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A New General Definition of Deformable Fractional Derivative with Some Applications

Mohamed Dilmi and Mohamed Benallia

Abstract: This paper extends the existing research on fractional derivatives by introducing a generalization of the concept of the deformable derivative (see [14]). This new derivative generalizes the ordinary derivative, maintaining equivalence since the existence of one implies the existence of the other. The definition is given as follows

$$\mathcal{D}^{\rho}_{\Upsilon(\cdot),\sigma(\cdot)}\vartheta(x) = \lim_{\epsilon\to 0}\frac{e^{\epsilon\sigma(x)}\vartheta\big(x+\epsilon\Upsilon\,(x)^{1-\rho}\big)-\vartheta(x)}{\epsilon},$$

where $\sigma(\cdot)$ and $\Upsilon(\cdot)$ are two functions that satisfy some conditions and $0<\rho<1$.

We develop the foundational properties of the generalized deformable derivative and establish versions of Rolle's and the Mean Value theorems. We also introduce the concept of the generalized deformable integral through the fundamental theorem of calculus, exploring its properties such as the inverse, linearity, and the integration by parts technique. As practical examples, we address and solve several fractional differential equations.

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1. Introduction

The concept of Fractional derivatives and fractional calculus has a long history, dating back to a conversation on September 30, 1695, between L'Hospital and Leibniz regarding the interpretation of the symbol: $\dfrac{d^{\rho}y}{dx^{\rho}}$ with $\rho = \dfrac{1}{2}$. Since then, numerous mathematicians, including Riemann, Liouville, Caputo, Grunwald and Letnikov, have made significant contributions to its development (see [3], [12]). Recent studies have demonstrated that fractional calculus plays a crucial role in modeling various real-world scenarios in engineering and the sciences. It has found broad application across mathematics, mechanics, chemical engineering and other scientific and technical fields (see [9], [8]). Fractional derivatives have been defined in multiple ways in the literature, including the Caputo, Hadamard, Riesz, Weyl, Caputo-Fabrizio and Hilfer-Katugampola formulations [13].

Since the 1960s, certain differential operators have appeared which are called local fractional derivatives. Recently, several new local limit-based definitions of the so-called conformal derivative have been formulated. For instance, Khalil et al. in [7] introduced a fractional derivative based on the limit approach, referring to it as the conformable fractional derivative

$$T_{\rho}\vartheta(x) = \lim_{\epsilon \to 0} \frac{\vartheta\left(x + \epsilon x^{1-\rho}\right) - \vartheta(x)}{\epsilon},$$

where ϑ a real function, $\rho \in]0,1[$.

However, their definition does not accommodate zero or negative numbers. Katugampola introduced a new derivative in [6], defined by

$$D^{\rho}\vartheta(x) = \lim_{\epsilon \to 0} \frac{\vartheta\left(xe^{\epsilon x^{-\rho}}\right) - \vartheta(x)}{\epsilon}.$$

Atangana and Goufo [2], invented the beta operator, which has been used in problems involving the asymptotic method. The Beta operator is given by the following formula

$${}_{0}^{A}D_{x}^{\beta}\left(\vartheta(x)\right) = \lim_{\epsilon \to 0} \frac{\vartheta\Big(x + \epsilon\big(x + 1/\Gamma(\beta)\big)^{1-\beta}\Big) - \vartheta(x)}{\epsilon}, \quad \beta \in]0,1[\,.$$

In 2016, Almeida et al. [1], generalized the beta operator and the conformable fractional derivative, by introducing a new type of fractional derivative with a kernel in the following way

$$\vartheta^{(\rho)}(x) = \lim_{\epsilon \to 0} \frac{\vartheta\left(x + \epsilon\Upsilon(x)^{1-\rho}\right) - \vartheta(x)}{\epsilon},$$

where the kernel $\Upsilon : [a,b] \to \mathbb{R}$ is a continuous, nonnegative map. In 2018, Nápoles Valdés et al. [11], proposed a definition for a nonconformable fractional derivative, denoted as $N_F^{\rho}\vartheta(x)$ and defined as follows

$$N_F^{\rho}\vartheta(x) = \lim_{\epsilon \to 0} \frac{\vartheta\left(x + \epsilon F(x,\rho)\right) - \vartheta(x)}{\epsilon},$$

where $F(\cdot,\cdot)$ is an absolutely continuous function that depends on $x>0$ and $\rho\in\,]0,1]$.

In their paper, Zulfeqarr et al. [14], introduced a new concept called the deformable derivative, using a limit approach similar to that of the standard derivative. They termed it 'deformable' due to its intrinsic property of continuously deforming a function into its derivative. The definition is given as follows

$$\mathcal{D}^{\rho}\vartheta\left(x\right)=\lim_{\epsilon\to 0}\frac{\left(1+\epsilon\beta\right)\vartheta\left(x+\epsilon\rho\right)-\vartheta\left(x\right)}{\epsilon},\quad \rho\in[0,1]\ \text{ and } \rho+\beta=1.$$

This paper aims to present a new generalized definition of a non-conformable fractional derivative, extending the conventional notion of a derivative at a specific point x. Furthermore, it seeks to generalize several results from prior research [1], [6], [7] and [14].

The paper is organized as follows: Section 2 introduces a new general definition of the local fractional derivative, which depends on an unspecified kernel. We then derive the fundamental properties of this fractional derivative, including the Product Rule, Quotient Rule and Chain Rule. In Sections 3, we introduce the generalized fractional integral with some of its properties. In Section 4, we prove some important theorems about deformable derivatives, including Rolle's theorem and the Mean Value Theorem. In Section 5, we give some applications to fractional differential equations.

2. The general fractional derivative definition

In this section, we introduce the main definition of the paper and elucidate its relationship with traditional differentiation. This relationship allows for the direct derivation of many fundamental properties of the fractional derivative.

Definition 2.1. Let $\Upsilon:[a,b]\longrightarrow\mathbb{R}$, and $\sigma:[a,b]\longrightarrow\mathbb{R}$ be two continuous maps such that nonnegative $\Upsilon(x)\neq 0$; whenever $x>a$: Given a function $\vartheta:[a,b]\longrightarrow\mathbb{R}$ and a real number $\rho\in\,]0,1[$, we define the generalized deformable derivative of ϑ of order ρ as follows

$$\mathcal{D}^{\rho}_{\Upsilon(\cdot),\sigma(\cdot)}\vartheta(x)=\lim_{\epsilon\to 0}\frac{e^{\epsilon\sigma(x)}\vartheta\left(x+\epsilon\Upsilon\left(x\right)^{1-\rho}\right)-\vartheta(x)}{\epsilon},\tag{2.1}$$

for $x\in\,]a,b[$ and $\rho\in\,]0,1[$. If ϑ is ρ-differentiable at $x=a$, and $\lim_{x\to a^+}\mathcal{D}^{\rho}_{\Upsilon(\cdot),\sigma(\cdot)}\vartheta(x)$ exists, then

$$\mathcal{D}^{\rho}_{\Upsilon(\cdot),\sigma(\cdot)}\vartheta(a)=\lim_{x\to a^+}\mathcal{D}^{\rho}_{\Upsilon(\cdot),\sigma(\cdot)}\vartheta(x).$$

Remark 2.2.

- For $\sigma(\cdot):=0$ and $\Upsilon(\cdot):=1$, we get the classical derivation $\mathcal{D}^{\rho}_{1,0}\vartheta(x)=\vartheta'(x)$.
- When $\sigma(\cdot):=0$, this definition reduces to the standard derivative of a function, as given in [1].

- When $\sigma(\cdot) := 0$ and $\Upsilon(x) := x$, this definition reduces to the standard derivative of a function, as given in [7].
- When $\sigma(\cdot) := \beta$ and $\Upsilon(\cdot) := \lambda^{1/1-\rho}$ where β and λ are constants such that $\beta + \lambda = 1$, this definition reduces to the standard derivative of a function, as given in [14].

Theorem 2.3. *If a function ϑ is differentiable at a point $x \in]a,b[$, then it is also ρ-differentiable at that point for any $\rho \in]0,1[$. Furthermore, we have*

$$\mathcal{D}^{\rho}_{\Upsilon(\cdot),\sigma(\cdot)}\vartheta(x) = \sigma(x)\vartheta(x) + \Upsilon(x)^{1-\rho}\vartheta'(x). \tag{2.2}$$

Proof. By definition, we have

$$\mathcal{D}^{\rho}_{\Upsilon(\cdot),\sigma(\cdot)}\vartheta(x) = \lim_{\epsilon\to 0}\frac{e^{\epsilon\sigma(x)}\vartheta\left(x+\epsilon\Upsilon(x)^{1-\rho}\right)-\vartheta(x)}{\epsilon}.$$

On the other hand, we note that

$$e^{\epsilon\sigma(x)} \simeq 1 + \epsilon\sigma(x) + O\left(\epsilon^2\right).$$

So, we conclude that

$$\begin{aligned}\mathcal{D}^{\rho}_{\Upsilon(\cdot),\sigma(\cdot)}\vartheta(x) &= \lim_{\epsilon\to 0}\frac{\left(1+\epsilon\sigma(x)+O\left(\epsilon^2\right)\right)\vartheta\left(x+\epsilon\Upsilon(x)^{1-\rho}\right)-\vartheta(x)}{\epsilon}\\ &= \Upsilon(x)^{1-\rho}\lim_{\epsilon\to 0}\frac{\vartheta\left(x+\epsilon\Upsilon(x)^{1-\rho}\right)-\vartheta(x)}{\epsilon\Upsilon(x)^{1-\rho}}\\ &+ \lim_{\epsilon\to 0}\left(\sigma(x)+O(\epsilon)\right)\vartheta\left(x+\epsilon\Upsilon(x)^{1-\rho}\right)\\ &= \Upsilon(x)^{1-\rho}\vartheta'(x) + \lim_{\epsilon\to 0}\sigma(x)\vartheta\left(x+\epsilon\Upsilon(x)^{1-\rho}\right).\end{aligned}$$

□

Lemma 2.4. *If ϑ is ρ-differentiable in $x \in]a,b[$ for some ρ, then ϑ is locally bounded there.*

Proof. Assume ϑ is ρ-differentiable at x. Then, there exists a positive number δ such that

$$\left|e^{\epsilon\sigma(x)}\vartheta(x+\epsilon\Upsilon(x)^{1-\rho})-\vartheta(x)-\epsilon\mathcal{D}^{\rho}_{\Upsilon(\cdot),\sigma(\cdot)}\vartheta(x)\right| \le |\epsilon|, \quad \text{for} \quad |\epsilon|<\delta,$$

this implies that

$$e^{\epsilon\sigma(x)}\left|\vartheta\left(x+\epsilon\Upsilon(x)^{1-\rho}\right)\right| \le |\epsilon| + \left|\vartheta(x)+\epsilon\mathcal{D}^{\rho}_{\Upsilon(\cdot),\sigma(\cdot)}\vartheta(x)\right|, \quad \text{for} \quad |\epsilon|<\delta,$$

from it we find

$$e^{\epsilon\sigma(x)}\left|\vartheta\left(x+\epsilon\Upsilon(x)^{1-\rho}\right)\right| \le |\epsilon| + |\vartheta(x)| + |\epsilon|\left|\mathcal{D}^{\rho}_{\Upsilon(\cdot),\sigma(\cdot)}\vartheta(x)\right|, \quad \text{for} \quad |\epsilon|<\delta,$$

then, we get

$$\left|\vartheta\left(x+\epsilon\Upsilon\left(x\right)^{1-\rho}\right)\right| \leq \frac{|\vartheta(x)|+|\epsilon|\left(1+\left|\mathcal{D}^{\rho}_{\Upsilon(\cdot),\sigma(\cdot)}\vartheta(x)\right|\right)}{e^{\epsilon\sigma(x)}}, \quad \text{for} \quad |\epsilon|<\delta.$$

So, there exist positive numbers M and δ, such that

$$\left|\vartheta\left(x+\epsilon\Upsilon\left(x\right)^{1-\rho}\right)\right| \leq M, \quad \text{for} \ |\epsilon|<\delta,$$

where δ is selected small enough to ensure that $x+\epsilon\Upsilon\left(x\right)^{1-\rho} \in \left]a,b\right[$. This ensures that ϑ is locally bounded at x. □

Theorem 2.5. *If a function $\vartheta : [a,b] \to \mathbb{R}$ is ρ-differentiable at $x>a$, for $\rho \in \left]0,1\right[$, then ϑ must be continuous at x.*

Proof. We know that

$$\vartheta\Big(x+\epsilon\Upsilon\left(x\right)^{1-\rho}\Big)-\vartheta(x)=\mathcal{D}^{\rho}_{\Upsilon(\cdot),\sigma(\cdot)}\vartheta(x)\cdot\epsilon+\vartheta\Big(x+\epsilon\Upsilon\left(x\right)^{1-\rho}\Big)-e^{\epsilon\sigma(x)}\vartheta\Big(x+\epsilon\Upsilon\left(x\right)^{1-\rho}\Big),$$

then, we have

$$\begin{aligned}\lim_{\epsilon\to 0}\left(\vartheta\left(x+\epsilon\Upsilon\left(x\right)^{1-\rho}\right)-\vartheta(x)\right) &= \mathcal{D}^{\rho}_{\Upsilon(\cdot),\sigma(\cdot)}\vartheta(x)\cdot 0 \\ &\quad +\lim_{\epsilon\to 0}\left(\vartheta\left(x+\epsilon\Upsilon\left(x\right)^{1-\rho}\right)-e^{\epsilon\sigma(x)}\vartheta\left(x+\epsilon\Upsilon\left(x\right)^{1-\rho}\right)\right),\end{aligned}$$

this implies that

$$\begin{aligned}\lim_{\epsilon\to 0}\left(\vartheta\left(x+\epsilon\Upsilon\left(x\right)^{1-\rho}\right)-\vartheta(x)\right) &= \lim_{\epsilon\to 0}\Big(\vartheta\left(x+\epsilon\Upsilon\left(x\right)^{1-\rho}\right) \\ &- \left(1+\epsilon\sigma\left(x\right)+O\left(\epsilon^{2}\right)\right)\vartheta\left(x+\epsilon\Upsilon\left(x\right)^{1-\rho}\right)\Big),\end{aligned}$$

so, we get

$$\lim_{\epsilon\to 0}\left(\vartheta\left(x+\epsilon\Upsilon\left(x\right)^{1-\rho}\right)-\vartheta(x)\right)=\lim_{\epsilon\to 0}\left(\epsilon\sigma\left(x\right)+O\left(\epsilon^{2}\right)\right)\vartheta\left(x+\epsilon\Upsilon\left(x\right)^{1-\rho}\right).$$

Now, setting $h:=\epsilon\Upsilon\left(x\right)^{1-\rho}$ and using Lemma 2.4, we find

$$\lim_{h\to 0}\left(\vartheta\left(x+h\right)-\vartheta\left(x\right)\right)=0.$$

This completes the proof. □

Corollary 2.6. *Any ρ-differentiable function ϑ defined on $[a,b]$ is also differentiable.*

Proof. Using the classical derivative definition, we find

$$\begin{aligned}\vartheta'(x) &= \lim_{\epsilon\to 0}\frac{\vartheta\left(x+\epsilon\Upsilon(x)^{1-\rho}\right)-\vartheta(x)}{\epsilon\Upsilon(x)^{1-\rho}} \\ &= \Upsilon(x)^{\rho-1}\lim_{\epsilon\to 0}\frac{e^{\epsilon\sigma(x)}\vartheta\left(x+\epsilon\Upsilon(x)^{1-\rho}\right)-\vartheta(x)}{\epsilon} \\ &+ \Upsilon(x)^{\rho-1}\lim_{\epsilon\to 0}\frac{\left(1-e^{\epsilon\sigma(x)}\right)\vartheta\left(x+\epsilon\Upsilon(x)^{1-\rho}\right)}{\epsilon} \\ &= \Upsilon(x)^{\rho-1}\mathcal{D}^{\rho}_{\Upsilon(\cdot),\sigma(\cdot)}\vartheta(x)-\Upsilon(x)^{\rho-1}\lim_{\epsilon\to 0}\left(\sigma(x)+O(\epsilon)\right)\vartheta\left(x+\epsilon\Upsilon(x)^{1-\rho}\right).\end{aligned}$$

By using hypothesis and Theorem 2.5, we get the result done. □

Theorem 2.7. *Consider a function ϑ defined on $[a,b]$. For any ρ in the interval $]0,1[$, ϑ being ρ-differentiable is equivalent to ϑ being differentiable.*

Proof. It is concluded from Theorem 2.3 and Corollary above. □

Next, we explore the case where ρ lies in the interval $]n,n+1[$, for some $n\in\mathbb{N}$. We have the following definition.

Definition 2.8. Let ϑ be n-times differentiable at $x\in\,]a,b[$. For any $\rho\in\,]n,n+1[$, we naturally extend the concept of the deformable derivative and define it using the following limit

$$\mathcal{D}^{\rho}_{\Upsilon(\cdot),\sigma(\cdot)}\vartheta(x) := \lim_{\epsilon\to 0}\frac{e^{\epsilon\sigma(x)}\vartheta^{(n)}\left(x+\epsilon\Upsilon(x)^{n+1-\rho}\right)-\vartheta^{(n)}(x)}{\epsilon}.$$

Remark 2.9. If $\vartheta^{(n+1)}$ exists, we have

$$\mathcal{D}^{\rho}_{\Upsilon(\cdot),\sigma(\cdot)}\vartheta(x) = \Upsilon(x)^{n+1-\rho}\,\vartheta^{(n+1)}(x)+\sigma(x)\,\vartheta^{(n)}(x).$$

We now outline several properties of the generalized deformable derivative.

Theorem 2.10. *Consider $\rho\in\,]0,1[$. If ϑ, g are both ρ-differentiable at a point $x>0$. Then,*

1. $\mathcal{D}^{\rho}_{\Upsilon(\cdot),\sigma(\cdot)}\left(\mu\vartheta+\nu g\right) = \mu\,\mathcal{D}^{\rho}_{\Upsilon(\cdot),\sigma(\cdot)}\vartheta+\nu\,\mathcal{D}^{\rho}_{\Upsilon(\cdot),\sigma(\cdot)}g$, *for all $\mu,\nu\in\mathbb{R}$.*
2. $\mathcal{D}^{\rho}_{\Upsilon(\cdot),\sigma(\cdot)}(C) = \sigma(x)\,C$, *for all constant functions $\vartheta(\cdot)=C$.*
3. $\mathcal{D}^{\rho}_{\Upsilon(\cdot),\sigma(\cdot)}(\vartheta g) = g\,\mathcal{D}^{\rho}_{\Upsilon(\cdot),\sigma(\cdot)}\vartheta+\vartheta\,\mathcal{D}^{\rho}_{\Upsilon(\cdot),\sigma(\cdot)}g-\sigma(\cdot)\,\vartheta g.$
4. $\mathcal{D}^{\rho}_{\Upsilon(\cdot),\sigma(\cdot)}(\vartheta/g) = \dfrac{g\,\mathcal{D}^{\rho}_{\Upsilon(\cdot),\sigma(\cdot)}(\vartheta)-\vartheta\,\mathcal{D}^{\rho}_{\Upsilon(\cdot),\sigma(\cdot)}(g)}{g^2}+\sigma(\cdot)\vartheta/g$, *for all g is a non-zero function.*

5. *Non-commutativity*

$$\mathcal{D}^{\rho_1}_{\Upsilon(\cdot),\sigma_1(\cdot)}\big(\mathcal{D}^{\rho_2}_{\Upsilon(\cdot),\sigma_2(\cdot)}\vartheta\big) = \mathcal{D}^{\rho_2}_{\Upsilon(\cdot),\sigma_2(\cdot)}\big(\mathcal{D}^{\rho_1}_{\Upsilon(\cdot),\sigma_2(\cdot)}\vartheta\big) + \big(\Upsilon(\cdot)^{1-\rho_1}\left(\Upsilon(\cdot)^{1-\rho_2}\right)' - \Upsilon(\cdot)^{1-\rho_2}\left(\Upsilon(\cdot)^{1-\rho_1}\right)'\big)\vartheta' + \left(\sigma_1'(\cdot)\,\Upsilon(\cdot)^{1-\rho_2} - \sigma_2'(\cdot)\,\Upsilon(\cdot)^{1-\rho_1}\right)\vartheta.$$

6. $\mathcal{D}^{\rho}_{\Upsilon(\cdot),\sigma(\cdot)}(\vartheta \circ g) = \Upsilon(\cdot)^{1-\rho}\, g'(\vartheta' \circ g) + \sigma(x)\,(\vartheta \circ g).$

Proof.
(1) From definition, we have

$$\mathcal{D}^{\rho}_{\Upsilon(\cdot),\sigma(\cdot)}(\mu\vartheta + \nu g) = \sigma(\cdot)(\mu\vartheta + \nu g) + \Upsilon(\cdot)^{1-\rho}(\mu\vartheta + \nu g)' = \mu\, \mathcal{D}^{\rho}_{\Upsilon(\cdot),\sigma(\cdot)}\vartheta + \nu\, \mathcal{D}^{\rho}_{\Upsilon(\cdot),\sigma(\cdot)} g\ .$$

(2) Is evident from equality (2.2).
(3) Product rule

$$\mathcal{D}^{\rho}_{\Upsilon(\cdot),\sigma(\cdot)}(\vartheta g) = \sigma(\cdot)(\vartheta g) + \Upsilon(\cdot)^{1-\rho}(\vartheta g' + \vartheta' g) = g\, \mathcal{D}^{\rho}_{\Upsilon(\cdot),\sigma(\cdot)}\vartheta + \vartheta\, \mathcal{D}^{\rho}_{\Upsilon(\cdot),\sigma(\cdot)} g - \sigma(\cdot)\,\vartheta g.$$

(4) Quotient rule

$$\begin{aligned}\mathcal{D}^{\rho}_{\Upsilon(\cdot),\sigma(\cdot)}(\vartheta/g) &= \sigma(\cdot)\vartheta/g + \Upsilon(\cdot)^{1-\rho}\left(\frac{\vartheta' g - \vartheta g'}{g^2}\right)\\ &= \frac{\left(\sigma(\cdot)\vartheta + \Upsilon(\cdot)^{1-\rho}\vartheta'\right) g - \left(\sigma(\cdot) g + \Upsilon(\cdot)^{1-\rho} g'\right)\vartheta + \sigma(\cdot)\vartheta g}{g^2}\\ &= \frac{g\, \mathcal{D}^{\rho}_{\Upsilon(\cdot),\sigma(\cdot)}\vartheta - \vartheta\, \mathcal{D}^{\rho}_{\Upsilon(\cdot),\sigma(\cdot)} g}{g^2} + \sigma(\cdot)\,\vartheta/g.\end{aligned}$$

(5) We prove the non-commutativity as follows

$$\begin{aligned}\mathcal{D}^{\rho_1}_{\Upsilon(\cdot),\sigma_1(\cdot)}\big(\mathcal{D}^{\rho_2}_{\Upsilon(\cdot),\sigma_2(\cdot)}\vartheta\big) &= \mathcal{D}^{\rho_1}_{\Upsilon(\cdot),\sigma_1(\cdot)}\big(\Upsilon(\cdot)^{1-\rho_2}\vartheta' + \sigma_2(\cdot)\vartheta\big)\\ &= \Upsilon(\cdot)^{1-\rho_1}\left(\Upsilon(\cdot)^{1-\rho_2}\vartheta' + \sigma_2(\cdot)\vartheta\right)' + \sigma_1(\cdot)\left(\Upsilon(\cdot)^{1-\rho_2}\vartheta' + \sigma_2(\cdot)\vartheta\right)\\ &= \Upsilon(\cdot)^{1-\rho_1}\left(\left(\Upsilon(\cdot)^{1-\rho_2}\right)'\vartheta' + \Upsilon(\cdot)^{1-\rho_2}\vartheta'' + \sigma_2'(\cdot)\vartheta + \sigma_2(\cdot)\vartheta'\right)\\ &\quad + \sigma_1(\cdot)\Upsilon(\cdot)^{1-\rho_2}\vartheta' + \sigma_1(\cdot)\sigma_2(\cdot)\vartheta\\ &= \mathcal{D}^{\rho_2}_{\Upsilon(\cdot),\sigma_2(\cdot)}\big(\mathcal{D}^{\rho_1}_{\Upsilon(\cdot),\sigma_2(\cdot)}\vartheta\big) + \left(\sigma_1'(\cdot)\Upsilon(\cdot)^{1-\rho_2} - \sigma_2'(\cdot)\Upsilon(\cdot)^{1-\rho_1}\right)\vartheta\\ &\quad + \left(\Upsilon(\cdot)^{1-\rho_1}\left(\Upsilon(\cdot)^{1-\rho_2}\right)' - \Upsilon(\cdot)^{1-\rho_2}\left(\Upsilon(\cdot)^{1-\rho_1}\right)'\right)\vartheta'.\end{aligned}$$

(6) Chain rule

$$\begin{aligned}\mathcal{D}^{\rho}_{\Upsilon(\cdot),\sigma(\cdot)}(\vartheta\circ g) &= \sigma(\cdot)\vartheta(g)+\Upsilon(\cdot)^{1-\rho}g'\vartheta'(g)\\ &= \Upsilon(\cdot)^{1-\rho}g'(\vartheta'\circ g)+\sigma(\cdot)(\vartheta\circ g).\end{aligned}$$

□

The generalized deformable fractional derivative of certain functions.

Proposition 2.11.

1. $\mathcal{D}^{\rho}_{\Upsilon(\cdot),\sigma(\cdot)}(x^n) = nx^{n-1}\Upsilon(x)^{1-\rho}+\sigma(x)x^n$, *for all* $n\in\mathbb{R}$.
2. $\mathcal{D}^{\rho}_{\Upsilon(\cdot),\sigma(\cdot)}(e^x) = \left(\Upsilon(x)^{1-\rho}+\sigma(x)\right)e^x$.
3. $\mathcal{D}^{\rho}_{\Upsilon(\cdot),\sigma(\cdot)}(\cos(x)) = -\Upsilon(x)^{1-\rho}\sin(x)+\sigma(x)\cos(x)$.
4. $\mathcal{D}^{\rho}_{\Upsilon(\cdot),\sigma(\cdot)}(\sin(x)) = \Upsilon(x)^{1-\rho}\cos(x)+\sigma(x)\sin(x)$.
5. $\mathcal{D}^{\rho}_{\Upsilon(\cdot),\sigma(\cdot)}(\log(x)) = \Upsilon(x)^{1-\rho}x^{-1}+\sigma(x)\log(x)$.

3. Deformable fractional integral

In fractional calculus, the fractional integral, which serves as the inverse of the fractional derivative, is just as crucial as the fractional derivative itself. Now, we introduce the generalized fractional integral as the inverse operator for the deformable derivative. Throughout this section, we assume all functions to be continuous.

Definition 3.1 (Deformable Fractional Integral)**.** Let $x\in[a,b]$ and ϑ be a function defined on $]a,x]$. Then, the ρ-fractional integral of ϑ is defined by

$$ {}^{x}_{a}\mathcal{I}^{\rho}_{\Upsilon(\cdot),\sigma(\cdot)}\vartheta(x) := \int_a^x \frac{e^{\int_t^x \frac{-\sigma(s)}{\Upsilon(s)^{1-\rho}}ds}}{\Upsilon(x)^{1-\rho}}\vartheta(x)\,dx, \tag{3.1}$$

if the Riemann improper integral exists.

Remark 3.2. If $\sigma(\cdot):=0$ and $\Upsilon(x):=x-t$, we get

$$ {}^{t}_{a}\mathcal{I}^{\rho}_{\Upsilon(\cdot),\sigma(\cdot)}\vartheta(x) = \Gamma(\rho)\ {}^{RL}I^{\rho}_{a}\vartheta(x)\ ,$$

with ${}^{RL}I^{\rho}_{a}\vartheta(\cdot)$ is Riemann-Liouville fractional integral.

It is noteworthy that the ρ-fractional derivative and the ρ-fractional integral are inverses of each other, as illustrated by the following result.

Theorem 3.3 (Inverse property)**.** *Let $\rho \in]0,1[$ and let ϑ be a continuous function for which ${}_a^x\mathcal{I}^{\rho}_{\Upsilon(\cdot),\sigma(\cdot)}\vartheta(x)$ exists. Then*

$$\mathcal{D}^{\rho}_{\Upsilon(\cdot),\sigma(\cdot)}\left({}_a^x\mathcal{I}^{\rho}_{\Upsilon(\cdot),\sigma(\cdot)}\vartheta(x)\right) = \vartheta(x), \qquad \text{for } x \geq a,$$

and

$${}_a^x\mathcal{I}^{\rho}_{\Upsilon(\cdot),\sigma(\cdot)}(\mathcal{D}^{\rho}_{\Upsilon(\cdot),\sigma(\cdot)}\vartheta(x)) = \vartheta(x) - e^{\int_a^x \frac{-\sigma(s)}{\Upsilon(s)^{1-\rho}}ds}\vartheta(a), \qquad \text{for } x \geq a. \tag{3.2}$$

Proof. Let $P(x)$ be a continuous function over $[a,b]$. Since ϑ is given to be continuous so ${}_a^t\mathcal{I}^{\rho}_{\Upsilon(\cdot),\sigma(\cdot)}\vartheta(x)$ is ρ-differentiable.

If we set $P(x) := {}_a^x\mathcal{I}^{\rho}_{\Upsilon(\cdot),\sigma(\cdot)}\vartheta(x)$, then we have

$$\begin{aligned}\mathcal{D}^{\rho}_{\Upsilon(\cdot),\sigma(\cdot)}\left({}_a^x\mathcal{I}^{\rho}_{\Upsilon(\cdot),\sigma(\cdot)}\vartheta(x)\right) &= \mathcal{D}^{\rho}_{\Upsilon(\cdot),\sigma(\cdot)}P(x)\\ &= \Upsilon(x)^{1-\rho}P'(x) + \sigma(x)P(x).\end{aligned}$$

We know that a particular solution of the differential equation

$$\Upsilon(x)^{1-\rho}P'(x) + \sigma(x)P(x) = \vartheta(x),$$

is given as

$$P(x) = \int_a^x \frac{e^{\int_t^x \frac{-\sigma(s)}{\Upsilon(s)^{1-\rho}}ds}}{\Upsilon(t)^{1-\rho}}\vartheta(t)\,dt.$$

For the second part, we have

$$P(x) := \mathcal{D}^{\rho}_{\Upsilon(\cdot),\sigma(\cdot)}\vartheta(x) = \Upsilon(x)^{1-\rho}\vartheta'(x) + \sigma(x)\vartheta(x).$$

By integrating both sides, we notice that

$$\begin{aligned}{}_a^x\mathcal{I}^{\rho}_{\Upsilon(\cdot),\sigma(\cdot)}P(x) &= {}_a^x\mathcal{I}^{\rho}_{\Upsilon(\cdot),\sigma(\cdot)}\left(\Upsilon(x)^{1-\rho}\vartheta'(x)\right) + {}_a^x\mathcal{I}^{\rho}_{\Upsilon(\cdot),\sigma(\cdot)}\left(\sigma(x)\vartheta(x)\right)\\ &= e^{\int_0^x \frac{-\sigma(s)}{\Upsilon(s)^{1-\rho}}ds}\int_a^x e^{\int_0^t \frac{\sigma(s)}{\Upsilon(s)^{1-\rho}}ds}\vartheta'(t)\,dt\\ &+ e^{\int_0^x \frac{-\sigma(s)}{\Upsilon(s)^{1-\rho}}ds}\int_a^x \frac{\sigma(t)\,e^{\int_0^t \frac{\sigma(s)}{\Upsilon(s)^{1-\rho}}ds}}{\Upsilon(t)^{1-\rho}}\vartheta(t)\,dt.\end{aligned}$$

On the other hand, we have

$$\int_a^x e^{\int_0^t \frac{\sigma(s)}{\Upsilon(s)^{1-\rho}}ds}\vartheta'(t)\,dt = \left(e^{\int_0^t \frac{\sigma(s)}{\Upsilon(s)^{1-\rho}}ds}\vartheta(t)\right]_a^x - \int_a^x \frac{\sigma(t)}{\Upsilon(t)^{1-\rho}}e^{\int_0^t \frac{\sigma(s)}{\Upsilon(s)^{1-\rho}}ds}\vartheta(t)\,dt.$$

Then, form this we get (3.2). □

Theorem 3.4. *The deformable integral ${}_a^x\mathcal{I}^{\rho}_{\Upsilon(\cdot),\sigma(\cdot)}(\cdot)$ exhibits the following properties:*

(a) *Linearity:* ${}_a^x\mathcal{I}^{\rho}_{\Upsilon(\cdot),\sigma(\cdot)}(\mu\vartheta + \nu g) = \mu\, {}_a^x\mathcal{I}^{\rho}_{\Upsilon(\cdot),\sigma(\cdot)}\vartheta + \nu\, {}_a^x\mathcal{I}^{\rho}_{\Upsilon(\cdot),\sigma(\cdot)}g$, *for all* $\mu,\nu \in \mathbb{R}$.

(*b*) *Integration by parts*

$$ {}_a^b\mathcal{I}^{\rho}_{\Upsilon(\cdot),\sigma(\cdot)}\left(\vartheta\mathcal{D}^{\rho}_{\Upsilon(\cdot),\sigma(\cdot)}g\right)=\int_a^b e^{\int_t^b \frac{-\sigma(s)}{\Upsilon(s)^{1-\rho}}ds}\left(\vartheta g\right)'(t)\,dt-{}_a^b\mathcal{I}^{\rho}_{\Upsilon(\cdot),\sigma(\cdot)}\left(g\mathcal{D}^{\rho}_{\Upsilon(\cdot),\sigma(\cdot)}\vartheta\right). $$

Proof. Linearity readily follows from definition (3.1). For the formula of integration by parts, we have from Theorem 2.10

$$ \mathcal{D}^{\rho}_{\Upsilon(\cdot),\sigma(\cdot)}\left(\vartheta g\right)=g\mathcal{D}^{\rho}_{\Upsilon(\cdot),\sigma(\cdot)}\vartheta+\vartheta\mathcal{D}^{\rho}_{\Upsilon(\cdot),\sigma(\cdot)}g-\sigma\left(\cdot\right)\vartheta g, $$

by integrating both sides, we find

$$ \begin{aligned} {}_a^b\mathcal{I}^{\rho}_{\Upsilon(\cdot),\sigma(\cdot)}\left(\mathcal{D}^{\rho}_{\Upsilon(\cdot),\sigma(\cdot)}\vartheta g\right) &= {}_a^b\mathcal{I}^{\rho}_{\Upsilon(\cdot),\sigma(\cdot)}\left(g\mathcal{D}^{\rho}_{\Upsilon(\cdot),\sigma(\cdot)}\vartheta\right)+{}_a^b\mathcal{I}^{\rho}_{\Upsilon(\cdot),\sigma(\cdot)}\left(\vartheta\mathcal{D}^{\rho}_{\Upsilon(\cdot),\sigma(\cdot)}g\right)\\ &\quad-{}_a^b\mathcal{I}^{\rho}_{\Upsilon(\cdot),\sigma(\cdot)}\left(\sigma\left(\cdot\right)\vartheta g\right), \end{aligned} $$

using the formula (3.2), we obtain

$$ \begin{aligned} \left(\vartheta g\right)(b)-e^{\int_a^b \frac{-\sigma(s)}{\Upsilon(s)^{1-\rho}}ds}\left(\vartheta g\right)(a) &= {}_a^b\mathcal{I}^{\rho}_{\Upsilon(\cdot),\sigma(\cdot)}\left(g\mathcal{D}^{\rho}_{\Upsilon(\cdot),\sigma(\cdot)}\vartheta\right)+{}_a^b\mathcal{I}^{\rho}_{\Upsilon(\cdot),\sigma(\cdot)}\left(\vartheta\mathcal{D}^{\rho}_{\Upsilon(\cdot),\sigma(\cdot)}g\right)\\ &\quad-{}_a^b\mathcal{I}^{\rho}_{\Upsilon(\cdot),\sigma(\cdot)}\left(\sigma\left(\cdot\right)\left(\vartheta g\right)\right), \end{aligned} $$

this implies that

$$ \begin{aligned} {}_a^b\mathcal{I}^{\rho}_{\Upsilon(\cdot),\sigma(\cdot)}\left(\vartheta\mathcal{D}^{\rho}_{\Upsilon(\cdot),\sigma(\cdot)}g\right) &= \left(\vartheta g\right)(b)-e^{\int_a^b \frac{-\sigma(s)}{\Upsilon(s)^{1-\rho}}ds}\left(\vartheta g\right)(a)-{}_a^b\mathcal{I}^{\rho}_{\Upsilon(\cdot),\sigma(\cdot)}\left(g\mathcal{D}^{\rho}_{\Upsilon(\cdot),\sigma(\cdot)}\vartheta\right)\\ &\quad+{}_a^b\mathcal{I}^{\rho}_{\Upsilon(\cdot),\sigma(\cdot)}\left(\sigma\left(\cdot\right)\vartheta g\right). \end{aligned} \tag{3.3} $$

On the other hand, we have

$$ \begin{aligned} {}_a^b\mathcal{I}^{\rho}_{\Upsilon(\cdot),\sigma(\cdot)}\left(\sigma\left(\cdot\right)\vartheta g\right) &= \int_a^b \frac{\sigma\left(x\right)e^{\int_t^b \frac{-\sigma(s)}{\Upsilon(s)^{1-\rho}}ds}}{\Upsilon\left(t\right)^{1-\rho}}\left(\vartheta g\right)(t)\,dt\\ &= \left[-e^{\int_t^b \frac{-\sigma(s)}{\Upsilon(s)^{1-\rho}}ds}\left(\vartheta g\right)(t)\right]_a^b+\int_a^b e^{\int_t^b \frac{-\sigma(s)}{\Upsilon(s)^{1-\rho}}ds}\left(\vartheta g\right)'(t)\,dt, \end{aligned} $$

then

$$ {}_a^b\mathcal{I}^{\rho}_{\Upsilon(\cdot),\sigma(\cdot)}\left(\sigma\left(\cdot\right)\vartheta g\right)=-\left(\vartheta g\right)(b)+e^{\int_a^b \frac{-\sigma(s)}{\Upsilon(s)^{1-\rho}}ds}\left(\vartheta g\right)(a)+\int_a^b e^{\int_t^b \frac{-\sigma(s)}{\Upsilon(s)^{1-\rho}}ds}\left(\vartheta g\right)'(t)\,dt. \tag{3.4} $$

Now, by substituting formula (3.4) into formula (3.3), we find

$$ {}_a^b\mathcal{I}^{\rho}_{\Upsilon(\cdot),\sigma(\cdot)}\left(\vartheta\mathcal{D}^{\rho}_{\Upsilon(\cdot),\sigma(\cdot)}g\right)=\int_a^b e^{\int_t^b \frac{-\sigma(s)}{\Upsilon(s)^{1-\rho}}ds}\left(\vartheta g\right)'(t)\,dt-{}_a^b\mathcal{I}^{\rho}_{\Upsilon(\cdot),\sigma(\cdot)}\left(g\mathcal{D}^{\rho}_{\Upsilon(\cdot),\sigma(\cdot)}\vartheta\right). $$

□

Now we present the integration of some functions.

Proposition 3.5. *Let $\Upsilon(x) := x$ and $\sigma(x) := -x^{1-\rho}$, we have*

1. ${}_{0}^{x}\mathcal{I}^{\rho}_{x,-x^{1-\rho}}\lambda = \lambda e^{x}\gamma(\rho, x)$, *where $\gamma(\cdot,\cdot)$ is incomplete gamma function.*
2. ${}_{0}^{x}\mathcal{I}^{\rho}_{x,-x^{1-\rho}}e^{x} = \dfrac{e^{x}}{\rho}x^{\rho}$.
3. ${}_{0}^{x}\mathcal{I}^{\rho}_{x,-x^{1-\rho}}(x^{n}) = e^{x}\gamma(\rho+n, x)$, $\forall n \in \mathbb{N}^{*}$.
4. ${}_{0}^{x}\mathcal{I}^{\rho}_{x,-x^{1-\rho}}e^{x}\log(x) = \dfrac{e^{x}}{\rho}\big(x^{\rho}\log(x) - x^{\rho}/\rho\big)$.

4. Significant theorems regarding deformable derivatives

In this section, we prove Rolle's theorem and the Mean Value theorem for the generalized deformable fractional derivative.

Theorem 4.1 (Rolle's theorem for Deformable Fractional Differentiable Functions)**.** *Let $\vartheta : [a, b] \to \mathbb{R}$ be a function with the properties that*

- *ϑ is continuous on $[a, b]$.*
- *ϑ is ρ-differentiable on $]a, b[$ for some $\rho \in]0, 1[$.*
- *$\vartheta(a) = \vartheta(b)$.*

Then, there exists $c \in]a, b[$, such that $\mathcal{D}^{\rho}_{\Upsilon(\cdot),\sigma(\cdot)}\vartheta(c) = \sigma(c)\,\vartheta(c)$.

Proof. According the Theorem 2.7, we have ϑ is differentiable. On the other hand, as ϑ is continuous on $[a, b]$ and satisfies $\vartheta(a) = \vartheta(b)$, so there is $c \in]a, b[$, at which the function has a local extrema, this means that

$$\vartheta'(c) = 0,$$

this implies that

$$\Upsilon(c)^{1-\rho}\,\vartheta'(c) + \sigma(c)\,\vartheta(c) = \sigma(c)\,\vartheta(c).$$

Hence

$$\mathcal{D}^{\rho}_{\Upsilon(\cdot),\sigma(\cdot)}\vartheta(c) = \sigma(c)\,\vartheta(c)\,.$$

□

Theorem 4.2 (Mean Value Theorem for Deformable Fractional Differentiable Functions)**.** *Let $\vartheta : [a, b] \to \mathbb{R}$ be a function with the properties that*

- *ϑ is continuous on $[a, b]$.*
- *ϑ is ρ-differentiable on $]a, b[$ for some $\rho \in]0, 1[$.*

Then, there exists $c \in]a,b[$, such that

$$\mathcal{D}^{\rho}_{\Upsilon(\cdot),\sigma(\cdot)}\vartheta(c) = \sigma(c)\,\vartheta(c) + \frac{\vartheta(b)-\vartheta(a)}{b-a}\Upsilon(c)^{1-\rho}.$$

Proof. Let us consider the function

$$u(x) = \vartheta(x) - \vartheta(a) - \frac{\vartheta(b)-\vartheta(a)}{b-a}x.$$

Then, the function u meets the criteria of the fractional Rolle's theorem. Hence, $\exists c \in]a,b[$, such that

$$\mathcal{D}^{\rho}_{\Upsilon(\cdot),\sigma(\cdot)}u(c) = \sigma(c)\,u(c).$$

Using the fact that

$$\mathcal{D}^{\rho}_{\Upsilon(\cdot),\sigma(\cdot)}u(x) = \mathcal{D}^{\rho}_{\Upsilon(\cdot),\sigma(\cdot)}\vartheta(x) - \mathcal{D}^{\rho}_{\Upsilon(\cdot),\sigma(\cdot)}\vartheta(a) - \frac{\vartheta(b)-\vartheta(a)}{b-a}\mathcal{D}^{\rho}_{\Upsilon(\cdot),\sigma(\cdot)}x,$$

we get

$$\begin{aligned}\mathcal{D}^{\rho}_{\Upsilon(\cdot),\sigma(\cdot)}\vartheta(c) &= \mathcal{D}^{\rho}_{\Upsilon(\cdot),\sigma(\cdot)}\vartheta(a) + \frac{\vartheta(b)-\vartheta(a)}{b-a}\left(\Upsilon(c)^{1-\rho} + \sigma(c)\,c\right)\\ &\quad + \sigma(c)\left(\vartheta(c) - \vartheta(a) - \frac{\vartheta(b)-\vartheta(a)}{b-a}c\right),\end{aligned}$$

then

$$\mathcal{D}^{\rho}_{\Upsilon(\cdot),\sigma(\cdot)}\vartheta(c) = \sigma(c)\,\vartheta(c) + \frac{\vartheta(b)-\vartheta(a)}{b-a}\Upsilon(c)^{1-\rho}.$$

□

5. Applications to differential equations with deformable derivatives

Now, we solve several deformable differential equations using the generalized deformable derivative operator $\mathcal{D}^{\rho}_{\Upsilon(\cdot),\sigma(\cdot)}(\cdot)$. In the first examples, we discuss methods for solving both homogeneous and non-homogeneous linear differential equations. In the last, we address the Cauchy problem for nonlinear deformable differential equations.

5.1. Examples for linear deformable differential equations

Example 5.1. Let us examine the deformable differential equation

$$\mathcal{D}^{\rho}_{\Upsilon(\cdot),\sigma(\cdot)}y(x) + h(x)y(x) = 0,$$

where $h(x)$ is continuous. By applying the expression provided in (2.2), the equation transforms as follows

$$\Upsilon(x)^{1-\rho} y'(x) + (\sigma(x) + h(x))\, y(x) = 0,$$

this implies that

$$\frac{y'(x)}{y(x)} = \frac{\sigma(x) + h(x)}{\Upsilon(x)^{1-\rho}},$$

by integration we get

$$y(x) = ce^{\int \frac{\sigma(x)+h(x)}{\Upsilon(x)^{1-\rho}} dx},$$

where c represents an arbitrary constant.

Example 5.2. Now, we examine a non-homogeneous linear deformable equation

$$\mathcal{D}^{\rho}_{\Upsilon(\cdot),\sigma(\cdot)} y(x) - \sigma(x)\, y(x) + (1-\rho)\, \Upsilon'(x)\, \Upsilon(x)^{-\rho}\, y(x) = x,$$

this means that

$$\Upsilon(x)^{1-\rho} y'(x) + (1-\rho)\, \Upsilon'(x)\, \Upsilon(x)^{-\rho}\, y(x) = x,$$

using integration we find

$$\Upsilon(x)^{1-\rho} y(x) = \frac{1}{2}x^2 + c,$$

then

$$y(x) = \frac{1}{\Upsilon(x)^{1-\rho}} \Big(\frac{1}{2}x^2 + c\Big),$$

where c represents an arbitrary constant.

Example 5.3. Let us consider the following non-homogeneous linear deformable equation

$$\mathcal{D}^{\rho}_{\Upsilon(\cdot),\sigma(\cdot)} y(x) + (1-\rho)\, \Upsilon'(x)\, \Upsilon(x)^{-\rho}\, y(x) + y'(x) \int \sigma(x)\, dx = e^x,$$

the solution is

$$y(x) = \frac{ce^x}{\Upsilon(x)^{1-\rho} + \int \sigma(x)\, dx}.$$

Example 5.4. We consider the following deformable problem

$$\begin{cases} \mathcal{D}^{1/2}_{t^2,1}\Big(\mathcal{D}^{1/2}_{t,1/2\sqrt{t}} y(x)\Big) = x; \qquad x \geq 1, \\ \mathcal{D}^{1/2}_{1,1/2} y(1) = 0, \end{cases} \tag{5.1}$$

we have

$$\mathcal{D}^{1/2}_{x,1/2\sqrt{x}} y(x) = x^{1/2} y'(x) + \frac{1}{2\sqrt{x}} y(x).$$

So the first equation of (5.1) becomes as follows

$$x\Big(x^{1/2}y'(x)+\frac{1}{2\sqrt{x}}y(x)\Big)'+\Big(x^{1/2}y'(x)+\frac{1}{2\sqrt{x}}y(x)\Big)=x,$$

this implies that

$$\Big[x\Big(x^{1/2}y'(x)+\frac{1}{2\sqrt{x}}y(x)\Big)\Big]'=x.$$

Now, by integration we find

$$x\Big(x^{1/2}y'(x)+\frac{1}{2\sqrt{x}}y(x)\Big)=\frac{x^2}{2}+c,$$

using the fact that $\mathcal{D}^{1/2}_{1,1/2}y(1)=0$, we get $c=-1/2$. Then we have

$$\Big(x^{1/2}y'(x)+\frac{1}{2\sqrt{x}}y(x)\Big)=\frac{1}{2}\Big(x-\frac{1}{x}\Big),$$

from this, we get

$$y\left(x\right)=\frac{1}{4}x^{1/2}-\frac{1}{2}\frac{\log\left(x\right)}{\sqrt{x}}+\frac{c}{\sqrt{x}},$$

where c is arbitrary constant.

5.2. Cauchy problem for nonlinear deformable differential equations

Example 5.5. We consider the following Cauchy problem

$$\begin{cases} \mathcal{D}^{\rho}_{\Upsilon(\cdot),\sigma(\cdot)}y(x)=\vartheta\left(x,y(x)\right),\ x\in\left]0,T\right], \\ y(0)=y_0. \end{cases} \tag{5.2}$$

To study this problem, we denote by $\mathcal{H}:=C([0,T];\mathbb{R})$ the Banach space of all real-valued continuous functions defined on $[0,T]$. The norm in this space will be denoted by $\|y\|_\infty:=\sup\limits_{x\in[0,T]}|y(x)|$. We also use the following notations

$$\mathcal{B}_r=\left\{y\in\mathcal{H}:\quad \|y\|_\infty\leq r\right\},$$

and

$$\lambda_0:=\min_{x\in[0,T]}\left|\frac{\sigma\left(x\right)}{\Upsilon\left(x\right)^{1-\rho}}\right|,\ \ \lambda_1:=\max_{x\in[0,T]}\left|\frac{\sigma\left(x\right)}{\Upsilon\left(x\right)^{1-\rho}}\right|,\ \ \Upsilon_0:=\min_{x\in[0,T]}\Upsilon\left(x\right)^{1-\rho}.$$

Proposition 5.6. *The system* (5.2) *is equivalent to the following integral equation*

$$y\left(x\right)=y_0e^{\int_0^x\frac{-\sigma(s)}{\Upsilon(s)^{1-\rho}}ds}+e^{\int_0^x\frac{-\sigma(s)}{\Upsilon(s)^{1-\rho}}ds}\int_0^x\frac{e^{\int_0^s\frac{\sigma(\tau)}{\Upsilon(\tau)^{1-\rho}}d\tau}}{\Upsilon\left(s\right)^{1-\rho}}\vartheta\left(s,y(s)\right)ds. \tag{5.3}$$

Proof. By integrating both sides, we find

$$ {}_{0}^{x}\mathcal{I}_{\Upsilon(\cdot),\sigma(\cdot)}^{\rho}\left(\mathcal{D}_{\Upsilon(\cdot),\sigma(\cdot)}^{\rho}y(x)\right)={}_{0}^{t}\mathcal{I}_{\Upsilon(\cdot),\sigma(\cdot)}^{\rho}\,\vartheta\left(x,y(x)\right), $$

the formula (3.2), gives

$$ y(x)-y\left(0\right)e^{\int_{0}^{x}\frac{-\sigma(s)}{\Upsilon(s)^{1-\rho}}ds}=e^{\int_{0}^{x}\frac{-\sigma(s)}{\Upsilon(s)^{1-\rho}}ds}\int_{0}^{x}\frac{e^{\int_{0}^{s}\frac{\sigma(\tau)}{\Upsilon(\tau)^{1-\rho}}d\tau}}{\Upsilon\left(s\right)^{1-\rho}}\vartheta\left(s,y(s)\right)ds. $$

Using $y(0)=y_0$, we get

$$ y(x)=y_{0}e^{\int_{0}^{x}\frac{-\sigma(s)}{\Upsilon(s)^{1-\rho}}ds}+e^{\int_{0}^{x}\frac{-\sigma(s)}{\Upsilon(s)^{1-\rho}}ds}\int_{0}^{x}\frac{e^{\int_{0}^{s}\frac{\sigma(\tau)}{\Upsilon(\tau)^{1-\rho}}d\tau}}{\Upsilon\left(s\right)^{1-\rho}}\vartheta\left(s,y(s)\right)ds. $$

Conversely, assuming (5.3) and applying the deformable derivative operator $\mathcal{D}_{\Upsilon(\cdot),\sigma(\cdot)}^{\rho}\left(\cdot\right)$ to both sides of the equation, we get

$$ \begin{aligned} \mathcal{D}_{\Upsilon(\cdot),\sigma(\cdot)}^{\rho}y\left(x\right)&=\mathcal{D}_{\Upsilon(\cdot),\sigma(\cdot)}^{\rho}\left(y_{0}e^{\int_{0}^{x}\frac{-\sigma(s)}{\Upsilon(s)^{1-\rho}}ds}\right)\\ &\quad+\mathcal{D}_{\Upsilon(\cdot),\sigma(\cdot)}^{\rho}\left(e^{\int_{0}^{x}\frac{-\sigma(s)}{\Upsilon(s)^{1-\rho}}ds}\int_{0}^{x}\frac{e^{\int_{0}^{s}\frac{\sigma(\tau)}{\Upsilon(\tau)^{1-\rho}}d\tau}}{\Upsilon\left(s\right)^{1-\rho}}\vartheta\left(s,y(s)\right)ds\right)\\ &=y_{0}\Upsilon\left(x\right)^{1-\rho}\left(\frac{-\sigma\left(x\right)}{\Upsilon\left(x\right)^{1-\rho}}\right)e^{\int_{0}^{x}\frac{-\sigma(s)}{\Upsilon(s)^{1-\rho}}ds}+y_{0}\sigma\left(x\right)e^{\int_{0}^{x}\frac{-\sigma(s)}{\Upsilon(s)^{1-\rho}}ds}\\ &\quad+\Upsilon\left(x\right)^{1-\rho}\left(\frac{-\sigma\left(x\right)}{\Upsilon\left(x\right)^{1-\rho}}e^{\int_{0}^{x}\frac{-\sigma(s)}{\Upsilon(s)^{1-\rho}}ds}\int_{0}^{x}\frac{e^{\int_{0}^{s}\frac{\sigma(\tau)}{\Upsilon(\tau)^{1-\rho}}d\tau}}{\Upsilon\left(s\right)^{1-\rho}}\vartheta\left(s,y(s)\right)ds\right)\\ &\quad+\Upsilon\left(x\right)^{1-\rho}\left(e^{\int_{0}^{x}\frac{-\sigma(s)}{\Upsilon(s)^{1-\rho}}ds}.\frac{e^{\int_{0}^{x}\frac{\sigma(s)}{\Upsilon(s)^{1-\rho}}ds}}{\Upsilon\left(x\right)^{1-\rho}}\vartheta\left(x,y(x)\right)\right)\\ &\quad+\sigma\left(x\right)e^{\int_{0}^{x}\frac{-\sigma(s)}{\Upsilon(s)^{1-\rho}}ds}\int_{0}^{x}\frac{e^{\int_{0}^{s}\frac{\sigma(\tau)}{\Upsilon(\tau)^{1-\rho}}d\tau}}{\Upsilon\left(s\right)^{1-\rho}}\vartheta\left(s,y(s)\right)ds\\ &=\vartheta\left(x,y(x)\right). \end{aligned} $$

□

Theorem 5.7. *Under the following assumptions*

- $\vartheta:[0,T]\times\mathbb{R}\longrightarrow\mathbb{R}$ *is continuous.*
- *There exists a constant* $\mu>0$*, such that for all functions* $v,y:[0,T]\longrightarrow\mathbb{R}$*, we have*

$$ \left|\vartheta(x,v\left(x\right))-\vartheta(x,y\left(x\right))\right|\leq\mu\left|v\left(x\right)-y\left(x\right)\right|\ \textit{ for all } x\in\left[0,T\right]. $$

- $\mu e^{(\lambda_1-\lambda_0)T} < \lambda_1 \Upsilon_0$ *and* $\dfrac{\lambda_1 \Upsilon_0 |y_0| + Me^{(\lambda_1-\lambda_0)T}}{\lambda_1 \Upsilon_0 + \mu e^{(\lambda_1-\lambda_0)T}} < r$, *where* $M := \sup_{x\in[0,T]} |\vartheta(x,0)|$.

The Cauchy problem (5.2) *has a unique solution.*

Proof. Firstly, we define the mapping $\mathcal{P} : \mathcal{H} \longrightarrow \mathcal{H}$ by

$$\mathcal{P}\left(y(s)\right) = y_0 e^{\int_0^x \frac{-\sigma(s)}{\Upsilon(s)^{1-\rho}} ds} + e^{\int_0^x \frac{-\sigma(s)}{\Upsilon(s)^{1-\rho}} ds} \int_0^x \frac{e^{\int_0^s \frac{\sigma(\tau)}{\Upsilon(\tau)^{1-\rho}} d\tau}}{\Upsilon(s)^{1-\rho}} \vartheta\left(s, y(s)\right) ds,$$

then, we show that $\mathcal{P}\left(\mathcal{B}_r\right) \subset \mathcal{B}_r$. Let $y \in \mathcal{B}_r$, we have

$$\begin{aligned}
|\mathcal{P}\left(y(t)\right)| &\leq \left| y_0 e^{\int_0^x \frac{-\sigma(s)}{\Upsilon(s)^{1-\rho}} ds} \right| + e^{\int_0^x \frac{-\sigma(s)}{\Upsilon(s)^{1-\rho}} ds} \int_0^x \frac{e^{\int_0^s \frac{\sigma(\tau)}{\Upsilon(\tau)^{1-\rho}} d\tau}}{\Upsilon(s)^{1-\rho}} \left|\vartheta\left(s, y(s)\right)\right| ds \\
&\leq |y_0| e^{-\lambda_0 x} + e^{\int_0^x \frac{-\sigma(s)}{\Upsilon(s)^{1-\rho}} ds} \int_0^x \frac{e^{\int_0^s \frac{\sigma(\tau)}{\Upsilon(\tau)^{1-\rho}} d\tau}}{\Upsilon(s)^{1-\rho}} \left(\left|\vartheta\left(s, y(s)\right) - \vartheta\left(s, 0\right)\right| + \left|\vartheta\left(s, 0\right)\right|\right) ds \\
&\leq |y_0| e^{-\lambda_0 x} + \frac{e^{-\lambda_0 x}}{\Upsilon_0} \left(\mu r + M\right) \int_0^x e^{\lambda_1 s} ds \\
&\leq |y_0| e^{-\lambda_0 x} + \frac{\left(\mu r + M\right)}{\Upsilon_0} \frac{\left(e^{(\lambda_1-\lambda_0)x} - 1\right)}{\lambda_1} \\
&\leq |y_0| + \frac{\left(\mu r + M\right) e^{(\lambda_1-\lambda_0)T}}{\lambda_1 \Upsilon_0} \leq r.
\end{aligned}$$

Now, for all $v, y \in \mathcal{B}_r$, we get

$$\begin{aligned}
|\mathcal{P}\left(v(t)\right) - \mathcal{P}\left(y(t)\right)| &\leq e^{\int_0^x \frac{-\sigma(s)}{\Upsilon(s)^{1-\rho}} ds} \int_0^x \frac{e^{\int_0^s \frac{\sigma(\tau)}{\Upsilon(\tau)^{1-\rho}} d\tau}}{\Upsilon(s)^{1-\rho}} \left|\vartheta\left(s, v(s)\right) - \vartheta\left(s, y(s)\right)\right| ds \\
&\leq \frac{\mu e^{-\lambda_0 x}}{\Upsilon_0} \|v - y\|_\infty \int_0^x e^{\int_0^s \frac{\sigma(\tau)}{\Upsilon(\tau)^{1-\rho}} d\tau} ds \\
&\leq \frac{\mu e^{-\lambda_0 x}}{\Upsilon_0} \|v - y\|_\infty \int_0^x e^{\lambda_1 s} ds \\
&\leq \frac{\mu e^{(\lambda_1-\lambda_0)T}}{\lambda_1 \Upsilon_0} \|v - y\|_\infty ,
\end{aligned}$$

this means that

$$\|\mathcal{P}\left(v(\cdot)\right) - \mathcal{P}\left(y(\cdot)\right)\|_\infty \leq \frac{\mu e^{(\lambda_1-\lambda_0)T}}{\lambda_1 \Upsilon_0} \|v - y\|_\infty .$$

Since $\dfrac{\mu e^{(\lambda_1-\lambda_0)T}}{\lambda_1 \Upsilon_0} < 1$ the mapping $\mathcal{P}$ is a contraction. Thus, $\mathcal{P}$ has a unique fixed point. This concludes the proof. □

Conclusion

In this research paper, we introduced a new generalized definition of the deformable fractional derivative, which unifies and extends several well-known local fractional derivatives, including those proposed by Khalil, Katugampola, Almeida, and Zulfeqarr. The proposed operator is constructed using two kernel functions $\sigma(\cdot)$ and $\Upsilon(\cdot)$, offering a flexible mathematical framework capable of modeling a wide range of dynamic behaviors. This framework also facilitates the analysis and resolution of complex differential equations that arise in various physical, chemical, and biological contexts.

We rigorously established the fundamental properties of the proposed operator, including linearity, product and quotient rules, and the chain rule. Additionally, we demonstrated its consistency with classical differentiation and proved that the corresponding deformable integral acts as its inverse, thereby strengthening its theoretical foundations.

To illustrate the applicability of this new operator, we provided explicit solutions to several classes of fractional differential equations, both linear and nonlinear. These results are supported by existence and uniqueness theorems for the associated Cauchy problems, highlighting the effectiveness and versatility of the generalized deformable derivative in solving complex mathematical models.

Moreover, this work opens several promising directions for future research. For instance, new generalized local derivatives may be constructed using alternative kernel functions, such as those found in recent works like ([10], [4], [5]).

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Optimal Control Problem with Infinite Constrains on the State

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Abstract: This article contributes to a deeper understanding of Pontryagin's maximum principle within the realm of optimal control problems that are subject to an infinite number of constraints on the state variable. We delve into the application of the Dubovitskii-Milyutin theory, which employs conical approximations around pivotal elements such as the objective function, the system of differential equations, and the constraints on both control and state variables. This theoretical framework provides a robust methodological approach to tackle the complexities introduced by infinite constraints. Furthermore, the inclusion of a detailed illustrative example not only elucidates the theoretical constructs but also underscores the practical applicability and relevance of this results. This example serves to bridge the gap between abstract theory and practical implementation, demonstrating how our findings can be employed to solve real-world problems characterized by an infinite constraint structure.
In honor to Dr. Zoltan Varga

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1. Setting the Problem and the Introduction

With this background, we now present the problem studied in this work:

Problem 1.1.

$$\int_0^T \Theta(z(t),\, v(t),\, t)dt \longrightarrow \text{loc min}\,. \tag{1.1}$$

$$(z,\, v) \in E := C^n[0,\, T] \times L^r_\infty[0,\, T], \tag{1.2}$$

$$\dot{z}(t) = \Psi(z(t),\, v(t),\, t), \qquad z(0) = z_0 \tag{1.3}$$

$$z(T) = z_1;\;\, z_1,\, z_0 \in \mathbb{R}^n, \tag{1.4}$$

$$v(t) \in V, \quad t \in [0,\, T], \quad a.e., \tag{1.5}$$

$$g(z(t),\, t,\, \alpha) \leq 0 \qquad (\alpha \in \mathfrak{A},\;\, t \in [0,\, T]), \tag{1.6}$$

where $n,\, r \in \mathbb{N}$ and $T \in \mathbb{R}_+$ are fixed, and the functions Ψ, Θ, g are defined as follow

$$\begin{array}{rcl} \Psi & : & \mathbb{R}^n \times \mathbb{R}^r \times [0,\, T] \longrightarrow \mathbb{R}^n, \\ \Theta & : & \mathbb{R}^n \times \mathbb{R}^r \times [0,\, T] \longrightarrow \mathbb{R}, \\ g & : & \mathbb{R}^n \times [0,\, T] \times \mathfrak{A} \longrightarrow \mathbb{R}. \end{array}$$

Now, we define the following linear spaces $\mathrm{C}^n[0,T] = \mathrm{C}([0,T];\mathbb{R}^n)$ and L^r_∞ by:

$$\mathrm{C}^n[0,T] \;\; = \;\; \{z : [0,T] \to \mathbb{R}^n : z \;\text{ is a continuous function}\},$$

equipped with the norm defined as follows

$$\|z\| = \sup_{t\in[0,T]} \|z(t)\|_{\mathbb{R}^n},$$

and we consider the classical Banach space $L^r_\infty = L_\infty([0,T];\mathbb{R}^r)$ of essentially bounded measurable functions (measurable functions which are bounded except on a set of measure zero or bounded almost everywhere (a.e)) endowed with essential supremum norm defined as follows:

$$\|l\|_{L^\infty} = \inf\{C \,:\, \|l(t)\|_{\mathbb{R}^r} \leq C, \text{ a.e in } [0,T]\},$$

where we will not distinguish between two functions that are equal almost everywhere.

Hypotheses

a) Θ is a continuous functions whose partial derivatives Θ_z, Θ_v are smooth enough functions on compact subsets of $\mathbb{R}^n \times \mathbb{R}^r \times\, [0,\, T]$.

b) $V \subset \mathbb{R}^r$ is convex and closed set with $\mathrm{int}(V) \neq \emptyset$.

c) $\mathfrak{A}$ is a compact topological space.

d) g is a continuous and convex in z variable with $g(z_0,\, 0,\, \alpha) < 0,\;\; g(z_1,\, T,\, \alpha) < 0$ $(\alpha \in \mathfrak{A})$ where $z_0,\, z_1 \in \mathbb{R}^n$ are fixed. Besides g has continuous derivative with respect to its first variable g_z, such that $g_z(z,\, t,\, \alpha) \neq 0$ when $g(z,\, t,\, \alpha) = 0$.

e) For all finite set $\mathfrak{A}^* \subset \mathfrak{A}$ there exists $h_0 \in \mathbb{R}^n$ such that

$$g_z(z,\, t,\, \alpha)h_0 < 0 \qquad (\alpha \in \mathfrak{A}^*,\;\; t \in [0,\, T],\;\; z \in \mathbb{R}^n).$$

Theorem 1.1. *Suppose that conditions a) - e) are fulfilled. Let* $(z^\circ,\, v^\circ) \in E$ *be a solution of the Problem 1.1 such that*

i) $\mathfrak{A}_0 = \{\alpha \in \mathfrak{A} \,/\, g(z^\circ(t),\, t,\, \alpha) = 0,\;\; \text{for some } t \in [0,\, T]\}$ *is finite.*

ii) For the set

$$H := \{(z,\, v) \in E \,/\, g(z(t),\, t,\, \alpha) \leq 0 \quad (t \in [0,\, T],\;\; \alpha \in \mathfrak{A}/\mathfrak{A}_0)\},$$

we have that $(z^\circ,\, v^\circ) \in int(H)$.
Then

I) there exists $m \in \mathbb{N}$ *and a non-negative Borel measures* μ_{α_i} *on* $[0,\, T]$ $(i = 1,\, 2, \ldots,\, m)$ *with support in*

$$R_{\alpha_i} := \{t \in [0,\, T] \,/\, g(z^\circ(t),\, t,\, \alpha_i) = 0\}, \quad \alpha_i \in \mathfrak{A}_0 \quad (i = 1, \ldots,\, m).$$

II) There exists $\varrho_0 \geq 0$ *and a function* $\eta \in L_1^n[0,\, T] = L_1([0,T];\mathbb{R}^n)$ *such that* ϱ_0 *and* η *are not simultaneously zero. Moreover,* η *is solution of the integral equation*

$$\begin{aligned} \eta(t) &= -a + \int_t^T [(-\Psi_z^*(z^\circ,\, v^\circ,\, \tau)\eta(\tau) + \varrho_0\, \Theta_z(z^\circ,\, v^\circ,\, \tau)]d\tau \\ &+ \sum_{i=1}^m \int_t^T g_z(z^\circ(\tau),\, \tau,\, \alpha_i)d\,\mu_{\alpha_i}(\tau). \end{aligned} \tag{1.7}$$

and also, for all $\mathbf{v} \in V$ *and almost all* $t \in [0,\, T]$ *it follows*

$$\langle -\Psi_v^*(z^\circ(t),\, v^\circ(t),\, t)\eta(t) + \varrho_0\, \Theta_v(z^\circ(t),\, v^\circ(t),\, t),\, \mathbf{v} - v^\circ(t)\rangle \geq 0.$$

Now, we will make a brief review of the Pontryagin's maximum principle (PMP) and the Dubovitskii-Milyutin (DM) theory: This work focuses on establishing Pontryagin's maximum principle (1962, [25]) for optimal control problems with infinite restrictions on the state variable. Infinite restrictions on the state variable add complexity to the problem and require specialized techniques to address. To tackle this, we utilize the Dubovitskii-Milyutin theory ([5]), a theoretical framework that involves conical approximations around essential elements such as the objective function, the system of differential equations, and control and state restrictions.

PMP typically generates a set of conditions that are necessary for optimality but may not be sufficient in all cases. To ensure the sufficiency of this principle, certain additional conditions must be met. This work delves into these conditions and provides a comprehensive analysis of their implications.

One of the significant contributions of this work is the application of the DM theory to handle an infinite family of functions where restrictions on the state variable are imposed. This extension represents a novel approach, allowing us to formulate and solve optimal control problems with a broader set of state variable restrictions.

Additionally, an example is presented to demonstrate the applicability of the derived principle, showcasing the practical relevance of our theoretical findings. While other optimization theories can be closely applied, such as the Lofee-Tihomirov theory ([9]), offer first-order necessary conditions for **soft convex** problems using Lagrangian multipliers, this work specifically addresses the gap in the literature concerning optimal control problems with infinite state variable restrictions. The PM proved here could be applied to address the complex dynamics and constraints of underactuated mechanical systems ([27]), and to enhance high-resolution image processing techniques using neural networks ([26]).

In conclusion, this research not only extends the application of DM theory to more complex constraints but also provides a robust framework for solving optimal control problems with infinite restrictions, thus contributing to the advancement of the field.

The DM theory has a rich history of application in the study of optimal control problems, with notable contributions documented in works such as [3, 5, 7, 8, 10, 13, 20], particularly in [13], where the author focus on impulsive optimal control problems. Numerous works in the literature, as highlighted in [21, 23, 24], have successfully utilized the DM theory for various optimal control scenarios.

2. Preliminaries Results

In this section, we provide a concise overview of the key findings of the DM theory. The general optimization problem is formulated, including the restrictions, and cones of approximation to the problem data are constructed, encompassing the objective function and the restrictions. The optimization condition, expressed by the Euler-Lagrange (EL) equation, is presented in terms of the dual of the approximation cones.

The fundamental results outlined here are based on established principles of DM theory, with detailed proofs available in [6, 13]. For the sake of brevity, we will focus on presenting the proof the original result Theorem 2.10, related to the directional derivative of the maximum of a family of continuous and convex functions. The objective is to offer a succinct but complete summary, emphasizing the aspects that provide novelty or clarification to the material presented.

2.1. Cones and Dual Cones

Let E be a locally convex topological linear space, and denote its dual space by E^* (the space of continuous linear functionals defined on E).

Definition 2.1. (See [6]) A set $\mathfrak{K} \subset E$ is a *cone with apex at zero* , if

$$\beta\,\mathfrak{K} = \mathfrak{K} \qquad (\beta > 0).$$

Definition 2.2. (See [6])

$$\mathfrak{K}^{+} = \{\mathfrak{F} \in E^* \,/\, \mathfrak{F}(z) \geq 0, \quad \forall\, z \in \mathfrak{K}\},$$

is called the *dual cone* of $\mathfrak{K}$.

Lemma 2.3. *Let $\mathfrak{K}_\alpha \subset E$ $(\alpha \in \mathfrak{A})$ be convex cones $w-$closed, then*

$$\left(\bigcap_{\alpha \in \mathfrak{A}} \mathfrak{K}_\alpha\right)^{+} = \overline{\sum_{\alpha \in \mathfrak{A}} \mathfrak{K}_\alpha^{+}} \qquad (w^* - \textit{closure}).$$

Lemma 2.4. *Let $\mathfrak{K}_1,\, \mathfrak{K}_2, \ldots, \mathfrak{K}_n \subset E$ be open convex cones such that*

$$\bigcap_{i=1}^{n} \mathfrak{K}_i \neq \emptyset.$$

Then

$$\left(\bigcap_{i=1}^{n} \mathfrak{K}_i\right)^{+} = \sum_{i=1}^{n} \mathfrak{K}_i^{+}.$$

Theorem 2.5 (DM)**.** *Let $\mathfrak{K}_1,\, \mathfrak{K}_2, \ldots, \mathfrak{K}_{n+1} \subset E$ be convex cones with apex at zero, with $\mathfrak{K}_1,\, \mathfrak{K}_2, \ldots, \mathfrak{K}_n$ open. Then*

$$\bigcap_{i=1}^{n+1} \mathfrak{K}_i = \emptyset$$

if and only if there are $\mathfrak{F}_i \in \mathfrak{K}_i^{+}$ $(i = 1, 2, \ldots, n+1)$, not all zero such that

$$\mathfrak{F}_1 + \mathfrak{F}_2 + \cdots + \mathfrak{F}_n + \mathfrak{F}_{n+1} = 0.$$

2.2. Cones of Decay Vectors

In this subsection, we explicitly compute the cones of decay vectors for some functions.

Definition 2.6. (See [6]) A vector $h \in E$ is called a *vector of decay direction* of $\mathfrak{L} : E \longrightarrow \mathbb{R}$ at the point $z^\circ \in E$, if there exists a neighborhood $\mathfrak{U}$ of the vector h, numbers $\alpha = \alpha(\mathfrak{L},\, z^\circ,\, h) < 0$ and $\varepsilon_0 \in \mathbb{R}_+$, such that for all $\varepsilon \in (0,\, \varepsilon_0)$ and all $\overline{h} \in \mathfrak{U}$ the following inequality holds

$$\mathfrak{L}(z^\circ + \varepsilon\overline{h}) \leq \mathfrak{L}(z^\circ) + \varepsilon\,\alpha.$$

Definition 2.7. (See [6]) Let E be a linear space and $\mathfrak{L} : E \longrightarrow \mathbb{R}$ a function. Then, we shall say that $\mathfrak{L}$ has directional derivative in $z^\circ \in E$ on the direction of $h \in E$ if the following limit there exists:

$$\lim_{\varepsilon \to 0^+} \frac{\mathfrak{L}(z^\circ + \varepsilon\, h) - \mathfrak{L}(z^\circ)}{\varepsilon} =: \mathfrak{L}'(z^\circ,\, h). \tag{2.1}$$

For $z^\circ \in E$, we denote by $\mathfrak{K}_d = \mathfrak{K}_d(\mathfrak{L},\, z^\circ)$ the cone of decay direction.

Theorem 2.8 (See [6, p. 48])**.** *If E is a Banach space and $\mathfrak{L}$ is Fréchet–differentiable at $z^\circ \in E$, then*

$$\mathfrak{K}_d(\mathfrak{L},\, z^\circ) = \{h \in E / \mathfrak{L}'(z^\circ)h < 0\},$$

where $\mathfrak{L}'(z^\circ)$ is the Fréchet's derivative of $\mathfrak{L}$ at z°.

Theorem 2.9 (See [6, p. 45])**.** *Let $\mathfrak{L} : E \longrightarrow \mathbb{R}$ be a continuous and convex function in a topological linear space E, and let $z^\circ \in E$. Then $\mathfrak{L}$ has a directional derivative in all directions at z°, and we also have that*

a) $\mathfrak{L}'(z^\circ,\, h) = \inf\left\{ \frac{\mathfrak{L}(z^\circ + \varepsilon\, h) - \mathfrak{L}(z^\circ)}{\varepsilon} / \varepsilon \in \mathbb{R}_+ \right\}$,

b) $\mathfrak{K}_d(\mathfrak{L},\, z^\circ) = \{h \in E / \mathfrak{L}'(z^\circ,\, h) < 0\}$.

Theorem 2.10. *Let $\mathfrak{A}$ be a compact topological space and E a linear topological space. Suppose that $l : E \times \mathfrak{A} \longrightarrow \mathbb{R}$ is a continuous function, and for all $\alpha \in \mathfrak{A}, l(\cdot,\, \alpha)$ is convex. Let us define $\mathfrak{L} : E \longrightarrow \mathbb{R}$ as follows*

$$\mathfrak{L}(z) := \max_{\alpha \in \mathfrak{A}} l(z,\, \alpha).$$

Then, for all $z^\circ,\, h \in E$ there exists $\mathfrak{L}'(z^\circ,\, h)$ and

$$\mathfrak{L}'(z^\circ,\, h) = \max_{\alpha \in \mathfrak{A}(z^\circ)} l'(z^\circ,\, h,\, \alpha)$$

where

$$\mathfrak{A}(z^\circ) = \{\alpha \in \mathfrak{A} \,/\, \mathfrak{L}(z^\circ) = l(z^\circ,\, \alpha)\}.$$

Proof. Let $z^\circ,\, h \in E$. Then since $\mathfrak{L}$ and $l(\cdot,\, \alpha)$ are continuous and convex functions, by Theorem 2.9 there exist $\mathfrak{L}'(z^\circ,\, h)$ and $l'(z^\circ,\, h,\, \alpha)$. Moreover, if we define $G : \mathbb{R}_{+0} \longrightarrow \mathbb{R}$ and $g : \mathbb{R}_{+0} \times \mathfrak{A} \longrightarrow \mathbb{R}$ as follows

$$\begin{aligned} G(\lambda) &:= \mathfrak{L}(z^\circ + \lambda\, h), \\ g(\lambda,\, \alpha) &:= l(z^\circ + \lambda\, h,\, \alpha), \end{aligned}$$

then

$$\mathfrak{L}'(z^\circ,\, h) = G'(0) \qquad \text{and} \qquad l'(z^\circ,\, h,\, \alpha) = g'(0,\, \alpha) \quad (\alpha \in \mathfrak{A}).$$

Now, let us define for all $z \in E$ the set $\mathfrak{A}(z)$ as follows:

$$\mathfrak{A}(z) := \{\alpha \in \mathfrak{A} / \mathfrak{L}(z) = l(z,\, \alpha)\}.$$

Let $\alpha \in \mathfrak{A}(z)$ and $\alpha_0 \in \mathfrak{A}(z^\circ)$. Hence

$$\begin{aligned} l(z,\, \alpha) - l(z^\circ,\, \alpha) &\geq \mathfrak{L}(z) - \mathfrak{L}(z^\circ) \\ &\geq l(z,\, \alpha_0) - l(z^\circ,\, \alpha_0). \end{aligned}$$

Thus, if $z = z_0 + \lambda\, h$, we get that

$$\begin{aligned} \tfrac{g(\lambda,\alpha) - g(0,\,\alpha)}{\lambda} &\geq \tfrac{G(\lambda) - G(0)}{\lambda} \\ &\geq \tfrac{g(\lambda,\,\alpha_0) - g(0,\,\alpha_0)}{\lambda} \qquad (\lambda \in \mathbb{R}_+). \end{aligned} \tag{2.2}$$

Let us define $\mathfrak{A}_\lambda := \mathfrak{A}(z^\circ + \lambda\, h), \quad (\lambda \in \mathbb{R}_{+0})$. Then, by the convexity and continuity of $l(\cdot,\, \alpha)$, from item $a)$ of Theorem 2.9, we get

$$\frac{G(\lambda) - G(0)}{\lambda} \geq g'(0,\, \alpha_0) \qquad (\alpha_0 \in \mathfrak{A}_0),$$

which implies

$$G'(0) \geq g'(0,\, \alpha_0) \qquad (\alpha_0 \in \mathfrak{A}_0).$$

Then

$$G'(0) \geq \sup_{\alpha_0 \in \mathfrak{A}_0} g'(0,\, \alpha_0),$$

and from the inequality (2.2), we obtain

$$G'(0) \leq \frac{g(\lambda,\, \alpha) - g(0,\, \alpha)}{\lambda} \qquad (\lambda \in \mathbb{R}_+,\ \alpha \in \mathfrak{A}_\lambda),$$

which implies that

$$G'(0) \leq \max\left\{\frac{g(\lambda,\, \alpha) - g(0,\, \alpha)}{\lambda} \,/\, \alpha \in \mathfrak{A}_\lambda\right\} \qquad (\lambda \in \mathbb{R}_+).$$

Now, let $\lambda_n \in \mathbb{R}_+$ $(n \in \mathbb{N})$ be a sequence such that $(\lambda_n) \to 0$, then by the compactness of $\mathfrak{A}_\lambda$ and the continuity of g, there exists $\alpha_n \in \mathfrak{A}_{\lambda_n}$ $(n \in \mathbb{N})$ such that

$$\begin{aligned} G'(0) &\leq \tfrac{g(\lambda_n,\,\alpha_n) - g(0,\,\alpha_n)}{\lambda_n} \\ &= \max\left\{\tfrac{g(\lambda_n,\,\alpha) - g(0,\,\alpha)}{\lambda_n} \,/\, \alpha \in \mathfrak{A}_{\lambda_n}\right\}. \end{aligned}$$

Then, since $\mathfrak{A}$ is compact, we can suppose that $(\alpha_n) \to \alpha_0 \in \mathfrak{A}$. Then $\alpha_0 \in \mathfrak{A}_0$; in fact, if $\alpha_n \in \mathfrak{A}_{\lambda_n}$ $(n \in \mathbb{N})$, we get that

$$\mathfrak{L}(z^\circ + \lambda_n\, h) = l(z^\circ + \lambda_n\, h,\, \alpha_n).$$

Thus, by the continuity
$$\mathfrak{L}(z^\circ) = l(z^\circ,\, \alpha_0),$$
which implies that $\alpha_0 \in \mathfrak{A}_0$. If $\delta \in \mathbb{R}_+$ and $\lambda_n \in [0,\, \delta]$ $(n \in \mathbb{N})$, then by the convexity of $g(\cdot,\, \alpha_n)$ it follows that for all $n \in \mathbb{N}$

$$lG'(0) \le \frac{g(\lambda_n,\, \alpha_n) - g(0,\, \alpha_n)}{\lambda_n} \le \frac{g(\delta,\, \alpha_n) - g(0,\, \alpha_n)}{\delta}.$$

Hence, the sequence

$$\frac{g(\lambda_n,\, \alpha_n) - g(0,\, \alpha_n)}{\lambda_n} \qquad (n \in \mathbb{N})$$

is bounded.

Then, for a subsequence of $(\lambda_n,\, \alpha_n)$, $n \ge 1$, denoted in the same way, there exists $\mu \in \mathbb{R}$ such that

$$\lim_{n\to\infty} \frac{g(\lambda_n,\, \alpha_n) - g(0,\, \alpha_n)}{\lambda_n} = \mu.$$

Let us consider $\lambda \in \mathbb{R}_+$, such that $\lambda_n < \lambda$ $(n \in \mathbb{N},\ n \ge k)$. Then

$$\frac{g(\lambda,\, \alpha_n) - g(0,\, \alpha_n)}{\lambda} \ge \frac{g(\lambda_n,\, \alpha_n) - g(0,\, \alpha_n)}{\lambda_n} \qquad (n \in \mathbb{N},\ n \ge k).$$

Passing to the limit when n tends to infinity, we get

$$\frac{g(\lambda,\, \alpha_0) - g(0,\, \alpha_0)}{\lambda} \ge \mu \qquad (\lambda \in \mathbb{R}_+),$$

which implies that

$$g'(0,\, \alpha_0) \ge \mu.$$

Thus
$$g'(0,\, \alpha_0) \ge \mu \ge G'(0) \ge \sup_{\alpha \in \mathfrak{A}_0} g'(0,\, \alpha).$$

Since $\mathfrak{A}_0 = \mathfrak{A}(z^\circ)$, we get that

$$G'(0) = \max_{\alpha \in \mathfrak{A}(z^\circ)} g'(0,\, \alpha),$$

i.e.,
$$\mathfrak{L}'(z^\circ,\, h) = \max_{\alpha \in \mathfrak{A}(z^\circ)} \mathfrak{L}'(z^\circ,\, h,\, \alpha).$$

□

2.3. Cones of Admissible Vectors

Now, we will define the admissible cone. The admissible cone comprises the vectors through which we can approach a predetermined point.

Definition 2.11. A vector $h \in E$ is called an *admissible vector to* $\mathfrak{Q} \subset E$ *in the point* $z^\circ \in \mathfrak{Q}$, if there is a neighborhood $\mathfrak{U}$ of the vector h and $\varepsilon_0 \in \mathbb{R}_+$, such that for all $\varepsilon \in (0, \varepsilon_0)$ and all $\overline{h} \in \mathfrak{U}$, we have that

$$z^\circ + \varepsilon \overline{h} \in \mathfrak{Q}.$$

The admissible cone will be denoted by $\mathfrak{K}_a = \mathfrak{K}_a(\mathfrak{Q}, z^\circ)$.

Theorem 2.12. *(See [6, p. 59]). If* $\mathfrak{Q}$ *is an arbitrary convex set with* $int(\mathfrak{Q}) \neq \emptyset$, *then*

$$\mathfrak{K}_a = \{h \in E/h = \lambda(z - z^\circ),\ z \in int(\mathfrak{Q}),\ \lambda \in \mathbb{R}_+\}.$$

2.4. Cones of Tangent Vectors

In this section, we highlight the Lyusternik Theorem, a potent tool for computing the cone of tangent vectors. This theorem is crucial to our analysis, as it facilitates the determination of vectors tangent to a given point.

Definition 2.13. A vector $h \in E$ is called a tangent vector to $\mathfrak{Q} \subset E$ in the point z°, if there are $\varepsilon_0 \in \mathbb{R}_+$ and a function $\theta : [0, \varepsilon] \longrightarrow E$, such that

$$\lim_{\varepsilon \to 0^+} \frac{\theta(\varepsilon)}{\varepsilon} = 0,$$

and

$$z^\circ + \varepsilon h + \theta(\varepsilon) \in \mathfrak{Q} \qquad (\varepsilon \in (0, \varepsilon_0)).$$

The set of all the tangent vectors to $\mathfrak{Q}$ in z° is a cone with apex at zero, which will be denoted by $\mathfrak{K}_T := \mathfrak{K}_T(\mathfrak{Q}, z^\circ)$, and it will be called *tangent cone.*

Theorem 2.14 (Lyusternik-See [6])**.** *Let* E_1, E_2 *be Banach spaces, and suppose that*

a) $z^\circ \in E_1,\ P : E_1 \longrightarrow E_2$ *is Fréchet's differentiable at* z°.

b) $P'(z^\circ) : E_1 \longrightarrow E_2$ *is surjective.*

Then, the cone of tangent vectors $\mathfrak{K}_T$ *to the set* $\mathfrak{Q} := \{z \in E_1/P(z) = 0\}$ *at the point* $z^\circ \in \mathfrak{Q}$, *is given by*

$$\mathfrak{K}_T = Ker\ P'(z^\circ).$$

The proof of the aforementioned theorem, which is not simple, can be found in [9, p. 30].

2.5. Dubovitskii-Milyutin Theorem

Let us consider a function $\mathfrak{L} : E \longrightarrow \mathbb{R}$, and $\mathfrak{Q}_i \subset E \quad (i = 1, 2, \dots, n+1)$ such that $\operatorname{int}(\mathfrak{Q}_i) \neq \emptyset \ \ (i = 1, 2, \dots, n)$. Set the following problem

$$\begin{cases} \mathfrak{L}(z) \longrightarrow \operatorname{loc\,min} \\ \\ z \in \mathfrak{Q}_i, \quad (i = 1, 2, \dots, n+1). \end{cases} \tag{2.3}$$

Remark 2.15. The sets $\mathfrak{Q}_i$, $(i = 1, 2, \dots, n)$ usually are given by restrictions of inequality type, and $\mathfrak{Q}_{n+1}$ by restrictions of equality type, and in general $\operatorname{int}(\mathfrak{Q}_{n+1}) = \emptyset$.

Theorem 2.16 (DM)**.** *Let $z^\circ \in E$ be a solution of problem (2.3), and assume that:*

a) $\mathfrak{K}_0$ is the decay cone of $\mathfrak{L}$ at z°.

b) $\mathfrak{K}_i$ are the admissible cones to $\mathfrak{Q}_i$ at $z^\circ \in \mathfrak{Q}_i$ $(i = 1, 2, \dots, n)$.

c) $\mathfrak{K}_{n+1}$ is the tangent cone to $\mathfrak{Q}_{n+1}$ at z°.

If $\mathfrak{K}_i \quad (i = 0, 1, 2, \dots, n+1)$ are convex, then there exist functions $\mathfrak{F}_i \in \mathfrak{K}_i^+$, $(i = 0, 1, \dots, n+1)$ not all zero such that

$$\mathfrak{F}_0 + \mathfrak{F}_1 + \cdots + \mathfrak{F}_{n+1} = 0 \tag{2.4}$$

Equation (2.4) is called the **Abstract EL Equation**.

Remark 2.17. Occasionally, it is crucial to verify that $\mathfrak{F}_0 \neq 0$; an analysis of the proof of Theorem 2.16 reveals that a sufficient condition for this is that

$$\bigcap_{i=1}^{n+1} \mathfrak{K}_i \neq \emptyset.$$

Plan to apply the DM Theorem to specific problems:

i) Identify the decay vectors.

ii) Identify the admissible vectors.

iii) Identify the tangent vectors.

iv) Construct the dual cones.

We will now address steps (i) - (iv). The necessary optimality condition stated in Theorem 2.16 is, under certain conditions, also sufficient.

3. Proof of the Main Theorem 1.1

Proof. Let $\overline{\mathfrak{L}} : E \longrightarrow \mathbb{R}$ be a function defined as follows

$$\overline{\mathfrak{L}}(z,\, v) = \int_0^T \Theta(z(t),\, v(t),\, t)dt,$$

and let $\mathfrak{Q} := \mathfrak{Q}_1 \cap \mathfrak{Q}_2 \cap \mathfrak{Q}_3$, where $\mathfrak{Q}_1,\, \mathfrak{Q}_2,\, \mathfrak{Q}_3$ are given by pairs sets $(z,\, v) \in E$, which satisfy (1.3)–(1.4),(1.5) and (1.6) respectively.

Then, Problem 1.1 is equivalent to

$$\begin{cases} \overline{\mathfrak{L}}(z,\, v) \longrightarrow \operatorname{loc\,min}, \\ (z,\, v) \in \mathfrak{Q}. \end{cases}$$

a) Analysis of the function $\overline{\mathfrak{L}}$.

Let $\mathfrak{K}_0 := \mathfrak{K}_d(\overline{\mathfrak{L}},\, (z^\circ,\, v^\circ))$ be the decay cone of $\overline{\mathfrak{L}}$ in the point $(z^\circ,\, v^\circ)$, then by Theorem 2.8, we have that

$$\mathfrak{K}_0 = \{(z,\, v) \in E \,/\, \overline{\mathfrak{L}}'(z^\circ,\, v^\circ)(z,\, v) < 0\}.$$

Suppose for a moment that $\mathfrak{K}_0 \neq \emptyset$, then, trivially, we obtain

$$\mathfrak{K}_0^+ = \{-\varrho_0\, \overline{\mathfrak{L}}'(z^\circ,\, v^\circ) \,/\, \varrho_0 \geq 0\}.$$

By example 9.2 [6, p. 62], we obtain that

$$\overline{\mathfrak{L}}'(z^\circ,\, v^\circ)(z,\, v) = \int_0^T [\Theta_z(z^\circ,\, v^\circ,\, t)z(t) + \Theta_v(z^\circ,\, v^\circ,\, t)v(t)]dt \qquad ((z,\, v) \in E).$$

Therefore, for all $\mathfrak{F}_0 \in \mathfrak{K}_0^+$, there exists $\varrho_0 \geq 0$ such that

$$\mathfrak{F}_0(z,\, v) = -\varrho_0 \int_0^T [\Theta_z(z^\circ,\, v^\circ,\, t)z(t) + \Theta_v(z^\circ,\, v^\circ,\, t)v(t)]dt \qquad ((z,\, v) \in E).$$

b) Analysis of the Restriction $\mathfrak{Q}_1$.

Let's determine the tangent cone to $\mathfrak{Q}_1$ at the point $(z^\circ,\, v^\circ)$

$$\mathfrak{K}_1 := \mathfrak{K}_T(\mathfrak{Q}_1,\, (z^\circ,\, v^\circ)).$$

Assume that the system

$$\dot{z}(z) = \Psi_z(z^\circ(t),\, v^\circ(t),\, t)z(t) + \Psi_v(z^\circ(t),\, v^\circ(t),\, t)v(t) \tag{3.1}$$

is controllable (see [12]), then in view of Theorem 2.14, we find that

$$\begin{aligned} \mathfrak{K}_1 &= \Big\{(z,\, v) \in E \,/\, z(t) = \textstyle\int_0^T [\Psi_z(z^\circ(\tau),\, v^\circ(\tau),\, \tau) z(\tau) \\ &+ \Psi_v(z^\circ(\tau),\, v^\circ(\tau),\, t) v(\tau)] d\tau, \quad z(T) = 0 \quad (t \in [0,\, T])\Big\}. \end{aligned}$$

Now, let us calculate $\mathfrak{K}_1^+$. To do so, we shall consider the following linear spaces

$$\begin{aligned} L_1 &:= \Big\{(z,\, v) \in E \,/\, z(t) = \textstyle\int_0^t [\Psi_z(z^\circ,\, v^\circ,\, \tau) z(\tau) \\ &+ \Psi_v(z^\circ,\, v^\circ,\, (\tau))] v(\tau)) d\tau, \quad (t \in [0,\, T])\Big\}, \\ L_2 &:= \{(z,\, v) \in E \,/\, z(T) = 0\}. \end{aligned}$$

Hence

$$\mathfrak{K}_1 = L_1 \cap L_2.$$

Then, by Proposition 2.40 from [13], we have that $\mathfrak{F}_{12} \in L_2^+$ if, and only if, there exists $a \in \mathbb{R}^n$ such that

$$\mathfrak{F}_{12}(z,\, v) = \langle a,\, z(T)\rangle \qquad ((z,\, v) \in E).$$

Moreover, by Lemma 2.5 from [13], it follows that $L_1^+ + L_2^+$ is w^*- closed; then by Lemma 2.3 we obtain that

$$\mathfrak{K}_1^+ = L_1^+ + L_2^+.$$

Therefore, $\mathfrak{F}_1 \in \mathfrak{K}_1^+$ if, and only if, $\mathfrak{F}_1 = \mathfrak{F}_{11} + \mathfrak{F}_{12}, \quad \mathfrak{F}_{11} \in L_1^+, \quad \mathfrak{F}_{12} \in\in L_2^+$.

c) Analysis of Restriction $\mathfrak{Q}_2$.

Let us examine the set

$$\mathfrak{Q}_2' := \{v \in L_\infty^r[0,\, T] \,/\, v(t) \in V, \quad \forall\, t \in [0,\, T], \quad a.e.\}.$$

Then $\mathfrak{Q}_2 = C^n[0,\, T] \times \mathfrak{Q}_2'$. Moreover, by the hypothesis V is convex and closed, with $\operatorname{int}(V) = \emptyset$. So, the following statements hold

i) $\mathfrak{Q}_2$, $\mathfrak{Q}_2'$ are closed and convex.

ii) $\operatorname{int}(\mathfrak{Q}_2) \neq \emptyset, \quad \operatorname{int}(\mathfrak{Q}_2') \neq \emptyset$.

If we denote $\mathfrak{K}_2$ the admissible cone to $\mathfrak{Q}_2$ at $(z^\circ,\, v^\circ) \in \mathfrak{Q}_2$, then

$$\mathfrak{K}_2 = C^n[0,\, T] \times \mathfrak{K}_2',$$

where $\mathfrak{K}_2'$ is the admissible cone to $\mathfrak{Q}_2'$ at $v^\circ \in \mathfrak{Q}_2'$.

Therefore, for all $\mathfrak{F}_2 \in \mathfrak{K}_2^+$ there is $\mathfrak{F}_2' \in \mathfrak{K}_2'^+$ such that $\mathfrak{F}_2 = (0,\, \mathfrak{F}_2')$.

By Theorem 2.12 it follows that $\mathfrak{F}_2'$ is a support of $\mathfrak{Q}_2'$ at v°.

d) Analysis of Restriction $\mathfrak{Q}_3$.

Let

$$\mathfrak{Q}_3' := \{(z,\, v) \in E \,/\, g(z(t),\, t,\, \alpha) \leq 0 \quad (t \in [0,\, T], \quad \alpha \in \mathfrak{A}_0)\}.$$

Then $\mathfrak{Q}_3 = \mathfrak{Q}_3' \cap H$. Since $(z^\circ,\, v^\circ) \in \mathrm{int}(H)$, we have that

$$\mathfrak{K}_3 := \mathfrak{K}_a(\mathfrak{Q}_3,\, (z^\circ,\, v^\circ)) = \mathfrak{K}_a(\mathfrak{Q}_3',\, (z^\circ,\, v^\circ)).$$

Let us define the function as follows

$$\begin{aligned} l \quad &: \quad C^n\,[0,\, T] \times \mathfrak{A}_0 \longrightarrow \mathbb{R} \\ l(z,\, \alpha) \quad &:= \quad \max\nolimits_{t \in [0,\, T]}\, g(z(t),\, t,\, \alpha). \end{aligned}$$

Then, l is continuous, and convex function in its first variable. Hence, by example 7.5 from (See [6, p. 52]), we have that

$$l'(z^\circ;\, h,\, \alpha) = \max_{t \in R_\alpha} g_z(z^\circ(t),\, t,\, \alpha) \qquad (h \in C^n[0,\, T], \quad \alpha \in A_0),$$

where

$$R_\alpha := \{t \in [0,\, T] \,/\, g(z^\circ(t),\, t,\, \alpha) = l(z^\circ,\, \alpha)\}.$$

Then, since $\alpha \in \mathfrak{A}_0$, we obtain that

$$R_\alpha = \{t \in [0,\, T] \,/\, g(z^\circ(t),\, t,\, \alpha) = 0\}.$$

Now, we define the following function

$$\begin{aligned} \mathfrak{L} \quad &: \quad C^n\,[0,\, T] \longrightarrow \mathbb{R} \\ \mathfrak{L}(z) \quad &:= \quad \max\nolimits_{\alpha \in \mathfrak{A}_0}\, l(z,\, \alpha) \qquad (z \in C^n\,[0,\, T]). \end{aligned}$$

Then

$$\mathfrak{Q}_3' = \{(z,\, v) \in E \,/\, \mathfrak{L}(z) \leq 0\},$$

and, since $\mathfrak{L}$ is continuous and convex, by Theorem 2.10 we get that

$$\mathfrak{L}'(z^\circ;\, h) = \max_{\alpha\in\mathfrak{A}_0} l'(z^\circ;\, h,\, \alpha) \qquad (h \in C^n\,[0,\, T]).$$

In this case

$$\mathfrak{A}_0 = \{\alpha \in \mathfrak{A}\,/\,\mathfrak{L}(z^\circ) = l(z^\circ,\, \alpha)\}.$$

On the other hand,

$$\mathfrak{K}_3 = \mathfrak{K}_a(\mathfrak{Q}_3',\, (z^\circ,\, v^\circ)) \supset \mathfrak{K}_d(\mathfrak{L},\, (z^\circ,\, v^\circ)) =: \mathfrak{K}_d \tag{3.2}$$

But, by Theorem 2.9, we obtain that

$$\begin{aligned}\mathfrak{K}_d &= \{(h,\, u) \in E\,/\,\mathfrak{L}'(z^\circ,\, h) < 0\}\\ &= \{(h,\, u) \in E\,/\,g_z(z^\circ(t),\, t,\, \alpha)h(t) < 0 \quad (\alpha \in \mathfrak{A}_0,\ t \in R_\alpha)\}.\end{aligned}$$

Let us consider the following cones

$$\mathfrak{K}_\alpha := \{(h,\, u) \in E\,/\,g_z(z^\circ,\, t,\, \alpha)h(t) < 0 \quad (t \in R_\alpha)\} \quad (\alpha \in \mathfrak{A}_0).$$

Then

$$\mathfrak{K}_d = \bigcap_{\alpha\in\mathfrak{A}_0} \mathfrak{K}_\alpha.$$

Since $\mathfrak{A}_0$ is finite, then by condition e) there is $h_0 \in \mathbb{R}^n$ such that

$$g_z(z^\circ,\, t,\, \alpha)h_0 < 0 \quad (\alpha \in \mathfrak{A}_0,\ t \in [0,\, T]),$$

which implies that

$$\bigcap_{\alpha\in\mathfrak{A}_0} \operatorname{int}(K)_\alpha \neq \emptyset.$$

Therefore, by Lemma 2.4, it obtains

$$\mathfrak{K}_d^+ = \sum_{i=1}^m \mathfrak{K}_{\alpha_i}^+,$$

where m is the number of elements of $\mathfrak{A}_0$. From (3.2), we have that

$$\mathfrak{K}_3^+ \subset \mathfrak{K}_d^+ = \sum_{i=1}^m \mathfrak{K}_{\alpha_1}^+.$$

So, each $\mathfrak{F}_3 \in \mathfrak{K}_3^+$ is given by the form:

$$\mathfrak{F}_3 = \sum_{i=1}^m \mathfrak{F}_i, \qquad (\mathfrak{F}_i \in \mathfrak{K}_{\alpha_i}^+, \quad i = 1, 2, \ldots, m).$$

Moreover, by example 10.3 [6, p. 73], we have that for all $\mathfrak{F}_i \in \mathfrak{K}^+_{\alpha_i}$, there is a non–negative Borel measure μ_{α_i} on $[0, T]$ such that

$$\mathfrak{F}_i(z, v) = -\int_0^T g_z(z^\circ(t), t, \alpha_i)z(t)d\mu_{\alpha_i}(t) \quad ((z, v) \in E)$$

and μ_{α_i} has support in

$$R_{\alpha_i} = \{t \in [0, T] \,/\, g(z^\circ(t), t, \alpha_i) = 0\}.$$

e) Euler-Lagrange Equation.

It is evident that $\mathfrak{K}_0$, $\mathfrak{K}_1$, $\mathfrak{K}_2$, $\mathfrak{K}_3$ are convex cones. Then, by Theorem 2.16 there exist functionals $\mathfrak{F}_i \in \mathfrak{K}^+_i \quad (i = 0, 1, 2, 3)$ not all zero, so that

$$\mathfrak{F}_0 + \mathfrak{F}_1 + \mathfrak{F}_2 + \mathfrak{F}_3 = 0. \tag{3.3}$$

Equation (3.3) can be expressed as follows

$$\begin{aligned} - \quad & \varrho_0 \int_0^T [\Theta_z(z^\circ, v^\circ, t)z(t) + \Theta_v(z^\circ, v^\circ, t)v(t)]dt \\ + \quad & \mathfrak{F}_{11}(z, v) + \langle a, z(T)\rangle + \mathfrak{F}'_2(u) + \mathfrak{F}_3(z, v) = 0 \quad ((z, v) \in E). \end{aligned}$$

Now, for all $u \in L^r_\infty$ there exists $z \in C^n\,[0, T]$, solution of equation (3.1) with $z(0) = 0$. Then $(z, v) \in S_1$, and therefore $\mathfrak{F}_{11}(z, v) = 0$. Hence, EL Equation can be written as follows:

$$\begin{aligned} \mathfrak{F}'_2(u) \quad = \quad & \varrho_0 \int_0^T \Theta_z(z^\circ, v^\circ, t)z(t)dt + \varrho_0 \int_0^T \Theta_v(z^\circ, v^\circ, t)v(t)dt \\ - \quad & \langle a, z(T)\rangle + \sum_{i=1}^m \int_0^T g_z(z^\circ(t), t, \alpha_i)z(t)d\,\mu_{\alpha_i}(t) \quad ((z, v) \in E). \end{aligned}$$

Let η be the solution of equation (1.7), which means

$$\begin{aligned} - \quad & \eta(t) = -a + \int_t^T [(-\Psi^*_z(z^\circ, v^\circ, \tau)\eta(\tau) + \varrho_0\,\Theta_z(z^\circ, v^\circ, \tau)]d\tau \\ + \quad & \sum_{i=1}^m \int_t^T g_z(z^\circ(\tau), \tau, \alpha_i)d\,\mu_{\alpha_i}(\tau). \end{aligned}$$

This equation is a second-order Volterra type equation, which admits a unique solution $\eta \in S^n_1[0, T]$ (see [11, p. 519]).

Multiplying both sides of equation (1.7) by $\dot{z}$ and integrating from 0 to T, we obtain

$$
\begin{aligned}
- & \int_0^T \langle \dot{z},\, \eta(t)\rangle dt = -\int_0^T \langle a,\, \dot{z}(t)\rangle dt \\
+ & \int_0^T \left\langle \dot{z}(t),\, \int_t^T [(-\Psi_z^*(z^\circ,\, v^\circ,\, \tau)\eta(\tau) + \varrho_0\,\Theta_z(z^\circ,\, v^\circ,\, \tau)]d\tau \right\rangle dt \\
+ & \int_0^T \sum_{i=1}^m \left\langle \dot{z}(t),\, \int_t^T g_z(z^\circ(\tau),\, \tau,\, \alpha_i) d\,\mu_{\alpha_i}(\tau) \right\rangle dt.
\end{aligned}
$$

Since

$$
\dot{z}(t) = \Psi_z(z^\circ(t),\, v^\circ(t),\, t)z(t) + \Psi_v(z^\circ(t),\, v^\circ(t),\, t)v(t), \quad z(0) = 0,
$$

then

$$
\langle \dot{z}(t) - \Psi_v(z^\circ(t),\, v^\circ(t),\, t)v(t), \eta(t)\rangle = \langle \Psi_z(z^\circ(t),\, v^\circ(t),\, t)z(t), \eta(t)\rangle.
$$

Then, the expression given above can be reformulated as follows:

$$
\begin{aligned}
- & \int_0^T \langle \dot{z}(t),\, \eta(t)\rangle dt = -\langle a,\, z(T)\rangle - \int_0^T \langle \dot{z}(t),\, \eta(t)\rangle dt \\
+ & \int_0^T \langle \Psi_v^*(z^\circ(t),\, v^\circ(t),\, t)\eta(\tau), v(t)\rangle dt + \varrho_0 \int_0^T \langle z(t),\, \Theta_z(z^\circ,\, v^\circ,\, t)\rangle dt \\
+ & \int_0^T \sum_{i=1}^m \left\langle \dot{z}(t),\, \int_t^T g_z(z^\circ(\tau),\, \tau,\, \alpha_i) d\,\mu_{\alpha_i}(\tau) \right\rangle dt.
\end{aligned}
$$

The third term on the right can be simplified by applying the integration by parts method for the Stieltjes-Integral and the fact that $g(z_0,\, t,\, \alpha) < 0$, $g(z_1,\, t,\, \alpha) < 0$ $(t \in [0,T], \quad \alpha \in \mathfrak{A})$. That is $0 \notin R_\alpha$, $T \notin R_\alpha$, then $\mu_\alpha(0) = \mu_\alpha(T) = 0$. Thus

$$
\begin{aligned}
& \sum_{i=1}^m \int_0^T \left\langle \dot{z}(t),\, \int_t^T g_z(z^\circ(\tau),\, \tau,\, \alpha_i) d\,\mu_{\alpha_i}(\tau) \right\rangle dt \\
& \qquad = \sum_{i=1}^m \int_0^T g_z(z^\circ(t),\, t,\, \alpha_i) z(t) d\,\mu_{\alpha_i}(t).
\end{aligned}
$$

Then

$$
\begin{aligned}
& \varrho_0 \int_0^T \Theta_z(z^\circ(\tau),\, v^\circ,\, t)z(t)d\,t + \sum_{i=1}^m \int_0^T g_z(z^\circ(t),\, t,\, \alpha_i)z(t)d\,\mu_{\alpha_i}(t) - \\
& -\langle a,\, z(T)\rangle = -\int_0^T \langle \Psi_v^*(z^\circ,\, v^\circ,\, t)\eta(t),\, v(t)\rangle dt. \qquad (3.4)
\end{aligned}
$$

Then, by EL equation (3.2), we obtain that

$$\mathfrak{F}_2'(t) = \int_0^T \langle -\Psi_v^*(z^\circ(t),\, v^\circ(t),\, t)\eta(t) + \varrho_0\, \Theta_v(z^\circ(t),\, v^\circ(t),\, t)v(t)\rangle dt \tag{3.5}$$

for all $v \in L_\infty^r[0,\, T]$. Since $\mathfrak{F}_2'$ is a support of $\mathfrak{Q}_2'$ at the point $v^\circ \in \mathfrak{Q}_2'$, from example 10.5 [6, p. 76], it follows that

$$\langle -\Psi_v^*(z^\circ(t),\, v^\circ(t),\, t)\eta(t) + \varrho_0\, \Theta_v(z^\circ(t),\, v^\circ(t),\, t),\, \mathbf{v} - v^\circ(t)\rangle \geq 0,$$

for all $\mathbf{v} \in V$ and almost all $t \in [0,\, T]$.
Now, we will see that the case $\varrho_0 = 0,\ \eta = 0$ is not possible. In fact, if $\eta = 0$, then $\eta(T) = a = 0$. Thus

$$\mathfrak{F}_{12}(z,\, v) = \langle a,\, z(T)\rangle = 0 \qquad ((z,\, v) \in E),$$

that is $\mathfrak{F}_{12} \equiv 0$. So, from equation (1.7), and the fact that $\varrho_0 = 0$, we obtain that

$$\sum_{i=1}^{m} \int_t^T g_z(z^\circ(\tau),\, \tau,\, \alpha_i) d\, \mu_{\alpha_i}(\tau) = 0 \qquad (t \in [0,\, T]),$$

which implies that $\mathfrak{F}_3 = 0$. Also, from (3.5), we have that $\mathfrak{F}_2'(u) = 0$ $(v \in L_\infty^r[0,\, T])$, then from EL equation it follows that $\mathfrak{F}_{11} = 0$, where

$$\mathfrak{F}_1 = \mathfrak{F}_{11} + \mathfrak{F}_{12} = 0,$$

which contradicts the statement of Theorem 2.16.

At this point, we have introduced two additional assumptions:

Firstly, we assumed that $\mathfrak{K}_0 \neq \emptyset$. Secondly, we have supposed that the system

$$\dot{z} = \Psi_z(z^\circ,\, v^\circ,\, t)z(t) + \Psi_v(z^\circ,\, v^\circ,\ ,t)v(t)$$

is controllable.

We shall now establish that these assumptions are superfluous. Indeed, if $\mathfrak{K}_0 = \emptyset$, then by definition of $\mathfrak{K}_0$, we have that

$$\int_0^T [\Theta_z(z^\circ(t),\, v^\circ(t),\, t)z(t) + \Theta_v(z^\circ(t),\, v^\circ(t),\, t)v(t)]dt = 0 \quad ((z,\, v) \in E).$$

Let us put $\mu_{\alpha_i} = 0 \quad (i = 1,\, 2, \ldots,\, m),\ \varrho_0 = 1,\ \eta(t) = a = 0$, then, from equation (3.4), we have that

$$\int_0^T \Theta_z(z^\circ,\, v^\circ,\, t)z(t) = -\int_0^T \Psi_v^\star(z^\circ,\, v^\circ,\, t)\eta(t)v(t)dt,$$

for all $(z,\, v)$ such that z is solution of equation the (3.1). Then

$$\int_0^T [-\Psi_v^*(z^\circ(t),\, v^\circ(t),\, t)\eta(t)v(t) + \Theta_v(z^\circ(t),\, v^\circ(t),\, t)v(t)]dt = 0\ (v \in L_\infty^r[0,\, T])$$

this leads to the conclusion that

$$\langle -\Psi_v^*(z^\circ, v^\circ, t)\eta(t) + \Theta_v(z^\circ, v^\circ, t), \mathbf{v} - v^\circ(t)\rangle = 0,$$

for all $\mathbf{v} \in V$ and almost all $t \in [0, T]$.

Assuming system (3.1) is not controllable, then according to an equivalence to the definition of controllability outlined in [6, 12, 14, 15], there is a non-trivial function $\eta \in C^n[0, T]$ that is solution of

$$\dot{\eta}(t) = -\Psi_z^*(z^\circ(t), v^\circ(t), t)\eta(t),$$

such that, for all $t \in [0, T]$ it follows that

$$\Psi_v^*(z^\circ(t), v^\circ(t), t)\eta(t) = 0.$$

By taking $\varrho_0 = 0,\ \mu_{\alpha_i} = 0 \quad (i = 1, 2, \dots, m)$, we get that η is solution of (1.7), and therefore

$$\langle -\Psi_v^*(z^\circ(t), v^\circ(t), t)\eta(t), \mathbf{v} - v^\circ(t)\rangle \geq 0,$$

for all $\mathbf{v} \in V$ and almost all $t \in [0, T]$.

Thus, the proof of Theorem 1.1 is now fully complete.

□

4. Sufficient Condition of Optimality

The necessary condition for optimality presented in Theorem 1.1 (Maximum Principle), given certain additional conditions, is also sufficient. Specifically, let us examine the specific case of Problem 1.1 where the differential equation is linear.

Problem 4.1.

$$\int_0^T \Theta(z(t), v(t), t)dt \longrightarrow \text{loc min}. \tag{4.1}$$

$$(z, v) \in E := C^n[0, T] \times L^r_\infty[0, T],$$

$$\dot{z}(t) = \Lambda(t)z(t) + \mathfrak{B}(t)v(t), \tag{4.2}$$

$$z(0) = z_0, \quad z(T) = z_1; \qquad z_1, z_0 \in \mathbb{R}^n, \tag{4.3}$$

$$v(t) \in V, \qquad \forall\, t \in [0, T], \tag{4.4}$$

$$g(z(t), t, \alpha) \leq 0, \qquad (\alpha \in \mathfrak{A},\ t \in [0, T]), \tag{4.5}$$

where $\Lambda(\cdot) : [0, T] \longrightarrow \mathbb{R}^{n\times n}$, $\mathfrak{B}(\cdot) : [0, T] \longrightarrow \mathbb{R}^{n\times r}$ are continuous matrix functions. Let $(z^\circ, v^\circ) \in E$ be satisfying the conditions (4.2)– (4.5).

Theorem 4.1. *Let us suppose that only the conditions $a) - d)$, $ii)$, $I)$ and $II)$ from Theorem 1.1 are satisfied.*

Furthermore, let us assume the following:

A) The system (4.2) is controllable.

B) There exists $\widetilde{v} \in L^r_\infty\,[0,\, T]$ such that $\widetilde{v}(t) \in int(V), \quad \forall\, t \in [0,\, T]$.

C) The corresponding solution to $\widetilde{v}$, of equation (4.2), $\widetilde{z}$ satisfies $\widetilde{z}(T) = z_1$ and $g(\widetilde{z}(t),\, t,\, \alpha) < 0, \quad (\alpha \in \mathfrak{A} \quad t \in [0,\, T])$.

D) Θ is a convex function in its two first variables.

Then $(z^\circ,\, v^\circ)$ is global solution of Problem 4.1.

Proof. Let us define the function $\overline{\mathfrak{L}} : E \longrightarrow \mathbb{R}$ as follows

$$\overline{\mathfrak{L}}(z,\, v) = \int_0^T \Theta(z(t),\, v(t),\, t)dt.$$

Let us consider $\mathfrak{Q} := \mathfrak{Q}_1 \cap \mathfrak{Q}_2 \cap \mathfrak{Q}_3$, where $\mathfrak{Q}_1$ is given by (4.2)–(4.3), $\mathfrak{Q}_2$ by (4.4), $\mathfrak{Q}_3$ by (4.5) as in the Theorem 1.1.

Then, Problem 4.1 is equivalent to:

$$\begin{cases} \overline{\mathfrak{L}}(z,\, v) \longrightarrow \operatorname{loc}\min, \\ (z,\, v) \in \mathfrak{Q}. \end{cases}$$

Now, we consider

$$\mathfrak{A}_0 := \{\alpha \in \mathfrak{A} \,/\, g(z^\circ(t),\, t,\, \alpha) = 0, \text{ for some } \ t \in [0,\, T]\},$$

and

$$H := \{(z,\, v) \in E \,/\, g(z(t),\, t,\, \alpha) \le 0 \quad (\alpha \in \mathfrak{A}/\mathfrak{A}_0,\ t \in [0,\, T])\}.$$

Hence, from $ii)$ $(z^\circ,\, v^\circ) \in \operatorname{int}(H)$.

It is clear that $\mathfrak{Q}_i \quad (i = 1,\, 2,\, 3)$ are convex sets, and from the conditions $(C)-(D)$ we have that $\overline{\mathfrak{L}}$ is convex, and $(\widetilde{z},\, \widetilde{v}) \in \operatorname{int}(\mathfrak{Q}_2) \cap \operatorname{int}(\mathfrak{Q}_3) \cap \mathfrak{Q}_1$.

Thus, by Theorem 2.17 from [13] it follows that:

$(z^\circ,\, v^\circ)$ is a minimum point of $\mathfrak{L}$ at $\mathfrak{Q}$ if, and only if, there are $\mathfrak{F}_i \in \mathfrak{K}_i^+$ $(i = 0,\, 1,\, 2,\, 3)$, not all zero such that

$$\mathfrak{F}_0 + \mathfrak{F}_1 + \mathfrak{F}_2 + \mathfrak{F}_3 = 0.$$

Here, $\mathfrak{K}_i \quad (i = 0,\, 1,\, 2,\, 3)$ are cones defined as in the Theorem 1.1, (except that $\mathfrak{A}_0$ can be infinite and condition e) of Theorem 1.1 does not have to hold anymore).

Let $\mathfrak{K}_3 = \mathfrak{K}_a\,(\mathfrak{Q}_3,\,(z^\circ,\,v^\circ))$ be the admissible cone to $\mathfrak{Q}_3$ at the point $(z^\circ,\,v^\circ)$. Then

$$\mathfrak{K}_3 \supset \mathfrak{K}_d(\mathfrak{L},\,(z^\circ,\,v^\circ)) =: \mathfrak{K}_d = \bigcap_{\alpha\in\mathfrak{A}_0} \mathfrak{K}_\alpha,$$

where

$$\mathfrak{K}_\alpha := \{(z,\,v) \in E\,/\,g_z(z^\circ(t),\,t,\,\alpha)z(t) < 0 \quad (t \in R_\alpha)\},$$

and

$$R_\alpha := \{t \in [0,\,T]\,/\,g(z^\circ(t),\,t,\,\alpha) = 0\} \qquad (\alpha \in \mathfrak{A}_0).$$

Then, by Lemma 2.3, we have that

$$\mathfrak{K}_3^+ \subset \overline{\sum_{\alpha\in\mathfrak{A}_0} \mathfrak{K}_\alpha^+} = \mathfrak{K}_d^+.$$

So, each $\mathfrak{F}_\alpha \in \mathfrak{K}_\alpha^+$ has the following form

$$\mathfrak{F}_\alpha(z,\,v) = -\int_0^T g_z(z^\circ(t),\,t,\,\alpha)z(t)d\mu_\alpha(t) \qquad ((z,\,v) \in E).$$

Here $\mu_\alpha \quad (\alpha \in \mathfrak{A}_0)$ is a non negative Borel measures with support on R_α.

Now, suppose that the Maximum Principle of Theorem 1.1 holds. That is, there are $\varrho_0 \geq 0$, $a \in \mathbb{R}^n$, $m \in \mathbb{N}$ and non-negative Borel measures $\mu_{\alpha_i} \quad (\alpha_i \in \mathfrak{A}_0$ $i = 1,\,2,\ldots,\,m)$ with support on R_{α_i}; and also, a function $\eta \in L_1^n[0,\,T]$ that is solution of the following integral equation

$$\begin{aligned} -\eta(t) &= -a + \int_t^T(-\Lambda^*(\tau)\eta(\tau)) + \varrho_0\,\Theta_z(z^\circ(\tau),\,v(\tau),\,\tau)d\tau + \\ &+ \sum_{i=1}^m \int_t^T g_z(z^\circ(\tau),\,\tau,\,\alpha_i)d\,\mu_{\alpha_i}, \end{aligned} \tag{4.6}$$

where both ϱ_0 and η are non-zero, and for every $\mathbf{v} \in V$ and almost every $t \in [0,\,,T]$, the following holds

$$\langle -\mathfrak{B}^*(t)\eta(t) + \varrho_0\,\Theta_v(z^\circ(t),\,v^\circ(t),\,t),\,\mathbf{v} - v^\circ(t)\rangle \geq 0. \tag{4.7}$$

To demonstrate the theorem, it is enough to show that there exist $\mathfrak{F}_i \in \mathfrak{K}_i^+ \quad (i = 0,\,1,\,2,\,3)$ not all zero, such that $\mathfrak{F}_0 + \mathfrak{F}_1 + \mathfrak{F}_2 + \mathfrak{F}_3 = 0$; for which we define the following set

$$\mathfrak{Q}_2' = \{v \in L_\infty^r\,/\,v(t) \in V, \quad \forall\,t \in [0,\,T], \quad a.e.\}.$$

and functionals

$$\begin{aligned} \mathfrak{F}_2' &: L_\infty^r \longrightarrow \mathbb{R},\ \mathfrak{F}_2 : E \longrightarrow \mathbb{R} \\ \mathfrak{F}_2'(v) &:= \int_0^T \langle -B^*(t)\eta(t) + \varrho_0\,\Theta_v(z^\circ(t),\,v^\circ(t),\,t),\,v(t)\rangle dt, \\ \mathfrak{F}_2 &:= (0,\,\mathfrak{F}_2'). \end{aligned}$$

Then, from (4.7), we get that

$$\mathfrak{F}_2'(v) \geq \mathfrak{F}_2'(v^\circ) \qquad (v \in \mathfrak{Q}_2').$$

So, $\mathfrak{F}_2'$ is a support of $\mathfrak{Q}_2'$ at v°. Hence $\mathfrak{F}_2 = (0, \mathfrak{F}_2') \in \mathfrak{K}_2^+$. Let us define the functional $\mathfrak{F}_{11} : E \longrightarrow \mathbb{R}$ as follows

$$\mathfrak{F}_{11}(z, v) := \varrho_0 \int_0^T [\Theta_z(z^\circ(t), v^\circ(t), t)z(t) + \Theta_v(z^\circ(t), v^\circ(t), t)v(t)]dt -$$

$$-\mathfrak{F}_2'(u) - \langle a, z(T)\rangle + \sum_{i=1}^m \int_0^T g_z(z^\circ(t), t, \alpha_i)z(t)d\,\mu_{\alpha_i}(t).$$

Now, we will see that $\mathfrak{F}_{11} \in L_1^+$, where

$$S_1 = \left\{(z, v) \,/\, z(t) = \int_0^t [\Lambda(\tau)z(\tau) + \mathfrak{B}(\tau)v(\tau)]d\tau \qquad (t \in [0, T])\right\},$$

as in the Theorem 1.1. In fact, suppose that $(z, v) \in S_1$, then multiplying both sides of the equation (4.6) by $\dot{z}$ and integrating from 0 to T, we obtain that

$$\varrho_0 \int_0^T [\Theta_z(z^\circ(t), v^\circ(t), t)z(t)dt + \sum_{i=1}^m \int_0^T g_z(z^\circ(t), t, \alpha_i)z(t)d\,\mu_{\alpha_i}(t) -$$

$$-\langle a, z(T)\rangle = -\int_0^T \langle \mathfrak{B}^*(t)\eta(t), v(t)\rangle dt.$$

Then

$$\mathfrak{F}_{11}(z, v) = -\mathfrak{F}_2'(u) - \int_0^T \langle \mathfrak{B}^*(t)\eta(t), v(t)\rangle dt + \varrho_0 \int_0^T \Theta_v(z^\circ(t), v^\circ(t), t)v(t)dt.$$

Therefore

$$\mathfrak{F}_{11}(z, v) = -\mathfrak{F}_2'(u) + \mathfrak{F}_2'(u) = 0.$$

Thus $\mathfrak{F}_{11} \in L_1^+$.

Next, we will introduce the following functionals

$$\mathfrak{F}_0, \mathfrak{F}_1, \mathfrak{F}_3; E \longrightarrow \mathbb{R},$$

by

$$\begin{aligned}
\mathfrak{F}_0(z, v) &:= \varrho_0 \int_0^T [\Theta_z(z^\circ(t), v^\circ(t), t)z(t) + \Theta_v(z^\circ(t), v^\circ(t), t)v(t)]dt \\
\mathfrak{F}_1(z, v) &:= \mathfrak{F}_{11}(z, v) + \langle a, z(T)\rangle, \\
\mathfrak{F}_3(z, v) &:= \sum_{i=1}^m \int_0^T g_z(z^\circ(t), t, \alpha_i)z(t)d\,\mu_{\alpha_i}(t).
\end{aligned}$$

Then $\mathfrak{F}_0 \in \mathfrak{K}_0^+$, $\mathfrak{F}_1 \in \mathfrak{K}_1^+$, $\mathfrak{F}_3 \in \mathfrak{K}_3^+$, and also

$$\mathfrak{F}_0 + \mathfrak{F}_1 + \mathfrak{F}_2 + \mathfrak{F}_3 = 0,$$

not all these functionals are zero, because by hypothesis ϱ_0 and η are not both zero. From the convexity conditions, it follows the global-minimality of (z°, v°) $\square$

5. Example

Next, we will present an example to demonstrate the application of the primary result from this study. In this regard, we provide the following two previous propositions.

Proposition 5.1. *Let $z_0 \in \mathbb{R}^n_+$ and $\Lambda = (a_{ij})_{n\times n}$ a real matrix, such that $a_{ij} > 0$ $(i \neq j, i,\, j = 1,\, 2, \ldots,\, n)$. Then*

$$e^{\Lambda t}\, z_0 \in \mathbb{R}^n_+, \qquad (t \in \mathbb{R}).$$

The proof of above proposition is trivial.

Let $V \subset \mathbb{R}^r$ be a set, then we define the set $\mathfrak{Q}_V$ as follows:

$$\mathfrak{Q}_V := \{v \in L^r_\infty\,[0,\, T] \,/\, v(t) \in V,\ \forall\, t \in [0,\, T], \quad a.e.\}.$$

Proposition 5.2. *Let $z_0 \in \mathbb{R}^n_+$, and $\mathfrak{B} = (b_{ij})_{h\times r}$ a real matrix. Then, there exists $V \subset \mathbb{R}^r$ convex and closed, with $int(V) \neq \emptyset$ such that*

$$\left(e^{\Lambda t}\, z_0 + \int_0^T e^{\Lambda(t-s)} \mathfrak{B}\, v(s) ds\right) \in \mathbb{R}^n_+ \qquad (u \in \mathfrak{Q}_V,\ \forall\, t \in [0,\, T], a.e.).$$

Proof. Let $\{e_1,\, e_2, \ldots,\, e_n\}$ the canonical basis of $\mathbb{R}^n$, and define

$$\begin{aligned} \alpha_i &:= \min_{t\in[0,T]} \langle e_i,\, e^{\Lambda t}\, z_0\rangle \quad (i = 1,\, 2, \ldots,\, n), \\ \overline{V} &:= (\alpha_1,\, \alpha_2, \ldots,\, \alpha_n). \end{aligned}$$

Then, by Proposition 5.1 it follows that $V \in \mathbb{R}^n_+$.

Let $\delta := \min\{\alpha_i \,/\, i = 1,\, 2, \ldots,\, n\}$; then for all $z \in \mathbb{R}^n$ such that $\|z\| < \delta$, we have that $\overline{V} + z \in \mathbb{R}^n_+$.

Let us consider

$$\mathfrak{K}_1 := \max_{t\in[0,\,T]} \|e^{\Lambda t}\|, \quad \mathfrak{K}_2 := \max_{t\in[0,\,T]} \|e^{-\Lambda t}\|.$$

Then

$$\left|\int_0^T e^{\Lambda(t-s)}\,\mathfrak{B}\,v(s)ds\right| < T\,\mathfrak{K}_1\,\mathfrak{K}_2\|\mathfrak{B}\|\|v\|_\infty, \quad \forall\, t\in[0,\,T], a.e.,$$

and taking

$$V := \left\{\mathbf{v}\in\mathbb{R}^r\,/\,|\mathbf{v}| \leq \frac{\delta}{T\mathfrak{K}_1\mathfrak{K}_2\|\mathfrak{B}\|}\right\}.$$

□

Next, we shall consider the following example where Theorem 1.1 is applicable:

Example 5.3. Let $n = 2,\ r = 1$ and suppose that Θ meets the same conditions as specified in Problem 1.1. Additionally, let us consider

$$\begin{aligned} \mathfrak{B} &= \begin{pmatrix} b_{11} \\ b_{12} \end{pmatrix}, \qquad \Lambda = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}; \quad a_{12} > 0, \quad a_{21} > 0 \\ V &:= \left\{\mathbf{v}\in R\,/\,|\mathbf{v}| \leq \tfrac{\delta}{T\mathfrak{K}_1\mathfrak{K}_2\|\mathfrak{B}\|}\right\}, \end{aligned}$$

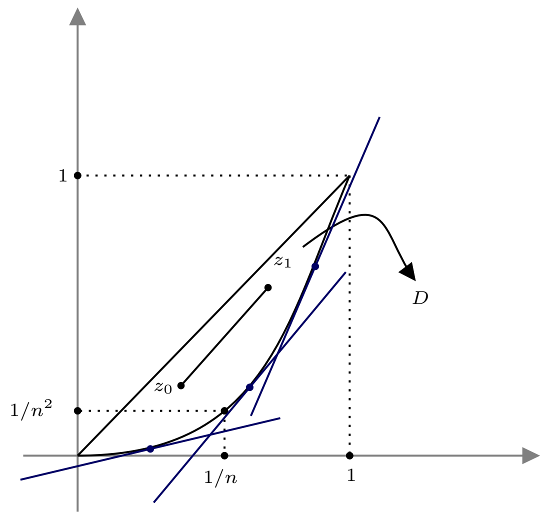
where δ, $\mathfrak{K}_1$, $\mathfrak{K}_2$ are defined as in Proposition 5.2.
Let us consider the following problem

$$\int_0^T \Theta(z(t),\,v(t),\,t)dt \longrightarrow \text{loc min}$$

$$(z,\,v)\in C^2\,[0,\,T]\times L_\infty[0,\,T]$$

$$\begin{aligned} \dot z(t) &= \Lambda z(t) + \mathfrak{B}v(t) \\ z(0) &= z_0,\ z(T) = z_1; \quad z_0,\,z_1\in\mathbb{R}^2_+, \\ v(t)\in V, & \qquad \forall\, t\in[0,\,T], \quad a.e. \\ z(t)\in D & \qquad (t\in[0,\,T]), \end{aligned}$$

where $D\subset\mathbb{R}^2_{+0}$ is a convex-polyhedron given by the following figure



Blue lines are tangent lines.

where the vertices of D are given by

$$V_\infty := (0,\, 0), \quad V_n := \left(\frac{1}{n},\, \frac{1}{n^2}\right) \qquad (n \in \mathbb{N}).$$

The normals outside of each edge are given by:

$$\mathfrak{N}_\infty \;:=\; (0,\, -1), \quad \mathfrak{N}_0 = \tfrac{1}{\sqrt{2}}(-1,\, 1),$$

$$\mathfrak{N}_n \;=\; \tfrac{1}{\sqrt{n^2+4}}(2,\, -n) \quad (n \in \mathbb{N}\backslash\{0\}).$$

Let us consider $\overline{\mathbb{N}} = \mathbb{N} \cup \{\infty\}$ a compactification of the natural numbers, taking as an ideal point ∞. Then, the function $g : \mathbb{R}^2 \times \overline{\mathbb{N}} \longrightarrow \mathbb{R}$ defined as follows

$$g(z,\, n) := \begin{cases} \langle \mathfrak{N}_n,\, z\rangle - \langle \mathfrak{N}_n,\, V_n\rangle & \text{if} \quad n \in \mathbb{N}, \\ \langle \mathfrak{N}_\infty,\, z\rangle - \langle \mathfrak{N}_\infty,\, V_\infty\rangle & \text{if} \quad n = \infty, \end{cases}$$

is continuous and convex in z.

Then $z \in D$ if, and only if, $g(z,\, n) \leq 0 \quad (n \in \overline{\mathbb{N}})$.

Let $(z^\circ,\, v^\circ) \in E = C^2[0,\, T] \times L_\infty[0,\, T]$ be a solution of the above problem, then the set

$$\mathfrak{A}_0 = \{n \in \overline{\mathbb{N}} \,/\, g(z^\circ(t),\, n) = 0 \,\text{for some}\;\, t \in [0,\, T]\}$$

is finite. In fact, by Proposition 5.2 it follows that

$$z^\circ(t) = e^{\Lambda t} z_0 + \int_0^T e^{\lambda(t-s)} \mathfrak{B}\, v^\circ(s) ds \in \mathbb{R}^2_+ \qquad (t \in [0,\, T]).$$

Then, there exists $N \in \mathbb{N}$ such that

$$z^\circ(t) - \left(\frac{1}{N},\, \frac{1}{N^2}\right) \in \mathbb{R}^2_+ \qquad (t \in [0,\, T]).$$

Then, since $\left(\frac{1}{n},\, \frac{1}{n^2}\right) \to (0,\, 0)$ it follows that

$$z^\circ(t) - \left(\frac{1}{n},\, \frac{1}{n^2}\right) \in \mathbb{R}^2_+ \qquad (n \geq N),\ (t \in [0,\, T]),$$

thus, $\mathfrak{A}_0$ is finite. Moreover, if

$$H := \{(z,\, v) \in E \,/\, g(z(t),\, n) \leq 0 \qquad (t \in [0,\, T],\ n \in \overline{\mathbb{N}} \backslash \mathfrak{A}_0)\},$$

hence

$$(z^\circ,\, v^\circ) \in \operatorname{int}(H).$$

Thus, all the conditions of Theorem 1.1 are fulfilled. Then, there exist $\varrho_0 \in \mathbb{R}_+$, $a \in \mathbb{R}^2$, $m \in \mathbb{N}$ and non–negative measures $\mu_{m_i} \quad (i = 1, 2, \ldots, m)$ with support in

$$R_{m_i} = \{t \in [0,\, T] \,/\, g(z^\circ(t),\, m_i) = 0\} \qquad (m_i \in \mathfrak{A}_0),$$

and there is a function $\eta \in L^2_1\,[0,\, T]$ solution of the equation

$$\begin{aligned} -\eta(t) = &- a + \int_t^T (\Lambda^* \eta(\tau) + \varrho_0\, \Theta_z(z^\circ(\tau),\, v^\circ(\tau),\, \tau))) d\tau \\ &+ \sum_{i=1}^m \int_t^T g_z(z^\circ(\tau),\, m_i) d\mu_{m_i}(\tau), \end{aligned} \tag{5.1}$$

such that ϱ_0 and η are not both zero, and for all $\mathbf{v} \in M$ and almost all $t \in [0,\, T]$, we have that

$$\langle -\mathfrak{B}^*\, \eta(t) + \varrho_0\, \Theta_v(z^\circ(t),\, v^\circ(t),\, t),\, \mathbf{v} - v^\circ(t)\rangle \geq 0$$

or equivalently

$$\begin{aligned} &\max_{\mathbf{v} \in V} \langle B^*\, \eta(t) - \varrho_0\, \Theta_v(z^\circ(t),\, v^\circ(t),\, t),\, \mathbf{v}\rangle \\ &= \langle B^*\, \eta(t) - \varrho_0\, \Theta_v(z^\circ(t),\, v^\circ(t),\, t),\, v^\circ(t)\rangle \end{aligned} \tag{5.2}$$

for almost all $t \in [0,\, T]$.
Let us examine the specific case in which

$$\Theta(z,\, \mathbf{v}) = \mathfrak{C}\mathbf{v} \qquad ((z,\, \mathbf{v},\, t) \in \mathbb{R}^2 \times \mathbb{R} \times [0,\, T]),$$

and let us determine how the controls $v \in S_\infty[0,\,T]$ that solve the problem should be:

$$\begin{aligned} -\eta(t) &= -a - \int_t^T \Lambda^*\eta(\tau)d\tau + \sum_{i=1}^m \mathfrak{N}_{m_i}\int_t^T d\mu_{m_i}(\tau), \\ R_{m_i} &= \{t \in [0,\,T]\,/\,g(z^\circ(t),\,m_i) = 0\}, \\ \max(B^*\eta(t) - \varrho_0\,\mathfrak{C})\mathbf{v} &= (\mathfrak{B}^*\eta(t) - \varrho_0\,\mathfrak{C})v^\circ(t), \quad \mathbf{v} \in [-\rho,\,\rho] \end{aligned}$$

for almost all $t \in [0,\,T]$, where $\rho = \delta/\mathfrak{K}_1\,\mathfrak{K}_2\,\|\mathfrak{B}\|T$.

Let

$$\begin{aligned} \mathfrak{N}_{\mathfrak{B}^*} &:= \{z \in \mathbb{R}^2\,/\,\mathfrak{B}^*z - \varrho_0\,\mathfrak{C} = 0\}, \\ H_1 &:= \{t \in [0,\,T]\,/\,\eta(t) \notin \mathfrak{N}_{\mathfrak{B}^*}\}, \end{aligned}$$

then $v^\circ(t) := \rho\ \mathrm{sig}\ (\mathfrak{B}^*\eta(t) - \varrho_0\,\mathfrak{C}) \quad \text{if} \quad t \in H_1$.

This implies that the optimal control must be of the 'bang-bang' type over the set H_1.

6. Open Problems

In this section, we introduce an open problem that holds the potential to serve as a subject for future research endeavors or even a Ph.D. thesis for interested students. This open problem pertains to an optimal control scenario that concurrently involves impulses and restrictions on a state variable. Our primary objective is to delve into the analysis of the following optimal control problem, which is slated for exploration in our upcoming research:

Problem 6.1.

$$\int_0^T \Theta(z(t),\,v(t),\,t)dt \longrightarrow \text{loc min}\,. \tag{6.1}$$

$$(z,\,v) \in E := \mathcal{PW}([0,T];\mathbb{R}^n) \times L_\infty^r[0,\,T], \tag{6.2}$$

$$\dot z(t) = \Psi(z(t),\,v(t),\,t), \quad z(0) = z_0 \tag{6.3}$$

$$G_i(z(T)) = 0, \quad i = 1,2,\ldots,q \le n, \tag{6.4}$$

$$z(t_k^+) = z(t_k^-) + \mathcal{J}_k(z(t_k)), \quad k = 1,2,3,\ldots,p, \tag{6.5}$$

$$v(t) \in V, \quad t \in [0,\,T], \quad a.e., \tag{6.6}$$

$$g(z(t),\,t,\,\alpha) \le 0 \qquad (\alpha \in \mathfrak{A},\ t \in [0,\,T]). \tag{6.7}$$

7. Conclusion and Final Remark

In this paper, we have addressed the optimal control problem with infinite constraints on the state variable. Utilizing the Dubovitskii-Milyutin (DM) theory, we have extended Pontryagin's Maximum Principle (PMP) to encompass scenarios where an infinite family of constraints is imposed on the state variable. This extension represents a novel contribution to the field of optimal control theory, offering a framework for addressing more complex and realistic problems. Our findings demonstrate that the DM theory, when applied to such an extended set of constraints, provides a robust method for deriving necessary conditions for optimality. The inclusion of an illustrative example underscores the practical applicability of our theoretical results, showing how these principles can be used to solve real-world problems with infinite constraints.

The results presented in this paper pave the way for future research in several directions. Firstly, the extension of PMP using the DM theory could be further explored in other types of dynamic systems, particularly those with non-linear dynamics and more complex constraints. Secondly, there is potential to apply these findings to various engineering fields, such as robotics and aerospace, where systems often operate under numerous state constraints. Furthermore, the methods developed here could be integrated with other optimization techniques, such as those involving neural networks and high-resolution image processing, to enhance their efficiency and effectiveness. The bridging of control theory with modern computational methods holds promise for significant advancements in both theoretical and applied domains.

In conclusion, our work not only contributes to the theoretical foundation of optimal control with infinite state constraints but also opens new avenues for practical applications and interdisciplinary research. We hope that these contributions will inspire further exploration and innovation in the field. Notably, this work was motivated by [13, 15], which highlights the importance of exploring optimal control problems with impulses depending on the state using the DM theory. An important aspect in several papers, including references [1, 2, 4, 14, 15, 22], is the robustness of the controllability of linear systems in the presence of disturbances such as impulses, delays and non-local conditions. Existing literature suggests that the controllability of linear systems tends to be robust even when subject to these perturbations. Starting from this understanding, this work has proved the persistence of PMP under infinite restrictions on the state variable. With this in mind, there is an intriguing proposal that this principle can also remain invariant when non-local conditions are introduced.

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Pulses and Stability for Impulsive Bell Replication Model

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Abstract: In this paper, the classical Bell replication model is extended to an impulsive analogue incorporating a variable-time impulsive system to analyze the dynamics of antigen–antibody interactions. The model investigates the interaction between antigens and antibody complexes through pulse phenomena to elucidate mechanisms by which antigens can be effectively neutralized via drug action and antibodies reinforced through vaccination. Employing a combination of mathematical modeling, stability theory, and immunological principles, sufficient conditions for the occurrence of pulse phenomena and system stability are established. It is shown that, under appropriate parameter constraints, the solution of the system is asymptotically stable. The analysis further reveals that the inclusion of impulses in the antigen–antibody dynamics enhances antibody efficacy against antigenic invasion. From a theoretical perspective, the results provide a framework for understanding pulse-mediated immune responses and offer potential insights relevant to the design of effective therapeutic and vaccination strategies.

AMS Subject Classification: 34A37, 92-10, 92C32.
Keywords and Phrases: Impulse systems; Variable times; Pulses; Bell equation; Stability; Antigens; Antibodies and immunology.

1. Introduction

Impulsive differential equations (IDEs) are systems characterized by rapid changes in the form of jumps and shocks that take place for a short moment compared to that of the evolution time for the whole system ([3,8,10,11]). Many real-life problems can be modelled using impulsive systems. These systems have several applications in geophysics, electrodynamics, medicine, finance, environmental pollution and the military. ([9-11,14]). The pulse phenomenon is about solutions of a given variable times impulsive system hitting a hyperspace several times [3, 4, 10]. Modelling and Simulation of variable-times impulsive systems have potential applications in the military, especially in the design of missiles with impulsive trajectories. It can also be applied in the study of the dynamics of the epidemic and the game of pursuing and capture.

Computational modelling of an antibody-antigen complex is gaining research ground of late therefore, exploring the use of impulsive systems is gradually becoming inevitable in future- medical research ([7, 13-15]).

The research interest in this paper is to consider the impulsive analogue of the Bell model to gain insight into the human immune response system [10, 15, 16]. This involves studying the activities of self-replicating antigens such as bacteria, viruses and other foreign agents that invade living organism cells.

We note that models for studying the interaction between antigen and antibody cells as a typical prey-predator population model were considered in [5, 15].

Crucial to our study are β-cells and T-cells. The β-cells are a variety of white blood cells (lymphocytes) responsible for the production of proteins called antibodies. The antibody protects the body from invasion by substances that the body finds to be foreign. These foreign substances are called antigens. T-cells and macrophages must act on the antigen before the β -cells are triggered (hormonal response) ([1, 2, 6]).

Antibodies are important immune-regulatory proteins that have a negative influence on lymphocyte activation. The human defence mechanism should be able to predict the right concentration of the antigens in the blood and release the appropriate dose of β-cells and other antibodies to fight off the antigens.

In immunology, lymphocytes can bind to cells that have surface antigens and collaborate with specific antigen-recognizing and binding cells for the immune responses to be initiated ([2]). Conditions that allow beating or pulse effect support the continuous interaction between the antibody and the antigen, thereby offering the antibody the opportunity to attack and consume the antigens at the predicted capture zone at the prescribed capture time. The idea of pulses from an immunology point of view means that there is a period that depends on the trajectories of the solution to the model wherein the antibodies have infinitely many chances to capture the antigens and destroy them. In the immunology context, it simply means there is a period and a region where the antibodies and the β-cells are mobilized to fight-off the antigens. Geometrically, this means that the trajectories of the solution of the model hit the hypersurface several times.

In this paper, we will determine the condition for the absence of pulses and use quantitative techniques to obtain stable results.

2. Preliminaries, Notations and Definitions

Notations 1

u	Concentration of the antigen in the blood plasma
v	Concentration of the antibody in the blood plasma
w	Concentration of the β -cells present in the blood plasma
s	Source constant term
$\lambda_1, \alpha_1, \lambda_2, \alpha_2$	Positive constants
λ_3	Rate of decay of Lymphocytes
α_3	Maximum proliferation rate of β -cells when simulated
φ	Maximum of $\beta\ - cells$ possible
n	Number of receptors on a β -cell
k	maximum production rate for the antibody
$\Re^n$	n-dimensional Euclidean space with elements $x = (x_1, x_2, \cdots x_n)$ and the norm defined as $\lvert x \rvert = (\sum_{k=1}^{n} x_k^2)^{\frac{1}{2}}$
$t_1(x), t_2(x), ..., t_k(x) \in \Re_+ = [0, +\infty)$	Variable impulsive moments (times) such that $0 < t_0(x) < t_1(x) < t_2(x) < ... < t_k(x)$ and $\lim_{k\to\infty} t_k\ (x) = +\infty$
$C(\Re^+, \Re^n)$	Set of continuous functions in $\Re^+$taking values in $\Re^n$
$C'(\Re^+, \Re^n)$	Set of continuous functions together with its first derivative in $\Re^+$ and taking values in $\Re^n$

Table 1

Notations 2
$PC(\Re_+, \Re^n)$ is the set of all continuous functions on R_+ except for at most a discrete set of points in R_+. $PC(\Re_+, \Re^n)$ together with the Euclidean norm sum form a Banach space.
Consider the impulsive system

$$\begin{aligned} z' &= f(t, z), t \neq \tau_k(z), \\ \Delta z &= I_k\left(z\left(t_k\right)\right), t = \tau_k(z). \end{aligned} \tag{1}$$

Where $I_k\left(z\left(t_k\right)\right) = z\left(t_k^+\right) - z\left(t_k^-\right), z\left(t_k^-\right) = z\left(t_k\right)$. $f : \Re^+ \times \Omega \to \Re^n, \Omega \subset \Re^n$ is open, $I_k : \Omega \to R^n$ is the impulse function characterizing the process and t_k are the moments at which the impulses take place. The impulses in the impulsive system described by

the equation (1), are said to be non-fixed moments for the impulsive system. That is, the impulsive moments describing the system depend on the solution of the overall system.
Solutions of variable times (non-fixed moments) impulsive systems starting at different points will have different discontinuities. And the solution may hit a given hypersurface several times. When this happens, the pulse or beating phenomenon is said to have occurred. Systems with variable (non-fixed) moments offer more difficult problems when compared with fixed moments problems ([3, 4, 8, 10, 12]).

Definition 1. Consider the hypersurface: $\delta_k = \Big\{ t \in \mathfrak{R}^+ \times \mathfrak{R}^n : 1 - \frac{\partial t_k(x)}{\partial x} f(t,x) = 0,$ $t = t_k(x),\ k = 0, 1, 2, 3 \ldots \Big\}$. We will say that the set of trajectory $(t, x(t))$ of the solutions to the equation (1) hits the hypersurface δ_k, l times; if $x = x(t)$ is the solution of equation (1) such that $t = t_i(x), i = 1, 2, 3, \ldots$.

Crucial to our study is the set δ_k and the following impulsive comparison equation:

$$\begin{aligned} u^{\cdot} &= g(t, u(t)), t \neq \gamma_k(u), k = 0, 1, 2, 3, \ldots \\ u(t_k^+) &= u(t) + \psi_k(u(t)), t_k = \gamma_k(u) \\ u(t_0^+) &= u_0. \end{aligned} \tag{2}$$

Where $u \in \mathfrak{R}^+$, γ_k and ψ_k are smooth functions.
Let $u\,(t) = u(t, t_0, u_0) \in PC(\mathfrak{R}^+, \mathfrak{R}^2)$ be the solution of the comparison system in equation (2) passing through (t_0, u_0).

Definition 2. We shall say that any function $z\,(t) = z\,(t, t_0, z_0) \in PC\,(\mathfrak{R}^+, \mathfrak{R}^n)$ is the solution to equation (1) passing through (t_0, z_0) if it satisfies the equation (1) along with the initial condition

$$z(t_0) = z(t_0, t_0 + 0, z_0) = z_0.$$

Definition 3. We say that the condition (B) is satisfied if: there exist constants $\rho_1 > 0$, $k_1 > 0, k_2 > 0$ and $k_3 > 0$ such that
(B_1) $|t_k(z_1) - t_k(z_2)| \leq k_1 |z_1 - z_2|^{\rho_1}$
(B_2) $|f(t, z_1) - f(t, z_2)| \leq k_2 |z_1 - z_2|$
(B_3) $|I(z_1) - I(z_2)| \leq k_3 |z_1 - z_2|$,
where $z_i = (u_i, v_i, z_i),\ i = 1, 2$ and $|z_1 - z_2| = \sqrt{(u_1 - u_2)^2 + (v_1 - v_2)^2 + (z_1 - z_2)^2}$.

Assumptions
Assume that interaction and competition are existing between the antigens and the antibodies. In every encounter, the antibody chooses a path to avoid the antigen. We denote the interaction functions by $f_i(u, v, w),\ i = 1, 2, 3$.
We will investigate the conditions for the beating effect to have taken place for the model. It can be shown that equation (1) satisfies all the conditions ($B_1 - B_3$) and that the solution is uniquely determined in the given interval (see [11-13]).
There are many versions of Bell's immune humorral response models. Stephen J. M. (see [16]) modified the two-dimensional prey-predation model of Pimbley [9] to have

a third equation governing the concentration of β-cells with a logistic term added. We revisit this model in a generalized form by introducing impulses and a function for regulating the hormonal and immune response in the desired way.

2.1. Statement of the Problem

Consider the impulsive analogue of the hormonal and immune response system, which is the modified Bell's replication model for the antigens antibodies given as follows:

$$\begin{aligned}
\frac{du}{dt} &= u\left[\lambda_1 + k\lambda_1 v - k\left(\alpha_1 - \lambda_1\right)u + kn\lambda_1 w\right] + f_1(u,v,w), t \neq t_k(u,v,w) \\
\frac{dv}{dt} &= v\left[-\lambda_2 - k\left(\lambda_1 + \lambda_2\right)u - k\lambda_2 v - kn\lambda_2 w\right] + k\lambda uw + f_2(u,v,w), t \neq t_k(u,v,w) \\
\frac{dw}{dt} &= w\left[-\lambda_3 + k\left(\lambda_1 - \lambda_3\right)u - k\lambda_3 v - kn\lambda_3 w - \frac{k\alpha_3}{\varphi}uw\right] + f_3(u,v,w) \\
&+ s(1 + k(u + v + nw)), t \neq t_k(u,v,w), k = 0,1,2... \\
& \Delta u(t = t_k(u,v,w)) = I_1(u) \\
& \Delta v(t = t_k(u,v,w)) = I_2(v) \\
& 0 < t_0(z) < t_1(z) < \cdots < t_n(z), \lim_{k\to\infty} t_k(z) + \infty,\ z = (u,v,w).
\end{aligned} \tag{3}$$

For some uniform $t_k(z) : t_{k+1}(z) - t_k(z) \geq T,\ k = 0,1,2,...$ and subject to $u\left(t_0 + 0\right) = u_0,\ v\left(t_0 + 0\right) = v_0, w\left(t_0 + 0\right)$.
In practice, $f(u,v,w)$ is a function for altering or excluding a specific pathway to achieve a desired medical result as suggested by the experiments.

Remark 1. $I_1(u) = 0, I_2(v) = 0,\ f_i(u,v,w) = 0,\ i = 1,2,3$. The equation (3) reduces to the classical Bell's equation without impulses.

The simple choice of $0 \leq u < 1, 0 \leq v < 1$ and $0 \leq w < 1$ allows the absence of a beating phenomenon. This simply allows the antigens to flourish and the antibodies become ineffective. Drugs and Vaccine developers must carry out experiments to determine the region to avoid the beating effect during drug/vaccine administration. Let the region of the capture of antigen by the antibody be S_k and the capture time be t. Therefore, there are infinitely many ways the antigens can be captured. Without loss of generality, for this study, we will make use of

$$s_k = (t,z) : t = t_k(z),\ z = au^2 + bv^2 + cw^2, \tag{4}$$

where (u,v,w) is the solution to equation (3).

$$\begin{aligned}
\delta_k &= (t,x) \in \Re^+ \times \Omega,\ \Omega \in \Re^n, t = t_k\left(z\right) = \frac{z^{-k}}{2k},\ k = 1,2,\ldots \lambda = const., \\
& z \geq (ks)^{\frac{1}{\lambda}}\ , 0 < s < 1.
\end{aligned}$$

Definition 4. If the solution of equation (3) hits the hypersurface, that is, it hits the capture region several times, then the antibody has infinitely many chances to capture the antigen then we say 'beating effect phenomenon' or 'pulse phenomenon' has taken place. However, the antigen must avoid this by moving away from this zone to avoid being captured and consumed.

Remark 2. Once the antigen enters into the system, then the β-cells and antibodies must be deployed in such a way to attack the antigen. The antigen should avoid the comfort zones of antibodies by preventing pulses to take place for it to propagate effectively in the blood plasma of its victim.

Remark 3. Furthermore, if the antigen must be captured, it should be once and in such a way that it may lead to destruction. In this case, 'no beating effect' is observed.

3. Main Results

3.1. Methods

1. Quasi-equilibrium points
Consider the quasi-equilibrium set as follows

$$E_q = \{ (u,v,w) : \dot{u} = 0, \dot{v} = 0, \dot{w} = 0, \text{for } \dot{t}=\tau_k \;, \dot{u} = c_1, v = c_2, \dot{w} = c_3, k = 0,1,2,... \} \;,$$

$$E_q = \{ (u,v,w) \in PC(R,R^3) : \dot{u}=0, \dot{v}=0, \dot{w}=0 \} \bigcap \{ I_1(u)=c_1, I_2(u)=c_2, I_3(u)=c_3 \}.$$

We can find the quasi-equilibrium points to the model when the interaction among the antibodies, antigen and β-cells are complainer, hence, $f_i, i = 1,2$ can be chosen to be

$$f_1(u,v,w) = a_1u + a_2v + a_3w + d_1,$$
$$f_2(u,v,w) = b_1u + b_2v + b_3w + d_2,$$
$$f_3(u,v,w) = c_1u + c_2v + c_3w + d_3,$$

where a_i, b_i, c_i, d_i for $i= 1,2,3$ are constants.

The following result is on Beating Effects and Applications. We state without proof.

Lemma 1. *(See Theorem 2.1 in [8]) Assume that*

(i) For any (t_0, z_0) in $[t_0, \infty)$ for $\Delta z = 0$;

(ii) $\frac{\partial t_k(z)}{\partial z} f(t,z) < 1$;

(iii) $\left(\frac{\partial t_k}{\partial z}\left[z + sI_k(z)\right]\right) I_k(z) < 0, 0 \leq s \leq 1$ and

$$t_{k+1}(z + I_k(z)) > t_k(z).$$

Then a solution of equation (1) exists on $[t_0, \infty)$ and meets every surface $s_k : t = t_k(z)$ only once.

Lemma 2. *(See [10]) Let the following condition be satisfied.*

(i) Condition B.

(ii) $t_k(z + sI(z)) \leq t_k(z), k = 1, 2, 3, \ldots, t > t_0.$

Then a solution of equation (1) exists on $[t_0, 0)$ *and meets every surface* $s_k : t = t_k(x)$ *only once. The continuity of* f *and* I *and the conditions (ii) and (iii) guarantee the existence of solution of IVP (1) and the Hölder's continuity condition replaces the differentiability of* $t_k(x)$ *in Lemma 1.*

2. Equilibrium points

When $\Delta u = 0, \Delta v = 0,$ and $\Delta w = 0$ in the equation (3) then

$$\begin{aligned}
&u\left[\lambda_1 + k\lambda_1 v - k\left(\alpha_1 - \lambda_1\right)u + kn\lambda_1 w\right] + f_1(u,v,w) = 0\\
&v\left[-\lambda_2 - k\left(\lambda_1 + \lambda_2\right)u - k\lambda_2 v - kn\lambda_2 w\right] + k\lambda uw + f_2(u,v,w) = 0\\
&w\left[-\lambda_3 + k\left(\lambda_1 - \lambda_3\right)u - k\lambda_3 v - kn\lambda_3 w - \frac{k\alpha_3}{\varphi}uw\right]\\
&+f_3(u,v,w) + s(1 + k(u + v + nw)) = 0. \qquad (5)
\end{aligned}$$

Where

$$f_1(u,v,w) = a_1u + a_2v + a_3w + d_1,$$
$$f_2(u,v,w) = b_1u + b_2v + b_3w + d_2,$$

and

$$f_3(u,v,w) = c_1u + c_2v + c_3w + d_3.$$

We consider the following equilibrium set of points

$$E = \{E(u,0,0), E(0,v,0), E(0,0,w), E(u,v,w)\}.$$

Where $E(u,0,0)$ involves setting $v = 0$ and $w = 0$ in equation (5) we get the following equations:

$$\begin{aligned}
a_1u + \lambda_1u - k(a_1 - \lambda_1)u^2 + a_1u + d_1 = 0\\
b_1u + d_2 = 0\\
c_1u + s(ku + 1) + d_3 = 0.
\end{aligned}$$

Now solving for u in the above equilibrium equations we get

$$u = -\frac{-a_1 - \lambda_1 + \sqrt{4ka_1d_1 - 4kd_1\lambda_1 + a_1^2 + 2a_1\lambda_1 + \lambda_1^2}}{2k(a_1 - \lambda_1)}$$

$$u = \frac{a_1 + \lambda_1 + \sqrt{4ka_1d_1 - 4kd_1\lambda_1 + a_1^2 + 2a_1\lambda_1 + \lambda_1^2}}{2k(a_1 - \lambda_1)}$$

$$u = -\frac{d_2}{b_1}$$

$$u = -\frac{s + d_3}{sk + c_1}.$$

Next, to solve $E(0,v,0)$ involves setting $u=0, v=0$ in equation (3) to get

$$a_2 v + d_1 = 0$$

$$c_2 + s(ku+1) + d_3 = 0$$

$$-kn\lambda_2 v^2 + b_2 v - \lambda_2 v + d_2 = 0$$

The solutions of the above equilibrium equations are given as

$$v = \frac{b_2 - \lambda_2 + \sqrt{4knd_2\lambda_2 + b_2^2 - 2\lambda_2 b_2 + \lambda_2^2}}{2kn\lambda_2}$$

$$v = -\frac{-b_2 + \lambda_2 + \sqrt{4knd_2\lambda_2 + b_2^2 - 2b_2\lambda_2 + \lambda_2^2}}{2kn\lambda_2}$$

$$v = -\frac{d_1}{a_2}$$

$$v = -\frac{s + d_3}{sk + c_2}.$$

In the same vein, $E(0,0,w)$ can be obtained by setting $u=0, v=0$ in equation (3) and we have

$$a_3 w + d_1 = 0$$

$$b_3 w + d_2 = 0$$

$$-kn\lambda_3 w^2 + sknw + c_3 w - \lambda_3 w + s - d_3 = 0.$$

And the solution of the above equilibrium equations being

$$w = -\frac{-d_1}{a_3}$$

$$w = -\frac{-d_2}{b_3}$$

$$\begin{aligned} w &= \frac{1}{2kn\lambda_3}\Big[skn + c_3 - \lambda_3 \\ &\quad + \sqrt{4kn\lambda_3 s - 4kn\lambda_3 d_3 + kn^2 s^2 + 2knsc_3 - 2kns\lambda_3 + c_3^2 - 2c_3\lambda_3 + \lambda_3^2}\Big] \\ w &= -\frac{1}{2kn\lambda_3}\Big[-skn - c_3 + \lambda_3 \\ &\quad + \sqrt{4kn\lambda_3 s - 4kn\lambda_3 d_3 + kn^2 s^2 + 2knsc_3 - 2kns\lambda_3 + c_3^2 - 2c_3\lambda_3 + \lambda_3^2}\Big]. \end{aligned}$$

For $E(u, v, w)$, generally is intractable, except through numerical approximation when the numerical values of the parameters are known. To apply the pulse phenomenon to the model, we will make use of Lemma 1 and 2 as follows:
We will investigate the conditions for the occurrence of pulses in the model. Let us return to equation (3). Applying Lemma 1 for the absence of the beating then yields

$$2a\lambda u^2+2ak\lambda_1 u^3-ak\left(\alpha_1-\lambda_1\right)u^2v+2akn\lambda_1 u^2 w+f_1<1-2b\lambda_2 v^2-2kb\left(\alpha_2+\lambda_2\right)v^2u$$
$$-2bk\lambda_2 v^3-2bkn\lambda_2 v^2 w+2bk\lambda_2 vw+f_2<1$$
$$-2cw^2\lambda_3+2w^2ck\left(\alpha_3-\lambda_1\right)u-2ckn\lambda_3 w^3-\frac{2ck\alpha_3}{\varphi}vw^3+2bsw\left(1+k\left(u+v+nw\right)\right)$$
$$+f_3<1. \tag{IBC1}$$

In addition, let the following conditions be satisfied:

$$\begin{aligned}
&\partial t_k\left(u+s^* I_1(u)\right)<0; 0\le s^*<1\\
&\partial t_k\left(v+s^* I_2(v)\right)I_2(v)<0; 0\le s^*<1\\
&t_{k+1}\left(u+I_1(u)\right)>t_k(u)\\
&t_{k+1}\left(v+I_2(v)\right)>t_k(v),
\end{aligned} \tag{IBC2}$$

for $0<t_1<t_2<\cdots<t_k<\cdots$, $\lim_{k\to\infty} t_k=+\infty$.
The equilibrium equation (5) can be transformed by setting

$$\begin{aligned}
&A+Cv+D^2w=0,\\
&E+Gv+Hw=0, \text{ and}\\
&M+Ku+zv=0.
\end{aligned} \tag{6}$$

Assume that $k\neq 0, g\neq 0, c\neq 0, dg-h\neq 0$ and solving the above system of equations in (6), we get

$$\begin{aligned}
u&=\frac{cdfg-cdgm-ahz+chm}{kc(dg-h)}\\
v&=\frac{ah-cdf}{c(dg-h)}\\
w&=\frac{cf-ag}{c(dg-h)}.
\end{aligned} \tag{7}$$

Therefore, we need to solve the above inequality together with conditions imposed on the impulse function (IBC2) to obtain the following results:

Theorem 1. *Let*

1. $N=-2c\lambda_3, M=2ckb(\alpha_3-\lambda_3), Y=-2ckn\lambda_3, Z=\frac{-2ck\alpha_3}{\varphi}, s=0,\ f_{30}=\max_{(u,v,w)} f_2$.
2. $A=2a\lambda, B=2ak\lambda_1, C=-ak(\alpha_1-\lambda_1), D=2akn\lambda_1$ *and* $f_{10}=\max_{(u,v,z)} f_1$.

3. $E = -2b\lambda_2, F = -2akb(\alpha_2 - \lambda_1), G = -2bkn\lambda_2, H = -2bk\lambda_2, f_{20} = \max_{(u,v,w)} f_2$.

Let the conditions in (IBC1) and (IBC2) be satisfied.
Then

$$\begin{array}{c} \text{signum}(\frac{1-f_{10}}{B})^{2/3}u < \text{signum}(\frac{1-f_{10}}{B})^{2/3}(\frac{1-f_{10}}{B})^{1/3} \\ w < \frac{1-f_{20}-Ets}{Ft^2} \\ f_{30} < 1 \\ s,t = 1,2,3,\cdots \end{array}$$

Proof. By substituting the values of N, M, Y, Z and f_{30} in (IBC1), we have the following system of inequalities.

$$\begin{array}{c} u^3 - \frac{1-f_{10}}{B} < 1 \\ Fuw + Euv + f_{20} < 1 \\ f_{30} < 1. \end{array}$$

And solving the inequality equation we get

$$\begin{array}{c} \text{signum}(\frac{1-f_{10}}{B})^{2/3}u < \text{signum}(\frac{1-f_{10}}{B})^{2/3}(\frac{1-f_{10}}{B})^{1/3} \\ w < \frac{1-f_{20}-Ets}{Ft^2} \\ f_{30} < 1 \\ s,t = 1,2,3,\cdots \end{array},$$

where v takes any arbitrary value. This ends the proof. □

Theorem 1 gives us the bound for concentration of antigen in the blood plasma and the corresponding concentration for the β -cells for the absence of pulses.

Theorem 2. *Assume that*

1. *there exist constants t^*, k_2 and k_3 such that*

$$t^* = \max\left[1, |\lambda_1|, k, k|\lambda_1 - \lambda_2|, n\right],$$

$$k_2 = \max\left[1, |\lambda_2|, k, k|(\lambda_2 + \lambda_1)|, kn\lambda_2\right]$$

and

$$k_3 = \max\left[1, |\lambda_3|, k, k|(\lambda_1 - \lambda_3)|, kn\lambda_3\right].$$

2. *the conditions in (IBC2) are satisfied.*

3. *$F(t,z) = (f_i)^T \in J \times \Re^n, i = 1,2,3$ where f_i are locally lipschitzian with respect to z in the manifold $\Omega \subseteq S_i \bigcap \delta_i$.*

Then the solution of the equation (3) meets the hypersurface δ_i only once.

Proof. Let $\delta_k = (t, x) \in \Re^n \times \Omega, t = t_k(z) = \frac{z^{-k}}{2k}, k = 1, 2, \ldots \lambda = const., z \geq (ks)^{\frac{1}{\lambda}}$, $0 < s < 1$. Then $t_{k+1}(z) - t_k(z) = \left(2^{-k-1} - 2^{-k}\right) z^{\lambda} > 0$

$$t_k(z + sI) = 2^{-k} \frac{1}{(z + sI)^{\lambda}} \leq t_k(z).$$

Furthermore, $t_k(z_1) - t_k(z_2) \leq k_1 |z_1 - z_2|^{\delta}, z_1, z_2 \in \ \Omega \subseteq S_i \bigcap \delta_i$.
Applying the Lemma 2 we have

$$|F(z_1) - F(z_2)| \leq |f_1 \ (z_1) - f_1 \ (z_2)| + |f_2 \ (z_1) - f_2 \ (z_2)| + |f_3 \ (z_1) - f_3 \ (z_2)|.$$

But

$$\begin{aligned} |f_1 \ (z_1) - f_1 \ (z_2)| &\leq |\lambda_1| \, |u_1 - u_2| + k\, |\lambda_1| \, |v_1 - v_2| \\ &\quad + k(\lambda_2 - \lambda_1) \, |w_1 - w_2| + kn\lambda_1 \, |w_1 - w_2| \\ &\quad + |f_1 \ (z_1) - f_1 \ (z_2)| \\ &\leq t^* \left[|u_1 - u_2| + |v_1 - v_2| + |w_1 - w_2|\right] + L_1 \, |z_1 - z_2|. \end{aligned}$$

Similarly, we can prove that

$$|f_1 \ (z_1) - f_1 \ (z_2)| \leq (k_2 + L_2) \, |z_1 - z_2|,$$

$$|f_2 \ (z_1) - f_2 \ (z_2)| \leq (k_3 + L_2) \, |z_1 - z_2|.$$

Therefore,

$$|F(z_1) - F(z_2)| \leq \sum_{i=1}^{3} (k_i + L_i) \, |z_1 - z_2|, \ \ z_1, z_2 \in \ \Omega \subseteq S_i \bigcap \delta_i.$$

If we take $I(z) = \frac{z^2}{2k}$,

$$|I(z_1) - I(z_2)| \leq \frac{1}{2k} \left(|z_1| + |z_2|\right) \left(|z_1 - z_2|\right).$$

The proof follows immediately by Lemma 1. □

Theorem 3. *Let* $\theta = \lambda(\lambda_2 + nk_1c_1)$, $K = \frac{|\lambda w_0|}{nk\lambda_3 w_0 + \lambda}$ *and there exist constants* $N_1 > 0$, $N_2 > 0$ *such that* $|f_1(u, v, w)| \leq N_1$, $|f_2(u, v, w)| \leq N_2$, *and*

$$\varphi(t, w_0) = \frac{\lambda w_0}{e^{\chi t} nk\lambda_3 w_0 - \lambda knw_0 + \lambda e^{\lambda t}}.$$

Then

$$|\varphi(t, w_0)| = \left| \frac{\lambda w_0}{e^{\chi t} nk\lambda_3 w_0 - \lambda knw_0 + \lambda e^{\lambda t}} \right| \leq Ke^{-\lambda t}$$

$$|r_1(t)| \leq Ke^{-\lambda t} + KM \int_0^t e^{-\lambda s} ds + kw_0 \sum_{i=0}^{\infty} ke^{-\lambda t_i} \left(1 - w_i^2\right),$$

$$|r_2\,(t)\,| \leq (c_1 v_0 + \frac{N_2 - c_1 v_0}{\theta}) exp - \theta t + \cdots + \frac{c_1 v_0}{\theta} \quad + c_1 v_0 \sum_0^{\infty} \left(1 + v_i^2\right) exp - \theta t_i$$

$$|r_3\,(t)\,| \leq \ \mathfrak{P} + \mathfrak{P} N_3 t + \sum_{i=0}^{\infty} \mathfrak{P} \left(1 + w_i^2\right),$$

where

$$\mathfrak{P} := \left(\frac{kn\lambda_1 c_1^3 v_0^3 (N_2 - c_1 v_0)(1 - exp - 3\theta t)}{\theta^3}\right) \times \left[\, \tfrac{c_1 v_0}{\theta} + c_1 v_0 \textstyle\sum_0^{\infty} \left(1 + v_i^2\right) exp - \theta t_i \,\right]$$

$$\times exp - \lambda_1 z dz |u|_0.$$

and $r_1\,(t)\,, r_2\,(t)$ *and* $r_3\,(t)$ *are maximum solutions of the equation (3) respectively which is in fact the solution of the comparison equation (2).*

Proof. Maximum solution to the equation (3) can be obtain from the following estimates:

$$\frac{du}{dt} \leq \lambda_1 u + k\lambda_1 uv + kn\lambda_1 uv^2 + f_1(u, v, w)$$

$$\frac{du}{dt} \leq -\lambda_2 v + k\lambda uw + f_2(u, v, w)$$

$$\frac{du}{dt} \leq -\lambda w - k\lambda_3 w^3 + f_3(u, v, w)$$

$$\Delta u \leq (1 + u_i)$$

$$\Delta v \leq (1 + v_i)$$

$$\Delta w \leq (1 + w_i).$$

Solving the third inequality together with the impulse in the above inequality, we get the maximal solution for $u(t)$ as

$$r_1(t) = w(t) = \varphi(t, w_0) + \int_0^t \varphi(s, w_0) f_1(u, v, v) ds + \sum_{i=0}^{\infty} \varphi(t_i, w_0)(1 + u_i),$$

where $\varphi(t, w_0) = \dfrac{\lambda w_0}{e^{\chi t} nk\lambda_3 w_0 - \lambda knw_0 + \lambda e^{\lambda t}}$.
Let $V(t) = v_0 exp(\int_0^t nk_1 r_1(z) dz - \lambda_2) dz$ then

$$r_2(t) = V(t) + \int_0^t V(s) f_2(u, v, w) ds + \sum_{i=0}^{\infty} V(t_i)(1 + v_i^2).$$

Suppose that $U\,(t) \;=\; \left(\int_0^t kn\lambda_1 V\,(z_1)\, r_2\,(z_1)\, e^{-\lambda_1 z_1} dz_1 + u_0\right) e^{\lambda_1 t}$ then it is not difficult to show that

$$r_3(t) = U(t) + \int_0^t U(s) f_3(u, v, w) ds + \sum_{i=0}^{\infty} U(t_i)(1 + w_i^2).$$

Estimation:
Let $\theta = \lambda(\lambda_2 + nk_1c_1)$ and $K = \frac{|\lambda w_0|}{nk\lambda_3 w_0 + \lambda}$ then

$$|\varphi(t, w_0)| = \left| \frac{\lambda w_0}{e^{\chi t} nk\lambda_3 w_0 - \lambda knw_0 + \lambda e^{\lambda t}} \right| \leq Ke^{-\lambda t}$$

$$|r_1(t)| \leq Ke^{-\lambda t} + KM \int_0^t e^{-\lambda s} ds + kw_0 \sum_{i=0}^{\infty} ke^{-\lambda t_i}(1 - w_i^2)$$

$$\begin{aligned} |V(t)| &\leq c_1 v_0 exp(\int_0^t nk_1 |r_1(z)| dz - \lambda_2) dz \\ &\leq c_1 v_0 exp - \theta t; \ (1 - exp\lambda t \approx -\lambda t) \end{aligned}$$

$$\begin{aligned} |r_2(t)| &\leq |V(t)| + \int_0^t |V(t)| \, |f_2(u, v, w) ds + \sum_{i=0}^{\infty} |V(t)| \left(1 + v_i^2\right) \\ &\leq (c_1 v_0 + \frac{N_2}{\theta}) exp - \theta t + c_1 v_0 \int_0^t exp - \theta s ds + c_1 v_0 \sum_{i=0}^{\infty} \left(1 + v_i^2\right) \ exp - \theta t_i \\ &= (c_1 v_0 + \frac{N_2 - c_1 v_0}{\theta}) exp - \theta t + \frac{c_1 v_0}{\theta} \ + c_1 v_0 \sum_{i=0}^{\infty} \left(1 + v_i^2\right) exp - \theta t_i. \end{aligned}$$

Therefore,

$$\begin{aligned} |U(t)| \leq \ & kn\lambda_1 \left(\frac{c_1 v_0 (N_2 - c_1 v_0)}{\theta} \right) \int_0^t c_1^2 v_1^2 exp - 3\theta z \\ & \times \left[\frac{c_1 v_0}{\theta} \ + \ c_1 v_0 \sum_{i=0}^{\infty} \left(\ 1 + \ v_i^2 \right) exp - \theta t_i \right] \\ & \times exp - \lambda_1 z dz |u|_0 = \mathfrak{P}(\text{say}) \end{aligned}$$

$$\mathfrak{P} := \left(\frac{kn\lambda_1 c_1^3 v_0^3 (N_2 - c_1 v_0)(1 - exp - 3\theta t)}{\theta^3} \right) \times \left[\ \tfrac{c_1 v_0}{\theta} + c_1 v_0 \textstyle\sum_{i=0}^{\infty} \left(1 + v_i^2\right) \ exp - \theta t_i \ \right]$$
$$\times exp - \lambda_1 z dz |u|_0.$$

Then the estimation for $r_3(t)$ is as follows:

$$|r_3(t)| \leq \ \mathfrak{P} + \mathfrak{P} N_3 t + \sum_{i=0}^{\infty} \mathfrak{P}(1 + w_i^2).$$

This ends the proof. □

Theorem 4. ***(Stability)*** *Let the following conditions be satisfied:*

1. *The conditions in Theorem 3.*

2. There exist finite numbers $L_i \ \geq 0, k = 1, 2, 3$ such that

$$\lim_{n\to\infty}\sum_{i=0}^{n} ke^{-\lambda t_i}(1-u_i^2) \ = L_1, u_i \geq \frac{1}{\sqrt{2}}, \lambda > 0.$$

$$\lim_{n\to\infty}\sum_{i=0}^{n} ke^{-\lambda t_i}(1-v_i^2) \ = L_2, \ v_i \geq \frac{1}{\sqrt{2}}, \lambda > 0.$$

$$\lim_{n\to\infty}\sum_{i=0}^{n} ke^{-\lambda t_i}(1-w_i^2) \ = L_3, \ w_i \geq \frac{1}{\sqrt{2}}, \lambda > 0.$$

Then the solution of equation (1) is asymptotically stable.

Proof By the apriori estimate, since u, v, w are the solutions to the model passing through $u\,(0) = u_0, v\,(0) = v_0, w\,(0) = w_0$. Therefore, by standard results in literature (See [8,15]),

$$|u| \leq |r_1\,(t)|\,, \ |v| \leq |r_2\,(t)|\,, \ |w| \leq |r_3(t)|.$$

Let $|v_0| \leq |w_0| \leq |u_0|$. Thus

$$|u| \leq \ Ke^{-\lambda t} + KM\int_0^t e^{-\lambda s}ds + kw_0L_3$$

$$= k\left(1-\frac{m}{\lambda}\right)e^{-\lambda t} + \frac{km}{\lambda} + k^2w_0L.$$

Therefore, for every $\epsilon_3 > 0$ there exist $\delta_1 = \delta_1\,(\epsilon_3, w_0) > 0$ such that $|u_0| < \delta_1 \implies |u| < \epsilon_3$ for $t \geq t_0 + T_1(\eta)$ in this case $T_1\,(\eta) = \frac{1}{\lambda}\ln|\left[\frac{\lambda}{\lambda-m}\left(\epsilon_3 - \frac{km}{\lambda} + k^2L\delta_1\right)\right]|$. Similarly, for every $\epsilon_2 > 0$ there exists $\delta_2 = \delta_2\,(\epsilon_2, v_0) > 0$ such that $|v_0| < \delta_2 \implies |v| < \epsilon_2$ for $t \geq t_0 + T_2(\eta)$. In this case,

$$T_2\,(\eta) = \frac{1}{\theta(\ c_1v_0 + \frac{N_2-c_1v_0}{\theta})}\ln\left|\ \left[\epsilon_2 - \frac{C_1}{\theta}\delta_2 - C_1\delta_2L\right]\right|.$$

And finally, for every $\epsilon_1 > 0$ there exist $\delta_3 = \delta_3\,(\epsilon_1, u_0) > 0$ such that $|w_0| < \delta_3 \implies |w| < \epsilon_1$ for $t \geq t_0 + T_3(\eta)$. In this case,

$$T_3\,(\eta) = \ln\left|\left[1 - \frac{\delta_3}{\frac{kn\lambda_1c_1^3v_0^3(N_2-c_1v_0)}{\theta^3}\left(\frac{C_1}{\theta}\delta_3 + C_1L_3\delta_3\right)}\right]\ \right|.$$

Then for every $\epsilon > 0$ such that $\epsilon = \min(1, \epsilon_1, \epsilon_2, \epsilon_3)$, we choose $\delta = \min\,(\delta_1, \delta_2, \delta_3)$ and take $T = \max\,[T_1\,(\eta)\,, T_2\,(\eta)\,, T_3\,(\eta)]$. Therefore the solution of the equation (1) is asymptotically stable but not uniformly stable since T depends on δ_3.

Applications of pulse phenomenon

Military application: Consider a case study of battle between two warring nations A and B. Suppose that nation B launches a missile or drone against nation A. To prevent it from hitting the target in the territory of A is problem from a game of pursuit and capture. Pulse phenomenon can be utilized to counteract the missile or the drone from hitting target. This is done by predicting the trajectory of missile or drone from B by creating a dynamic hypersurface capture region where the counter missile from A can have the opportunity to hit the B missiles several times and destroy them. However, the missile from B must have an impulsive trajectory to avoid being intercepted by moving away from this capture region or else will be intercepted and destroyed. In the same vein, we look at the interaction between antigens and antibodies and the complex battle between two warring biological entities. Capturing and destroying antigens are the targets of antibodies. Pulse phenomenon must be invoked to provide a pathway for developing effective drugs and vaccines. Therefore, designing missiles with effective drugs trajectory and cross checking the efficiency of drugs and vaccines in future research will strategically need geometric modelling techniques incorporating variable times impulsive systems, and associated hyperspaces and pulse phenomenon. Most Research modeling of infectious diseases and immunology do not incorporate pulse phenomenon to investigate the behavior of the underlying variables in the models. They may do, but their focus does not extend to the clinical implications of pulse effects.

Conclusion

Computational modelling of antibody-antigen complexes is gaining research ground of late. This paper considered the concept of the absence of beating to analyse the behaviour of antigens and antibodies, especially beta-cells using the impulsive analogue Bell's equation. Using the comparison principle, the solution to the model is asymptotically stable under certain conditions but not uniformly stable. As a recommendation, the pulses can be explored in the design of missiles with impulsive trajectory and cross-checking the efficiencies of drugs and vaccines in future research using geometric modelling techniques.

Statement of Competing Interests

The authors declare that there is no conflict of interest.

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Best Strategy in Multi-agent Symmetric Dilemma Game with a Fixed Fine for Total Defiance

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ABSTRACT: A multi-agent symmetric dilemma game is studied, in which the agent possesses the two pure strategies of defiance and compliance. For total defiance each agent is fined for three conditional units, while the cost of complying with system requirements is set to one conditional unit. Whichever the number of agents is, every agent has the best defiance strategy, at which the agents' losses are identically minimized. This strategy is a mixed one, which is a nonzero probability of defiance. The probability can be approximated by using the bisection method with any accuracy desired, where the left and right endpoints for the method are 0 and the reciprocal of the number of agents decreased by 1, respectively.

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1. Multi-agent dilemma games

Dilemma games are used to model strategic interaction amongst two or more agents (or personified entities), whose action space is extremely narrowed down to just two options [12]. These options may be opposite or antagonistic by nature, but not always [1]. In general, they are mutually complementary, where selecting one option excludes the other one [2]. The classical dilemma game is the prisoner's dilemma with its cooperation and defection options left for two agents [11]. This game has been widely used to model strategic behavior in many fields such as economics [12], business [2], ecology [7], sociology [11], biology [1], networking [6], traffic engineering [12], computer science [8], and even artificial intelligence [15].

The options to defy or comply with system requirements (suggestions, conditions, restrictions, etc.) are the general dilemma game strategies [13]. While any dilemma game being a finite one has a solution in either pure or mixed strategies, the solution type is a matter of continuing discussions and debates [11]. This is reasoned by that the dilemma game is a non-cooperative one, so a game solution must be attractive and acceptable with similar thoroughness for every agent. The Nash equilibrium solution is formally such one, but in practice of symmetric games (where every agent has the same consequences of one's actions), when there are two equilibria with asymmetric payoffs, the selection of the best strategy among those two shakes the expected equilibrium [14]. This could result in a new game, a metagame whose pure strategies are those equilibria, but agents naturally search for simpler decisions. The simplest decision is to apply a strategy, which would be the same for every agent and ensure the maximized profit. Unlike the general Nash equilibrium solution type, such a strategy provides fairness of payoffs. Recent studies in sociology and psychology confirm that symmetry and fairness attract agents more than equilibrium whose either asymmetry or multiplicity makes it unstable and subsequently repelling [10].

2. Motivation and objective

Two-agent dilemma games, known better as prisoner's dilemma games [2], are thoroughly studied with a lot of aspects concerning the agent's capabilities, dispositions, inclinations, propensities, etc. [4]. Three-agent dilemma games have been studied as well [3], although not such widely as prisoner's dilemma games. The main purpose is to determine the agent's best strategy which would allow to fairly minimize expenses of every agent under a set of definite fines imposed on agents whenever they defy system requirements. This has been done for dilemma games with two [11], three [7], and four agents [9]. Nevertheless, the general case of the number of agents has not been solved yet.

Hence, the objective is to find the most attractive and acceptable strategy for every agent in the multi-agent symmetric dilemma game with a fixed fine of three conditional units for total defiance (ignorance of system requirements). The cost of complying with system requirements is set to one conditional unit. The agent's payoff is refined to the loss that includes possible cost of compliance and fine for defiance. The objective will be completed upon accomplishing the following tasks. First, the dilemma game is formalized. Second, the agent's losses are defined for all pure strategy situations. Then pure strategy equilibrium and efficient situations are to be determined with a purpose to find out whether the dilemma game has an appropriate equilibrium or efficient solution in pure strategies. Next, the agent's expected loss in the game mixed extension is deduced, whereupon the expected loss is minimized by any number of agents. Finally, a conclusion on the minimum point, if any, is made with an outlook for possible further research or supplement.

3. Dilemma strategies and game

Consider an N-agent dilemma game, $N \in \mathbb{N}\backslash\{1,\ 2\}$, in which every agent has the same consequences of one's actions. Therefore, this game is symmetric. Denote by x_i a pure strategy of agent i, $i = \overline{1,\ N}$, where [7]

$$x_i \in \{0,\ 1\}\,. \tag{1}$$

Strategy $x_i = 0$ is compliance (agent i complies with system requirements), and strategy $x_i = 1$ is defiance (agent i defies system requirements). In a pure strategy situation $\{x_i\}_{i=1}^{N}$ the loss of agent i is a nonnegative value

$$L_i\left(\{x_k\}_{k=1}^{N}\right). \tag{2}$$

With (1) and (2), the dilemma game is defined on an N-dimensional lattice

$$\mathop{\times}_{i=1}^{N} \{0,\ 1\}\,. \tag{3}$$

Formally, the game is

$$\left\langle \{\{0,\ 1\}\}_{i=1}^{N}\,,\ \left\{L_i\left(\{x_k\}_{k=1}^{N}\right)\right\}_{i=1}^{N} \right\rangle \tag{4}$$

whose loss functions (2), $i = \overline{1,\ N}$, are yet to be defined by taking into account the symmetry of game (4).

4. Agent's losses

As game (4) is symmetric, it is proper to define all the pure strategy situations that are symmetric. Owing to (3), there are only two such situations.

Definition 1. Pure strategy situation

$$\{0\}_{i=1}^{N} \tag{5}$$

is called the total compliance situation.

In total compliance situation (5) every agent loses a conditional unit:

$$L_i\left(\{0\}_{k=1}^{N}\right) = 1 \ \ \forall\, i = \overline{1,\ N}. \tag{6}$$

This unit is the cost of compliance.

Definition 2. Pure strategy situation

$$\{1\}_{i=1}^{N} \tag{7}$$

is called the total defiance situation.

In total defiance situation (7) every agent loses three conditional units [9]:

$$L_i \left(\{1\}_{k=1}^{N} \right) = 3 \;\; \forall\, i = \overline{1,\ N}. \tag{8}$$

This is the cost of total defiance, by which every agent is properly fined for defying regardless of whether there is a collusion or not.

The fine is not imposed if only one agent defies. Hence, the respective situations must be given a definition as well.

Definition 3. Pure strategy situation

$$D_1\left(k_1\right) = \left\{ \{x_k\}_{k=1}^{N} : x_k = 0 \;\forall\, k \in \left\{\overline{1,\ N}\right\} \setminus \{k_1\} \text{ and } x_{k_1} = 1 \right\} \\ \text{by } \; k_1 \in \left\{\overline{1,\ N}\right\} \tag{9}$$

is called a 1-defiance situation.

It is clear that there are N 1-defiance situations

$$\left\{ D_1\left(k_1\right) \right\}_{k_1=1}^{N} \tag{10}$$

in game (4). If an agent complies in a 1-defiance situation, the agent loses a conditional unit. Otherwise, if an ("impudent") agent defies in an 1-defiance situation, the agent's loss is 0. So,

$$L_i\left(D_1\left(k_1\right)\right) = 1 - x_i \;\; \forall\, i = \overline{1,\ N} \;\text{ and }\; \forall\, k_1 = \overline{1,\ N} \tag{11}$$

in 1-defiance situation (9) with the "impudent" agent k_1, whose loss, obviously, is

$$L_{k_1}\left(D_1\left(k_1\right)\right) = 0. \tag{12}$$

If more than just one agent defies, the fine is imposed regardless of whether the agent defies or complies. The subset of such situations is much wider than subset (10).

Definition 4. Every pure strategy situation of subset

$$\bar{\mathscr{C}} = \left\{ \{x_k\}_{k=1}^{N} : x_{k_0} = 0 \;\forall\, k_0 \in \{j_i\}_{i=1}^{N_0} \text{ and } x_{k_1} = 1 \;\forall\, k_1 \in \{j_i\}_{i=N_0+1}^{N} \\ \text{by } 1 \leqslant N_0 < N - 1 \right\} \tag{13}$$

is called a non-compliance situation.

In a non-compliance situation $\bar{C} \in \bar{\mathscr{C}}$ by (13), where N_0 agents comply and $N - N_0$ agents defy, the loss of agent i is

$$L_i\left(\bar{C}\right) = 4 - x_i \;\; \forall\, i = \overline{1,\ N} \;\text{ and }\; \forall\, \bar{C} \in \bar{\mathscr{C}}. \tag{14}$$

Obviously, subset (13) does not include 1-defiance situations. Overall, game (4) has 2^N pure strategy situations: total compliance situation (5), total defiance situation (7), N 1-defiance situations (10), and $2^N - 2 - N$ non-compliance situations of subset (13).

5. Pure strategy equilibrium and efficient situations

Now, it is about to find pure strategy equilibria in dilemma game (4) with agents' losses (6), (8), (11), (14). The purpose is to ascertain whether the game has a symmetric pure strategy equilibrium or not.

Theorem 1. *Dilemma game (4) with agents' losses (6), (8), (11), (14) has* $N+1$ *pure strategy equilibria, which are total defiance situation (7) and* N *1-defiance situations (10).*

Proof. Suppose that an agent in total defiance situation (7) with losses (8) switches from defiance to compliance. Then, due to (14), the agent's loss in the respective non-compliance situation from subset (13) increases by 1. Therefore, total defiance situation (7) is an equilibrium situation.

Suppose that agent k_1 in total compliance situation (5) with losses (6) switches from compliance to defiance. Then, owing to (12), the agent's loss in the respective 1-defiance situation (9) drops down to 0. Therefore, total compliance situation (5) is not an equilibrium situation. This means also that 1-defiance situation (9) is acceptable for "impudent" agent k_1. If a complying agent in 1-defiance situation (9) switches to defiance, then, due to (14), the agent's loss in the respective non-compliance situation from subset (13) with the two defying agents increases by 2. Therefore, every 1-defiance situation out of subset (10) is an equilibrium situation.

In a non-compliance situation there are at least two defying agents and at most $N-1$ defying agents which lose three conditional units each. So, at least one agent complies and at most $N-2$ agents comply. Suppose that a single-complying agent switches to defiance. Then the former non-compliance situation is changed into total defiance situation (7). If there are two or more complying agents in a non-compliance situation and one of them switches to defiance, then the former non-compliance situation is changed into another non-compliance situation, in which the loss of the switching-to-defiance agent decreases by 1. Therefore, any non-compliance situation out of subset (13) is not an equilibrium situation. □

As any 1-defiance situation is asymmetric, according to Theorem 1, the single symmetric equilibrium situation is total defiance situation (7). Nevertheless, total defiance situation (7) is absolutely unattractive, even repelling, situation due to the agent's loss is higher than that in any 1-defiance situation, being asymmetric, though (regardless of the strategy the agent applies). Moreover, the agents' summed loss in every 1-defiance situation is equal to $N-1$, whereas the summed loss in total defiance situation (7) is always higher being equal to $3N$. This is why dilemma game (4) does not have an appropriate equilibrium solution in pure strategies.

It is also worth mentioning that every 1-defiance situation is a Pareto efficient situation. Apart from subset (10), game (4) does not have any other efficient situations. Indeed, total defiance situation (7) losses are higher than those in any 1-defiance situation, and the agent's loss in total compliance situation (5) is higher than the loss of an "impudent" agent in a 1-defiance situation. Hence, any symmetric pure strategy situation is not efficient. Owing to (14), any of the remaining $2^N - 2 - N$ pure strategy situations being non-compliance ones is dominated by every 1-defiance situation.

6. Expected losses in mixed extension

In the mixed extension of dilemma game (4), denote by p_i the probability of defiance of agent i (i.e. agent i defies with probability p_i and thus complies with probability $1 - p_i$). Then

$$\{p_i\}_{i=1}^{N} \tag{15}$$

is a mixed strategy situation in this game, where probability p_i is a mixed strategy of agent i. It is worth noting that mixed strategies $p_i = 0$ and $p_i = 1$ correspond to pure strategies $x_i = 0$ and $x_i = 1$, respectively.

The expected loss of agent i in situation (15) is

$$l_i\left(\{p_k\}_{k=1}^{N}\right) = \sum_{\substack{x_j \in \{0,\,1\} \\ j=\overline{1,\,N}}} \prod_{k=1}^{N} p_k^{x_k} (1 - p_k)^{1 - x_k} \cdot L_i\left(\{x_j\}_{j=1}^{N}\right) \text{ for } i = \overline{1,\,N}. \tag{16}$$

Inasmuch as the agents will search for a solution among symmetric mixed strategy situations, then it is proper to further consider a symmetric mixed strategy situation

$$\{p\}_{k=1}^{N} \tag{17}$$

with a probability p of every agent's defiance. This simplifies expected loss (16) of agent i in situation (17) to

$$l_i\left(\{p\}_{k=1}^{N}\right) = l(p) = \sum_{\substack{x_j \in \{0,\,1\} \\ j=\overline{1,\,N}}} \prod_{k=1}^{N} p^{x_k} (1 - p)^{1 - x_k} \cdot L_i\left(\{x_j\}_{j=1}^{N}\right). \tag{18}$$

In terms of denotation of (18) that expresses the loss of every agent via function $l(p)$, losses

$$l(0) = 1 \text{ and } l(1) = 3 \tag{19}$$

due to (6) and (8). The best strategy of every agent should be a probability p^* such that

$$l(p^*) = \min_{p \in [0;\,1]} l(p). \tag{20}$$

In addition, the best strategy must satisfy inequality

$$l(p^*) < 1 \tag{21}$$

unless total compliance situation (5) is the best strategy.

Theorem 2. *Expected loss (18) of every agent in dilemma game (4) with agents' losses (6), (8), (11), (14) can be written as*

$$l\left(p\right)=3p^{N}+ \\ +\sum_{h=2}^{N-1}\frac{\left(4N-h\right)\left(N-1\right)!}{\left(N-h\right)!\cdot h!}\cdot p^{h}\left(1-p\right)^{N-h}+ \\ +\left(N-1\right)\cdot p\left(1-p\right)^{N-1}+\left(1-p\right)^{N}. \tag{22}$$

Proof. The sum in (18) has 2^N summands, each corresponding to a pure strategy situation $\{x_i\}_{i=1}^{N}$ with its respective agent's loss. The summand corresponding to total defiance situation (7) is

$$\prod_{k=1}^{N}p^{1}\left(1-p\right)^{1-1}\cdot L_i\left(\{1\}_{j=1}^{N}\right)=3p^{N}. \tag{23}$$

The summand corresponding to total compliance situation (5) is

$$\prod_{k=1}^{N}p^{0}\left(1-p\right)^{1-0}\cdot L_i\left(\{0\}_{j=1}^{N}\right)=\left(1-p\right)^{N}. \tag{24}$$

The remaining 2^N-2 summands depend on the number of zeros (compliances) and ones (defiances) in a pure strategy situation. As expected loss (18) is the same for every agent, it is sufficient to continue considering the summands for one agent. When the agent complies, there are $N-1$ summands corresponding to 1-defiance situations, in which this agent is not the "impudent" one:

$$\left(N-1\right)\cdot p\left(1-p\right)^{N-1}. \tag{25}$$

While the agent still complies in a non-compliance situation (hence, there are at least two defying agents in a pure strategy situation), this agent's loss is 4 and there are 1 to $N-2$ ones in such non-compliance situations, for which the sum of the respective summands is

$$\sum_{h=2}^{N-1}4\cdot\frac{\left(N-1\right)!}{\left(N-1-h\right)!\cdot h!}\cdot p^{h}\left(1-p\right)^{N-h}. \tag{26}$$

When this agent defies in the 1-defiance situation, not included into the abovementioned $N-1$ 1-defiance situations, the agent's loss is 0. When this agent defies in a non-compliance situation, the agent's loss is 3 and there are 2 to $N-1$ ones in such situations, for which the sum of the respective summands is

$$\sum_{h=1}^{N-2}3\cdot\frac{\left(N-1\right)!}{\left(N-h-1\right)!\cdot h!}\cdot p^{h+1}\left(1-p\right)^{N-h-1}. \tag{27}$$

The two sums in (26), (27) can be unified by changing the running index in (27) to $h = \overline{2,\ N-1}$:

$$\sum_{h=2}^{N-1} 4 \cdot \frac{(N-1)!}{(N-1-h)! \cdot h!} \cdot p^h (1-p)^{N-h} +$$
$$+ \sum_{h=1}^{N-2} 3 \cdot \frac{(N-1)!}{(N-h-1)! \cdot h!} \cdot p^{h+1} (1-p)^{N-h-1} =$$
$$= \sum_{h=2}^{N-1} \left(4 \cdot \frac{(N-1)!}{(N-1-h)! \cdot h!} + 3 \cdot \frac{(N-1)!}{(N-h)! \cdot (h-1)!} \right) \cdot p^h (1-p)^{N-h} =$$
$$= \sum_{h=2}^{N-1} \left(4 \cdot (N-h) \cdot \frac{(N-1)!}{(N-h)! \cdot h!} + 3h \cdot \frac{(N-1)!}{(N-h)! \cdot h!} \right) \cdot p^h (1-p)^{N-h} =$$
$$= \sum_{h=2}^{N-1} \frac{(4N-h)(N-1)!}{(N-h)! \cdot h!} \cdot p^h (1-p)^{N-h}. \tag{28}$$

Adding up (23) — (28), expression (22) of expected loss (18) is finally obtained. □

Although Theorem 2 gives an explicit expression of the agent's loss, it would be very hard to work with the loss function presented as (22). Thankfully, expected loss (22) can be dramatically simplified.

Theorem 3. *Expected loss (22) of every agent in dilemma game (4) with agents' losses (6), (8), (11), (14) is*

$$l(p) = 4 - 3 \cdot (1-p)^N - 3Np(1-p)^{N-1} - p. \tag{29}$$

Proof. The second term in (22) is a binomial-like sum, which can be broken into two parts:

$$\sum_{h=2}^{N-1} \frac{(4N-h)(N-1)!}{(N-h)! \cdot h!} \cdot p^h (1-p)^{N-h} =$$
$$= \sum_{h=2}^{N-1} \frac{4N(N-1)!}{(N-h)! \cdot h!} \cdot p^h (1-p)^{N-h} - \sum_{h=2}^{N-1} \frac{h(N-1)!}{(N-h)! \cdot h!} \cdot p^h (1-p)^{N-h} =$$
$$= \sum_{h=2}^{N-1} \frac{4N!}{(N-h)! \cdot h!} \cdot p^h (1-p)^{N-h} -$$
$$- \sum_{h=2}^{N-1} \frac{(N-1)!}{(N-h)! \cdot (h-1)!} \cdot p^h (1-p)^{N-h}. \tag{30}$$

Note that binomial sum

$$\sum_{h=0}^{N} \frac{N!}{(N-h)! \cdot h!} \cdot p^h (1-p)^{N-h} = (p + (1-p))^N = 1. \tag{31}$$

Then the first term in (30), using (31), is

$$4 \cdot \sum_{h=2}^{N-1} \frac{N!}{(N-h)! \cdot h!} \cdot p^h (1-p)^{N-h} =$$
$$= 4 \cdot \left(1 - \frac{N!}{N! \cdot 0!} \cdot p^0 (1-p)^{N-0} - \frac{N!}{(N-1)! \cdot 1!} \cdot p^1 (1-p)^{N-1} - \right.$$
$$\left. - \frac{N!}{(N-N)! \cdot N!} \cdot p^N (1-p)^{N-N} \right) =$$
$$= 4 \cdot \left(1 - (1-p)^N - Np(1-p)^{N-1} - p^N\right). \tag{32}$$

The second term in (30) is reduced to an incomplete binomial sum by substitution $h - 1 = k$:

$$\sum_{h=2}^{N-1} \frac{(N-1)!}{(N-h)! \cdot (h-1)!} \cdot p^h (1-p)^{N-h} =$$
$$= \sum_{k=1}^{N-2} \frac{(N-1)!}{(N-k-1)! \cdot k!} \cdot p^{k+1} (1-p)^{N-k-1} =$$
$$= p \sum_{k=1}^{N-2} \frac{(N-1)!}{(N-k-1)! \cdot k!} \cdot p^k (1-p)^{N-k-1} =$$
$$= p \left(1 - \frac{(N-1)!}{(N-1)! \cdot 0!} \cdot p^0 (1-p)^{N-0-1} - \frac{(N-1)!}{0! \cdot (N-1)!} \cdot p^{N-1} (1-p)^0 \right) =$$
$$= p \left(1 - (1-p)^{N-1} - p^{N-1}\right). \tag{33}$$

With (32) and (33) plugged into (30) expected loss (22) becomes

$$l(p) = 3p^N + 4 \cdot \left(1 - (1-p)^N - Np(1-p)^{N-1} - p^N\right) -$$
$$-p\left(1 - (1-p)^{N-1} - p^{N-1}\right) + (N-1) \cdot p(1-p)^{N-1} + (1-p)^N =$$
$$= 3p^N + 4 - 4 \cdot (1-p)^N - 4Np(1-p)^{N-1} - 4p^N -$$
$$-p + p(1-p)^{N-1} + p^N + (N-1) \cdot p(1-p)^{N-1} + (1-p)^N =$$
$$= 4 - 3 \cdot (1-p)^N - 3Np(1-p)^{N-1} - p,$$

completing the proof. □

Obviously, owing to Theorem 3, the presentation of the agent's expected loss in symmetric situation (17) as function (29) is a way more convenient to analyze than the loss presentation by (22).

7. Loss minimum

Before starting to solve minimization problem (20) and seeing whether inequality (21) holds, it is proper to know how many minima expected loss (29) has.

Theorem 4. *Expected loss (29) of every agent in dilemma game (4) with agents' losses (6), (8), (11), (14) has a single minimum and a single maximum which belong to interval* (0; 1).

Proof. The first derivative of expected loss (29) is

$$\frac{dl}{dp} = 3N\left(1-p\right)^{N-1} - 3N\left(1-p\right)^{N-1} + 3Np\left(N-1\right)\left(1-p\right)^{N-2} - 1 =$$
$$= 3Np\left(N-1\right)\left(1-p\right)^{N-2} - 1. \tag{34}$$

Find zeros of function (34):

$$3Np\left(N-1\right)\left(1-p\right)^{N-2} - 1 = 0 \tag{35}$$

if

$$p\left(1-p\right)^{N-2} = \frac{1}{3N\left(N-1\right)}. \tag{36}$$

Denote the left side in (36) by a function

$$\psi\left(p\right) = p\left(1-p\right)^{N-2}. \tag{37}$$

The first derivative of function (37) is

$$\frac{d\psi}{dp} = \left(1-p\right)^{N-2} - p\left(N-2\right)\left(1-p\right)^{N-3} =$$
$$= \left(1-p\right)^{N-3}\left(1-p-pN+2p\right) = \left(1-p\right)^{N-3}\left(1+p-pN\right). \tag{38}$$

Find zeros of function (38):

$$\left(1-p\right)^{N-3}\left(1+p-pN\right) = 0$$

if

$$p = 1 \tag{39}$$

or

$$1 + p - pN = 0,$$

whence

$$p = \frac{1}{N-1}. \tag{40}$$

Hence, function (37) having zeros at $p = 0$ and (39) has two stationary points — (39) and (40). The second derivative of function (37) is

$$\frac{d^2\psi}{dp^2} = -\left(N-3\right)\left(1-p\right)^{N-4}\left(1+p-pN\right) + \left(1-p\right)^{N-3}\left(1-N\right) =$$

$$= (1-p)^{N-4}\left(3 - N + 3p - pN - 3pN + pN^2 + 1 - p - N + pN\right) =$$
$$= (1-p)^{N-4}\left(4 - 2N + 2p - 3pN + pN^2\right). \quad (41)$$

It is clear that

$$\left.\frac{d^2\psi}{dp^2}\right|_{p=1} = 0,$$

but

$$\left.\frac{d^2\psi}{dp^2}\right|_{p=\frac{1}{N-1}} = \left(1 - \frac{1}{N-1}\right)^{N-4}\left(4 - 2N + \frac{2}{N-1} - \frac{3N}{N-1} + \frac{N^2}{N-1}\right) =$$
$$= \left(\frac{N-2}{N-1}\right)^{N-4} \cdot \frac{4N - 4 - 2N^2 + 2N + 2 - 3N + N^2}{N-1} =$$
$$= \left(\frac{N-2}{N-1}\right)^{N-4} \cdot \frac{3N - 2 - N^2}{N-1} =$$
$$= -\left(\frac{N-2}{N-1}\right)^{N-4} \cdot \frac{(N-2)(N-1)}{N-1} =$$
$$= -\frac{(N-2)^{N-3}}{(N-1)^{N-4}} < 0. \quad (42)$$

Inequality (42) means that (40) is the single maximum point of function (37). At this maximum point function (37) reaches maximum value

$$\psi\left(\frac{1}{N-1}\right) = \frac{(N-2)^{N-2}}{(N-1)^{N-1}}. \quad (43)$$

Now, relationship between maximum (43) and the right side in (36) should be investigated. The ratio of maximum (43) to the right side in (36) is

$$\frac{(N-2)^{N-2}}{(N-1)^{N-1}} \cdot 3N(N-1) = 3N \cdot \left(\frac{N-2}{N-1}\right)^{N-2}. \quad (44)$$

Analyze function

$$\zeta(N) = \left(\frac{N-2}{N-1}\right)^{N-2}. \quad (45)$$

The first derivative of function (45) is

$$\frac{d\zeta}{dN} = \left(\frac{N-2}{N-1}\right)^{N-2} \cdot \left(\ln\left(\frac{N-2}{N-1}\right) + \left(\frac{1}{N-1} - \frac{N-2}{(N-1)^2}\right)(N-1)\right) =$$
$$= \left(\frac{N-2}{N-1}\right)^{N-2} \cdot \left(\ln\left(\frac{N-2}{N-1}\right) + \frac{1}{N-1}\right). \quad (46)$$

In (46),

$$\ln\left(\frac{N-2}{N-1}\right)+\frac{1}{N-1}<0 \tag{47}$$

if

$$\frac{1}{N-1}<\ln\left(\frac{N-1}{N-2}\right),$$

whence

$$e^{\frac{1}{N-1}}<\frac{N-1}{N-2}. \tag{48}$$

The term in the left side of inequality (48) has the expansion into the Maclaurin series:

$$e^{\frac{1}{N-1}}=\sum_{n=0}^{\infty}\frac{1}{n!\cdot(N-1)^n}. \tag{49}$$

The term in the right side of inequality (48) can be re-written as

$$\frac{N-1}{N-2}=(N-1)\cdot\frac{1}{(N-1)-1}=$$
$$=(N-1)\cdot\frac{1}{N-1}\cdot\frac{1}{1-\dfrac{1}{N-1}}=\frac{1}{1-\dfrac{1}{N-1}}. \tag{50}$$

Using the geometric series expansion

$$\frac{1}{1-x}=\sum_{n=0}^{\infty}x^n \quad \text{for} \quad |x|<1,$$

where

$$\left|\frac{1}{N-1}\right|<1,$$

the last term in (50) has the expansion into the Maclaurin series as well:

$$\frac{N-1}{N-2}=\frac{1}{1-\dfrac{1}{N-1}}=\sum_{n=0}^{\infty}\frac{1}{(N-1)^n}. \tag{51}$$

Term-by-term comparison of series (49) to series (51) for $N\in\mathbb{N}\setminus\{1,\ 2\}$ allows to conclude that inequality (48) is true. Hence, inequality (47) holds as well and, therefore, first derivative (46) of function (45) is ever negative, so function (45) is decreasing. Its maximum value is

$$\max_{N\geqslant 3}\zeta(N)=\zeta(3)=\frac{1}{2}$$

and its minimum value is never reached, but

$$\inf_{N\geqslant 3}\zeta(N)=\lim_{N\to\infty}\zeta(N)=\lim_{N\to\infty}\left(\frac{N-2}{N-1}\right)^{N-2}=$$

$$= \lim_{N\to\infty}\left(1-\frac{1}{N-1}\right)^{N-2} = \lim_{N\to\infty}\left(\left(1-\frac{1}{N-1}\right)^{N-1}\cdot\left(1-\frac{1}{N-1}\right)^{-1}\right) =$$

$$= \left(\lim_{N\to\infty}\left(1-\frac{1}{N-1}\right)^{N-1}\right)\cdot\left(\lim_{N\to\infty}\left(1-\frac{1}{N-1}\right)^{-1}\right) =$$

$$= e^{-1}\cdot 1 = e^{-1},$$

whence value (44) satisfies inequality

$$3N\cdot\left(\frac{N-2}{N-1}\right)^{N-2} > \frac{3N}{e} > 1. \tag{52}$$

Inequality (52) means that maximum value (43) of function (37) is greater than the right side in (36):

$$\frac{(N-2)^{N-2}}{(N-1)^{N-1}} > \frac{1}{3N\,(N-1)} > 0. \tag{53}$$

Inequality (53) along with the two zeros of polynomial function (37) at $p = 0$ and (39), between which the single maximum point of function (37) exists, implies that equation (36) must have only two roots belonging to interval (0; 1). Therefore, equation (35) has two roots existing within interval (0; 1) and first derivative (34) thus has only two zeros existing within interval (0; 1). So, expected loss (29) has only two extrema. Inasmuch as it is a polynomial, these extrema are a single minimum and a single maximum within interval (0; 1). □

Now, having Theorem 4, it is far easier to find the expected loss minimum quality and ascertain the relative location of the single minimum point p^*.

Theorem 5. *The single minimum of expected loss (29) of every agent in dilemma game (4) with agents' losses (6), (8), (11), (14) satisfies inequality (21), and the minimum point is such that*

$$0 < p^* < \frac{1}{N-1} \leqslant \frac{1}{2}. \tag{54}$$

Proof. Inasmuch as inequality (53) holds, and polynomial equation (35) with respect to p has only two roots $p^{(1)}$ and $p^{(2)}$ within interval (0; 1), where

$$0 < p^{(1)} < p^{(2)} < 1, \tag{55}$$

then inequalities

$$p\,(1-p)^{N-2} < \frac{1}{3N\,(N-1)} \quad \text{by} \quad p \in \left[0;\ p^{(1)}\right) \tag{56}$$

and

$$p\,(1-p)^{N-2} > \frac{1}{3N\,(N-1)} \quad \text{by} \quad p \in \left(p^{(1)};\ p^{(2)}\right) \tag{57}$$

and

$$p\left(1-p\right)^{N-2} < \frac{1}{3N\left(N-1\right)} \ \text{ by } \ p \in \left(p^{(2)};\ 1\right] \tag{58}$$

hold as well. Inequalities (56) — (58) imply that inequalities

$$\frac{dl}{dp} < 0 \ \text{ by } \ p \in \left[0;\ p^{(1)}\right) \tag{59}$$

and

$$\frac{dl}{dp} > 0 \ \text{ by } \ p \in \left(p^{(1)};\ p^{(2)}\right) \tag{60}$$

and

$$\frac{dl}{dp} < 0 \ \text{ by } \ p \in \left(p^{(2)};\ 1\right] \tag{61}$$

hold, where the pair of inequalities (59) and (60) mean that point $p^* = p^{(1)}$ is the single minimum point of expected loss (29), and the pair of inequalities (60) and (61) mean that point $p^{(2)}$ is its single maximum point. Hence, minimum point $p^{(1)}$ lies between 0 and (40), where the latter is such that

$$\frac{1}{N-1} \in \left(p^{(1)};\ p^{(2)}\right).$$

This is followed by inequality (54). Besides, inequality (21) holds owing to (19) and (20) for polynomial function (29). □

To find the loss minimum for a given number of agents N, it is sufficient to solve a polynomial equation (35), which has just two real roots

$$p^{(1)} \in \left(0;\ \frac{1}{N-1}\right)$$

and

$$p^{(2)} \in \left(\frac{1}{N-1};\ 1\right)$$

within interval (0; 1) owing to Theorem 5. The lesser root is the minimum point being the agent's best strategy, and the minimum loss is

$$l\left(p^*\right) = l\left(p^{(1)}\right) = 4 - 3 \cdot \left(1 - p^{(1)}\right)^N - 3Np^{(1)}\left(1 - p^{(1)}\right)^{N-1} - p^{(1)}.$$

For instance, in the three-agent symmetric dilemma game, equation (35) is just a quadratic one

$$18p\left(1-p\right) - 1 = 0,$$

whose roots are

$$p^{(1)} = \frac{3-\sqrt{7}}{6} = p^*, \ \ p^{(2)} = \frac{3+\sqrt{7}}{6},$$

and the minimum loss is

$$l\left(p^{*}\right)=l\left(p^{(1)}\right)=\frac{36-7\sqrt{7}}{18}$$

by

$$0.05904144<p^{(1)}=\frac{3-\sqrt{7}}{6}<0.05904145.$$

In the four-agent symmetric dilemma game, equation (35) is just a cubic one

$$36p\left(1-p\right)^{2}-1=0. \tag{62}$$

Equation (62) is easily transformed into a depressed cubic equation [5], which has three real roots, two of which belong to interval (0; 1). They are those $p^{(1)}$ and $p^{(2)}$. Nonetheless, Theorem 5 allows to localize the loss minimum point in an N-agent dilemma game (4) with agents' losses (6), (8), (11), (14), for any $N\in\mathbb{N}\backslash\{1,\,2\}$, in an easier way, by using the bisection method. Indeed,

$$\left.\frac{dl}{dp}\right|_{p=0}=-1<0 \tag{63}$$

and

$$\left.\frac{dl}{dp}\right|_{p=\frac{1}{N-1}}>0 \tag{64}$$

owing to (34) and (35) — (37), (43), (53), so loss minimum point p^{*} can be approximated by using the bisection method on first derivative (34) with any accuracy desired. Thus, loss minimum point p^{*} in the four-agent symmetric dilemma game, belonging to interval $\left(0;\ \frac{1}{3}\right)$, is approximately determined in 38 iterations with an accuracy of 10^{-12}:

$$0.02949164<p^{*}<0.029491641, \tag{65}$$

where the exact value is

$$p^{*}=\frac{2}{3}-\frac{2}{3}\cos\left(\frac{1}{3}\arccos\left(\frac{5}{8}\right)\right)$$

in this game. However, it is quite obvious that estimation (65) is more than sufficient for practical implementation.

8. Conclusion

The multi-agent symmetric dilemma game with a fixed fine of three conditional units for total defiance has a perfect solution, which is identically attractive and acceptable for every agent. The solution is to defy with a nonzero probability, whose value can be approximated by the bisection method with respect to the first derivative of the loss function, where the probability lies between 0 and the reciprocal of the number of agents decreased by 1. The exact value of the probability is the lesser zero of the first derivative of the loss function, that is the lesser root of the respective polynomial equation. The equation has only two real roots belonging to interval (0; 1), whichever the number of agents is.

Further research should be directed towards studying the general case of total defiance fining. Besides, the presented research must be supplemented with an analysis of the best strategy trend depending on the number of agents. Although Theorem 5 clearly states that, as the number of agents increases, the open interval of searching for the best defiance probability narrows (to the left, because the left endpoint of this interval, i. e. point 0, remains the same), it is to be proved yet that the probability decreases. Moreover, it would be interesting to obtain the pattern of the decrement.

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A Structure Theorem for Order Martingales

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Abstract: We prove that an order martingale is the sequential order limit of a suitable family of projection band-type regular martingales.

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Keywords and Phrases: Vector lattice; Projection band; Order martingale.

1. Introduction

Initiated by R. DeMarr and G. Stoica, martingales in vector lattices (or Riesz spaces) were developed and studied in recent years by C.C.A. Labuschagne, J.J. Grobler, B.A. Watson, W.C. Kuo, V.G. Troitsky et al. In the original version (see [3]), a sequence $(x_n)_{n\geq 1}$ in a vector lattice $(\mathcal{X}, \leq)$ is called an *order martingale* if, for a family $(\mathcal{E}_n)_{n\geq 1}$ of projection bands in $\mathcal{X}$, we have $E_n(x_{n+1}) = x_n$ for every $n \geq 1$. (Recall that a projection band $\mathcal{E}$ in a vector lattice $\mathcal{X}$ satisfies the direct sum condition $\mathcal{X} = \mathcal{E} \oplus \mathcal{E}^{\perp}$, where $\mathcal{E}^{\perp} := \{y \in \mathcal{X}, |y| \wedge |x| = 0 \text{ for all } x \in \mathcal{E}\}$ is the order orthogonal of $\mathcal{E}$, see [1], p. 39; and the projection band $\mathcal{E}$ generates a linear, non-negative, order bounded and continuous projection operator, denoted by E). In other words, the next value x_{n+1} in the sequence is equal to the present observed value x_n even given knowledge of all prior observed values gathered in $\mathcal{E}_n$. Typical examples are the *regular order martingales* $(E_n(x))_{n\geq 1}$, for any $x \in \mathcal{X}$ and any family of projection bands $(\mathcal{E}_n)_{n\geq 1}$.

The limit laws obtained in [6] show that the asymptotic behaviour of martingales in the order topology require appropriate normalizations and boundedness conditions involving their increments. The limit laws for *regular* order martingales (cf. [4]) are much nicer, in that no normalizations and no restrictions on their increments are required. This phenomenon led us to study the inner structure of order martingales, and the purpose of this paper is to prove that such martingales can be written as the sequential order limit of a suitable family of projection band-type regular martingales.

2. Main result

To formulate and prove our main result, we need some preparatory material. The sequence $(x_i)_{i\geq 1}$ in a vector lattice $\mathcal{X}$ is said to converge in order to $x \in \mathcal{X}$ as $i \to \infty$, in short, $x = (\mathrm{o}) - \lim\limits_{i\to\infty} x_i$, whenever $|x_i - x| \leq v_i$ for all $i \geq 1$, where the sequence $(v_i)_{i\geq 1}$ in $\mathcal{X}$ decreases towards 0 as $i \to \infty$. Here, $|a| = a_+ - a_-$, where a_+ and a_- are the positive and negative parts of a, respectively. A vector lattice is called σ-complete if any countable bounded subset has $\vee$ and $\wedge$; an element is called a weak unit of $\mathcal{X}$ if its order orthogonal is $\{0\}$. For a sequence $(y_t)_{t\in\mathbb{R}}$ with $y_t \in \mathcal{X}$ we denote

$$\Phi\big((y_t)_{t\in\mathbb{R}}\big) := (\mathrm{o}) - \lim_{||\Delta_i||\to 0} \sum_{j\in\mathbb{Z}} r_j^i \big(y_{u_{j+1}^i} - y_{u_j^i}\big), \tag{1}$$

provided the limit on the right hand side exists. The sum on the right hand side is computed with respect to a sequence of partitions $\Delta_i = (u_j^i)_{j\in\mathbb{Z}}$ of $\mathbb{R}$ and any intermediary points $r_j^i \in [u_j^i, u_{j+1}^i)$, whereas the limit is taken in the sequential order topology of $\mathcal{X}$ as the mesh $||\Delta_i||$ of the partition Δ_i goes to zero as $i \to \infty$.

Theorem 2.1. *If $(x_n)_{n\geq 1}$ is an order martingale in a σ-complete vector lattice $\mathcal{X}$ with weak unit, then there exists a family of regular martingales $(y_{t,n})_{n\geq 1}$, $t \in \mathbb{R}$, such that*

$$x_n = \Phi\big((y_{t,n})_{t\in\mathbb{R}}\big) \text{ for all } n \geq 1. \tag{2}$$

Proof. Similar to the definition of projection band operators E, we define the new operators $E^{\perp}$ and $E^{\perp\perp}$, the latter being generated by the second order orthogonal of $\mathcal{E}$, i.e., $\mathcal{E}^{\perp\perp} := (\mathcal{E}^{\perp})^{\perp}$. It is easy to see that $\mathcal{E}^{\perp\perp}$ is always a projection band for any subset $\mathcal{E}$ of $\mathcal{X}$; in particular, for any $x \in \mathcal{X}$, we shall denote by $x^{\perp\perp}$ the projection operator generated by the *principal projection band* $\{x\}^{\perp\perp}$, see [1], p. 96.

If we denote by $\mathbf{1}$ the weak unit in $\mathcal{X}$ we then have, from Freudenthal's spectral theorem (the unbounded case), see [1]. p. 258:

$$\left| x - \sum_{j\in\mathbb{Z}} r_j^i \left(\big(u_{j+1}^i \cdot \mathbf{1} - x\big)_+^{\perp\perp}(\mathbf{1}) - \big(u_j^i \cdot \mathbf{1} - x\big)_+^{\perp\perp}(\mathbf{1}) \right) \right| \leq ||\Delta_i|| \cdot \mathbf{1}$$

for any $x \in \mathcal{X}$, any sequence of partitions $\Delta_i = (u_j^i)_{j\in\mathbb{Z}}$ of $\mathbb{R}$ and any intermediary points $r_j^i \in [u_j^i, u_{j+1}^i)$. Thus,

$$x = (\mathrm{o}) - \lim_{||\Delta_i||\to 0} \sum_{j\in\mathbb{Z}} r_j^i \left(\big(u_{j+1}^i \cdot \mathbf{1} - x\big)_+^{\perp\perp}(\mathbf{1}) - \big(u_j^i \cdot \mathbf{1} - x\big)_+^{\perp\perp}(\mathbf{1}) \right), \tag{3}$$

in the sense that the (o)-limit in equation (3) exists as the mesh $||\Delta_i|| \to 0$, and its value is precisely x. Comparing to formula (1), equation (3) gives, for any $x \in \mathcal{X}$:

$$x = \Phi\big((y_t)_{t\in\mathbb{R}}\big), \text{ where } y_t := (t \cdot \mathbf{1} - x)_+^{\perp\perp}(\mathbf{1}), t \in \mathbb{R}. \tag{4}$$

We now write the martingale $(x_n)_{n\geq 1}$ as follows: $x_n = -\big((x_n)_-\big) - \big(-(x_n)_+\big) =: x_n^1 - x_n^2$ for $n \geq 1$, and further apply equation (4) to both x_n^i, $i = 1, 2$, to deduce that $x_n^i = \Phi\big((y_{t,n}^i)_{t\in\mathbb{R}}\big)$, where $y_{t,n}^i := (t \cdot \mathbf{1} - x_n^i)_+^{\perp\perp}(\mathbf{1}), t \in \mathbb{R}$. As Φ is a linear operator, we obtain formula (2), with $y_{t,n} = y_{t,n}^1 - y_{t,n}^2$.

Moreover, the sequences $(x_n^i)_{n\geq 1}$, $i = 1, 2$ are negative martingales, hence they are decreasing (cf. [3]). We then have $t \cdot \mathbf{1} - x_n^i \leq t \cdot \mathbf{1} - x_{n+1}^i$, that is, $(t \cdot \mathbf{1} - x_n^i)_+^{\perp\perp} \subseteq (t \cdot \mathbf{1} - x_{n+1}^i)^{\perp\perp}$ and, for $i = 1, 2$ and every $t \in \mathbb{R}$,

$$(t \cdot \mathbf{1} - x_n^i)_+^{\perp\perp}\big((t \cdot \mathbf{1} - x_{n+1}^i)^{\perp\perp}(\mathbf{1})\big) = (t \cdot \mathbf{1} - x_n^i)_+^{\perp\perp}(\mathbf{1}). \tag{5}$$

Equation (5) shows that, for every $t \in \mathbb{R}$, the sequence $n \to (t\cdot\mathbf{1} - x_n^i)_+^{\perp\perp}(\mathbf{1})$ is a *regular* order martingale with respect to the family $\big((t \cdot \mathbf{1} - x_{n+1}^i)_+^{\perp\perp}\big)_{n\geq 1}$ of projection bands, for $i = 1, 2$, as required. □

Example 2.2.

(*i*) On the vector lattice $\mathcal{X}$ of real functions on $[0, \infty)$ endowed with pointwise ordering, we define a sequence of projection bands by $\mathcal{E}_n := \{f \in \mathcal{X}, f{=}0 \text{ on } [n, \infty)\}$, $n \geq 1$. Note that $\mathcal{E}_n^{\perp\perp} = \mathcal{E}_n$, $n \geq 1$, hence the corresponding martingales are given by the following sequences:

$$(f_n)_{n\geq 1}, \quad f_n = f\chi_{[0,n)} \text{ for any } f \in \mathcal{X},$$

where $\chi_A(\cdot)$ denotes the indicator function of a set A. Indeed, we have: $E_n(f_{n+1}) = f_{n+1}\chi_{[0,n)} = f_n$, $n \geq 1$. These martingales are not necessarily regular ones (unless we impose topological conditions on f). According to Theorem 2.1, their structure is given via the operator Φ and the family of regular martingales therein:

$$f_n = \Phi\big((y_{t,n})_{t\in\mathbb{R}}\big), \quad y_{t,n} = \big(t\chi_{[0,\infty)} - f(\cdot)\chi_{[0,n)}\big)_+^{\perp\perp}\big(\chi_{[0,\infty)}\big), t \in \mathbb{R}. \tag{6}$$

(*ii*) On the same vector lattice as in (i), fix $g \in \mathcal{X}$ and define a sequence of projection bands by $\mathcal{E}_n := \{g_n\}^{\perp\perp}$, i.e., the second order orthogonal of $g_n \in \mathcal{X}$, where $g_n := \max\{n - g, 0\}, n \geq 1$. One can easily see that $E_n(f) = f\chi_{\{g<n\}}$ for any $f \in \mathcal{X}$, hence the corresponding martingales are given by the following sequences:

$$(f_n)_{n\geq 1}, \quad f_n = f\chi_{\{g<n\}} \text{ for any } f \in \mathcal{X}.$$

According to Theorem 2.1, their structure is described by:

$$f_n = \Phi\big((y_{t,n})_{t\in\mathbb{R}}\big), \quad y_{t,n} = \big(t\chi_{[0,\infty)} - f\chi_{\{g<n\}}\big)_+^{\perp\perp}\big(\chi_{[0,\infty)}\big), t \in \mathbb{R}. \tag{7}$$

Remark 2.3. We saw in the proof of the Theorem 2.1 that the required regular martingales are explicitly constructed, namely $y_{t,n}^i := (t \cdot \mathbf{1} - x_n^i)_+^{\perp\perp}(\mathbf{1}), t \in \mathbb{R}$. As a matter of continuity (or lack thereof), note that, although the sequences $n \to y_{t,n}^i$ are order convergent (cf. [3]), the operator $\Phi(\cdot)$ *is not order continuous*. Otherwise, it would follow that any martingale $(x_n)_{n\geq 1}$ with $x_n = \Phi\big((y_{t,n})_{t\in\mathbb{R}}\big)$ is order convergent - but we know (see [4]) that the regular martingales are the only order convergent ones.

The following consequence of Theorem 2.1 requires a new definition, see [4]: a sequence $(x_n)_{n\geq 1}$ in a vector lattice $(\mathcal{X}, \leq)$ is called an *order submartingale* if, for a family $(\mathcal{E}_n)_{n\geq 1}$ of projection bands in $\mathcal{X}$, we have $x_n \in \mathcal{F}_n$ and $x_n \leq E_n(x_{n+1})$ for every $n \geq 1$. Any martingale is a submartingale; conversely, submartingales relate to martingales via their Riesz and Doob-Meyer decompositions, see Theorems 2.8. and 2.13. in [4]. Note that for order martingales, the condition $x_n \in \mathcal{F}_n$ is automatic.

Corollary 2.4. *The conclusion of the Theorem 2.1 holds, under the same hypotheses, for order bounded below submartingales.*

Proof. The proof of the Theorem 2.1 works for sequences that can be written as the difference of two decreasing sequences. As our order submartingale is bounded below, we have $x_n \geq a$ for some $a \in \mathcal{X}$. We then have $x_n = E_n(a) + c_n$, $n \geq 1$, where the first term on the right hand side is a regular order martingale, and the second term on the right hand side is an increasing sequence, to which Theorem 2.1 applies. Indeed, from $x_n \geq a$ it follows that $x_n \geq E_n(a)$, i.e., $c_n \geq 0$, hence $E(c_n) \leq c_n$ for any projection band $\mathcal{E}$. Then, $E_n(c_{n+1}) = E_n(x_{n+1}) - E_n(E_{n+1}(a)) = E_n(x_{n+1}) - E_n(a)$, so $c_n = x_n - E_n(a) \leq E_n(x_{n+1}) - E_n(a) = E_n(c_{n+1}) \leq c_{n+1}$. □

Example 2.5. The order submartingales in Example 2.2 (i) and (ii) above are given by the following sequences $(f_n)_{n\geq 1}$:

$$f_n = f\chi_{[0,n)} \text{ for any } f \in \mathcal{X} \text{ such that } f \geq 0 \text{ on } [n, n+1), n \geq 1$$

and, respectively,

$$f_n = f(\cdot)\chi_{\{g<n\}} \text{ for all } f \in \mathcal{X} \text{ such that } f \geq 0 \text{ on the set } \{n \leq g < n+1), n \geq 1.$$

According to our result, their structure is described, in both cases, via the operator Φ defined in (1): $f_n = \Phi\big((y_{t,n})_{t\in\mathbb{R}}\big)$ for all $n \geq 1$, and where the family $\{y_{t,n}\}$ of regular order martingales is given by formulae (6) and (7), respectively.

Remark 2.6. The relationship between general and regular martingales is as follows (see [3] and [4]): Let $(x_n)_{n\geq 1}$ be a martingale with respect to a family $(\mathcal{E}_n)_{n\geq 1}$ of projection bands in an order complete vector lattice $\mathcal{X}$. Then the following are equivalent:

(a) $(x_n)_{n\geq 1}$ is (o)-bounded (above and below);

(b) $(x_n)_{n\geq 1}$ is (o)-convergent;

(c) $x_n = E_n(x)$ for some $x \in \mathcal{X}$, i.e., $(x_n)_{n\geq 1}$ is a regular martingale.

In this case, (o)-$\lim x_n = E_\infty(x)$, where E_∞ is the projection operator generated by the projection band $\mathcal{E}_\infty := (\cup_{n\geq 1}\mathcal{E}_n)^{\perp\perp}$.

If, in addition, $\mathcal{X}$ is a normed Kantorovich-Banach space, then (see [5]) the above statements (a)-(c) are equivalent to each of the following:

(d) $(x_n)_{n\geq 1}$ is norm convergent;

(e) $(x_n)_{n\geq 1}$ is relatively compact in the weak topology on $\mathcal{X}$.

It is useful to our reader to relate order convergence to concrete Riesz spaces. For instance (see, e.g., [5] and [6]), in the lattice of measurable functions, the order convergence coincides with the almost everywhere convergence (which is not topological); in L^p probability spaces, $1 \leq p \leq \infty$, order convergence is equivalent to almost everywhere convergence *plus* a uniform boundedness condition; and in L^∞ probability spaces, order convergence is equivalent to almost everywhere convergence *plus* a condition of uniform boundedness by a constant function.

Our construct is built upon a Riemann-Stieltjes scheme adapted to order martingales. It would be interesting to check if the Lebesgue-type scheme adapted by P.A. Meyer to Itô stochastic calculus (see [2]) works as well in the latticial context.

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