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On the Derivative of a Polynomial with Prescribed Zeros

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ABSTRACT: For a polynomial $p(z) = a_n \prod_{t=1}^{n}(z - z_t)$ of degree n having all its zeros in $|z| \leq K$, $K \geq 1$ it is known that

$$\max_{|z|=1} |p'(z)| \geq \frac{2}{1+K^n} \left\{ \sum_{t=1}^{n} \frac{K}{K+|z_t|} \right\} \max_{|z|=1} |p(z)| \; .$$

By assuming a possible zero of order m, $0 \leq m \leq n-4$, at $z = 0$, of $p(z)$ for $n \geq k+m+1$ with integer $k \geq 3$ we have obtained a new refinement of the known result.

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1. Introduction and statement of results

For an arbitrary polynomial $f(z)$ let $M(f,r) = \max_{|z|=r} |f(z)|$ and $m(f,r) = \min_{|z|=r} |f(z)|$. Further let $p(z) = \sum_{j=0}^{n} a_j z^j$ be a polynomial of degree n. Concerning the estimate of $|p'(z)|$ on $|z| \leq 1$ we have the following result due to Turán [12].

Theorem 1.1. *If $p(z)$ is a polynomial of degree n, having all its zeros in $|z| \leq 1$ then*

$$M(p',1) \geq \frac{n}{2} M(p,1).$$

The result is sharp with equality for the polynomial $p(z)$ having all its zeros on $|z| = 1$.

More generally, for the polynomial having all its zeros in $|z| \leq K$, $(K \leq 1)$, Malik [10] proved:

Theorem 1.2. *If $p(z)$ be a polynomial of degree n, having all its zeros in $|z| \leq K$, $(K \leq 1)$ then*

$$M(p',1) \geq \frac{n}{1+K} M(p,1) \; .$$

The result is sharp with equality for the polynomial $p(z) = (z+K)^n$.

And for the polynomial having all its zeros in $|z| \leq K$, $(K \geq 1)$, Govil [6] proved:

Theorem 1.3. *If $p(z)$ is a polynomial of degree n, having all its zeros in $|z| \leq K$, $(K \geq 1)$ then*

$$M(p',1) \geq \frac{n}{1+K^n} M(p,1) \ .$$

The result is sharp with equality for the polynomial $p(z) = z^n + K^n$.

By using the coefficients a_n, a_{n-1}, of the polynomial $p(z)$, Govil et al. [7] obtained the following refinement of Theorem 1.2.

Theorem 1.4. *If $p(z) = \sum_{j=0}^{n} a_j z^j$ is a polynomial of degree n, having all its zeros in $|z| \leq K$, $(K \leq 1)$ then*

$$M(p',1) \geq n \frac{n|a_n| + |a_{n-1}|}{(1+K^2)n|a_n| + 2|a_{n-1}|} M(p,1) \ .$$

And Aziz [1] used the moduli of all the zeros of the polynomial $p(z)$ to obtain the following refinement of Theorem 1.3.

Theorem 1.5. *If all the zeros of the polynomial $p(z) = a_n \prod_{j=1}^{n}(z - z_j)$, of degree n lie in $|z| \leq K$, $(K \geq 1)$ then*

$$M(p',1) \geq \frac{2}{1+K^n} \Big(\sum_{j=1}^{n} \frac{K}{K + |z_j|} \Big) M(p,1) \ .$$

The result is best possible with equality for the polynomial $p(z) = z^n + K^n$.

Later Govil [8] used certain coefficients as well as moduli of all the zeros, of the polynomial $p(z)$ to obtain the following refinement of Theorem 1.5.

Theorem 1.6. *Let $p(z) = \sum_{j=0}^{n} a_j z^j = a_n \prod_{t=1}^{n}(z - z_t)$ be a polynomial of degree n, $(n \geq 2)$, $|z_t| \leq K_t$, $1 \leq t \leq n$ and let $K = \max(K_1, K_2, \dots, K_n) \geq 1$. Then for $n > 2$*

$$\begin{aligned} M(p',1) \ \ \geq \ \ & \frac{2}{1+K^n} \Big(\sum_{t=1}^{n} \frac{K}{K+K_t} \Big) M(p,1) \\ & + \frac{2|a_{n-1}|}{(1+K^n)} \Big(\sum_{t=1}^{n} \frac{1}{K+K_t} \Big) \Big(\frac{K^n - 1}{n} - \frac{K^{n-2} - 1}{n-2} \Big) + |a_1| \Big(1 - \frac{1}{K^2} \Big), \end{aligned}$$

and

$$M(p',1) \geq \frac{2}{1+K^n} \Big(\sum_{t=1}^{n} \frac{K}{K+K_t} \Big) M(p,1) + \frac{(K-1)^n}{1+K^n} |a_1| \Big(\sum_{t=1}^{n} \frac{1}{K+K_t} \Big)$$
$$+ |a_1| \Big(1 - \frac{1}{K} \Big), \ n = 2.$$

The result is best possible with equality for the polynomial $p(z) = z^n + K^n$.

Dewan et al. [3] also obtained a result similar to Theorem 1.6 for $n \geq 3$.

In this paper by assuming a possible zero of order m, $0 \leq m \leq n-4$, at $z=0$, of $p(z)$ we have obtained a new refinement of Theorem 1.5, similar to Theorem 1.6 for $n \geq k+m+1$ with integer $k \geq 3$. More precisely we have proved

Theorem 1.7. *Let $p(z)$ be a polynomial of degree n such that*

$$p(z) = a_n \prod_{t=1}^{n}(z-z_t) = \sum_{j=0}^{n} a_j z^j, |z_t| \leq K_t, 1 \leq t \leq n$$

$$\text{and } K = \max(K_1, K_2, \ldots, K_n) \geq 1, \tag{1.1}$$

$$= z^m p_1(z),\ p_1(0) \neq 0,\ 0 \leq m \leq n-4, \tag{1.2}$$

with

$$n \geq k+m+1, (k \geq 3). \tag{1.3}$$

Then

$$\begin{aligned}
M(p',1) \geq{}& \frac{2}{1+K^{n-m}}\Big(\sum_{t=1}^{n}\frac{K}{K+K_t}\Big)M(p,1) \\
&+ \frac{1}{K^n}\Big(\sum_{t=1}^{n}\frac{K}{K+K_t}\Big)\frac{K^{n-m}-1}{K^{n-m}+1}m(p,K) \\
&+ \frac{2}{1+K^{n-m}}\Big(\sum_{t=1}^{n}\frac{K}{K+K_t}\Big)\Bigg[\frac{1}{K}\cdot\frac{2}{\overline{n-m-1}+2}\cdot\frac{1!|a_{n-1}|}{n-m} \\
&\times \left\{K^{n-m}-1-\binom{n-m}{1}(K-1)\right\} \\
&+ \frac{1}{K^2}\cdot\frac{2}{\overline{n-m-2}+2}\cdot\frac{2!|a_{n-2}|}{(n-m)(n-m-1)} \\
&\times \left\{K^{n-m}-1-\binom{n-m}{1}(K-1)-\binom{n-m}{2}(K-1)^2\right\} \\
&+ \frac{1}{K^3}\cdot\frac{2}{\overline{n-m-3}+2}\cdot\frac{3!|a_{n-3}|}{(n-m)(n-m-1)(n-m-2)} \\
&\times \left\{K^{n-m}-1-\binom{n-m}{1}(K-1)-\binom{n-m}{2}(K-1)^2-\binom{n-m}{3}(K-1)^3\right\}+\ldots \\
&+ \frac{1}{K^{k-1}}\cdot\frac{2}{\overline{n-m-(k-1)}+2}\cdot\frac{(k-1)!|a_{n-(k-1)}|}{(n-m)(n-m-1)\ldots(n-m-\overline{k-2})} \\
&\times \left\{K^{n-m}-1-\binom{n-m}{1}(K-1)-\binom{n-m}{2}(K-1)^2-\ldots-\binom{n-m}{k-1}(K-1)^{k-1}\right\}
\end{aligned}$$

$$+ \frac{k!|a_{n-k}|}{K^k}\Bigg(\frac{1}{(n-m)(n-m-1)\dots(n-m-\overline{k-1})}$$

$$\times \left\{K^{n-m}-1-\binom{n-m}{1}(K-1)-\binom{n-m}{2}(K-1)^2-\dots-\binom{n-m}{k-1}(K-1)^{k-1}\right\}$$

$$-\frac{1}{(n-m-2)(n-m-3)\dots(n-m-\overline{k+1})}$$

$$\times \Bigg\{K^{n-m-2}-1-\binom{n-m-2}{1}(K-1)-\binom{n-m-2}{2}(K-1)^2-\dots$$

$$-\binom{n-m-2}{k-1}(K-1)^{k-1}\Bigg\}\Bigg)\Bigg]$$

$$+\frac{1}{K^{n-1}}\Bigg[\frac{2|a_1|}{n+1}(K^{n-1}-1)+\frac{2}{\overline{n-2}+2}.\frac{2!|a_2|}{n-1}\left\{K^{n-1}-1-\binom{n-1}{1}(K-1)\right\}$$

$$+\frac{2}{\overline{n-3}+2}.\frac{3!|a_3|}{(n-1)(n-2)}\left\{K^{n-1}-1-\binom{n-1}{1}(K-1)-\binom{n-1}{2}(K-1)^2\right\}$$

$$+\frac{2}{\overline{n-4}+2}.\frac{4!|a_4|}{(n-1)(n-2)(n-3)}$$

$$\times \left\{K^{n-1}-1-\binom{n-1}{1}(K-1)-\binom{n-1}{2}(K-1)^2-\binom{n-1}{3}(K-1)^3\right\}+\dots$$

$$+\frac{2}{\overline{n-(k+m-1)}+2}.\frac{(k+m-1)!|a_{k+m-1}|}{(n-1)(n-2)\dots(n-\overline{k+m-2})}$$

$$\times \left\{K^{n-1}-1-\binom{n-1}{1}(K-1)-\binom{n-1}{2}(K-1)^2-\dots-\binom{n-1}{k+m-2}(K-1)^{k+m-2}\right\}$$

$$+(k+m)!|a_{k+m}|\Bigg(\frac{1}{(n-1)(n-2)\dots(n-\overline{k+m-1})}$$

$$\times \left\{K^{n-1}-1-\binom{n-1}{1}(K-1)-\binom{n-1}{2}(K-1)^2-\dots-\binom{n-1}{k+m-2}(K-1)^{k+m-2}\right\}$$

$$-\frac{1}{(n-3)(n-4)\dots(n-\overline{k+m+1})}$$

$$\times \left\{K^{n-3}-1-\binom{n-3}{1}(K-1)-\binom{n-3}{2}(K-1)^2-\dots-\binom{n-3}{k+m-2}(K-1)^{k+m-2}\right\}\Bigg)\Bigg],$$

$$n > k+m+1, \quad (k \geq 3) \tag{1.4}$$

and

$$M(p',1) \geq \frac{2}{1+K^{k+1}}\Big(\sum_{t=1}^{k+m+1}\frac{K}{K+K_t}\Big)M(p,1) + \frac{1}{K^{k+m+1}}\Big(\sum_{t=1}^{k+m+1}\frac{K}{K+K_t}\Big)$$

$$\times \frac{K^{k+1}-1}{K^{k+1}+1}m(p,K) + \frac{2}{1+K^{k+1}}\Big(\sum_{t=1}^{k+m+1}\frac{K}{K+K_t}\Big)$$

$$\times \Bigg[\frac{1}{K}.\frac{2}{k+2}.\frac{1!|a_{k+m}|}{k+1}\Big\{K^{k+1}-1-\binom{k+1}{1}(K-1)\Big\}$$

$$+ \frac{1}{K^2}.\frac{2}{\overline{k-1}+2}.\frac{2!|a_{k+m-1}|}{(k+1)k}\Big\{K^{k+1}-1-\binom{k+1}{1}(K-1)-\binom{k+1}{2}(K-1)^2\Big\}$$

$$+ \frac{1}{K^3}.\frac{2}{\overline{k-2}+2}.\frac{3!|a_{k+m-2}|}{(k+1)k(k-1)}$$

$$\times \Big\{K^{k+1}-1-\binom{k+1}{1}(K-1)-\binom{k+1}{2}(K-1)^2-\binom{k+1}{3}(K-1)^3\Big\}+\dots$$

$$+ \frac{1}{K^{k-1}}.\frac{2}{2+2}.\frac{(k-1)!|a_{m+2}|}{(k+1)k\dots 3}$$

$$\times \Big\{K^{k+1}-1-\binom{k+1}{1}(K-1)-\binom{k+1}{2}(K-1)^2-\dots-\binom{k+1}{k-1}(K-1)^{k-1}\Big\}$$

$$+ \frac{1}{K^k}.\frac{|a_{m+1}|}{k+1}.(K-1)^{k+1}\Bigg] + \frac{1}{K^{k+m}}\Big[\frac{2|a_1|}{k+m+2}(K-1) + \frac{2}{\overline{k+m-1}+2}.\frac{2!|a_2|}{k+m}$$

$$\times \Big\{K^{k+m}-1-\binom{k+m}{1}(K-1)\Big\} + \frac{2}{\overline{k+m-2}+2}.\frac{3!|a_3|}{(k+m)(k+m-1)}$$

$$\times \Big\{K^{k+m}-1-\binom{k+m}{1}(K-1)-\binom{k+m}{2}(K-1)^2\Big\}$$

$$+ \frac{2}{\overline{k+m-3}+2}.\frac{4!|a_4|}{(k+m)(k+m-1)(k+m-2)}$$

$$\times \left\{K^{k+m} - 1 - \binom{k+m}{1}(K-1) - \binom{k+m}{2}(K-1)^2 - \binom{k+m}{3}(K-1)^3\right\} + \ldots$$

$$+ \frac{2}{2+2}.\frac{(k+m-1)!|a_{k+m-1}|}{(k+m)(k+m-1)\ldots 3}$$

$$\times \left\{K^{k+m} - 1 - \binom{k+m}{1}(K-1) - \binom{k+m}{2}(K-1)^2 - \ldots - \binom{k+m}{k+m-2}(K-1)^{k+m-2}\right\}$$

$$+ |a_{k+m}|(K-1)^{k+m}\Big], \qquad n = k+m+1, (k \geq 3). \tag{1.5}$$

Result is best possible and equality holds in (1.4) and (1.5) for $p(z) = z^n + K^n$.

Since corresponding to each of m zeros at 0, one can take

$$K_t = 0 \ ,$$

thereby implying

$$\frac{K}{K+K_t} = 1$$

and since corresponding to each of remaining $(n-m)$ zeros, we have

$$\frac{K}{K+K_t} \geq 1/2 \text{ , by } (1.1),$$

Theorem 1.7 gives, in particular, the following statement.

Corollary 1.8. *Let $p(z) = \sum_{j=0}^{n} a_j z^j$ be a polynomial of degree n, having all its zeros in $|z| \leq K$, $(K \geq 1)$ such that*

$$p(z) = z^m p_1(z), \ p_1(0) \neq 0, \ 0 \leq m \leq n-4, \tag{1.6}$$

with

$$n \geq k+m+1, (k \geq 3) \ .$$

Then

$$M(p',1) \geq \frac{n+m}{1+K^{n-m}} M(p,1) + \frac{n+m}{2K^n}.\frac{K^{n-m}-1}{K^{n-m}+1} m(p,K)$$

$$+ \frac{n+m}{1+K^{n-m}}\left[\frac{1}{K} \cdot \frac{2}{\overline{\overline{n-m-1}}+2} \cdot \frac{1!|a_{n-1}|}{n-m}\left\{K^{n-m} - 1 - \binom{n-m}{1}(K-1)\right\}\right.$$

$$+ \frac{1}{K^2}.\frac{2}{\overline{\overline{n-m-2}}+2}.\frac{2!|a_{n-2}|}{(n-m)(n-m-1)}$$

$$\times \left\{K^{n-m} - 1 - \binom{n-m}{1}(K-1) - \binom{n-m}{2}(K-1)^2\right\}$$

$$+\ \frac{1}{K^3}\cdot\frac{2}{\overline{n-m-3}+2}\cdot\frac{3!|a_{n-3}|}{(n-m)(n-m-1)(n-m-2)}$$

$$\times\left\{K^{n-m}-1-\binom{n-m}{1}(K-1)-\binom{n-m}{2}(K-1)^2-\binom{n-m}{3}(K-1)^3\right\}+\dots$$

$$+\ \frac{1}{K^{k-1}}.\frac{2}{\overline{n-m-(k-1)}+2}.\frac{(k-1)!|a_{n-(k-1)}|}{(n-m)(n-m-1)\dots(n-m-\overline{k-2})}$$

$$\times\ \left\{K^{n-m}-1-\binom{n-m}{1}(K-1)-\binom{n-m}{2}(K-1)^2-\dots-\binom{n-m}{k-1}(K-1)^{k-1}\right\}$$

$$+\ \frac{k!|a_{n-k}|}{K^k}\Bigg(\frac{1}{(n-m)(n-m-1)\dots(n-m-\overline{k-1})}$$

$$\times\ \left\{K^{n-m}-1-\binom{n-m}{1}(K-1)-\binom{n-m}{2}(K-1)^2-\dots-\binom{n-m}{k-1}(K-1)^{k-1}\right\}$$

$$-\ \frac{1}{(n-m-2)(n-m-3)\dots(n-m-\overline{k+1})}$$

$$\times\ \Bigg\{K^{n-m-2}-1-\binom{n-m-2}{1}(K-1)-\binom{n-m-2}{2}(K-1)^2-\dots$$

$$-\binom{n-m-2}{k-1}(K-1)^{k-1}\Bigg\}\Bigg)\Bigg]$$

$$+\ \frac{1}{K^{n-1}}\left[\frac{2|a_1|}{n+1}(K^{n-1}-1)+\frac{2}{\overline{n-2}+2}.\frac{2!|a_2|}{n-1}\left\{K^{n-1}-1-\binom{n-1}{1}(K-1)\right\}\right.$$

$$+\ \frac{2}{\overline{n-3}+2}.\frac{3!|a_3|}{(n-1)(n-2)}\left\{K^{n-1}-1-\binom{n-1}{1}(K-1)-\binom{n-1}{2}(K-1)^2\right\}$$

$$+\ \frac{2}{\overline{n-4}+2}.\frac{4!|a_4|}{(n-1)(n-2)(n-3)}$$

$$\times\ \left\{K^{n-1}-1-\binom{n-1}{1}(K-1)-\binom{n-1}{2}(K-1)^2-\binom{n-1}{3}(K-1)^3\right\}+\dots$$

$$+\ \frac{2}{\overline{n-(k+m-1)}+2}.\frac{(k+m-1)!|a_{k+m-1}|}{(n-1)(n-2)\dots(n-\overline{k+m-2})}$$

$$\times\ \left\{K^{n-1}-1-\binom{n-1}{1}(K-1)-\binom{n-1}{2}(K-1)^2-\dots-\binom{n-1}{k+m-2}(K-1)^{k+m-2}\right\}$$

$$+\ (k+m)!|a_{k+m}|\left(\frac{1}{(n-1)(n-2)\dots(n-\overline{k+m-1})}\right.$$

$$\times\ \left\{K^{n-1}-1-\binom{n-1}{1}(K-1)-\binom{n-1}{2}(K-1)^2-\dots-\binom{n-1}{k+m-2}(K-1)^{k+m-2}\right\}$$

$$-\ \frac{1}{(n-3)(n-4)\dots(n-\overline{k+m+1})}$$

$$\times\ \left\{K^{n-3}-1-\binom{n-3}{1}(K-1)-\binom{n-3}{2}(K-1)^2-\dots\right.$$

$$\left.\left.\left.-\ \binom{n-3}{k+m-2}(K-1)^{k+m-2}\right\}\right)\right],$$

$$n>k+m+1,\quad (k\geq 3)$$

and

$$M(p',1)\geq\frac{k+2m+1}{1+K^{k+1}}M(p,1)+\frac{k+2m+1}{2K^{k+m+1}}.\frac{K^{k+1}-1}{K^{k+1}+1}m(p,K)+\frac{k+2m+1}{1+K^{k+1}}$$

$$\times\ \left[\frac{1}{K}.\frac{2}{k+2}.\frac{1!|a_{k+m}|}{k+1}\left\{K^{k+1}-1-\binom{k+1}{1}(K-1)\right\}+\frac{1}{K^2}.\frac{2}{\overline{k-1}+2}.\frac{2!|a_{k+m-1}|}{(k+1)k}\right.$$

$$\times\ \left\{K^{k+1}-1-\binom{k+1}{1}(K-1)-\binom{k+1}{2}(K-1)^2\right\}+\frac{1}{K^3}.\frac{2}{\overline{k-2}+2}.\frac{3!|a_{k+m-2}|}{(k+1)k(k-1)}$$

$$\times\ \left\{K^{k+1}-1-\binom{k+1}{1}(K-1)-\binom{k+1}{2}(K-1)^2-\binom{k+1}{3}(K-1)^3\right\}+\dots$$

$$+\ \frac{1}{K^{k-1}}.\frac{2}{2+2}.\frac{(k-1)!|a_{m+2}|}{(k+1)k\dots 3}$$

$$\times\ \left\{K^{k+1}-1-\binom{k+1}{1}(K-1)-\binom{k+1}{2}(K-1)^2-\dots-\binom{k+1}{k-1}(K-1)^{k-1}\right\}$$

$$\left.+\ \frac{1}{K^k}.\frac{|a_{m+1}|}{k+1}(K-1)^{k+1}\right]+\frac{1}{K^{k+m}}\left[\frac{2|a_1|}{k+m+2}(K-1)+\frac{2}{\overline{k+m-1}+2}.\frac{2!|a_2|}{k+m}\right.$$

$$\times \left\{K^{k+m} - 1 - \binom{k+m}{1}(K-1)\right\} + \frac{2}{k+m-2+2}.\frac{3!|a_3|}{(k+m)(k+m-1)}$$

$$\times \left\{K^{k+m} - 1 - \binom{k+m}{1}(K-1) - \binom{k+m}{2}(K-1)^2\right\}$$

$$+ \frac{2}{k+m-3+2}.\frac{4!|a_4|}{(k+m)(k+m-1)(k+m-2)}$$

$$\times \left\{K^{k+m} - 1 - \binom{k+m}{1}(K-1) - \binom{k+m}{2}(K-1)^2 - \binom{k+m}{3}(K-1)^3\right\} + \ldots$$

$$+ \frac{2}{2+2}.\frac{(k+m-1)!|a_{k+m-1}|}{(k+m)(k+m-1)\ldots 3}$$

$$\times \left\{K^{k+m} - 1 - \binom{k+m}{1}(K-1) - \binom{k+m}{2}(K-1)^2 - \ldots - \binom{k+m}{k+m-2}(K-1)^{k+m-2}\right\}$$

$$\left. + |a_{k+m}|(K-1)^{k+m}\right], \quad n = k+m+1, \quad (k \geq 3).$$

Result is best possible with equality for the polynomial $p(z) = z^n + K^n$.

Remark 1.9. Corollary 1.8 is similar to the results ([8, Corollary] and [3, Corollary]). Further Corollary 1.8 is a refinement of Theorem 1.3 for $n \geq k+m+1$.

2. Lemmas

For the proof of Theorem 1.7 we require the following lemmas.

Lemma 2.1. *If* $p(z) = a_n \prod_{t=1}^{n}(z - z_t)$ *is a polynomial of degree* n *such that* $|z_t| \leq 1, 1 \leq t \leq n$ *then*

$$M(p', 1) \geq \Big(\sum_{t=1}^{n} \frac{1}{1+|z_t|}\Big) M(p, 1).$$

Result is best possible with equality for the polynomial $p(z)$ *whose all zeros are positive.*

This lemma is due to Giroux et al. [5].

Lemma 2.2. *If* $p(z) = \sum_{j=0}^{n} a_j z^j$ *is a polynomial of degree* n *then*

$$M(p, R) \leq R^n M(p, 1) - \frac{2|a_0|}{n+2}(R^n - 1) - |a_1|\Big(\frac{R^n - 1}{n} - \frac{R^{n-2} - 1}{n-2}\Big), R \geq 1 \quad (2.1)$$

for $n > 2$ *and*

$$M(p, R) \leq R^2 M(p, 1) - \frac{|a_0|}{2}(R^2 - 1) - \frac{|a_1|}{2}(R-1)^2, R \geq 1 \text{ for } n = 2. \quad (2.2)$$

This lemma is due to Dewan et al. [3].

Lemma 2.3. *Let $p(z)$ be a polynomial of degree at most n. Then*

$$M(p',1) \le nM(p,1) - \epsilon_n |p(0)|,$$

where $\epsilon_n = 2n/(n+2)$ if $n \ge 2$, where as $\epsilon_1 = 1$. The coefficient of $|p(0)|$ is the best possible for each n.

This lemma is due to Frappier et al. [4].

Lemma 2.4. *Let $p(z) = \sum_{j=0}^{n} a_j z^j$ be a polynomial of degree $n(\ge k)$, $(k \ge 3)$. Then*

$$M(p,R) \le R^n M(p,1) - \frac{2|a_0|}{n+2}(R^n - 1) - \frac{2}{\overline{n-1}+2}.\frac{1!|a_1|}{n}\left\{(R^n-1) - \binom{n}{1}(R-1)\right\}$$

$$- \frac{2}{\overline{n-2}+2}.\frac{2!|a_2|}{n(n-1)}\left\{(R^n-1) - \binom{n}{1}(R-1) - \binom{n}{2}(R-1)^2\right\}$$

$$- \frac{2}{\overline{n-3}+2}.\frac{3!|a_3|}{n(n-1)(n-2)}\left\{(R^n-1) - \binom{n}{1}(R-1)\right.$$

$$\left. - \binom{n}{2}(R-1)^2 - \binom{n}{3}(R-1)^3\right\} - \ldots - \frac{2}{\overline{n-(k-2)}+2}.\frac{(k-2)!|a_{k-2}|}{n(n-1)\ldots(n-\overline{k-3})}$$

$$\times \left\{(R^n-1) - \binom{n}{1}(R-1) - \binom{n}{2}(R-1)^2 - \ldots - \binom{n}{k-2}(R-1)^{k-2}\right\}$$

$$- (k-1)!|a_{k-1}|\left[\frac{1}{n(n-1)\ldots(n-\overline{k-2})}\right.$$

$$\times \left\{(R^n-1) - \binom{n}{1}(R-1) - \binom{n}{2}(R-1)^2 - \ldots - \binom{n}{k-2}(R-1)^{k-2}\right\}$$

$$- \frac{1}{(n-2)(n-3)\ldots(n-k)}$$

$$\left. \times \left\{(R^{n-2}-1) - \binom{n-2}{1}(R-1) - \binom{n-2}{2}(R-1)^2 - \ldots - \binom{n-2}{k-2}(R-1)^{k-2}\right\}\right],$$

$$R \ge 1 \text{ and } n > k, \quad (k \ge 3) \tag{2.3}$$

and

$$M(p,R) \leq R^k M(p,1) - \frac{2|a_0|}{k+2}(R^k-1) - \frac{2}{\overline{k-1}+2}.\frac{1!|a_1|}{k}\left\{(R^k-1) - \binom{k}{1}(R-1)\right\}$$

$$- \frac{2}{\overline{k-2}+2}.\frac{2!|a_2|}{k(k-1)}\left\{(R^k-1) - \binom{k}{1}(R-1) - \binom{k}{2}(R-1)^2\right\}$$

$$- \frac{2}{\overline{k-3}+2}.\frac{3!|a_3|}{k(k-1)(k-2)}\left\{(R^k-1) - \binom{k}{1}(R-1) - \binom{k}{2}(R-1)^2 - \binom{k}{3}(R-1)^3\right\} - \dots$$

$$- \frac{2}{2+2}.\frac{(k-2)!|a_{k-2}|}{k(k-1)\dots 4.3}\left\{(R^k-1) - \binom{k}{1}(R-1) - \binom{k}{2}(R-1)^2 - \dots\right.$$

$$\left. - \binom{k}{k-2}(R-1)^{k-2}\right\} - \frac{(k-1)!|a_{k-1}|}{k!}(R-1)^k, \qquad R \geq 1 \text{ and } n = k, (k \geq 3). \quad (2.4)$$

Lemma 2.4 is best posssible and equality holds in (2.3) and (2.4) for $p(z) = \lambda z^n$.

Proof of Lemma 2.4. We will prove inequalities (2.3) and (2.4) by mathematical induction. Accordingly for a polynomial $p(z)$ of degree

$$n > 3,$$

we have

$$\begin{aligned} |p(Re^{i\phi}) - p(e^{i\phi})| &= \left|\int_1^R p'(te^{i\phi})e^{i\phi}dt\right|, \ 0 \leq \phi \leq 2\pi \ , \\ &\leq \int_1^R M(p',t)dt \ , \end{aligned}$$

which implies that

$$M(p,R) \leq M(p,1) + \int_1^R M(p',t)dt \qquad (2.5)$$

and further as $p(z)$ is a polynomial of degree $n(>3)$, $p'(z)$ will be a polynomial of degree $(n-1)$, (>2) and therefore we can apply inequality (2.1), of Lemma 2.2, to polynomial $p'(z) = a_1 + 2a_2 z + \ldots + na_n z^{n-1}$, thereby helping us to rewrite (2.5), in the form

$$M(p,R) \leq M(p,1) + M(p',1)\int_1^R t^{n-1}dt - \frac{2|a_1|}{n+1}\int_1^R (t^{n-1}-1)dt - 2|a_2|$$

$$\times \int_1^R \Big(\frac{t^{n-1}-1}{n-1} - \frac{t^{n-3}-1}{n-3}\Big)dt,$$

$$\leq R^n M(p,1) - \frac{2|a_0|}{n+2}(R^n-1) - \frac{2}{n+1}.\frac{1!|a_1|}{n}\Big\{(R^n-1) - \binom{n}{1}(R-1)\Big\}$$

$$- 2!|a_2|\Big[\frac{1}{n(n-1)}\Big\{(R^n-1) - \binom{n}{1}(R-1)\Big\}$$

$$- \frac{1}{(n-2)(n-3)}\Big\{(R^{n-2}-1) - \binom{n-2}{1}(R-1)\Big\}\Big],$$

by Lemma 2.3.

This proves inequality (2.3) for polynomial $p(z)$ of degree $n(>k)$, with $k=3$. We can similarly prove inequality (2.4) for polynomial $p(z)$ of degree $n(=k)$, with $k=3$, by continuing in the same manner with one change:

inequality (2.2), of Lemma 2.2, to polynomial $p'(z)$ of degree $(n-1),(=2)$,
instead of inequality (2.1), of Lemma 2.2, to polynomial $p'(z)$ of degree $(n-1),(>2)$.

Now we assume that inequality (2.3) is true for a polynomial $p(z)$ of degree $n(>k)$, with certain arbitrarily chosen fixed $k(\geq 3)$. Then for a polynomial $p(z)$ of degree $n(>k+1)$, with fixed $k(\geq 3)$, inequality (2.5) will obviosuly be true and as $p'(z)$ will be a polynomial of degree $(n-1)$, $(>k)$, with fixed $k(\geq 3)$, we can apply inequality (2.3) to polynomial

$$p'(z) = a_1 + 2a_2z + 3a_3z^2 + 4a_4z^3 + \ldots + (k-1)a_{k-1}z^{k-2} + ka_kz^{k-1} + \ldots + na_nz^{n-1},$$

thereby helping us to rewrite (2.5) presently, in the form

$$M(p,R) \leq M(p,1) + M(p',1)\int_1^R t^{n-1}dt - \frac{2|a_1|}{n+1}\int_1^R (t^{n-1}-1)dt$$

$$- \frac{2}{\overline{n-2}+2}.\frac{1!2|a_2|}{n-1}\int_1^R \Big\{(t^{n-1}-1) - \binom{n-1}{1}(t-1)\Big\}dt$$

$$- \frac{2}{\overline{n-3}+2}.\frac{2!3|a_3|}{(n-1)(n-2)}\int_1^R \Big\{(t^{n-1}-1) - \binom{n-1}{1}(t-1) - \binom{n-1}{2}(t-1)^2\Big\}dt$$

$$- \frac{2}{\overline{n-4}+2}.\frac{3!4|a_4|}{(n-1)(n-2)(n-3)}$$

$$\times \int_1^R \Big\{(t^{n-1}-1) - \binom{n-1}{1}(t-1) - \binom{n-1}{2}(t-1)^2 - \binom{n-1}{3}(t-1)^3\Big\}dt - \ldots$$

$$- \frac{2}{\overline{n-1-(k-2)+2}} . \frac{(k-2)!(k-1)|a_{k-1}|}{(n-1)(n-2)\dots(n-1-\overline{k-3})}$$

$$\times \int_1^R \left\{ (t^{n-1}-1) - \binom{n-1}{1}(t-1) - \binom{n-1}{2}(t-1)^2 - \dots - \binom{n-1}{k-2}(t-1)^{k-2} \right\} dt$$

$$- (k-1)!k|a_k| \int_1^R \left[\frac{1}{(n-1)(n-2)\dots(n-1-\overline{k-2})} \right.$$

$$\times \left\{ (t^{n-1}-1) - \binom{n-1}{1}(t-1) - \binom{n-1}{2}(t-1)^2 - \dots - \binom{n-1}{k-2}(t-1)^{k-2} \right\}$$

$$- \frac{1}{(n-3)(n-4)\dots(n-1-k)}$$

$$\left. \times \left\{ (t^{n-3}-1) - \binom{n-3}{1}(t-1) - \binom{n-3}{2}(t-1)^2 - \dots - \binom{n-3}{k-2}(t-1)^{k-2} \right\} \right] dt$$

$$\leq R^n M(p,1) - \frac{2|a_0|}{n+2}(R^n-1) - \frac{2}{\overline{n-1}+2} . \frac{1!|a_1|}{n} \left\{ (R^n-1) - \binom{n}{1}(R-1) \right\}$$

$$- \frac{2}{\overline{n-2}+2} . \frac{2!|a_2|}{n(n-1)} \left\{ (R^n-1) - \binom{n}{1}(R-1) - \binom{n}{2}(R-1)^2 \right\}$$

$$- \frac{2}{\overline{n-3}+2} . \frac{3!|a_3|}{n(n-1)(n-2)} \left\{ (R^n-1) - \binom{n}{1}(R-1) - \binom{n}{2}(R-1)^2 - \binom{n}{3}(R-1)^3 \right\}$$

$$- \frac{2}{\overline{n-4}+2} . \frac{4!|a_4|}{n(n-1)(n-2)(n-3)}$$

$$\times \left\{ (R^n-1) - \binom{n}{1}(R-1) - \binom{n}{2}(R-1)^2 - \binom{n}{3}(R-1)^3 - \binom{n}{4}(R-1)^4 \right\} - \dots$$

$$- \frac{2}{\overline{n-(k-1)}+2} . \frac{(k-1)!|a_{k-1}|}{n(n-1)\dots(n-\overline{k-2})}$$

$$\times \left\{ (R^n-1) - \binom{n}{1}(R-1) - \binom{n}{2}(R-1)^2 - \dots - \binom{n}{k-1}(R-1)^{k-1} \right\}$$

$$- k!|a_k|\Big[\frac{1}{n(n-1)\dots(n-\overline{k-1})}$$

$$\times \left\{(R^n-1)-\binom{n}{1}(R-1)-\binom{n}{2}(R-1)^2-\dots-\binom{n}{k-1}(R-1)^{k-1}\right\}$$

$$-\frac{1}{(n-2)(n-3)\dots(n-\overline{k+1})}$$

$$\times \left\{(R^{n-2}-1)-\binom{n-2}{1}(R-1)-\binom{n-2}{2}(R-1)^2-\dots-\binom{n-2}{k-1}(R-1)^{k-1}\right\}\Bigg], \quad (2.6)$$

by Lemma 2.3.

This proves inequality (2.3) for a polynomial $p(z)$ of degree $n(> k+1)$, with fixed $k(\geq 3)$ under the assumption that inequality (2.3) is true for a polynomial $p(z)$ of degree $n(> k)$, with certain arbitrarily chosen fixed $k(\geq 3)$. Earlier we have shown that (2.3) is true for a polynomial $p(z)$ of degree $n(> k)$, with $k = 3$. This therefore completes the proof of inequality (2.3) for a polynomial $p(z)$ of degree $n(> k)$, with $k(\geq 3)$. Again as we have proved inequality (2.3) for a polynomial $p(z)$ of degree $n(> k+1)$, with fixed $k(\geq 3)$ under the assumption that inequality (2.3) is true for a polynomial $p(z)$ of degree $n(> k)$, with certain arbitrarily chosen fixed $k(\geq 3)$, we can similarly prove inequality (2.4) for a polynomial $p(z)$ of degree $n(= k+1)$, with fixed $k(\geq 3)$ under the assumption that inequality (2.4) is true for a polynomial $p(z)$ of degree $n(= k)$, with certain arbitrarily chosen fixed $k(\geq 3)$, by continuing in the same manner with one change:

inequality (2.4) to polynomial $p'(z)$ of degree $(n-1), (= k)$,

instead of inequality (2.3) to polynomial $p'(z)$ of degree $(n-1), (> k)$,

and further we have shown earlier that inequality (2.4) is true for a polynomial $p(z)$ of degree $n(= k)$, with $k = 3$. This therefore completes the proof of inequality (2.4) for a polynomial $p(z)$ of degree $n(= k)$, with $k(\geq 3)$. This completes the proof of Lemma 2.4.

Remark 2.5. Lemma 2.4 is a refinement of well known result (see [11, Problem III 269, p. 158])

$$M(p, R) \leq R^n M(p, 1), \quad R \geq 1 \; .$$

Remark 2.6. Lemma 2.4, along with results ([4, inequality (1.5)] and [3, Lemma 3]) suggests an inequality for $M(p, R)$ in terms of $M(p, 1)$ and most of the available coefficients of the polynomial.

Lemma 2.7. *If $p(z)$ is a polynomial of degree n, having no zeros in $|z| < K_0, K_0 \geq 1$ then*

$$M(p', 1) \leq \frac{n}{1+K_0}\left\{M(p, 1) - m(p, K_0)\right\} \; .$$

The result is best possible and equality holds for $p(z) = (z + K)^n$.

This lemma is due to Govil [9].

Lemma 2.8. *Let $p(z)=\sum_{j=0}^{n} a_j z^j$ be a polynomial of degree $n(\geq k+1),(k\geq 3)$, having no zeros in $|z|<K_0, K_0\geq 1$. Then*

$$M(p,R)\leq \frac{R^n+K_0}{1+K_0}M(p,1)-\frac{R^n-1}{1+K_0}m(p,K_0)-\frac{2}{\overline{n-1}+2}.\frac{1!|a_1|}{n}$$

$$\times\left\{(R^n-1)-\binom{n}{1}(R-1)\right\}-\frac{2}{\overline{n-2}+2}.\frac{2!|a_2|}{n(n-1)}$$

$$\times\left\{(R^n-1)-\binom{n}{1}(R-1)-\binom{n}{2}(R-1)^2\right\}$$

$$-\frac{2}{\overline{n-3}+2}.\frac{3!|a_3|}{n(n-1)(n-2)}$$

$$\times\left\{(R^n-1)-\binom{n}{1}(R-1)-\binom{n}{2}(R-1)^2-\binom{n}{3}(R-1)^3\right\}-\dots$$

$$-\frac{2}{\overline{n-(k-1)}+2}.\frac{(k-1)!|a_{k-1}|}{n(n-1)\dots(n-\overline{k-2})}$$

$$\times\left\{(R^n-1)-\binom{n}{1}(R-1)-\binom{n}{2}(R-1)^2-\dots-\binom{n}{k-1}(R-1)^{k-1}\right\}$$

$$-\ k!|a_k|\Big[\frac{1}{n(n-1)\dots(n-\overline{k-1})}$$

$$\times\left\{(R^n-1)-\binom{n}{1}(R-1)-\binom{n}{2}(R-1)^2-\dots-\binom{n}{k-1}(R-1)^{k-1}\right\}$$

$$-\frac{1}{(n-2)(n-3)\dots(n-\overline{k+1})}$$

$$\times\left\{(R^{n-2}-1)-\binom{n-2}{1}(R-1)-\binom{n-2}{2}(R-1)^2-\dots-\binom{n-2}{k-1}(R-1)^{k-1}\right\},$$

$$\textit{for } R\geq 1 \quad \textit{and } n>k+1,(k\geq 3)\ , \tag{2.7}$$

and

$$M(p,R)\leq \frac{R^{k+1}+K_0}{1+K_0}M(p,1)-\frac{R^{k+1}-1}{1+K_0}m(p,K_0)-\frac{2}{\overline{(k+1)-1}+2}.\frac{1!|a_1|}{k+1}$$

$$\times \left\{(R^{k+1}-1)-\binom{k+1}{1}(R-1)\right\}-\frac{2}{(k+1)-2+2}.\frac{2!|a_2|}{(k+1)(\overline{k+1}-1)}$$

$$\times \left\{(R^{k+1}-1)-\binom{k+1}{1}(R-1)-\binom{k+1}{2}(R-1)^2\right\}$$

$$-\frac{2}{(k+1)-3+2}.\frac{3!|a_3|}{(k+1)(\overline{k+1}-1)(\overline{k+1}-2)}$$

$$\times \left\{(R^{k+1}-1)-\binom{k+1}{1}(R-1)-\binom{k+1}{2}(R-1)^2-\binom{k+1}{3}(R-1)^3\right\}-\dots$$

$$-\frac{2}{2+2}.\frac{(k-1)!|a_{k-1}|}{(k+1)(\overline{k+1}-1)\dots4.3}$$

$$\times \left\{(R^{k+1}-1)-\binom{k+1}{1}(R-1)-\binom{k+1}{2}(R-1)^2-\dots-\binom{k+1}{k-1}(R-1)^{k-1}\right\}$$

$$-\frac{k!|a_k|}{(k+1)!}(R-1)^{k+1}\ , \quad R\geq 1 \textit{ for } n=k+1,\ (k\geq 3). \tag{2.8}$$

Proof of Lemma 2.8. As we had proved inequality (2.6) for a polynomial $p(z)$ of degree $n(>k+1)$, with fixed $k(\geq 3)$, under the assumption that ineq. (2.3) is true for a polynomial $p(z)$ of degree $n(>k)$, with certain arbitrarily chosen fixed $k(\geq 3)$, we can similarly prove inequality (2.7), (as ineq. (2.3) is now known to be true), with one change:

Lemma 2.7 instead of Lemma 2.3.

Further as we have proved ineq. (2.7), we can similarly prove ineq. (2.8), with changes:

(i) ineq. (2.4) instead of ineq. (2.3),
(ii) $n(=k)$, instead of $n(>k)$,
n(=k+1), instead of $n(>k+1)$ and
$(n-1),(=k)$, instead of $(n-1),(>k)$.

This completes the proof of Lemma 2.8.

3. Proof of Theorem 1.7

Well from (1.1) and (1.2) we can say that for $m\geq 1$, the coefficients $a_0,a_1,\dots,a_{m-1}$ will all be zero. Further $T(z)=p(Kz)$ is a polynomial of degree n, having all its zeros $z_t/K,(1\leq t\leq n)$, in $|z|\leq 1$ and therefore by Lemma 2.1

$$M(T',1)\geq\Big(\sum_{t=1}^{n}\frac{K}{K+|z_t|}\Big)M(T,1),$$

i.e.

$$KM(p',K) \geq \Big(\sum_{t=1}^{n} \frac{K}{K+K_t}\Big) M(p,K), \quad \text{by (1.1)} . \tag{3.1}$$

Now we first prove (1.4). As

$$p'(z) = a_1 + 2a_2 z + 3a_3 z^2 + \ldots + n a_n z^{n-1},$$

is a polynomial of degree $(n-1)$, $(> k+m)$, by (1.4) and

$$k+m \geq 3, \quad \text{by (1.4) and (1.2)},$$

we can apply ineq. (2.3), (Lemma 2.4), to $p'(z)$, with $R = K$, thereby giving

$$M(p',K) \leq K^{n-1} M(p',1) - \frac{2|a_1|}{n+1}(K^{n-1}-1) - \frac{2}{\overline{\overline{n-2}+2}} . \frac{2!|a_2|}{n-1}$$

$$\times \left\{ K^{n-1} - 1 - \binom{n-1}{1}(K-1) \right\}$$

$$- \frac{2}{\overline{\overline{n-3}+2}} . \frac{3!|a_3|}{(n-1)(n-2)}$$

$$\times \left\{ K^{n-1} - 1 - \binom{n-1}{1}(K-1) - \binom{n-1}{2}(K-1)^2 \right\}$$

$$- \frac{2}{\overline{\overline{n-4}+2}} . \frac{4!|a_4|}{(n-1)(n-2)(n-3)}$$

$$\times \left\{ K^{n-1} - 1 - \binom{n-1}{1}(K-1) - \binom{n-1}{2}(K-1)^2 - \binom{n-1}{3}(K-1)^3 \right\} - \ldots$$

$$- \frac{2}{\overline{\overline{n-(k+m-1)}+2}} . \frac{(k+m-1)!|a_{k+m-1}|}{(n-1)(n-2)\ldots(n-\overline{k+m-2})}$$

$$\times \left\{ K^{n-1} - 1 - \binom{n-1}{1}(K-1) - \binom{n-1}{2}(K-1)^2 - \ldots - \binom{n-1}{k+m-2}(K-1)^{k+m-2} \right\}$$

$$- (k+m)!|a_{k+m}| \Bigg(\frac{1}{(n-1)(n-2)\ldots(n-\overline{k+m-1})}$$

$$\times \left\{ K^{n-1} - 1 - \binom{n-1}{1}(K-1) - \binom{n-1}{2}(K-1)^2 - \ldots - \binom{n-1}{k+m-2}(K-1)^{k+m-2} \right\}$$

$$- \frac{1}{(n-3)(n-4)\ldots(n-\overline{k+m+1})}$$

$$\times \left\{K^{n-3} - 1 - \binom{n-3}{1}(K-1) - \binom{n-3}{2}(K-1)^2 - \ldots \right.$$

$$\left. - \binom{n-3}{k+m-2}(K-1)^{k+m-2}\right\}\Bigg). \tag{3.2}$$

It should be noted here that in (3.2), among the coefficients $a_1, a_2, \ldots, a_{k+m}$, the coefficients $a_1, a_2, \ldots, a_{m-1}$ will all be zero for $m > 1$, as told earlier. Further by (1.1) and (1.2) we can say that

$$p_1(z) = a_m + a_{m+1}z + \ldots + a_n z^{n-m} \tag{3.3}$$

is a polynomial of degree $(n-m)$, having all its zeros in $|z| \leq K$ and therefore

$$P(z) = p_1(Kz) \tag{3.4}$$

is a polynomial of degree $(n-m)$, having all its zeros in $|z| \leq 1$, thereby implying that

$$\begin{aligned} Q(z) &= z^{n-m}\overline{P(1/\overline{z})} \\ &= z^{n-m}\overline{p_1(K/\overline{z})} \text{ (by (3.4))} \\ &= \overline{a_n}K^{n-m} + \overline{a_{n-1}}K^{n-m-1}z + \ldots \\ &+ \overline{a_{m+1}}Kz^{n-m-1} + \overline{a_m}z^{n-m} \text{ (by (3.3))} \end{aligned} \tag{3.5}$$

is a polynomial of degree $(n-m)$, $(> k+1)$, $(k \geq 3)$, (by (1.3)), having no zeros in $|z| < 1$. Accordingly we can apply ineq. (2.7), (Lemma 2.8), to $Q(z)$, with $K_0 = 1$ and $R = K$, thereby giving

$$M(Q,K) \leq \frac{K^{n-m}+1}{2}M(Q,1) - \frac{K^{n-m}-1}{2}m(Q,1)$$

$$- \frac{2}{\overline{n-m-1}+2}.\frac{1!K^{n-m-1}|a_{n-1}|}{n-m}\left\{K^{n-m} - 1 - \binom{n-m}{1}(K-1)\right\}$$

$$- \frac{2}{\overline{n-m-2}+2}.\frac{2!K^{n-m-2}|a_{n-2}|}{(n-m)(n-m-1)}$$

$$\times \left\{K^{n-m} - 1 - \binom{n-m}{1}(K-1) - \binom{n-m}{2}(K-1)^2\right\}$$

$$- \frac{2}{\overline{n-m-3}+2}.\frac{3!K^{n-m-3}|a_{n-3}|}{(n-m)(n-m-1)(n-m-2)}$$

$$\times \left\{K^{n-m} - 1 - \binom{n-m}{1}(K-1) - \binom{n-m}{2}(K-1)^2 - \binom{n-m}{3}(K-1)^3\right\} - \ldots$$

$$- \frac{2}{\overline{n-m-(k-1)}+2}.\frac{(k-1)!K^{n-m-(k-1)}|a_{n-(k-1)}|}{(n-m)(n-m-1)\ldots(n-m-\overline{k-2})}$$

$$\times \left\{K^{n-m} - 1 - \binom{n-m}{1}(K-1) - \binom{n-m}{2}(K-1)^2 - \ldots - \binom{n-m}{k-1}(K-1)^{k-1}\right\}$$

$$- k!K^{n-m-k}|a_{n-k}|\Bigg[\frac{1}{(n-m)(n-m-1)\ldots(n-m-\overline{k-1})}$$

$$\times \left\{K^{n-m} - 1 - \binom{n-m}{1}(K-1) - \binom{n-m}{2}(K-1)^2 - \ldots - \binom{n-m}{k-1}(K-1)^{k-1}\right\}$$

$$- \frac{1}{(n-m-2)(n-m-3)\ldots(n-m-\overline{k+1})}$$

$$\times \left\{K^{n-m-2} - 1 - \binom{n-m-2}{1}(K-1) - \binom{n-m-2}{2}(K-1)^2 - \ldots\right.$$

$$\left. - \binom{n-m-2}{k-1}(K-1)^{k-1}\right\}\Bigg]. \tag{3.6}$$

Now by (3.5) and (1.2) we get

$$\left.\begin{array}{l} M(Q,K) = K^{n-m}M(p,1), \\ M(Q,1) = \frac{1}{K^m}M(p,K), \\ m(Q,1) = \frac{1}{K^m}m(p,K), \end{array}\right\},$$

which, on being used in (3.6), implies that

$$M(p,K) \geq \frac{2K^n}{1+K^{n-m}}M(p,1) + \frac{K^{n-m}-1}{K^{n-m}+1}m(p,K) + \frac{2K^m}{K^{n-m}+1}$$

$$\times \left(\frac{2}{\overline{\overline{n-m-1}}+2}.\frac{1!K^{n-m-1}|a_{n-1}|}{n-m}\left\{K^{n-m} - 1 - \binom{n-m}{1}(K-1)\right\}\right.$$

$$+ \frac{2}{\overline{\overline{n-m-2}}+2}.\frac{2!K^{n-m-2}|a_{n-2}|}{(n-m)(n-m-1)}$$

$$\times \left\{K^{n-m} - 1 - \binom{n-m}{1}(K-1) - \binom{n-m}{2}(K-1)^2\right\}$$

$$+ \frac{2}{\overline{\overline{n-m-3}}+2}.\frac{3!K^{n-m-3}|a_{n-3}|}{(n-m)(n-m-1)(n-m-2)}$$

$$\times \left\{K^{n-m} - 1 - \binom{n-m}{1}(K-1) - \binom{n-m}{2}(K-1)^2 - \binom{n-m}{3}(K-1)^3\right\} + \ldots$$

$$+ \frac{2}{\overline{n-m-(k-1)}+2}.\frac{(k-1)!K^{n-m-(k-1)}|a_{n-(k-1)}|}{(n-m)(n-m-1)\dots(n-m-\overline{k-2})}$$

$$\times \left\{K^{n-m} - 1 - \binom{n-m}{1}(K-1) - \binom{n-m}{2}(K-1)^2 - \dots - \binom{n-m}{k-1}(K-1)^{k-1}\right\}$$

$$+ k!K^{n-m-k}|a_{n-k}|\Bigg[\frac{1}{(n-m)(n-m-1)\dots(n-m-\overline{k-1})}$$

$$\times \left\{K^{n-m} - 1 - \binom{n-m}{1}(K-1) - \binom{n-m}{2}(K-1)^2 - \dots - \binom{n-m}{k-1}(K-1)^{k-1}\right\}$$

$$- \frac{1}{(n-m-2)(n-m-3)\dots(n-m-\overline{k+1})}$$

$$\times \Bigg\{K^{n-m-2} - 1 - \binom{n-m-2}{1}(K-1) - \binom{n-m-2}{2}(K-1)^2 - \dots$$

$$- \binom{n-m-2}{k-1}(K-1)^{k-1}\Bigg\}\Bigg]\Bigg). \tag{3.7}$$

Finally on using inequalities (3.7) and (3.2) in ineq. (3.1) we get

$$K^nM(p',1) - \frac{2K|a_1|}{n+1}(K^{n-1}-1) - \frac{2K}{\overline{n-2}+2}.\frac{2!|a_2|}{n-1}\left\{K^{n-1} - 1 - \binom{n-1}{1}(K-1)\right\}$$

$$- \frac{2K}{\overline{n-3}+2}.\frac{3!|a_3|}{(n-1)(n-2)}\left\{K^{n-1} - 1 - \binom{n-1}{1}(K-1) - \binom{n-1}{2}(K-1)^2\right\}$$

$$- \frac{2K}{\overline{n-4}+2}.\frac{4!|a_4|}{(n-1)(n-2)(n-3)}$$

$$\times \left\{K^{n-1} - 1 - \binom{n-1}{1}(K-1) - \binom{n-1}{2}(K-1)^2 - \binom{n-1}{3}(K-1)^3\right\} - \dots$$

$$- \frac{2K}{\overline{n-(k+m-1)}+2}.\frac{(k+m-1)!|a_{k+m-1}|}{(n-1)(n-2)\dots(n-\overline{k+m-2})}$$

$$\times \left\{K^{n-1} - 1 - \binom{n-1}{1}(K-1) - \binom{n-1}{2}(K-1)^2 - \dots - \binom{n-1}{k+m-2}(K-1)^{k+m-2}\right\}$$

$$- (k+m)!|a_{k+m}|K\Bigg(\frac{1}{(n-1)(n-2)\dots(n-\overline{k+m-1})}$$

$$\times\left\{K^{n-1}-1-\binom{n-1}{1}(K-1)-\binom{n-1}{2}(K-1)^2-\dots-\binom{n-1}{k+m-2}(K-1)^{k+m-2}\right\}$$

$$-\frac{1}{(n-3)(n-4)\dots(n-\overline{k+m+1})}$$

$$\times\left\{K^{n-3}-1-\binom{n-3}{1}(K-1)-\binom{n-3}{2}(K-1)^2-\dots-\binom{n-3}{k+m-2}(K-1)^{k+m-2}\right\}\Bigg)$$

$$\geq \frac{2K^n}{1+K^{n-m}}\Big(\sum_{t=1}^{n}\frac{K}{K+K_t}\Big)M(p,1)+\frac{K^{n-m}-1}{K^{n-m}+1}\Big(\sum_{t=1}^{n}\frac{K}{K+K_t}\Big)m(p,K)$$

$$+\frac{2K^m}{K^{n-m}+1}\Big(\sum_{t=1}^{n}\frac{K}{K+K_t}\Big)\Bigg(\frac{2}{\overline{n-m-1}+2}.\frac{1!K^{n-m-1}|a_{n-1}|}{n-m}$$

$$\times\left\{K^{n-m}-1-\binom{n-m}{1}(K-1)\right\}+\frac{2}{\overline{n-m-2}+2}.\frac{2!K^{n-m-2}|a_{n-2}|}{(n-m)(n-m-1)}$$

$$\times\left\{K^{n-m}-1-\binom{n-m}{1}(K-1)-\binom{n-m}{2}(K-1)^2\right\}$$

$$+\frac{2}{\overline{n-m-3}+2}.\frac{3!K^{n-m-3}|a_{n-3}|}{(n-m)(n-m-1)(n-m-2)}$$

$$\times\left\{K^{n-m}-1-\binom{n-m}{1}(K-1)-\binom{n-m}{2}(K-1)^2-\binom{n-m}{3}(K-1)^3\right\}+\dots$$

$$+\frac{2}{\overline{n-m-(k-1)}+2}.\frac{(k-1)!K^{n-m-(k-1)}|a_{n-(k-1)}|}{(n-m)(n-m-1)\dots(n-m-\overline{k-2})}$$

$$\times\left\{K^{n-m}-1-\binom{n-m}{1}(K-1)-\binom{n-m}{2}(K-1)^2-\dots-\binom{n-m}{k-1}(K-1)^{k-1}\right\}$$

$$+k!K^{n-m-k}|a_{n-k}|\Bigg[\frac{1}{(n-m)(n-m-1)\dots(n-m-\overline{k-1})}$$

$$\times \left\{K^{n-m} - 1 - \binom{n-m}{1}(K-1) - \binom{n-m}{2}(K-1)^2 - \ldots - \binom{n-m}{k-1}(K-1)^{k-1}\right\}$$

$$- \frac{1}{(n-m-2)(n-m-3)\ldots(n-m-\overline{k+1})}$$

$$\times \left\{K^{n-m-2} - 1 - \binom{n-m-2}{1}(K-1) - \binom{n-m-2}{2}(K-1)^2 - \ldots\right.$$

$$\left. - \binom{n-m-2}{k-1}(K-1)^{k-1}\right\}\Bigg]\Bigg)$$

and ineq. (1.4) follows.

As we have proved ineq. (1.4), we can similarly prove ineq. (1.5), with changes:
(i) $(n-1),\ (=k+m),$ instead of $(n-1), (>k+m)$,
(ii) ineq. (2.4), (Lemma 2.4), instead of ineq. (2.3), (Lemma 2.4),
(iii) $(n-m),\ (=k+1),$ instead of $(n-m),\ (>k+1)$,
(iv) ineq. (2.8), (Lemma 2.8), instead of ineq. (2.7), (Lemma 2.8).
This completes the proof of Theorem 1.7.

Remark 3.1. In Theorem 1.7 possibilities

$$\left.\begin{array}{l} n-m > k+1, \\ n-m = k+1, \end{array}\right\}, k \geq 3$$

are considered and for remaining possibilities

$$\left.\begin{array}{l} n-m > k+1, \\ n-m = k+1, \end{array}\right\}, k = 2, 1, 0,$$

similar results can be obtained in a similar manner by using the results ([4, ineq. (1.5)], [3, inequalities (2.3), (2.4) and (2.5)]), Lemma 2.4, the result [2, Theorem 3], the result

$$\begin{aligned} M(p,R) &\leq \frac{R^n+1}{2}M(p,1) - \frac{R^n-1}{2}m(p,1) - |a_1|\left\{\frac{R^n-1}{n} - \frac{R^{n-2}-1}{n-2}\right\}, n > 2, \\ M(p,R) &\leq \frac{R^2+1}{2}M(p,1) - \frac{R^2-1}{2}m(p,1) - |a_1|\frac{(R-1)^2}{2}, n = 2 \end{aligned}$$

(obtained similar to the proof of Lemma 2.8 (with $K_0 = 1$), by using the results ([4, ineq. (1.5)], [3, ineq. (2.3)] and [2, Theorem 2]) and the result [3, Lemma 6]).

4. Importance of our results

Theorem 1.3 follows trivially from Corollary 1.8 by taking only first term on right hand side of inequality sign in main inequalities, (as the remaining part on right hand side of inequality sign is non-negative) and Theorem 1.5 follows trivially from Theorem 1.7 (with $K_t = |z_t|, 1 \le t \le n$) by taking only first term on right hand side of inequality sign in (1.4) and (1.5), (as the remaining part on right hand side of inequality sign is non-negative), $n \ge k+m+1$, (integer $k \ge 3$ and m (= order of possible zero of $p(z)$ at $z = 0$) $\le n-4$).

By taking first two terms on right hand side of inequality sign in (1.4) and (1.5), one obtains the following new result (as the remaining part on right hand side of inequality sign is non-negative).

Under the hypotheses of Theorem 1.7

$$M(p',1) \ge \frac{2}{1+K^{n-m}}\Big(\sum_{t=1}^{n}\frac{K}{K+K_t}\Big)M(p,1)+\frac{1}{K^n}\Big(\sum_{t=1}^{n}\frac{K}{K+K_t}\Big)\frac{K^{n-m}-1}{K^{n-m}+1}m(p,K).$$

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