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Some seminormed difference sequence spaces defined by a Musielak-Orlicz function over n-normed spaces

Kuldip Raj and Sunil K. Sharma

Abstract: In the present paper we introduced some seminormed difference sequence spaces combining lacunary sequences and Musielak-Orlicz function $\mathcal{M} = (M_k)$ over n-normed spaces and examine some topological properties and inclusion relations between resulting sequence spaces.

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1 Introduction and preliminaries

The concept of 2-normed spaces was initially developed by Gähler [6] in the mid of 1960's, while that of n-normed spaces one can see in Misiak [17]. Since then, many others have studied this concept and obtained various results, see Gunawan ([7], [8]) and Gunawan and Mashadi [9] and many others. Let $n \in \mathbb{N}$ and X be a linear space over the field $\mathbb{K}$, where $\mathbb{K}$ is field of real or complex numbers of dimension d, where $d \geq n \geq 2$. A real valued function $||\cdot, \cdots, \cdot||$ on X^n satisfying the following four conditions:

1. $||x_1, x_2, \cdots, x_n|| = 0$ if and only if $x_1, x_2, \cdots, x_n$ are linearly dependent in X;
2. $||x_1, x_2, \cdots, x_n||$ is invariant under permutation;
3. $||\alpha x_1, x_2, \cdots, x_n|| = |\alpha| \ ||x_1, x_2, \cdots, x_n||$ for any $\alpha \in \mathbb{K}$, and
4. $||x + x', x_2, \cdots, x_n|| \leq ||x, x_2, \cdots, x_n|| + ||x', x_2, \cdots, x_n||$

is called a n-norm on X, and the pair $(X, ||\cdot, \cdots, \cdot||)$ is called a n-normed space over the field $\mathbb{K}$.
For example, we may take $X = \mathbb{R}^n$ being equipped with the Euclidean n-norm

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$||x_1, x_2, \cdots, x_n||_E =$ the volume of the n-dimensional parallelopiped spanned by the vectors $x_1, x_2, \cdots, x_n$ which may be given explicitly by the formula

$$||x_1, x_2, \cdots, x_n||_E = |\det(x_{ij})|,$$

where $x_i = (x_{i1}, x_{i2}, \cdots, x_{in}) \in \mathbb{R}^n$ for each $i = 1, 2, \cdots, n$. Let $(X, ||\cdot, \cdots, \cdot||)$ be an n-normed space of dimension $d \geq n \geq 2$ and $\{a_1, a_2, \cdots, a_n\}$ be linearly independent set in X. Then the following function $||\cdot, \cdots, \cdot||_\infty$ on X^{n-1} defined by

$$||x_1, x_2, \cdots, x_{n-1}||_\infty = \max\{||x_1, x_2, \cdots, x_{n-1}, a_i|| : i = 1, 2, \cdots, n\}$$

defines an $(n-1)$-norm on X with respect to $\{a_1, a_2, \cdots, a_n\}$.
A sequence (x_k) in a n-normed space $(X, ||\cdot, \cdots, \cdot||)$ is said to converge to some $L \in X$ if

$$\lim_{k\to\infty} ||x_k - L, z_1, \cdots, z_{n-1}|| = 0 \text{ for every } z_1, \cdots, z_{n-1} \in X.$$

A sequence (x_k) in a n-normed space $(X, ||\cdot, \cdots, \cdot||)$ is said to be Cauchy if

$$\lim_{k,i\to\infty} ||x_k - x_i, z_1, \cdots, z_{n-1}|| = 0 \text{ for every } z_1, \cdots, z_{n-1} \in X.$$

If every Cauchy sequence in X converges to some $L \in X$, then X is said to be complete with respect to the n-norm. Any complete n-normed space is said to be n-Banach space.
An Orlicz function M is a function, which is continuous, non-decreasing and convex with $M(0) = 0$, $M(x) > 0$ for $x > 0$ and $M(x) \longrightarrow \infty$ as $x \longrightarrow \infty$.
Lindenstrauss and Tzafriri [12] used the idea of Orlicz function to define the following sequence space. Let w be the space of all real or complex sequences $x = (x_k)$, then

$$\ell_M = \Big\{x \in w : \sum_{k=1}^{\infty} M\Big(\frac{|x_k|}{\rho}\Big) < \infty\Big\}$$

which is called as an Orlicz sequence space. The space ℓ_M is a Banach space with the norm

$$||x|| = \inf\Big\{\rho > 0 : \sum_{k=1}^{\infty} M\Big(\frac{|x_k|}{\rho}\Big) \leq 1\Big\}.$$

It is shown in [12] that every Orlicz sequence space ℓ_M contains a subspace isomorphic to $\ell_p (p \geq 1)$. The Δ_2-condition is equivalent to $M(Lx) \leq kLM(x)$ for all values of $x \geq 0$, and for $L > 1$. A sequence $\mathcal{M} = (M_k)$ of Orlicz functions is called a Musielak-Orlicz function see ([16], [20]). A sequence $\mathcal{N} = (N_k)$ is defined by

$$N_k(v) = \sup\{|v|u - (M_k) : u \geq 0\}, \ k = 1, 2, \cdots$$

is called the complementary function of a Musielak-Orlicz function $\mathcal{M}$. For a given Musielak-Orlicz function $\mathcal{M}$, the Musielak-Orlicz sequence space $t_{\mathcal{M}}$ and its subspace $h_{\mathcal{M}}$ are defined as follows

$$t_{\mathcal{M}} = \Big\{x \in w : I_{\mathcal{M}}(cx) < \infty \text{ for some } c > 0\Big\},$$

$$h_{\mathcal{M}} = \Big\{x \in w : I_{\mathcal{M}}(cx) < \infty \text{ for all } c > 0\Big\},$$

where $I_{\mathcal{M}}$ is a convex modular defined by

$$I_{\mathcal{M}}(x) = \sum_{k=1}^{\infty}(M_k)(x_k), x = (x_k) \in t_{\mathcal{M}}.$$

We consider $t_{\mathcal{M}}$ equipped with the Luxemburg norm

$$||x|| = \inf\Big\{k > 0 : I_{\mathcal{M}}\Big(\frac{x}{k}\Big) \le 1\Big\}$$

or equipped with the Orlicz norm

$$||x||^0 = \inf\Big\{\frac{1}{k}\Big(1 + I_{\mathcal{M}}(kx)\Big) : k > 0\Big\}.$$

Let ℓ_∞, c and c_0 denotes the sequence spaces of bounded, convergent and null sequences $x = (x_k)$ respectively. A sequence $x = (x_k) \in \ell_\infty$ is said to be almost convergent if all Banach limits of $x = (x_k)$ coincide. In [13], it was shown that

$$\hat{c} = \Big\{x = (x_k) : \lim_{n\to\infty} \frac{1}{n}\sum_{k=1}^{n} x_{k+s} \text{ exists, uniformly in } s\Big\}.$$

In ([14], [15]) Maddox defined strongly almost convergent sequences. Recall that a sequence $x = (x_k)$ is strongly almost convergent if there is a number L such that

$$\lim_{n\to\infty} \frac{1}{n}\sum_{k=1}^{n} |x_{k+s} - L| = 0, \text{ uniformly in } s.$$

By a lacunary sequence $\theta = (i_r)$, $r = 0, 1, 2, \cdots$, where $i_0 = 0$, we shall mean an increasing sequence of non-negative integers $g_r = (i_r - i_{r-1}) \to \infty$ $(r \to \infty)$. The intervals determined by θ are denoted by $I_r = (i_{r-1}, i_r]$ and the ratio i_r/i_{r-1} will be denoted by q_r. The space of lacunary strongly convergent sequences N_θ was defined by Freedman et. al [5] as follows:

$$N_\theta = \Big\{x = (x_k) : \lim_{r\to\infty} \frac{1}{g_r}\sum_{k\in I_r} |x_k - L| = 0 \text{ for some } L\Big\}.$$

The notion of difference sequence spaces was introduced by Kızmaz [11], who studied the difference sequence spaces $l_\infty(\Delta)$, $c(\Delta)$ and $c_0(\Delta)$. The notion was further generalized by Et and Çolak [4] by introducing the spaces $l_\infty(\Delta^n)$, $c(\Delta^n)$ and $c_0(\Delta^n)$. Let m, n be non-negative integers, then for $Z = c, c_0$ and l_∞, we have sequence spaces

$$Z(\Delta_n^m) = \{x = (x_k) \in w : (\Delta_n^m x_k) \in Z\}$$

for $Z = c, c_0$ and l_∞ where $\Delta_n^m x = (\Delta_n^m x_k) = (\Delta_n^{m-1} x_k - \Delta_n^{m-1} x_k)$ and $\Delta^0 x_k = x_k$ for all $k \in \mathbb{N}$, which is equivalent to the following binomial representation

$$\Delta_n^m x_k = \sum_{v=0}^{m} (-1)^v \binom{m}{v} x_{k+nv}.$$

Taking $n = 1$, we get the spaces $l_\infty(\Delta^m)$, $c(\Delta^m)$ and $c_0(\Delta^m)$ studied by Et and Çolak [4]. Taking $m = 1$, $n = 1$, we get the spaces $l_\infty(\Delta)$, $c(\Delta)$ and $c_0(\Delta)$ studied by Kızmaz [11]. Let X be a linear metric space. A function $p : X \to \mathbb{R}$ is called paranorm, if

1. $p(x) \geq 0$ for all $x \in X$,
2. $p(-x) = p(x)$ for all $x \in X$,
3. $p(x + y) \leq p(x) + p(y)$ for all $x, y \in X$,
4. if (λ_n) is a sequence of scalars with $\lambda_n \to \lambda$ as $n \to \infty$ and (x_n) is a sequence of vectors with $p(x_n - x) \to 0$ as $n \to \infty$, then $p(\lambda_n x_n - \lambda x) \to 0$ as $n \to \infty$.

A paranorm p for which $p(x) = 0$ implies $x = 0$ is called total paranorm and the pair (X, p) is called a total paranormed space. It is well known that the metric of any linear metric space is given by some total paranorm (see [26], Theorem 10.4.2, pp. 183). For more details about sequence spaces see ([1], [2], [3], [18], [19], [21], [22], [23], [24], [25]) and references therein.
Let M be an Orlicz function and $p = (p_k)$ be any sequence of strictly positive real numbers. Güngor and Et [10] defined the following sequence spaces:

$$[c, M, p](\Delta^m) = \Big\{x = (x_k) : \lim_{n\to\infty} \frac{1}{n}\sum_{k=1}^{n}\Big[M\Big(\frac{|\Delta^m x_{k+s} - L|}{\rho}\Big)\Big]^{p_k} = 0,$$

$$\text{uniformly in } s, \text{ for some } \rho > 0 \text{ and } L > 0\Big\},$$

$$[c, M, p]_0(\Delta^m) = \Big\{x = (x_k) : \lim_{n\to\infty} \frac{1}{n}\sum_{k=1}^{n}\Big[M\Big(\frac{|\Delta^m x_{k+s}|}{\rho}\Big)\Big]^{p_k} = 0,$$

$$\text{uniformly in } s, \text{ for some } \rho > 0 \Big\},$$

$$[c, M, p]_\infty(\Delta^m) = \Big\{x = (x_k) : \sup_{n,s} \frac{1}{n}\sum_{k=1}^{n}\Big[M\Big(\frac{|\Delta^m x_{k+s}|}{\rho}\Big)\Big]^{p_k} < \infty \text{ for some } \rho > 0\Big\}.$$

Let $\mathcal{M} = (M_k)$ be a Musielak-Orlicz function and X be a seminormed space, seminormed by $q = (q_k)$. Let $p = (p_k)$ be any bounded sequence of positive real numbers and $u = (u_k)$ be any sequence of strictly positive real numbers. In this paper we define the following sequence spaces:
$[\, c, \mathcal{M}, p, ||\cdot, \cdots, \cdot||\,]^\theta(\Delta^m_n, u, q) =$

$$\Big\{x = (x_k) \in w(n{-}X) : \lim_{r\to\infty} \frac{1}{g_r}\sum_{k\in I_r}\Big[M_k\Big(q_k\Big(||\frac{u_k\Delta^m_n x_{k+s} - L}{\rho}, z_1, \cdots, z_{n-1}||\Big)\Big)\Big]^{p_k} = 0,$$

$$\text{uniformly in } s, \ z_1, \cdots, z_{n-1} \in X \text{ for some } L \text{ and } \rho > 0\Big\},$$

$[\, c, \mathcal{M}, p, ||\cdot, \cdots, \cdot||\,]_0^{\theta}(\Delta_n^m, u, q) =$

$$\Big\{x = (x_k) \in w(n-X) : \lim_{r\to\infty} \frac{1}{g_r} \sum_{k\in I_r} \Big[M_k\Big(q_k\Big(||\frac{u_k\Delta_n^m x_{k+s}}{\rho}, z_1, \cdots, z_{n-1}||\Big)\Big)\Big]^{p_k} = 0,$$

$$\text{uniformly in } s, \; z_1, \cdots, z_{n-1} \in X \text{ for some } \rho > 0 \Big\},$$

$[\, c, \mathcal{M}, p, ||\cdot, \cdots, \cdot||\,]_\infty^{\theta}(\Delta_n^m, u, q) =$

$$\Big\{x = (x_k) \in w(n-X) : \sup_{r,s} \frac{1}{g_r} \sum_{k=1}^{n} \Big[M_k\Big(q_k\Big(||\frac{u_k\Delta_n^m x_{k+s}}{\rho}, z_1, \cdots, z_{n-1}||\Big)\Big)\Big]^{p_k} < \infty,$$

$$\text{uniformly in } s, \; z_1, \cdots, z_{n-1} \in X \text{ for some } \rho > 0\Big\}.$$

When, $\mathcal{M}(x) = x$, we get
$[\, c, p, ||\cdot, \cdots, \cdot||\,]^{\theta}(\Delta_n^m, u, q) =$

$$\Big\{x = (x_k) \in w(n-X) : \quad \lim_{r\to\infty} \frac{1}{g_r} \sum_{k\in I_r} \Big(q_k\Big(||\frac{u_k\Delta_n^m x_{k+s} - L}{\rho}, z_1, \cdots, z_{n-1}||\Big)\Big)^{p_k} = 0,$$

$$\text{uniformly in } s, \; z_1, \cdots, z_{n-1} \in X \text{ for some } L \text{ and } \rho > 0\Big\},$$

$[\, c, p, ||\cdot, \cdots, \cdot||\,]_0^{\theta}(\Delta_n^m, u, q) =$

$$\Big\{x = (x_k) \in w(n-X) : \lim_{r\to\infty} \frac{1}{g_r} \sum_{k\in I_r} \Big(q_k\Big(||u_k\frac{\Delta_n^m x_{k+s}}{\rho}, z_1, \cdots, z_{n-1}||\Big)\Big)^{p_k} = 0,$$

$$\text{uniformly in } s, z_1, \cdots, z_{n-1} \in X \quad \text{for some } \rho > 0 \Big\},$$

$[\, c, p, ||\cdot, \cdots, \cdot||\,]_\infty^{\theta}(\Delta_n^m, u, q) =$

$$\Big\{x = (x_k) \in w(n-X) : \sup_{r,s} \frac{1}{g_r} \sum_{k=1}^{n} \Big(q_k\Big(||\frac{u_k\Delta_n^m x_{k+s}}{\rho}, z_1, \cdots, z_{n-1}||\Big)\Big)^{p_k} < \infty,$$

$$z_1, \cdots, z_{n-1} \in X \text{ for some } \rho > 0\Big\}.$$

If we take $p_k = 1$ for all k, then we get
$[\, c, \mathcal{M}, ||\cdot, \cdots, \cdot||\,]^{\theta}(\Delta_n^m, u, q) =$

$$\Big\{x = (x_k) \in w(n-X) : \lim_{r\to\infty} \frac{1}{g_r} \sum_{k\in I_r} \Big[M_k\Big(q_k\Big(||\frac{u_k\Delta_n^m x_{k+s} - L}{\rho}, z_1, \cdots, z_{n-1}||\Big)\Big)\Big] = 0,$$

$$\text{uniformly in } s, \; z_1, \cdots, z_{n-1} \in X \text{ for some } L \text{ and } \rho > 0\Big\},$$

$[\, c, \mathcal{M}, ||\cdot, \cdots, \cdot||\,]_0^{\theta}(\Delta_n^m, u, q) =$

$$\Big\{x = (x_k) \in w(n-X) : \lim_{r\to\infty} \frac{1}{g_r} \sum_{k\in I_r} \Big[M_k\Big(q_k\Big(||\frac{u_k\Delta_n^m x_{k+s}}{\rho}, z_1, \cdots, z_{n-1}||\Big)\Big)\Big] = 0,$$

$$\text{uniformly in } s, \ \ z_1, \cdots, z_{n-1} \in X \ \text{ for some } \ \rho > 0 \ \Big\},$$

$[\, c, \mathcal{M}, ||\cdot, \cdots, \cdot||\,]_\infty^{\theta}(\Delta_n^m, u, q) =$

$$\Big\{x = (x_k) \in w(n-X) : \sup_{r,s} \frac{1}{g_r} \sum_{k=1}^{n} \Big[M_k\Big(q_k\Big(||\frac{u_k\Delta_n^m x_{k+s}}{\rho}, z_1, \cdots, z_{n-1}||\Big)\Big)\Big] < \infty,$$

$$z_1, \cdots, z_{n-1} \in X \ \text{ for some } \ \rho > 0\Big\}.$$

The following inequality will be used throughout the paper. If $0 \le p_k \le \sup p_k = H$, $D = \max(1, 2^{H-1})$ then

$$|a_k + b_k|^{p_k} \le D\{|a_k|^{p_k} + |b_k|^{p_k}\} \tag{1.1}$$

for all k and $a_k, b_k \in \mathbb{C}$. Also $|a|^{p_k} \le \max(1, |a|^H)$ for all $a \in \mathbb{C}$.
The main aim of this paper is to study some seminormed difference sequence spaces defined by a Musielak-Orlicz function over n-normed space. We also make an effort to study some topological properties and prove some inclusion relations between these spaces.

2 Main Results

Theorem 2.1 *Let $\mathcal{M} = (M_k)$ be a Musielak-Orlicz function, $p = (p_k)$ be a bounded sequence of positive real numbers and $u = (u_k)$ be any sequence of strictly positive real numbers. Then the spaces $[\, c, \mathcal{M}, p, ||\cdot, \cdots, \cdot||\,]^{\theta}(\Delta_n^m, u, q)$, $[\, c, \mathcal{M}, p, ||\cdot, \cdots, \cdot||\,]_0^{\theta}(\Delta_n^m, u, q)$ and $[\, c, \mathcal{M}, p, ||\cdot, \cdots, \cdot||\,]_\infty^{\theta}(\Delta_n^m, u, q)$ are linear over the field of complex numbers $\mathbb{C}$.*

Proof. Let $x = (x_k),\ y = (y_k) \in [\, c, \mathcal{M}, p, ||\cdot, \cdots, \cdot||\,]_0^{\theta}(\Delta_n^m, u, q)$ and $\alpha, \beta \in \mathbb{C}$. Then there exists positive numbers ρ_1 and ρ_2 such that

$$\lim_{r\longrightarrow\infty} \frac{1}{g_r} \sum_{k\in I_r} \Big[M_k\Big(q_k\Big(||\frac{u_k\Delta_n^m x_{k+s}}{\rho_1}, z_1, \cdots, z_{n-1}||\Big)\Big)\Big]^{p_k} = 0, \ \text{ uniformly in } \ s,$$

and

$$\lim_{r\longrightarrow\infty} \frac{1}{g_r} \sum_{k\in I_r} \Big[M_k\Big(q_k\Big(||\frac{u_k\Delta_n^m x_{k+s}}{\rho_2}, z_1, \cdots, z_{n-1}||\Big)\Big)\Big]^{p_k} = 0, \ \text{ uniformly in } \ s.$$

Let $\rho_3 = \max(2|\alpha|\rho_1, 2|\beta|\rho_2)$. Since $\mathcal{M} = (M_k)$ is non-decreasing convex function, by using inequality (1.1), we have

$$\begin{aligned}
&\frac{1}{g_r}\sum_{k\in I_r}\Big[M_k\Big(q_k\Big(||\frac{u_k\Delta_n^m(\alpha x_{k+s}+\beta y_{k+s})}{\rho_3}, z_1, \cdots, z_{n-1}||\Big)\Big)\Big]^{p_k}\\
&\qquad \le D\frac{1}{g_r}\sum_{k\in I_r}\frac{1}{2^{p_k}}\Big[M_k\Big(q_k\Big(||\frac{u_k\Delta_n^m(x_{k+s})}{\rho_1}, z_1, \cdots, z_{n-1}||\Big)\Big)\Big]^{p_k}\\
&\qquad\quad + D\frac{1}{g_r}\sum_{k\in I_r}\frac{1}{2^{p_k}}\Big[M_k\Big(q_k\Big(||\frac{u_k\Delta_n^m(y_{k+s})}{\rho_2}, z_1, \cdots, z_{n-1}||\Big)\Big)\Big]^{p_k}\\
&\qquad \le D\frac{1}{g_r}\sum_{k\in I_r}\Big[M_k\Big(q_k\Big(||\frac{u_k\Delta_n^m(x_{k+s})}{\rho_1}, z_1, \cdots, z_{n-1}||\Big)\Big)\Big]^{p_k}\\
&\qquad\quad + D\frac{1}{g_r}\sum_{k\in I_r}\Big[M_k\Big(q_k\Big(||\frac{u_k\Delta_n^m(y_{k+s})}{\rho_1}, z_1, \cdots, z_{n-1}||\Big)\Big)\Big]^{p_k}\\
&\qquad \longrightarrow 0 \ \text{ as } \ r\longrightarrow\infty, \ \text{ uniformly in } \ s.
\end{aligned}$$

Thus, we have $\alpha x + \beta y \in [\, c, \mathcal{M}, p, ||\cdot, \cdots, \cdot||\,]_0^\theta(\Delta_n^m, u, q)$.

Hence $[\, c, \mathcal{M}, p, ||\cdot, \cdots, \cdot||\,]_0^\theta(\Delta_n^m, u, q)$ is a linear space. Similarly, we can prove that $[\, c, \mathcal{M}, p, ||\cdot, \cdots, \cdot||\,]^\theta(\Delta_n^m, u, q)$ and $[\, c, \mathcal{M}, p, ||\cdot, \cdots, \cdot||\,]_\infty^\theta(\Delta_n^m, u, q)$ are linear spaces. ∎

Theorem 2.2 *For any Musielak-Orlicz function $\mathcal{M} = (M_k)$, $p = (p_k)$ be a bounded sequence of positive real numbers and $u = (u_k)$ be any sequence of strictly positive real numbers, the space $[\, c, \mathcal{M}, p, ||\cdot, \cdots, \cdot||\,]_0^\theta(\Delta_n^m, u, q)$ is a topological linear space paranormed by*

$$g(x) = \inf\Big\{\rho^{\frac{p_r}{K}} : \Big(\frac{1}{g_r}\sum_{k\in I_r}\Big[M_k\Big(q_k\Big(||\frac{u_k\Delta_n^m x_{k+s}}{\rho}, z_1, \cdots, z_{n-1}||\Big)\Big)\Big]^{p_k}\Big)^{\frac{1}{K}} \le 1, r, s\in\mathbb{N}\Big\},$$

where $K = \max(1, \sup_k p_k < \infty)$.

Proof. Clearly $g(x) \ge 0$ for $x = (x_k) \in [\, c, \mathcal{M}, ||\cdot, \cdots, \cdot||\,]_0^\theta(\Delta_n^m, u, q)$. Since $M_k(0) = 0$, we get $g(0) = 0$. Again, if $g(x) = 0$, then

$$\inf\Big\{\rho^{\frac{p_r}{K}} : \Big(\frac{1}{g_r}\sum_{k\in I_r}\Big[M_k\Big(q_k\Big(||\frac{u_k\Delta_n^m x_{k+s}}{\rho}, z_1, \cdots, z_{n-1}||\Big)\Big)\Big]^{p_k}\Big)^{\frac{1}{K}} \le 1, r, s\in\mathbb{N}\Big\} = 0.$$

This implies that for a given $\epsilon > 0$, there exists some $\rho_\epsilon (0 < \rho_\epsilon < \epsilon)$ such that

$$\Big(\frac{1}{g_r}\sum_{k\in I_r}\Big[M_k\Big(q_k\Big(||\frac{u_k\Delta_n^m x_{k+s}}{\rho_\epsilon}, z_1, \cdots, z_{n-1}||\Big)\Big)\Big]^{p_k}\Big)^{\frac{1}{K}} \le 1.$$

Thus

$$\Big(\tfrac{1}{g_r}\sum_{k\in I_r}\Big[M_k\Big(q_k\Big(||\frac{u_k\Delta_n^m x_{k+s}}{\epsilon},z_1,\cdots,z_{n-1}||\Big)\Big)\Big]^{p_k}\Big)^{\frac{1}{K}}$$

$$\begin{aligned} &\le \Big(\frac{1}{g_r}\sum_{k\in I_r}\Big[M_k\Big(q_k\Big(||\frac{\Delta^m x_{k+s}}{\rho_\epsilon},z_1,\cdots,z_{n-1}||\Big)\Big)\Big]^{p_k}\Big)^{\frac{1}{K}} \\ &\le 1, \end{aligned}$$

for each r and s. Suppose that $x_k \neq 0$ for each $k \in N$. This implies that $\Delta_n^m x_{k+s} \neq 0$, for each $k, s \in N$. Let $\epsilon \longrightarrow 0$, then $q_k\Big(||\frac{u_k\Delta_n^m x_{k+s}}{\epsilon},z_1,\cdots,z_{n-1}||\Big) \longrightarrow \infty$. It follows that

$$\Big(\frac{1}{g_r}\sum_{k\in I_r}\Big[M_k\Big(q_k\Big(||\frac{u_k\Delta_n^m x_{k+s}}{\epsilon},z_1,\cdots,z_{n-1}||\Big)\Big)\Big]^{p_k}\Big)^{\frac{1}{K}} \longrightarrow \infty,$$

which is a contradiction. Therefore, $\Delta_n^m x_{k+s} = 0$ for each k and s and thus $x_k = 0$ for each $k \in N$. Let $\rho_1 > 0$ and $\rho_2 > 0$ be such that

$$\Big(\frac{1}{g_r}\sum_{k\in I_r}\Big[M_k\Big(q_k\Big(||\frac{u_k\Delta_n^m x_{k+s}}{\rho_1},z_1,\cdots,z_{n-1}||\Big)\Big)\Big]^{p_k}\Big)^{\frac{1}{K}} \le 1$$

and

$$\Big(\frac{1}{g_r}\sum_{k\in I_r}\Big[M_k\Big(q_k\Big(||\frac{u_k\Delta_n^m x_{k+s}}{\rho_2},z_1,\cdots,z_{n-1}||\Big)\Big)\Big]^{p_k}\Big)^{\frac{1}{K}} \le 1$$

for each r and s. Let $\rho = \rho_1 + \rho_2$. Then, by Minkowski's inequality, we have

$$\begin{aligned} &\Big(\frac{1}{g_r}\sum_{k\in I_r}\Big[M_k\Big(q_k\Big(||\frac{u_k\Delta_n^m(x_{k+s}+y_{k+s})}{\rho},z_1,\cdots,z_{n-1}||\Big)\Big)\Big]^{p_k}\Big)^{\frac{1}{K}} \\ &\le \Big(\sum_{k\in I_r}\Big[\frac{\rho_1}{\rho_1+\rho_2}M_k\Big(q_k\Big(||\frac{u_k\Delta_n^m(x_{k+s})}{\rho_1},z_1,\cdots,z_{n-1}||\Big)\Big) \\ &\quad + \frac{\rho_2}{\rho_1+\rho_2}M_k\Big(q_k\Big(||\frac{u_k\Delta_n^m(y_{k+s})}{\rho_2},z_1,\cdots,z_{n-1}||\Big)\Big)\Big]^{p^k}\Big)^{\frac{1}{K}} \\ &\le \Big(\frac{\rho_1}{\rho_1+\rho_2}\Big)\Big(\frac{1}{g_r}\sum_{k\in I_r}\Big[M_k\Big(q_k\Big(||\frac{u_k\Delta_n^m(x_{k+s})}{\rho_1},z_1,\cdots,z_{n-1}||\Big)\Big)\Big]^{p_k}\Big)^{\frac{1}{K}} \\ &\quad + \Big(\frac{\rho_2}{\rho_1+\rho_2}\Big)\Big(\frac{1}{g_r}\sum_{k\in I_r}\Big[M_k\Big(q_k\Big(||\frac{u_k\Delta_n^m(y_{k+s})}{\rho_2},z_1,\cdots,z_{n-1}||\Big)\Big)\Big]^{p_k}\Big)^{\frac{1}{K}} \\ &\le 1 \end{aligned}$$

Since $\rho' s$ are non-negative, so we have

$$g(x+y)$$

$$= \inf\Big\{\rho^{\frac{p_r}{K}} : \Big(\frac{1}{g_r}\sum_{k\in I_r}\Big[M_k\Big(q_k\Big(||\frac{u_k\Delta_n^m(x_{k+s}+y_{k+s})}{\rho}, z_1, \cdots, z_{n-1}||\Big)\Big)\Big]^{p_k}\Big)^{\frac{1}{K}} \le 1, r,s\in\mathbb{N}\Big\},$$

$$\le \inf\Big\{\rho_1^{\frac{p_r}{K}} : \Big(\frac{1}{g_r}\sum_{k\in I_r}\Big[M_k\Big(q_k\Big(||\frac{u_k\Delta_n^m(x_{k+s})}{\rho_1}, z_1, \cdots, z_{n-1}||\Big)\Big)\Big]^{p_k}\Big)^{\frac{1}{K}} \le 1, r,s\in\mathbb{N}\Big\}$$

$$+ \inf\Big\{\rho_2^{\frac{p_r}{K}} : \Big(\frac{1}{g_r}\sum_{k\in I_r}\Big[M_k\Big(q_k\Big(||\frac{\Delta^m(y_{k+s})}{\rho_2}, z_1, \cdots, z_{n-1}||\Big)\Big)\Big]^{p_k}\Big)^{\frac{1}{K}} \le 1, r,s\in\mathbb{N}\Big\}.$$

Therefore,

$$g(x+y) \le g(x) + g(y).$$

Finally, we prove that the scalar multiplication is continuous. Let λ be any complex number. By definition,

$$g(\lambda x) = \inf\Big\{\rho^{\frac{p_r}{K}} : \Big(\frac{1}{g_r}\sum_{k\in I_r}\Big[M_k\Big(q_k\Big(||\frac{u_k\Delta_n^m\lambda x_{k+s}}{\rho}, z_1, \cdots, z_{n-1}||\Big)\Big)\Big]^{p_k}\Big)^{\frac{1}{K}} \le 1, r,s\in\mathbb{N}\Big\}.$$

Then

$$g(\lambda x) = \inf\Big\{(|\lambda| t)^{\frac{p_r}{K}} : \Big(\frac{1}{g_r}\sum_{k\in I_r}\Big[M_k\Big(q_k\Big(||\frac{\Delta^m x_{k+s}}{t}, z_1, \cdots, z_{n-1}||\Big)\Big)\Big]^{p_k}\Big)^{\frac{1}{K}} \le 1, r,s\in\mathbb{N}\Big\},$$

where $t = \frac{\rho}{|\lambda|}$. Since $|\lambda|^{p_r} \le \max(1, |\lambda|^{\sup p_r})$, we have
$g(\lambda x) \le \max(1, |\lambda|^{\sup p_r})$

$$\inf\Big\{t^{\frac{p_r}{K}} : \Big(\frac{1}{g_r}\sum_{k\in I_r}\Big[M_k\Big(q_k\Big(||\frac{u_k\Delta_n^m x_{k+s}}{t}, z_1, \cdots, z_{n-1}||\Big)\Big)\Big]^{p_k}\Big)^{\frac{1}{K}} \le 1, r,s\in\mathbb{N}\Big\}.$$

So, the fact that scalar multiplication is continuous follows from the above inequality. This completes the proof of the theorem.■

Theorem 2.3 *Let $\mathcal{M} = (M_k)$ be a Musielak-Orlicz function. If $\sup_k [M_k(x)]^{p_k} < \infty$ for all fixed $x > 0$, then $[\, c, \mathcal{M}, p, ||\cdot, \cdots, \cdot||\,]_0^\theta(\Delta_n^m, u, q) \subset [\, c, \mathcal{M}, p, ||\cdot, \cdots, \cdot||\,]_\infty^\theta(\Delta_n^m, u, q)$.*

Proof. Let $x = (x_k) \in [\, c, \mathcal{M}, p, ||\cdot, \cdots, \cdot||\,]_0^\theta(\Delta_n^m, u, q)$. There exists some positive ρ_1 such that

$$\lim_{r\to\infty}\frac{1}{g_r}\sum_{k\in I_r}\Big[M_k\Big(q_k\Big(||\frac{\Delta_n^m x_{k+s}}{\rho_1}, z_1, \cdots, z_{n-1}||\Big)\Big)\Big]^{p_k} = 0, \text{ uniformly in } s.$$

Define $\rho = 2\rho_1$. Since $\mathcal{M} = (M_k)$ is non-decreasing and convex, by using inequality(1.1), we have

$$\sup_{r,s} \frac{1}{g_r} \sum_{k \in I_r} \Big[M_k\Big(q_k\Big(||\frac{u_k \Delta_n^m x_{k+s}}{\rho}, z_1, \cdots, z_{n-1}||\Big)\Big)\Big]^{p_k}$$

$$\leq D \sup_{r,s} \frac{1}{g_r} \sum_{k \in I_r} \Big[\frac{1}{2^{p_k}} M_k\Big(q_k\Big(||\frac{u_k \Delta_n^m x_{k+s} - L}{\rho_1}, z_1, \cdots, z_{n-1}||\Big)\Big)\Big]^{p_k}$$
$$+ D \sup_{r,s} \frac{1}{g_r} \sum_{k \in I_r} \Big[\frac{1}{2^{p_k}} M_k\Big(q_k\Big(||\frac{L}{\rho_1}, z_1, \cdots, z_{n-1}||\Big)\Big)\Big]^{p_k}$$
$$\leq D \sup_{r,s} \frac{1}{g_r} \sum_{k \in I_r} \Big[M_k\Big(q_k\Big(||\frac{u_k \Delta_n^m x_{k+s} - L}{\rho_1}, z_1, \cdots, z_{n-1}||\Big)\Big]^{p_k}$$
$$+ D \sup_{r,s} \frac{1}{g_r} \sum_{k \in I_r} \Big[M_k\Big(q_k\Big(||\frac{L}{\rho_1}, z_1, \cdots, z_{n-1}||\Big)\Big)\Big]^{p_k}$$
$$< \infty.$$

Hence $x = (x_k) \in [\, c, \mathcal{M}, p, ||\cdot, \cdots, \cdot||\,]_\infty^\theta(\Delta_n^m, u, q)$. ■

Theorem 2.4 *If* $0 < \inf p_k = h \leq p_k \leq \sup p_k = H < \infty$ *and* $\mathcal{M} = (M_k)$, $\mathcal{M}' = (M_k')$ *be two Musielak-Orlicz functions satisfying* Δ_2-*condition, then we have*

(i)$[\, c, \mathcal{M}', p, ||\cdot, \cdots, \cdot||\,]_0^\theta(\Delta_n^m, u, q) \subset [\, c, \mathcal{M} \circ \mathcal{M}', p, ||\cdot, \cdots, \cdot||\,]_0^\theta(\Delta_n^m, u, q)$,

(ii)$[\, c, \mathcal{M}', p, ||\cdot, \cdots, \cdot||\,]^\theta(\Delta_n^m, u, q) \subset [\, c, \mathcal{M} \circ \mathcal{M}', p, ||\cdot, \cdots, \cdot||\,]^\theta(\Delta_n^m, u, q)$,

(iii)$[\, c, \mathcal{M}', p, ||\cdot, \cdots, \cdot||\,]_\infty^\theta(\Delta_n^m, u, q) \subset [\, c, \mathcal{M} \circ \mathcal{M}', p, ||\cdot, \cdots, \cdot||\,]_\infty^\theta(\Delta_n^m, u, q)$.

Proof. Let $x = (x_k) \in [c, \mathcal{M}', p, ||\cdot, \cdots, \cdot||]^\theta(\Delta_n^m, u, q)$. Then we have

$$\lim_{r \to \infty} \frac{1}{g_r} \sum_{k \in I_r} \Big[M_k'\Big(q_k\Big(||\frac{u_k \Delta_n^m x_{k+s} - L}{\rho}, z_1, \cdots, z_{n-1}||\Big)\Big)\Big]^{p_k} = 0,$$

uniformly in s for some L.

Let $\epsilon > 0$ and choose δ with $0 < \delta < 1$ such that $M_k(t) < \epsilon$ for $0 \leq t \leq \delta$. Let

$$y_{k,s} = M_k'\Big(q_k\Big(||\frac{u_k \Delta_n^m x_{k+s} - L}{\rho}, z_1, \cdots, z_{n-1}||\Big)\Big) \text{ for all } k, s \in \mathbb{N}.$$

We can write

$$\frac{1}{g_r} \sum_{k \in I_r} [M_k(y_{k,s})]^{p_k} = \frac{1}{g_r} \sum_{k \in I_r, y_{k,s} \leq \delta} [M_k(y_{k,s})]^{p_k} + \frac{1}{g_r} \sum_{k \in I_r, y_{k,s} > \delta} [M_k(y_{k,s})]^{p_k}.$$

Since $\mathcal{M} = (M_k)$ satisfies Δ_2-condition, we have

$$\frac{1}{g_r} \sum_{k \in I_r, y_{k,s} \leq \delta} [M_k(y_{k,s})]^{p_k} \leq [M_k(1)]^H \frac{1}{g_r} \sum_{k \in I_r, y_{k,s} \leq \delta} [M_k(y_{k,s})]^{p_k}$$
$$\leq [M_k(2)]^H \frac{1}{g_r} \sum_{k \in I_r, y_{k,s} \leq \delta} [M_k(y_{k,s})]^{p_k} \qquad (2.1)$$

For $y_{k,s} > \delta$

$$y_{k,s} < \frac{y_{k,s}}{\delta} < 1 + \frac{y_{k,s}}{\delta}.$$

Since $\mathcal{M} = (M_k)$ is non-decreasing and convex, it follows that

$$M_k(y_{k,s}) < M_k\Big(1 + \frac{y_{k,s}}{\delta}\Big) < \frac{1}{2}M_k(2) + \frac{1}{2}M_k\Big(\frac{2y_{k,s}}{\delta}\Big).$$

Since (M_k) satisfies Δ_2-condition, we can write

$$M_k(y_{k,s}) < \frac{1}{2}T\frac{y_{k,s}}{\delta}M_k(2) + \frac{1}{2}T\frac{y_{k,s}}{\delta}M_k(2) = T\frac{y_{k,s}}{\delta}M_k(2).$$

Hence,

$$1g_r \sum_{k\in I_r, y_{k,s}>\delta} [M_k(y_{k,s})]^{p_k} \leq \max\Big(1, \Big(\frac{TM_k(2)}{\delta}\Big)^H\Big)\frac{1}{g_r} \sum_{k\in I_r, y_{k,s}>\delta} [(y_{k,s})]^{p_k} \qquad (2.2)$$

from equations (2.1) and (2.2), we have

$$x = (x_k) \in [\, c, \mathcal{M}\circ\mathcal{M}', p, ||\cdot, \cdots, \cdot||]_0^\theta(\Delta_n^m, u, q).$$

This completes the proof of (i). Similarly, we can prove that

$$[\, c, \mathcal{M}', p, ||\cdot, \cdots, \cdot||]_0^\theta(\Delta_n^m, u, q) \subset [\, c, \mathcal{M}\circ\mathcal{M}', ||\cdot, \cdots, \cdot||\,]_0^\theta(\Delta_n^m, u, q)$$

and

$$[\, c, \mathcal{M}', p, ||\cdot, \cdots, \cdot||\,]_\infty^\theta(\Delta_n^m, u, q) \subset [\, c, \mathcal{M}\circ\mathcal{M}', p, ||\cdot, \cdots, \cdot||\,]_\infty^\theta(\Delta_n^m, u, q).$$

■

Corollary 2.5 *If* $0 < \inf p_k = h \leq p_k \leq \sup p_k = H < \infty$ *and* $\mathcal{M} = (M_k)$ *be Musielak-Orlicz function satisfying* Δ_2*- condition, then we have*

$$[\, c, p, ||\cdot, \cdots, \cdot||\,]_0^\theta(\Delta_n^m, u, q) \subset [\, c, \mathcal{M}, p, ||\cdot, \cdots, \cdot||\,]_0^\theta(\Delta_n^m, u, q)$$

and

$$[\, c, p, ||\cdot, \cdots, \cdot||\,]_\infty^\theta(\Delta_n^m, u, q) \subset [\, c, \mathcal{M}, p, ||\cdot, \cdots, \cdot||\,]_\infty^\theta(\Delta_n^m, u, q).$$

Proof. Taking $\mathcal{M}'(x) = x$ in the above theorem, we get the required result.■

Theorem 2.6 *If* $\mathcal{M} = (M_k)$ *be the Musielak-Orlicz function, then the following statements are equivalent:*
(i) $[\, c, p, ||\cdot, \cdots, \cdot||\,]_\infty^\theta(\Delta_n^m, u, q) \subset [c, \mathcal{M}, p, ||\cdot, \cdots, \cdot||\,]_\infty^\theta(\Delta_n^m, u, q)$,
(ii) $[c, p, ||\cdot, \cdots, \cdot||\,]_0^\theta(\Delta_n^m, u, q) \subset [\, c, \mathcal{M}, p, ||\cdot, \cdots, \cdot||\,]_\infty^\theta(\Delta_n^m, u, q)$,
(iii) $\sup_r \frac{1}{g_r} \sum_{k\in I_r} [M_k(\frac{t}{\rho})]^{p_k} < \infty \ \ (t, \rho > 0)$.

Proof. (i) $\Rightarrow$ (ii) The proof is obvious in view of the fact that

$$[\, c, p, ||\cdot, \cdots, \cdot||\,]_0^{\theta}(\Delta_n^m, u, q) \subset [\, c, p, ||\cdot, \cdots, \cdot||\,]_{\infty}^{\theta}(\Delta_n^m, u, q).$$

(ii) $\Rightarrow$(iii) Let $[\, c, p, ||\cdot, \cdots, \cdot||\,]_0^{\theta}(\Delta_n^m, u, q) \subset [\, c, \mathcal{M}, p, ||\cdot, \cdots, \cdot||\,]_{\infty}^{\theta}(\Delta_n^m, u, q)\cdot$ Suppose that (iii) does not hold. Then for some $t,\ \rho > 0$

$$\sup_r \frac{1}{g_r} \sum_{k \in I_r} [M_k(\frac{t}{\rho})]^{p_k} = \infty$$

and therefore we can find a subinterval $I_{r(j)}$ of the set of interval I_r such that

$$1 g_{r(j)} \sum_{k \in I_{r(j)}} \Big[M_k\Big(\frac{j^{-1}}{\rho}\Big)\Big]^{p_k} > j, \quad j = 1, 2, \tag{2.3}$$

Define the sequence $x = (x_k)$ by

$$\Delta^m x_{k+s} = \begin{cases} j^{-1}, & k \in I_{r(j)} \\ 0, & k \notin I_{r(j)} \end{cases} \quad \text{for all } \ s \in \mathbb{N}.$$

Then $x = (x_k) \in [c, p, ||\cdot, \cdots, \cdot||]_0^{\theta}(\Delta_n^m, u, q)$ but by equation(2.3), $x = (x_k) \notin [c, \mathcal{M}, p, ||\cdot, \cdots, \cdot||\,]_{\infty}^{\theta}(\Delta_n^m, u, q)$, which contradicts (ii). Hence (iii) must hold.
(iii) $\Rightarrow$ (i) Let (iii) hold and $x = (x_k) \in [c, p, ||\cdot, \cdots, \cdot||\,]_{\infty}^{\theta}(\Delta_n^m, u, q)$. Suppose that $x = (x_k) \notin [c, \mathcal{M}, p, ||\cdot, \cdots, \cdot||\,]_{\infty}^{\theta}(\Delta_n^m, u, q)$. Then

$$\sup_{r,s} \frac{1}{g_r} \sum_{k \in I_r} \Big[M_k\Big(||\frac{\Delta^m x_{k+s}}{\rho}, z_1, \cdots, z_{n-1}||\Big)\Big]^{p_k} = \infty. \tag{2.4}$$

Let $t = q_k\Big(||u_k \Delta_n^m x_{k+s}, z_1, \cdots, z_{n-1}||\Big)$ for each k and fixed s, then by equations(2.4)

$$\sup_r \frac{1}{g_r} \sum_{k \in I_r} \Big[M_k\Big(\frac{t}{\rho}\Big)\Big] = \infty,$$

which contradicts (iii). Hence (i) must hold.■

Theorem 2.7 *Let $1 \le p_k \le \sup p_k < \infty$ and $\mathcal{M} = (M_k)$ be a Musielak Orlicz function. Then the following statements are equivalent:*
(i) $[\, c, \mathcal{M}, p, ||\cdot, \cdots, \cdot||\,]_0^{\theta}(\Delta_n^m, u, q) \subset [c, p, ||\cdot, \cdots, \cdot||]_0^{\theta}(\Delta_n^m, u, q)$,
(ii) $[\, c, \mathcal{M}, p, ||\cdot, \cdots, \cdot||\,]_0^{\theta}(\Delta_n^m, u, q) \subset [\, c, p, ||\cdot, \cdots, \cdot||\,]_{\infty}^{\theta}(\Delta_n^m, u, q)$,
(iii) $\inf_r \frac{1}{g_r} \sum_{k \in I_r} \Big[M_k\Big(\frac{t}{\rho}\Big)\Big]^{p_k} > 0 \ \ (t, \rho > 0).$

Proof. (i) $\Rightarrow$ (ii) It is trivial.
(ii) $\Rightarrow$ (iii) Let (ii) hold. Suppose that (iii) does not hold. Then

$$\inf_r \frac{1}{g_r} \sum_{k \in I_r} \Big[M_k\Big(\frac{t}{\rho}\Big)\Big]^{p_k} = 0 \ \ (t, \rho > 0),$$

so we can find a subinterval $I_{r(j)}$ of the set of interval I_r such that

$$\frac{1}{g_{r(j)}}\sum_{k\in I_{r(j)}}\Big[M_k\Big(\frac{j}{\rho}\Big)\Big]^{p_k} < j^{-1}, \ \ j=1,2, \tag{2.5}$$

Define the sequence $x=(x_k)$ by

$$\Delta^m x_{k+s} = \begin{cases} j, & k\in I_{r(j)} \\ 0, & k\notin I_{r(j)} \quad \text{for all} \ \ s\in\mathbb{N}. \end{cases}$$

Thus by equation(2.5), $x=(x_k)\in[\,c,\mathcal{M},p,||\cdot,\cdots,\cdot||]_0^\theta(\Delta_n^m,u,q)$, hence
$x=(x_k)\notin[c,p,||\cdot,\cdots,\cdot||\,]_\infty^\theta(\Delta_n^m,u,q)$, which contradicts (ii). Hence (iii) must hold.

(iii) $\Rightarrow$ (i) Let (iii) hold and suppose that $x=(x_k)\in[\,c,\mathcal{M},p,||\cdot,\cdots,\cdot||\,]_0^\theta(\Delta_n^m,u,q)$, i.e,

$$\lim_{r\to\infty}\frac{1}{g_r}\sum_{k\in I_r}\Big[M_k\Big(q_k\Big(||\frac{u_k\Delta_n^m x_{k+s}}{\rho},z_1,\cdots,z_{n-1}||\Big)\Big)\Big]^{p_k}=0, \tag{2.6}$$

uniformly in s, for some $\rho>0$.

Again, suppose that $x=(x_k)\notin[\,c,p,||\cdot,\cdots,\cdot||\,]_0^\theta(\Delta_n^m,u,q)$. Then, for some number $\epsilon>0$ and a subinterval $I_{r(j)}$ of the set of interval I_r, we have

$$||u_k\Delta_n^m x_{k+s},z_1,\cdots,z_{n-1}||\geq\epsilon$$

for all $k\in\mathbb{N}$ and some $s\geq s_0$. Then, from the properties of the Orlicz function, we can write

$$M_k\Big(q_k\Big(||\frac{u_k\Delta_n^m x_{k+s}}{\rho},z_1,\cdots,z_{n-1}||\Big)\Big)^{p_k}\geq M_k\Big(\frac{\epsilon}{\rho}\Big)^{p_k}$$

and consequently by (2.6)

$$\lim_{r\to\infty}\frac{1}{g_r}\sum_{k\in I_r}\Big[M_k\Big(\frac{\epsilon}{\rho}\Big)\Big]^{p_k}=0,$$

which contradicts (iii). Hence (i) must hold.■

Theorem 2.8 *Let* $0<p_k\leq q_k$ *for all* $k\in\mathbb{N}$ *and* $\Big(\frac{q_k}{p_k}\Big)$ *be bounded. Then,*

$$[\,c,\mathcal{M},q,||\cdot,\cdots,\cdot||\,]^\theta(\Delta_n^m,u,q)\subset[\,c,\mathcal{M},p,||\cdot,\cdots,\cdot||\,]^\theta(\Delta_n^m,u,q).$$

Proof. Let $x\in[\,c,\mathcal{M},q,||\cdot,\cdots,\cdot||\,]^\theta(\Delta_n^m,u,q)$. Write

$$t_k=\Big[M_k\Big(q_k\Big(||u_k\frac{\Delta_n^m x_{k+s}-L}{\rho},z_1,\cdots,z_{n-1}||\Big)\Big)\Big]^{q_k}$$

and $\mu_k = \frac{p_k}{q_k}$ for all $k \in \mathbb{N}$. Then $0 < \mu_k \leq 1$ for $k \in \mathbb{N}$. Take $0 < \mu < \mu_k$ for $k \in \mathbb{N}$. Define the sequences (u_k) and (v_k) as follows: For $t_k \geq 1$, let $u_k = t_k$ and $v_k = 0$ and for $t_k < 1$, let $u_k = 0$ and $v_k = t_k$. Then clearly for all $k \in \mathbb{N}$, we have

$$t_k = u_k + v_k, \qquad t_k^{\mu_k} = u_k^{\mu_k} + v_k^{\mu_k}$$

Now it follows that $u_k^{\mu_k} \leq u_k \leq t_k$ and $v_k^{\mu_k} \leq v_k^{\mu}$. Therefore,

$$\begin{aligned} \frac{1}{g_r} \sum_{k \in I_r} t_k^{\mu_k} &= \frac{1}{g_r} \sum_{k \in I_r} (u_k^{\mu_k} + v_k^{\mu_k}) \\ &\leq \frac{1}{g_r} \sum_{k \in I_r} t_k + \frac{1}{g_r} \sum_{k \in I_r} v_k^{\mu}. \end{aligned}$$

Now for each k,

$$\begin{aligned} \frac{1}{g_r} \sum_{k \in I_r} v_k^{\mu} &= \sum_{k \in I_r} \Big(\frac{1}{g_r} v_k\Big)^{\mu} \Big(\frac{1}{g_r}\Big)^{1-\mu} \\ &\leq \Big(\sum_{k \in I_r} \Big[\Big(\frac{1}{g_r} v_k\Big)^{\mu}\Big]^{\frac{1}{\mu}}\Big)^{\mu} \Big(\sum_{k \in I_r} \Big[\Big(\frac{1}{g_r}\Big)^{1-\mu}\Big]^{\frac{1}{1-\mu}}\Big)^{1-\mu} \\ &= \Big(\frac{1}{g_r} \sum_{k \in I_r} v_k\Big)^{\mu} \end{aligned}$$

and so

$$\frac{1}{g_r} \sum_{k \in I_r} t_k^{\mu_k} \leq \frac{1}{g_r} \sum_{k \in I_r} t_k + \Big(\frac{1}{g_r} \sum_{k \in I_r} v_k\Big)^{\mu}.$$

Hence $x \in [\, c, \mathcal{M}, p, ||\cdot, \cdots, \cdot||\,]^{\theta}(\Delta_n^m, u, q)$. ■

Theorem 2.9 *(a) If* $0 < \inf p_k \leq p_k \leq 1$ *for all* $k \in \mathbb{N}$, *then*

$$[\, c, \mathcal{M}, p, ||\cdot, \cdots, \cdot||\,]^{\theta}(\Delta_n^m, u, q) \subset [\, c, \mathcal{M}, ||\cdot, \cdots, \cdot||\,]^{\theta}(\Delta_n^m, u, q).$$

(b) If $1 \leq p_k \leq \sup p_k < \infty$ *for all* $k \in \mathbb{N}$. *Then*

$$[\, c, \mathcal{M}, ||\cdot, \cdots, \cdot||\,]^{\theta}(\Delta_n^m, u, q) \subset [\, c, \mathcal{M}, p, ||\cdot, \cdots, \cdot||\,]^{\theta}(\Delta_n^m, u, q).$$

Proof. (a) Let $x \in [\, c, \mathcal{M}, p, ||\cdot, \cdots, \cdot||\,]^{\theta}(\Delta_n^m, u, q)$, then

$$\lim_{r \to \infty} \frac{1}{g_r} \sum_{k \in I_r} \Big[M_k\Big(q_k\Big(||\frac{u_k \Delta_n^m x_{k+s} - L}{\rho}, z_1, \cdots, z_{n-1}||\Big)\Big]^{p_k} = 0.$$

Since $0 < \inf p_k \leq p_k \leq 1$. This implies that

$$\lim_{r\to\infty} \frac{1}{g_r} \sum_{k\in I_r} \Big[M_k\Big(q_k\Big(||\frac{u_k\Delta_n^m x_{k+s} - L}{\rho}, z_1, \cdots, z_{n-1}||\Big)\Big)\Big]$$
$$\leq \lim_{r\to\infty} \frac{1}{g_r} \sum_{k\in I_r} \Big[M_k\Big(q_k\Big(||\frac{u_k\Delta_n^m x_{k+s} - L}{\rho}, z_1, \cdots, z_{n-1}||\Big)\Big)\Big]^{p_k},$$

therefore, $\lim_{r\to\infty} \frac{1}{g_r} \sum_{k\in I_r} \Big[M_k\Big(q_k\Big(||\frac{u_k\Delta_n^m x_{k+s} - L}{\rho}, z_1, \cdots, z_{n-1}||\Big)\Big)\Big] = 0$. This shows that $x \in [\, c, \mathcal{M}, ||\cdot, \cdots, \cdot||\,]^\theta(\Delta_n^m, u, q)\cdot$ Therefore,

$$[\, c, \mathcal{M}, p, ||\cdot, \cdots, \cdot||\,]^\theta(\Delta_n^m, u, q) \subset [\, c, \mathcal{M}, ||\cdot, \cdots, \cdot||\,]^\theta(\Delta_n^m, u, q).$$

This completes the proof.

(b) Let $p_k \geq 1$for each k and $\sup p_k < \infty$. Let $x \in [\, c, p, ||\cdot, \cdots, \cdot||\,]^\theta(\Delta_n^m, u, q)$. Then for each $\epsilon > 0$ there exists a positive integer N such that

$$\lim_{r\to\infty} \frac{1}{g_r} \sum_{k\in I_r} \Big[M_k\Big(q_k\Big(||\frac{u_k\Delta_n^m x_{k+s} - L}{\rho}, z_1, \cdots, z_{n-1}||\Big)\Big)\Big]^{p_k} = 0 < 1.$$

Since $1 \leq p_k \leq \sup p_k < \infty$, we have

$$\lim_{r\to\infty} \frac{1}{g_r} \sum_{k\in I_r} \Big[M_k\Big(q_k\Big(||\frac{u_k\Delta_n^m x_{k+s} - L}{\rho}, z_1, \cdots, z_{n-1}||\Big)\Big)\Big]^{p_k}$$
$$\leq \lim_{r\to\infty} \frac{1}{g_r} \sum_{k\in I_r} \Big[M_k\Big(q_k\Big(||\frac{u_k\Delta^m x_{k+s} - L}{\rho}, z_1, \cdots, z_{n-1}||\Big)\Big)\Big]$$
$$= 0$$
$$< 1.$$

Therefore $x \in [\, c, \mathcal{M}, p, ||\cdot, \cdots, \cdot||\,]^\theta(\Delta_n^m, u, q)$. ∎

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Kuldip Raj - corresponding author
email: kuldeepraj68@rediffmail.com
School of Mathematics
Shri Mata Vaishno Devi University
Katra-182320, J&K, India

Sunil K. Sharma
email: sunilksharma42@gmail.com
Department of mathematics
Model Institute of Engineering & Technology
Kot Bhalwal-181122, J&K, INDIA

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