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# On a study of double gai sequence space

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ABSTRACT: Let $\chi^2$ denote the space of all prime sense double gai sequences and $\Lambda^2$ the space of all prime sense double analytic sequences. This paper is devoted to the general properties of $\chi^2$.

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## 1. Introduction

Throughout $w, \chi$ and $\Lambda$ denote the classes of all, gai and analytic scalar valued single sequences, respectively. We write $w^2$ for the set of all complex sequences $(x_{mn})$, where $m, n \in \mathbb{N}$, the set of positive integers. Then, $w^2$ is a linear space under the coordinate wise addition and scalar multiplication.

Some initial works on double sequence spaces is found in Bromwich[4]. Later on, they were investigated by Hardy[8], Moricz[12], Moricz and Rhoades[13], Basarir and Solankan[2], Tripathy[20], Colak and Turkmenoglu[6], Turkmenoglu[22], and many others.

Let us define the following sets of double sequences:

$$\mathcal{M}_u(t) := \left\{(x_{mn}) \in w^2 : sup_{m,n\in N} |x_{mn}|^{t_{mn}} < \infty\right\},$$
$$\mathcal{C}_p(t) := \left\{(x_{mn}) \in w^2 : p - lim_{m,n\to\infty} |x_{mn} - l|^{t_{mn}} = 1\, for\, some\, l \in \mathbb{C}\right\},$$
$$\mathcal{C}_{0p}(t) := \left\{(x_{mn}) \in w^2 : p - lim_{m,n\to\infty} |x_{mn}|^{t_{mn}} = 1\right\},$$
$$\mathcal{L}_u(t) := \left\{(x_{mn}) \in w^2 : \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} |x_{mn}|^{t_{mn}} < \infty\right\},$$
$$\mathcal{C}_{bp}(t) := \mathcal{C}_p(t)$$

where $t = (t_{mn})$ is the sequence of strictly positive reals $t_{mn}$ for all $m, n \in \mathbb{N}$ and $p - lim_{m,n\to\infty}$ denotes the limit in the Pringsheim's sense. In the case $t_{mn} = 1$ for all $m, n \in \mathbb{N}; \mathcal{M}_u (t), \mathcal{C}_p (t), \mathcal{C}_{0p} (t), \mathcal{L}_u (t), \mathcal{C}_{bp} (t)$ and $\mathcal{C}_{0bp} (t)$ reduce to the sets $\mathcal{M}_u, \mathcal{C}_p, \mathcal{C}_{0p}, \mathcal{L}_u, \mathcal{C}_{bp}$ and $\mathcal{C}_{0bp}$, respectively.

Now, we may summarize the knowledge given in some document related to the double sequence spaces. Gökhan and Colak [27,28] have proved that $\mathcal{M}_u (t)$ and $\mathcal{C}_p (t), \mathcal{C}_{bp} (t)$ are complete paranormed spaces of double sequences and gave the $\alpha-, \beta-, \gamma-$ duals of the spaces $\mathcal{M}_u (t)$ and $\mathcal{C}_{bp} (t)$. Quite recently, in her PhD thesis, Zelter [29] has essentially studied both the theory of topological double sequence spaces and the theory of summability of double sequences. Mursaleen and Edely [30] have recently introduced the statistical convergence and Cauchy for double sequences and given the relation between statistical convergent and strongly Cesàro summable double sequences. Nextly, Mursaleen [31] and Mursaleen and Edely [32] have defined the almost strong regularity of matrices for double sequences and applied these matrices to establish a core theorem and introduced the $M-$core for double sequences and determined those four dimensional matrices transforming every bounded double sequences $x = (x_{jk})$ into one whose core is a subset of the $M-$core of $x$. More recently, Altay and Basar [33] have defined the spaces $\mathcal{BS}, \mathcal{BS} (t), \mathcal{CS}_p, \mathcal{CS}_{bp}, \mathcal{CS}_r$ and $\mathcal{BV}$ of double sequences consisting of all double series whose sequence of partial sums are in the spaces $\mathcal{M}_u, \mathcal{M}_u (t), \mathcal{C}_p, \mathcal{C}_{bp}, \mathcal{C}_r$ and $\mathcal{L}_u$, respectively, and also examined some properties of those sequence spaces and determined the $\alpha-$ duals of the spaces $\mathcal{BS}, \mathcal{BV}, \mathcal{CS}_{bp}$ and the $\beta (\vartheta) -$ duals of the spaces $\mathcal{CS}_{bp}$ and $\mathcal{CS}_r$ of double series. Quite recently Basar and Sever [34] have introduced the Banach space $\mathcal{L}_q$ of double sequences corresponding to the well-known space $\ell_q$ of single sequences and examined some properties of the space $\mathcal{L}_q$. Quite recently Subramanian and Misra [35] have studied the space $\chi^2_M (p, q, u)$ of double sequences and gave some inclusion relations. We need the following inequality in the sequel of the paper. For $a, b, \geq 0$ and $0 < p < 1$, we have

$$(a + b)^p \leq a^p + b^p \tag{1}$$

The double series $\sum_{m,n=1}^{\infty} x_{mn}$ is called convergent if and only if the double sequence $(s_{mn})$ is convergent, where $s_{mn} = \sum_{i,j=1}^{m,n} x_{ij} (m, n \in \mathbb{N})$ (see[1]).

A sequence $x = (x_{mn})$is said to be double analytic if $sup_{mn} |x_{mn}|^{1/m+n} < \infty$. The vector space of all double analytic sequences will be denoted by $\Lambda^2$. A sequence $x = (x_{mn})$ is called double gai sequence if $((m + n)! |x_{mn}|)^{1/m+n} \to 0$ as $m, n \to \infty$. The double gai sequences will be denoted by $\chi^2$. Let $\phi = \{allfinitesequences\}$.

Consider a double sequence $x = (x_{ij})$. The $(m, n)^{th}$ section $x^{[m,n]}$ of the sequence is defined by $x^{[m,n]} = \sum_{i,j=0}^{m,n} x_{ij} \Im_{ij}$ for all $m, n \in \mathbb{N}$; where $\Im_{ij}$ denotes the double sequence whose only non zero term is a $\frac{1}{(i+j)!}$ in the $(i, j)^{th}$ place for each $i, j \in \mathbb{N}$.

An FK-space(or a metric space)$X$ is said to have AK property if $(\Im_{mn})$ is a Schauder basis for $X$. Or equivalently $x^{[m,n]} \to x$.

An FDK-space is a double sequence space endowed with a complete metrizable; locally convex topology under which the coordinate mappings $x = (x_k) \to (x_{mn})(m, n \in$

$\mathbb{N}$) are also continuous.

If $X$ is a sequence space, we give the following definitions:

(i) $X^{'}=$ the continuous dual of $X$;
(ii) $X^{\alpha}=\left\{a=(a_{mn}):\sum_{m,n=1}^{\infty}|a_{mn}x_{mn}|<\infty,\, for\, each\, x\in X\right\};$
(iii) $X^{\beta}=\left\{a=(a_{mn}):\sum_{m,n=1}^{\infty}a_{mn}x_{mn}\, is\, convegent,\, foreach\, x\in X\right\};$
(iv) $X^{\gamma}=\left\{a=(a_{mn}):sup_{mn}\geq 1\left|\sum_{m,n=1}^{M,N}a_{mn}x_{mn}\right|<\infty, foreachx\in X\right\};$
(v) $let\, X\, beanFK-space\supset\phi;\, then\, X^{f}=\left\{f(\Im_{mn}):f\in X^{'}\right\};$
(vi) quad $X^{\delta}=\left\{a=(a_{mn}):sup_{mn}|a_{mn}x_{mn}|^{1/m+n}<\infty,\, foreach\, x\in X\right\};$

$X^{\alpha}.X^{\beta},X^{\gamma}$ are called $\alpha-(orK\ddot{o}the-Toeplitz)$dual of $X,\beta-(or\, generalized-K\ddot{o}the-Toeplitz)\, dual\, of X,\gamma-\, dual\, of\, X,\, \delta-\, dual\, of X\, respectively.X^{\alpha}$ is defined by Gupta and Kamptan [24]. It is clear that $X^{\alpha}\subset X^{\beta}$ and $X^{\alpha}\subset X^{\gamma}$, but $X^{\alpha}\subset X^{\gamma}$ does not hold, since the sequence of partial sums of a double convergent series need not to be bounded.

The notion of difference sequence spaces (for single sequences) was introduced by Kizmaz [36] as follows

$$Z(\Delta)=\{x=(x_k)\in w:(\Delta x_k)\in Z\}$$

for $Z=c,c_0$ and $\ell_{\infty}$, where $\Delta x_k=x_k-x_{k+1}$ for all $k\in\mathbb{N}$. Here $w,c,c_0$ and $\ell_{\infty}$ denote the classes of all, convergent,null and bounded sclar valued single sequences respectively. The above spaces are Banach spaces normed by

$$\|x\|=|x_1|+sup_{k\geq 1}|\Delta x_k|$$

Later on the notion was further investigated by many others. We now introduce the following difference double sequence spaces defined by

$$Z(\Delta)=\left\{x=(x_{mn})\in w^2:(\Delta x_{mn})\in Z\right\}$$

where $Z=\Lambda^2,\chi^2$ and $\Delta x_{mn}=(x_{mn}-x_{mn+1})-(x_{m+1n}-x_{m+1n+1})=x_{mn}-x_{mn+1}-x_{m+1n}+x_{m+1n+1}$ for all $m,n\in\mathbb{N}$

## 2. Definitions and Preliminaries

A sequence $x=(x_{mn})$is said to be double analytic if

$$sup_{mn}|x_{mn}|^{1/m+n}<\infty.$$

The vector space of all double analytic sequences is usually denoted by $\Lambda^2$. A sequence $x=(x_{mn})$ is called double entire sequence if $|x_{mn}|^{\,1/m+n}\to 0$ as $m,n\to\infty$. The vector space of double entire sequences is usually denoted by $\Gamma^2$. A sequence $x=(x_{mn})$ is called double gai sequence if $((m+n)!\,|x_{mn}|)^{1/m+n}\to 0$ as $m,n\to\infty$. The vector space of double gai sequences is usually denoted by $\chi^2$. The space $\chi^2$ is a metric space with the metric

$$d(x,y) = sup_{m,n}\left\{((m+n)!\,|x_{mn}-y_{mn}|)^{1/m+n} : m,n : 1,2,3,...\right\} \tag{2}$$

for all $x = \{x_{mn}\}$ and $y = \{y_{mn}\}$ in $\chi^2$.

# 3. Main Results

**Proposition 3.1** *$\chi^2$ has monotone metric.*

**Proof**: We know that

$$d(x,y) = sup_{mn}\left\{((m+n)!\,|x_{mn}-y_{mn}|)^{1/m+n} : m,n : 1,2,3,...\right\}$$
$$d(x^n,y^n) = sup_{n,n}\left\{((2n)!\,|x_{nn}-y_{nn}|)^{1/2n}\right\}$$

and

$$d(x^m,y^m) = sup_{m,m}\left\{((2m)!\,|x_{mm}-y_{mm}|)^{1/2m}\right\}$$

Let $m > n$. Then

$$sup_{m,m}\left\{((2m)!\,|x_{mm}-y_{mm}|)^{1/2m}\right\} \geq sup_{n,n}\left\{((2n)!\,|x_{nn}-y_{nn}|)^{1/2n}\right\}$$
$$d(x^m,y^m) \geq d(x^n,y^n), \qquad m > n \tag{3}$$

Also $\{d(x^n,x^n) : n = 1,2,3,...\}$ is monotonically increasing bounded by $d(x,y)$. For such a sequence

$$sup_{n,n}\left\{((2n!)\,|x^{nn}-y^{nn}|)^{1/2n}\right\} = n \overset{lim}{\to} \infty\, d(x^n,y^n) = d(x,y) \tag{4}$$

From(3) and (4) it follows that $d(x,y) = sup_{mn}\left\{((m+n)!\,|x_{mn}-y_{mn}|)^{1/m+n}\right\}$ is a monotone metric for $\chi^2$. This completes the proof.

**Proposition 3.2** *The dual space of $\chi^2$ is $\Lambda^2$.In other words $(\chi^2)^* = \Lambda^2$.*

**Proof:** We recall that

$$\Im_{mn} = \begin{pmatrix} 0, & 0, & ...0, & 0, & ... \\ 0, & 0, & ...0, & 0, & ... \\ . & & & & \\ . & & & & \\ . & & & & \\ 0, & 0, & ...\frac{1}{(m+n)!}, & 0, & ... \\ 0, & 0, & ...0, & 0, & ... \end{pmatrix}$$

with $\frac{1}{(m+n)!}$ in the $(m,n)th$ position and zero's else where. With

$$x = \Im_{mn}, (|x_{mn}|)^{1/m+n}$$

$$= \begin{pmatrix} 0^{1/2}, . & . & . & 0^{1/1+n} \\ . & & & \\ . & & & \\ . & & & \\ 0^{1/m+1}, & (\frac{1}{(m+n)!})^{1/m+n}, & . & 0^{1/m+n+1} \\ & (m,n)^{th} & & \\ 0^{1/m+2}, & . & . & 0^{1/m+n+2} \end{pmatrix}$$

$$= \begin{pmatrix} 0, . & . & . & 0 \\ . & & & \\ . & & & \\ . & & & \\ 0, & \left(\frac{1}{(m+n)!}\right)^{1/m+n}, & . & 0 \\ & (m,n)^{th} & & \\ 0, & . & . & 0 \end{pmatrix}$$

which is a double gai sequence. Hence $\Im_{mn} \in \chi^2. We have f(x) = \sum_{m,n=1}^{\infty} x_{mn} y_{mn}$. With $x \in \chi^2$ and $f \in (\chi^2)^*$ the dual space of $\chi^2$. Take $x = (x_{mn}) = \Im_{mn} \in \chi^2$.Then

$$|y_{mn}| \leq \|f\| \, d(\Im_{mn}, 0) < \infty \quad \forall m, n \tag{5}$$

Thus $(y_{mn})$ is a bounded sequence and hence an double analytic sequence. In other words $y \in \Lambda^2$. Therefore $(\chi^2)^* = \Lambda^2$.This completes the proof.

**Proposition 3.3** $\chi^2$ *is separable.*

**Proof:**It is routine verification. Therefore omit the proof.

**Proposition 3.4** $\Lambda^2$ *is not separable.*

**Proof:**Since $|x_{mn}|^{1/m+n} \to 0 \, as \, m,n \to \infty$,so it may so happen that first row or column may not be convergent, even may not be bounded. Let $S$ be the set that has double sequences such that the first row is built up of sequences of zeros and ones. Then $S$ will be uncountable. Consider open balls of radius $3^{-1}$ units. Then these open balls will not cover $\Lambda^2$.Hence $\Lambda^2$is not separable. This completes the proof.

**Proposition 3.5** $\chi^2$*is not reflexive.*

**Proof:**$\chi^2$is separable by Proposition 3.3. But $(\chi^2)^* = \Lambda^2$, by Proposition 3.2. Since $\Lambda^2$ is not separable, by Proposition 3.4. Therefore $\chi^2$ is not reflexive. This completes the proof.

**Proposition 3.6** $\chi^2$ *is not an inner product space as such not a Hilbert space.*

**Proof:** Let us take

$$x = x_{mn} = \begin{pmatrix} 1/2!, & 1/3!, & 0, & 0, & ... \\ 0, & 0, & 0, & 0, & ... \\ . & & & & \\ . & & & & \\ . & & & & \end{pmatrix}$$

and

$$y = y_{mn} = \begin{pmatrix} 1/2!, & -1/3!, & 0, & 0, & \dots \\ 0, & 0, & 0, & 0, & \\ \vdots & \vdots & \vdots & & \end{pmatrix}$$

$$d(x,0) = sup \begin{pmatrix} (2!\,|x_{11} - 0|)^{1/2}, & (3!\,|x_{12} - 0|)^{1/3}, & ... \\ (3!\,|x_{21} - 0|)^{1/3}, & (4!\,|x_{22} - 0|)^{1/4}, & ... \\ \vdots & \vdots & \vdots \end{pmatrix}$$

$$= sup \begin{pmatrix} (2!\,|1/2! - 0|)^{1/2}, & (3!\,|1/3! - 0|)^{1/3}, & ... \\ 0, & 0, & ... \\ \vdots & \vdots & \vdots \end{pmatrix}$$

$$= \sup \begin{pmatrix} (1)^{1/2}, & (1)^{1/3}, & 0, & \dots \\ 0, & 0, & 0, & \dots \\ \vdots & \vdots & \vdots & \vdots \end{pmatrix}$$

$$d(x,0) = 1.$$

Similarly $d(x,0) = 1$. Hence $d(x,0) = d(y,0) = 1$

$$x + y = \begin{pmatrix} 1/2!, & 1/3!, & 0 & ,0 & ... \\ 0, & 0, & 0, & 0, & ... \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0, & 0, & 0, & 0, & ... \end{pmatrix} + \begin{pmatrix} 1/2!, & -1/3!, & 0, & 0 & ... \\ 0, & 0, & 0, & 0, & \dots \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0, & 0, & 0, & 0, & \dots \end{pmatrix}$$

$$= \begin{pmatrix} 1, & 0, & 0, & 0 & \dots \\ 0, & 0, & 0, & 0, & \dots \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0, & 0, & 0, & 0, & \dots \end{pmatrix}$$

$$d(x+y, x+y) = sup \Big\{ ((m+n)!\,(|x_{mn} + y_{mn}| - |x_{mn} - y_{mn}|))^{1/m+n} \\ : m, n = 1, 2, 3, ... \}$$

$$d(x_{mn}+y_{mn},0)=\sup\begin{pmatrix}(2!\,|x_{11}+y_{11}|)^{1/2}, & (3!\,|x_{12}+y_{12}|)^{1/3}, & \ldots\\ \vdots & \vdots & \end{pmatrix}$$

$$=\sup\begin{pmatrix}(2!\,|1/2!+1/2!|)^{1/2}, & (3!\,|1/3!-1/3!|)^{1/3}, & \ldots\\ \vdots & \vdots & \end{pmatrix}$$

$$=\sup\begin{pmatrix}(2)^{1/2}, & 0, & \ldots\\ 0, & 0, & \ldots\\ \vdots & & \end{pmatrix}=\sup\begin{pmatrix}1.414, & 0, & \ldots\\ 0, & 0, & \ldots\\ \vdots & \vdots & \end{pmatrix}=1.414$$

Therefore $d(x+y,0)=1.414$. Similarly $d(x-y,0)=1.26$

By parellogram law,

$$[d(x+y,0)]^2+[d(x-y,0)]^2=2\left[(d(x,0))^2+(d(0,y))^2\right] \implies$$
$$(1.414)^2+1.26^2=2\left[1^2+1^2\right] \implies$$
$$3.586996=4.$$

Hence it is not satisfied by the law. Therefore $\chi^2$ is not an inner product space. Assume that $\chi^2$ is a Hilbert space. But then $\chi^2$ would satisfy reflexivity condition. [Theorem 4.6.6 [42]] . Proposition 3.5, $\chi^2$ is not reflexive. Thus $\chi^2$ is not a Hilbert space. This completes the proof.

**Proposition 3.7** *$\chi^2$ is rotund.*

**Proof:** Let us take

$$x=x_{mn}=\begin{pmatrix}1/2!, & 0, & 0, & 0 & \ldots\\ 0, & 0, & 0, & 0, & \ldots\\ \vdots & \vdots & \vdots & \vdots & \end{pmatrix} \quad \text{and} \quad y=y_{mn}=\begin{pmatrix}1/2!, & 0, & 0, & 0, & \ldots\\ 0, & 0, & 0, & 0, & \ldots\\ \vdots & \vdots & \vdots & \vdots & \end{pmatrix}$$

Then $x=(x_{mn})$ and $y=(y_{mn})$ are in $\chi^2$. Also

$$d(x,y)=$$
$$\sup\begin{pmatrix}(2!\,|x_{11}-y_{11}|)^{\frac{1}{1}}, & ... & ((n+1)!\,|x_{1n}-y_{1n}|)^{\frac{1}{1+n}}, & 0, & \ldots\\ \vdots & \vdots & & & \\ ((m+1)!\,|x_{m1}-y_{m1}|)^{\frac{1}{m+1}}, & \ldots & ((m+n)!\,|x_{mn}-y_{mn}|)^{\frac{1}{m+n}}, & 0, & \ldots\\ 0, & ... & 0, & & ...\end{pmatrix}$$

Therefore

$$d(x,0)=sup\begin{pmatrix}1, & 0, & 0, & 0 & \ldots\\ 0, & 0, & 0, & 0, & \ldots\\ \vdots & \vdots & & \vdots & \vdots\\ 0, & 0, & 0, & 0, & \ldots\end{pmatrix}, \qquad d(0,y)=1.$$

Obviously $x = (x_{mn}) \neq y = (y_{mn})$. But

$$(x_{mn}) + (y_{mn}) = \begin{pmatrix} 1/2!, & 0, & 0 & ,0 & \dots \\ 0, & 0, & 0, & 0, & \dots \\ \vdots & \vdots & \vdots & \vdots & \end{pmatrix} + \begin{pmatrix} 1/2!, & 0, & 0, & 0 & \dots \\ 0, & 0, & 0, & 0, & \dots \\ \vdots & \vdots & \vdots & \vdots & \end{pmatrix}$$

$$= \begin{pmatrix} 1, & 0, & 0, & 0 & \dots \\ 0, & 0, & 0, & 0, & \dots \\ \vdots & \vdots & \vdots & \vdots & \end{pmatrix}$$

$$d(\frac{x_{mn} + y_{mn}}{2}, 0)$$

$$= \sup \begin{pmatrix} \frac{(2!|x_{11}+y_{11}|)^{1/2}}{2}, & \dots & \frac{((1+n)!|x_{1n}+y_{1n}|)^{1/n+1}}{2}, & 0, & \dots \\ \vdots & \ddots & \vdots & \vdots & \vdots \\ \frac{((m+1)!|x_{m1}+y_{m1}|)^{1/m+1}}{2}, & \dots, & \frac{((m+n)!|x_{mn}+y_{mn}|)^{1/m+n}}{2}, & 0, & \dots \end{pmatrix}$$

$$d(\frac{x_{mn} + y_{mn}}{2}, 0) = \sup \begin{pmatrix} (2^{1/2})/2, & 0, & 0, & 0 & \dots \\ 0, & 0, & 0, & 0, & \dots \\ \vdots & \vdots & \vdots & \vdots & \end{pmatrix} = 0.71.$$

Therefore $\chi^2$ is rotund. This completes the proof.

**Proposition 3.8** *Weak convergence and strong convergence are equivalent in* $\chi^2$.

**Proof: Step1**: Always strong convergence implies weak convergence.
**Step2:** So it is enough to show that weakly convergence implies strongly convergence in $\chi^2 . y^{(\eta)}$ tends to weakly in $\chi^2$, where $(y_{mn}^{(\eta)}) = y^{(\eta)}$ and $y = (y_{mn})$. Take any $x = (x_{mn}) \in \chi^2$ and

$$f(z) = \sum_{m,n=1}^{\infty} ((m+n)! \left| z_{mn} x_{mn} \right|)^{1/m+n} \; for each \, z = (z_{mn}) \in \chi^2 \tag{6}$$

Then $f \in (\chi^2)^*$ by Proposition 3.2. By hypothesis $f(y^\eta) \to f(y) \, as \, \eta \to \infty$.

$$f\left(y^{(\eta)} - y\right) \to 0 \quad \text{as} \quad \eta \to \infty. \quad \Longrightarrow \tag{7}$$

$$\sum_{m,n=1}^{\infty} \left( \left| y_{mn}^{(\eta)} - y_{mn} \right|^{1/m+n} ((m+n)!)^{1/m+n} \left| x_{mn} \right|^{1/m+n} \right) \to 0 \quad \text{as} \quad \eta \to \infty.$$

By using (6) and (7) we get since $x = (x_{mn}) \in \Lambda^2$ we have

$$\sum_{m,n=1}^{\infty} \left| x_{mn} \right|^{1/m+n} < \infty \quad \text{for all} \quad x \in \Lambda^2.$$

$$\Longrightarrow \quad \sum_{m,n=1}^{\infty} \left( (m+n)! \left| y_{mn}^{(\eta)} - y_{mn} \right| \right)^{1/m+n} \to 0 \quad \text{as} \quad \eta \to \infty.$$

$$\Longrightarrow \quad \overset{sup}{mn} \left( (m+n)! \left| (y_{mn}^{(\eta)} - y_{mn}), 0 \right| \right)^{1/m+n} \to 0 \quad \text{as} \quad \eta \to \infty.$$

$$\Longrightarrow \quad \overset{sup}{mn} \left( (m+n)! \left| y_{mn}^{(\eta)} - y_{mn} \right| \right)^{1/m+n} \to 0 \quad \text{as} \quad \eta \to \infty.$$

$$\Longrightarrow \quad d\left( \left( y^{(\eta)} - y \right), 0 \right) \to 0 \quad \text{as} \quad \eta \to \infty.$$

$$\Longrightarrow \quad d\left( y^{(\eta)} - y \right) \to 0 \quad \text{as} \quad \eta \to \infty.$$

This completes the proof.

**Proposition 3.9** *There exists an infinite matrix $A$ for which $\chi_A^2 = \chi^2$.*

**Proof:** Consider the matrix

$$\begin{pmatrix} 2!y_{11}, & 3!y_{12}, & ..., & (1+n)!y_{1n}, & 0, & 0 & ... \\ 3!y_{21}, & 4!y_{22}, & ..., & (2+n)!y_{2n}, & 0, & 0 & ... \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \\ (m+1)!y_{m1}, & (m+2)!y_{m2}, & ..., & (m+n)!y_{mn}, & 0, & 0 & ... \\ 0, & 0, & ..., & 0, & 0, & 0 & ... \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \end{pmatrix}$$

$$= \begin{pmatrix} 1, & 0, & 0, & ... & & \\ 1, & 0, & 0, & ... & & \\ 0, & 1, & 0, & ... & & \\ 0, & 1, & 0, & ... & & \\ 0, & 1, & 0, & ... & & \\ 0, & 1, & 0, & ... & & \\ 0, & 0, & 1, & 0, & ... & \\ 0, & 0, & 1, & 0, & ... & \\ 0, & 0, & 1, & 0, & ... & \\ 0, & 0, & 1, & 0, & ... & \\ 0, & 0, & 1, & 0, & ... & \\ 0, & 0, & 1, & 0, & ... & \\ 0, & 0, & 1, & 0, & ... & \\ 0, & 0, & 1, & 0, & ... & \\ . & & & & & \\ . & & & & & \\ . & & & & & \end{pmatrix} \begin{pmatrix} 2!x_{11}, & 3!x_{12}, & ..., & (1+n)!x_{1n}, & 0 & ... \\ 3!x_{21}, & 4!x_{22}, & ..., & (2+n)!x_{2n}, & 0 & ... \\ \vdots & \vdots & \vdots & \vdots & \vdots & \\ (m+1)!x_{m1}, & (m+2)!x_{m2}, & ..., & (m+n)!x_{mn}, & 0, & ... \\ 0, & 0, & \ldots, & 0, & 0 & ... \\ \vdots & \vdots & \vdots & \vdots & \vdots & \end{pmatrix}$$

$$\begin{pmatrix} 2!y_{11}, & 3!y_{12}, & ..., & (1+n)!y_{1n}, & 0, & 0 & ... \\ 3!y_{21}, & 4!y_{22}, & ..., & (2+n)!y_{2n}, & 0, & 0 & ... \\ . \\ . \\ . \\ (m+1)!y_{m1}, & (m+2)!y_{m2}, & ..., & (m+n)!y_{mn}, & 0, & 0 & ... \\ 0, & 0, & ..., & 0, & 0, & 0 & ... \\ . \\ . \\ . \end{pmatrix}$$

$$= \begin{pmatrix} 2!x_{11}, ..., & (1+n)!x_{1n}, & 0, ... \\ 2!x_{11}, ..., & (1+n)!x_{1n}, & 0, ... \\ 3!x_{21}, ..., & (2+n)!x_{2n}, & 0, ... \\ 3!x_{21}, ..., & (2+n)!x_{2n}, & 0, ... \\ 3!x_{21}, ..., & (2+n)!x_{2n}, & 0, ... \\ 3!x_{21}, ..., & (2+n)!x_{2n}, & 0, ... \\ 4!x_{31}, ..., & (3+n)!x_{3n}, & 0, ... \\ 4!x_{31}, ..., & (3+n)!x_{3n}, & 0, ... \\ 4!x_{31}, ..., & (3+n)!x_{3n}, & 0, ... \\ 4!x_{31}, ..., & (3+n)!x_{3n}, & 0, ... \\ 4!x_{31}, ..., & (3+n)!x_{3n}, & 0, ... \\ 4!x_{31}, ..., & (3+n)!x_{3n}, & 0, ... \\ 4!x_{31}, ..., & (3+n)!x_{3n}, & 0, ... \\ 4!x_{31}, ..., & (3+n)!x_{3n}, & 0, ... \\ \vdots & \vdots & \vdots \end{pmatrix}$$

$$\begin{aligned} 2!y_{11}, ..., (1+n)!y_{1n} &= 2!x_{11}, ..., (1+n)!x_{1n} \\ 3!y_{21}, ..., (2+n)!y_{2n} &= 2!x_{11}, ..., (1+n)!x_{1n} \\ 4!y_{31}, ..., (3+n)!y_{3n} &= 3!x_{21}, ..., (2+n)!x_{2n} \\ 5!y_{41}, ..., (4+n)!y_{4n} &= 3!x_{21}, ..., (2+n)!x_{2n} \\ 6!y_{51}, ..., (5+n)!y_{5n} &= 3!x_{21}, ..., (2+n)!x_{2n} \\ 7!y_{61}, ..., (6+n)!y_{6n} &= 3!x_{21}, ..., (2+n)!x_{2n} \\ &\vdots \end{aligned}$$

and so on. For any $x = (x_{mn}) \in \chi^2$.

$$\left|(Ax)_{mn}\right| = m, n \overset{lim}{\to} \infty \left((m+n)! \left|\Sigma x_{mn}\right|\right)^{1/m+n} \leq d(x, 0)$$

where metric is taken $\chi^2$.

$$[d(x,0)]_{\chi^2_A} \leq [d(x,0)]_{\chi^2} \tag{8}$$

**Conversely,** Given $x \in [d(x,0)]_{\chi^2_A}$ fix any $m, n$ then,

$$
\begin{aligned}
& \underset{m,n \to \infty}{lim} \left((m+n)! \left|x_{mn}\right|\right)^{1/m+n} \leq (Ax)_{mn}. \\
\Longrightarrow \quad & \underset{m,n \to \infty}{lim} \left((m+n)! \left|x_{mn}\right|\right)^{1/m+n} \leq [d(x,0)]_{\chi^2_A} \\
& [d(x,0)]_{\chi^2} \leq [d(x,0)]_{\chi^2_A} .
\end{aligned}
$$

Therefore the matrix $A = (x^{\ell k}_{mn})$ for whcich the summability field $[d(x,0)]_{\chi^2} = [d(x,0)]_{\chi^2_A}$ is given by

$$
A = \begin{pmatrix}
1, & 0, & 0, & ... & \\
1, & 0, & 0, & ... & \\
0, & 1, & 0, & ... & \\
0, & 1, & 0, & ... & \\
0, & 1, & 0, & ... & \\
0, & 1, & 0, & ... & \\
0, & 0, & 1, & 0, & ... \\
0, & 0, & 1, & 0, & ... \\
0, & 0, & 1, & 0, & ... \\
0, & 0, & 1, & 0, & ... \\
0, & 0, & 1, & 0, & ... \\
0, & 0, & 1, & 0, & ... \\
0, & 0, & 1, & 0, & ... \\
0, & 0, & 1, & 0, & ... \\
\vdots & & & &
\end{pmatrix}
$$

**//Program for generalization:**

```
#include ⟨iostream.h⟩
#include ⟨conio.h⟩
#include ⟨math.h⟩
#include ⟨fstream.h⟩
void main()
{
clrscr() ;
int m,n,i,nn=0,j,count=1,k,1pp,abc;
ofstream fout,fout 1;
fout.open(”aa1.txt”);
fout1.open(”aa2.txt”);
cout << ”enter the value of m:”;
cin>> m;
for(i=1;i<=m;i++)
{
nn=nn+pow(2,i);
}
```

```
i=0
while(count<=nn)
{
cout<< " − ";
fout<< " − ";
for(abc=1;abc<=m+2;abc++)
{
cout<< " ";
fout<< " ";
}
cout<< " − \n";
fout<< " − "\n;
for(j = 1; j <= m; j + +)
{
for(k=1;k<=pow(2,j);k++)
{
for(pp=1;pp<=3;pp++)
{
fout1<< count + pp << "!Y" << count << "," << pp << " ";
}
fout1<< "...("‘ << count << " + n)!Y" << count << ", n = ";
cout<< " | ";
fout<< " | ";
for(int q=1;q<=m+1;q++)
{
if(q==j)
{
cout<< "1";
fout<< "1";
}
else
{
cout<< "0";
fout<< "0";
}
}
for(int l=1;l<=3;l++)
{
foutl<< j + 1 << "!X" << "j" << "," <<l<< "";
}
fout1<< "...("‘ << j << "′ + n)!X" << j << "n";
cout<< "... | \n";
fout<< "... | \n";
fout1<< "... | \n";
count++;
```

```
}
}
}
cout<< " · \n · \n · \n";
fout<< " · \n · \n · \n";
cout<< " | −";
fout<< " | −";
for(abc=1;abc<<=m+1;abc++)
{
cout<< " ";
fout<< " ";
}
cout<< "− | ";
fout<< "− | ";
fout1<< ".\n.\n.\n";
fout.close();
fout1.close();
getch();
}
```

**SAMPLE INPUT/OUTPUT:**

Enter the value of m=3

$$\begin{pmatrix} 1, & 0, & 0, & ... & \\ 1, & 0, & 0, & ... & \\ 0, & 1, & 0, & ... & \\ 0, & 1, & 0, & ... & \\ 0, & 1, & 0, & ... & \\ 0, & 1, & 0, & ... & \\ 0, & 0, & 1, & 0, & ... \\ 0, & 0, & 1, & 0, & ... \\ 0, & 0, & 1, & 0, & ... \\ 0, & 0, & 1, & 0, & ... \\ 0, & 0, & 1, & 0, & ... \\ 0, & 0, & 1, & 0, & ... \\ 0, & 0, & 1, & 0, & ... \\ 0, & 0, & 1, & 0, & ... \\ \vdots & & & & \end{pmatrix}$$

$2!Y_{1,1}, ..., (1+n)!Y_{1,n} = 2!X_{1,1}, ..., (1+n)!X_{1,n}$
$3!Y_{2,1}, ..., (2+n)!Y_{2,n} = 2!X_{1,1}, ..., (1+n)!X_{1,n}$
$4!Y_{3,1}, ..., (3+n)!Y_{3,n} = 3!X_{2,1}, ..., (2+n)!X_{2,n}$
$5!Y_{4,1}, ..., (4+n)!Y_{4,n} = 3!X_{2,1}, ..., (2+n)!X_{2,n}$
$6!Y_{5,1}, ..., (5+n)!Y_{5,n} = 3!X_{2,1}, ..., (2+n)!X_{2,n}$
$7!Y_{6,1}, ..., (6+n)!Y_{6,n} = 3!X_{2,1}, ..., (2+n)!X_{2,n}$

$8!Y_{7,1},...,(7+n)!Y_{7,n} = 4!X_{3,1},...,(3+n)!X_{3,n}$
$9!Y_{8,1},...,(8+n)!Y_{8,n} = 4!X_{3,1},...,(3+n)!X_{3,n}$
$10!Y_{9,1},...,(9+n)!Y_{9,n} = 4!X_{3,1},...,(3+n)!X_{3,n}$
$11!Y_{10,1},...,(10+n)!Y_{10,n} = 4!X_{3,1},...,(3+n)!X_{3,n}$
$12!Y_{11,1},...,(11+n)!Y_{11,n} = 4!X_{3,1},...,(3+n)!X_{3,n}$
$13!Y_{12,1},...,(12+n)!Y_{12,n} = 4!X_{3,1},...,(3+n)!X_{3,n}$
$14!Y_{13,1},...,(13+n)!Y_{13,n} = 4!X_{3,1},...,(3+n)!X_{3,n}$
$15!Y_{14,1},...,(14+n)!Y_{14,n} = 4!X_{3,1},...,(3+n)!X_{3,n}$
$\vdots$

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