq V(G) - N_G^{mp}(S)$ be any (possibly empty) imp-set of G. Because $J \subseteq V(G) - N_G^{mp}[J] \subseteq \sum\limits_{j=k+1}^{n-1} V(M_j)$ so it immediately follows that $|J| \leq n - k - 1$. Moreover, since $J \cap N_G^{mp}[S] = \emptyset$, then $S \cup J$ is an imp-set of G and there is an imp-set J such that $S \cup J$ is a maximal imp-set in G and $|S \cup J| \leq n - 1 < \alpha_{mp}(G)$, which gives a final contradiction. $\qquad\square$

3 Parameters of independence domination and irredundance in edge-coloured graphs products

It is often easy to work with graphs whose structure can be characterized in terms of smaller and simpler graphs, so many of the existing results come from the study of products of graphs. The operations on graphs allow us to build several families of graphs and in a large family of considered sets can be characterized in therms of smaller and simpler graphs.

In this paper we study edge-coloured G-join $\sigma(\alpha, G)$ of graphs which preserves considered parameters with respect to related parameters in the product factors. Let G be and edge-coloured graph on $V(G) = \{x_1, ..., x_n\}$, $n \geq 2$ and $\alpha = (G_i)_{i \in \{1, ..., n\}}$ be a sequence of vertex disjoint edge-coloured graphs on $V(G_i) = \{y_1, ..., y_{p_i}\}$, $p_i \geq 1$, $i = 1, ..., n$. Then the G-join of the graph G and the sequence α is the graph $\sigma(\alpha, G)$ such that $V(\sigma(\alpha, G)) = \bigcup\limits_{i=1}^{n} (\{x_i\} \times \{V(G_i))$ and $E(\sigma(\alpha, G)) = \{(x_s, y_j^s)(x_q, y_t^q)$-coloured ψ; $(x_s = x_q$ and $y_j^s y_t^s \in E(G_s)$-coloured $\psi)$ or $(x_s x_q \in E(G)$-coloured $\psi)\}$. By G_i^c we mean a copy of G_i in $\sigma(\alpha, G)$. It may be noted that if all graphs from the sequence α have the same vertex set, then from the G-join we obtain the generalized lexicographic product of the graph G and the sequence of graphs G_i, i.e. $\sigma(\alpha, G) = G[G_1, ..., G_n]$. If all graphs from the sequence α are isomorphic to the same graph H, then we obtain the classical product of graphs, namely the composition $G[H]$ of the graph G and H.

Let $X \subseteq V(G)$ and $X = \{x_{t_1}, ..., x_{t_k}\}$, $1 \leq k \leq n$. If $G_i = \begin{cases} 2K_1 & and\ i = t_j,\ j = 1, ..., k \\ K_1 & otherwise, \end{cases}$

then $\sigma(\alpha, G)$ is the duplication G^X, see [11, 12, 13].

Independent sets and dominating sets in G-join of digraphs were studied in [11, 12, 13, 1, 2]. Recently interesting concept of H-kernels in G-join of digraphs were studied in [9]. It generalize imp-sets and dmp-sets in edge coloured graphs.

Imp-sets and dmp-sets in G-join of digraphs were studied in [9]. Using the same method we can prove similar results for maximal imp-sets and minimal dmp-sets.

Theorem 3.1. *Let G be an edge-coloured graph on n vertices, $n \geq 2$ and α be a sequence of vertex disjoint edge-coloured graphs G_i, $i = 1, ..., n$. A subset $S^* \subset V(\sigma(\alpha, G))$ is a maximal imp-set of $\sigma(\alpha, G)$ if and only if $S \subset V(G)$ is a maximal imp-set of G such that $S^* = \bigcup_{i \in \mathcal{I}} S_i$, where $\mathcal{I} = \{i; x_i \in S\}$ and $S_i \subseteq V(G_i^c)$ and S_i is an arbitrary nonempty 1-element subset of $V(G_i^c)$, for every $i \in \mathcal{I}$.*

Theorem 3.2. *Let G be an edge-coloured graph on n vertices, $n \geq 2$ and α be a sequence of vertex disjoint edge-coloured graphs G_i, $i = 1, ..., n$. A subset $Q^* \subset V(\sigma(\alpha, G))$ is a minimal dmp-set of $\sigma(\alpha, G)$ if and only if $Q \subseteq V(G)$ is a minimal dmp-set of G such that $Q^* = \bigcup_{i \in \mathcal{I}} Q_i$, where $\mathcal{I} = \{i; x_i \in Q\}$, $Q_i \subseteq V(G_i^c)$ and Q_i is an arbitrary nonempty 1-element subset of $V(G_i^c)$, for every $i \in \mathcal{I}$.*

For irmp-sets we prove an analogous theorem.

Theorem 3.3. *Let G be an edge-coloured graph on n vertices, $n \geq 2$ and α be a sequence of vertex disjoint edge-coloured graphs G_i, $i = 1, ..., n$. A subset $X^* \subset V(\sigma(\alpha, G))$ is a maximal irmp-set of $\sigma(\alpha, G)$ if and only if X is a maximal irmp-set of G such that $X^* = \bigcup_{i \in \mathcal{I}} X_i$, where $\mathcal{I} = \{i; x_i \in X\}$ and X_i is an arbitrary nonempty subset of $V(G_i^c)$, for every $i \in \mathcal{I}$.*

Proof. Let X^* be a maximal irmp-set of $\sigma(\alpha, G)$. Denote $X = \{x_i \in V(G); X^* \cap V(G_i^c) \neq \emptyset\}$. First we shall prove that X is not an irmp-set of G. This means that there is a vertex $x_i \in X$ such that $I_G^{mp}(x_i, X) = \emptyset$. Hence by the definition of $\sigma(\alpha, G)$ and the set X we have that $X^* \cap V(G_i^c) \neq \emptyset$ and for every (x_i, y_t^i), $1 \leq t \leq p_i$ holds $I_G^{mp}((x_i, y_p^i), X^*) = \emptyset$. Consequently every (x_i, y_t^i), $1 \leq t \leq p_i$, is a redundant by monochromatic paths, contradicting the irredundance by monochromatic paths of X^*. Now we will prove that X is maximal. Suppose on contrary that X is not a maximal irmp-set of G. Then there is $x_t \in (V(G) \setminus X)$ such that $X \cup \{x_t\}$ is an irmp-set of G. Hence for every (x_t, y_m), $1 \leq m \leq p_t$ the set $X^* \cup \{(x_t, y_m)\}$ would be a greater irmp-set of $\sigma(\alpha, G)$, a contradicting the maximality of X^*. Clearly $X^* = \bigcup_{i \in \mathcal{I}} X_i$, where $\mathcal{I} = \{i; x_i \in X\}$. The definition of $\sigma(\alpha, G)$ implies that for every two vertices from each copy G_i^c, $i = 1, ..., n$ there is a monochromatic path between them in $\sigma(\alpha, G)$. Let $X_i \subset V(G_i^c)$. If $|X_i| \geq 2$, then for an arbitrary subset $Y \subseteq X_i$, where $|Y| \geq 2$ and for each $(x_i, y_p^i), (x_i, y_q^i)$ holds $N_{\sigma(\alpha, G)}[(x_i, y_p^i)] = N_{\sigma(\alpha, G)}[(x_i, y_q^i)]$. Consequently one vertex from copy G_i^c can belong to irmp-set of $\sigma(\alpha, G)$. So X_i is an 1-element set containing arbitrary vertex from $V(G_i^c)$, for every $i \in \mathcal{I}$.

Let $X \subseteq V(G)$ be a maximal irmp-set of G and let X_i, where $i \in \mathcal{I}$ and $\mathcal{I} = \{i; x_i \in X\}$ be an 1-element set containing an arbitrary vertex from $V(G_i^c)$. We will prove that $X^* = \bigcup_{i \in \mathcal{I}} X_i$ is a maximal irmp-set of $\sigma(\alpha, G)$. It is obvious from the definition of $\sigma(\alpha, G)$ that X^* is an irmp-set of $\sigma(\alpha, G)$. Assume on the contrary that X^* is not a maximal irmp-set of $\sigma(\alpha, G)$. Then there is $(x_t, y_m^t) \in (V(\sigma(\alpha, G) \setminus X^*)$

such that the set $X^* \cup \{(x_t, y_m^t)\}$ is an irmp-set of $\sigma(\alpha, G)$. Consequently (x_t, y_m^t) is not a redundant by monochromatic paths in X^*. The definition of X^* implies that $x_t \notin X$ in other case we get a contradiction with the assumption of S_t, $t \in \mathcal{I}$. Moreover the definition of $\sigma(\alpha, G)$ gives that x_t is not redundant by monochromatic paths in X. So $X \cup \{x_t\}$ is an irmp-set of G, a contradiction with the maximality of X.

Thus the Theorem is proved. $\qquad\qquad\square$

From the above theorems immediately follows the following results for parameters of independence, domination ond irredundence by monochromatic paths in $\sigma(\alpha, G)$.

Theorem 3.4. *Let $G, G_1, ..., G_n$ be edge-coloured graphs. Then*

1. $\alpha_{mp}(\sigma(\alpha, G)) = \alpha_{mp}(G)$

2. $i_{mp}(\sigma(\alpha, G)) = i_{mp}(G)$

3. $\gamma_{mp}(\sigma(\alpha, G)) = \gamma_{mp}(G)$

4. $\Gamma_{mp}(\sigma(\alpha, G)) = \Gamma_{mp}(G)$

5. $ir_{mp}(\sigma(\alpha, G)) = ir_{mp}(G)$

6. $IR_{mp}(\sigma(\alpha, G)) = IR_{mp}(G)$

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