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Boolean Algebra of One-Point Local Compactifications

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Abstract: For a given locally compact Hausdorff space we introduce a Boolean algebra structure on the family of all its one-point local compactifications.

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1. Introduction

Every locally compact, noncompact Hausdorff space X has a well known one-point compactification (Alexandroff compactification, [1]). In this paper we consider the set $\mathcal{B}(X)$ of all one-point local compactifications of X up to an equivalence. We prove that $\mathcal{B}(X)$ is a partially ordered set such that the order $\leqslant$ induces a Boolean algebra. Moreover, the elements 0 and 1 of $\mathcal{B}(X)$ are respectively X and ωX. Then we focus on describing the algebra we get using topological notions and convergence and we provide examples by computing the algebra in some special cases. We also note the connection with the topic of ends of manifolds (see [2, pages 110-118]), as for a noncompact, connected, second countable manifold L with n ends, $n < \infty$, we have $|\mathcal{B}(L)| = 2^n$.

2. Notation and terminology

- Throughout the paper, ZFC is assumed.
- Given a locally compact Hausdorff space X we denote by ωX a one-point compactification of X if X is not compact and X otherwise,

- a clopen set is a set that is both closed and open,
- if Y is a one-point local compactification different from X, the unique point of $Y \backslash X$ will be denoted by ∞_Y,
- a filter $\mathcal{F}$ of open sets in a topological space X is a non-empty family of sets open in X such that $\emptyset \notin \mathcal{F}$ and, for all $V_1, V_2 \in \mathcal{F}$ and an open $V \subset X$ we have $V_1 \cap V_2 \in \mathcal{F} \Rightarrow V \in \mathcal{F}$.

3. Main results

Definition 1. If X is a locally compact Hausdorff space, we call (Y, f) an at most one-point local compactification of X iff Y is a locally compact Hausdorff and $f : X \to Y$ is a homeomorphic embedding such that $f(X)$ is dense in Y and $|Y \setminus f(X)| \leqslant 1$. If (Y, f) is an at most one point local compactification of X and $|Y \setminus f(X)| = 1$, we call (Y, f) a one-point local compactiication of X.
For simplicity, we say that Y is a/an (at most) one-point local compactification of X iff (Y, id_X) is a/an (at most) one-point local compactification of X.

Definition 2. Let X be a locally compact Hausdorff space, (Y_1, f_1) and (Y_2, f_2) its at most one-point local compactifications. We will write $(Y_1, f_1) \leqslant (Y_2, f_2)$ (or, for simplicity, $Y_1 \leqslant Y_2$) iff one of the following conditions apply:

- $f_1(X) = Y_1$
- $Y_1 = f_1(X) \cup \{\infty_{Y_1}\}$, $Y_2 = f_2(X) \cup \{\infty_{Y_2}\}$ and the function

  $$Y_1 \ni x \mapsto \begin{cases} f_2(f_1^{-1}(x)), & x \in f_1(X) \\ \infty_{Y_2}, & x = \infty_{Y_1} \end{cases} \in Y_2$$

  is continuous.

Note that $\leqslant$ is reflexive and transitive, with $0 = X$ and $1 = \omega X$. We can define an equivalence relation $\equiv$ by

$$(Y_1, f_1) \equiv (Y_2, f_2) \text{ iff } (Y_1, f_1) \leqslant (Y_2, f_2) \text{ and } (Y_2, f_2) \leqslant (Y_1, f_1),$$

or, for simplicity,

$$Y_1 \equiv Y_2 \text{ iff } Y_1 \leqslant Y_2 \text{ and } Y_2 \leqslant Y_1.$$

We also define

$$\mathcal{B}(X) := {}^{\{Y\text{—one-point local compactification of } X\}}\!/\!_{\equiv}.$$

From now on instead of an equivalence class of Y in $\mathcal{B}(X)$ we will just write Y.

We are now ready to state the first result where we will prove that $\mathcal{B}(X)$ ordered by $\leqslant$ is a Boolean algebra, by showing that it is in fact order isomorphic to a much simpler one.

Theorem 1. *Given a locally compact Hausdorff space X, $\mathcal{B}(X)$ is a partially ordered space with a lattice such that the order $\leqslant$ induces a Boolean algebra, i.e., for Y_1, Y_2 one-point local compactifications of X:*

- $Y_1 \vee Y_2 = \sup_{\leqslant}\{Y_1, Y_2\}$,
- $Y_1 \wedge Y_2 = \inf_{\leqslant}\{Y_1, Y_2\}$,
- $0 = X$,
- $1 = \omega X$,
- *for any space $Y \in \mathcal{B}(X)$ there exists a unique space $\backslash Y \in \mathcal{B}(X) : Y \wedge \backslash Y = 0$, $Y \vee \backslash Y = 1$.*

In particular, $0 = 1$ *iff X is compact.*

Proof. First consider βX, a Čech–Stone compactification of X. We define $\mathcal{A}(X) := \{F \subset \beta X \setminus X : F \text{ clopen in } \beta X \setminus X\}$ (note that $\beta X \setminus X$ is compact). $\mathcal{A}(X)$ with standard set operations is a Boolean algebra. We will show an isomorphism between $\mathcal{B}(X)$ and $\mathcal{A}(X)$, proving that $\mathcal{B}(X)$ is also a Boolean algebra.

To this end, we will define $f : \mathcal{B}(X) \to \mathcal{A}(X)$. If X is compact, both $\mathcal{B}(X)$ and $\mathcal{A}(X)$ are trivial, therefore assume that X is not compact. Consider a clopen in $\beta X \setminus X$ set F such that $\emptyset \neq F \neq \beta X \setminus X$. We can now identify F and $(\beta X \setminus X) \setminus F$ with points, getting a compact space $X \cup \{\{F\}\} \cup \{\{(\beta X \setminus X) \setminus F\}\}$. Its subspace $X \cup \{\{F\}\}$ is then a one-point local compactification of X. Conversely, for any one-point local compactification Y of X there exists a unique clopen in $\beta X \setminus X$ set F_Y such that Y is equivalent with $X \cup \{\{F_Y\}\}$ (from the universal property of βX). We define $f(X) = \emptyset$ and for every one-point local compactification Y of X we put $f(Y) = F_Y$, where Y is the unique clopen in $\beta X \setminus X$ set such that Y is equivalent to $X \cup \{\{F_Y\}\}$. It can be easily seen that for one-point local compactifications Y_1, Y_2 of X we have $Y_1 \leqslant Y_2$ iff $F_{Y_1} \subset F_{Y_2}$, so f preserves the partial order and is indeed an isomorphism. Furthermore, for one-point local compactifications Y_1, Y_2 of X we have:

1. $Y_1 \vee Y_2 = X \cup \{\{F_{Y_1} \cup F_{Y_2}\}\}$.
2. $Y_1 \wedge Y_2 = X \cup \{\{F_{Y_1} \cap F_{Y_2}\}\}$ if $F_{Y_1} \cap F_{Y_2} \neq \emptyset$ and $Y_1 \wedge Y_2 = X$ otherwise.
3. $\backslash Y = X \cup \{\{(\beta X \setminus X) \setminus F_Y\}\}$ for $\emptyset \neq F_Y \neq \beta X \setminus X$.

□

Remark 1. The proof of Theorem 1 shows that $\mathcal{B}(X)$ is isomorphic (as a Boolean algebra) to the algebra of all clopen subsets of the remainder $\beta X \setminus X$ of X. One easily concludes that the Stone space of $\mathcal{B}(X)$ is homeomorphic to the space of all connected components of $\beta X \setminus X$ (that is, the space obtained from $\beta X \setminus X$ by identifying points that lie in a common connected component).

Now that we know that $\mathcal{B}(X)$ is a Boolean algebra, we will focus on describing it without using $\mathcal{A}(X)$. If we add a point $\{\infty_Y\}$ to a locally compact Hausdorff space X to get its one-point local compactification Y, we only need to know the neighborhood basis at $\{\infty_Y\}$ to know its topology. To this end, let us introduce the following characterization. For simplicity, we will also use one more definition.

Definition 3. Let X be a locally compact Hausdorff space, Y its one-point local compactification. Then

$$\tau(Y) := \{U \backslash \{\infty_Y\} : U \text{ open neighborhood of } \infty_Y \text{ in } Y\}.$$

$\tau(Y)$ uniquely determines $Y \neq X$, $Y \in \mathcal{B}(X)$.

Proposition 1. *Let X be a locally compact Hausdorff space, $Y_1, Y_2 \in \mathcal{B}(X)$, $Y_1, Y_2 \neq 0$, $Y_1, Y_2 \neq 1$.*

1. *$\tau(Y_1 \wedge Y_2) = \{U_1 \cap U_2 : U_1 \in \tau(Y_1), U_2 \in \tau(Y_2)\}$, provided that the sets $U_1 \cap U_2$ are nonempty for all $U_1 \in \tau(Y_1), U_2 \in \tau(Y_2)$ and $Y_1 \wedge Y_2 = 0$ otherwise.*
2. *$\tau(Y_1 \vee Y_2) = \{U_1 \cup U_2 : U_1 \in \tau(Y_1), U_2 \in \tau(Y_2)\} = \tau(Y_1) \cap \tau(Y_2)$.*
3. *$\tau(\backslash Y_1) = \{X \backslash F : F \subset X$, for any $U \in \tau(Y_1)$ $F \backslash U$ compact$\}$.*

Or, in terms of convergence:

(a) *A net $(x_\gamma) \subset X$ in $Y_1 \wedge Y_2$ is convergent to $\infty_{Y_1 \wedge Y_2}$ iff (x_γ) is convergent to ∞_{Y_1} in Y_1 and to ∞_{Y_2} in Y_2, and $Y_1 \wedge Y_2 = 0$ if there is no such net.*

(b) *A net $(x_\gamma) \subset X$ in $Y_1 \vee Y_2$ is convergent to $\infty_{Y_1 \vee Y_2}$ iff every subnet of (x_γ) has a subnet convergent to ∞_{Y_1} in Y_1 or to ∞_{Y_2} in Y_2.*

(c) *A net $(x_\gamma) \subset X$ in $\backslash Y_1$ is convergent to $\infty_{\backslash Y_1}$ iff (x_γ) has no convergent subnets in Y_1.*

Proof. Again, let βX be a Čech–Stone compactification of X.

Note that if Y is a one-point local compactification of X and F_Y is a clopen set in $\beta X \setminus X$ such that Y is equivalent with $X \cup \{\{F_Y\}\}$, then

$$\tau(Y) = \{X \cap U : U \supset F_Y \text{ and } U \text{ open in } \beta X\}. \tag{*}$$

Following this notation consider F_{Y_1} and F_{Y_2} such that Y_1 and Y_2 are equivalent to $X \cup \{\{F_{Y_1}\}\}$ and $X \cup \{\{F_{Y_2}\}\}$ respectively.

Property (2) follows easily from (*).

To see that $\{U_1 \cup U_2 : U_1 \in \tau(Y_1), U_2 \in \tau(Y_2)\} = \tau(Y_1) \cap \tau(Y_2)$, take any $U_1 \in \tau(Y_1), U_2 \in \tau(Y_2)$. $U_2 = (U_2 \cup \{\infty_{Y_2}\}) \cap X$ is open in X, and thus open in Y_1. $U_1 \cup \{\infty_{Y_1}\}$ is also open in Y_1 and thus so is $U_1 \cup \{\infty_{Y_1}\} \cup U_2$. Similarly, $U_1 \cup \{\infty_{Y_2}\} \cup U_2$ is open in Y_2. The reverse inclusion is trivial.

We turn to (1). If $F_{Y_1} \cap F_{Y_1} = \emptyset$ we have $Y_1 \wedge Y_2 = 0$, assume the contrary. Consider U open in βX such that $F_{Y_1} \cap F_{Y_1} \subset U$ and take V_1, V_2 open in βX such that $V_1 \cap V_2 = \emptyset$, and we have $F_{Y_1} \setminus U \subset V_1$ and $F_{Y_2} \setminus U \subset V_2$. Then $U_1 := V_1 \cup U$ and $U_2 := V_2 \cup U$ are open (in βX) supersets of respectively F_{Y_1} and F_{Y_2} such that $U_1 \cap U_2 = U$, which gives us (1).

We are left with (3). To see that

$$\tau(\setminus Y_1) \subset \{X \setminus F : F \subset X, \text{ for any } U \in \tau(Y_1)\ F \setminus U \text{ compact}\},$$

consider V open in βX such that $(\beta X \setminus X) \setminus F_{Y_1} \subset V$ and take any U open in βX such that $F_{Y_1} \subset U$. Then $(X \setminus V) \setminus U = X \setminus (U \cup V) = \beta X \setminus (U \cup V)$ is a closed subset of βX contained in X and therefore compact.

For the reverse inclusion, let V_0 and W_0 be open sets with disjoint closures in βX such that $(\beta X \setminus X) \setminus F_{Y_1} \subset V_0$ and $F_{Y_1} \subset W_0$. Consider $F \subset X$ such that for any $U \in \tau(Y_1)$ the set $F \setminus U$ is compact. Take any $x \in X$ and its closed (in X) neighborhood G such that G is compact. Then $X \setminus G \in \tau(Y_1)$, so $F \cap G$ is compact. Since x and its neighborhood G were arbitrary, this implies that F is closed in X (since if we take x from the boundary of F, we get that it must be in F). Similarly, since $F \cap \overline{V_0} \subset F \setminus W_0$ and $W_0 \cap X \in \tau(Y_1)$, we get that $F \cap \overline{V_0}$ is compact which implies that $F \cup F_{Y_1}$ is closed in βX. Therefore we have $X \setminus F = X \cap (\beta X \setminus (F \cup F_0)) \in \tau(\setminus Y_1)$ which ends the proof of (3).

Properties (a) – (c) follow easily from (1) – (3). □

On the other hand, one can wonder when a family $\mathcal{F}$ of sets open in a locally compact Hausdorff space X induces its one-point local compactifiaction. The following proposition answers that question.

Proposition 2. *Let $\mathcal{F}$ be a filter of open sets in a locally compact Hausdorff space X. Then $\mathcal{F}$ induces a one-point local compactification Y of X such that $\tau(Y) = \mathcal{F}$ iff:*

1. *$\bigcap \mathcal{F} = \emptyset$,*
2. *there exists $U \in \mathcal{F}$ such that for every $V \in \mathcal{F}$, $\overline{U} \setminus V$ is compact,*
3. *for every $U \in \mathcal{F}$ there exists $V \in \mathcal{F}$ such that $\overline{V} \subset U$.*

Proof. It follows from the definition of $\tau(Y)$ and the definition of a locally compact Hausdorff space that those conditions are necessary. We will prove that they are also sufficient. We take $Y := X \cup \{\infty_Y\}$. A set is open in Y iff it is open in X or it is of the form $U \cup \{\infty_Y\}$ for some $U \in \mathcal{F}$. It follows from (1) and (3) that the topology defined like that is Hausdorff. It remains to show that Y is locally compact. Take $U \in \mathcal{F}$ such that for every $V \in \mathcal{F}$ $\overline{U} \setminus V$ is compact and assume that $\overline{U}$ (closure taken in Y) is not compact. It follows that there exists a net $(x_\gamma) \subset \overline{U}$ with no convergent subnets. In particular, (x_γ) is not convergent to ∞_Y, so there exists V_1 a neighborhood of ∞_Y and (y_γ) a subnet of (x_γ) such that $(y_\gamma) \subset \overline{U} \setminus V_1$ with no convergent subnets, a contradiction. □

We will now provide a characterization for $\mathcal{B}(\mathbb{R}^n)$. To this end, we will need facts about n-point Hausdorff compactifications (see [5] or [3, Theorem 6.8]).

Theorem 2 (Theorem 2.1 in [5]). *The following statements concerning a space X are equivalent:*

1. *X has a N-point compactification.*

2. *X is locally compact and contains a compact subset K whose complement is the union of N mutually disjoint, open subsets $\{G_i\}_{i=1}^N$ such that $K \cup G_i$ is not compact for each i.*

3. *X is locally compact and contains a compact subset K whose complement is the union of N mutually disjoint, open subsets $\{G_i\}_{i=1}^N$ such that $K \cup G_i$ is contained in no compact subset for each i.*

Using this, we can prove the following facts.

Lemma 1. *Let X be a locally compact, noncompact Hausdorff space such that for any $K \subset X$ compact there exists K_0 compact such that $K \subset K_0$ and $X \backslash K_0$ has exactly n connected components (for some fixed $n \in \mathbb{N}$ independent of the choice of K), all of them are open and have noncompact (in X) closures. Then X has an n-point Hausdorff compactification and does not have an $(n+1)$-point Hausdorff compactification.*

Lemma 2. *Let $n \in \mathbb{N}$ and X be a Hausdorff topological space that has an n-point Hausdorff compactification and does not have an $(n+1)$-point Hausdorff compactification. Then X is locally compact and $|\mathcal{B}(X)| = 2^n$.*

We will start with Lemma 1.

Proof. Applying the assumption of the lemma to the empty set we get that there exists $n \in \mathbb{N}$ and K_0 compact such that $X \backslash K_0$ has exactly n connected components, let us denote them by $G_1, \dots, G_n$. Therefore (by [5]) X has an n-point Hausdorff compactification. Suppose that X has an $(n+1)$-Hausdorff compactification. Again by [5], there exist $H_1, \dots, H_{n+1}$ such that $K_1 := X \setminus \bigcup_{i=1}^{n+1} H_i$ is compact, but for each i the set $K_1 \cup H_i$ is not compact. Applying the assumption of the lemma again, this time to K_1, we get that there exists a compact set K_2 such that $K_1 \subset K_2$ and $X \setminus K_2$ has n connected components, let us denote them by $V_1, \dots, V_n$. Then there exist $i_0 \in \{1, \dots, n\}$ and $j_1, j_2 \in \{1, \dots, n+1\}$ such that $j_1 \neq j_2$ and H_{i_0} has nonempty intersection with both V_{j_1}, V_{j_2}, so it cannot be connected, a contradiction. □

Now we turn to Lemma 2.

Proof. Since X has an n-point Hausdorff compactification, but does not have an $n+1$-point Hausdorff compactification, $\beta X \setminus X$ has exactly n connected components. From the proof of Theorem 1 we know that $|\mathcal{B}(X)| = |\mathcal{A}(X)|$. Each element of $\mathcal{A}(X)$ is a union of some connected components of $\beta X \setminus X$, so $|\mathcal{B}(X)| = |\mathcal{A}(X)| = 2^n$. □

Remark 2. Note that if we assume that if X is a locally compact space such that $|\mathcal{B}(X)| = 2^n$, we also get that X has an n-point Hausdorff compactification and does not have an $(n+1)$-point Hausdorff compactification (see also [3, Theorem 6.32]).

From the above lemmas we immediately get the following.

Corollary 1.

- $\mathcal{B}(\mathbb{R}) = \{\mathbb{R}, [-\infty, \infty), (-\infty, \infty], \mathbb{S}^1\}$.
- $\mathcal{B}(\mathbb{R}^n) = \{\mathbb{R}^n, \mathbb{S}^n\}$ *for* $n \geqslant 2$.

We will now define the end of manifolds, as seen in [2].

Definition 4. Let L be a noncompact, connected manifold. Denote by $\{K_\alpha\}_{\alpha\in\mathcal{K}}$ the family of all compact subsets of L. We consider descending chains

$$U_{\alpha_1} \supsetneq U_{\alpha_2} \supsetneq \cdots \supsetneq U_{\alpha_n} \supsetneq \cdots$$

where each U_{α_k} is a connected component of $L\backslash K_{\alpha_k}$, has noncompact closure in L, satisfies $U_{\alpha_k} \supsetneq \overline{U_{\alpha_{k+1}}}$ and

$$\bigcap_{k=1}^{\infty} U_{\alpha_k} = \emptyset.$$

We say that two such chains $\mathcal{U} = \{U_{\alpha_k}\}_{k=1}^{\infty}$ and $\mathcal{V} = \{U_{\beta_k}\}_{k=1}^{\infty}$ are equivalent ($\mathcal{U} \sim \mathcal{V}$) if for each $k \geqslant 1$ there is $n > k$ such that $U_{\alpha_k} \supset V_{\beta_n}$ and $V_{\beta_k} \supset U_{\alpha_n}$. It is easy to check that $\sim$ is an equivalence relation. If

$$\mathcal{U} = \{U_{\alpha_1} \supsetneq U_{\alpha_2} \supsetneq \cdots \supsetneq U_{\alpha_n} \supsetneq \cdots\}$$

is as above, we call its equivalence class under $\sim$ an *end* of L.

Corollary 2.
If L is a noncompact, connected, second countable manifold with n ends, $n < \infty$, then $|\mathcal{B}(L)| = 2^n$.

Proof. Let

$$\mathcal{U}_1 = \{U^1_{\alpha_1} \supsetneq U^1_{\alpha_2} \supsetneq \cdots\}$$
$$\vdots$$
$$\mathcal{U}_n = \{U^n_{\alpha_1} \supsetneq U^n_{\alpha_2} \supsetneq \cdots\}$$

be representatives of the ends of L.

For every $k \in \{1, 2, \ldots\}, l \in \{1, 2, \ldots, n\}$ let $K^l_{\alpha_k}$ be a compact set such that $U^l_{\alpha_k}$ is a connected component of $L\backslash K^l_{\alpha_k}$. We will show that by taking subsequences of $\mathcal{U}_2, \ldots, \mathcal{U}_n$ we can assume that $U^{l_2}_{\alpha_k} \subset L\backslash K^{l_1}_{\alpha_k}$ for every $l_2 > l_1$ (note that a subsequence of a representative of an end is a representative of the same end).

Consider $K^1_{\alpha_1}$. Then $\{L\backslash\overline{U^2_{\alpha_1}}, L\backslash\overline{U^2_{\alpha_2}}, \ldots\}$ is an open cover of $K^1_{\alpha_1}$ so there exists $N_1 > 0$ such that $K^1_{\alpha_1} \subset L\backslash\overline{U^2_{\alpha_{N_1}}} \subset L\backslash U^2_{\alpha_{N_1}}$. Therefore $U^2_{\alpha_{N_1}} \subset L\backslash K^1_{\alpha_1}$. Similarly, for each $m > 1$, we can define $N_m > N_{m-1}$ such that $U^2_{\alpha_{N_m}} \subset L\backslash K^1_{\alpha_m}$. Replacing $U^2_{\alpha_m}$ by $U^2_{\alpha_{N_m}}$ for each $m > 0$ we get a subsequence we want for $\mathcal{U}_2$. Now we proceed similarly for $\mathcal{U}_3, \ldots, \mathcal{U}_n$.

We will now show that by again taking subsequences we can assume that for every $l_1 \neq l_2$ we have $U^{l_1}_{\alpha_1} \cap U^{l_2}_{\alpha_1} = \emptyset$. Assume the contrary. Then, without loss of generality, for each $k > 0$ we have $U^1_{\alpha_k} \cap U^2_{\alpha_k} \neq \emptyset$. Since $U^2_{\alpha_k} \subset L \backslash K^1_{\alpha_k}$, the set $U^2_{\alpha_k}$ is connected, $U^1_{\alpha_k}$ is a connected component of $L \backslash K^1_{\alpha_k}$ and $U^1_{\alpha_k} \cap U^2_{\alpha_k} \neq \emptyset$, it follows that $U^2_{\alpha_k} \subset U^1_{\alpha_k}$ for each $k > 0$. Now consider $K^2_{\alpha_k}$. As before, there exists $N_k > k$ such that $K^2_{\alpha_k} \subset L \backslash U^1_{\alpha_{N_k}}$. It follows that $U^1_{\alpha_{N_k}} \subset L \backslash K^2_{\alpha_k}$. If $U^1_{\alpha_{N_k}} \not\subset U^2_{\alpha_k}$ then $U^1_{\alpha_{N_k}} \cap U^2_{\alpha_k} = \emptyset$, so $U^1_{\alpha_{N_k}} \cap U^2_{\alpha_{N_k}} = \emptyset$ (since $U^2_{\alpha_{N_k}} \subset U^2_{\alpha_k}$). Therefore $U^1_{\alpha_{N_k}} \subset U^2_{\alpha_k}$ and so $\mathcal{U}_1$ and $\mathcal{U}_2$ are representatives of the same end, a contradiction.

Now our aim is to use Lemmas 1 and 2, which will end the proof. To this end, we will construct a family of compact sets $\{K_j\}_{j=1}^{\infty}$. We will need some properties of manifolds, namely that a second countable manifold is metrizable and that the one-point compactification of a connected manifold is locally connected (see [4] or [6, page 104]). Let $\omega L = L \cup \{\infty\}$ be the one-point compactification of L. Since L is second countable we can choose a countable basis of its topology $\mathcal{B} = \{B_1, B_2, B_3, \ldots\}$ consisting of open sets with compact closures. Take $A_1 := K^1_{\alpha_1} \cup \ldots \cup K^n_{\alpha_1} \cup \overline{B_1}$. Let K_1 be a compact superset of A_1 such that $\omega L \backslash K_1$ is connected (it exists because ωL is locally connected). Note that connected components of $L \backslash K_1$ are all open and have noncompact (in L) closures (because ∞ is in the closure taken in ωL of every one of them). Again, because L is locally compact we can take an open set A_2 with compact closure such that $K_1 \cup \overline{B_2} \subset A_2$. Let K_2 be a compact superset of A_2 such that $\omega L \backslash K_2$ is connected. As before, all connected components of $L \backslash K_2$ are open and have noncompact (in L) closures. Moreover, each of them is contained together with its closure in some connected component of $L \backslash K_1$. Note that since $\omega L \backslash K_2$ has non-empty intersection with every connected component of $L \backslash K_1$ (because ∞ is in the closure taken in ωL of every one of them), for every connected component of $L \backslash K_1$ there is at least one connected component of $L \backslash K_2$ contained in it. Continuing in this manner, we get $\{K_j\}_{j=1}^{\infty}$. Note that K_j is contained in the interior of K_{j+1} for each $j \geqslant 1$ and $\bigcup_{j=1}^{\infty} K_j = L$. Moreover, when j increases the number of connected components of $L \backslash K_j$ either increases or stays the same. Consider a connected component U_1 of $L \setminus K_1$. We want to show that $U_1 \cap U^i_{\alpha_1} \neq \emptyset$ for some i. Indeed, otherwise by choosing a connected component U_2 of $U_1 \setminus K_2$, then a connected U_3 of $U_2 \setminus K_3$ etc. we would get a representative of an end that is not among $\mathcal{U}_1, \ldots, \mathcal{U}_n$, a contradiction. Suppose that $U_1 \cap U^1_{\alpha_1} \neq \emptyset$. Since $K^1_{\alpha_1} \subset K_1$ and $U^1_{\alpha_1}, U_1$ are connected components of their complements we get $U_1 \subset U^1_{\alpha_1}$. The sets $U^i_{\alpha_1}$ are pairwise disjoint, so $L \setminus K_1$ has at least n connected components. Moreover, the number of connected components of $L \backslash K_j$ cannot increase past n for any j. Indeed, if we had at least $n+1$ connected components of $L \setminus K_j$ for some j, we could construct at least $n+1$ different ends (similarly as before) which again contradicts the fast that $\mathcal{U}_1, \ldots, \mathcal{U}_n$ are all of the ends in L. Lemma 1 ends the proof. □

From this and Remark 2 we also get the following.

Corollary 3. *If L is a noncompact, connected, second countable manifold with n ends, $n < \infty$, then L has an n-point Hausdorff compactification and does not have an $(n+1)$-point Hausdorff compactification.*

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References

[1] P. Alexandroff, *Über die Metrisation der im Kleinen kompakten topologischen Räume*, Math. Ann. 92 (1924) 294–301.

[2] A. Candel, L. Conlon, *Foliations I*, Amer. Math. Soc., 1999.

[3] R. Chandler, *Hausdorff Compactifications*, Marcel Dekker, New York, 1976.

[4] J. De Groot, R.H. McDowell, *Locally connected spaces and their compactifications*, Illinois J. Math. 11 Issue 3 (1967) 353-364.

[5] K.D. Magill Jr., *N-point compactifications*, Am. Math. Mon. 72 (1965) 1075–1081.

[6] R.L. Wilder, *Topology of manifolds*, Amer. Math. Soc. Colloquium Publications 42 (1949).

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