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Certain classes of multivalent functions with negative coefficients defined by using a differential operator

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Submitted by: *Jan Stankiewicz*

ABSTRACT: In this paper, we investigate the various important properties and characteristics of the subclasses $S_n(p, q, \alpha, \beta)$ and $C_n(p, q, \alpha, \beta)$ of multivalent functions with negative coefficients defined by using a differential operator. We also derive many results for the modified Hadamard products of functions belonging to the classes $S_n(p, q, \alpha, \beta)$ and $C_n(p, q, \alpha, \beta)$. Finally several applications involving an integral operator and certain fractional calculus operators are also considered

AMS Subject Classification: *30C45*
Key Words and Phrases: *Multivalent functions, differential operator, modified-Hadamard product, fractional calculus*

1. Introduction

Let $T(n, p)$ denote the class of functions of the form :

$$f(z) = z^p - \sum_{k=n+p}^{\infty} a_k z^k \quad (a_k \geq 0; p, n \in N = \{1, 2, ...\}), \tag{1.1}$$

which are analytic and p-valent in the open unit disc $U = \{z : |z| < 1\}$. A function $f(z) \in T(n, p)$ is said to be p-valently starlike of order α if it satisfies the inequality:

$$\text{Re}\left\{\frac{zf'(z)}{f(z)}\right\} > \alpha \quad (z \in U;\ 0 \leq \alpha < p; p \in N). \tag{1.2}$$

We denote by $T_n^*(p, \alpha)$ the class of all p-valently starlike functions of order α. Also a function $f(z) \in T(n, p)$ is said to be p-valently convex of order α if it satisfies the inequality:

$$\text{Re}\left\{1 + \frac{zf''(z)}{f'(z)}\right\} > \alpha \quad (z \in U;\ 0 \leq \alpha < p; p \in N). \tag{1.3}$$

We denote by $C_n(p,\alpha)$ the class of all p-valently convex functions of order α. We note that (see for example Duren [4] and Goodman [5])

$$f(z)\in C_n(p,\alpha)\Longleftrightarrow \frac{zf^{'}(z)}{p}\in T_n^*(p,\alpha)\quad (\,0\le\alpha<p;p\in N). \tag{1.4}$$

The classes $T_n^*(p,\alpha)$ and $C_n(p,\alpha)$ are studied by Owa [12].

For each $f(z)\in T(n,p)$, we have (see [3])

$$f^{(q)}(z)=\frac{p!}{(p-q)!}z^{p-q}-\sum_{k=n+p}^{\infty}\frac{k!}{(k-q)!}a_kz^{k-q}\quad (q\in N_0=N\cup\{0\};p>q). \tag{1.5}$$

The main purpose of the present paper is to investigate various intersting properties and characteristics of functions belonging to two subclasses $S_n(p,q,\alpha,\beta)$ and $C_n(p,q,\alpha,\beta)$ of the class $T(n,p)$, which consist (respectivaly) of p-valently starlike functions of order α and type β and p-valently convex functions of order α and type $\beta\,(0\le\alpha<p-q;p\in N;q\in N_0;p>q;0<\beta\le1)$. Indeed we have

$$S_n(p,q,\alpha,\beta)=\left\{\ f(z)\in T(n,p):\ \left|\frac{\frac{zf^{(1+q)}(z)}{f^{(q)}(z)}-(p-q)}{\frac{zf^{(1+q)}(z)}{f^{(q)}(z)}+(p-q-2\alpha)}\right|<\beta\,,\ \ z\in U\right\} \tag{1.6}$$

and

$$C_n(p,q,\alpha,\beta)=\left\{\ f(z)\in T(n,p)\ :\ \left|\frac{(1+\frac{zf^{(2+q)}(z)}{f^{(1+q)}(z)})-(p-q)}{(1+\frac{zf^{(2+q)}(z)}{f^{(1+q)}(z)})+(p-q-2\alpha)}\right|<\beta\,,\quad z\in U\right\}. \tag{1.7}$$

It follows from (1.6) and (1.7) that

$$f^{(q)}(z)\in C_n(p,q,\alpha,\beta)\Leftrightarrow\frac{zf^{(1+q)}(z)}{(p-q)}\in S_n(p,q,\alpha,\beta). \tag{1.8}$$

We note that, by specializing the parameters n,p,q,α and β, we obtain the following subclasses studied by various authors:

(i) $S_n(p,q,\alpha,1)=S_n(p,q,\alpha)$ and $C_n(p,q,\alpha,1)=C_n(p,q,\alpha)$ (Chen et al. [2]);

(ii) $S_n(p,0,\alpha,1)=\begin{cases} T_n^*(p,\alpha) & \text{(Owa [12])}\\ T_\alpha(p,n) & \text{(Yamakawa [19])}\end{cases}\qquad(0\le\alpha<p;p,n\in N)$

(iii) $C_n(p,0,\alpha,1)=\begin{cases} C_n(p,\alpha) & \text{(Owa [12])}\\ CT_\alpha(p,n) & \text{(Yamakawa [19])}\end{cases}\qquad(0\le\alpha<p;p,n\in N)$

(iv) $S_1(p,0,\alpha,1) = T^*(p,\alpha)$ and $C_1(p,0,\alpha,1) = C(p,\alpha)$
$(0 \leq \alpha < p; p \in N)$ (Owa [11]) and Salagean et al. [13]);

(v) $S_1(p,0,\alpha,\beta) = S^*(p,\alpha,\beta)$ and $C_1(p,0,\alpha,\beta) = C^*(p,\alpha,\beta)$
$(0 \leq \alpha < p; p \in N; 0 \leq \beta < 1)$ (Hossen [7]);

(vi) $S_1(1,0,\alpha,\beta) = T^*(\alpha,\beta)$ and $C_1(1,0,\alpha,\beta) = C(\alpha,\beta)$
$(0 \leq \alpha < 1; 0 < \beta \leq 1)$ (Gupta and Jain [6]);

(vii) $S_n(1,0,\alpha,1) = T_\alpha(n)$ and $C_n(1,0,\alpha,1) = C_\alpha(n)$
$(0 \leq \alpha < 1; n \in N)$ (Srivastava et al. [18]).

In our present paper, we shall make use of the familiar integral operator $J_{c,p}$ defined by (cf. [1], [8] and [9] ; see also [17])

$$(J_{c,p}f)(z) = \frac{c+p}{z^c} \int_0^z t^{c-1} f(t)dt \tag{1.9}$$

$$(f(z) \in T(n,p);\ c > -p; p \in N)$$

as well as the fractional calculus operator D_z^μ for which it is well known that (see, for details, [10] and [15] ; see also Section 5 below)

$$D_z^\mu\{z^\rho\} = \frac{\Gamma(\rho+1)}{\Gamma(\rho+1-\mu)} z^{\rho-\mu} \quad (\rho > -1; \mu \in R) \tag{1.10}$$

in terms of Gamma functions.

2. Coefficient estimates

Theorem 1. *Let the function $f(z) \in T(n,p)$ be given by (1.1). Then $f(z) \in S_n(p,q,\alpha,\beta)$ if and only if*

$$\sum_{k=n+p}^{\infty} \{(k-p) + \beta[(k-p) + 2(p-q-\alpha)]\}\, \delta(k,q) a_k \leq 2\beta(p-q-\alpha)\delta(p,q) \tag{2.1}$$

$(0 \leq \alpha < p-q; p,n \in N; q \in N_0; p > q)$, *where*

$$\delta(p,q) = \frac{p!}{(p-q)!} = \begin{cases} p(p-1)...(p-q+1) & (q \neq 0) \\ 1 & (q = 0)\,. \end{cases} \tag{2.2}$$

Proof. Assume that the inequality (2.1) holds true, we find from (1.1) and (2.1) that

$$\left|zf^{(1+q)}(z)-(p-q)f^{(q)}(z)\right|-\beta\ \left|zf^{(1+q)}(z)+(p-q-2\alpha)f^{(q)}(z)\right|$$

$$=\left|-\sum_{k=n+p}^{\infty}(k-p)\delta(k,q)a_kz^{k-q}\right|$$

$$-\beta\left|2(p-q-\alpha)\delta(p,q)z^{p-q}-\sum_{k=n+p}^{\infty}\left[(k-p)+2(p-q-\alpha)\right]\delta(k,q)a_kz^{k-q}\right|$$

$$\leq\sum_{k=n+p}^{\infty}\left\{(k-p)+\beta\left[(k-p)+2(p-q-\alpha)\right]\right\}\delta(k,q)a_k-2\beta(p-q-\alpha)\delta(p,q)\leq 0$$

$(z\in U)$. Hence, by the maximum modulus theorem, we have $f(z)\in S_n(p,q,\alpha,\beta)$.

Conversely, let $f(z)\in S_n(p,q,\alpha,\beta)$ be given by (1.1). Then from (1.1) and (1.6), we find that

$$\left|\frac{\dfrac{zf^{(1+q)}(z)}{f^{(q)}(z)}-(p-q)}{\dfrac{zf^{(1+q)}(z)}{f^{(q)}(z)}+(p-q-2\alpha)}\right| \tag{2.3}$$

$$=\left\{\frac{\sum\limits_{k=n+p}^{\infty}(k-p)\delta(k,q)a_kz^{k-q}}{2(p-q-\alpha)\delta(p,q)z^{p-q}-\sum\limits_{k=n+p}^{\infty}\left[(k-p)+2(p-q-\alpha)\right]\delta(k,q)a_kz^{k-q}}\right\}<\beta$$

$(z\in U)$. Now, since $|\mathrm{Re}(z)|\leq|z|$ for all z, we have

$$\mathrm{Re}\left\{\frac{\sum\limits_{k=n+p}^{\infty}(k-p)\delta(k,q)z^{k-q}}{2(p-q-\alpha)\delta(p,q)z^{p-q}-\sum\limits_{k=n+p}^{\infty}\left[(k-p)+2(p-q-\alpha)\right]\delta(k,q)a_kz^{k-q}}\right\}<\beta. \tag{2.4}$$

Now choose values of z on the real axis so that $\dfrac{zf^{(1+q)}(z)}{f^{(q)}(z)}$ is real. Then, upon clearing the denominator in (2.4) and letting $z\to 1^-$ through real values, we get

$$\sum_{k=n+p}^{\infty}(k-p)\delta(k,q)a_k\leq\beta\left\{2(p-q-\alpha)\delta(p,q)-\sum_{k=n+p}^{\infty}\left[(k-p)+2(p-q-\alpha)\right]\delta(k,q)a_k\right\}.$$

This gives the required condition.

Corollary 1. *Let the function $f(z)$ defined by (1.1) be in the class $S_n(p,q,\alpha,\beta)$. Then*

$$a_k \le \frac{2\beta(p-q-\alpha)\delta(p,q)}{\{(k-p)+\beta\left[(k-p)+2(p-q-\alpha)\right]\}\,\delta(k,q)} \tag{2.5}$$

$(k \ge n+p; p,n \in N; q \in N_0; p > q)$.

The result is sharp for the function $f(z)$ given by

$$f(z) = z^p - \frac{2\beta(p-q-\alpha)\delta(p,q)}{\{(k-p)+\beta\left[(k-p)+2(p-q-\alpha)\right]\}\,\delta(k,q)} z^k \tag{2.6}$$

$(k \ge n+p; p,n \in N; q \in N_0; p > q)$.

From Theorem 1 and using (1.8), we can prove the following theorem.

Theorem 2. *Let the function $f(z) \in T(n,p)$ be given by (1.1). Then $f(z) \in C_n(p,q,\alpha,\beta)$ if and only if*

$$\sum_{k=n+p}^{\infty} \left(\frac{k-q}{p-q}\right)\{(k-p)+\beta\left[(k-p)+2(p-q-\alpha)\right]\}\,\delta(k,q)a_k \le 2\beta(p-q-\alpha)\delta(p,q). \tag{2.7}$$

Corollary 2. *Let the function $f(z)$ defined by (1.1) be in the class $C_n(p,q,\alpha,\beta)$. Then*

$$a_k \le \frac{2\beta(p-q-\alpha)\delta(p,q)}{\left(\frac{k-q}{p-q}\right)\{(k-p)+\beta\left[(k-p)+2(p-q-\alpha)\right]\}\,\delta(k,q)} \tag{2.8}$$

$(k \ge n+p; p,n \in N; q \in N_0; p > q)$.

The result is sharp for the function $f(z)$ given by

$$f(z) = z^p - \frac{2\beta(p-q-\alpha)\delta(p,q)}{\left(\frac{k-q}{p-q}\right)\{(k-p)+\beta\left[(k-p)+2(p-q-\alpha)\right]\}\,\delta(k,q)} z^k \tag{2.9}$$

$(k \ge n+p; p,n \in N; q \in N_0; p > q)$.

3. Distortion theorems

Theorem 3. *If a function $f(z)$ defined by (1.1) is in the class $S_n(p,q,\alpha,\beta)$, then*

$$\begin{aligned} &\left\{\frac{p!}{(p-j)!} - \frac{2\beta(p-q-\alpha)\delta(p,q)(n+p-q)!}{\{n+\beta[n+2(p-q-\alpha)]\}\,(n+p-j)!}\,|z|^n\right\}|z|^{p-j} \\ \le\ & \left|f^{(j)}(z)\right| \\ \le\ & \left\{\frac{p!}{(p-j)!} + \frac{2\beta(p-q-\alpha)\delta(p,q)(n+p-q)!}{\{n+\beta[n+2(p-q-\alpha)]\}\,(n+p-j)!}\,|z|^n\right\}|z|^{p-j} \end{aligned} \tag{3.1}$$

$(z \in U; 0 \leq \alpha < p-q; p,n \in N; q,j \in N_0; p > \max\{q,j\})$.

The result is sharp for the function $f(z)$ given by

$$f(z) = z^p - \frac{2\beta(p-q-\alpha)\delta(p,q)}{\{n+\beta[n+2(p-q-\alpha)]\}\,\delta(n+p,q)} z^{n+p} \tag{3.2}$$

$(p,n \in N; q \in N_0; p > q)$.

Proof. *Since the sequence $\{\delta(k,q)\}(k \geq n+p)$ is nondecreasing, where $\delta(k,q)$ is defined by (2.2), in view of Theorem 1, we have*

$$\frac{\{n+\beta[n+2(p-q-\alpha)]\}\,\delta(n+p,q)}{2\beta(p-q-\alpha)\delta(p,q)(n+p)!} \sum_{k=n+p}^{\infty} k!a_k$$

$$\leq \sum_{k=n+p}^{\infty} \frac{\{(k-p)+\beta\,[(k-p)+2(p-q-\alpha)]\}\,\delta(k,q)}{2\beta(p-q-\alpha)\delta(p,q)} a_k \leq 1$$

which readily yields

$$\sum_{k=j+p}^{\infty} k!a_k \leq \frac{2\beta(p-q-\alpha)\delta(p,q)(n+p-q)!}{\{n+\beta[n+2(p-q-\alpha)]\}}. \tag{3.3}$$

Now, by differentiating both of (1.1) j *times, we obtain*

$$f^{(j)}(z) = \frac{p!}{(p-j)!} z^{p-j} - \sum_{k=n+p}^{\infty} \frac{k!}{(k-j)!} a_k z^{k-j} \tag{3.4}$$

$(k \geq n+p; p,n \in N; q,j \in N_0; p > \max\{q,j\})$.

Theorem 2 follows readily from (3.3) *and* (3.4).

Finally, it is easy to see that the bounds in (3.1) *are attained for the function $f(z)$ given by* (3.2).

Theorem 4. *If a function $f(z)$ defined by* (1.1) *is in the class $C_n(p,q,\alpha,\beta)$, then*

$$\left\{\frac{1}{(p-j)!} - \frac{2\beta(p-q-\alpha)(n+p-q-1)!}{(p-q-1)!\,\{n+\beta[n+2(p-q-\alpha)]\}\,(n+p-j)!}\,|z|^n\right\} p!\,|z|^{p-j} \tag{3.5}$$

$$\leq \left|f^{(j)}(z)\right|$$

$$\leq \left\{\frac{1}{(p-j)!} + \frac{2\beta(p-q-\alpha)(n+p-q-1)!}{(p-q-1)!\,\{n+\beta[n+2(p-q-\alpha)]\}\,(n+p-j)!}\,|z|^n\right\} p!\,|z|^{p-j}$$

$(z \in U; 0 \leq \alpha < p-q; p,n \in N; q,j \in N_0; p > \max\{q,j\})$.

The result is sharp for the function $f(z)$ given by

$$f(z) = z^p - \frac{2\beta(p-q-\alpha)\delta(p,q)}{(\frac{n+p-q}{p-q})\,\{n+\beta[n+2(p-q-\alpha)]\}\,\delta(n+p,q)} z^{n+p} \tag{3.6}$$

$(p,n \in N; q \in N_0; p > q)$.

4. Modified Hadamard products

For the functions $f_\nu(z)(\nu = 1, 2)$ *given by*

$$f_\nu(z) = z^p - \sum_{k=n+p}^{\infty} a_{k,\nu} z^k \quad (a_{k,\nu} \geq 0; \nu = 1, 2) \tag{4.1}$$

we denote by $(f_1 \circledast f_2)(z)$ *the modified Hadamard product (or convolution) of the functions* $f_1(z)$ *and* $f_2(z)$ *defined by*

$$(f_1 \circledast f_2)(z) = z^p - \sum_{k=n+p}^{\infty} a_{k,1}.a_{k,2} z^k. \tag{4.2}$$

Theorem 5. *Let the functions* $f_\nu(z)(\nu = 1, 2)$ *defined by (4.1) be in the class* $S_n(p, q, \alpha, \beta)$*. Then* $(f_1 \circledast f_2)(z) \in S_n(p, q, \gamma, \beta)$*, where*

$$\gamma = (p - q) - \frac{2\beta(1+\beta)n(p-q-\alpha)^2\delta(p, q)}{\{n + \beta[n + 2(p-q-\alpha)]\}^2 \delta(n+p, q) - 4\beta^2(p-q-\alpha)^2\delta(p, q)}. \tag{4.3}$$

The result is sharp for the functions $f_\nu(z)(\nu = 1, 2)$ *given by*

$$f_\nu(z) = z^p - \frac{2\beta(p-q-\alpha)\delta(p, q)}{\{n + \beta[n + 2(p-q-\alpha)]\}\delta(n+p, q)} z^{n+p} \quad (\nu = 1, 2). \tag{4.4}$$

Proof. *Emloying the technique used earlier by Schild and Silverman [14], we need to find the largest* γ *such that*

$$\sum_{k=n+p}^{\infty} \frac{\{(k-p) + \beta[(k-p) + 2(p-q-\gamma)]\}\delta(k, q)}{2\beta(p-q-\gamma)\delta(p, q)} a_{k,1}.a_{k,2} \leq 1 \tag{4.5}$$

$(f_\nu(z) \in S_n(p, q, \alpha, \beta) \ \ (\nu = 1, 2))$.

Since $f_\nu(z) \in S_n(p, q, \alpha, \beta)(\nu = 1, 2)$*, we readily see that*

$$\sum_{k=n+p}^{\infty} \frac{\{(k-p) + \beta[(k-p) + 2(p-q-\alpha)]\}\delta(k, q)}{2\beta(p-q-\alpha)\delta(p, q)} a_{k,\nu} \leq 1 \ \ (\nu = 1, 2). \tag{4.6}$$

Therefore, by the Cauchy - Schwarz inequality, we obtain

$$\sum_{k=n+p}^{\infty} \frac{\{(k-p) + \beta[(k-p) + 2(p-q-\alpha)]\}\delta(k, q)}{2\beta(p-q-\alpha)\delta(p, q)} \sqrt{a_{k,1}.a_{k,2}} \leq 1. \tag{4.7}$$

Thus we only need to show that

$$\frac{\{(k-p) + \beta[(k-p) + 2(p-q-\gamma)]\}}{(p-q-\gamma)} a_{k,1}.a_{k,2} \tag{4.8}$$

$$\leq \frac{\{(k-p)+\beta[(k-p)+2(p-q-\alpha)]\}}{(p-q-\alpha)}\sqrt{a_{k,1}.a_{k,2}}$$

$(k \geq n+p; p, n \in N)$, or, equivalently, that

$$\sqrt{a_{k,1}.a_{k,2}} \leq \frac{(p-q-\gamma)\{(k-p)+\beta[(k-p)+2(p-q-\alpha)]\}}{(p-q-\alpha)\{(k-p)+\beta[(k-p)+2(p-q-\gamma)]\}} \tag{4.9}$$

$(k \geq n+p; p, n \in N)$. Hence, in light of the inequality (4.7), it is sufficient to prove that

$$\frac{2\beta(p-q-\alpha)\delta(p,q)}{\{(k-p)+\beta[(k-p)+2(p-q-\alpha)]\}\,\delta(k,q)} \leq \frac{(p-q-\gamma)\{(k-p)+\beta[(k-p)+2(p-q-\alpha)]\}}{(p-q-\alpha)\{(k-p)+\beta[(k-p)+2(p-q-\gamma)]\}} \tag{4.10}$$

$(k \geq n+p; p, n \in N)$. It follows from (4.10) that

$$\gamma \leq (p-q) - \frac{2\beta(1+\beta)(k-p)(p-q-\alpha)^2\delta(p,q)}{\{(k-p)+\beta[(k-p)+2(p-q-\alpha)]\}^2\,\delta(k,q)-4\beta^2(p-q-\alpha)^2\delta(p,q)} \tag{4.11}$$

$(k \geq n+p; p, n \in N)$. Now, defining the function $G(k)$ by

$$G(k) = (p-q) - \frac{2\beta(1+\beta)(k-p)(p-q-\alpha)^2\delta(p,q)}{\{(k-p)+\beta[(k-p)+2(p-q-\alpha)]\}^2\,\delta(k,q)-4\beta^2(p-q-\alpha)^2\delta(p,q)} \tag{4.12}$$

$k \geq n+p; p, n \in N)$,we see that $G(k)$ is an increasing function of k. Therefore, we conclude that

$$\gamma \leq G(n+p) = (p-q) - \frac{2\beta(1+\beta)n(p-q-\alpha)^2\delta(p,q)}{\{n+\beta[n+2(p-q-\alpha)]\}^2\,\delta(n+p,q)-4\beta^2(p-q-\alpha)^2\delta(p,q)} \tag{4.13}$$

which evidently completes the proof of Theorem 5.

Putting $\beta = 1$ Theorem 5, we obtain

Corollary 3. *Let the functions $f_\nu(z)(\nu = 1,2)$ defined by (4.1) be in the class $S_n(p,q,\alpha)$. Then $(f_1 \circledast f_2)(z) \in S_n(p,q,\gamma)$, where*

$$\gamma = (p-q) - \frac{n(p-q-\alpha)^2\delta(p,q)}{(n+p-q-\alpha)^2\delta(n+p,q)-(p-q-\alpha)^2\delta(p,q)}\,. \tag{4.14}$$

The result is sharp.

Remark 1. *We note that the result obtained by Chen et al. [2, Theorem 5] is not correct. The correct result is given by (4.14).*

Using arguments similar to those in the proof of Theorem 5, we obtain the following results.

Theorem 6. *Let the function $f_1(z)$ defined by (4.1) be in the class $S_n(p,q,\alpha,\beta)$. Suppose also that the function $f_2(z)$ defined by (4.1) be in the class $S_n(p,q,\gamma,\beta)$. Then $(f_1 \circledast f_2)(z) \in S_n(p,q,\zeta,\beta)$, where*

$$\zeta = (p-q) - \frac{2\beta(1+\beta)n(p-q-\alpha)(p-q-\gamma)\delta(p,q)}{\{n+\beta[n+2(p-q-\alpha)]\}\{n+\beta[n+2(p-q-\gamma)]\}\delta(n+p,q)-\Omega} \tag{4.15}$$

$(\Omega = 4\beta^2(p-q-\alpha)(p-q-\gamma)\delta(p,q))$.

This result is sharp for the functions $f_\nu(z)(\nu=1,2)$ given by

$$f_1(z) = z^p - \frac{2\beta(p-q-\alpha)\delta(p,q)}{\{n+\beta[n+2(p-q-\alpha)]\}\delta(n+p,q)} z^{n+p} \quad (p,n \in N) \tag{4.16}$$

and

$$f_2(z) = z^p - \frac{2\beta(p-q-\gamma)\delta(p,q)}{\{n+\beta[n+2(p-q-\gamma)]\}\delta(n+p,q)} z^{n+p} \quad (p,n \in N). \tag{4.17}$$

Theorem 7. *Let the functions $f_\nu(z)(\nu = 1,2)$ defined by (4.1) be in the class $C_n(p,q,\alpha,\beta)$. Then $(f_1 \circledast f_2)(z) \in C_n(p,q,\gamma,\beta)$, where*

$$\gamma = (p-q) - \frac{2\beta(1+\beta)n(p-q-\alpha)^2\delta(p,q)}{(\frac{n+p-q}{p-q})\{n+\beta[n+2(p-q-\alpha)]\}^2\delta(n+p,q)-4\beta^2(p-q-\alpha)^2\delta(p,q)}. \tag{4.18}$$

The result is sharp for the functions $f_\nu(z)(\nu=1,2)$ given by

$$f_\nu(z) = z^p - \frac{2\beta(p-q-\alpha)\delta(p,q)}{(\frac{n+p-q}{p-q})\{n+\beta[n+2(p-q-\alpha)]\}\delta(n+p,q)} z^{n+p} \tag{4.19}$$

$(\nu = 1,2)$.

Remark 2. *Putting $\beta = 1$ in Theorem 7, we obtain*

Corollary 4. *Let the functions $f_\nu(z)(\nu = 1,2)$ defined by (4.1) be in the class $C_n(p,q,\alpha)$. Then $(f_1 \circledast f_2)(z) \in C_n(p,q,\gamma)$, where*

$$\gamma = (p-q) - \frac{n(p-q-\alpha)^2\delta(p,q+1)}{(n+p-q-\alpha)^2\delta(n+p,q+1)-(p-q-\alpha)^2\delta(p,q+1)}. \tag{4.20}$$

The result is sharp.

Remark 3. *We note that the result obtained by Chen et al. [2, Theorem 6] is not correct. The correct result is given by* (4.20).

Theorem 8. *Let the functions* $f_\nu(z)(\nu = 1,2)$ *defined by* (4.1) *be in the class* $S_n(p,q,\alpha,\beta)$. *Then the function*

$$h(z) = z^p - \sum_{k=n+p}^{\infty} (a_{k,1}^2 + a_{k,2}^2) z^k \tag{4.21}$$

belongs to the class $S_n(p,q,\xi,\beta)$, *where*

$$\xi = (p-q) - \frac{4\beta(1+\beta)n(p-q-\alpha)^2\delta(p,q)}{\{n+\beta[n+2(p-q-\alpha)]\}^2\,\delta(n+p,q) - 8\beta^2(p-q-\alpha)^2\delta(p,q)}. \tag{4.22}$$

The result is sharp for the functions $f_\nu(z)(\nu = 1,2)$ *defined by* (4.4).

Theorem 9. *Let the functions* $f_\nu(z)(\nu = 1,2)$ *defined by* (4.1) *be in the class* $C_n(p,q,\alpha,\beta)$. *Then the function h(z) defined by* (4.21) *belongs to the class* $C_n(p,q,\alpha,\xi)$, *where*

$$\xi = (p-q) - \frac{4\beta(1+\beta)n(p-q-\alpha)^2\delta(p,q)}{(\frac{n+p-q}{p-q})\,\{n+\beta[n+2(p-q-\alpha)]\}^2\,\delta(n+p,q) - 8\beta^2(p-q-\alpha)^2\delta(p,q)}. \tag{4.23}$$

The result is sharp for the functions $f_\nu(z)(\nu = 1,2)$ *defined by* (4.19).

5. Applications of fractional calculus

Various operators of fractional calculus (that is, fractional integral and fractional derivatives) have been studied in the literature rather extensively (cf., e.g., [3] , [10], [16] and [17]; see also the various references cited therein). For our present investigation, we recall the following definitions.

Definition 1. *The fractional integral of order* μ *is defined, for a function* $f(z)$, *by*

$$D_z^{-\mu} f(z) = \frac{1}{\Gamma(\mu)} \int_0^z \frac{f(\zeta)}{(z-\zeta)^{1-\mu}} d\zeta \quad (\mu > 0), \tag{5.1}$$

where the function $f(z)$ *is analytic in a simply- connected domain of the complex z - plane containing the origin and the multiplicity of* $(z-\zeta)^{\mu-1}$ *is removed by requiring* $log(z-\zeta)$ *to be real when* $z-\zeta > 0$.

Definition 2. *The fractional derivative of order* μ *is defined, for a function* $f(z)$, *by*

$$D_z^{\mu} f(z) = \frac{1}{\Gamma(1-\mu)} \int_0^z \frac{f(\zeta)}{(z-\zeta)^{\mu}} d\zeta \quad (0 \le \mu < 1), \tag{5.2}$$

where the function $f(z)$ is constrained, and the multiplicity of $(z-\zeta)^{-\mu}$ is removed, as in Definition 1.

Definition 3. *Under the hypotheses of Definition 2, the fractional derivative of order $n+\mu$ is defined, for a function $f(z)$, by*

$$D_z^{n+\mu} f(z) = \frac{d^n}{dz^n}\left\{D_z^{\mu} f(z)\right\} \quad (0 \leq \mu < 1; n \in N_0). \tag{5.3}$$

In this section, we shall investigate the growth and distortion properties of functions in the classes $S_n(p,q,\alpha,\beta)$ and $C_n(p,q,\alpha,\beta)$, involving the operators $J_{c,p}$ and D_z^{μ}. In order to derive our results, we need the following lemma given by Chen et al. [3].

Lemma 1. *(see Chen et al. [3]). Let the function $f(z)$ defined by (1.1). Then*

$$D_z^{\mu}\left\{(J_{c,p}f)(z)\right\} = \frac{\Gamma(p+1)}{\Gamma(p+1-\mu)} z^{p-\mu} - \sum_{k=n+p}^{\infty} \frac{(c+p)\Gamma(k+1)}{(c+k)\Gamma(k+1-\mu)} a_k z^{k-\mu} \tag{5.4}$$

$(\mu \in R; c > -p; p, n \in N)$ and

$$J_{c,p}(D_z^{\mu}\left\{f(z)\right\}) = \frac{(c+p)\Gamma(p+1)}{(c+p-\mu)\Gamma(p+1-\mu)} z^{p-\mu} \tag{5.5}$$

$$- \sum_{k=n+p}^{\infty} \frac{(c+p)\Gamma(k+1)}{(c+k-\mu)\Gamma(k+1-\mu)} a_k z^{k-\mu}$$

$(\mu \in R; c > -p; p, n \in N)$, provided that no zeros appear in the denominators in (5.4) and (5.5).

Theorem 8. *Let the function $f(z)$ defined by (1.1) be in the class $S_n(p,q,\alpha,\beta)$. Then*

$$\left|D_z^{-\mu}\left\{(J_{c,p}f)(z)\right\}\right| \geq \left\{\frac{\Gamma(p+1)}{\Gamma(p+1+\mu)}\right. \tag{5.6}$$

$$\left. - \frac{(c+p)\Gamma(n+p+1)2\beta(p-q-\alpha)\delta(p,q)}{(c+n+p)\Gamma(n+p+1+\mu)\left\{n+\beta[n+2(p-q-\alpha)\right\}\delta(n+p,q)} |z|^n\right\} |z|^{p+\mu}$$

$(z \in U; 0 \leq \alpha < p-q; \mu > 0; c > -p; p, n \in N, q \in N_0; p > q)$ and

$$\left|D_z^{-\mu}\left\{(J_{c,p}f)(z)\right\}\right| \leq \left\{\frac{\Gamma(p+1)}{\Gamma(p+1+\mu)}\right. \tag{5.7}$$

$$\left. + \frac{(c+p)\Gamma(n+p+1)2\beta(p-q-\alpha)\delta(p,q)}{(c+n+p)\Gamma(n+p+1+\mu)\left\{n+\beta[n+2(p-q-\alpha)\right\}\delta(n+p,q)} |z|^n\right\} |z|^{p+\mu}$$

$(z \in U; 0 \leq \alpha < p-q; \mu > 0; c > -p; p, n \in N, q \in N_0; p > q)$.

Each of the assertions (5.6) and (5.7) is sharp.

Proof. *In view of Theorem 1, we have*

$$\frac{\{n+\beta[n+2(p-q-\alpha)\}\,\delta(n+p,q)}{2\beta(p-q-\alpha)\delta(p,q)}\sum_{k=n+p}^{\infty} a_k \leq \tag{5.8}$$

$$\sum_{k=n+p}^{\infty}\frac{\{(k-p)+\beta[(k-p)2(p-q-\alpha)\}\,\delta(k,q)}{2\beta(p-q-\alpha)\delta(p,q)}a_k \leq 1,$$

which readily yields

$$\sum_{k=n+p}^{\infty} a_k \leq \frac{2\beta(p-q-\alpha)\delta(p,q)}{\{n+\beta[n+2(p-q-\alpha)\}\,\delta(n+p,q)}. \tag{5.9}$$

Consider the function $F(z)$ *defined in* U *by*

$$\begin{aligned} F(z) &= \frac{\Gamma(p+1+\mu)}{\Gamma(p+1)} z^{-\mu} D_z^{-\mu}\left\{(J_{c,p}f)(z)\right\} \\ &= z^p - \sum_{k=n+p}^{\infty}\frac{(c+p)\Gamma(k+1)\Gamma(p+1+\mu)}{(c+k)\Gamma(k+1+\mu)\Gamma(p+1)}a_k z^k \\ &= z^p - \sum_{k=n+p}^{\infty}\Phi(k)a_k z^k \quad (z\in U) \end{aligned}$$

where

$$\Phi(k) = \frac{(c+p)\Gamma(k+1)\Gamma(p+1+\mu)}{(c+k)\Gamma(k+1+\mu)\Gamma(p+1)} \quad (k\geq n+p; p,n\in N; \mu>0)\ . \tag{5.10}$$

Since $\Phi(k)$ *is a decreasing function of* k *when* $\mu>0$, *we get*

$$0<\Phi(k)\leq\Phi(n+p)=\frac{(c+p)\Gamma(n+p+1)\Gamma(p+1+\mu)}{(c+n+p)\Gamma(n+p+1+\mu)\Gamma(p+1)} \tag{5.11}$$

$(c>-p; p,n\in N;\mu>0)$. *Thus, by using (5.9) and (5.11), we deduce that*

$$|F(z)| \geq |z|^p - \Phi(n+p)\,|z|^{n+p}\sum_{k=n+p}^{\infty}a_k$$

$$\geq |z|^p - \frac{(c+p)\Gamma(n+p+1)\Gamma(p+1+\mu)2\beta(p-q-\alpha)\delta(p,q)}{(c+n+p)\Gamma(n+p+1+\mu)\Gamma(p+1)\left\{n+\beta[n+2(p-q-\alpha)]\right\}\delta(n+p,q)}\,|z|^{n+p}$$

$(z\in U)$ *and*

$$|F(z)| \leq |z|^p + \Phi(n+p)\,|z|^{n+p}\sum_{k=n+p}^{\infty}a_k$$

$$\leq |z|^p + \frac{(c+p)\Gamma(n+p+1)\Gamma(p+1+\mu)2\beta(p-q-\alpha)\delta(p,q)}{(c+n+p)\Gamma(n+p+1+\mu)\Gamma(p+1)\left\{n+\beta[n+2(p-q-\alpha)]\right\}\delta(n+p,q)}|z|^{n+p}$$

$(z \in U)$, which yield the inequalities (5.6) and (5.7) of Theorem 10. The equalities in (5.6) and (5.7) are attained for the function $f(z)$ given by

$$D_z^{-\mu}\left\{(J_{c,p}f)(z)\right\} = \left\{\frac{\Gamma(p+1)}{\Gamma(p+1+\mu)}\right.$$

$$\left. - \frac{(c+p)\Gamma(n+p+1)2\beta(p-q-\alpha)\delta(p,q)}{(c+n+p)\Gamma(n+p+1+\mu)\left\{n+\beta[n+2(p-q-\alpha)]\right\}\delta(n+p,q)} z^n\right\} z^{p+\mu} \quad (5.12)$$

or, equivalently, by

$$(J_{c,p}f)(z) = z^p - \frac{(c+p)2\beta(p-q-\alpha)\delta(p,q)}{(c+n+p)\left\{n+\beta[n+2(p-q-\alpha)]\right\}\delta(n+p,q)} z^{n+p}. \quad (5.13)$$

Thus we complete the proof of Theorem 10.

Theorem 10. *Let the function $f(z)$ defined by (1.1) be in the class $S_n(p,q,\alpha,\beta)$. Then*

$$|D_z^{\mu}\left\{(J_{c,p}f)(z)\right\}| \geq \left\{\frac{\Gamma(p+1)}{\Gamma(p+1-\mu)}\right. \quad (5.14)$$

$$\left. - \frac{(c+p)\Gamma(n+p+1)2\beta(p-q-\alpha)\delta(p,q)}{(c+n+p)\Gamma(n+p+1-\mu)\left\{n+\beta[n+2(p-q-\alpha)]\right\}\delta(n+p,q)}|z|^n\right\}|z|^{p-\mu}$$

$(z \in U; 0 \leq \alpha < p-q; 0 \leq \mu < 1; c > -p; p,n \in N; q \in N_0; p > q)$ and

$$|D_z^{\mu}\left\{(J_{c,p}f)(z)\right\}| \leq \left\{\frac{\Gamma(p+1)}{\Gamma(p+1-\mu)}\right. \quad (5.15)$$

$$\left. + \frac{(c+p)\Gamma(n+p+1)2\beta(p-q-\alpha)\delta(p,q)}{(c+n+p)\Gamma(n+p+1-\mu)\left\{n+\beta[n+2(p-q-\alpha)]\right\}\delta(n+p,q)}|z|^n\right\}|z|^{p-\mu}$$

$(z \in U; 0 \leq \alpha < p-q; 0 \leq \mu < 1; c > -p; p,n \in N; q \in N_0; p > q)$.

Each of the assertions (5.14) and (5.15) is sharp.

Proof. *It follows from Theorem 1, that*

$$\sum_{k=n+p}^{\infty} k a_k \leq \frac{(n+p)2\beta(p-q-\alpha)\delta(p,q)}{\left\{n+\beta[n+2(p-q-\alpha)]\right\}\delta(n+p,q)}. \quad (5.16)$$

We consider the function $H(z)$ defined in U by

$$\begin{aligned} H(z) &= \frac{\Gamma(p+1-\mu)}{\Gamma(p+1)} z^{\mu} D_z^{\mu}\left\{(J_{c,p}f)(z)\right\} \\ &= z^p - \sum_{k=n+p}^{\infty} \Psi(k) k a_k z^k \quad (z \in U), \end{aligned}$$

where, for convenience,

$$\Psi(k) = \frac{(c+p)\Gamma(k)(p+1-\mu)}{(c+k)\Gamma(k+1-\mu)\Gamma(p+1)} \quad (k \geq n+p; p,n \in N; 0 \leq \mu < 1).$$

Since $\Psi(k)$ *is a decreasing function of* k *when* $\mu < 1$, *we find that*

$$0 < \Psi(k) \leq \Psi(n+p) = \frac{(c+p)\Gamma(n+p)\Gamma(p+1-\mu)}{(c+n+p)\Gamma(n+p+1-\mu)\Gamma(p+1)} \tag{5.17}$$

$(c > -p; p,n \in N; 0 \leq \mu < 1)$.

Consequently, with the aid of (5.16) *and* (5.17), *we find that*

$$|H(z)| \geq |z|^p - \Psi(n+p)\,|z|^{n+p} \sum_{k=n+p}^{\infty} k a_k$$

$$\geq |z|^p - \frac{(c+p)\Gamma(n+p+1)\Gamma(p+1-\mu)2\beta(p-q-\alpha)\delta(p,q)}{(c+n+p)\Gamma(n+p+1-\mu)\Gamma(p+1)\left\{n+\beta[n+2(p-q-\alpha)]\right\}\delta(n+p,q)}\,|z|^{n+p}$$

$(z \in U)$, *and*

$$|H(z)| \leq |z|^p + \Psi(n+p)\,|z|^{n+p} \sum_{k=n+p}^{\infty} k a_k$$

$$\leq |z|^p + \frac{(c+p)\Gamma(n+p+1)\Gamma(p+1-\mu)2\beta(p-q-\alpha)\delta(p,q)}{(c+n+p)\Gamma(n+p+1-\mu)\Gamma(p+1)\left\{n+\beta[n+2(p-q-\alpha)]\right\}\delta(n+p,q)}\,|z|^{n+p}$$

$(z \in U)$ *which yield the inequalities* (5.14) *and* (5.15) *of Theorem 11. The equalities in* (5.14) *and* (5.15) *are attained for the function* $f(z)$ *given by*

$$D_z^{\mu}\left\{(J_{c,p}f)(z)\right\} = \left\{\frac{\Gamma(p+1)}{\Gamma(p+1-\mu)}\right.$$

$$\left. - \frac{(c+p)\Gamma(n+p+1)2\beta(p-q-\alpha)\delta(p,q)}{(c+n+p)\Gamma(n+p+1-\mu)\left\{n+\beta[n+2(p-q-\alpha)]\right\}\delta(n+p,q)} z^n \right\} z^{p+\mu} \tag{5.18}$$

or for the function $(J_{c,p}f)(z)$ *given by* (5.13). *The proof of Theorem 11 is thus completed.*

Theorem 11. *Let the function* $f(z)$ *defined by* (1.1) *be the class* $C_n(p,q,\alpha,\beta)$. *Then for* $z \in U; 0 \leq \alpha < p-q; \mu > 0; c > -p; p,n \in N; q \in N_0$ *and* $p > q$, *we have*

$$\left|D_z^{-\mu}\left\{(J_{c,p}f)(z)\right\}\right| \geq \left\{\frac{\Gamma(p+1)}{\Gamma(p+1+\mu)}\right. \tag{5.19}$$

$$\left. - \frac{(c+p)\Gamma(n+p+1)2\beta(p-q-\alpha)\delta(p,q+1)}{(c+n+p)\Gamma(n+p+1+\mu)\left\{n+\beta[n+2(p-q-\alpha)]\right\}\delta(n+p,q+1)}\,|z|^n \right\} |z|^{p+\mu}, \tag{1}$$

and

$$\left|D_z^{-\mu}\left\{(J_{c,p}f)(z)\right\}\right| \leq \left\{\frac{\Gamma(p+1)}{\Gamma(p+1+\mu)}\right. \tag{5.20}$$

$$\left.+\frac{(c+p)\Gamma(n+p+1)2\beta(p-q-\alpha)\delta(p,q+1)}{(c+n+p)\Gamma(n+p+1+\mu)\left\{n+\beta[n+2(p-q-\alpha)]\right\}\delta(n+p,q+1)}|z|^n\right\}|z|^{p+\mu}.$$

(2)

Also for $z \in U.; 0 \leq \alpha < p-q; 0 \leq \mu < 1; c > -p; p, n \in N; q \in N_0$ *and* $p > q$, *we have*

$$|D_z^{\mu}\left\{(J_{c,p}f)(z)\right\}| \geq \left\{\frac{\Gamma(p+1)}{\Gamma(p+1-\mu)}\right.$$

$$\left.-\frac{(c+p)\Gamma(n+p+1)2\beta(p-q-\alpha)\delta(p,q+1)}{(c+n+p)\Gamma(n+p+1-\mu)\left\{n+\beta[n+2(p-q-\alpha)]\right\}\delta(n+p,q+1)}|z|^n\right\}|z|^{p-\mu} \tag{5.21}$$

and

$$|D_z^{\mu}\left\{(J_{c,p}f)(z)\right\}| \leq \left\{\frac{\Gamma(p+1)}{\Gamma(p+1-\mu)}\right.$$

$$\left.+\frac{(c+p)\Gamma(n+p+1)2\beta(p-q-\alpha)\delta(p,q+1)}{(c+n+p)\Gamma(n+p+1-\mu)\left\{n+\beta[n+2(p-q-\alpha)]\right\}\delta(n+p,q+1)}|z|^n\right\}|z|^{p-\mu}. \tag{5.22}$$

The equalities (5.19), (5.20), (5.21) *and* (5.22) *are attained for the function* $f(z)$ *given by*

$$(J_{c,p}f)(z) = z^p - \frac{(c+p)2\beta(p-q-\alpha)\delta(p,q+1)}{(c+n+p)\left\{n+\beta[n+2(p-q-\alpha)]\right\}\delta(n+p,q+1)}z^{n+p}. \tag{5.23}$$

Remark 4. *Putting* $\beta = 1$ *in Theorems 10, 11 and 12, we obtain the corresponding results for the classes* $S_n(p,q,\alpha)$ *and* $C_n(p,q,\alpha)$, *respectively.*

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References

[1] S. D. Bernardi, *Convex and starlike univalent functions*, Trans. Amer. Math. *Soc. 135(1969), 429-446.*

[2] M.-P. Chen, H. Irmak and H. M. Srivastava, *Some multivalent functions with negative coefficients,* defined by using differential operator, PanAmer. Math. J. *6(1996), no.2, 55-64.*

[3] M.-P. Chen, H. Irmak and H. M. Srivastava, *Some families of multivalently analytic functions with negative coefficients*, J. Math. Anal. Appl. *214(1997), 674-490.*

[4] P. L. Duren, Univalent Functions, *Grundlehen der Mathematischen Wissenschaften 259*, Springer- Verlag, New York, Berlin, Heidelberg, and Tokoyo, 1983.

[5] A. W. Goodman, Univalent Functions, Vols. I and II, *Polygonal Publishing House*, Washington, New Jersey, 1983.

[6] V. P. Gupta and P. K. Jain, *Certain classes of univalent functions with negative coefficients*, Bull. Austral. Math. Soc. 14(1976), 409-416.

[7] H. M. Hossen, *Ouasi- Hadamard product of certain p-valent functions*, Demomstratio Math. 33(2000), no.2,177-281.

[8] R. J. Libera, *Some classes of regular univalent functions*, Proc. Amer. Math. Soc. 16(1969), 755-758.

[9] A. E. Livingston, *On the radius of univalence of cerain analytic functions*, Proc. Amer. Math. Soc. 17(1966), 352-357.

[10] S. Owa, *On the distortion theorems. I*, Kyungpook Math. J. 18(1978), 55-59.

[11] S. Owa, *On certain classes of p-valent functions with negative coefficients*, Simon Stevin 59 (1985), 385-402.

[12] S. Owa, *The quasi-Hadamard products of certain analytic functions in: H. M.* Srivastava and S. Owa (Eds.) Current Topics in Analytic Function Theory, World Scientific Publishing Company, Singapore, New Jersey, Lnodon, and Hong Kong, 1992, 234-251.

[13] G. S. Salagean, H. M. Hossen and M. K. Aouf, *On certain classes of p-valent functions with negative coefficients. II*, Studia Univ. Babes-Bolyai 69 (2004), no.1, 77-85.

[14] A. Schild and H. Silverman, *Convolultions, of univalent functions with negative coefficients*, Ann. Univ. Mariae- Curie Sklodowska Sect. A 29(1975),99-107.

[15] H. M. Srivastava and M.K.Aouf, *A certain fractional derivative operator and its applications to a new class of analytic and multivalent functions with negative coefficients. I and II*, J. Math. Anal. Appl. 171(1992), 1-13; ibid. 192(1995),973-688.

[16] H. M. Srivastava and S. Owa (Editors), *Univalent Functions, Fractional Calculus, and Their Applivations*, Halsted Press (Ellis Horwood Limited, Chichester), John Wiley and Sons, New York, Chichester, Brisbane and Toronto, 1989.

[17] H. M. Srivastava and S. Owa (Editors), *Current Topics in Analytic Function Theory*, World Scientific Publishing Company, Singapore, New Jersey, London and Hong Kong, 1992.

[18] *H. M. Srivastava, S. Owa and S. K. Chatterjea,* *A note on certain classes of starlike functions*, *Rend. Sem. Mat. Univ. Padova 77(1987), 115-124.*

[19] *R. Yamakawa,* *Certain subclasses of p-valently starlike functions with negative coefficients*, *in : H. M. Srivastava and S. Owa (Eds.) Current Topics in Analytic Function Theory, World Scientific Publishing Company, Singapore, New Jersey, London and Hong Kong, 1992, 393-402.*

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Certain class of analytic functions associated with the wright generalized hypergeometric function

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ABSTRACT: Using the Wright's generalized hypergeometric function, we introduce a new class $W(q, s; A, B, \lambda)$ of analytic functions with negative coefficients. In this paper we investigate coefficient estimates, distortion theorem and the radii of convexity and starlikeness

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1. Introduction

Let D denote the class of functions $f(z)$ of the form :

$$f(z) = z + \sum_{k=2}^{\infty} a_k z^k \ , \tag{1}$$

which are analytic in $U = U(1)$, where $U(r) = \{z : z \in C \text{ and } |z| < r\}$.

If $f(z)$ and $g(z)$ are analytic in U, we say that $f(z)$ is subordinate to $g(z)$, written symbolically as follows :

$$f \prec g \quad \text{or} \quad f(z) \prec g(z) \qquad (z \in U)\,,$$

if there exists a Schwarz function $w(z)$ in U such that $f(z) = g(w(z))\,(z \in U)$.

A function $f(z)$ belonging to the class D is said to be convex in $U(r)$ if and only if

$$\text{Re}\left\{1 + \frac{z\,f''(z)}{f'(z)}\right\} > 0 \quad (z \in U(r);\ 0 < r \leq 1)\,.$$

A function $f(z)$ belonging to the class D is said to be starlike in $U(r)$ if and only if

$$\operatorname{Re}\left\{\frac{z\,f'(z)}{f(z)}\right\} > 0 \quad (z \in U(r);\ 0 < r \leq 1)\,.$$

We denote by S^c the class of all functions in D which are convex in U and by S^* we denote the class of all functions in D which are starlike in U.

For analytic functions $f(z) = \sum_{k=0}^{\infty} a_k z^k$ and $g(z) = \sum_{k=0}^{\infty} b_k z^k$, by $(f*g)(z)$ we denote the Hadamard product (or convolution) of $f(z)$ and $g(z)$, defined by

$$(f*g)(z) = \sum_{k=0}^{\infty} a_k b_k z^k\ .$$

Let $\mathcal{B}$ be a subclass of the class D. We define the radius of starlikeness $R^*(\mathcal{B})$ and the radius of convexity $R^c(\mathcal{B})$ for the class $\mathcal{B}$ by

$$\begin{aligned} R^*(\mathcal{B}) &= \inf_{f\in\mathcal{B}} \left(\sup\{r \in (0,1] : f \ \text{ is starlike of order } 0 \ \text{ in } U(r)\}\right), \\ R^c(\mathcal{B}) &= \inf_{f\in\mathcal{B}} \left(\sup\{r \in (0,1] : f \ \text{ is convex in } U(r)\}\right), \end{aligned}$$

respectively.

Let $\alpha_1, A_1, ..., \alpha_q, A_q$ and $\beta_1, B_1, ..., \beta_s, B_s (q, s \in N = \{1, 2, ...\})$ be positive real parameters such that

$$1 + \sum_{k=1}^{s} B_k - \sum_{k=1}^{q} A_k \geq 0\ .$$

The Wright generalized hypergeometric function [15] (see also [6])

$${}_q\Psi_s[(\alpha_1, A_1), ..., (\alpha_q, A_q); (\beta_1, B_1), ..., (\beta_s, B_s); z] = {}_q\Psi_s[(\alpha_n, A_n)_{1,q}; (\beta_n, B_n)_{1,s}; z]$$

is defined by

$$\begin{aligned} &{}_q\Psi_s[(\alpha_k, A_k)_{1,q}; (\beta_k, B_k)_{1,s}; z] \\ &\quad = \sum_{k=0}^{\infty} \left\{\prod_{n=1}^{q} \Gamma(\alpha_n + kA_n)\right\} \left\{\prod_{n=1}^{s} \Gamma(\beta_n + kB_n)\right\}^{-1} \frac{z^k}{k!} \quad (z \in U)\,. \end{aligned}$$

If $A_n = 1 (n = 1, ..., q)$ and $B_n = 1 (n = 1, ..., s)$, we have the relationship :

$$\Omega\, {}_q\Psi_s[(\alpha_{n,1})_{1,q}; (\beta_{n,1})_{1,s}; z] = {}_qF_s(\alpha_1, ..., \alpha_q;\ \beta_1, ..., \beta_s; z)\,,$$

where ${}_qF_s(\alpha_1, ..., \alpha_q;\ \beta_1, ..., \beta_s; z)$ is the generalized hypergeometric function (see for details [2], [3], [4], [5] and [7]) and

$$\Omega = \left(\prod_{n=1}^{q} \Gamma(\alpha_n)\right)^{-1} \left(\prod_{n=1}^{s} \Gamma(\beta_n)\right). \tag{2}$$

The Wright generalized hypergeometric functions were invoked in the geometric function theory (see [1], [2], [3], [8], [9] and [10]) .

By using the generalized hypergeometric function Dziok and Srivastava [3] introduced a linear operator. In [1] Dziok and Raina extended the linear operator by using the Wright generalized hypergeometric function.

First we define a function ${}_q\phi_s[(\alpha_n, A_n)_{1,q}; (\beta_n, B_n)_{1,s}; z]$ by

$$ {}_q\phi_s[(\alpha_n, A_n)_{1,q}; (\beta_n, B_n)_{1,s}; z] = \Omega z {}_q\Psi_s[(\alpha_n, A_n)_{1,q}; (\beta_n, B_n)_{1,s}; z] $$

and consider the following linear operator

$$ \theta[(\alpha_n, A_n)_{1,q}; (\beta_n, B_n)_{1,s}] : D \to D \ , $$

defined by the convolution

$$ \theta[(\alpha_n, A_n)_{1,q}; (\beta_n, B_n)_{1,s}]f(z) = {}_q\phi_s[(\alpha_n, A_n)_{1,q}; (\beta_n, B_n)_{1,s}; z] * f(z) \, . $$

We observe that, for a function $f(z)$ of the form (1), we have

$$ \theta[(\alpha_n, A_n)_{1,q}; (\beta_n, B_n)_{1,s}]f(z) = z + \sum_{k=2}^{\infty} \Omega \sigma_k(\alpha_1) a_k z^k \ , \tag{3} $$

where Ω is given by (2) and $\sigma_k(\alpha_1)$ is defined by

$$ \sigma_k(\alpha_1) = \frac{\Gamma(\alpha_1 + A_1(k-1)) ......\Gamma(\alpha_q + A_q(k-1))}{\Gamma(\beta_1 + B_1(k-1)) ....\Gamma(\beta_s + B_s(k-1))(k-1)!} \, . \tag{4} $$

We note that :

If $A_n = 1 (n = 1, ..., q)$, $B_n = 1 (n = 1, ..., s)$, $q = 2$ and $s = 1$, we have

(i) $\theta[n+1, 1; 1]f(z) = D^n f(z)$ $(n \in N_0 = \{0, 1, ...\})$, where $D^n f(z)$ is the $n-th$ order Ruscheweyh derivative of $f(z)$ (see [13]);

(ii) $\theta[2, 1; 2-\phi]f(z) = \Omega^{\phi} f(z) = \Gamma(2-\phi) z^{\phi} D_z^{\phi} f(z)$ $(\varphi \in R; \varphi \neq 2, 3, 4, ...; f \in D)$, where the operator $\Omega^{\phi} f(z)$ was introduced by Owa and Srivastava [11].

If, for convenience, we write

$$ \theta[\alpha_1]f(z) = \theta[(\alpha_1, A_1), ..., (\alpha_q, A_q); (\beta_1, B_1), ..., (\beta_s, B_s)]f(z) \ , $$

then one can easily verify from the definition (3) that

$$ zA_1(\theta[\alpha_1]f(z))' = \alpha_1 \theta[\alpha_1 + 1]f(z) - (\alpha_1 - A_1)\theta[\alpha_1]f(z) \, . \tag{5} $$

The linear operator $\theta[\alpha_1]$ was introduced by Dziok and Raina [1].

Let us denote by $V(q, s; A, B, \lambda)$ the class of functions of the form (1) which also satisfy the following condition:

$$ \frac{1}{(1-\lambda)} \left(\alpha_1 \frac{\theta[\alpha_1 + 1]f(z)}{\theta[\alpha_1]f(z)} + A_1(1-\lambda) - \alpha_1 \right) \prec A_1 \frac{1+Az}{1+Bz} $$

$$ (0 \leq B \leq 1; -B \leq A < B; 0 \leq \lambda < 1) \, , $$

or, by using (5), if it satisfies the following condition:

$$\frac{1}{(1-\lambda)}\left(\frac{z(\theta[\alpha_1]f(z))'}{\theta[\alpha_1]f(z)}-\lambda\right)\prec\frac{1+Az}{1+Bz}$$

or, equivalently, if

$$\left|\frac{\frac{z(\theta[\alpha_1]f(z))'}{\theta[\alpha_1]f(z)}-1}{B\frac{z(\theta[\alpha_1]f(z))'}{\theta[\alpha_1]f(z)}-[B+(A-B)(1-\lambda)]}\right|<1 \quad (z\in U)\,. \tag{6}$$

Let T denote the subclass of D consisting of functions of the form:

$$f(z)=z-\sum_{k=2}^{\infty}a_kz^k \quad (a_k\geq 0) \tag{7}$$

Further, we define the class $W(q,s;A,B,\lambda)$ by

$$W(q,s;A,B,\lambda)=V(q,s;A,B,\lambda)\cap T\ .$$

In particular, for $q=s+1$ and $\alpha_{s+1}=A_{s+1}=1$, we write $W(s;A,B,\lambda)=W(s+1,s;A,B,\lambda)$. The class $W(q,s;A,B,0)=W(q,s;A,B)$ was studied by Dziok and Raina [1].

If $A_n=1(n=1,...,q)$ and $B_n=1(n=1,...,s)$, then we note that:
(i) $W(q,s;A,B,0)=V_2^1(q,s;A,B)$ (Dziok and Srivastava [3]);
(ii) For $\alpha_1=n+1,\alpha_2=1$ and $\beta_1=1$, we have:

$$W(2,1;-\rho,\rho,\lambda) = T_n(\lambda,\rho)=\left\{f\in T:\ \left|\frac{\frac{z(D^nf(z))'}{D^nf(z)}-1}{\frac{z(D^nf(z))'}{D^nf(z)}+1-2\lambda}\right|<\rho\ ,\right.$$
$$\left.(z\in U,0\leq\lambda<1,0<\rho\leq 1,n\in N_0)\right\}.$$

The class $T_n(\lambda,\rho)$ was studied by Patel and Acharya [12];
(ii) For $\alpha_1=2,\alpha_2=1$ and $\beta_1=2-\phi(\phi\in R;\phi\neq 2,3,4,...)$, we have:

$$W(2,1;-\rho,\rho,\lambda) = T^{\phi}(\lambda,\rho)=\left\{f\in T:\ \left|\frac{\frac{z(\Omega^{\phi}f(z))'}{\Omega^{\phi}f(z)}-1}{\frac{z(\Omega^{\phi}f(z))'}{\Omega^{\phi}f(z)}+1-2\lambda}\right|<\rho\ ,\right.$$
$$\left.(z\in U,0\leq\lambda<1,0<\rho\leq 1,\phi\in R(\neq 2,3,...))\right\}.$$

2. Coefficient estimates

Theorem 1 *Let a function $f(z)$ of the form (7) belongs to the class D and let Ω $\sigma_k(\alpha_1)$ be defined by (2) and (4), respectively. If*

$$\sum_{k=2}^{\infty}\Omega\delta_k\,|a_k|\leq(B-A)(p-\lambda), \tag{8}$$

where

$$\delta_k = [(1+B)(k-1) + (B-A)(1-\lambda)]\sigma_k(\alpha_1)\,, \tag{9}$$

then $f(z) \in W(q,s;A,B,\lambda)$.

Proof. Let $z \in U$. If (8) holds, we find from (7) that

$$\begin{aligned}
& - \left| z(\theta[\alpha_1]f(z))' - \theta[\alpha_1]f(z) \right| - \left| Bz(\theta[\alpha_1]f(z))' \right. \\
& \left. - \left[B + (A-B)(1-\lambda)\right]\theta[\alpha_1]f(z) \right| = \left| - \sum_{k=2}^{\infty}(k-1)\Omega\sigma_k(\alpha_1)a_k z^k \right| \\
& - \left| (B-A)(1-\lambda)\,z - \sum_{k=2}^{\infty}[B(k-1) + (B-A)(1-\lambda)]\Omega\sigma_k(\alpha_1)a_k z^k \right| \\
\leq \;& \sum_{k=2}^{\infty}(k-1)\Omega\sigma_k(\alpha_1)\,|a_k|\,r^k - \{(B-A)(1-\lambda)r - \\
& \left. \sum_{k=2}^{\infty}[B(k-1) + (B-A)(1-\lambda)]\,\Omega\sigma_k(\alpha_1)\,|a_k|\,r^k \right\} \\
= \;& r\left\{ \sum_{k=2}^{\infty}[(1+B)(k-1) + (B-A)(1-\lambda)]\Omega\sigma_k(\alpha_1)\,|a_k|\,r^{k-1} - (B-A)(1-\lambda) \right\} \\
< \;& \sum_{k=2}^{\infty}\Omega\delta_k\,|a_k| - (B-A)(1-\lambda) \leq 0\,.
\end{aligned}$$

Thus we have condition (6) and $f(z) \in W(q,s;A,B,\lambda)$. ■

Theorem 2 *A function* $f(z)$ *of the form (7) belongs to the class* $W(q,s;A,B,\lambda)$ *if and only if*

$$\sum_{k=2}^{\infty}\Omega\delta_k a_k \leq (B-A)(p-\lambda), \tag{10}$$

where δ_k *is defined by (9).*

Proof. By Theorem 1 we have that (10) is the sufficient condition for the class $W(q,s;A,B,\lambda)$. Let now $f(z) \in W(q,s;A,B,\lambda)$ be given by (7). Then, from (6) and (7), we have

$$\begin{aligned}
& \left| \frac{\frac{z(\theta[\alpha_1]f(z))'}{\theta[\alpha_1]f(z)} - 1}{B\frac{z(\theta[\alpha_1]f(z))'}{\theta[\alpha_1]f(z)} - [B + (A-B)(1-\lambda)]} \right| \\
= \;& \left| \frac{\sum_{k=2}^{\infty}(k-1)\Omega\sigma_k(\alpha_1)a_k z^{k-1}}{(B-A)(1-\lambda) - \sum_{k=2}^{\infty}[B(k-1) + (B-A)(1-\lambda)]\Omega\sigma_k(\alpha_1)a_k z^{k-1}} \right| < 1 \\
& (z \in U)\,,
\end{aligned}$$

where Ω and $\sigma_k(\alpha_1)$ are defined by (2) and (4), respectively. Putting $z = r\,(0 \le r < 1)$, we obtain

$$\sum_{k=2}^{\infty}(k - 1)\Omega\sigma_k(\alpha_1)a_k r^{k-1} < (B-A)(1-\lambda) - \sum_{k=2}^{\infty}[B(k-1)+(B-A)(1-\lambda)]\Omega\sigma_k(\alpha_1)a_k r^{k-1}\,,$$

which, upon letting $r \to 1^-$, readily yields the assertion (10). This completes the proof of Theorem 2. ■

Since the expression δ_k defined by (9) is a decreasing function with respect to $\beta_n, B_n (n = 1, ..., s)$ and an increasing function with respect to α_ℓ, $A_\ell (\ell = 1, ..., q)$, from Theorem 2, we obtain :

Corollary 1 *If $\ell \in \{1, ..., q\}$; $j \in \{1, ..., s\}$, $0 \le \alpha'_\ell \le \alpha_\ell$, $0 < A'_\ell \le A_\ell$ and $0 \le \beta_j \le \beta'_j$, $0 < B_\ell \le B'_\ell$, then the class $W(q, s; A, B, \lambda)$ (for the parameters $(\alpha_n, A_n)_{1,q}; (\beta_n, B_n)_{1,s}$) is included in the class $W(q, s; A, B, \lambda)$ for the parameters*

$$(\alpha_1, A_1)..., (\alpha_{\ell-1}, A_{\ell-1}), (\alpha'_\ell, A'_\ell), (\alpha_{\ell+1}, A_{\ell+1}), ..., (\alpha_q, A_q)$$

and

$$(\beta_1, B_1)..., (\beta_{j-1}, B_{j-1}), (\beta'_j, B'_j), (\beta_{j+1}, B_{j+1}), ..., (\beta_s, B_s)\ .$$

From Theorem 2, we also have the following corollary.

Corollary 2 *If a function $f(z)$ of the form (7) belongs to the class $W(q, s; A, B, \lambda)$, then*

$$a_k \le \frac{(B-A)(1-\lambda)}{\Omega\delta_k} \quad (k \ge 2)\ .$$

The result is sharp, the functions $f_k(z)$ of the form :

$$f_k(z) = z - \frac{(B-A)(1-\lambda)}{\Omega\delta_k}\ z^k \quad (k \ge 2) \tag{11}$$

being the extremal functions.

Let $f(z)$ be defined by (7) and for $A = -1$ and $B = 1$, the condition (6) is equivalent to

$$\theta[\alpha_1]f(z) \in T^*(\lambda) \quad (0 \le \lambda < 1)\ ,$$

where $T^*(\lambda)$ is the class of starlike functions of order $\lambda (0 \le \lambda < 1)$ with negative coefficients, was studied by Silverman [14]. Thus we have the following lemma :

Lemma 1 *If $\alpha_n = \beta_n$ and $A_n = B_n (n = 1, ..., s)$ then*

$$W(s; -1, 1, \lambda) \subset T^*(\lambda) \quad (0 \le \lambda < 1)\ .$$

By the definition of the class $W(q,s;A,B,\lambda)$, we have the following lemma.

Lemma 2 *If* $A_1 \le A_2, B_1 \ge B_2$ *and* $0 \le \lambda_1 \le \lambda_2 < 1$ *then*

$$W(q,s;A_1,B_1,\lambda_2) \subset W(q,s;A_2,B_2,\lambda_1) \subset W(q,s;-1,1,0)\ .$$

Remark 1 *Throught our paper we use* Ω *and* δ_k*, where* Ω *and* δ_k *are defined by (2) and (9), respectively.*

3. Distortion theorem

Theorem 3 *Let a function* $f(z)$ *of the form (7) belong to the class* $W(q,s;A,B,\lambda)$ *If the sequence* $\{\delta_k\}$ *is nondecreasing, then*

$$r - \frac{(B-A)(1-\lambda)}{\Omega\delta_2}\, r^2 \le |f(z)| \le r + \frac{(B-A)(1-\lambda)}{\Omega\delta_2}\, r^2 \quad (|z| = r < 1). \tag{12}$$

If the sequence $\left\{\frac{\delta_k}{k}\right\}$ *is nondecreasing, then*

$$1 - \frac{2(B-A)(1-\lambda)}{\Omega\delta_2}\, r \le |f^{'}(z)| \le 1 + \frac{2(B-A)(1-\lambda)}{\Omega\delta_2}\, r \quad (|z| = r < 1). \tag{13}$$

The result is sharp, with the extremal function $f(z)$ *given by*

$$f(z) = z - \frac{(B-A)(1-\lambda)}{\Omega\delta_2} z^2\ . \tag{14}$$

Proof. Let a function $f(z)$ of the form (7) belong to the class $W(q,s;A,B,\lambda)$. If the sequence $\{\delta_k\}$ is nondecreasing and positive, by Theorem 2, we have

$$\sum_{k=2}^{\infty} a_k \le \frac{(B-A)(1-\lambda)}{\Omega\delta_2}\ , \tag{15}$$

and if the sequence $\left\{\frac{\delta_k}{k}\right\}$ is nondecreasing and positive, by Theorem 2, we have

$$\sum_{k=2}^{\infty} k\, a_k \le \frac{2(B-A)(1-\lambda)}{\Omega\delta_2}\ . \tag{16}$$

Making use of the conditions (15) and (16), in conjunction with the definition (7), we readily obtain the assertions (12) and (13) of Theorem 3 ■

Corollary 3 *Let a function* $f(z)$ *of the form* (7) *belong to the class* $W(s;A,B,\lambda)$*. If* $\beta_n \le \alpha_n, B_n \le A_n (n = 1,2,...,s)$*, then the assertions (12) and (13) hold true.*

Proof. If $q = s$ and $\beta_n \le \alpha_n, B_n \le A_n (n = 1,2,...,s)$, then the sequences $\{\delta_k\}$ and $\left\{\frac{\delta_k}{k}\right\}$ are nondecreasing. Thus, by Theorem 3, we have Corollary 3. ■

4. The radii of convexity and starlikeness

Theorem 4 *The radius of starlikeness for the class $W(q,s;A,B,\lambda)$ is given by*

$$R^*(W(q,s;A,B,\lambda)) = \inf_{k\geq 2}\left[\frac{\Omega\delta_k}{k(B-A)(1-\lambda)}\right]^{\frac{1}{k-1}}. \tag{17}$$

The result is sharp.

Proof. It is sufficient to show that

$$\left|\frac{zf'(z)}{f(z)}-1\right| < 1 \quad (z\in U(r); 0<r\leq 1). \tag{18}$$

Since

$$\left|\frac{zf'(z)}{f(z)}-1\right| = \left|\frac{\sum_{k=2}^{\infty}(k-1)a_kz^k}{z+\sum_{k=2}^{\infty}a_kz^k}\right| \leq \frac{\sum_{k=2}^{\infty}(k-1)a_k|z|^{k-1}}{1-\sum_{k=2}^{\infty}a_k|z|^{k-1}},$$

putting $|z|=r$, the condition (18) is true if

$$\sum_{n=2}^{\infty} ka_k\, r^{k-1} \leq 1 \ . \tag{19}$$

By Theorem 2, we have

$$\sum_{k=2}^{\infty}\frac{\Omega\delta_k}{(B-A)(1-\lambda)}\ a_k \leq 1 \ .$$

Thus the condition (19) is true if

$$k\, r^{k-1} \leq \frac{\Omega\delta_k}{(B-A)(1-\lambda)} \quad (k\geq 2),$$

that is, if

$$r \leq \left(\frac{\Omega\delta_k}{k(B-A)(1-\lambda)}\right)^{\frac{1}{k-1}} \quad (k\geq 2).$$

It follows that any function $f(z)\in W(q,s;A,B,\lambda)$ is starlike in the disc $U(R^*(W(q,s;A,B,\lambda)))$, where $R^*(W(q,s;A,B,\lambda))$ is defined by (17). ■

Corollary 4

$$R^*(W(s;A,B,\lambda)) = \begin{cases} 1 & (\alpha_k\geq\beta_k, A_k\geq B_k;\ k=1,...,s) \\ \min_{k\geq 2}\left(\frac{\Omega\delta_k}{k(B-A)(1-\lambda)}\right)^{\frac{1}{k-1}} & (\alpha_k<\beta_k, A_k<B_k;\ k=1,...,s). \end{cases}$$

The result is sharp.

Proof. From Corollary 1, Lemma 1 and Lemma 2, we have

$$W(s; A, B, \lambda) \subset T^*(\lambda) \quad (\alpha_n \geq \beta_n, A_n \geq B_n;\ n = 1, ..., s).$$

By Theorem 3, any function $f(z) \in W(s; A, B, \lambda)$ is starlike in the disc $U(r)$, where

$$r = \inf_{k \geq 2} (d_k)^{\frac{1}{k-1}} \left(d_k = \frac{\Omega \delta_k}{k(B - A)(1 - \lambda)} \right).$$

Since, for $\alpha_n < \beta_n, A_n < B_n (\, n = 1, ..., s)$, we have $\lim\limits_{k \to \infty} d_k = d < 1$,

$\lim\limits_{k \to \infty} (d_k)^{\frac{1}{k-1}} = 1$, and $d_k > 0 (k \geq 2)$, the infimum of the set $\left\{ (d_k)^{\frac{1}{k-1}} : k \geq 2 \right\}$ is realized for an element of this set for some $k = k_0$. Moreover, the function

$$f_{k_0}(z) = z - \frac{(B - A)(1 - \lambda)}{\Omega \delta_{k_0}} \, z^{k_0} \ ,$$

belongs to the class $W(s; A, B, \lambda)$, and for $z = (d_{k_0})^{\frac{1}{k_0} - 1}$, we have

$$\mathrm{Re} \left\{ \frac{z_0 \, f'_{k_0}(z)}{f_{k_0}(z)} \right\} = 0 \ .$$

Thus the result is sharp. ■

Theorem 5 *The radius of convexity for the class $W(q, s; A, B, \lambda)$ is given by*

$$R^c(W(q, s; A, B, \lambda)) = \inf_{k \geq 2} \left(\frac{\Omega \delta_k}{k^2 (B - A)(1 - \lambda)} \right)^{\frac{1}{k-1}} ,$$

The result is sharp.

Proof. The proof is analagous to that of Theorem 4, and we omit the details. ■

References

[1] J. Dziok and R. K. Raina, *Families of analytic functions associated with the Wright generalized hypergeometric function*, Demonstratio Math. 37(2004), no. 3, 533-542.-1mm

[2] J. Dziok, R.K. Raina and H. M. Srivastava, *Some classes of analytic functions associated with operators on Hilbert space involving Wright's generalized hypergeometric function*, Proc. of the Jangieon Math. Soc., 7(2004), 43-55.-1mm

[3] J. Dziok and H. M. Srivastava, *Classes of analytic functions with the generalized hypergeometric function*, Applied Math. Comput. 103(1999), 1-13.-1mm

[4] J. Dziok and H. M. Srivastava, *Certain subclasses of analytic functions associated with the generalized hypergeometric function*, Integral Transform. Spec. Funct. 14(2003), 7-18.-1mm

[5] A. Gangadharan, T. N. Shanmugam and H. M. Srivastava, *Generalized hypergeometric function associated with k-uniformly convex functions*, Comput. Math. Appl. 44(2002), no. 12, 1515-1526.-1mm

[6] P. W. Karlsson and H. M. Srivastava, *Multiple Gaussian Hypergeometric Series*, Halsted Press (Ellis Horwood Ltd., Chichester), John Wiley and Sons, New York, Chichester, Brisbane and London 1985.-1mm

[7] J. -L. Liu, *Strongly starlike functions associated with the Dziok-Srivastava operator*, Tamkang J. Math. 35(2004), no. 1, 37-42.-1mm

[8] T. S. Nahar and R. K. Raina, *A note on boundedness properties of Wright's generalized hypergeometric function*, Ann. Math. Blaise Pascal 4(1997), 83-95.-1mm

[9] T. S. Nahar and R. K. Raina, *On characterization of certain Wright's generalized hypergeometric functions involving certain subclasses of analytic functions*, Informatica 10(1999), 219-230.-1mm

[10] T. S. Nahar and R. K. Raina, *On univalent and starlike Wright's hypergeometric functions*, Rend. Sem. Math. Univ. Padova 95(1996), 11-22.-1mm

[11] S. Owa and H. M. Srivastava, *Univalent and starlike generalized hypergeometric functions*, Canad. J. Math. 39(1987), no. 5, 1057-1077.-1mm

[12] J. Patel and M. Acharya, *Certain subclasses of starlike functions with negative coefficients*, Bull. Cal. Math. Soc. 87(1995), 265-276.-1mm

[13] St. Ruscheweyh, *New criteria for univalent functions*, Proc. Amer. Math. Soc. 49.-1mm

[14] H. Silverman, *Univalent functions with negative coefficients*, Proc. Amer. Math. Soc. 51(1975), 109-116.-1mm

[15] E. M. Wright, *The asymptotic expansion of the generalized hypergeometric function*, Proc. London Math. Soc. 46(1946), 389-408.-1mm

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Fixed point theory for Volterra Kakutani Mönch maps

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ABSTRACT: New fixed point theorems for multivalued Volterra Kakutani Mönch maps between Fréchet spaces are presented. The proof relies on fixed point theory in Banach spaces and viewing a Fréchet space as the projective limit of a sequence of Banach spaces

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1. Introduction

This paper presents new fixed point theorems for multivalued Mönch type maps between Fréchet spaces. In the literature [1, 2, 3, 5, 6] one usually assumes the map F is defined on a subset X of a Fréchet space E and its restriction (again called F) is well defined on $\overline{X_n}$ (see Section 2). In general of course for Volterra operators the restriction is always defined on X_n and in most applications it is in fact defined on $\overline{X_n}$ and usually even on E_n (see Section 2). In this paper we make use of the fact that the restriction is well defined on X_n and we only assume it admits an extension (satisfying certain properties) on $\overline{X_n}$. We also show how easily one can extend fixed point theory in Banach spaces to fixed point theory in Fréchet spaces. In particular we obtain an applicable Leray-Schauder alternative in Fréchet spaces for Volterra Kakutani Mönch type operators. Also inward type maps are discussed.

Existence in Section 2 is based on a Leray–Schauder alternative for Kakutani Mönch maps [1, 6] which we state here for the convenience of the reader.

Theorem 1.1. *Let K be a closed convex subset of a Banach space X, U a relatively open subset of K, $x_0 \in U$ and suppose $F : \overline{U} \to CK(K)$ is a upper semicontinuous map (here $CK(K)$ denotes the family of nonempty convex compact subsets of K). Also assume the following conditions hold:*

$$(1.1) \qquad \begin{cases} M \subseteq \overline{U},\ M \subseteq co\,(\{x_0\} \cup F(M)) \ \ \text{with} \ \ \overline{M} = \overline{C} \ \ \text{and} \\ C \subseteq M \ \ \text{countable, implies} \ \ \overline{M} \ \ \text{is compact} \end{cases}$$

and

$$x \notin (1-\lambda)\{x_0\} + \lambda F x \quad \text{for} \quad x \in \overline{U} \setminus U \quad \text{and} \quad \lambda \in (0,1). \tag{1.2}$$

Then there exists a compact set $\sum$ *of* $\overline{U}$ *and a* $x \in \sum$ *with* $x \in F x$.

Also in Section 2 we will discuss inward Kakutani Mönch maps. Let Q be a subset of a Hausdorff topological space X and $x \in X$. The inward set $I_Q(x)$ is defined by

$$I_Q(x) = \{x + r\,(y-x) : \; y \in Q, \; r \geq 0\}.$$

If Q is convex and $x \in Q$ then

$$I_Q(x) = x + \{r\,(y-x) : \; y \in Q, \; r \geq 1\}.$$

In our next definition and theorem E is a Banach space, C a closed convex subset of E and U_0 a bounded open subset of E. We will let $U = U_0 \cap C$ and $0 \in U$. In our definitions $\overline{U}$ and ∂U denote the closure and the boundary of U in C respectively.

Definition 1.1. *We say* $F \in K(\overline{U}, E)$ *if* $F : \overline{U} \to CK(E)$ *is upper semicontinuous,* $F(\overline{U})$ *is bounded,* $F(x) \subseteq I_C(x)$ *for* $x \in \overline{U}$*, and if* $D \subseteq E$ *with* $D \subseteq co\,(\{0\} \cup F(D \cap U))$ *and* $\overline{D} = \overline{B}$ *with* $B \subseteq D$ *countable then* $\overline{D \cap U}$ *is compact.*

The following theorem [2, 5] will be needed in Section 2.

Theorem 1.2. *Let* E, C, U_0, U *be as before Definition 1.1,* $0 \in U$ *and* $F \in K(\overline{U}, E)$ *with*

$$x \notin \lambda F x \quad \text{for} \quad x \in \partial U \quad \text{and} \quad \lambda \in (0,1) \tag{1.3}$$

holding. Then there exists a compact set $\sum$ *of* $\overline{U}$ *and a* $x \in \sum$ *with* $x \in F x$.

Now let I be a directed set with order $\leq$ and let $\{E_\alpha\}_{\alpha \in I}$ be a family of locally convex spaces. For each $\alpha \in I$, $\beta \in I$ for which $\alpha \leq \beta$ let $\pi_{\alpha,\beta} : E_\beta \to E_\alpha$ be a continuous map. Then the set

$$\left\{ x = (x_\alpha) \in \prod_{\alpha \in I} E_\alpha : \; x_\alpha = \pi_{\alpha,\beta}(x_\beta) \; \forall\, \alpha,\, \beta \in I,\, \alpha \leq \beta \right\}$$

is a closed subset of $\prod_{\alpha \in I} E_\alpha$ and is called the projective limit of $\{E_\alpha\}_{\alpha \in I}$ and is denoted by $\lim_{\leftarrow} E_\alpha$ (or $\lim_{\leftarrow} \{E_\alpha, \pi_{\alpha,\beta}\}$ or the generalized intersection [4, pp. 439] $\cap_{\alpha \in I} E_\alpha$.)

2. Fixed point theory in Fréchet spaces.

Let $E = (E, \{|\cdot|_n\}_{n \in N})$ be a Fréchet space with the topology generated by a family of seminorms $\{|\cdot|_n : n \in N\}$; here $N = \{1, 2, ...\}$. We assume that the family of seminorms satisfies

$$|x|_1 \leq |x|_2 \leq |x|_3 \leq \ldots\ldots \quad \text{for every} \quad x \in E. \tag{2.1}$$

A subset X of E is bounded if for every $n \in N$ there exists $r_n > 0$ such that $|x|_n \leq r_n$ for all $x \in X$. For $r > 0$ and $x \in E$ we denote $B(x,r) = \{y \in E : |x-y|_n \leq r \, \forall n \in N\}$. To E we associate a sequence of Banach spaces $\{(\mathbf{E}_n, |\cdot|_n)\}$ described as follows. For every $n \in N$ we consider the equivalence relation $\sim_n$ defined by

$$x \sim_n y \quad \text{iff} \quad |x-y|_n = 0. \tag{2.2}$$

We denote by $\mathbf{E}^n = (E/\sim_n, |\cdot|_n)$ the quotient space, and by $(\mathbf{E}_n, |\cdot|_n)$ the completion of $\mathbf{E}^n$ with respect to $|\cdot|_n$ (the norm on $\mathbf{E}^n$ induced by $|\cdot|_n$ and its extension to $\mathbf{E}_n$ are still denoted by $|\cdot|_n$). This construction defines a continuous map $\mu_n : E \to \mathbf{E}_n$. Now since (2.1) is satisfied the seminorm $|\cdot|_n$ induces a seminorm on $\mathbf{E}_m$ for every $m \geq n$ (again this seminorm is denoted by $|\cdot|_n$). Also (2.2) defines an equivalence relation on $\mathbf{E}_m$ from which we obtain a continuous map $\mu_{n,m} : \mathbf{E}_m \to \mathbf{E}_n$ since $\mathbf{E}_m/\sim_n$ can be regarded as a subset of $\mathbf{E}_n$. Now $\mu_{n,m}\,\mu_{m,k} = \mu_{n,k}$ if $n \leq m \leq k$ and $\mu_n = \mu_{n,m}\,\mu_m$ if $n \leq m$. We now assume the following condition holds:

$$\begin{cases} \text{for each } n \in N, \ \text{ there exists a Banach space } (E_n, |\cdot|_n) \\ \text{and an isomorphism (between normed spaces) } j_n : \mathbf{E}_n \to E_n. \end{cases} \tag{2.3}$$

Remark 2.1. (i). *For convenience the norm on E_n is denoted by $|\cdot|_n$.*
(ii). In our applications $\mathbf{E}_n = \mathbf{E}^n$ for each $n \in N$.
(iii). Note if $x \in \mathbf{E}_n$ (or $\mathbf{E}^n$) then $x \in E$. However if $x \in E_n$ then x is not necessaily in E and in fact E_n is easier to use in applications (even though E_n is isomorphic to $\mathbf{E}_n$). For example if $E = C[0,\infty)$, then $\mathbf{E}^n$ consists of the class of functions in E which coincide on the interval $[0,n]$ *and* $E_n = C[0,n]$.

Finally we assume

$$\begin{cases} E_1 \supseteq E_2 \supseteq \ldots\ldots\ldots \text{ and for each } \ n \in N, \\ |j_n\,\mu_{n,n+1}\,j_{n+1}^{-1}\,x|_n \leq |x|_{n+1} \,\forall\, x \in E_{n+1} \end{cases} \tag{2.4}$$

(here we use the notation from [4] i.e. decreasing in the generalized sense). Let $\lim_{\leftarrow} E_n$ (or $\cap_1^\infty E_n$ where $\cap_1^\infty$ is the generalized intersection [4]) denote the projective limit of $\{E_n\}_{n\in N}$ (note $\pi_{n,m} = j_n\,\mu_{n,m}\,j_m^{-1} : E_m \to E_n$ for $m \geq n$) and note $\lim_{\leftarrow} E_n \cong E$, so for convenience we write $E = \lim_{\leftarrow} E_n$.

For each $X \subseteq E$ and each $n \in N$ we set $X_n = j_n\,\mu_n(X)$, and we let $\overline{X_n}$, $int\,X_n$ and ∂X_n denote respectively the closure, the interior and the boundary of X_n with respect to $|\cdot|_n$ in E_n. Also the pseudo-interior of X is defined by

$$pseudo-int\,(X) = \{x \in X : \ j_n\,\mu_n(x) \in \overline{X_n} \setminus \partial X_n \ \text{ for every } \ n \in N\}.$$

The set X is pseudo-open if $X = pseudo-int\,(X)$. For $r > 0$ and $x \in E_n$ we denote $B_n(x,r) = \{y \in E_n : |x-y|_n \leq r\}$.

Let $M \subseteq E$ and consider the map $F : M \to 2^E$. Assume for each $n \in N$ and $x \in M$ that $j_n\,\mu_n\,F\,(x)$ is closed. Let $n \in N$ and $M_n = j_n\,\mu_n(M)$. Since we only consider Volterra type operators we assume

$$\text{if } x,\, y \in E \text{ with } |x-y|_n = 0 \text{ then } H_n(F\,x, F\,y) = 0; \tag{2.5}$$

here H_n denotes the appropriate generalized Hausdorff distance (alternatively we could assume $\forall n \in N, \forall x,\, y \in M$ if $j_n\,\mu_n\,x = j_n\,\mu_n\,y$ then $j_n\,\mu_n\,F\,x = j_n\,\mu_n\,F\,y$ and of course here we do not need to assume that $j_n\,\mu_n\,F\,(x)$ is closed for each $n \in N$ and $x \in M$). Now (2.5) guarantees that we can define (a well defined) F_n on M_n as follows:

For $y \in M_n$ there exists a $x \in M$ with $y = j_n\,\mu_n(x)$ and we let

$$F_n\,y = j_n\,\mu_n\,F\,x$$

(we could of course call it $F\,y$ since it is clear in the situation we use it); note $F_n : M_n \to C(E_n)$ and note if there exists a $z \in M$ with $y = j_n\,\mu_n(z)$ then $j_n\,\mu_n\,F\,x = j_n\,\mu_n\,F\,z$ from (2.5) (here $C(E_n)$ denotes the family of nonempty closed subsets of E_n). In this paper we assume F_n will be defined on $\overline{M_n}$ i.e. we assume the F_n described above admits an extension (again we call it F_n) $F_n : \overline{M_n} \to 2^{E_n}$ (we will assume certain properties on the extension).

We now show how easily one can extend fixed point theory in Banach spaces to applicable fixed point theory in Fréchet spaces.

Theorem 2.1. *Let E and E_n be as described above and let $F : X \to 2^E$ where $X \subseteq E$. Also assume for each $n \in N$ and $x \in X$ that $j_n\,\mu_n\,F\,(x)$ is closed and also for each $n \in N$ that $F_n : \overline{X_n} \to 2^{E_n}$ is as described above. Suppose the following conditions are satisfied:*

$$x_0 \in pseudo - int\,(X) \tag{2.6}$$

$$\begin{cases} \text{for each } n \in N,\ F_n : \overline{int\,X_n} \to CK(E_n) \text{ is a upper} \\ \text{semicontinuous map} \end{cases} \tag{2.7}$$

$$\begin{cases} \text{for each } n \in N,\ M \subseteq \overline{int\,X_n} \text{ with} \\ M \subseteq co\,(\{j_n\,\mu_n(x_0)\} \cup F_n(M)) \text{ with } \overline{M} = \overline{C} \\ \text{and } C \subseteq M \text{ countable, implies } \overline{M} \text{ is compact} \end{cases} \tag{2.8}$$

$$\begin{cases} \text{for each } n \in N,\ y \notin (1-\lambda)\,j_n\,\mu_n(x_0) + \lambda\,F_n\,y \text{ in } E_n \\ \text{for all } \lambda \in (0,1] \text{ and } y \in \partial\,int\,X_n \end{cases} \tag{2.9}$$

and

$$\begin{cases} \text{for each } n \in \{2,3,....\} \text{ if } y \in int\,X_n \text{ solves } y \in F_n\,y \\ \text{in } E_n \text{ then } j_k\,\mu_{k,n}\,j_n^{-1}\,(y) \in int\,X_k \text{ for } k \in \{1,...,n-1\}. \end{cases} \tag{2.10}$$

Then F has a fixed point in E.

PROOF: For each $n \in N$ let $\sum_n = \{x \in \overline{int\, X_n} : x \in F_n\, x \text{ in } E_n\}$. From Theorem 1.1 there exists $y_n \in int\, X_n$ (note (2.9) holds with $\lambda \in (0,1]$) with $y_n \in F_n\, y_n$. Lets look at $\{y_n\}_{n \in N}$. Notice $y_1 \in int\, X_1$ and $j_1\, \mu_{1,k}\, j_k^{-1}\,(y_k) \in int\, X_1$ for $k \in N\backslash\{1\}$ from (2.10). Note $j_1\, \mu_{1,n}\, j_n^{-1}(y_n) \in F_1\,(j_1\, \mu_{1,n}\, j_n^{-1}(y_n))$ in E_1; to see note for $n \in N$ fixed there exists a $x \in E$ with $y_n = j_n\, \mu_n\,(x)$ so $j_n\, \mu_n\,(x) \in F_n\,(y_n) = j_n\, \mu_n\, F(x)$ on E_n so on E_1 we have

$$\begin{aligned} j_1\, \mu_{1,n}\, j_n^{-1}(y_n) &= j_1\, \mu_{1,n}\, j_n^{-1}\, j_n\, \mu_n\,(x) \in j_1\, \mu_{1,n}\, j_n^{-1}\, j_n\, \mu_n\, F(x) \\ &= j_1\, \mu_{1,n}\, \mu_n\, F(x) = j_1\, \mu_1\, F(x) = F_1(j_1\, \mu_1\,(x)) \\ &= F_1(j_1\, \mu_{1,n}\, j_n^{-1}\, j_n\, \mu_n\,(x)) = F_1\,(j_1\, \mu_{1,n}\, j_n^{-1}(y_n)). \end{aligned}$$

Thus $j_1\, \mu_{1,n}\, j_n^{-1}(y_n) \in F_1\,(j_1\, \mu_{1,n}\, j_n^{-1}(y_n))$ in E_1 and so $j_1\, \mu_{1,n}\, j_n^{-1}\,(y_n) \in \sum_1$ for $n \in N$. Now since $\sum_1$ is compact there is a subsequence $N_1^\star$ of N and a $z_1 \in \sum_1$ with $j_1\, \mu_{1,n}\, j_n^{-1}\,(y_n) \to z_1$ in E_1 as $n \to \infty$ in $N_1^\star$ and $z_1 \in F_1\, z_1$ since F_1 is upper semicontinuous. Also (2.9) implies $z_1 \in int\, X_1$. Let $N_1 = N_1^\star \setminus \{1\}$. Now $j_2\, \mu_{2,n}\, j_n^{-1}\,(y_n) \in int\, X_2$ for $n \in N_1$ and $\sum_2$ compact guarantees that there exists a subsequence $N_2^\star$ of N_1 and a $z_2 \in \sum_2$ with $j_2\, \mu_{2,n}\, j_n^{-1}\,(y_n) \to z_2$ in E_2 as $n \to \infty$ in $N_2^\star$ and $z_2 \in F_2\, z_2$. Also (2.9) implies $z_2 \in int\, X_2$. Note from (2.4) and the uniqueness of limits that $j_1\, \mu_{1,2}\, j_2^{-1}\, z_2 = z_1$ in E_1 since $N_2^\star \subseteq N_1$ (note $j_1\, \mu_{1,n}\, j_n^{-1}\,(y_n) = j_1\, \mu_{1,2}\, j_2^{-1}\, j_2\, \mu_{2,n}\, j_n^{-1}\,(y_n)$ for $n \in N_2^\star$). Let $N_2 = N_2^\star \setminus \{2\}$. Proceed inductively to obtain subsequences of integers

$$N_1^\star \supseteq N_2^\star \supseteq ......, \quad N_k^\star \subseteq \{k, k+1, ...\}$$

and $z_k \in \sum_k$ with $j_k\, \mu_{k,n}\, j_n^{-1}\,(y_n) \to z_k$ in E_k as $n \to \infty$ in $N_k^\star$ and $z_k \in F_k\, z_k$. Also (2.9) implies $z_k \in int\, X_k$. Note $j_k\, \mu_{k,k+1}\, j_{k+1}^{-1}\, z_{k+1} = z_k$ in E_k for $k \in \{1, 2, ...\}$. Also let $N_k = N_k^\star \setminus \{k\}$.

Fix $k \in N$. Now $z_k \in F_k\, z_k$ in E_k. Note as well that

$$\begin{aligned} z_k &= j_k\, \mu_{k,k+1}\, j_{k+1}^{-1}\, z_{k+1} = j_k\, \mu_{k,k+1}\, j_{k+1}^{-1}\, j_{k+1}\, \mu_{k+1,k+2}\, j_{k+2}^{-1}\, z_{k+2} \\ &= j_k\, \mu_{k,k+2}\, j_{k+2}^{-1}\, z_{k+2} = ... = j_k\, \mu_{k,m}\, j_m^{-1}\, z_m = \pi_{k,m}\, z_m \end{aligned}$$

for every $m \geq k$. We can do this for each $k \in N$. As a result $y = (z_k) \in \lim_{\leftarrow} E_n = E$ and also note $y \in X$ since $z_k \in int\, X_k$ for each $k \in N$. Thus for each $k \in N$ we have

$$j_k\, \mu_k\,(y) = z_k \in F_k\, z_k = j_k\, \mu_k\, F\, y \text{ in } E_k$$

so $y \in F\, y$ in E. □

Remark 2.2. Usually in our applications we have $\partial X_n = \partial\, int X_n$ (so $\overline{X_n} = \overline{int\, X_n}$). If X is a pseudo-open subset of E then for each $n \in N$ we have X_n is a open subset of E_n so $int\, X_n = X_n$. To see this note $X_n \subseteq \overline{X_n} \setminus \partial X_n$ since if $y \in X_n$ then there exists $x \in X$ with $y = j_n \mu_n(x)$ and this together with $X = pseudo - int\, X$ yields $j_n \mu_n(x) \in \overline{X_n} \setminus \partial X_n$ i.e. $y \in \overline{X_n} \setminus \partial X_n$. In addition notice

$$\overline{X_n} \setminus \partial X_n = (int\, X_n \cup \partial X_n) \setminus \partial X_n = int\, X_n \setminus \partial X_n = int\, X_n$$

since $int\, X_n \cap \partial X_n = \emptyset$. Consequently

$$X_n \subseteq \overline{X_n} \setminus \partial X_n = int\, X_n, \quad \text{so} \quad X_n = int\, X_n.$$

Remark 2.3. We can replace (2.10) in Theorem 2.1 with

$$\begin{cases} \text{for each } \ n \in \{2,3,....\} \ \text{ if } \ y \in int\, X_n \ \text{ solves } \ y \in F_n\, y \\ \text{in } \ E_n \ \text{ then } \ j_k\, \mu_{k,n}\, j_n^{-1}\,(y) \in X_k \ \text{ for } \ k \in \{1, ..., n-1\} \end{cases}$$

provided we adjust (2.7) and (2.8) appropriately (i.e. replace $int\, X_n$ with X_n).

Remark 2.4. It is possible to replace $\lambda \in (0,1]$ in (2.9) with $\lambda \in (0,1)$ provided in this case we take X to be a closed subset of E and (2.10) is changed to

$$(2.10)^\star \qquad \begin{cases} \text{for each } \ n \in \{2,3,....\} \ \text{ if } \ y \in \overline{int\, X_n} \ \text{ solves } \ y \in F_n\, y \\ \text{in } \ E_n \ \text{ then } \ j_k\, \mu_{k,n}\, j_n^{-1}\,(y) \in \overline{int\, X_k} \ \text{ for } \ k \in \{1, ..., n-1\}. \end{cases}$$

The proof follows as in Theorem 2.1 except in this case $y_n \in \overline{intX_n}$ and $z_k \in \overline{intX_k}$. Also from $y = (z_k) \in \lim_{\leftarrow} E_n = E$ and $\pi_{k,m}\,(y_m) \to z_k$ in E_k as $m \to \infty$ we can conclude that $y \in \overline{X} = X$ (note $q \in \overline{X}$ iff for every $k \in N$ there exists $(x_{k,m}) \in X$, $x_{k,m} = \pi_{k,n}\,(x_{n,m})$ for $n \geq k$ with $x_{k,m} \to j_k\, \mu_k\,(q)$ in E_k as $m \to \infty$). Thus $z_k = j_k\, \mu_k\,(y) \in X_k$ and so $j_k\, \mu_k\,(y) \in j_k\, \mu_k\, F\,(y)$ in E_k as before.

Note here also that $(2.10)^\star$ could be replaced by

$$\begin{cases} \text{for each } \ n \in \{2,3,....\} \ \text{ if } \ y \in \overline{int\, X_n} \ \text{ solves } \ y = F_n\, y \\ \text{in } \ E_n \ \text{ then } \ j_k\, \mu_{k,n}\, j_n^{-1}\,(y) \in \overline{X_k} \ \text{ for } \ k \in \{1, ..., n-1\} \end{cases}$$

provided we adjust (2.7) and (2.8) appropriately (i.e. replace $int\, X_n$ with X_n).

Essentially the same reasoning as in Theorem 2.1 (now using Theorem 1.2) establishes the following result.

Theorem 2.2. *Let E and E_n be as described in the beginning of Section 2, C a convex subset in E, V a pseudo-open bounded subset of E, $0 \in V \cap C$, and $F : Y \to 2^E$ with $Y \subseteq E$, and $\overline{U_n} = \overline{V_n \cap \overline{C_n}} \subseteq Y_n$ for each $n \in N$ (here $U_n = V_n \cap \overline{C_n}$). Also assume for each $n \in N$ and $x \in Y$ that $j_n\, \mu_n\, F\,(x)$ is closed and also for each $n \in N$ that $F_n : \overline{U_n} \to 2^{E_n}$ is as described above. Suppose the following conditions are satisfied:*

$$(2.11) \qquad \begin{cases} \textit{for each } \ n \in N, \ F_n : \overline{U_n} \to CK(E_n) \ \textit{ is} \\ \textit{upper semicontinuous and } \ F_n(\overline{U_n}) \ \textit{ is bounded;} \\ \textit{here } \overline{U_n} \ \textit{denotes the closure of } U_n \ \textit{in } \ \overline{C_n} \end{cases}$$

$$(2.12) \qquad \begin{cases} \textit{for each } \ n \in N, \ D \subseteq E_n \ \textit{ with} \\ D \subseteq co\,(\{j_n\, \mu_n(0)\} \cup F_n(D \cap U_n)) \ \textit{ and } \ \overline{D} = \overline{B} \\ \textit{with } \ B \subseteq D \ \textit{ countable, implies } \ \overline{D \cap U_n} \ \textit{ is compact} \end{cases}$$

$$(2.13) \qquad \textit{for each } \ n \in N, \ F_n(x) \subseteq I_{\overline{C_n}}\,(x) \ \textit{ for each } \ x \in \overline{U_n}$$

$$(2.14)\qquad \begin{cases} \textit{for each } n \in N,\ y \notin \lambda F_n\, y \ \textit{ in } E_n \textit{ for all} \\ \lambda \in (0,1] \textit{ and } y \in \partial\, U_n; \textit{ here } \partial U_n \\ \textit{denotes the boundary of } U_n \textit{ in } \overline{C_n} \end{cases}$$

and

$$(2.15)\qquad \begin{cases} \textit{for each } n \in \{2,3,....\} \textit{ if } y \in U_n \textit{ solves } y \in F_n\, y \textit{ in } E_n \\ \textit{then } j_k\, \mu_{k,n}\, j_n^{-1}(y) \in U_k \textit{ for } k \in \{1,...,n-1\}. \end{cases}$$

Then F has a fixed point in E.

Remark 2.5. Note in Theorem 2.2 if $x \in \overline{U_n}$ then $x \in Y_n$ so there exists a $y \in Y$ with $x = j_n\, \mu_n\,(y)$ and so $F_n\,(x) = j_n\, \mu_n\, F(y)$.

PROOF: Fix $n \in N$. Let $\sum_n = \{x \in \overline{U_n} : x \in F_n\, x \text{ in } E_n\}$. We would like to apply Theorem 1.2. To do so we need to show

$$(2.16)\qquad \overline{C_n} \text{ is convex}$$

and

$$(2.17)\qquad V_n \text{ is a bounded open subset of } E_n \text{ and } j_n\, \mu_n\,(0) \in U_n.$$

First we check (2.16). To see this let $\hat{x}$, $\hat{y} \in \mu_n(C)$ and $\lambda \in [0,1]$. Then for every $x \in \mu_n^{-1}(\hat{x})$ and $y \in \mu_n^{-1}(\hat{y})$ we have $\lambda x + (1-\lambda)y \in C$ since C is convex and so $\lambda\hat{x}+(1-\lambda)\hat{y} = \lambda\mu_n(x)+(1-\lambda)\mu_n(y)$. It is easy to check that $\lambda\mu_n(x)+(1-\lambda)\mu_n(y) = \mu_n(\lambda x + (1-\lambda)y)$ so as a result

$$\lambda\hat{x} + (1-\lambda)\hat{y} = \mu_n(\lambda x + (1-\lambda)y) \in \mu_n(C),$$

and so $\mu_n(C)$ is convex. Now since j_n is linear we have $C_n = j_n(\mu_n(C))$ is convex and as a result $\overline{C_n}$ is convex. Thus (2.16) holds.

Now since V is pseudo-open and $0 \in V$ then $j_n\, \mu_n\,(0) \in \textit{pseudo} - \textit{int}\, V$ so $j_n\, \mu_n(0) \in \overline{V_n} \setminus \partial V_n$ (here $\overline{V_n}$ and ∂V_n denote the closure and boundary of V_n in E_n respectively). Of course

$$\overline{V_n} \setminus \partial V_n = (V_n \cup \partial V_n) \setminus \partial V_n = V_n \setminus \partial V_n$$

so $j_n\, \mu_n\,(0) \in V_n \setminus \partial V_n$, and in particular $j_n\, \mu_n\,(0) \in V_n$. Thus $j_n\, \mu_n\,(0) \in V_n \cap \overline{C_n} = U_n$. Next notice V_n is bounded since V is bounded (note if $y \in V_n$ then there exists $x \in V$ with $y = j_n\mu_n(x)$). Finally notice V_n is open in E_n (see Remark 2.2) so (2.17) holds.

For each $n \in N$ (see Theorem 1.2) there exists $y_n \in U_n = V_n \cap \overline{C_n}$ with $y_n \in F_n\, y_n$. Lets look at $\{y_n\}_{n \in N}$. Notice $y_1 \in U_1$ and $j_1\, \mu_{1,k}\, j_k^{-1}\,(y_k) \in U_1$ for $k \in N\backslash\{1\}$ from (2.15). Also as in Theorem 2.1 we have $j_1\, \mu_{1,n}\, j_n^{-1}(y_n) \in F_1\,(j_1\, \mu_{1,n}\, j_n^{-1}(y_n))$ in E_1 and so $j_1\, \mu_{1,n}\, j_n^{-1}\,(y_n) \in \sum_1$ for $n \in N$. Now since $\sum_1$ is compact there is a subsequence $N_1^\star$ of N and a $z_1 \in \sum_1$ with $j_1\, \mu_{1,n}\, j_n^{-1}\,(y_n) \to z_1$ in E_1 as $n \to \infty$ in $N_1^\star$ and $z_1 \in F_1\, z_1$ since F_1 is upper semicontinuous. Also

(2.14) implies $z_1 \in U_1$. Let $N_1 = N_1^\star \setminus \{1\}$. Proceed inductively to obtain subsequences of integers

$$N_1^\star \supseteq N_2^\star \supseteq ......, \quad N_k^\star \subseteq \{k, k+1, ...\}$$

and $z_k \in \sum_k$ with $j_k\,\mu_{k,n}\,j_n^{-1}\,(y_n) \to z_k$ in E_k as $n \to \infty$ in $N_k^\star$ and $z_k \in F_k\,z_k$. Also (2.14) implies $z_k \in U_k$. Note $j_k\,\mu_{k,k+1}\,j_{k+1}^{-1}\,z_{k+1} = z_k$ in E_k for $k \in \{1, 2, ...\}$. Also let $N_k = N_k^\star \setminus \{k\}$.

Fix $k \in N$. Now $z_k \in F_k\,z_k$ in E_k. Note as well that

$$\begin{aligned} z_k &= j_k\,\mu_{k,k+1}\,j_{k+1}^{-1}\,z_{k+1} = j_k\,\mu_{k,k+1}\,j_{k+1}^{-1}\,j_{k+1}\,\mu_{k+1,k+2}\,j_{k+2}^{-1}\,z_{k+2} \\ &= j_k\,\mu_{k,k+2}\,j_{k+2}^{-1}\,z_{k+2} = ... = j_k\,\mu_{k,m}\,j_m^{-1}\,z_m = \pi_{k,m}\,z_m \end{aligned}$$

for every $m \geq k$. We can do this for each $k \in N$. As a result $y = (z_k) \in \lim_{\leftarrow} E_n = E$ and also note $z_k \in U_k \subseteq Y_k$ for each $k \in N$. Thus for each $k \in N$ we have

$$j_k\,\mu_k\,(y) = z_k \in F_k\,z_k = j_k\,\mu_k\,F\,y \quad \text{in} \quad E_k$$

so $y \in F\,y$ in E. □

Remark 2.6. In Theorem 2.2 it is possible to replace $\overline{\overline{C_n} \cap V_n} \subseteq Y_n$ with $\overline{\overline{C_n} \cap V_n}$ a subset of the closure of Y_n in E_n provided Y is a closed subset of E so in this case we could have $Y = C \cap \overline{V}$ if $\overline{\overline{C_n} \cap V_n}$ is a subset of the closure of $j_n\,\mu_n\,(C \cap \overline{V})$ in E_n and if C is closed. To see this note from $y = (z_k) \in \lim_{\leftarrow} E_n = E$ and $\pi_{k,m}\,(y_m) \to z_k$ in E_k as $m \to \infty$ we can conclude that $y \in \overline{Y} = Y$ (note $q \in \overline{Y}$ iff for every $k \in N$ there exists $(x_{k,m}) \in Y$, $x_{k,m} = \pi_{k,n}\,(x_{n,m})$ for $n \geq k$ with $x_{k,m} \to j_k\,\mu_k\,(q)$ in E_k as $m \to \infty$). Thus $z_k = j_k\,\mu_k\,(y) \in Y_k$ and so $j_k\,\mu_k\,(y) \in j_k\,\mu_k\,F\,(y)$ in E_k as before. Also it is easy to see with the above argument that $\lambda \in (0, 1]$ in (2.14) can be replaced by $\lambda \in (0, 1)$ again provided $\overline{\overline{C_n} \cap V_n}$ is a subset of the closure of Y_n in E_n and Y is a closed subset of E.

References

[1] R.P. Agarwal, J. H. Dshalalow and D. O'Regan, Fixed point theory for Mönch type maps defined on closed subsets of Fréchet spaces: the projective limit approach, *Int. Jour. Math. Math. Sciences,* **17**(2005), 2775-2782.

[2] R.P. Agarwal, J. H. Dshalalow and D. O'Regan, Leray-Schauder principles for inward Kakutani Mönch type maps, *Nonlinear Functional Analysis and Applications,* **10**(2005), 325–330.

[3] M. Frigon and D. O'Regan, A Leray-Schauder alternative for Mönch maps on closed subsets of Fréchet spaces, *Zeitschrift Anal. Anwendungen,* **21**(2001), 753–760.

[4] L.V. Kantorovich and G.P. Akilov, Functional analysis in normed spaces, *Pergamon Press*, Oxford, 1964.

[5] D. O'Regan, Leray-Schauder results for inward acyclic and approximable maps defined on Fréchet space, *Applied Math. Letters,* **19**(2006), 976–982.

[6] D. O'Regan and R. Precup, Fixed point theory for set valued maps and existence principles for integral inclusions, *Jour. Math. Anal. Appl.,* **245**(2000), 594–612.

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Application of two spectral methods to a problem of convection with uniform internal heat source

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ABSTRACT: Two methods based on Fourier series expansions (a Chandrasekhar functions - based method and a shifted Legendre polynomials - based method) are used to study analytically the eigenvalue problem governing the linear convection problem with an uniform internal heat source in a horizontal fluid layer bounded by two rigid walls. For each method some theoretical remarks are made. Numerical results are given and they are compared with some existing ones. Good agrement is found

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1. Problem setting

The effects of the presence in a fluid of an internal heat source have been experimentally, numerically and analytically investigated by researchers in many convection problems [6], [7], [8],[9]. The investigations concerned the effects of the heating and cooling rate. Various conditions were imposed on the lower and upper boundaries. The motion in the atmosphere or mantle convection are two among phenomena of natural convection induced by internal heat sources. They bifurcate from the conduction state as a result of its loss of stability. In spite of their importance, due to the occurrence of variable coefficients in the nonlinear partial differential equations governing the evolution of the perturbations around the basic equilibrium, so far these phenomena were treated mostly numerically and experimentally.

Herein a horizontal layer of viscous incompressible fluid with constant viscosity and thermal conductivity coefficients ν and k is considered [9]. In this context, the heat and hydrostatic transfer equations are [9]

$$\eta = k\frac{\partial^2\theta_B}{\partial z^2}, \tag{1}$$

$$\frac{dp_B}{dz} = -\rho_B g, \tag{2}$$

where $\eta = const.$ is the heating rate, θ_B, p_B and ρ_B are the potential temperature, pressure and density in the basic state, respectively. In the fluid, the temperature at all point varies at the same rate as the boundary temperature, so the problem is characterized by a constant potential temperature difference between the lower and the upper boundaries $\Delta\theta_B = \theta_{B_0} - \theta_{B_1}$. Taking into account (1) this leads to the following formula for the potential temperature distribution [9]

$$\theta_B = \theta_{B_0} - \frac{\Delta\theta_B}{h}\Big(z + \frac{h}{2}\Big) + \frac{\eta}{2k}\Big[z^2 - \Big(\frac{h^2}{2}\Big)^2\Big]. \tag{3}$$

In nondimensional variables the system of equations characterizing the problem is

$$\begin{cases} \dfrac{d\mathbf{U}}{dt} = -\nabla p' + \Delta\mathbf{U} + Gr\theta'\mathbf{k}, \\ \mathrm{div}\mathbf{U} = 0, \\ \dfrac{d\theta'}{dt} = (1 - Nz)\mathbf{U}\mathbf{k} + Pr^{-1}\Delta\theta', \end{cases} \tag{4}$$

where $\mathbf{U} = (u, v, w)$ is the velocity, θ' and p' are the temperature and pressure deviations from the basic state [9], Gr is the Grashof number, Pr is the Prandtl number and N is a nondimensional parameter characterizing the heating (cooling) rate of the layer.

The boundaries are assumed rigid and ideal heat conducting, so the boundary conditions read

$$\mathbf{U} = \theta' = 0 \quad \text{at} \quad z = -\frac{1}{2} \quad \text{and} \quad z = \frac{1}{2}. \tag{5}$$

In [9] the numerical investigations concerned the vertical distribution of the total heat fluxes and their individual components for small and moderate supercritical Rayleigh number in the presence of a uniform heat source.

The eigenvalue problem associated with the equations for a convection problem with an uniform internal heat source in a horizontal fluid layer bounded by two rigid walls was deduced in [2].

Consider the viscous incompressible fluid confined into a periodicity rectangular box $V : 0 \le x \le a_1,\ 0 \le y \le a_2,\ -\frac{1}{2} \le z \le \frac{1}{2}$ [4] bounded by two rigid horizontal walls. The corresponding eigenvalue problem [2] has the form

$$\begin{cases} (D^2 - a^2)^2 W - a^2 Ra\Theta = 0, \\ (D^2 - a^2)\Theta + (1 - Nz)W = 0. \end{cases} \tag{6}$$

with the boundary conditions

$$W = DW = \Theta = 0 \quad \text{at} \quad z = -\frac{1}{2} \quad \text{and} \quad z = \frac{1}{2}. \tag{7}$$

In (6) the Rayleigh number Ra represents the eigenvalue while (W, Θ) represents the corresponding eigenvector. The analytical study of this stability problem consists in finding the smallest eigenvalue, i.e. the critical value of the Rayleigh number at which the convection sets in.

In [2] the analytical study of the eigenvalue problem (6)-(7) was performed by means of a method from [1]. First the system (6)-(7) was written in a more convenient independent variable $x = z + \frac{1}{2}$. Then, two methods (one based on Fourier series expansions of the unknown functions and other a variational one) were used in order to find the smallest eigenvalue. Here, the analytical study in also based on Fourier series expansions of the unknown functions, but the expansion functions satisfy all boundary conditions.

Taking into account the form of the boundary conditions two methods are used and, for each of them, some analytical remarks on the chosen sets of expansion functions are presented.

2. A method based on Chandrasekhar functions

In this method, the unknown function W is expanded upon a complete set of orthogonal functions that satisfy all boundary conditions $\left(W = DW = 0 \text{ at } z = \pm\frac{1}{2}\right)$ and then, from $(6)_2$ we find the expression of the unknown function Θ. Replacing these expansions in $(6)_1$ and imposing the condition that the left-hand side of the obtained equation to be orthogonal to each function from the expansion set, we obtain an algebraic system of equations which leads us to the secular equation, yielding the critical value of the Rayleigh number.

When the normal component of the velocity and its derivative are zero at $z = -\frac{1}{2}$ and $z = \frac{1}{2}$, the classical set of complete orthogonal functions that satisfy these conditions are the Chandrasekhar sets of functions $\{C_n\}_{n\in\mathbb{N}}$, $\{S_n\}_{n\in\mathbb{N}}$[1]

$$C_n(z) = \frac{\cosh \lambda_n z}{\cosh \lambda_n/2} - \frac{\cos \lambda_n z}{\cos \lambda_n/2}, \tag{8}$$

$$S_n(z) = \frac{\sinh(\mu_n z)}{\sinh(\mu_n/2)} - \frac{\sin(\mu_n z)}{\sin(\mu_n/2)} \tag{9}$$

where λ_n and μ_n are the positive roots of the equations $\tanh\left(\frac{\lambda}{2}\right) + \tan\left(\frac{\lambda}{2}\right) = 0$ and $\coth\left(\frac{\mu}{2}\right) - \cot\left(\frac{\mu}{2}\right) = 0$. We have

$$\int_{-0.5}^{0.5} C_n(z)C_m(z)dz = \int_{-0.5}^{0.5} S_n(z)S_m(z)dz = \delta_{mn}.$$

By definition, the functions C_n and S_n and their derivatives vanish at $z = \pm\frac{1}{2}$ so the boundary conditions (7) are satisfied.

Let us consider $W = \sum_{n=1}^{\infty} W_n C_n(z)$. From $(6)_2$ we obtain the expression of the unknown function Θ,

$$\Theta = A\cosh az + B\sinh az + \frac{W_n \cosh\lambda_n z(Nz-1)}{(\lambda_n^2 - a^2)\cosh\lambda_n/2} - \frac{2\lambda_n N W_n}{(\lambda_n^2-a^2)^2\cosh\lambda_n/2}\cdot$$
$$\cdot\sinh\lambda_n z + \frac{(1-Nz)W_n\cos\lambda_n z}{(\lambda_n^2+a^2)\cos\lambda_{n/2}} - \frac{2\lambda_n N W_n}{(\lambda_n^2+a^2)^2\cos\lambda_n/2}\sin\lambda_n z,$$

where $A = \dfrac{2a^2 W_n}{(\lambda_n^2-a^2)(\lambda_n^2+a^2)\cosh a/2}$ and

$$B = \frac{8\lambda_n^3 N W_n a^2}{(\lambda_n^2-a^2)^2(\lambda_n^2+a^2)^2\cosh\lambda_n/2} - \frac{a^2 N W_n}{(\lambda_n^2-a^2)(\lambda_n^2+a^2)}.$$

However, in our case, replacing these expressions in $(6)_1$ and imposing the condition that the left-hand side of the obtained equation to be orthogonal to C_m, $m \in \mathbb{N}$, we obtain an expression in which the physical parameter N is missing. The mathematical explanation is that the chosen set of expansion functions introduced an extraparity (inexistent in the given problem), leading to the loss of one of the physical parameter, in this case the cooling (heating) rate N.

Remark 1. The physical parameter N also disappear when the expansion functions are S_n, $n = 1, 2, ...$

Another explanation could be the fact that we have no physical or mathematical reason to assume that W is either even or odd. The general form of W, $W(z) = \sum_{n=1}^{\infty} C_n(z)W_n^1 + S_n(z)W_n^2$, will be considered elsewhere.

3. A method based on shifted Legendre polynomials

In order to avoid the loss of N, we use a different set of orthogonal functions, namely a basis of shifted Legendre polynomials (SLP) on $[0, 1]$.

Let us modify the system (6) by a translation of the variable z, $x = z + \dfrac{1}{2}$, such that the eigenvalue problem becomes

$$\begin{cases} (D^2-a^2)^2 W - a^2 Ra\Theta = 0, \\ (D^2-a^2)\Theta + (N_1 - Nx)W = 0, \end{cases} \tag{10}$$

with $N_1 = 1 + \dfrac{N}{2}$ and the boundary conditions

$$W = DW = \Theta = 0 \quad \text{at} \quad x = 0 \quad \text{and} \quad 1. \tag{11}$$

Starting with the classical Legendre polynomials defined on $(-1, 1)$, let us introduce the complete sets of expansion functions. We are interested in expansion functions that satisfy all boundary conditions. Let $H_0^1(0,1)$, $H_0^2(0,1)$ be two Hilbert spaces [5]

$$H_0^1(0,1) = \{f|f, f' \in L^2(0,1), f(0) = f(1) = 0\},$$

$$H_0^2(0,1) = \{f|f, f', f'' \in L^2(0,1), f(0) = f(1) = f'(0) = f'(1) = 0\}$$

and denote by L_k the Legendre polynomials defined on $(-1,1)$. By means of them, we construct the SLP (denoted by us by Q_k) on (a,b), namely $Q_k(x) = L_k\Big(\dfrac{2x-a-b}{b-a}\Big)$. Taking $(a,b) = (0,1)$ we find that Q_k are orthogonal polynomials on the interval $(0,1)$, i.e. $\displaystyle\int_0^1 Q_iQ_jdx = \frac{1}{2i+1}\delta_{ij}$. Using the identity [5]

$$2(2i+1)Q_i(x) = Q'_{i+1}(x) - Q'_{i-1}(x). \tag{12}$$

we define the complete sets of orthogonal functions $\{\phi_i\}_{i=1,2,...} \subset H_0^1(0,1)$,

$$\phi_i(x) = \int_0^x Q_i(t)dt = \frac{Q_{i+1} - Q_{i-1}}{2(2i+1)},$$

satisfying boundary conditions $\phi_i(0) = \phi_i(1) = 0$ at $x = 0$ and 1 and $\{\beta_i\}_{i=1,2,...} \subset H_0^2(0,1)$,

$$\beta_i(x) = \int_0^x\int_0^s Q_{i+1}(t)dtds = \frac{1}{4}\Big[\frac{Q_{i+3} - Q_{i+1}}{(2i+3)(2i+5)} - \frac{Q_{i+1} - Q_{i-1}}{(2i+1)(2i+3)}\Big],$$

satisfying boundary conditions $\beta_i(0) = \beta_i(1) = \beta'_i(0) = \beta'_i(1) = 0$ at $x = 0$ and 1.

Remark 2. We could also work with SLP on $(a,b) = \Big(-\dfrac{1}{2}, \dfrac{1}{2}\Big)$. However, the choice $(a,b) = (0,1)$ leads us to simplified numerical evaluations.

The system (6) can be solved numerically by approximating the solution (W,Θ) by

$$W = \sum_{i=1}^n W_i\beta_i(x), \quad \Theta = \sum_{i=1}^n \Theta_i\phi_i(x) \tag{13}$$

with W_i and Θ_i the Fourier coefficients. In this way, the system (6) can be written in terms of the expansion functions only

$$\begin{cases} \sum\limits_{i=1}^n [W_i(D^2-a^2)^2\beta_i - a^2Ra\Theta_i\phi_i] = 0, \\ \sum\limits_{i=1}^n [\Theta_i(D^2-a^2)\phi_i + (N_1 - Nz)W_i\beta_i] = 0. \end{cases} \tag{14}$$

Multiplying the system (14) by the vector (β_k, ϕ_k) we obtain the algebraic system

$$\begin{cases} \sum\limits_{i=1}^n [W_i\Big((D^2-a^2)^2\beta_i, \beta_k\Big) - a^2Ra\Theta_i(\phi_i, \beta_k)] = 0, \\ \sum\limits_{i=1}^n [\Theta_i\Big((D^2-a^2)\phi_i, \phi_k\Big) + W_iN_1(\beta_i, \phi_k) - W_iN(z\beta_i, \phi_k)] = 0. \end{cases} \tag{15}$$

Taking into account the fact that the coefficients W_i, Θ_i are not all null, i.e. the Cramer determinant vanishes, the secular equation has the form

$$\begin{vmatrix} ((D^2-a^2)^2\beta_i,\beta_k) & -a^2Ra(\phi_i,\beta_k) \\ N_1(\beta_i,\phi_k)-N(z\beta_i,\phi_k) & ((D^2-a^2)\phi_i,\phi_k) \end{vmatrix} = 0. \quad (16)$$

The scalar products from (16) are given in the Appendix.

The system (10) has variable coefficients (functions of x). In this case, the following recurrence relation was used for the numerical study

$$2xQ_i = \frac{i+1}{2i+1}Q_{i+1} + Q_i + \frac{i}{2i+1}Q_{i-1}. \quad (17)$$

4. Numerical results

Taking $n = m = 1$ we obtained a first approximation of the Rayleigh number, which proved to be a good approximation compared to the one obtained in [2]. The obtained numerical results are presented in Table 1 in comparison with the results from [2].

N	a^2	$Ra - Fourier$	$R_a - var.meth.$	$Ra - Legendre$
0	9.711	1715.079324	1749.97575	1749.95727
1	9.711	1711.742588	1746.804944	1746.809422
2	9.711	1701.891001	1737.45025	1737.450242
1	10.0	1712.257687	1747.29100	1747.290998
4	10.0	1664.341789	1701.62704	1701.627037
4	12.0	1685.422373	1723.62407	1723.624047
8	12.0	1547.460446	1590.19681	1590.196769
9	12.0	1508.147637	1551.72378	1551.723746
10	12.0	1468.449223	1512.69203	1512.691998
12	12	1389.837162	1434.90396	1434.903926
16	12	1243.442054	1288.50149	1288.501459
10	9.0	1482.527042	1525.59302	1525.593072
11	9.0	1446.915467	1490.55802	1490.558078
12	9.00	1411.401914	1455.48233	1455.482384

Table 1. Numerical evaluations of the Rayleigh number for various values of the parameters N and a.

The disadvantage of this method is given by the fact that the approximations are limited by the difficult evaluation of the associated matrix for a large number of functions in the expansion sets. However, the expressions of the neutral manifolds are easy to obtain with this method.

When the wavenumber is kept constant an increase in the heating (cooling) rate parameter leads to a decreasing of the Rayleigh number. When $N = 0$ the problem reduces to the particular case of Rayleigh-Bénard convection and the numerical evaluation lead us to a value similar to the classical value for the Rayleigh number, i.e. $Ra = 1749.95727$ for $a = 3.117$.

5. Appendix

Let us give the expressions of the scalar products occurring in (16). Since in $(10)_1$ the expression $((D^2 - a^2)^2\beta_i, \beta_k)$ is written as

$$((D^2 - a^2)^2\beta_i, \beta_k) = (D^4\beta_i, \beta_k) - 2a^2(D^2\beta_i, \beta_k) + a^4(\beta_i, \beta_k)$$

let us simplify these products or simply evaluate them. Taking into account the definition of the scalar product on $L^2(0,1)$, i.e. $(f,g) = \int_0^1 fg dx$ and the boundary conditions satisfied by the expansion functions, we have

$$(D^4\beta_i, \beta_k) = (\beta_i'', \beta_k'') = \begin{cases} \dfrac{1}{2i+3} \text{if} i = k, \\ 0 \text{if} i \neq k \end{cases} \tag{18}$$

and

$$(D^2\beta_i, \beta_k) = -(\beta_i', \beta_k') = \begin{cases} -\dfrac{1}{2(2i+1)(2i+3)(2i+5)} \text{if} i = k, \\ \dfrac{1}{4(2i-1)(2i+1)(2i+3)} \text{if} i = k+2, \\ 0 \text{otherwise} \end{cases} \tag{19}$$

Given the fact that $(\beta_i, \beta_k) = \frac{1}{2}\Big(\frac{\phi_{i+2} - \phi_i}{2i+3}, \frac{\phi_{k+2} - \phi_k}{2k+3}\Big)$ we first evaluated the product (ϕ_i, ϕ_k) and we get

$$(\phi_i, \phi_k) = \begin{cases} \dfrac{1}{2(2i-1)(2i+1)(2i+3)} \text{if} i = k, \\ -\dfrac{1}{4(2i+1)(2i+3)(2i+5)} \text{if} i = k-2, \\ 0 \text{otherwise} \end{cases} \tag{20}$$

Using (20) we have

$$(\beta_i, \beta_k) = \begin{cases} \dfrac{3}{8(2i-1)(2i+1)(2i+3)(2i+5)(2i+7)} \text{if} i = k, \\ -\dfrac{1}{4(2i+1)(2i+3)(2i+5)(2i+7)(2i+9)} \text{if} i = k-2, \\ \dfrac{1}{16(2i+3)(2i+5)(2i+7)(2i+9)(2i+11)} \text{if} i = k-4, \\ 0 \text{otherwise} \end{cases} \tag{21}$$

We also used (20) to deduce (ϕ_i, β_k), i.e.

$$(\phi_i,\beta_k)=\begin{cases} -\dfrac{3}{8(2i-1)(2i+1)(2i+3)(2i+5)} \text{if} i=k, \\ \dfrac{3}{8(2i-3)(2i-1)(2i+1)(2i+3)} \text{if} i=k+2, \\ \dfrac{1}{8(2i+1)(2i+3)(2i+5)(2i+7)} \text{if} i=k-2, \\ -\dfrac{1}{8(2i-5)(2i-3)(2i-1)(2i+1)} \text{if} i=k+4, \\ 0 \text{otherwise} \end{cases} \tag{22}$$

Let us remark that $(\beta_i, \phi_k) = (\phi_k, \beta_i)$.

The computation of $(D^2\phi_i, \phi_k)$ was simplified by the expressions of the ϕ_i functions. We have

$$(D^2\phi_i,\phi_k)=-(Q_i,Q_k)=\begin{cases} -\dfrac{1}{2i+1} \text{if} i=k, \\ 0 \text{otherwise} \end{cases}. \tag{23}$$

All the obtained expressions (18) - (24) are based on the orthogonality relationship between the SLP. In deducing the expression below we also used the recurrence relation

(17)

$$
(z\beta_i, \phi_k) = \begin{cases}
-\dfrac{i+4}{16(2i+3)(2i+5)(2i+7)(2i+9)(2i+11)} \text{if} i = k-5, \\
-\dfrac{1}{16(2i+3)(2i+5)(2i+7)(2i+9)} \text{if} i = k-4, \\
\dfrac{1}{16(2i+1)(2i+3)(2i+5)(2i+9)} \text{if} i = k-3, \\
\dfrac{3}{16(2i+1)(2i+3)(2i+5)(2i+7)} \text{if} i = k-2, \\
-\dfrac{3}{16(2i-1)(2i+1)(2i+3)(2i+5)(2i+7)} \text{if} i = k-1, \\
-\dfrac{3}{16(2i-1)(2i+1)(2i+3)(2i+5)} \text{if} i = k, \\
-\dfrac{1}{16(2i-3)(2i+1)(2i+3)(2i+5)} \text{if} i = k+1, \\
\dfrac{1}{16(2i-3)(2i-1)(2i+1)(2i+3)} \text{if} i = k+2, \\
\dfrac{i+1}{16(2i-5)(2i-3)(2i-1)(2i+1)(2i+3)} \text{if} i = k+3 \\
0 \text{otherwise}
\end{cases} \tag{24}
$$

6. Conclusions

In this paper we performed an analytical study of the eigenvalue problem corresponding to a convection problem with uniform internal heat source. We pointed out some aspects of the spectral methods that we employed concerning the sets of the expansion functions that can be used to an analytical study of this problem. As in this case the expansion sets of Chandrasekhar functions introduced an extraparity they were not appropriate. However, for some other problems [3] their use proved to be successful. The method based on SLP lead to good numerical approximations. All numerical results obtained with this method are compared with the existing ones. The effect of the heating (cooling) rate on the values of the Rayleigh number is pointed out.

References

[1] Chandrasekar, S., *Hydrodynamic and hydromagnetic stability*, Oxford University Press, 1961.

[2] Dragomirescu I., Georgescu, A., *Stability bounds in a problem of convection with uniform internal heat source*, Proceedings of the 6^{th} ICNPAA, Ed. by Seenith Sivasundaram, Cambridge Scientific Publishers Ltd, 2007, 163-170

[3] Dragomirescu, I., *On the Chandrasekhar - Galerkin method in a convection problem for a micropolar fluid*, Univ. "Politehnica" of Timisoara, St.Bull.Math.-Phys., **50**(64), 2 (2005), 53-61.

[4] Georgescu, A., Mansutti, *Coincidence of the linear and non-linear stability bounds in a horizontal thermal convection problem*, Int. J. of Non-linear Mechanics, **34** (1999), 603-613.

[5] Hill, A.A., Straughan, B., *A Legendre spectral element method for eigenvalues in hydromagnetic stability*, J. of Computational and Applied Mathematics, **193** (2003), 363-381.

[6] Ming-I Char, Chiang, Ko-Ta, *Stability analysis of Bénard-Marangoni convection in fluids with internal heat generation*, J. Phys D: Appl Phys, **27**, 4 (1994), 748-755.

[7] Roberts, P.H., *Convection in horizontal layers with internal heat generation. Theory*, J. Fluid Mech, **30** (1967) 33-49.

[8] Tritton, D.J., Zarraga, M.N., *Convection in horizontal layers with internal heat generation. Experiments*, J. Fluid Mech, **30** (1967), 21-31.

[9] Vel'tishchev, N.F., *Convection in a horizontal fluid layer with a uniform heat source*, Fluid Dynamics, **39**, 2 (2004), 189-197.

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Classes of functions defined by subordination

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ABSTRACT: In the paper, we define classes of analytic functions, in terms of subordination. We present some inclusion relations for defined classes

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1. Introduction

Let $\mathcal{A}$ denote the class of functions which are *analytic* in $\mathcal{U} := \mathcal{U}(1)$, where

$$\mathcal{U}(r) := \{z : z \in \mathbf{C} \ \ and \ |z| < r\}.$$

By $\mathcal{A}_0$ we denote class of functions $f \in A$ of the form:

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n, \tag{1}$$

We say that a function $f \in \mathcal{A}$ is *subordinate* to a function $F \in \mathcal{A}$, and write $f(z) \prec F(z)$, if and only if there exists a function $\omega \in \mathcal{A}$,

$$\omega(0) = 0, \ |\omega(z)| < 1 \quad (z \in \mathcal{U}),$$

such that

$$f(z) = F(\omega(z)) \quad (z \in \mathcal{U}).$$

Moreover, we say that f is subordinate to F in $\mathcal{U}(r)$, if $f(r\,z) \prec F(r\,z)$. We shall write

$$f(z) \prec_r F(z)$$

in this case. In particular, if F is univalent in $\mathcal{U}$ we have the following equivalence (cf. [8]):

$$f(z) \prec F(z) \iff f(0) = F(0) \text{ and } f(\mathcal{U}) \subset F(\mathcal{U}).$$

A function f belonging to the class $\mathcal{A}$ is said to be *convex* in $\mathcal{U}(r)$ if and only if

$$\Re\left(1 + \frac{zf''(z)}{f'(z)}\right) > 0 \qquad (z \in \mathcal{U}(r);\ 0 < r \leqq 1). \tag{2}$$

By $f * g$ denote the *Hadamard product* (or *convolution*) of f, $g \in \mathcal{A}$, defined by

$$(f * g)(z) = \left(\sum_{n=1}^{\infty} a_n z^n\right) * \left(\sum_{n=1}^{\infty} b_n z^n\right) := \sum_{n=1}^{\infty} a_n b_n z^n.$$

Let λ be complex number. We consider the linear operator $D^\lambda : \mathcal{A} \to A$ defined by (see [3])

$$D^\lambda f(z) = (f * h_\lambda)(z),$$

where

$$h_\lambda(z) = \sum_{n=0}^{\infty} n^\lambda z^n \quad (z \in \mathcal{U}).$$

For a function $f \in \mathcal{A}_0$ of the form (1) we have

$$D^\lambda f(z) = z + \sum_{n=2}^{\infty} n^\lambda a_n z^n$$

and

$$D^{\lambda+1} f(z) = z\left[D^\lambda f(z)\right]'. \tag{3}$$

Let h be a function convex in $\mathcal{U}$ with $h(0) = 1$ and let t be complex number.

We denote by $\mathcal{V}\ (t, \lambda; h)$ the class of functions $f \in \mathcal{A}_0$ satisfying the following condition:

$$z^{-1}\left[(1-t)\, D^\lambda f(z) + tD^{\lambda+1} f(z)\right] \prec h(z), \tag{4}$$

in terms of subordination.

Moreover we define class $\mathcal{W}(t, \lambda; h)$ of functions $f \in \mathcal{A}_0$ satisfying the following condition:

$$\frac{(1-t)\, D^{\lambda+1} f(z) + tD^{\lambda+2} f(z)}{(1-t)\, D^\lambda f(z) + tD^{\lambda+1} f(z)} \prec h(z). \tag{5}$$

In particular for

$$h(z) = \frac{1 + Az}{1 + Bz} \quad (z \in \mathcal{U};\ 0 \leqq B \leqq 1,\ -B \leqq A < B)$$

we obtain the class

$$\mathcal{V}(t, \lambda; A, B) = \mathcal{V}\left(t, \lambda; \frac{1 + Az}{1 + Bz}\right)$$

which was studied by Dziok [3]. Moreover we denote

$$\mathcal{W}(t,\lambda;A,B)=\mathcal{W}\left(t,\lambda;\frac{1+Az}{1+Bz}\right).$$

For suitable chosen parameters t,λ,A,B classes defined above was investigated among overs by Stankiewicz *et al.* ([7], [11], [9] and [10]).

In the paper we present some inclusion relations for defined classes.

2. Main results

We shall need the following lemmas.

Lemma 1. [6] *Let w be a nonconstant function analytic in $\mathcal{U}(r)$ with $w(0)=0$. If*

$$|w(z_0)|=\max\{|w(z)|;\ |z|\le|z_0|\}\quad (z_0\in\mathcal{U}(r)),$$

then there exists a real number k $(k\ge 1)$, such that

$$z_0w'(z_0)=kw(z_0).$$

Lemma 2. [5] *Let h be a convex function in $\mathcal{U}$ with $h(0)=1$. If q is an analytic function in $\mathcal{U}$, $q(0)=1$ and*

$$q(z)+zq'(z)\prec h(z),$$

then

$$q(z)\prec h(z).$$

Lemma 3. [4] *Let h be a convex function in $\mathcal{U}(r)$ with $h(0)=1$. If q is an analytic function in $\mathcal{U}(r)$, $q(0)=1$ and*

$$q(z)+\frac{zq'(z)}{q(z)}\prec_r\frac{1+Az}{1+Bz},$$

then

$$q(z)\prec_r\frac{1+Az}{1+Bz}.$$

Making use of above lemmas , we get the following two theorem.

Theorem 1.

$$\mathcal{V}(t,\lambda+m;h)\subset\mathcal{V}(t,\lambda;h)\quad (m\in\mathbf{N}).$$

Proof. It is clear that it is sufficient to prove the theorem for $m=1$. Let a function f belong to the class $\mathcal{V}(t,\lambda+1;h)$ or equivalently

$$z^{-1}\left[(1-t)\,D^{\lambda+1}f(z)+tD^{\lambda+2}f(z)\right]\prec h(z),\tag{6}$$

It is sufficient to verify the condition (4). The function

$$q(z) = z^{-1}\left[(1-t)\,D^{\lambda}f(z) + tD^{\lambda+1}f(z)\right] \tag{7}$$

is analytic in $\mathcal{U}$ and $q(0)=1$. Taking the derivative of (7) we get

$$z^{-1}\left[(1-t)\,D^{\lambda+1}f(z) + tD^{\lambda+2}f(z)\right] = q(z) + zq'(z) \quad (z \in \mathcal{U})\,. \tag{8}$$

Thus by (6) we have

$$q(z) + zq'(z) \prec h(z).$$

Lemma 2 now yields

$$q(z) \prec h(z). \tag{9}$$

Thus by (7) $f \in \mathcal{V}(t,\lambda;h)$ and this proves Theorem 1.

Putting $h(z) = \frac{1+Az}{1+Bz}$ in Theorem 1 we obtain the following two corollary.

Corollary 1.

$$\mathcal{V}(t,\lambda+m;A,B) \subset \mathcal{V}(t,\lambda;A,B) \quad (m \in \mathbf{N}).$$

Theorem 2.

$$\mathcal{W}(t,\lambda{+}m;A,B) \subset \mathcal{W}(t,\lambda;A,B) \quad (m \in \mathbf{N}).$$

Proof. It is clear that it is sufficient to prove the theorem for $m=1$. Let a function f belong to the class $\mathcal{V}(a+1;A,B)$ or equivalently

$$\frac{(1-t)\,D^{\lambda+2}f(z) + tD^{\lambda+3}f(z)}{(1-t)\,D^{\lambda+1}f(z) + tD^{\lambda+2}f(z)} \prec \frac{1+Az}{1+Bz} \tag{10}$$

It is sufficient to verify condition (5). If we put

$$R = \sup\left\{r:\ (1-t)\,D^{\lambda}f(z) + tD^{\lambda+1}f(z) \neq 0,\ z \in \mathcal{U}(r)\right\},$$

then the function

$$q(z) = \frac{(1-t)\,D^{\lambda+1}f(z) + tD^{\lambda+2}f(z)}{(1-t)\,D^{\lambda}f(z) + tD^{\lambda+1}f(z)} \tag{11}$$

is analytic in $\mathcal{U}(R)$ and $q(0)=1$. Taking the logarithmic derivative of (11) and applying (3) we get

$$\frac{(1-t)\,D^{\lambda+2}f(z) + tD^{\lambda+3}f(z)}{(1-t)\,D^{\lambda+1}f(z) + tD^{\lambda+2}f(z)} = q(z) + \frac{zq'(z)}{q(z)} \quad (z \in \mathcal{U}(R))\,. \tag{12}$$

Thus by (10) we have

$$q(z) + \frac{zq'(z)}{q(z)} \prec_R \frac{1+Az}{1+Bz}.$$

Lemma 3 now yields

$$q(z) \prec_R \frac{1+Az}{1+Bz}. \tag{13}$$

By (11) it suffices to verify that $R = 1$. From (3), (11) and (13) we conclude that the function $H(z) = (1-t)\, D^{\lambda} f(z) + tD^{\lambda+1} f(z)$ is starlike in $\mathcal{U}(R)$ and consequently it is univalent in $\mathcal{U}(R)$. Thus we see that $H(z)$ cannot vanish on $|z| = R$ if $R < 1$. Hence $R = 1$ and this proves Theorem 1.

Using Lemma 1 we show the following sufficient conditions for the class $\mathcal{W}(t, \lambda; A, B)$.

Theorem 3. *Let t, λ, A, B be real numbers $0 \leqq B \leqq 1$, $-B \leqq A < 2AB - B$. If a function $f \in \mathcal{A}_0$ satisfies the following inequality:*

$$\left| \frac{(1-t)\, D^{\lambda+2} f(z) + tD^{\lambda+3} f(z)}{(1-t)\, D^{\lambda+1} f(z) + tD^{\lambda+2} f(z)} - 1 \right| < \frac{2\,(B-A) + A^2 - 3AB}{(1+B)\,(1-A)} \qquad (z \in \mathcal{U})\,, \tag{14}$$

then f belongs to the class $\mathcal{W}(t, \lambda; A, B)$.

Proof. Let a function f belong to the class $\mathcal{A}_0$. Putting

$$q(z) = \frac{1 + Aw(z)}{1 + Bw(z)} \qquad (z \in \mathcal{U}(R)) \tag{15}$$

in (12), we obtain

$$\frac{(1-t)\, D^{\lambda+2} f(z) + tD^{\lambda+3} f(z)}{(1-t)\, D^{\lambda+1} f(z) + tD^{\lambda+2} f(z)} = \frac{1 + Aw(z)}{1 + Bw(z)} + \frac{Azw'(z)}{1 + Aw(z)} - \frac{Bzw'(z)}{1 + Bw(z)}.$$

Consequently, we have

$$F(z) = w(z) \left\{ \frac{zw'(z)}{w(z)} \left(\frac{A}{1 + Aw(z)} - \frac{B}{1 + Bw(z)} \right) - \frac{B - A}{1 + Bw(z)} \right\}, \tag{16}$$

where

$$F(z) = \frac{(1-t)\, D^{\lambda+2} f(z) + tD^{\lambda+3} f(z)}{(1-t)\, D^{\lambda+1} f(z) + tD^{\lambda+2} f(z)} - 1.$$

By (5), (11) and (15) it is sufficient to verify that w is analytic in U and

$$|w(z)| < 1 \quad (z \in \mathcal{U}).$$

Now, suppose that there exists a point $z_0 \in \mathcal{U}(R)$, such that

$$|w(z_0)| = 1,\ |w(z)| < 1 \ \ (|z| < |z_0|).$$

Then, applying Lemma 1, we can write

$$z_0 w'(z_0) = kw(z_0),\ w(z_0) = e^{i\theta} \quad (k \geq 1).$$

Combining these with (16) , we obtain

$$\begin{aligned} |F(z_0)| &= \left| k \left(\frac{-A}{1 + Ae^{i\theta}} + \frac{B}{1 + Be^{i\theta}} \right) + \frac{B - A}{1 + Be^{i\theta}} \right| \\ &\geq k \operatorname{Re} \left(\frac{-A}{1 + Ae^{i\theta}} + \frac{B}{1 + Be^{i\theta}} \right) + \frac{B - A}{1 + B} \\ &\geq k \left(\frac{-A}{1 - A} + \frac{B}{1 + B} \right) + \frac{B - A}{1 + B} \geq \frac{2\,(B-A) + A^2 - 3AB}{(1+B)\,(1-A)}. \end{aligned}$$

Since this result contradicts (14) we conclude that w is the analytic function in $\mathcal{U}(R)$ and $|w(z)| < 1$ $(z \in \mathcal{U}(R))$. Applying the same methods as in the proof of Theorem 2 we obtain $R = 1$, which completes the proof of Theorem 3.

Putting $t = 0$, $A = 2\alpha - 1$ and $B = 1$ in Theorem 2 and 3 we obtain the following two corollaries.

Corollary 2. *Let* $0 \leq \alpha < 1$, $m \in \mathbf{N}$. *If a function* $f \in \mathcal{A}_0$ *satisfies the following inequality:*

$$\operatorname{Re}\left\{\frac{D^{\lambda+m+1}f(z)}{D^{\lambda+m}f(z)}\right\} > \alpha \quad (z \in \mathcal{U}),$$

then

$$\operatorname{Re}\left\{\frac{D^{\lambda+1}f(z)}{D^{\lambda}f(z)}\right\} > \alpha \quad (z \in \mathcal{U}).$$

Corollary 3. *Let* $m \in \mathbf{N}$, $0 \leq \alpha < 2/3$. *If a function* $f \in \mathcal{A}_0$ *satisfies the following inequality:*

$$\left|\frac{D^{\lambda+2}f(z)}{D^{\lambda+1}f(z)} - 1\right| < \frac{4 - 7\alpha + 2\alpha^2}{2(1-\alpha)} \quad (z \in \mathcal{U}),$$

then

$$\operatorname{Re}\left\{\frac{D^{\lambda+1}f(z)}{D^{\lambda}f(z)}\right\} > \alpha \quad (z \in \mathcal{U}).$$

Remark 2. Putting $\lambda = 0$ or $\lambda = 1$ and $m = 1$ in Corollary 2 and 3 we obtain the sufficient conditions for starlikeness of order α and convexity of order α, respectively.

References

[1] J. Dziok, *Applications of the Jack Lemma*, Acta Math. Hungar. 105(2004), 93-102.-1mm

[2] J. Dziok, *On some applications of the Briot-Bouquet differential subordination*, Journal of Mathematical Analysis and Applications, 328(2007), 295-301.-1mm

[3] J. Dziok, *Some parametric family of functions*, Journal of Mathematics and Applications, 29(2007).-1mm

[4] P.J. Eenigenburg, S. S. Miller, P. T. Mocanu and O.M. Reade, *Second order differential inequalities in the complex plane*, J. Math. Anal. Appl. 65(1978), 289-305.-1mm

[5] D.J. Hallenbeck, R.Ruscheweyh, *Subordination by convex functions*, Proc. Amer. Math. Soc., 52(1975), 191-195.-1mm

[6] I. S. Jack, *Functions starlike and convex of order α,* J. London Math. Soc. 3 (1971), 469-474.-1mm

[7] R. Jurasińska, J. Stankiewicz, , *Special subclasses ofd Caratheodory functions and their applications,* Folia Sci. Univ. Techn. Resoviensis, Math. 10(1991), 37-45.-1mm

[8] S. S. Miller, P. T. Mocanu, *Differential subordinations: theory and applications*, Series on Monographs and Textbooks in Pure and Applied Mathematics, Vol. 225, Marcal Dekker, New York, 2000.-1mm

[9] J. Stankiewicz, *Some Extremal Methods in the Theory of Univalent Functions*, Towarzystwo Naukowe, Rzeszów, 1986.-1mm

[10] J. Stankiewicz, *Some remarks of the classes R(A,B),* Folia Sci. Univ. Techn. Resoviensis, 18(1995), 91-98.-1mm

[11] J. Stankiewicz, L. Trojnar-Spelina, *Some parametric family of functions*, Folia Sci. Univ. Techn. Resoviensis,-1mm 14(1992), 45-54.

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On a new sequence space related to the Orlicz sequence space

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ABSTRACT: The space $m(\phi)$ was introduced by sargent[14]and further studied by Malkowsky and Mursaleen [8] and Mursaleen [9]. In this paper we extend this space to $m(M, \phi, p)$ and study some of its properties, inclusions relations and its relation with the Orlicz sequence space l_M

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1. Preliminaries and introduction

Let $\mathcal{C}$ denote the space whose elements are finite sets of distinct positive integers. Given any element σ of $\mathcal{C}$, we denote by $c(\sigma)$ the sequence $\{c_n(\sigma)\}$ which is such that $c_n(\sigma) = 1$ if $n \in \sigma, c_n(\sigma) = 0$ otherwise. Further

$$\mathcal{C}_s := \left\{ \sigma \in \mathcal{C} : \sum_{n=1}^{\infty} c_n(\sigma) \leq s \right\} (cf[9]),$$

the set of those σ whose support has cardinality at most s, and

$$\Phi := \left\{ \phi = (\phi_n) \in \omega : \phi_1 > 0, \Delta\phi_k \geq 0 \text{ and } \Delta\left(\frac{\phi_k}{k}\right) \leq 0 \quad (k = 1, 2, \dots) \right\},$$

where $\Delta\phi_n = \phi_n - \phi_{n-1}$; and ω is the set of all real or complex sequences $x = (x_k)$.

For $\phi \in \Phi$, we define the following sequence space, introduced by Sargent[14],and further studied by Malkowsky and mursaleen [8] and Mursaleen [9].

$$m(\phi) := \left\{ x = (x_n) \in \omega : \sup_{s\geq 1} \sup_{\sigma \in \mathcal{C}_s} \left(\frac{1}{\phi_s} \sum_{k\in\sigma} \mid x_k \mid \right) < \infty \right\}.$$

Recently the space $m(\phi)$ was extended to $m(\phi, p)$ by Tripathy and Sen[16] as follows:

$$m(\phi, p) := \left\{ x = (x_k) \in \omega : \sup_{s \geq 1} \sup_{\sigma \in \mathcal{C}_s} \frac{1}{\phi_s} \sum_{k \in \sigma} | x_k |^p < \infty \right\}.$$

A map $M : \mathbb{R} \to [0, +\infty]$ is said to be an Orlicz function if M is even, convex, left continuous on $\mathbb{R}_+$, continuous at zero, $M(0) = 0$ and $M(u) \to \infty$ as $u \to \infty$. If M takes value zero only at zero we will write $M > 0$ and if M takes only finite values we will write $M < \infty$. [1,4,6,7,10,13].

W.Orlicz [11] used the idea of orlicz function to construct the space (L^M) . Lindendstrauss and Tzafriri [5] used the idea of Orlicz function to define orlicz sequence space

$$\ell_M := \left\{ x \in \omega : \sum_{k=1}^{\infty} M\left(\frac{|x_k|}{\rho}\right) < \infty \text{ for some } \rho > 0 \right\}$$

in more detail . ℓ_M is a Banach space with the norm

$$||x|| := \inf\{\rho > 0 : \sum_{k=1}^{\infty} M\left(\frac{|x_k|}{\rho}\right) \leq 1\}$$

The space l_M is closely related to the space l_p , which is an Orlicz sequence space with $M(x) = x^p \ \ for \ \ 1 \leq p < \infty$.

The $\triangle_2$ - condition is equivalent to

$$M(Lx) \leq KLM(x), \text{ for all values of } x \geq 0, \text{ and for } L > 1.$$

An Orlicz function M can always be represented in the following integral form $M(x) = \int_0^x \eta(t)dt$, where η is known as the kernel of M , is right differentiable for $t \geq 0$,$\eta(0) = 0$, $\eta(t) > 0$, η is non-decreasing and $\eta(t) \to \infty$ as $t \to \infty$.
Note that an Orlicz function satisfies the inequality

$$M(\lambda x) \leq \lambda M(x) \ \ for\ all \ \ \lambda \ with\ 0 < \lambda < 1.$$

The study of Orlicz sequence spaces have been made recently by various authors (cf [2],[3],[12],[15]). In this paper we defined the following sequence space

$$m(M, \phi) := \left\{ x \in \omega : \sup_{s \geq 1} \sup_{\sigma \in \mathcal{C}_s} \frac{1}{\phi_s} \sum_{k \in \sigma} M\left(\frac{|x_k|}{\rho}\right) < \infty \ \ for\ some\ \rho > 0 \right\}$$

$$m(M, \phi, p) := \left\{ x \in \omega : \sup_{s \geq 1} \sup_{\sigma \in \mathcal{C}_s} \frac{1}{\phi_s} \sum_{k \in \sigma} M\left(\frac{|x_k|^p}{\rho}\right) < \infty \ \ for\ some\ \rho > 0 \right\}$$

Remark 1. (i) If $\phi_n = 1$ for all $n = 1, 2, \cdots$; then $m(M, \phi) = l_M$ and $m(M, \phi, p) = l_p(M)$.

(ii) If $\phi_n = n$ $(n = 1, 2, \cdots)$, then $m(M, \phi, p) = m(M, \phi) = l_\infty(M)$.

2. Topological Properties

Let E be a sequence space . Then E is called

(i) A sequence space E is said to be symmetric if $(x_n) \in E$ implies $(x_\pi(n)) \in E$, where $\pi(n)$ is a permutation of the elements of the elements of $\mathbb{N}$.

(ii) Solid (or normal), if $(\alpha_k x_k) \in E$, whenever $(x_k) \in E$ for all sequences of scalars (α_k) with $|\alpha_k| \leq 1$ for all $k \in \mathbb{N}$.

Lemma. *A sequence space E is solid implies E is monotone.*

Theorem 2.1. *$m(M, \phi, p)$ is a linear space.*

Routine verification.

Theorem 2.2. *The space $m(M, \phi, p)$ is a complete space.*

Proof. To show the completeness, suppose that (x^i) be a cauchy sequence in $m(M, \phi, p)$, where $x^i = (x^i_k) = (x^i_1, x^i_2, x^i_3, \cdots) \in m(M, \phi, p)$ for all $i \in \mathbb{N}$. Let $r > 0$ and x_0 be fixed . Then for each $\frac{\epsilon}{rx_0} > 0$, there exists a positive integer n_0 such that

$$g(x^i - x^j) < \frac{\epsilon}{rx_0}, \quad for\ all\ \ i, j \ \geq n_0$$

implies that

$$\inf\left\{\rho : \sup_{s\geq 1} \sup_{\sigma\in\mathcal{C}_s} \frac{1}{\phi_s} \sum_{k\in\sigma} M\left(\frac{|x^i_k - x^j_k|^p}{\rho}\right) \leq 1\right\} < \epsilon,\ for\ all\ i, j \geq n_0. \tag{2.2.1}$$

By (2.2.1) $for\ all\ i, j \ \geq n_0$, we have

$$\sup_{s\geq 1} \sup_{\sigma\in\mathcal{C}_s} \frac{1}{\phi_s} \sum_{k\in\sigma} M\left(\frac{|x^i_k - x^j_k|^p}{g(x^i - x^j)}\right) \leq 1$$

which implies that

$$\frac{1}{\phi_1} M\left(\frac{|x^i_k - x^j_k|^p}{g(x^i - x^j)}\right) \leq 1$$

$$\Rightarrow \ M\left(\frac{|x_k^i - x_k^j|^p}{g(x^i - x^j)}\right) \leq \phi_1 \ \ for\ all\ i, j \ \ \geq n_0,\ and\ k \in \mathbb{N}.$$

For $r > 0$ we have $\frac{rx_0}{2}\eta(\frac{x_0}{2}) \geq \phi_1$, where η is the kernel associated with M, such that

$$M\left(\frac{|x_k^i - x_k^j|^p}{g(x^i - x^j)}\right) \leq \frac{rx_0}{2}\eta(\frac{x_0}{2})$$

$$\Rightarrow \ |x_n^i - x_k^j| < \frac{rx_0}{2}.\frac{\epsilon}{rx_0} = \frac{\epsilon}{2}.$$

Hence $(x_k^i)_{i=1}^{\infty}$ is a Cauchy sequence in $\mathbb{R}$, which is complete. For each $k \in \mathbb{N}$, there exists $x_k \in \mathbb{R}$ such that $|x_k^i - x_k| \to 0$ as $i \to \infty$. By the continuity of M , we have

$$\sup_{s\geq 1}\sup_{\sigma\in\mathcal{C}_s}\frac{1}{\phi_s}\sum_{k\in\sigma}M\left(\frac{|x_k^i - \lim\limits_{j\to\infty} x_k^j|^p}{\rho}\right) \leq 1$$

$$\sup_{s\geq 1}\sup_{\sigma\in\mathcal{C}_s}\frac{1}{\phi_s}\sum_{k\in\sigma}M\left(\frac{|x_k^i - x_k|^p}{\rho}\right) \leq 1 \ \ for\ some\ \rho > 0.$$

Taking the infimum of such $\rho' s$, by (2.2.1) we get

$$\inf\left\{\rho : \sup_{s\geq 1}\sup_{\sigma\in\mathcal{C}_s}\frac{1}{\phi_s}\sum_{k\in\sigma}M\left(\frac{|x_k^i - x_k|^p}{\rho}\right) \leq 1\right\} < \epsilon,\ for\ all\ i, j \geq n_0.$$

Since $m(M, \phi, p)$ is a linear space and $(x^{(i)})$ and $(x - x^i)$ are in $m(M, \phi, p)$, then we have

$$(x) = (x^i) + (x - x^i) \in m(M, \phi, p)$$

Thus $m(M, \phi, p)$ is complete .
medskip
Theorem 2.3. *The space $m(M, \phi, p)$ is solid, symmetric and monotone.*

Proof. Let $x \in m(M, \phi, p)$. Then we have

$$\sup_{s\geq 1}\sup_{\sigma\in\mathcal{C}_s}\frac{1}{\phi_s}\sum_{k\in\sigma}M\left(\frac{|x_k|^p}{\rho}\right) < \infty. \tag{2.3.1}$$

Now let (λ_k) be a sequence of scalars with $|\lambda_k| \leq 1$ for all $k \in \mathbb{N}$. Then from (2.3.1) we have

$$\sum_{k\in\sigma}M\left(\frac{|\lambda_k x_k|^p}{\rho}\right) \leq \sum_{k\in\sigma}|\lambda_k| M\left(\frac{|x_k|^p}{\rho}\right)$$

$$\leq \sum_{k\in\sigma}M\left(\frac{|x_k|^p}{\rho}\right).$$

Hence $m(M, \phi, p)$ is solid.

The symmetricity of the space follows from the definition of the space $m(M, \phi, p)$. By the Lemma it follows that the space $m(M, \phi, p)$ is monotone.

3. Inclusions Relations

Theorem 3.1. *$m(M, \phi, p) \subseteq m(M, \psi, p)$ if and only if* $\sup\limits_{s\geq 1}\left(\frac{\phi_s}{\psi_s}\right) < \infty$.

Proof. Let $\sup\limits_{s\geq 1}\left(\frac{\phi_s}{\psi_s}\right) < \infty$ and $x \in m(M, \phi, p)$. Then

$$\sup_{s\geq 1}\sup_{\sigma\in\mathcal{C}_s}\frac{1}{\phi_s}\sum_{k\in\sigma}M\left(\frac{|x_k|^p}{\rho}\right) < \infty, \text{ for some } \rho > 0$$

This implies that

$$\sup_{s\geq 1}\sup_{\sigma\in\mathcal{C}_s}\frac{1}{\psi_s}\sum_{k\in\sigma}M\left(\frac{|x_k|^p}{\rho}\right) \leq \sup_{s\geq 1}\left(\frac{\phi_s}{\psi_s}\right)\sup_{s\geq 1}\sup_{\sigma\in\mathcal{C}_s}\frac{1}{\phi_s}\sum_{k\in\sigma}M\left(\frac{|x_k|^p}{\rho}\right)$$
$$< \infty.$$

Therefore $x \in m(M, \psi, p)$.

Conversely, let $m(M, \phi, p) \subseteq m(M, \psi, p)$ and suppose that $\sup\limits_{s\geq 1}\left(\frac{\phi_s}{\psi_s}\right) = \infty$. Then there exists a sequence (s_i) of naturals number such that

$$\lim_{i\to\infty}\left(\frac{\phi_{s_i}}{\psi_{s_i}}\right) = \infty.$$

Let $x \in m(M, \phi, p)$. Then there exists $\rho > 0$ such that

$$\sup_{s\geq 1}\sup_{\sigma\in\mathcal{C}_s}\frac{1}{\phi_s}\sum_{k\in\sigma}M\left(\frac{|x_k|^p}{\rho}\right) < \infty.$$

Now we have

$$\sup_{s\geq 1}\sup_{\sigma\in\mathcal{C}_s}\frac{1}{\psi_s}\sum_{k\in\sigma}M\left(\frac{|x_k|^p}{\rho}\right) \geq \sup_{i\geq 1}\left(\frac{\phi_{s_i}}{\psi_{s_i}}\right)\sup_{s\geq 1}\sup_{\sigma\in\mathcal{C}_s}\frac{1}{\phi_s}\sum_{k\in\sigma}M\left(\frac{|x_k|^p}{\rho}\right) = \infty.$$

Therefore $x \notin m(M, \psi, p)$. This contradict to $m(M, \phi, p) \subseteq m(M, \psi, p)$. Hence $\sup\limits_{s\geq 1}\left(\frac{\phi_s}{\psi_s}\right) < \infty$.

Theorem 3.2. *$l_p(M) \subseteq m(M, \phi, p) \subseteq l_\infty(M)$ for all $\phi \in \Phi$; where*

$$l_p(M) := \left\{x \in \omega : \sum_{k=1}^{\infty}M\left(\frac{|x_k|^p}{\rho}\right) < \infty \text{ for some } \rho > 0\right\}$$

and

$$l_\infty(M) := \left\{ x \in \omega : \sup_{k\geq 1} M\left(\frac{|x_k|}{\rho}\right) < \infty \ \ for\ some\ \rho > 0 \right\}.$$

Proof. Since $\phi_1 \leq \phi_s \leq s\phi_1$ for all for all $\phi \in \Phi$, that is $\phi_s^{-1} \leq \phi_1^{-1}$ and $\frac{\phi_s}{s} \leq \phi_1$, it follows that $\sup_{s\geq 1} \phi_s^{-1} < \infty$ and $\sup_{s\geq 1}(\frac{\phi_s}{s}) < \infty$. Hence from Theorem 3.1 and Remark 1, it follows that $l_p(M) \subseteq m(M,\phi,p) \subseteq l_\infty(M) \quad for\ \ all\ \ \phi \in \Phi$.

Theorem 3.3.(a) $m(M,\phi,p) = l_p(M) \ \ iff \ \ \lim_{s\to\infty} \phi_s < \infty$.
(b) $m(M,\phi,p) = l_\infty(M) \ \ iff \ \ \lim_{s\to\infty} (\frac{\phi_s}{s}) > 0$.

Proof. (a) If $\psi_s = 1 \ \ for\ all\ s$ in Theorem 3.1, then we get $m(M,\phi,p) \subseteq l_p(M)$ iff $\sup_{s\geq 1} \phi_s < \infty$ Hence by Theorem 3.2, we have (a), since (ϕ_s) is monotonic.

(b) Similarly, by Theorem 3.1 , we get $l_\infty(M) \subseteq m(M,\phi,p)$ iff $\sup_{s\geq 1}(\frac{s}{\phi_s}) < \infty$. Hence by Theorem 3.2, we have (b), since $(\frac{s}{\phi_s})$ is monotonic.

Corollary 3.4. $m(M,\phi) = l_M \ \ iff \ \ \lim_{s\to\infty} \phi_s < \infty$.

Theorem 3.5. Let M, M_1, M_2 be Orlicz functions each satisfy $\triangle_2$ - condition. Then
(i) $m(M_1,\phi,p) \subseteq m(M \circ M_1,\phi,p)$,
(ii)$m(M_1,\phi,p) \cap m(M_2,\phi,p) = m(M_1 + M_2,\phi,p)$.
Proof . (i) Suppose that $x \in m(M_1,\phi,p)$. Then there exists $\rho > 0$ such that

$$\sup_{s\geq 1} \sup_{\sigma\in\mathcal{C}_s} \frac{1}{\phi_s} \sum_{k\in\sigma} M_1\left(\frac{|x_k|^p}{\rho}\right) < \infty.$$

Now if we take $0 < \epsilon < 1$ and δ with $0 < \delta < 1$ then $M(t) < \epsilon$ for $0 \leq t < \delta$. Put $y_k = M\left(\frac{|x_k|^p}{\rho}\right)$ and for any $\sigma \in \mathcal{C}_s$ consider

$$\sum_{k\in\sigma} M(y_k) = \sum_1 M(y_k) + \sum_2 M(y_k),$$

where the first supremum is over $y_k \leq \delta$ and second is over $y_k > \delta$. We know that an Orlicz function satisfies the inequality $M(\lambda x) \leq \lambda M(x) \ \ for\ all\ \lambda\ with\ \ 0 < \lambda < 1$.

By above inequality we have

$$\sum_1 M(y_k) \leq M(1) \sum_1 y_k \leq M(2) \sum_1 y_k. \tag{3.5.1}$$

For $y_k > \delta$ we use the fact that

$$y_k < \frac{y_k}{\delta} < 1 + \left(\frac{y_k}{\delta}\right),$$

since M is convex, so

$$M(y_k) < M\left(1 + \frac{y_k}{\delta}\right) < \frac{1}{2}M(2) + \frac{1}{2}M\left(\frac{2y_k}{\delta}\right).$$

Since M satisfies $\triangle_2-$ condition, so we have

$$\begin{aligned} M(y_k) &< \frac{1}{2}K\frac{y_k}{\delta}M(2) + \frac{1}{2}K\frac{y_k}{\delta}M(2) \\ &= K\frac{y_k}{\delta}M(2). \end{aligned}$$

Hence ,

$$\sum_{2} M(y_k) \leq \max(1, K\delta^{-1}M(2)\sum_{2} y_k. \tag{3.5.2}$$

From (3.5.1) and (3.5.2) we have $(x_k) \in m(M \circ M_1, \phi, p)$.
Thus $m(M_1, \phi, p) \subseteq m(M \circ M_1, \phi, p)$.

(ii) Let $(x_k) \in m(M_1, \phi, p) \cap m(M_2, \phi, p)$. Then there exists $\rho > 0$ such that

$$\sup_{s\geq 1} \sup_{\sigma \in \mathcal{C}_s} \frac{1}{\phi_s} \sum_{k\in\sigma} M_1\left(\frac{|x_k|^p}{\rho}\right) < \infty$$

and

$$\sup_{s\geq 1} \sup_{\sigma \in \mathcal{C}_s} \frac{1}{\phi_s} \sum_{k\in\sigma} M_2\left(\frac{|x_k|^p}{\rho}\right) < \infty.$$

The remaining part of the proof follows from the equality

$$\sum_{k\in\sigma}(M_1 + M_2)\left[\left(\frac{|x_k|^p}{\rho}\right)\right] = \sum_{k\in\sigma} M_1\left[\left(\frac{|x_k|^p}{\rho}\right)\right]\sum_{k\in\sigma} M_2\left[\left(\frac{|x_k|^p}{\rho}\right)\right].$$

Put $M_1(x) = x$ in Theorem 3.5(i) We have the following result.
Corollary 3.6. Let M be an Orlicz function satisfying $\triangle_2-$ condition. Then $m(\phi, p) \subseteq m(M, \phi, p)$.

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References

[1] Chen, S. T, Geometry of Orlicz spaces, Dissertationes Math. (The Institute of Mathematics, Polish Academy of Sciences) (1996).

[2] Esi, A., and Et, M. , Some new sequence spaces defined by a sequnce of Orlicz functions, *Indian J. Pure Appl. Math.*, 31:967-972 (2000).

[3] Et, M., On some new Orlicz sequence spaces, *J. Analysis*, 9: 21-28 (2001).

[4] Krasnoselskii, M. A., and Rutickii, Ya. B, Convex functions and Orlicz spaces, (Gooningen: P. Nordhoff Ltd.) (1961) (translation).

[5] Lindenstrauss, J. and Tzafriri, L., On Orlicz sequence spaces, *Israel J. Math.*, 10: 379-390 (1971).

[6] Luxemburg, W. A., Banach function spaces, Thesis (Delft) (1955)

[7] Maligranda, L, Orlicz spaces and interpolation, Seminar in Math. 5, Campinas (1989)

[8] Malkowsky, E., and Mursaleen, Matrix transformations between FK-spaces and the sequence sapces $m(\phi)$ and $n(\phi)$, *J. Math. Anal. Appl.*, 196:659-665 (1995).

[9] Mursaleen, Some geometric properties of a sequence space related to ℓ^p, *Bull. Aust. Math. Soc.*, 67:343-347 (2003).

[10] Musielak J, Orlicz spaces and modular spaces, Lecture Notes in Math. 1034 (Springer- Verlag) (1983).

[11] Orlicz, W., Ü ber Raume (L^M), Bulletin International de l' Académie Polonaise des Sciences et des Letters, Série A, 93 - 107 (1936).

[12] Parashar, S.D., and Choudhary, B., Sequence spaces defined by Orlicz functions, *Indian J. Pure Appl. Math.*, 25:419-428 (1994).

[13] Rao, M. M, and Ren, Z. D, Theory of Orlicz spaces (New York, Basel, Hong Kong: Marcel Dekker Inc.) (1991).

[14] Sargent, W.L.C., Some sequence spaces related to the ℓ^p spaces, *J. London Math. Soc.*, 35:161-171 (1960).

[15] Tripathy, .B.C., and Mahanta, S., On a class of sequences related to the ℓ^p space defined by Orlicz functions, *Soochow J.Math.*, 29: 379-391(2003).

[16] Tripathy, B.C., and Sen,M.: On a new class of sequences related to the space l_p, *Tamkang J. Math.*, 33: 167-171 (2002).

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Domatic number of graph products

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ABSTRACT: A partition of $V(G)$, all of whose classes are dominating sets in G, is called a domatic partition of G. The maximum number of classes of a domatic partition of G is called the domatic number of G. In this paper we explore the bounds for the domatic numbers of the cartesian product, the strong product and the join of two graphs. The bounds are the best possible in the sense that there exist examples for which equalities are attained

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1. Introduction

By a graph G we mean a finite undirected graph without loops and multiple edges with the vertex set $V(G)$ and the edge set $E(G)$. A set $D \subseteq V(G)$ of vertices is a dominating set in G if every vertex not in D is adjacent to at least one vertex in D. A domatic partition of the graph G is a partition of $V(G)$ into pairwise disjoint dominating sets. The domatic number $d(G)$ of G is the maximum cardinality of a domatic partition of G. The domatic number was introduced by E. J. Cockayne and S. T. Hedetniemi in [3]. If G is not connected, then $d(G)$ equals the minimum of domatic numbers of its connected components. Because of this, throughout the paper G will always denote a connected graph. As usual, the minimum degree of G is denoted by $\delta(G)$, the maximum degree of G by $\Delta(G)$ and the domination number of G by $\gamma(G)$ (the minimum cardinality of a dominating set in G). Then the following simple relationships between these numbers hold.

Proposition 1 *For any graph G of order $n \geq 1$,*

a) [3] $d(G) \geq 2$ *if and only if G has no isolated vertex,*

b) [3] $d(G) \leq \delta(G) + 1$,

c) $d(G) \leq n/\gamma(G)$,

d) *if H is a spanning subgraph of G, then $d(H) \leq d(G)$.*

The last two results follow from the definition of the domatic number of a graph.

Moreover, note that if $d(G) = \delta(G) + 1$, then G is called *domatically full*. For example, in the literature it is known that

- every regular graph whose the domatic number divides its number of vertices
- every domatically 3-critical graph
- every block-cactus graph with the minimum degree at least 4
- every strongly chordal graph
- every graph with the minimum degree 1

is domatically full.

We recall here one of results which will be used in our investigations.

Theorem 1 [7] *A regular domatically full graph G of order n and $d(G) = d$ exists if and only if d divides n. Its structure is the following: The vertex set $V(G) = \bigcup_{i=1}^{d} V_i$, $V_i \cap V_j = \emptyset$, $|V_i| = n/d$ and the subgraph G_{ij} of G induced by $V_i \cup V_j$ is regular of degree 1 (for $i = 1, \ldots, d$; $j = 1, \ldots, d$; $i \neq j$).*

The *cartesian product* of the graphs G_1 and G_2 is the graph $G_1 \Box G_2$ such that $V(G_1 \Box G_2) = V(G_1) \times V(G_2)$ and $(x_1, y_1)(x_2, y_2) \in E(G_1 \Box G_2)$ whenever $x_1 = x_2$ and $y_1 y_2 \in E(G_2)$ or $y_1 = y_2$ and $x_1 x_2 \in E(G_1)$.

Instead of $K_1 \Box G_2$ we will write xG_2, where $\{x\} = V(K_1)$, similarly we put $G_1 y$ instead of $G_1 \Box K_1$.

The *strong product* of the graphs G_1 and G_2 is the graph $G_1 \boxtimes G_2$ such that $V(G_1 \boxtimes G_2) = V(G_1) \times V(G_2)$ and $(x_1, y_1)(x_2, y_2) \in E(G_1 \boxtimes G_2)$ whenever $x_1 = x_2$ and $y_1 y_2 \in E(G_2)$ or $y_1 = y_2$ and $x_1 x_2 \in E(G_1)$ or $x_1 x_2 \in E(G_1)$ and $y_1 y_2 \in E(G_2)$.

The *join* of two graphs G_1 and G_2 is the graph $G_1 + G_2$ defined as the disjoint union of graphs G_1 and G_2 with additional edges linking each vertex of G_1 with each vertex of G_2.

Standard notation is applied for the particular types of graphs, too, such as K_n (the complete graph on n vertices), P_n (the path on n vertices), C_n (the cycle on n vertices), $K_{m,n}$ (the complete bipartite graph), S_n (the star with n leaves).

It is immediately seen that

Proposition 2 [3],[4]

a) *$d(P_n) = 2$ and P_n is domatically full, for $n \geq 2$,*

b) *For $n \geq 3$, $d(C_n) = \begin{cases} 2, & \text{if } n \not\equiv 0 (\text{mod } 3); \\ 3, & \text{otherwise} \end{cases}$*

and C_n is domatically full if $n \equiv 0 (\text{mod } 3)$,

c) *$d(K_n) = n$ and K_n is domatically full, for $n \geq 2$,*

d) *$d(K_{m,n}) = \min\{m, n\}$, for $m, n \geq 2$,*

e) *$d(S_n) = 2$ and S_n is domatically full, for $n \geq 1$.*

For general concepts, not defined terms and symbols we refer the reader to [1], [4], [5] and [6].

Our aim is to determine upper and lower bounds for $d(G_1 \Box G_2)$, $d(G_1 \boxtimes G_2)$ and $d(G_1 + G_2)$. We also calculate these numbers for special graphs G_1 and G_2 mentioned above.

2. Domatic number of the cartesian product $G_1 \Box G_2$

In [2] it was calculated the domatic number of the cartesian product $P_n \Box P_m$, for $m, n \geq 2$.

Proposition 3 [2] *For $n, m \geq 2$,*

$$d(P_n \Box P_m) = \begin{cases} 2, & \text{if } m = n = 2 \text{ or } n = 4 \text{ and } m = 2 \text{ or} \\ & n = 2 \text{ and } m = 4, \text{ or} \\ 3, & \text{otherwise.} \end{cases}$$

We calculate this number for the cartesian product of two special graphs. Before proceeding we make a useful simple observation to help to do it.

Proposition 4 *For any two graphs G_1, G_2 we have*

$$\max\{d(G_1), d(G_2)\} \leq d(G_1 \Box G_2) \leq \delta(G_1) + \delta(G_2) + 1.$$

Corollary 1 *If $\delta(G_1) = 1$ and G_2 is domatically full, then*

$$d(G_2) \leq d(G_1 \Box G_2) \leq d(G_2) + 1.$$

From Corollary 1 and Theorem 1 it follows

Corollary 2 *Let G_1 be a graph with $\delta(G_1) = 1$ and let G_2 be regular domatically full. Then $d(G_1 \Box G_2) = d(G_2)$.*

Theorem 2 *Let G_1 be a graph with a spanning tree such that the distance between its each two leaves is even and G_2 be domatically full. If $\delta(G_i) = 1$, for $i = 1, 2$, then $G_1 \Box G_2$ is domatically full.*

Proof. To prove that the graph $G_1 \Box G_2$ is domatically full we must find its domatic partition of cardinality three, say $\{W_1, W_2, W_3\}$, because $\delta(G_1 \Box G_2) = \delta(G_1) + \delta(G_2) = 2$. Since $\delta(G_2) = 1$ and G_2 is domatically full, then the existance of a domatic partition $\{D_1, D_2\}$ of the graph G_2 is assured. Let $y \in V(G_2)$. Let T be a spanning tree of G_1 such that the distance between its each two leaves is even. Pick a leave $r \in T$. Put $(r, y) \in W_1$ whenever $y \in D_1$; otherwise $(r, y) \in W_2$. Now, let $u \in T, u \neq r$. If $d_T(u, r) \equiv 2 \pmod 4$ and $y \in D_1$, then $(u, y) \in W_2$; if $d_T(u, r) \equiv 2 \pmod 4$ and $y \in D_2$, then $(u, y) \in W_1$. If $d_T(u, r) \equiv 0 \pmod 4$ and $y \in D_i$, then $(u, y) \in W_i$, for $i = 1, 2$. In other cases $(u, y) \in W_3$. It is not difficult to see that the sets W_1, W_2, W_3 create a domatic partition of the graph $G_1 \Box G_2$ and the assertion holds. ■

Corollary 3 **a)** *For* $n, m \geq 1$, $d(S_n \Box S_m) = \begin{cases} 2, & \text{if } n = m = 1, \text{ or} \\ 3, & \text{otherwise,} \end{cases}$

b) $d(P_n \Box S_m) = 3$, *for* $n \geq 2$, $m \geq 1$.

The proof of the next result is based on the following lemma.

Lemma 1 *Let G be of order m, $m \geq 2$. If D is a dominating set in $K_n \Box G$, then $|D| \geq m$, $n \geq m$.*

Proof. We exhibit any graph G of order m, $m \geq 2$. Suppose on the contrary that there is a dominating set D in $K_n \Box G$ such that $|D| = k < m$. Then there would be a vertex $x \in V(K_n)$ such that $A = \{(x, y_j) \in V(K_n \Box G) : j = 1, \ldots, m\}$ and $A \cap D = \emptyset$. Moreover, there would be also a vertex $y \in V(G)$ such that $B = \{(x_i, y) \in V(K_n \Box G) : i = 1, \ldots, n\}$ and $B \cap D = \emptyset$. Therefore there is a vertex $(x, y) \in A \cap B \subseteq V(K_n \Box G)$ which is adjacent to no vertex in D. This contradicts the fact that D is the dominating set in $K_n \Box G$ and the assertion follows. ■

Theorem 3 *If G is of order m, then $d(K_n \Box G) = n$, for $n \geq m \geq 2$.*

Proof. Applying Proposition 4 and Proposition 2c), we conclude that $d(K_n \Box G) \geq \max\{n, d(G)\} = n$, with $n \geq m \geq d(G)$. It remains to prove that $d(K_n \Box G) \leq n$. Note that the set $D = \{(x_1, y_j) \in V(K_n \Box G) : j = 1, \ldots, m\}$ is a dominating set in the graph $K_n \Box G$. Indeed: $K_n y_j$ is a complete subgraph of $K_n \Box G$, for every $y_j \in V(G)$. Then every vertex $(x_i, y_j) \in V(K_n \Box G) - D$, for $i \in \{2, 3, \ldots, n\}$, $j \in \{1, 2, \ldots, m\}$, is adjacent to the vertex $(x_1, y_j) \in D$. Since $|D| = m$, hence $\gamma(K_n \Box G) \leq m$. Suppose $\gamma(K_n \Box G) < m$. This is certainly that there exists the dominating set D_1 in $K_n \Box G$, for which $|D_1| = m - 1$. On the other hand, according to Lemma 1, it must be $|D_1| \geq m$, a contradiction. Hence $\gamma(K_n \Box G) = m$. Therefore, from Proposition 1c), it follows that $d(K_n \Box G) \leq n$. Consequently, $d(K_n \Box G) = n$ and the theorem holds. ■

Corollary 4 *For* $n, m \geq 2$, $d(K_n \Box K_m) = \max\{n, m\}$.

Corollary 5 *For* $n, m \geq 2$,

$$d(P_n \Box K_m) = \begin{cases} 3, & \text{if } m = 2 \text{ and } n \in \{3, 5, 6, 7, 8, \ldots\}; \\ m, & \text{otherwise.} \end{cases}$$

Proof. From Theorem 3 we obtain that $d(P_n \Box K_m) = m$, for $m \geq n \geq 2$. Furthermore, by Proposition 3 we get $d(P_4 \Box K_2) = 2$ and $d(P_n \Box K_2) = 3$, if $n \in \{3, 5, 6, 7, 8, \ldots\}$.

Now, let $n > m \geq 3$. By Corollary 1 and Proposition 2c), we observe that $m \leq d(P_n \Box K_m) \leq m+1$. Suppose first $d(P_n \Box K_m) = m+1$ and let $\{W_1, \ldots, W_{m+1}\}$ be a domatic partition of the graph $P_n \Box K_m$. Without loss of generality let us assume that $(x_1, y_1) \in W_1$. Since $deg_{P_n \Box K_m}((x_1, y_1)) = m$, so we may suppose that $(x_1, y_j) \in W_j$, for $j = 2, \ldots, m$ and $(x_2, y_1) \in W_{m+1}$. Moreover, $deg_{P_n \Box K_m}((x_1, y_j)) = m$, for $j = 2, \ldots, m$, then it must be $(x_2, y_j) \in W_{m+1}$. This guarantees that $(x_3, y_1) \in \bigcap_{j=2}^{m} W_j$, otherwise the vertex (x_2, y_1) would not have any neighbour in at least one of the sets $W_2, \ldots, W_m$ and then such the set would not be a dominating set in $P_n \Box K_m$, a contradiction with the assumption. On the other hand, since $W_p \cap W_r = \emptyset$, for all $p \neq r$, hence $(x_3, y_1) \in \emptyset$, a contradiction. ■

Now we investigate the cartesian product of the path P_n and the cycle C_m.

Proposition 5 *Let $n \geq 2$, $m \geq 3$. If $m \equiv 0 \pmod 4$, then $d(P_n \Box C_m) = 4$.*

Proof. By Proposition 4 we obtain $d(P_n \Box C_m) \leq \delta(P_n) + \delta(C_m) + 1 = 4$. Let $m \equiv 0 \pmod 4$. It turns out that $V(P_n \Box C_m)$ can be partitioned into four subsets:

$D_1 = \{(x_i, y_j) : i \equiv 1 \pmod 2$ and $j \equiv 1 \pmod 4$, $1 \leq i \leq n$, $1 \leq j \leq m\} \cup \{(x_i, y_j) : i \equiv 0 \pmod 2$ and $j \equiv 3 \pmod 4$ $1 \leq i \leq n$, $1 \leq j \leq m\}$,

$D_2 = \{(x_i, y_j) : i \equiv 1 \pmod 2$ and $j \equiv 3 \pmod 4$, $1 \leq i \leq n$, $1 \leq j \leq m\} \cup \{(x_i, y_j) : i \equiv 0 \pmod 2$ and $j \equiv 1 \pmod 4$, $1 \leq i \leq n$, $1 \leq j \leq m\}$,

$D_3 = \{(x_i, y_j) : i \equiv 1 \pmod 2$ and $j \equiv 2 \pmod 4$, $1 \leq i \leq n$, $1 \leq j \leq m\} \cup \{(x_i, y_j) : i \equiv 0 \pmod 2$ and $j \equiv 0 \pmod 4$, $1 \leq i \leq n$, $1 \leq j \leq m\}$,

$D_4 = \{(x_i, y_j) : i \equiv 1 \pmod 2$ and $j \equiv 0 \pmod 4$, $1 \leq i \leq n$, $1 \leq j \leq m\} \cup \{(x_i, y_j) : i \equiv 0 \pmod 2$ and $j \equiv 2 \pmod 4$, $1 \leq i \leq n$, $1 \leq j \leq m\}$.

Evidently the sets D_1, D_2, D_3, D_4 are pairwise disjoint, $\bigcup_{i=1}^{4} D_i = V(P_n \Box C_m)$ and each of them is a dominating set in the graph $P_n \Box C_m$. In conclusion, $\{D_1, D_2, D_3, D_4\}$ is a domatic partition of $P_n \Box C_m$. ■

Proposition 6 *Let $m \geq 3$. Then $d(P_2 \Box C_m) = 4$ if and only if $m \equiv 0 \pmod 4$; otherwise $d(P_2 \Box C_m) = 3$.*

Proof. According to Proposition 5 we shall only prove the necessity of the first part of the proposition. Suppose that $d(P_2 \Box C_m) = 4$. Then there exists a domatic partition $\{D_1, D_2, D_3, D_4\}$ of $P_2 \Box C_m$. Without loss of generality suppose that $(x_1, y_1) \in D_1$. Then exactly one of the vertices (x_1, y_2), (x_1, y_m), (x_2, y_1) is in D_2, exactly one in D_3 and exactly one in D_4. This follows from the fact that $P_2 \Box C_m$ is a cubic graph. Supposing that $(x_1, y_2) \in D_2$, $(x_2, y_1) \in D_3$, it must be $(x_1, y_3) \in D_3$, $(x_2, y_2) \in D_4$, $(x_1, y_4) \in D_4$, $(x_2, y_3) \in D_1$, $(x_2, y_4) \in D_2$. Generally, we may prove that:
$(x_i, y_j) \in D_1$ if and and only if $i = 1$ and $j \equiv 1 \pmod 4$ or $i = 2$ and $j \equiv 3 \pmod 4$; $(x_i, y_j) \in D_2$ if and only if $i = 1$ and $j \equiv 2 \pmod 4$ or $i = 2$ and $j \equiv 0 \pmod 4$; $(x_i, y_j) \in D_3$ if and only if $i = 1$ and $j \equiv 3 \pmod 4$ or $i = 2$ and $j \equiv 1 \pmod 4$; $(x_i, y_j) \in D_4$ if and only if $i = 1$ and $j \equiv 0 \pmod 4$ or $i = 2$ and $j \equiv 2 \pmod 4$. Since

$(x_1, y_m) \in D_4$, hence $m \equiv 0 \pmod 4$. This means that m must be divisible by 4 and the necessity follows.

Now, we shall show that if $m \not\equiv 0 \pmod 4$, then $d(P_2 \Box C_m) = 3$. First, let us suppose that $m \not\equiv 0 \pmod 4$ and m is odd number. Construct a domatic partition $\{D_1, D_2, D_3\}$ of $P_2 \Box C_m$ as follows:

$D_1 = \{(x_i, y_j) : i = 1 \text{ and } j \equiv 1 \pmod 4 \text{ or } i = 2 \text{ and } j \equiv 3 \pmod 4),\ 1 \le j \le m\}$,

$D_2 = \{(x_i, y_j) : i = 1 \text{ and } j \equiv 3 \pmod 4 \text{ or } i = 2 \text{ and } j \equiv 1 \pmod 4),\ 1 \le j \le m\}$,

$D_3 = \{(x_i, y_j) : i \in \{1, 2\} \text{ and } j \equiv 0 \pmod 2),\ 1 \le j \le m\}$.

Let $m \not\equiv 0 \pmod 4$ and m be even number, then we can also construct a domatic partition of $P_2 \Box C_m$ in the following way. Namely,

$D_1 = \{(x_i, y_j) : i = 1$ and $j \equiv 1 \pmod 4$ and $1 \le j \le m/2$ or $i = 1$ and $j \equiv 0 \pmod 4$ and $m/2 < j \le m$ or $i = 2$ and $j \equiv 3 \pmod 4$ and $1 \le j \le m/2$ or $i = 2$ and $j \equiv 2 \pmod 4$ and $m/2 < j \le m\}$,

$D_2 = \{(x_i, y_j) : i = 1$ and $j \equiv 3 \pmod 4$ and $1 \le j \le m/2$ or $i = 1$ and $j \equiv 2 \pmod 4$ and $m/2 < j \le m$ or $i = 2$ and $j \equiv 1 \pmod 4$ and $1 \le j \le m/2$ or $i = 2$ and $j \equiv 0 \pmod 4$ and $m/2 < j \le m\}$,

$D_3 = \{(x_i, y_j) : i \in \{1, 2\}$ and $(j \equiv 0 \pmod 2$ and $1 \le j \le m/2$ or $j \equiv 1 \pmod 2$ and $m/2 < j \le m)\}$. ■

Now, we consider the cartesian product of two complete bipartite graphs.

Proposition 7 *Let $n \ge 2$. Then $d(K_{n,n} \Box K_{n,n}) = 2n$.*

Proof. First observe that the minimum degree of the graph $K_{n,n} \Box K_{n,n}$ is equal to $2n$. Further, using Proposition 1b), we conclude that $d(K_{n,n} \Box K_{n,n}) \le \delta(K_{n,n} \Box K_{n,n}) + 1 = 2n + 1$.

By Theorem 1 we know that a regular graph G is domatically full if and only if $d(G)$ divides the number of vertices of this graph. Since the graph $K_{n,n} \Box K_{n,n}$ has $4n^2$ vertices and it is regular of degree $2n$, its domatic number could be equal to $2n + 1$ if and only if $2n + 1$ divides $4n^2$. Unfortunately, it is not possible. Consequently, $d(K_{n,n} \Box K_{n,n}) \le 2n$. To complete the proof we construct a domatic partition of the graph $K_{n,n} \Box K_{n,n}$ using partite sets, say A_{11}, A_{12} of the first copy of the graph $K_{n,n}$ and A_{21}, A_{22} of the second copy of $K_{n,n}$. We can observe that $V(K_{n,n} \Box K_{n,n}) = A_{11} \times A_{21} \cup A_{11} \times A_{22} \cup A_{12} \times A_{21} \cup A_{12} \times A_{22}$. For convenience, let $B_1 = A_{11} \times A_{21}$, $B_2 = A_{11} \times A_{22}$, $B_3 = A_{12} \times A_{21}$, $B_4 = A_{12} \times A_{22}$ and v_{ij}^k denotes the vertex $(x_i, y_j) \in V(K_{n,n} \Box K_{n,n})$, which belongs to the set B_k, for $k = 1, 2, 3, 4$ and $i, j = 1, \ldots, n$. Let $F_1 = \{v_{ij}^k : k \in \{1, 3\}$ and $1 \le i \le n$ and $j = i\}$, $F_2 = \{v_{ij}^k : k \in \{2, 4\}$ and $1 \le i \le n$ and $j = i\}$, $M_p = \{v_{ij}^k : k \in \{1, 3\}$ and $1 \le i \le n$ and $j \equiv (i + p) \pmod n + 1\}$, for $p = 0, 1, \ldots, n - 2$ and $D_p = \{v_{ij}^k : k \in \{2, 4\}$ and $1 \le i \le n$ and $j \equiv (i + p) \pmod n + 1\}$, for $p = 0, 1, \ldots, n - 2$. The form of the sets F_1, F_2, M_p, D_p, for $p = 0, 1, \ldots, n - 2$, guarantees that they are pairwise disjoint and $F_1 \cup F_2 \cup \bigcup_{i=0}^{n-2} (M_i \cup D_i) = V(K_{n,n} \Box K_{n,n})$. Moreover, the sets are dominating sets in the graph $K_{n,n} \Box K_{n,n}$. Finally, $\{F_1, F_2, M_0, M_1, \ldots, M_{n-2}, D_0, D_1, \ldots, D_{n-2}\}$ is the domatic partition of $K_{n,n} \Box K_{n,n}$. Thus, the proposition follows. ■

3. Domatic number of the strong product$G_1 \boxtimes G_2$

In this section we estimate the domatic number of the strong product of two graphs. In particular, for some special factors of the product, its domatic number is calculated.

Proposition 8 *For any two graphs G_1, G_2 we have*

$$\max\{d(G_1), d(G_2)\} \leq d(G_1 \boxtimes G_2) \leq \delta(G_1) + \delta(G_2) + \delta(G_1)\delta(G_2) + 1.$$

Proof. By the definition of the strong product $G_1 \boxtimes G_2$ it follows immediately that $\delta(G_1 \boxtimes G_2) = \delta(G_1) + \delta(G_2) + \delta(G_1)\delta(G_2)$. Hence, by Proposition 1b), we have $d(G_1 \boxtimes G_2) \leq \delta(G_1)+ \delta(G_2) + \delta(G_1)\delta(G_2) + 1$, as required. Furthermore, it knows that $G_1 \Box G_2$ is the spanning subgraph of $G_1 \boxtimes G_2$. For this sake, from Proposition 1d) and Proposition 4 the lower bound follows. ∎

Now, we use this result to allow us to obtain some exact domatic numbers of the strong product.

Theorem 4 *Let G_1 be a graph with $\delta(G_1) = 1$ and let G_2 be domatically full. Then $G_1 \boxtimes G_2$ is domatically full.*

Proof. From Proposition 8 and by the assumption it is clear that $d(G_1 \boxtimes G_2) \leq 2d(G_2)$. We create a domatic partition of the graph $G_1 \boxtimes G_2$ with $2d(G_2)$ classes. Let $\{D_1, \ldots, D_{d(G_2)}\}$ be a domatic partition of the graph G_2. Take $y \in D_i$, for some $1 \leq i \leq d(G_2)$. Let T be a spanning tree in G_1 and pick a leave $r \in T$. Put $d = d_{G_1}(r, x)$, where $x \in V(G_1)$. If $d \equiv 0 (\mathrm{mod}\ 2)$, then $(x, y) \in W_i$; otherwise $(x, y) \in W_{i+d(G_2)}$. It is not difficult to see that $\{W_1, \ldots, W_{2d(G_2)}\}$ is a domatic partition of $G_1 \boxtimes G_2$. Consequently, the result is true. ∎

Corollary 6 **a)** *For $n, m \geq 2$, $d(P_n \boxtimes P_m) = 4$,*

b) *For $n \geq 2$, $m \geq 1$, $d(P_n \boxtimes S_m) = 4$,*

c) *For $n, m \geq 1$, $d(S_n \boxtimes S_m) = 4$.*

Theorem 5 *If G is domatically full, then $d(G \boxtimes K_m) = m \cdot d(G)$, for $m \geq 2$.*

Proof. By Proposition 8 we obtain that $d(G \boxtimes K_m) \leq m(\delta(G) + 1)$. Note, by Proposition 1d), that $\delta(G) = d(G) - 1$. This is certainly since G is domatically full. Then in a consequence $d(G \boxtimes K_m) \leq m \cdot d(G)$. Let $V(G) = \{x_1, \ldots, x_n\}$, $V(K_m) = \{y_1, \ldots, y_m\}$. Recall that Gy_i denotes the subgraph of $G \boxtimes K_m$ induced by $V(G) \times \{y_i\}$, for $1 \leq i \leq m$. Since $Gy_i \cong G$, then $d(Gy_i) = d(G)$. Therefore let $\{V_1^i, V_2^i, \ldots, V_{d(G)}^i\}$ be the domatic partition of Gy_i. Put $\mathcal{P} = \{V_1^1, V_2^1, \ldots, V_{d(G)}^1, V_1^2, V_2^2, \ldots, V_{d(G)}^2, \ldots, V_1^m, V_2^m, \ldots, V_{d(G)}^m\}$. We shall prove that $\mathcal{P}$ is the domatic partition of $G \boxtimes K_m$. Since $\{V_1^i, V_2^i, \ldots, V_{d(G)}^i\}$ is a domatic partition of Gy_i, for $i = 1, \ldots, m$ and $V(Gy_i) \cap V(Gy_j) = \emptyset$, for each $i, j \in \{1, \ldots, m\}$, $i \neq j$, then $V_k^i \cap V_l^j = \emptyset$, for $i \neq j$ or $k \neq l$, where $i, j = 1, \ldots, m$; $k, l = 1, \ldots, d(G)$. Moreover, $\bigcup_{i=1}^{m} \bigcup_{k=1}^{d(G)} V_k^i = V(G \boxtimes K_m)$.

Hence $\mathcal{P}$ is the partition of $G \boxtimes K_m$. It remains to prove that V_k^i is a dominating set in $G \boxtimes K_m$. To do it, consider the vertices belonging to the set $V(G \boxtimes K_m) \backslash V_k^i$, for a fixed i, $1 \leq i \leq m$ and k, $1 \leq k \leq d(G)$. Let $(x_p, y_q) \in V(G \boxtimes K_m) \backslash V_k^i$, $p \in \{1, \ldots, n\}$, $q \in \{1, \ldots, m\}$. There are three cases to discuss.

Case 1: Let $q = i$ and $(x_p, y_i) \in V(Gy_i) \backslash V_k^i$. Evidently, the vertex (x_p, y_i) is dominated by a vertex from V_k^i.

Case 2: If $q \neq i$ and $(x_p, y_q) \in V_k^q$, where $q \in \{1, \ldots, m\}$, then the vertex $(x_p, y_i) \in V_k^i$ dominates the vertex (x_p, y_q), since $y_i y_q \in E(K_m)$ and the edge $(x_p, y_i)(x_p, y_q)$ exists in $G \boxtimes K_m$.

Case 3: Let $(x_p, y_q) \in V_z^q$, $z \neq k$ and $q \neq i$. Then the vertex $(x_p, y_i) \in V_z^i \subset V(Gy_i) \subset V(G \boxtimes K_m)$ is dominated by a vertex of V_k^i say, by a vertex (x_r, y_i) (recall V_k^i is a dominating set in Gy_i). Moreover, $(x_p, y_i) \in V(Gy_i)$ is adjacent to $(x_p, y_q) \in V_z^q$, because of $y_q y_i \in E(K_m)$. Since $x_r x_p \in E(G)$ and $y_q y_i \in E(K_m)$, then by the definition of the strong product it follows that the vertex (x_p, y_q) is dominated by vertex (x_r, y_i) from V_k^i. Hence the set V_k^i, for $i = 1, \ldots, m$ and $k = 1, \ldots, d(G)$ is a dominating set in $G \boxtimes K_m$. Finally, the partition $\mathcal{P}$ is the domatic partition of $G \boxtimes K_m$. Moreover, its cardinality is equal to $m \cdot d(G)$. Hence, $d(G \boxtimes K_m) \geq m \cdot d(G)$, This completes the proof of the theorem. ∎

The above result in particular enables us to calculate domatic numbers of $G \boxtimes K_m$, for special domatically full graphs G. By Proposition 2 we may check easily that

Corollary 7 **a)** *For* $n, m \geq 2$, $d(P_n \boxtimes K_m) = 2m$,

b) *For* $n \geq 1$, $m \geq 2$, $d(S_n \boxtimes K_m) = 2m$,

c) *Let* $n \geq 3$, $m \geq 2$. *Then* $d(C_n \boxtimes K_m) = 3m$ *if and only if* $n \equiv 0 (\mathrm{mod}\ 3)$,

d) *For* $n, m \geq 2$, $d(K_n \boxtimes K_m) = nm$.

4. Domatic number of the join $G_1 + G_2$

Theorem 6 *For any two graphs* G_1, G_2 *we have*

$$\max\{d(G_1) + d(G_2), \min\{|V(G_1)|, |V(G_2)|\}\} \leq d(G_1 + G_2) \leq \min\{\delta(G_1) + |V(G_2)|, \delta(G_2) + |V(G_1)|\} + 1.$$

Proof. We put $V(G_1) = \{x_1, \ldots, x_n\}$, $V(G_2) = \{y_1, \ldots, y_m\}$ and assume without loss of generality that the minimum degrees $\delta(G_1), \delta(G_2)$ are realized by vertices x_k, y_l respectively. By the definition of the join $G_1 + G_2$, $\deg_{G_1+G_2}(x_k) = \delta(G_1) + |V(G_2)|$ and $\deg_{G_1+G_2}(y_l) = \delta(G_2) + |V(G_1)|$. Evidently, $\delta(G_1 + G_2) = \min\{\delta(G_1) + |V(G_2)|, \delta(G_2) + |V(G_1)|\}$. By Proposition 1b), we have $d(G_1 + G_2) \leq \min\{\delta(G_1) + |V(G_2)|, \delta(G_2) + |V(G_1)|\} + 1$. Consequently, the upper bound follows.

Now, we shall prove the lower bound. Let $\{V_1, \ldots, V_{d(G_1)}\}$ be a domatic partition of G_1 and $\{U_1, \ldots, U_{d(G_2)}\}$ be a domatic partition of G_2. The partition $\{V_1, \ldots, V_{d(G_1)}, U_1, \ldots, U_{d(G_2)}\}$ is the domatic partition of the join $G_1 + G_2$, where $d(G_1) + d(G_2) \leq$

$d(G_1+G_2)$. Indeed: each set V_i, for $i=1,\ldots,d(G_1)$ dominates the vertex set $V(G_2)$ of G_2 and simultaneously each set U_j, for $j=1,\ldots,d(G_2)$ dominates the vertex set $V(G_1)$ of G_1.

Moreover, the sets V_i, U_j, for $i=1,\ldots,d(G_1)$ and $j=1,\ldots,d(G_2)$ are pairwise disjoint and $\bigcup_{i=1}^{d(G_1)} V_i \cup \bigcup_{j=1}^{d(G_2)} U_j = V(G_1+G_2)$. Therefore

$$d(G_1+G_2) \geq d(G_1)+d(G_2). \tag{1}$$

On the other hand, each subset $\{x,y\} \subseteq V(G_1+G_2)$, where $x \in V(G_1)$, $y \in V(G_2)$ is the dominating set in the join G_1+G_2. Without loss of generality, let $|V(G_1)| \geq |V(G_2)|$. First, we claim that $|V(G_1)| = |V(G_2)|$. Then certainly there exists the domatic partition $\{\{x_i,y_i\} : i=1,\ldots,|V(G_2)|\}$ of the graph G_1+G_2. If $|V(G_1)| > |V(G_2)|$, then there exists the domatic partition

$$\left\{\{x_1,y_1\},\ldots,\{x_{|V(G_2)|-1},y_{|V(G_2)|-1}\}, V(G_1+G_2)\backslash\bigcup_{i=1}^{|V(G_2)|-1}\{x_i,y_i\}\right\}$$

of G_1+G_2. Hence $d(G_1+G_2) \geq |V(G_2)|$. Therefore, by the commutativity of the join,

$$d(G_1+G_2) \geq \min\{|V(G_1)|, |V(G_2)|\}. \tag{2}$$

Consequently, according to (1) and (2) we obtain the lower bound. ■

Some pairs of factors in G_1+G_2 for which the bounds in Theorem 6 are attained are given below.

By Theorem 6 it follows immediately.

Corollary 8 *Let G be domatically full and let $m \geq 1$. Then $d(G+K_m) = d(G)+m$.*

Theorem 7 *Let G_1, G_2 be given. If $|V(G_1)| = |V(G_2)|$ and $\Delta(G_i) < |V(G_i)|-1$, for $i=1,2$, then $d(G_1+G_2) = |V(G_1)|$.*

Proof. Our assumption $|V(G_1)| = |V(G_2)|$ and Theorem 6 imply that

$$d(G_1+G_2) \geq \max\{d(G_1)+d(G_2), |V(G_1)|\}. \tag{3}$$

Since $\Delta(G_i) < |V(G_i)|-1$, this means that there exists no dominating set D_i in G_i such that $|D_i| = 1$, for $i=1,2$. Namely, if it would be, say $D_1 = \{x_p\}$, then by the assumption $\deg_{G_1}(x_p) \leq \Delta(G_1) < |V(G_1)|-1$. Furthermore it would be at least one vertex in $V(G_1)\backslash\{x_p\}$ which could not be dominated by the set D_1. But this contradicts the fact that D_1 is a dominating set in G_1. Hence $|D_i| \geq 2$ and $\gamma(G_i) \geq 2$, for $i=1,2$, by commutativity of G_1+G_2. From this fact and Proposition 1c) we see that $d(G_i) \leq |V(G_i)|/2$, for $i=1,2$. Moreover,

$$d(G_1)+d(G_2) \leq |V(G_1)|. \tag{4}$$

By (3) and (4) it follows that $d(G_1+G_2) \geq \max\{d(G_1)+d(G_2), |V(G_1)|\} \geq |V(G_1)|$. On the other hand, because of $\gamma(G_i) \geq 2$, for $i=1,2$, then $\gamma(G_1+G_2) \geq 2$. Therefore by Proposition 1c), we have $d(G_1+G_2) \leq |V(G_1+G_2)|/\gamma(G_1+G_2) \leq 2|V(G_1)|/2 = |V(G_1)|$. Consequently, the assertion holds. ■

Corollary 9 *For $n \geq 4$, $d(P_n + P_n) = n$.*

The following observation we use to help to partite the vertex set of the join of two stars S_n, S_m.

Proposition 9 *Let $n, m \geq 2$ and put $V(S_n) = \{x_0\} \cup A$, $V(S_m) = \{y_0\} \cup B$, where $A = \{x_1, \ldots, x_n\}$, $B = \{y_1, \ldots, y_m\}$ with x_0, y_0 of degree n and m, respectively. Then $\{x_0\}, \{y_0\}, \{x_i, y_j\}_{i=1,\ldots,n; j=1,\ldots,m}, A, B$ are all possible minimal dominating sets in $S_n + S_m$.*

Corollary 10 *For $n, m \geq 2$, $d(S_n + S_m) = 2 + \min\{n, m\}$.*

Proof. By Theorem 7 we obtain $d(S_n + S_m) \leq \min\{n, m\} + 3$. Let us suppose that $n \geq m$. Then $d(S_n + S_m) \leq m + 3$. First, we claim that $d(S_n + S_m) = m + 3$. Without loss of generality we may assume that $x_0 \in D_1$ and $y_0 \in D_2$, where $\{D_1, D_2, \ldots, D_{m+3}\}$ is the domatic partition of $S_n + S_m$ (the proof in the case $x_0, y_0 \in D_1$ is analogous). Furthermore by Proposition 9, we may suppose $D_1 = \{x_0\}$ and $D_2 = \{y_0\}$. Since $|B| = m \geq 2$, there exists $k \in \{3, \ldots, m + 3\}$ such that $D_k \cap B = \emptyset$. Hence D_k is a subset of A. But by Proposition 9 it must be $D_k = A$. From this and by Proposition 9 we get $D_l = B$, for $l = 3, \ldots, m + 3$ with $l \neq k$, a contradiction (because the sets $D_3, \ldots, D_{m+3}$ are pairwise disjoint). All this together leads to a conclusion that $d(S_n + S_m) \leq m + 2$. To complete the proof we create a partition with $m + 2$ dominating sets in $G_1 + G_2$ in the following way: $D_1 = \{x_0\}$, $D_2 = \{y_0\}, D_{i+2} = \{x_i, y_i\}$, for $i = 1, \ldots, m-1$ and $D_{m+2} = \{x_m, x_{m+1}, \ldots, x_n, y_m\}$. ■

References

[1] C. Berge, *Graphs and Hypergraphs* (North-Holland, Amsterdam, 1973).

[2] G.J. Chang, *The domatic number problem*, Discrete Mathematics **125** (1994) 115-122.

[3] E.J. Cockayne, S.T. Hedetniemi, *Towards a theory of domination in graphs*, Networks **7** (1977) 247-261.

[4] E.J. Cockayne, *Domination of undirected graphs – a survey.* In: Theory and Applications of Graphs, Proc. Michigan 1976, ed. By Y.Alavi and D. R. Lick (Springer – Verlag Berlin-Heldelberg-New York 1978) 141-147.

[5] R. Diestel, *Graph Theory*, Springer-Verlang (1996).

[6] T.W. Haynes, S.T. Hedetniemi, P.J. Slater, *Fundaments of domination in graphs*, New York, Basel, Hong Kong, Marcel Dekker, Inc. (1998).

[7] B. Zelinka, *Domatically critical graphs*, Czech. Math. J. **30** (1980) 468-489.

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On certain properties of neighborhoods of analytic functions of complex order

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ABSTRACT: Let $\mathcal{A}(n)$ denote the class of functions of the form

$$f(z) = z - \sum_{k=n+1}^{\infty} a_k z^k, \quad (a_k \geq 0,\ k \in \mathbb{N}\backslash\{1\},\ n \in \mathbb{N} = \{1, 2, ...\})$$

which are analytic in the open unit disk $\mathcal{U} = \{z : |z| < 1\}$. In this note, the subclasses $\mathcal{S}_n(\beta, \gamma, a, c)$, $\mathcal{R}_n(\beta, \gamma, a, c; \mu)$, $\mathcal{S}_n^{\alpha}(\beta, \gamma, a, c)$ and $\mathcal{R}_n^{\alpha}(\beta, \gamma, a, c; \mu)$ of $\mathcal{A}(n)$ are defined and some properties of neighborhoods are studied for functions of complex order in these classes

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1. Introduction

Let $\mathcal{A}(n)$ denote the class of functions of the form

$$f(z) = z - \sum_{k=n+1}^{\infty} a_k z^k, \quad (a_k \geq 0,\ k \in \mathbb{N}\backslash\{1\},\ n \in \mathbb{N} = \{1, 2, ...\}) \tag{1}$$

which are analytic in the open unit disk $\mathcal{U} = \{z : |z| < 1\}$.
For any function $f(z) \in \mathcal{A}(n)$ and $\delta \geq 0$, we define,

$$\mathcal{N}_{n,\delta}(f) = \left\{ g \in \mathcal{A}(n) : g(z) = z - \sum_{k=n+1}^{\infty} b_k z^k \text{ and } \sum_{k=n+1}^{\infty} k|a_k - b_k| \leq \delta \right\} \tag{2}$$

which is the (n,δ) - neighborhood of $f(z)$.
For $e(z)=z$, we see that,

$$\mathcal{N}_{n,\delta}(e)=\left\{g\in\mathcal{A}(n):g(z)=z-\sum_{k=n+1}^{\infty}b_kz^k \text{ and } \sum_{k=n+1}^{\infty}k|b_k|\le\delta\right\}. \tag{3}$$

The concept of neighborhoods was first introduced by Goodman and then generalized by Ruscheweyh [8] .
In this paper, we discuss certain properties of (n,δ) - neighborhood for analytic functions of complex order in $\mathcal{U}$.
The subclass $\mathcal{S}_n^*(\gamma)$ of $\mathcal{A}(n)$, is the class of functions of complex order γ satisfying,

$$\Re\left\{1+\frac{1}{\gamma}\left[\frac{zf'(z)}{f(z)}-1\right]\right\}>0, \quad (z\in\mathcal{U},\ \gamma\in\mathbb{C}\backslash\{0\}). \tag{4}$$

The subclass $\mathcal{C}_n(\gamma)$ of $\mathcal{A}(n)$, is the class of functions of complex order γ satisfying,

$$\Re\left\{1+\frac{1}{\gamma}\frac{zf''(z)}{f'(z)}\right\}>0, \quad (z\in\mathcal{U},\ \gamma\in\mathbb{C}\backslash\{0\}). \tag{5}$$

The Hadamard product of two power series

$$f(z)=z+\sum_{k=2}^{\infty}a_kz^k \text{ and } g(z)=z+\sum_{k=2}^{\infty}b_kz^k$$

is defined as $(f*g)(z)=z+\sum_{k=2}^{\infty}a_kb_kz^k.$
In particular, we consider the convolution with the function $\phi(a,c)$ defined by

$$\phi(a,c;z)=z+\sum_{n=2}^{\infty}\frac{(a)_{n-1}}{(c)_{n-1}}z^n, \quad z\in\mathcal{U}, \quad c\ne 0,-1,-2,...$$

where,

$$(a)_n=\frac{\Gamma(a+n)}{\Gamma(a)}.$$

That is, $(a)_0=1,\ (a)_n=a(a+1)...(a+n-1), \quad n>1.$
The function $\phi(a,c)$ is an incomplete beta function related to the Gauss Hypergeometric function by

$$\phi(a,c;z)=z_2F_1(1,a,c;z).$$

It has an analytic continuation in the z-plane cut along the positive real line from 1 to ∞. We note that $\phi(a,1;z)=\dfrac{z}{(1-z)^a}$ and $\phi(2,1;z)$ is the Koebe function. Carlson and Shaffer defined a convolution operator involving an incomplete beta function as

$$L(a,c)f(z)=\phi(a,c;z)*f(z)=z-\sum_{n=2}^{\infty}\frac{(a)_{n-1}}{(c)_{n-1}}a_nz^n. \tag{6}$$

for a function $f(z) \in \mathcal{A}(n)$. Clearly, $L(a,c)$ maps onto itself and $L(c,c)$ is the identity operator. If $a = 0, -1, -2, ...,$ then $L(c,a)$ is an inverse of $L(a,c)$.
In particular, we have,

$$L(n+1,1)f(z) = \frac{z(z^{n-1}f(z))^{(n)}}{n!}, \quad (n \in \mathbb{N}_0 = \mathbb{N} \cup \{0\}).$$

The subclass $\mathcal{S}_n(\beta,\gamma,a,c)$, of $\mathcal{A}(n)$ is the class of functions $f(z)$ such that

$$\left| \frac{1}{\gamma} \left(\frac{z(L(a,c)f(z))'}{L(a,c)f(z)} - 1 \right) \right| < \beta, \tag{7}$$

where, $\gamma \in \mathbb{C}\backslash\{0\}$, $0 < \beta \leq 1$, $a > 0$ and $z \in \mathcal{U}$.
And let the subclass $\mathcal{R}_n(\beta,\gamma,a,c;\mu)$, of $\mathcal{A}(n)$ be the class of functions $f(z)$ such that

$$\left| \frac{1}{\gamma} \left((1-\mu)\frac{L(a,c)f(z)}{z} + \mu(L(a,c)f(z))' - 1 \right) \right| < \beta, \tag{8}$$

where, $\gamma \in \mathbb{C}\backslash\{0\}$, $0 < \beta \leq 1$, $a > 0$ and $z \in \mathcal{U}$.

2. Neighborhoods for classes $\mathcal{S}_n(\beta,\gamma,a,c)$ and $\mathcal{R}_n(\beta,\gamma,a,c;\mu)$

In this section, we obtain inclusion relations involving $\mathcal{N}_{n,\delta}$ for functions in the classes $\mathcal{S}_n(\beta,\gamma,a,c)$ and $\mathcal{R}_n(\beta,\gamma,a,c;\mu)$.

Lemma 1 *A function $f(z) \in \mathcal{S}_n(\beta,\gamma,a,c)$ if and only if*

$$\sum_{k=n+1}^{\infty} \frac{(a)_{k-1}}{(c)_{k-1}} (\beta|\gamma| + k - 1)a_k \leq \beta|\gamma|. \tag{9}$$

Proof. Let $f(z) \in \mathcal{S}_n(\beta,\gamma,a,c)$. Then by (6) we can write,

$$\Re\left\{ \frac{z(L(a,c)f(z))'}{L(a,c)f(z)} - 1 \right\} > -\beta|\gamma|, \quad (z \in \mathcal{U}). \tag{10}$$

Using (1) and (6), we have,

$$\Re\left\{ \frac{-\displaystyle\sum_{k=n+1}^{\infty} \frac{(a)_{k-1}}{(c)_{k-1}}(k-1)a_k z^k}{z - \displaystyle\sum_{k=n+1}^{\infty} \frac{(a)_{k-1}}{(c)_{k-1}} a_k z^k} \right\} > -\beta|\gamma|, \quad (z \in \mathcal{U}). \tag{11}$$

Letting $z \to 1$, through the real values, the inequality (11) yields the desired condition (9).
Conversely, by applying the hypothesis (9) and letting $|z| = 1$, we obtain,

$$\left|\frac{z(L(a,c)f(z))'}{L(a,c)f(z)}-1\right| = \left|\frac{\sum_{k=n+1}^{\infty}\frac{(a)_{k-1}}{(c)_{k-1}}(k-1)a_kz^k}{z-\sum_{k=n+1}^{\infty}\frac{(a)_{k-1}}{(c)_{k-1}}a_kz^k}\right|$$

$$\leq \frac{\beta|\gamma|\left(1-\sum_{k=n+1}^{\infty}\frac{(a)_{k-1}}{(c)_{k-1}}(k-1)a_k\right)}{1-\sum_{k=n+1}^{\infty}\frac{(a)_{k-1}}{(c)_{k-1}}a_k}$$

$$\leq \beta|\gamma|.$$

Hence, by the maximum modulus theorem, we have $f(z) \in \mathcal{S}_n(\beta,\gamma,a,c)$. Thus the proof is complete. ∎ On similar lines, we prove the following lemma.

Lemma 2 *A function* $f(z) \in \mathcal{R}_n(\beta,\gamma,a,c;\mu)$ *if and only if*

$$\sum_{k=n+1}^{\infty}\frac{(a)_{k-1}}{(c)_{k-1}}\left[\mu(k-1)+1\right]a_k \leq \beta|\gamma|. \tag{12}$$

Theorem 1 *If*

$$\delta = \frac{(n+1)\beta|\gamma|}{(\beta|\gamma|+n)\frac{(a)_n}{(c)_n}}, \quad (|\gamma|<1), \tag{13}$$

then, $\mathcal{S}_n(\beta,\gamma,a,c) \subset \mathcal{N}_{n,\delta}(e)$.

Proof. Let $f(z) \in \mathcal{S}_n(\beta,\gamma,a,c)$. By Lemma 1, we have,

$$(\beta|\gamma|+n)\frac{(a)_n}{(c)_n}\sum_{k=n+1}^{\infty}a_k \leq \beta|\gamma|.$$

which implies,

$$\sum_{k=n+1}^{\infty}a_k \leq \frac{\beta|\gamma|}{(\beta|\gamma|+n)\frac{(a)_n}{(c)_n}}. \tag{14}$$

Using (9) and(14) , we have,

$$\begin{aligned}
\frac{(a)_n}{(c)_n}\sum_{k=n+1}^{\infty} k a_k &\leq \beta|\gamma| + (1-\beta|\gamma|)\frac{(a)_n}{(c)_n}\sum_{k=n+1}^{\infty} a_k \\
&\leq \beta|\gamma| + (1-\beta|\gamma|)\frac{(a)_n}{(c)_n}\frac{\beta|\gamma|}{(\beta|\gamma|+n)\frac{(a)_n}{(c)_n}} \\
&\leq \frac{(n+1)\beta|\gamma|}{(\beta|\gamma|+n)\frac{(a)_n}{(c)_n}} = \delta.
\end{aligned}$$

That is,

$$\sum_{k=n+1}^{\infty} k a_k \leq \frac{(n+1)\beta|\gamma|}{(\beta|\gamma|+n)\frac{(a)_n}{(c)_n}} = \delta.$$

Thus, by the definition given by (3), $f(z) \in \mathcal{N}_{n,\delta}(e)$. This completes the proof. ■

Theorem 2 *If*

$$\delta = \frac{(n+1)\beta|\gamma|}{(\mu n+1)\frac{(a)_n}{(c)_n}}, \tag{15}$$

then, $\mathcal{R}_n(\beta, \gamma, a, c; \mu) \subset \mathcal{N}_{n,\delta}(e)$.

Proof. Let $f(z) \in \mathcal{R}_n(\beta, \gamma, a, c; \mu)$. Then, by Lemma 2, we have,

$$\frac{(a)_n}{(c)_n}(\mu n+1)\sum_{k=n+1}^{\infty} a_k \leq \beta|\gamma|,$$

which gives the following coefficient inequality:

$$\sum_{k=n+1}^{\infty} a_k \leq \frac{\beta|\gamma|}{(\mu n+1)\frac{(a)_n}{(c)_n}}. \tag{16}$$

Using (12) and (16), we also have,

$$\begin{aligned}
\mu\frac{(a)_n}{(c)_n}\sum_{k=n+1}^{\infty} k a_k &\leq \beta|\gamma| + (\mu-1)\frac{(a)_n}{(c)_n}\sum_{k=n+1}^{\infty} a_k \\
&\leq \beta|\gamma| + (\mu-1)\frac{(a)_n}{(c)_n}\frac{\beta|\gamma|}{(\mu n+1)\frac{(a)_n}{(c)_n}}.
\end{aligned}$$

That is,

$$\sum_{k=n+1}^{\infty} k a_k \leq \frac{(n+1)\beta|\gamma|}{(\mu n+1)\frac{(a)_n}{(c)_n}} = \delta.$$

Thus, by the definition given by (3), $f(z) \in \mathcal{N}_{n,\delta}(e)$. This completes the proof . ■

3. Neighborhoods for classes $\mathcal{S}_n^{\alpha}(\beta,\gamma,a,c)$ and $\mathcal{R}_n^{\alpha}(\beta,\gamma,a,c;\mu)$

In this section, we define the subclasses $\mathcal{S}_n^{\alpha}(\beta,\gamma,a,c)$ and $\mathcal{R}_n^{\alpha}(\beta,\gamma,a,c;\mu)$ of $\mathcal{A}(n)$ and neighborhoods of these classes are obtained.
For $0 \le \alpha < 1$ and $z \in \mathcal{U}$, a function $f(z) \in \mathcal{S}_n^{\alpha}(\beta,\gamma,a,c)$ if there exists a function $g(z) \in \mathcal{S}_n(\beta,\gamma,a,c)$ such that

$$\left|\frac{f(z)}{g(z)} - 1\right| < 1-\alpha. \tag{17}$$

For $0 \le \alpha < 1$ and $z \in \mathcal{U}$, a function $f(z) \in \mathcal{R}_n^{\alpha}(\beta,\gamma,a,c;\mu)$ if there exists a function $g(z) \in \mathcal{R}_n(\beta,\gamma,a,c;\mu)$ such that the inequality (17) holds true.

Theorem 3 *If* $g(z) \in \mathcal{S}_n(\beta,\gamma,a,c)$ *and*

$$\alpha = 1 - \frac{(\beta|\gamma|+n)\,\delta\,\dfrac{(a)_n}{(c)_n}}{(n+1)\left[(\beta|\gamma|+n)\dfrac{(a)_n}{(c)_n} - \beta|\gamma|\right]}, \tag{18}$$

then, $\mathcal{N}_{n,\delta}(g) \subset \mathcal{S}_n^{\alpha}(\beta,\gamma,a,c)$.

Proof. Let $f(z) \in \mathcal{N}_{n,\delta}(g)$. Then,

$$\sum_{k=n+1}^{\infty} k|a_k - b_k| \le \delta, \tag{19}$$

which yields the coefficient inequality,

$$\sum_{k=n+1}^{\infty} |a_k - b_k| \le \frac{\delta}{n+1}, \quad (n \in \mathbb{N}). \tag{20}$$

Since $g(z) \in \mathcal{S}_n(\beta,\gamma,a,c)$ by (14), we have,

$$\sum_{k=n+1}^{\infty} b_k \le \frac{\beta|\gamma|}{(\beta|\gamma|+n)\dfrac{(a)_n}{(c)_n}}, \tag{21}$$

so that,

$$\begin{aligned}\left|\frac{f(z)}{g(z)}-1\right| &< \frac{\sum\limits_{k=n+1}^{\infty}|a_k-b_k|}{1-\sum\limits_{k=n+1}^{\infty}b_k} \\ &\leq \frac{\delta}{n+1}\,\frac{(\beta|\gamma|+n)\dfrac{(a)_n}{(c)_n}}{\left[(\beta|\gamma|+n)\dfrac{(a)_n}{(c)_n}-\beta|\gamma|\right]} \\ &= 1-\alpha.\end{aligned}$$

Thus, by definition, $f(z) \in \mathcal{S}_n^{\alpha}(\beta,\gamma,a,c)$ for α given by (22). Thus the proof is complete. ■ On similar lines, we can prove the following theorem.

Theorem 4 *If* $g(z) \in \mathcal{R}_n(\beta,\gamma,a,c;\mu)$ *and*

$$\alpha = 1 - \frac{(\mu n+1)\,\delta\,\dfrac{(a)_n}{(c)_n}}{(n+1)\left[(\mu n+1)\dfrac{(a)_n}{(c)_n}-\beta|\gamma|\right]}, \tag{22}$$

then, $\mathcal{N}_{n,\delta}(g) \subset \mathcal{R}_n^{\alpha}(\beta,\gamma,a,c;\mu)$.

References

[1] O. Altintas and S. Owa, *Neighborhoods of certain analytic functions with negative coefficients*, Internat. J. Math. and Math. Sci., **19**(1996), 797-800.

[2] O. Altintas and Ö.Özkan and H. M. Srivastava, *Neighborhoods of a class of analytic functions with negative coefficients*, Appl. Math. Lett., **13**(3)(2000), 63-67.

[3] O. Altintas and Ö.Özkan and H. M. Srivastava, *Majorization by starlike functions of complex order*, *Complex Variable Theory Appl.,* **46**(2001), 207-218.

[4] O. Altintas and Ö.Özkan and H. M. Srivastava, *Neighborhoods of a certain family of multivalent functions with negative coefficients*, Comput. Math. Appl., **47**(2004).

[5] O. Altintas and H. M. Srivastava, *Some Majorization problems associated with p-valently starlike and convex functions of complex order*, East. Asian. Math. J., **17**(2001), 175-183.

[6] P. L. Duren, *Univalent functions*, Springer-Verlag, 1983.

[7] G. Murugusundaramoorthy and H. M. Srivastava, *Neighborhoods of certain classes of analytic functions of complex order*, Journal of Inequalities in Pure and Applied Mathematics ,Volume 5, Issue 2, Article 24, 2004.

[8] S. Ruscheweyh, *Neighborhoods of univalent functions*, Proc. Amer. Math. Soc., **81**(1981), 521-527.

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Some Criteria on Integral means for certain classes of functions with negative coefficients

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Abstract: Let T be the class of functions f with negative coefficients which are analytic and univalent in the open disk U with $f(0) = 0$ and $f'(0) = 1$. For the classes $T^\star(A, B)$ and $C(A, B), -1 \leq A < B \leq 1$ defined as subclasses of T, interesting results for integral means are discussed

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1. Introduction

Let A denote the class of normalized univalent functions of the form

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n \tag{1}$$

that are analytic in the open disc $U = \{z \ni |z| < 1\}$.
Define S to be the subclass of A consisting of all univalent functions $f \in U$. Suppose T as the subclass of functions of S of the form

$$f(z) = z - \sum_{n=2}^{\infty} a_n z^n, \ a_n \geq 0. \tag{2}$$

Let Ω be the class of functions $\omega(z)$ analytic in U such that
$\omega(0) = 0, |\omega(z)| < 1$.
For $f(z)$ and $g(z)$ in $A, f(z)$ is said to be subordinate to $g(z) \in U$ if there exists an analytic function $\omega(z) \in \Omega$ such that $f(z) = g(\omega(z))$. This subordination [1] is denoted by

$$f(z) \prec g(z). \tag{3}$$

Let $P_1(A, B)$ be the class of functions in U which are of the form

$$\frac{1 + A\omega(z)}{1 + B\omega(z)}, -1 \leq A < B \leq 1, \omega(z) \in \Omega.$$

Define

$$S_1^{\star}(A, B) = \{f(z)|f(z) \in S \text{ and} \frac{zf'(z)}{f(z)} \in P_1(A, B)\}$$

$$K_1(A, B) = \{f(z)|f(z) \in S \text{ and } \frac{(zf'(z))'}{f'(z)} \in P_1(A, B)\}$$

We further define,

$$T^{\star}(A, B) = \{f(z)|f(z) \in T \text{ and } \frac{zf'(z)}{f(z)} \in P_1(A, B)\}$$

$$C(A, B) = \{f(z)|f(z) \in T \text{ and } \frac{(zf'(z))'}{f'(z)} \in P_1(A, B)\}$$

We note that $f(z) \in C(A, B)$ if and only if $zf'(z) \in T^{\star}(A, B)$.
For $A = 2\alpha - 1, B = 1$ the class $T^{\star}(A, B)$ reduces to $T^{\star}(\alpha)$ introduced by Schild and Silverman [5] . Lakshma Reddy and Padmanabhan established the following results for the classes $T^{\star}(A, B)$ and $C(A, B)$ [4].

Lemma 1 *A function*

$$f(z) = z - \sum_{n=2}^{\infty} a_n z^n, \; a_n \geq 0$$

is in $T^{\star}(A, B)$ *if and only if*

$$\sum_{m=2}^{\infty} \frac{m(B + 1) - (A + 1)}{B - A} a_m \leq 1. \tag{4}$$

Lemma 2 *A function*

$$f(z) = z - \sum_{n=2}^{\infty} a_n z^n, \; a_n \geq 0$$

is in $C(A, B)$ *if and only if*

$$\sum_{m=2}^{\infty} m \left[\frac{m(B + 1) - (A + 1)}{B - A}\right] a_m \leq 1. \tag{5}$$

Extreme points of the classes $T^{\star}(A, B)$ and $C(A, B)$ are

$$f_1(z) = z \text{ and } f_n(z) = z - \frac{(B - A)}{n(B + 1) - (A + 1)} z^n \; (n \geq 2),$$

$$f_1(z) = z \text{ and } f_n(z) = z - \frac{(B - A)}{n[n(B + 1) - (A + 1)]} z^n \; (n \geq 2)$$

respectively. For subordinations, Littlewood [2] has given the following integral mean.

Theorem 1 *If $f(z)$ and $g(z)$ are analytic in U with $f(z) \prec g(z)$ then for $\lambda > 0$ and $|z| = r\,(0 < r < 1)$,*

$$\int_0^{2\pi} |f(re^{i\theta})|^\lambda d\theta \leq \int_0^{2\pi} |g(re^{i\theta})|^\lambda d\theta. \tag{6}$$

Applying Theorem 1 Owa, Pascu and Nishiwaki [3] proved the following results.

Theorem 2 *Let $f(z) \in T^\star, \lambda > 0$ and*

$$f_k(z) = z - \frac{z^k}{k}\,(k \geq 2).$$

If $f(z)$ satisfies

$$\sum_{j=0}^{k-3} \frac{j+1}{k}(a_{2k+j-1} + a_{k+j+1} - a_{k-j-1}) \geq 0, \textit{ for } k \geq 3 \tag{7}$$

and if there exists an analytic function $\omega(z) \in U$ given by

$$(\omega(z))^{k-1} = k(\sum_{n=2}^{\infty} a_n z^{n-1})$$

then, for $z = re^{i\theta}(0 < r < 1)$,

$$\int_0^{2\pi} |f(z)|^\lambda d\theta \leq \int_0^{2\pi} |f_k(z)|^\lambda d\theta \tag{8}$$

Corollary 1 *Let $f(z) \in T^\star$, $0 < \lambda \leq 2$ and*

$$f_k(z) = z - \frac{z^k}{k}\,(k \geq 2).$$

If $f(z)$ satisfies the conditions in Theorem 2,
then for $z = re^{i\theta}(0 < r < 1)$,

$$\begin{aligned} &\int_0^{2\pi} |f(z)^\lambda d\theta \\ &\leq 2\pi r^\lambda \left(1 + \frac{1}{k^2} r^{2(k-1)}\right)^{\frac{\lambda}{2}} \\ &< 2\pi \left(1 + \frac{1}{k^2}\right)^{\frac{\lambda}{2}}. \end{aligned} \tag{9}$$

Theorem 3 *Let $f(z) \in T^\star, \lambda > 0$ and*

$$f_k(z) = z - \frac{z^k}{k}\,(k \geq 2).$$

If there exists an analytic function $\omega(z) \in U$ *given by*

$$(\omega(z))^{k-1} = \sum_{n=2}^{\infty} n a_n z^{n-1}$$

then, for $z = re^{i\theta}(0 < r < 1)$,

$$\int_0^{2\pi} |f'(z)|^{\lambda} d\theta \leq \int_0^{2\pi} |f_k'(z)|^{\lambda} d\theta. \tag{10}$$

Corollary 2 *Let* $f(z) \in T^{\star}, 0 < \lambda \leq 2$ *and*

$$f_k(z) = z - \frac{z^k}{k}\,(k \geq 2).$$

If $f(z)$ *satisfies the conditions in Theorem 3,*
then for $z = re^{i\theta}\,(0 < r < 1)$,

$$\begin{aligned}&\int_0^{2\pi} |f'(z)^{\lambda} d\theta \\ &\leq 2\pi(1 + r^{2(k-1)})^{\frac{\lambda}{2}} \\ &< 2^{\frac{2+\lambda}{2}}\pi.\end{aligned}$$

Theorem 4 *Let* $f(z) \in C$, $\lambda > 0$ *and*

$$f_k(z) = z - \frac{z^k}{k^2}\,(k \geq 2).$$

If $f(z)$ *satisfies*

$$\sum_{j=2}^{k-1} \frac{(k+j)(k-j)}{k^2}(a_{2k-j} - a_j) \geq 0\, for\, k \geq 3, \tag{11}$$

and if there exists an analytic function $\omega(z) \in U$ *given by*

$$(\omega(z))^{k-1} = k^2 \sum_{n=2}^{\infty} a_n z^{n-1}$$

then for $z = re^{i\theta}(0 < r < 1)$,

$$\int_0^{2\pi} |f(z)|^{\lambda} d\theta \leq \int_0^{2\pi} |f_k(z)|^{\lambda} d\theta. \tag{12}$$

Corollary 3 *Let* $f(z) \in C$, $0 < \lambda \leq 2$ *and*

$$f_k(z) = z - \frac{z^k}{k}\,(k \geq 2).$$

If $f(z)$ satisfies the conditions in Theorem 4,
then for $z = re^{i\theta}(0 < r < 1)$,

$$\int_0^{2\pi} |f(z)^\lambda d\theta$$
$$\leq 2\pi r^\lambda \left(1 + \frac{1}{k^4} r^{2(k-1)}\right)^{\frac{\lambda}{2}}$$
$$< 2\pi \left(1 + \frac{1}{k^4}\right)^{\frac{\lambda}{2}}.$$

Theorem 5 *Let $f(z) \in C, \lambda > 0$ and*

$$f_k(z) = z - \frac{z^k}{k^2}\,(k \geq 2).$$

If $f(z)$ satisfies

$$\sum_{j=2}^{2k-2} j(k-j)a_j \leq 0 \tag{13}$$

and if there exists an analytic function $\omega(z) \in U$ given by

$$(\omega(z))^{k-1} = k \sum_{n=2}^{\infty} n a_n z^{n-1},$$

then, for $z = re^{i\theta}\,(0 < r < 1)$,

$$\int_0^{2\pi} |f'(z)|^\lambda d\theta \leq \int_0^{2\pi} |f_k'(z)|^\lambda d\theta. \tag{14}$$

Corollary 4 *Let $f(z) \in C, 0 < \lambda \leq 2$ and*

$$f_k(z) = z - \frac{z^k}{k}\,(k \geq 2).$$

If $f(z)$ satisfies the conditions in Theorem 3,
then for $z = re^{i\theta}\,(0 < r < 1)$,

$$\int_0^{2\pi} |f'(z)^\lambda d\theta$$
$$\leq 2\pi(1 + \frac{1}{k} r^{2(k-1)})^{\frac{\lambda}{2}}$$
$$< 2\pi(1 + \frac{1}{k})^{\frac{\lambda}{2}}.$$

2. Generalization Results

We prove the following results for integral means for the classes $T^\star(A,B)$ and $C(A,B)$ which generalize the above results.

Theorem 6 *Let* $f(z) \in T^\star(A,B)$, $\lambda > 0$ *and*

$$f_k(z) = z - \frac{(B-A)z^k}{[k(B+1)] - (A+1)}\ (k \geq 2).$$

If $f(z)$ *satisfies*

$$\sum_{j=0}^{k-3} \frac{j+1}{k}(a_{2k+j-1} + a_{k+j+1} - a_{k-j-1}) \geq 0 \tag{15}$$

for $k \geq 3$, *and if there exists an analytic function*

$$(\omega(z))^{k-1} = \frac{k(B+1)-(A+1)}{B-A}\sum_{n=2}^{\infty} a_n z^{n-1}$$

then, for $z = re^{i\theta}\ (0 < r < 1)$,

$$\int_0^{2\pi} |f(z)|^\lambda d\theta \leq \int_0^{2\pi} |f_k(z)|^\lambda d\theta \tag{16}$$

Proof. For $f(z) \in T^\star(A,B)$, we have to show that

$$\int_0^{2\pi} \left|1 - \sum_{n=2}^{\infty} a_n z^{n-1}\right| \leq \int_0^{2\pi} \left|1 - \frac{(B-A)z^{k-1}}{k(B+1)-(A+1)}\right| d\theta$$

By Theorem 1, it suffices to prove that

$$1 - \sum_{n=2}^{\infty} a_n z^{n-1} \prec 1 - \frac{(B-A)}{k(B+1)-(A+1)} z^{k-1}$$

Let us define the function $\omega(z)$ by

$$1 - \sum_{n=2}^{\infty} a_n z^{n-1} = 1 - \frac{(B-A)}{k(B+1)-(A+1)}(\omega(z))^{k-1} \tag{17}$$

It follows from (17) that

$$\begin{aligned} |\omega(z)|^{k-1} &= \left|\frac{k(B+1)-(A+1)}{B-A}\sum_{n=2}^{\infty} a_n z^{n-1}\right| \\ &\leq |z| \left(\sum_{n=2}^{\infty} \frac{k(B+1)-(A+1)}{B-A} a_n\right) \end{aligned}$$

Thus, we need to show that

$$\sum_{n=2}^{\infty} \frac{k(B+1)-(A+1)}{B-A} a_n \le \sum_{n=2}^{\infty} \frac{n(B+1)-(A+1)}{B-A} a_n$$

Equivalently, we only show that

$$\sum_{n=2}^{\infty} k a_n \le \sum_{n=2}^{\infty} n a_n$$

$$\sum_{n=2}^{\infty} a_n \le \frac{1}{k} \sum_{n=2}^{\infty} n a_n$$

Consider,

$$\begin{aligned}
\frac{1}{k}\sum_{n=2}^{\infty} n a_n &= \left(1-\frac{k-2}{k}\right) a_2 + \left(1-\frac{k-3}{k}\right) a_3 + ... + \left(1-\frac{2}{k}\right) a_{k-2} \\
&+ \left(1-\frac{1}{k}\right) a_{k-1} + a_k + \left(1+\frac{1}{k}\right) a_{k+1} + \left(1+\frac{2}{k}\right) a_{k+2} \\
&+ ... + \left(1+\frac{k+1}{k}\right) a_{2k+1} + \left(1+\frac{k+2}{k}\right) a_{2k+2} + ... \\
&= \frac{k-2}{k}(a_{2k-2}-a_2) + \frac{k-3}{k}(a_{2k-3}-a_3) + ... \\
&+ \frac{2}{k}(a_{k+2}-a_{k-2}) + \frac{1}{k}(a_{k+1}-a_{k-1}) + \left(1+\frac{k-1}{k}\right) a_{2k-1} \\
&+ \left(1+\frac{k}{k}\right) a_{2k} + \left(1+\frac{k+1}{k}\right) a_{2k+1} + ... + \sum_{n=2}^{2k-2} a_n.
\end{aligned}$$

Since

$$1+\frac{k+j}{k} \ge 1+\frac{2+j}{k},\ (j=-1,0,1....)$$

we have

$$\begin{aligned}
\frac{1}{k}\sum_{n=2}^{\infty} na_n \geq\ & \frac{k-2}{k}(a_{2k-2}-a_2)+\frac{k-3}{k}(a_{2k-3}-a_3)+...\\
&+\frac{2}{k}(a_{k+2}-a_{k-2})+\frac{1}{k}(a_{k+1}-a_{k-1})\\
&+\left(1+\frac{1}{k}\right)a_{2k-1}+\left(1+\frac{2}{k}\right)a_{2k}+...\\
&+\left(1+\frac{k-3}{k}\right)a_{3k-5}+\left(1+\frac{k-2}{k}\right)a_{3k-4}+...+\sum_{n=2}^{2k-2}a_n\\
&\geq \frac{1}{k}(a_{2k-1}+a_{k+1}-a_{k-1})+\frac{2}{k}(a_{2k}+a_{k+2}-a_{k-2})+...\\
&+\frac{k-2}{k}(a_{3k-4}+a_{2k-2}-a_2)+\sum_{n=2}^{\infty}a_n\\
&=\sum_{j=0}^{k-3}\frac{j+1}{k}(a_{2k+j-1}+a_{k+j+1}-a_{k-j-1})+\sum_{n=2}^{\infty}a_n\\
&\geq \sum_{n=2}^{\infty}a_n
\end{aligned} \tag{18}$$

since

$$\sum_{j=0}^{k-3}\frac{j+1}{k}(a_{2k+j-1}+a_{k+j+1}-a_{k-j-1}) \geq 0.$$

Hence, we observe that the function $\omega(z)$ defined by (17) is analytic in U with $\omega(0)=0$, $|\omega(z)|<1$, $(z\in U)$. Thus we have proved the theorem. ■

Remark 1 *For the parametric values* $A=-1, B=1$
we get Theorem 2.
Taking $A=-1, B=1, k=2$ *we have the following result by Silverman [6]:*
Suppose that $f(z)\in T^{\star}, \lambda>0$ *and* $f_2(z)=z-\dfrac{z^2}{2}$.
Then for $z=re^{i\theta}$ $(0<r<1)$,

$$\int_0^{2\pi}|f(z)|^{\lambda}d\theta \leq \int_0^{2\pi}|f_2(z)|^{\lambda}d\theta \tag{19}$$

Corollary 5 *Let* $f(z)\in T^{\star}(A,B)$, $0<\lambda\leq 2$ *and*

$$f_k(z)=z-\frac{(B-A)}{k(B+1)-(A+1)}z^k\ (k\geq 2).$$

If $f(z)$ satisfies the conditions in Theorem 6, then for $z = re^{i\theta}$ $(0 < r < 1)$,

$$\int_0^{2\pi} |f(z)|^\lambda d\theta \le \left[1 + \left(\frac{B-A}{k(B+1)-(A+1)}\right)^2 r^{2(k-1)}\right]^{\frac{\lambda}{2}} < \left[1 + \left(\frac{B-A}{k(B+1)-(A+1)}\right)^2\right]^{\frac{\lambda}{2}} \tag{20}$$

Proof. We have

$$\int_0^{2\pi} |f_k(z)|^\lambda d\theta = \int_0^{2\pi} |z|^k \left|\frac{B-A}{k(B+1)-(A+1)} z^{k-1}\right|^\lambda d\theta$$

Applying Hölder's Inequality for $0 < \lambda < 2$, we obtain

$$\begin{aligned}
&\int_0^{2\pi} |z|^\lambda \left|\frac{B-A}{k(B+1)-(A+1)} z^{k-1}\right|^\lambda d\theta \\
&\le \left(\int_0^{2\pi} (|z|^\lambda)^{\frac{2}{2-\lambda}} d\theta\right)^{\frac{2\lambda}{2}} \left(\int_0^{2\pi} \left(\left|1 - \frac{(B-A)}{k(B+1)-(A+1)} z^{k-1}\right|^\lambda\right)^{\frac{2}{\lambda}} d\theta\right)^{\frac{\lambda}{2}} \\
&= \left(\int_0^{2\pi} |z|^{\frac{2\lambda}{2-\lambda}} d\theta\right)^{\frac{2-\lambda}{2}} \left(\int_0^{2\pi} \left|1 - \frac{B-A}{k(B+1)-(A+1)} z^{k-1}\right|^2 d\theta\right)^{\frac{\lambda}{2}} \\
&= 2\pi r^{\frac{2\lambda}{2-\lambda}} \left(2\pi \left(\frac{B-A}{k(B+1)-(A+1)}\right)^2 r^{2(k-1)}\right)^{\frac{\lambda}{2}} \\
&= 2\pi r^\lambda \left(1 + \left(\frac{B-A}{k(B+1)-(A+1)}\right) r^{2(k-1)}\right)^{\frac{\lambda}{2}} \\
&< 2\pi \left(1 + \left(\frac{B-A}{k(B+1)-(A+1)}\right)^2\right)^{\frac{\lambda}{2}}
\end{aligned}$$

Further, it is clear for $\lambda = 2$. ■

Theorem 7 *Let $f(z) \in T^\star(A,B), \lambda > 0$ and*

$$f_k(z) = z - \frac{(B-A)}{k(B+1)-(A+1)} z^k \ (k \ge 2).$$

If there exists an analytic function $\omega(z) \in U$ given by

$$(\omega(z))^{k-1} = \sum_{n=2}^{\infty} \frac{n(B+1)-(A+1)}{B-A} a_n z^{n-1},$$

then, for $z = re^{i\theta}$ $(0 < r < 1)$,

$$\int_0^{2\pi} |f'(z)|^\lambda d\theta \leq \int_0^{2\pi} |f_k'(z)|^\lambda d\theta. \tag{21}$$

Proof. For $f(z) \in T^\star(A, B)$, it is sufficient to show that

$$1 - \sum_{n=2}^{\infty} \frac{n(B+1) - (A+1)}{B - A} a_n z^{n-1} \prec 1 - z^{k-1} \tag{22}$$

Let the function $\omega(z)$ be defined by

$$1 - \sum_{n=2}^{\infty} \frac{n(B+1) - (A+1)}{B - A} a_n z^{n-1} = 1 - (\omega(z))^{k-1} \tag{23}$$

Equivalently $\omega(z)$ is defined by

$$(\omega(z))^{k-1} = \sum_{n=2}^{\infty} \frac{n(B+1) - (A+1)}{B - A} a_n z^{n-1}$$

Since $f(z)$ satisfies

$$\sum_{n=2}^{\infty} \frac{n(B+1) - (A+1)}{B - A} \leq 1,$$

the function $\omega(z)$ is analytic in U, $\omega(0) = 0$
and $|\omega(z)| < 1\,(z \in U)$. ■

Remark 2 *Parametric values* $A = -1, B = 1$ *yield Theorem 3.*
For $A = -1, B = 1, k = 2$ *we obtain the following result by Silverman [6] :*
If $f(z) \in T^\star$, $\lambda > 0$ *and* $f_2(z) = z - \frac{z^2}{2}$,
then , for $z = re^{i\theta}$ $(0 < r < 1)$

$$\int_0^{2\pi} |f'(z)|^\lambda d\theta \leq \int_0^{2\pi} |f_2'(z)|^\lambda d\theta \tag{24}$$

Using Holder's inequality for Theorem 7 we have

Corollary 6 *Let* $f(z) \in T^\star(A, B)$, $0 < \lambda \leq 2$ *and*

$$f_k(z) = z - \frac{B - A}{k(B+1) - (A+1)} z^k \; (k \geq 2).$$

If $f(z)$ *satisfies conditions of Theorem 7, then for*
$z = re^{i\theta}$ $(0 < r < 1)$

$$\int_0^{2\pi} |f'(z)|^\lambda d\theta < 2\pi \left(1 + r^{2\left(\frac{k(B+1)-(A+1)}{B-A} - 1\right)}\right)^{\frac{\lambda}{2}} < 2^{\frac{2+\lambda}{2}} \pi$$

Now we discuss the integral means for functions in the class $C(A,B)$

Theorem 8 *Let* $f(z) \in C(A,B)$, $\lambda > 0$ *and*

$$f_k(z) = z - \frac{(B-A)z^k}{k[k(B+1)-(A+1)]}\,(k \geq 2)$$

If $f(z)$ *satisfies*

$$\sum_{j=2}^{k-1} \frac{(k+j)(k-j)}{k^2}(a_{2k-j} - a_j) \geq 0\, for\, k \geq 0, \tag{25}$$

and if there exists an analytic function $\omega(z) \in U$ *given by*

$$(\omega(z))^{k-1} = \frac{k[k(B+1)-(A+1)]}{B-A}\sum_{n=2}^{\infty} a_n z^{n-1}$$

then, for $z = re^{i\theta}$ $(0 < r < 1)$,

$$\int_0^{2\pi} |f(z)|^\lambda d\theta \leq \int_0^{2\pi} |f_k(z)|^\lambda \theta \tag{26}$$

Proof. It is sufficient to show that

$$1 - \sum_{n=2}^{\infty} a_n z^{n-1} \prec 1 - \frac{(B-A)}{k[k(B+1)-(A+1)]} z^{k-1}$$

by theorem 1, define the function $\omega(z)$ by

$$1 - \sum_{n=2}^{\infty} a_n z^{n-1} = 1 - \frac{B-A}{k[k(B+1)-(A+1)]}(\omega(z))^{k-1} \tag{27}$$

or by

$$(\omega(z))^{k-1} = \frac{k[k(B+1)-(A+1)]}{B-A}\sum_{n=2}^{\infty} a_n z^{n-1}$$

We need to show that

$$\sum_{n=2}^{\infty} a_n \leq \frac{B-A}{k[k(B+1)-(A+1)]}\left(\sum_{n=2}^{\infty} \frac{(n(B+1)-(A+1))}{B-A}\right) a_n$$

Using the same technique as in the proof of Theorem 6 we see that

$$\begin{aligned}
&\frac{B-A}{k[k(B+1)-(A+1)]}\sum_{n=2}^{\infty} \frac{n[n(B+1)-(A+1)]}{B-A} a_n \\
&\geq \sum_{j=2}^{k-1} \frac{(k+j)(k-j)}{k^2}(a_{2k-j} - a_j) + \sum_{n=2}^{\infty} a_n \\
&\geq \sum_{n=2}^{\infty} a_n
\end{aligned}$$

■

Remark 3 *For $A = -1, B = 1$ we get Theorem 3*

Corollary 7 *Let $f(z) \in C(A, B)$, $0 < \lambda \le 2$ and*

$$f_k(z) = z - \frac{(B-A)}{k[k(B+1)-(A+1)]} z^k \; (k \ge 2)$$

If $f(z)$ satisfies the condition in Theorem 8, then for $k \ge 3$ and $z = re^{i\theta}$ $(0 < r < 1)$

$$\begin{aligned} &\int_0^{2\pi} |f(z)|^\lambda d\theta \\ &\le 2\pi r^\lambda \left(1 + \left(\frac{B-A}{k[k(B+1)-(A+1)]}\right)^4 r^{2(k-1)}\right)^{\frac{\lambda}{2}} \\ &< 2\pi \left(1 + \left(\frac{B-A}{k[k(B+1)-(A+1)]}\right)^4\right)^{\frac{\lambda}{2}} \end{aligned}$$

$$\begin{aligned} &\int_0^{2\pi} |f(z)|^\lambda d\theta \\ &\le 2\pi r^\lambda \left(1 + \left(\frac{B-A}{k[k(B+1)-(A+1)]} r^{2(k-1)}\right)^4\right)^{\frac{\lambda}{2}} \\ &< 2\pi \left(1 + \left(\frac{B-A}{k[k(B+1)-(A+1)]}\right)^4\right)^{\frac{\lambda}{2}} \end{aligned}$$

Theorem 9 *Let $f(z) \in C(A, B)$, $\lambda > 0$ and*

$$f_k(z) = z - \frac{B-A}{k[k(B+1)-(A+1)]} z^k \; (k \ge 2)$$

If $f(z)$ satisfies

$$\sum_{j=2}^{2k-2} j(k-j)a_j \le 0, \tag{28}$$

and if there exists an analytic function

$$(\omega(z))^{k-1} = \frac{k(B+1)-(A+1)}{B-A} \sum_{n=2}^{\infty} \frac{n(B+1)-(A+1)}{B-A} a_n z^{n-1},$$

then for $z = re^{i\theta}$ $(0 < r < 1)$,

$$\int_0^{2\pi} |f'(z)|^\lambda d\theta \le \int_0^{2\pi} |f_k'(z)|^\lambda d\theta. \tag{29}$$

Remark 4 *Taking $A = -1, B = 1$ we obtain Theorem 5*

Corollary 8 *Let $f(z) \in C(A, B)$, $0 < \lambda \leq 2$, and*

$$f_k(z) = z - \frac{B - A}{k[k(B+1) - (A+1)]} z^k \; (k \geq 2).$$

If $f(z)$ satisfies the condition in Theorem 9, then for $k \geq 2$, and $z = re^{i\theta}$ $(0 < r < 1)$,

$$\int_0^{2\pi} |f'(z)|^\lambda \theta$$
$$\leq 2\pi \left(1 + \frac{B - A}{k(B+1) - (A+1)} r^{2(k-1)}\right)^{\frac{\lambda}{2}}$$
$$< 2\pi \left(1 + \frac{B - A}{k(B+1) - (A+1)}\right)^{\frac{\lambda}{2}}$$

References

[1] P.L. Duren, Univalent functions,Springer - Verlag, Newyork (1983).

[2] Littlewood, On inequalities in the theory of functions, Proc. London. Math.Soc. $23(1925), 481 - 519$

[3] S.Owa, M. Pascu, D. Yagi AND J. Nishiwaki, Integral means for starlike and convex functions with negative coefficients, J. Inequal. Pure Appl. Math., Vol 6 Issue , Art 50, (2005)

[4] K.S. Padmanabhan AND G. Lakshma Reddy, On Analytic functions with reference to the Bemardi Integral Operator, Bull. Aust. Math. Soc. $25(1982), 387 - 396$

[5] H. Silverman AND A Schild, Convolutions of Univalent functions with negative coefficients, Ann. Univ. Mariae, Curie $29(1975)(78 - 82)$

[6] H. Silverman, Integral means for Univalent functions with negative coefficients, Houston J. Math.,$23(1997)169 - 174$.

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Convolution properties of univalent functions defined by generalized Sâlâgean operator

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Abstract: In our present work we obtained some interesting properties of convolution using the generalized Sâlâgean operator on univalent functions with missing coefficients, also we investigate other results by making use of subordination concept

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1. Introduction

Let $\mathcal{A}$ denote the class of functions of the form

$$f(z) = z + \sum_{k=2}^{\infty} a_k z^k \tag{1}$$

analytic in the *open* unit disk

$$\mathcal{U} = \{z : |z| < 1\}.$$

For a function $f(z)$ in $\mathcal{A}$, due to Al-Oboudi [1]we define the following generalized Sâlâgean differential operator

$$\mathcal{D}^0 f(z) = f(z) \tag{2}$$

$$\mathcal{D}^1 f(z) = (1-\lambda)f(z) + \lambda z f'(z) = D_\lambda f(z), \quad \lambda \geq 0 \tag{3}$$

$$\mathcal{D}^n f(z) = \mathcal{D}_\lambda(D^{n-1} f(z)). \tag{4}$$

From (3) and (4) we note that

$$D^n f(z) = z + \sum_{k=2}^{\infty}(1 + (k-1)\lambda)^n a_k z^k, \tag{5}$$

when $\lambda = 1$, we have Sâlâgean's operator [7].

Denote by T [9] , the subclass of $\mathcal{A}$ consisting of functions of the form

$$f(z) = z - \sum_{k=2}^{\infty} a_k z^k \quad (a_k \geq 0). \tag{6}$$

Let $P(A, B, \alpha)$ be the subclass $\mathcal{A}$ satisfying the condition

$$f(z) \prec \frac{1 + ((1-\alpha)A + \alpha B)z}{1 + Bz}, \tag{7}$$

where $-1 \leq A < B \leq 1, 0 \leq \alpha < 1$ and "$\prec$" stands for subordination.

Denote by T_2^* the subclass of T consisting of functions of the form

$$f(z) = z - \sum_{k=2}^{\infty} a_{2k} z^{2k}. \tag{8}$$

Motivated by the works of Joshi [3] and Naik [5] and using the tecniques of Silverman and Berman [8], Padmanabhan and Ganeshan [6] and others [2,4], we define new subclasses of T and T_2^* as

$$\mathcal{A}(n, m, \gamma, \lambda, A, B, \alpha) = \left\{ f : f \in T : \frac{(1-\gamma)z(D^n f(z))' + \gamma z(D^{n+m} f(z))'}{(1-\gamma)D^n f(z) + \gamma D^{n+m} f(z)} \in P(A, B, \alpha) \right\} \tag{9}$$

$$T_2^*(n, m, \gamma, \lambda, A, B, \alpha) = \left\{ f : f \in T_2^* : \frac{(1-\gamma)z(D_*^n f(z))' + \gamma z(D_*^{n+m} f(z))'}{(1-\gamma)D_*^n f(z) + \gamma D_*^{n+m} f(z)} \in P(A, B, \alpha) \right\} \tag{10}$$

where $n, m \in I\!N \cup \{0\}, 0 \leq \gamma \leq 1, \lambda \geq 0, -1 \leq A < B \leq 1, 0 \leq \alpha < 1, D^n f(z)$ is defined by (5) and $D_*^n f(z) = z + \sum_{k=2}^{\infty}(1 + (2k-1)\lambda)^n a_{2k} z^{2k}$.

Specializing the parameter γ we can define the following subclasses as a particular case of our new class

$$S(n, m, \lambda, A, B, \alpha) = \mathcal{A}(n, m, 0, \lambda, A, B, \alpha) \tag{11}$$

$$K(n, m, \lambda, A, B, \alpha) = \mathcal{A}(n, m, 1, \lambda, A, B, \alpha) \tag{12}$$

$$S_2^*(n, m, \lambda, A, B, \alpha) = T_2^*(n, m, 0, \lambda, A, B, \alpha) \tag{13}$$

and

$$K_2^*(n, m, \lambda, A, B, \alpha) = T_2^*(n, m, 1, \lambda, A, B, \alpha) \tag{14}$$

We remark that by specializing the parameters n, m, γ, α and λ,
(i) $T_2^*(0, 0, 0, 1, A, B, 0) = T_2^*(A, B)$ and (ii) $T_2^*(0, 1, 1, 1, A, B, 0) = C_2(A, B)$ our new subclasses reduce to the subclasses studied in [5].

In this paper, we obtain the coefficient inequalities and convolution properties for univalent functions with negative coefficients of the form (8) in our new class. Further we state some interesting results as corollaries which are new and not found in the literature.

2. Main Results

Let $f(z) = z - \sum_{k=2}^{\infty} a_{2k} z^{2k}$ and $g(z) = z - \sum_{k=2}^{\infty} b_{2k} z^{2k}$, with $a_{2k} \geq 0, b_{2k} \geq 0$ then the convolution is defined by

$$f(z) * g(z) = z - \sum_{k=2}^{\infty} a_{2k} b_{2k} z^{2k}. \tag{15}$$

For proving our convolution results ,first we shall prove the following Lemma.
Lemma *Let $f(z)$ be of the form (8), then $f(z)$ belongs to $T_2^*(n,m,\gamma,\lambda,A,B,\alpha)$ if and only if*

$$\sum_{k=2}^{\infty} \frac{[(2k-1)+(2k-\alpha)B-(1-\alpha)A](1+(2k-1)\lambda)^n[1-\gamma+\gamma(1+(2k-1)\lambda)^m]}{(B-A)(1-\alpha)} a_{2k} \leq 1 \tag{16}$$

where $a_{2k} \geq 0, n \geq 0, m \geq 0, 0 \leq \gamma \leq 1, -1 \leq A < B \leq 1, 0 \leq \alpha < 1, \lambda \geq 0$.
Proof. Since $f(z) \in T_2^*(n,m,\gamma,\lambda,A,B,\alpha)$, then by (10) we have

$$\frac{(1-\gamma)z(D_*^n f(z))' + \gamma z(D_*^{n+m} f(z))'}{(1-\gamma)D_*^n f(z) + \gamma D_*^{n+m} f(z)}$$
$$= \frac{z - \sum_{k=2}^{\infty} 2kX^n(1-\gamma+\gamma X^m)a_{2k}z^{2k}}{z - \sum_{k=2}^{\infty} X^n(1-\gamma+\gamma X^m)a_{2k}z^{2k}} \prec \frac{1+((1-\alpha)A+\alpha B)z}{1+Bz}$$

where $X = 1 + (2k-1)\lambda$.

Now, by definition of subordination, there exists $w(z)$ which is analytic function in $\mathcal{U}$ with $w(0) = 0, |w(z)| < 1$ in $\mathcal{U}$ such that

$$\frac{z - \sum_{k=2}^{\infty} 2kX^n(1-\gamma+\gamma X^m)a_{2k}z^{2k}}{z - \sum_{k=2}^{\infty} X^n(1-\gamma+\gamma X^m)a_{2k}z^{2k}} = \frac{1+((1-\alpha)A+\alpha B)w(z)}{1+Bw(z)}$$

then by simple calculations we obtain

$$w(z) = \frac{\sum_{k=2}^{\infty}(2k-1)X^n(1-\gamma+\gamma X^m)a_{2k}z^{2k-1}}{B-(1-\alpha)A-\alpha B - \sum_{k=2}^{\infty}((2k-\alpha)B-(1-\alpha)A)X^n(1-\gamma+\gamma X^m)a_{2k}z^{2k-1}}$$

then by noting $|w(z)| < 1$, we get

$$\left| \frac{\sum\limits_{k=2}^{\infty}(2k-1)X^n(1-\gamma+\gamma X^m)a_{2k}z^{2k-1}}{(B-A)(1-\alpha)A-\sum\limits_{k=2}^{\infty}((2k-\alpha)B-(1-\alpha)A)X^n(1-\gamma+\gamma X^m)a_{2k}z^{2k-1}} \right| < 1.$$

Letting $z \to 1^-$, we have

$$\frac{\sum\limits_{k=2}^{\infty}((2k-1)+(2k-\alpha)B-(1-\alpha)A)(1+(2k-1)\lambda)^n(1-\gamma+\gamma(1+(2k-1)\lambda)^m)}{(B-A)(1-\alpha)}a_{2k} \leq 1.$$

Which completes the proof of Lemma.

Theorem 1 *If* $f(z) = z - \sum\limits_{k=2}^{\infty} a_{2k}z^{2k}$ *and* $g(z) = z - \sum\limits_{k=2}^{\infty} b_{2k}z^{2k}$ *where* $a_{2k} \geq 0, b_{2k} \geq 0$ *such that* $f(z), g(z) \in T_2^*(n, m, \gamma, \lambda, A, B, \alpha)$*, then* $q(z) = z - \sum\limits_{k=2}^{\infty} a_{2k}b_{2k}z^{2k}$ *belongs to* $T_2^*(n, m, \gamma, \lambda, A_1, B_1, \alpha)$ *with* $-1 \leq A < B \leq 1$*, where* $A_1 \leq 1-2j, B_1 \geq \frac{j+A_1}{1-j}$*,*

$$j = \frac{3(1-\alpha)(B-A)^2}{(3+(4-\alpha)B-(1-\alpha)A)^2(1+3\lambda)^n(1-\gamma+\gamma(1-3\lambda)^m)-(B-A)^2(1-\alpha)^2}$$

and $n \geq 0, m \geq 0, 0 \leq \gamma \leq 1, \lambda \geq 0, 0 \leq \alpha < 1$.

Proof. We have by Lemma

$$\sum_{k=2}^{\infty}[(2k-1)+(2k-\alpha)B-(1-\alpha)A]X^n(1-\gamma+\gamma X^m)[(B-A)(1-\alpha)]^{-1}a_{2k} \leq 1 \quad (17)$$

and

$$\sum_{k=2}^{\infty}[(2k-1)+(2k-\alpha)B-(1-\alpha)A]X^n(1-\gamma+\gamma X^m)[(B-A)(1-\alpha)]^{-1}b_{2k} \leq 1 \quad (18)$$

where $X = 1+(2k-1)\lambda$.

We want to find A_1, B_1 such that

$$-1 \leq A_1 < B_1 \leq 1 \quad \text{for} \quad q(z) \in T_2^*(n, m, \gamma, \lambda, A_1, B_1, \alpha,)$$

that is

$$\sum_{k=2}^{\infty}[(2k-1)+(2k-\alpha)B_1-(1-\alpha)A_1]X^n(1-\gamma+\gamma X^m)[(B_1-A_1)(1-\alpha)]^{-1}a_{2k}b_{2k} \leq 1. \quad (19)$$

By using Cauchy-Schwarz inequality, we get

$$\sum_{k=2}^{\infty} V(a_{2k}b_{2k})^{1/2} \le (\sum_{k=2}^{\infty} Va_{2k})^{1/2}(\sum_{k=2}^{\infty} Vb_{2k})^{1/2} \le 1 \tag{20}$$

where

$$V = [(2k-1)+(2k-\alpha)B-(1-\alpha)A]X^n(1-\gamma+\gamma X^m)[(B-A)(1-\alpha)]^{-1}. \tag{21}$$

If $V_1 a_{2k}b_{2k} \le V(a_{2k}b_{2k})^{1/2}$, then (19) is true, where

$$V_1 = [(2k-1)+(2k-\alpha)B_1-(1-\alpha)A_1]X^n(1-\gamma+\gamma X^m)[(B_1-A_1)(1-\alpha)]^{-1} \tag{22}$$

therefore, $V_1(a_{2k}b_{2k})^{1/2} \le V, k = 2, 3, 4, \cdots$.

In view of (20), we obtain

$$(a_{2k}b_{2k})^{1/2} \le V^{-1}. \tag{23}$$

Thus, we must find V_1, such that

$$V_1 = V^2, \tag{24}$$

that is,

$$((2k-1)+(2k-\alpha)B_1-(1-\alpha)A_1)X^n(1-\gamma+\gamma X^m) \le V^2((B_1-A_1)(1-\alpha)) \tag{25}$$

then

$$A_1 = \frac{V^2(1-\alpha)B_1 - ((2k-1)+(2k-\alpha)B_1)X^n(1-\gamma+\gamma X^m)}{(1-\alpha)(V^2 - X^n(1-\gamma+\gamma X^m))}. \tag{26}$$

It is clear that $V^2 \ge X^n(1-\gamma+\gamma X^m)$ for $k \ge 1$.

From (26) we can get

$$\frac{B_1-A_1}{B_1+1} \ge \frac{(2k-1)X^n(1-\gamma+\gamma X^m)}{(1-\alpha)(V^2 - X^n(1-\gamma+\gamma X^m))} \quad \text{for} \quad k \ge 2. \tag{27}$$

The right hand side of (27) is decreasing as k is increasing, then it has maximum for $k = 2$, thus (27) is true if

$$\begin{aligned} &\frac{B_1-A_1}{B_1+1} \\ \ge\ & \frac{3(1-\alpha)(B-A)^2}{(3+(4-\alpha)B-(1-\alpha)A)^2(1+3\lambda)^n(1-\gamma+\gamma(1+3\lambda)^m)-(B-A)^2(1-\alpha)^2} \\ =\ & j \end{aligned} \tag{28}$$

We can see that $j < 1$. Fixing A_1 in (28), we have

$$B_1 \ge \frac{j+A_1}{1-j} \tag{29}$$

and $-1 \le A_1 < B_1 \le 1$. Which completes the proof of theorem ■

Corollary 1 *If* $f(z) = z - \sum_{k=2}^{\infty} a_{2k} z^{2k}$ *and* $g(z) = z - \sum_{k=2}^{\infty} b_{2k} z^{2k}$, *where* $a_{2k} \geq 0, b_{2k} \geq 0$ *and* $f(z), g(z) \in S_2^*(n, \lambda, A, B, \alpha)$, *then* $q(z) = z - \sum_{k=2}^{\infty} a_{2k} b_{2k} z^{2k}$ *belongs to* $S_2^*(n, \lambda, A_1, B_1, \alpha)$ *with* $-1 \leq A_1 < B_1 \leq 1$ *and* $A_1 \leq 1 - 2j_1, B_1 \geq \frac{A_1 + j_1}{1 - j_1}$,

$$j_1 = \frac{3(1-\alpha)(B-A)^2}{(3 + (4-\alpha)B - (1-\alpha)A)^2(1+3\lambda)^n - (B-A)^2(1-\alpha)^2}.$$

Theorem 2 *Let* $f(z) \in T_2^*(n, m, \gamma, \lambda, A, B, \alpha)$ *and* $g(z) \in T_2^*(n, m, \gamma, \lambda, C, D, \alpha)$, *then* $f(z) * g(z) \in T_2^*(n, m, \gamma, \lambda, E, F, \alpha)$, *where* $E \leq 1 - 2j$ *and* $F \geq \frac{E+j}{1-j}$ *with*

$$j = [3(1-\alpha)(B-A)(D-C)]/[(3 + (4-\alpha)B - (1-\alpha)A)(3 + (4-\alpha)D - (1-\alpha)C) (1+3\lambda)^n(1-\gamma+\gamma(1+3\lambda)^m) - (B-A)(D-C)(1-\alpha)^2].$$

Proof. By virtue of Theorem 1, we require that

$$\begin{aligned} & \frac{(2k(F+1) - (1 - \alpha F + (1-\alpha)E)X^n(1-\gamma+\gamma X^m)}{(F-E)(1-\alpha)} \\ \leq \; & \frac{(2k(B+1) - (1 + \alpha B + (1-\alpha)A)X^n(1-\gamma+\gamma X^m)}{(B-A)(1-\alpha)} \times \\ & \frac{2k(D+1) - (1 + \alpha D + (1-\alpha)C)X^n(1-\gamma+\gamma X^m)}{(D-C)(1-\alpha)} = d \end{aligned} \tag{30}$$

where $X = (1 + (2k-1)\lambda), \quad \lambda \geq 0$, then by simple calculations, we have

$$\frac{F-E}{F+1} \geq \frac{(2k-1)X^n(1-\gamma+\gamma X^m)}{(1-\alpha)(d - X^n(1-\gamma+\gamma X^m))}. \tag{31}$$

The right hand side of (31) is decreasing as k is increasing and it has maximum for $k = 2$, then we obtain

$$\begin{aligned} \frac{F-E}{F+1} \geq \; & [3(1-\alpha)(B-A)(D-C)]/[(3 + (4-\alpha)B - (1-\alpha)A) \times \\ & (3 + (4-\alpha)D - (1-\alpha)C)(1+3\lambda)^n(1-\gamma+\gamma(1+3\lambda)^m) \\ - \; & (1-\alpha)^2(B-A)(D-C)] = j. \end{aligned}$$

It's clear that $j < 1$. Now fixing E in the last expression, we get $F \geq \frac{E+j}{1-j}$, so $F \leq 1$ and $E \leq 1 - 2j$. ■

Corollary 2 *Let* $f(z) \in S_2^*(n, \lambda, A, B, \alpha)$ *and* $g(z) \in S_2^*(n, \lambda, C, D, \alpha)$, *then* $f(z) * g(z) \in S_2^*(n, \lambda, E, F, \alpha)$ *where* $E \leq 1 - 2j_1$ *and* $F \geq \frac{E+j_1}{1-j_1}$ *with*

$$\begin{aligned} j_1 = \; & [3(1-\alpha)(B-A)(D-C)]/[(3 + (4-\alpha)B - (1-\alpha)A) \times \\ & (3 + (4-\alpha)D - (1-\alpha)C)(1+3\lambda)^n - (1-\alpha)^2(B-A)(D-C)]. \end{aligned}$$

Corollary 3 *Let* $f(z) \in K_2^*(n,m,\lambda,A,B,\alpha)$ *and* $g(z) \in K_2^*(n,m,\lambda,C,D,\alpha)$*, then* $f(z) * g(z) \in K_2^*(n,m,\lambda,E,F,\alpha)$ *where* $E \le 1-2j_3$ *and* $F \ge \frac{E+j_3}{1-j_3}$ *with*

$$\begin{aligned} j_3 &= [3(1-\alpha)(B-A)(D-C)]/[(3+(4-\alpha)B-(1-\alpha)A)\times \\ &\quad (3+(4-\alpha)D-(1-\alpha)C)(1+3\lambda)^{m+n}-(1-\alpha)^2(B-A)(D-C)]. \end{aligned}$$

Theorem 3 *Let* $f(z) = z - \sum_{k=2}^{\infty} a_{2k}z^{2k}, a_{2k} \ge 0$ *belong to* $T_2^*(n,m,\gamma,\lambda,A,B,\alpha)$ *and* $g(z) = z - \sum_{k=2}^{\infty} b_{2k}z^{2k}$ *with* $|b_{2i}| \le 1$ *for* $i \ge 1$*, then* $f * g \in T(n,m,\gamma,\lambda,A,B,\alpha)$*.*

Proof. By assumption we have

$$\sum_{k=2}^{\infty} \frac{[(2k-1)+(2k-\alpha)B-(1-\alpha)A](1+(2k-1)\lambda)^n[1-\gamma+\gamma(1+(2k-1)\lambda)^m]}{(B-A)(1-\alpha)} a_{2k} \le 1$$

and since $|b_{2i}| \le 1$ for $i \ge 1$, then

$$\sum_{k=2}^{\infty} \frac{[(2k-1)+(2k-\alpha)B-(1-\alpha)A](1+(2k-1)\lambda)^n[1-\gamma+\gamma(1+(2k-1)\lambda)^m]}{(B-A)(1-\alpha)} a_{2k}b_{2k} \le 1$$

$$\sum_{k=2}^{\infty} \frac{[(2k-1)+(2k-\alpha)B-(1-\alpha)A](1+(2k-1)\lambda)^n[1-\gamma+\gamma(1+(2k-1)\lambda)^m]}{(B-A)(1-\alpha)} a_{2k}|b_{2k}| \le 1.$$

That is $f(z) * g(z) = z - \sum_{k=2}^{\infty} a_{2k}b_{2k}z^{2k} \in T(n,m,\gamma,\lambda,A,B,\alpha)$. ■

Corollary 4 *Let* $f(z) = z - \sum_{k=2}^{\infty} a_{2k}z^{2k}, a_{2k} \ge 0$ *belongs to* $S_2^*(n,\lambda,A,B,\alpha)$ *and* $g(z) = z - \sum_{k=2}^{\infty} b_{2k}z^{2k}$ *with* $|b_{2i}| \le 1$ *for* $i \ge 1$*, then* $f * g \in S(n,\lambda,A,B,\alpha)$*.*

Corollary 5 *Let* $f(z) = z - \sum_{k=2}^{\infty} a_{2k}z^{2k}, a_{2k} \ge 0$ *belongs to* $K_2^*(n,m,\lambda,A,B,\alpha)$ *and* $g(z) = z - \sum_{k=2}^{\infty} b_{2k}z^{2k}$ *with* $|b_{2i}| \le 1$ *for* $i \ge 1$*, then* $f * g \in K(n,m,\lambda,A,B,\alpha)$*.*

Theorem 4 *Let* $f, g \in T_2^*(n,m,\gamma,\lambda,A,B,\alpha)$*, then* $q(z) = z - \sum_{k=2}^{\infty}(a_{2k}^2+b_{2k}^2) \in T_2^*(n,m,\gamma,\lambda,A_1,B_1,\alpha)$*, where* $A_1 \le 1-2j$ *and* $B_1 \ge \frac{A_1+j}{1-j}$ *with*

$$j = \frac{6(1-\alpha)(B-A)^2}{(3+(4-\alpha)B-(1-\alpha)A)^2(1+3\lambda)^n(1-\gamma+\gamma(1+3\lambda)^m)-2(B-A)^2(1-\alpha)^2}.$$

Proof. By assumption, we have

$$\sum_{k=2}^{\infty} \frac{[(2k-1)+(2k-\alpha)B-(1-\alpha)A]X^n(1-\gamma+\gamma X^m)}{(B-A)(1-\alpha)} a_{2k} \le 1$$

$$\sum_{k=2}^{\infty} \frac{[(2k-1)+(2k-\alpha)B-(1-\alpha)A]X^n(1-\gamma+\gamma X^m)}{(B-A)(1-\alpha)} b_{2k} \leq 1$$

where $X = 1+(2k-1)\lambda$. Thus

$$\sum_{k=2}^{\infty} \left(\frac{[(2k-1)+(2k-\alpha)B-(1-\alpha)A]X^n(1-\gamma+\gamma X^m)}{(B-A)(1-\alpha)} a_{2k} \right)^2$$
$$\leq \left(\sum_{k=2}^{\infty} \frac{[(2k-1)+(2k-\alpha)B-(1-\alpha)A]X^n(1-\gamma+\gamma X^m)}{(B-A)(1-\alpha)} a_{2k} \right)^2 \leq 1$$

and so,

$$\sum_{k=2}^{\infty} \left(\frac{[(2k-1)+(2k-\alpha)B-(1-\alpha)A]X^n(1-\gamma+\gamma X^m)}{(B-A)(1-\alpha)} b_{2k} \right)^2 \leq 1 \tag{32}$$

then we may write

$$\sum_{k=2}^{\infty} \frac{1}{2} \left(\frac{[(2k-1)+(2k-\alpha)B-(1-\alpha)A]X^n(1-\gamma+\gamma X^m)}{(B-A)(1-\alpha)} \right)^2 (a_{2k}^2+b_{2k}^2) \leq 1 \tag{33}$$

Therefore, in view of (32) the inequality (33) holds if

$$\frac{[(2k-1)+(2k-\alpha)B_1-(1-\alpha)A_1]X^n(1-\gamma+\gamma X^m)}{(B_1-A_1)(1-\alpha)}$$
$$\leq \frac{1}{2} \left(\frac{[(2k-1)+(2k-\alpha)B-(1-\alpha)A]X^n(1-\gamma+\gamma X^m)}{(B-A)(1-\alpha)} \right)^2 = \frac{V^2}{2}$$

and by simplification, the last inequality gives

$$\frac{B_1-A_1}{B_1+1} \geq \frac{2(2k-1)X^n(1-\gamma+\gamma X^m)}{(1-\alpha)(V^2-2X^n(1-\gamma+\gamma X^m)}. \tag{34}$$

The right hand side of (34) is decreasing as k is increasing and if we put $k=2$, we obtain

$$\frac{B_1-A_1}{B_1+1} \geq [6(1-\alpha)(B-A)^2]/[(3+(4-\alpha)B-(1-\alpha)A)^2(1+3\lambda)^n$$
$$(1-\gamma+\gamma(1+3\lambda)^m)-2(B-A)^2(1-\alpha)^2] = j.$$

Now fixing A_1, we have $B_1 \geq \frac{A_1+j}{1-j}$ and $B_1 \leq 1$ gives us $A_1 \leq 1-2j$. ■

Corollary 6 *Let* $f, g \in S_2^*(n,\lambda,A,B,\alpha)$, *then* $q(z) = z - \sum_{k=2}^{\infty}(a_{2k}+b_{2k})^2 z^{2k} \in S_2^*(n,\lambda,A_1,B_1,\alpha)$, *where* $A_1 \leq 1-2j_1$, *and* $B_1 \geq \frac{A_1+j_1}{1-j_1}$ *with*

$$j_1 = \frac{6(1-\alpha)(B-A)^2}{(3+(4-\alpha)B-(1-\alpha)A)^2(1+3\lambda)^n-2(B-A)^2(1-\alpha)^2}.$$

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References

[1] F. M. Al-Oboudi, *On univalent functions defined by generalized Sâlâgean operator*, JMMS, 27 (2004), 1429-1436.

[2] M. K. Aouf, H. M. Hossen and A. Y. Lashin, *On certain families of analytic functions with negative coefficients*, J. Pure Appl. Math., 31 (8) (2000), 999-1015.

[3] S. B. Joshi , *A study of univalent and multivalent functions*, Ph.D. Thesis, Shivaji University, Kolhapur, India (1994).

[4] G.Murugusundaramoorthy and K.G.Subramanian, *Univalent solutions of Briot - Bouquet differential equations*, SEAMS (2005), 297-309.

[5] U. H. Naik , *Topics in univalent and multivalent functions in geometric function theory*, Ph.D. Thesis, Shivaji University, Kolhapur, India (1997).

[6] K. S. Padmanabhan and M. S. Ganeshan, *Convolution of certain classes of univalent functions with negative coefficients*, Indian J. Pure Appl. Math., 19 (9) (1988), 880-889.

[7] G. S. Sâlâgean, *Subclasses of univalent functions, Complex Analysis - Fifth Romanian - Finnish Seminar*, Part 1 (Bucharest, 1981), Lecture Notes in Math., 1013 (1983), 362-372.

[8] H. Silverman and R. D. Berman, *Coefficient inequalities for a subclass of starlike functions*, J. Math. Anal. Appl. 107(1) (1985), 197-205.

[9] H. Silverman, *Univalent functions with negative coefficients*, Proc. Amer. Math. Soc. ,**51**, (1975), 109 - 116.

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A class of harmonic starlike functions with respect to other points defined by Dziok-Srivastava operator

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ABSTRACT: Making use of Dziok-Srivastava operator we introduced a new class of complex-valued harmonic functions which are orientation preserving , univalent and starlike with respect to other points. We investigate the coefficient bounds,distortion inequalities, extreme points and inclusion results for the generalized class of functions

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1. Introduction

A continuous function $f = u+iv$ is a complex- valued harmonic function in a complex domain Ω if both u and v are real and harmonic in Ω. In any simply connected domain $D \subset \Omega$ we can write $f = h+\overline{g}$ where h and g are analytic in D. We call h the analytic part and g the co-analytic part of f. A necessary and sufficient condition for f to be locally univalent and orientation preserving in D is that $|h'(z)| > |g'(z)|$ in D (see [2]).

Denote by $\mathcal{H}$ the family of functions

$$f = h + \overline{g} \tag{1}$$

which are harmonic univalent and orientation preserving in the open unit disc $U = \{z : |z| < 1\}$ so that f is normalized by $f(0) = h(0) = f_z(0) - 1 = 0$. Thus , for $f = h + \overline{g} \in \mathcal{H}$, we may express the analytic functions h and g in the forms

$$h(z) = z + \sum_{n=2}^{\infty} a_n z^n, g(z) = \sum_{n=1}^{\infty} b_n z^n, (0 \leq b_1 < 1).$$

Hence

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n + \overline{\sum_{n=1}^{\infty} b_n z^n}, \ \ |b_1| < 1. \tag{2}$$

We note that the family $\mathcal{H}$ of orientation preserving,normalized harmonic univalent functions reduces to the well known class S of normalized univalent functions if the co-analytic part of $f = h + \overline{g}$ is identically zero that is $g \equiv 0$. Due to Silverman[11] we denote $\overline{\mathcal{H}}$ the subclass of $\mathcal{H}$ consists harmonic functions $f = h + \overline{g}$ of the form

$$f(z) = z - \sum_{n=2}^{\infty} a_n z^n + \overline{\sum_{n=1}^{\infty} b_n z^n}, \ \ |b_1| < 1. \tag{3}$$

Let the hadamard product (or convolution) of two power series $\phi(z) = z + \sum\limits_{n=2}^{\infty} \phi_n z^n$ and $\psi(z) = z + \sum\limits_{n=2}^{\infty} \psi_n z^n$ be defined by

$$(\phi * \psi)(z) = \phi(z) * \psi(z) = z + \sum_{n=2}^{\infty} \phi_n \psi_n z^n.$$

For complex parameters $\alpha_1, \ldots, \alpha_l$ and $\beta_1, \ldots, \beta_m$ $(\beta_j \neq 0, -1, \ldots; j = 1, 2, \ldots, m)$ the *generalized hypergeometric function* ${}_lF_m(z)$ is defined by

$$\begin{aligned} {}_lF_m(z) \equiv {}_lF_m(\alpha_1, \ldots \alpha_l; \beta_1, \ldots, \beta_m; z) &:= \sum_{n=0}^{\infty} \frac{(\alpha_1)_n \ldots (\alpha_l)_n}{(\beta_1)_n \ldots (\beta_m)_n} \frac{z^n}{n!} \\ (l \leq m+1; \ l, m \in N_0 &:= N \cup \{0\}; z \in U) \end{aligned} \tag{4}$$

where N denotes the set of all positive integers and $(a)_n$ is the Pochhammer symbol defined by

$$(a)_n = \begin{cases} 1, & n = 0 \\ a(a+1)(a+2)\ldots(a+n-1), & n \in N. \end{cases} \tag{5}$$

For positive real values of $\alpha_1, \ldots, \alpha_l$ and $\beta_1, \ldots, \beta_m$ $(\beta_j \neq 0, -1, \ldots; j = 1, 2, \ldots, m)$, let

$$H(\alpha_1, \ldots \alpha_l; \beta_1, \ldots, \beta_m) : S \to S$$

be a linear operator defined by

$$\begin{aligned} [(H(\alpha_1, \ldots \alpha_l; \beta_1, \ldots, \beta_m))(\phi)](z) &= z \ {}_lF_m(\alpha_1, \alpha_2, \ldots \alpha_l; \beta_1, \beta_2 \ldots, \beta_m; z) * \phi(z) \\ &= z + \sum_{n=2}^{\infty} \Gamma(\alpha_1, n) \ \phi_n \ z^n \end{aligned} \tag{6}$$

where

$$\Gamma(\alpha_1, n) = \left| \frac{(\alpha_1)_{n-1} \ldots (\alpha_l)_{n-1}}{(\beta_1)_{n-1} \ldots (\beta_m)_{n-1}} \frac{1}{(n-1)!} \right| \tag{7}$$

$\alpha_i > 0, (i = 1, 2, ...l), \beta_j > 0, (j = 1, 2, ...m), l \leq m + 1;\ l, m \in N_0 = N \cup \{0\}$.

For notational simplicity, we use a shorter notation $H^l_m[\alpha_1]$ for $H(\alpha_1, \ldots \alpha_l; \beta_1, \ldots, \beta_m)$ in the sequel. It follows from (6) that

$$H^1_0[1]\phi(z) = \phi(z), H^1_0[2]\phi(z) = z\phi'(z)$$

The linear operator $H^l_m[\alpha_1]$ is called Dziok-Srivastava operator (see [5]),which contains such well known operators as the Hohlov linear operator, Saitho generalized linear operator, the Carlson-Shaffer linear operator, the Ruscheweyh derivative operator as well as its generalized versions, the Bernardi-Libera-Livingston operator, and the Srivastave-Owa fractional derivative operator. One may refer to [4], [5] and [12] for more details concerning these operators(see[3, 8, 9, 10]. Applying the Dziok-Srivastava operator to the harmonic functions $f = h + \overline{g}$ given by (1) we get

$$H^l_m[\alpha_1]f(z) = H^l_m[\alpha_1]h(z) + \overline{H^l_m[\alpha_1]g(z)} \tag{8}$$

Motivated by Jahangiri etal.[6, 7]and Ahujha and Jahangiri[1], we define a new subclass $\mathcal{HS}_s([\alpha_1], \gamma)$ of $\mathcal{H}$ that are starlike with respect to other points .

For $0 \leq \gamma < 1$, we let $\mathcal{HS}_s([\alpha_1], \gamma)$ a subclass of $\mathcal{H}$ of the form $f = h + \overline{g}$ given by (2) and satisfying the analytic criteria

$$\text{Re}\ \left\{\frac{2z(H^l_m[\alpha_1]f(z))'}{z'[H^l_m[\alpha_1]f(z) - H^l_m[\alpha_1]f(-z)]}\right\} > \gamma \tag{9}$$

where $H^l_m[\alpha_1]f(z)$ as given in (8),$z' = \frac{\partial}{\partial\theta}(z = re^{i\theta})$ and $z \in U$.

We also let $\overline{\mathcal{H}}\mathcal{S}_s([\alpha_1], \gamma) = \mathcal{HS}_s([\alpha_1], \gamma) \bigcap \overline{\mathcal{H}}$.

The family $\mathcal{HS}_s([\alpha_1], \gamma)$ is of special interest because for suitable choices of l, m and $[\alpha_1]$ we can state the following. From (8) we note that

(i) $H^1_0([1])f(z) = f(z)$ hence we define a class $\mathcal{HS}_s(\gamma)$ satisfying the criteria

$$\text{Re}\ \left\{\frac{2z(f(z))'}{z'[f(z) - f(-z)]}\right\} > \gamma,\ \ (0 \leq \gamma < 1).$$

(ii) $H^2_1([a, 1; c]) = \mathcal{L}(a, c)f(z)$, hence we define a class $\mathcal{HS}_s(a, c; \gamma)$satisfying the criteria

$$\text{Re}\ \left\{\frac{2z(\mathcal{L}(a, c)f(z))'}{z'[\mathcal{L}(a, c)f(z) - \mathcal{L}(a, c)f(-z)]}\right\} > \gamma,\ \ (0 \leq \gamma < 1).$$

where $\mathcal{L}(a, c)$ is the Carlson - Shaffer operator[4] .

(iii) $H^2_1([\lambda + 1, 1; 1]) = D^\lambda f(z)$, hence we define a class $\mathcal{HS}_s(\lambda, \gamma)$ satisfying the criteria

$$\text{Re}\ \left\{\frac{2z(D^\lambda f(z))'}{z'[D^\lambda f(z) - D^\lambda f(-z)]}\right\} > \gamma,\ \ (0 \leq \gamma < 1).$$

where $D^\lambda(\lambda > -1)$ is the Ruscheweyh derivative operator[10] .

(iv) $\mathcal{H}^2_1([2, 1; 2 - \mu]) = \Omega^\mu_z f(z)$ we define another class $\mathcal{HS}_s(\mu, \gamma)$ satisfying the condition

$$\text{Re}\ \left\{\frac{2z(\Omega^\mu_z f(z))'}{z'[\Omega^\mu_z f(z) - \Omega^\mu_z f(-z)]}\right\} > \gamma\ \ (0 \leq \gamma < 1).$$

given by

$$\Omega_z^\mu f(z) = \Gamma(2-\mu) z^\mu D_z^\mu f(z) (0 \leq \mu < 1) \ ,$$

where Ω_z^μ is the Srivastava-Owa fractional derivative operator [12].

In this paper, we obtained coefficient conditions for the classes $\mathcal{HS}_s([\alpha_1], \gamma)$ and $\overline{\mathcal{HS}}_s([\alpha_1], \gamma)$. A representation theorem, inclusion properties and distortion bounds for the class $\overline{\mathcal{HS}}_s([\alpha_1], \gamma)$ are also established.

2. Coefficient Bounds

In our first theorem, we obtain a sufficient coefficient bound for harmonic functions in $\mathcal{HS}_s([\alpha_1], \gamma)$.

Theorem 1 *Let $f = h + g$ be given by (2).If*

$$\sum_{n=2}^{\infty} \frac{[2n - \gamma(1-(-1)^n)]\Gamma(\alpha_1, n)}{2(1-\gamma)} |a_n| + \sum_{n=1}^{\infty} \frac{[2n + \gamma(1-(-1)^n)]\Gamma(\alpha_1, n)}{2(1-\gamma)} |b_n| \leq 1 \qquad (10)$$

where $a_1 = 1, 0 \leq \gamma < 1$ and $z \in U$. Then $f(z) \in \mathcal{HS}_s([\alpha_1], \gamma)$.

Proof. According the condition (9), we only need to show that if (10) holds, then

$$Re \left\{ \frac{2z(H_m^l[\alpha_1]f(z))'}{z'[H_m^l[\alpha_1]f(z) - H_m^l[\alpha_1]f(-z)]} \right\} = \text{Re}\, \frac{A(z)}{B(z)} \geq \gamma,$$

where

$$A(z) = 2z(H_m^l[\alpha_1]f(z))' = 2[z + \sum_{n=2}^{\infty} n\Gamma(\alpha_1, n) a_n z^n - \sum_{n=1}^{\infty} n\Gamma(\alpha_1, n) \overline{b}_n \overline{z}^n]$$

and

$$\begin{aligned} B(z) &= z'[H_m^l[\alpha_1]f(z) - H_m^l[\alpha_1]f(-z)] \\ &= 2z + \sum_{n=2}^{\infty} [1-(-1)^n]\Gamma(\alpha_1, n) a_n z^n + \sum_{n=1}^{\infty} [1-(-1)^n]\Gamma(\alpha_1, n) \overline{b}_n \overline{z}^n. \end{aligned}$$

Using the fact that Re $\{w(z)\} \geq \gamma$ if and only if $|1 - \gamma + w| \geq |1 + \gamma - w|$.
That is,

$$|A(z) + (1-\gamma)B(z)| - |A(z) - (1+\gamma)B(z)| \geq 0.$$

Substituting for $A(z)$ and $B(z)$ we get

$$\begin{aligned}
& |A(z)+(1-\gamma)B(z)|-|A(z)-(1+\gamma)B(z)| \\
= & \ | \ [2+2(1-\gamma)]z+\sum_{n=2}^{\infty}\{2n+(1-\gamma)[1-(-1)^n]\}\Gamma(\alpha_1,n)a_n z^n \\
& -\sum_{n=1}^{\infty}\{2n-(1-\gamma)[1-(-1)^n]\}\Gamma(\alpha_1,n)\overline{b}_n \ \overline{z}^n \ | \\
& -| \ [2-2(1+\gamma)]z+\sum_{n=2}^{\infty}\{2n-(1+\gamma)[1-(-1)^n]\}\Gamma(\alpha_1,n)a_n z^n \\
& -\sum_{n=1}^{\infty}\{2n+(1+\gamma)[1-(-1)^n]\}\Gamma(\alpha_1,n)\overline{b}_n\overline{z}^n \ | \\
\geq & \ [2+2(1-\gamma)]|z|-\sum_{n=2}^{\infty}\{2n+(1-\gamma)[1-(-1)^n]\}\Gamma(\alpha_1,n)|a_n||z|^n \\
& -\sum_{n=1}^{\infty}\{2n-(1-\gamma)[1-(-1)^n]\}\Gamma(\alpha_1,n)|b_n| \ |z|^n \\
& -2\gamma|z|-\sum_{n=2}^{\infty}\{2n-(1+\gamma)[1-(-1)^n]\}\Gamma(\alpha_1,n)|a_n| \ |z|^n \\
& -\sum_{n=1}^{\infty}\{2n+(1+\gamma)[1-(-1)^n]\Gamma(\alpha_1,n)|b_n| \ |z|^n \\
\geq & \ 4(1-\gamma)|z|\left\{1-\sum_{n=1}^{\infty}\Gamma(\alpha_1,n)\left[\frac{2n-\gamma[1-(-1)^n]}{2(1-\gamma)}|a_n|\right.\right. \\
& \left.\left.-\frac{2n+\gamma[1-(-1)^n]}{2(1-\gamma)}|b_n|\right]|z|^{n-1}\right\} \\
\geq & \ 0,
\end{aligned}$$

by (10). ■ The harmonic functions

$$f(z)=z+\sum_{n=2}^{\infty}\frac{2(1-\gamma)}{\Gamma(\alpha_1,n)\{2n-\gamma[1-(-1)^n]\}}x_n z^n+\sum_{n=1}^{\infty}\frac{2(1-\gamma)}{\Gamma(\alpha_1,n)\{2n+\gamma[1-(-1)^n]\}}\overline{y}_n\overline{z}^n,$$

where $\sum\limits_{n=2}^{\infty}|x_n|+\sum\limits_{n=1}^{\infty}|y_n|=1$, shows that the coefficient bound given by (10) is sharp. The functions of the form (2) are in $\mathcal{HS}_s([\alpha_1],\gamma)$ because

$$\sum_{n=2}^{\infty}\frac{\{2n-\gamma[1-(-1)^n]\}}{2(1-\gamma)}\Gamma(\alpha_1,n)|a_n| \ +\sum_{n=1}^{\infty}\frac{\{2n+\gamma[1-(-1)^n]\}}{2(1-\gamma)}\Gamma(\alpha_1,n)|b_n|\ |z|^n$$
$$=\sum_{n=2}^{\infty}|x_n|+\sum_{n=1}^{\infty}|y_n|=1.$$

The following theorem establishes that such coefficient bounds cannot be improved further .

Theorem 2 *Let $f=h+\overline{g}$ be given by (3).Then $f\in\overline{\mathcal{H}}S_s([\alpha_1],\gamma)$ if and only if*

$$\sum_{n=2}^{\infty}\frac{[2n-\gamma-(1-(-1)^n)]}{2(1-\gamma)}\Gamma(\alpha_1,n)|a_n|+\sum_{n=1}^{\infty}\frac{[2n+\gamma(1-(-1)^n)]}{2(1-\gamma)}\Gamma(\alpha_1,n)|b_n|\leq 1. \tag{11}$$

where $a_1=1, 0\leq\gamma<1$ and $z\in U$. **Proof.** Since $\overline{\mathcal{H}}S_s([\alpha_1],\gamma)\subset\mathcal{HS}_s([\alpha_1],\gamma)$, we only need to prove the "only if" part of the theorem. For the *only if* part, we assume that $f(z)\in\overline{\mathcal{H}}S_s([\alpha_1],\gamma)$. For functions $f(z)$ of the form (3) we notice that the condition (9) is equivalent to

$$\begin{aligned}
&\operatorname{Re}\left\{\frac{2[z(H_m^l[\alpha_1]h(z))'-\overline{z(H_m^l[\alpha_1]g(z))'}]}{H_m^l[\alpha_1]h(z)+\overline{H_m^l[\alpha_1]g(z)}-H_m^l[\alpha_1]h(z)-\overline{H_m^l[\alpha_1]g(-z)}}-\gamma\right\}\\
=\ &\operatorname{Re}\ \left\{\left[2(1-\gamma)-\sum_{n=2}^{\infty}[2n-\gamma(1-(-1)^n)]\Gamma(\alpha_1,n)|a_n|z^{n-1}\right.\right.\\
&\left.-\frac{\overline{z}}{z}\sum_{n=1}^{\infty}[2n+\gamma(1-(-1)^n)]\Gamma(\alpha_1,n)|b_n|\overline{z}^{n-1}\right]\Big/\\
&\left.\left[2-\sum_{n=2}^{\infty}(1-(-1)^n)\Gamma(\alpha_1,n)|a_n|z^{n-1}+\frac{\overline{z}}{z}\sum_{n=1}^{\infty}(1-(-1)^n)\Gamma(\alpha_1,n)|b_n|\overline{z}^{n-1}\right]\right\}\\
\geq\ &0.
\end{aligned}$$

The above required condition must hold for all values of z in U. Upon choosing the values of z on the positive real axis where $0\leq z=r<1$, we have

$$\begin{aligned}
&\left\{\left[2(1-\gamma)-\sum_{n=2}^{\infty}[2n-\gamma(1-(-1)^n)]\Gamma(\alpha_1,n)|a_n|r^{n-1}\right.\right.\\
&\left.-\sum_{n=1}^{\infty}[2n+\gamma(1-(-1)^n)]\Gamma(\alpha_1,n)|b_n|r^{n-1}\right]\Big/\\
&\left.\left[2-\sum_{n=2}^{\infty}(1-(-1)^n)\Gamma(\alpha_1,n)|a_n|r^{n-1}+\sum_{n=1}^{\infty}(1-(-1)^n)\Gamma(\alpha_1,n)|b_n|r^{n-1}\right]\right\}\geq 0.
\end{aligned}$$

If the condition (11) does not hold, then the numerator in (**??**) is negative for r sufficiently close to 1. Hence, there exist $z_0 = r_0$ in (0,1) for which the quotient of (**??**) is negative. This contradicts the required condition for $f(z) \in \overline{\mathcal{H}}S_s([\alpha_1], \gamma)$. ■

From the above theorem ,for suitable choices of l, m and $[\alpha_1]$ we state the necessary and sufficient conditions for the various subclasses as corollaries.

Corollary 1 *For* $a_1 = 1, 0 \leq \gamma < 1,\ f = h + \overline{g} \in \overline{\mathcal{H}}S_s(\gamma)$ *if and only if*

$$\sum_{n=2}^{\infty} \frac{[2n - \gamma - (1 - (-1)^n)]}{2(1-\gamma)}|a_n| + \sum_{n=1}^{\infty} \frac{[2n + \gamma(1 - (-1)^n)]}{2(1-\gamma)}|b_n| \leq 1. \tag{12}$$

Corollary 2 *For* $a_1 = 1, 0 \leq \gamma < 1,\ f = h + \overline{g} \in \overline{\mathcal{H}}S_s(a, c; \gamma)$ *if and only if*

$$\sum_{n=2}^{\infty} \frac{[2n - \gamma - (1 - (-1)^n)]}{2(1-\gamma)} \frac{(a)_n}{(b)_n}|a_n| + \sum_{n=1}^{\infty} \frac{[2n + \gamma(1 - (-1)^n)]}{2(1-\gamma)} \frac{(a)_n}{(b)_n}|b_n| \leq 1. \tag{13}$$

where $(a)_n$ *is given by (5)*

Corollary 3 *For* $a_1 = 1, 0 \leq \gamma < 1,\ f = h + \overline{g} \in \overline{\mathcal{H}}S_s(\lambda, \gamma)$ *if and only if*

$$\sum_{n=2}^{\infty} \frac{[2n - \gamma - (1 - (-1)^n)]}{2(1-\gamma)} C(\lambda, n)|a_n| + \sum_{n=1}^{\infty} \frac{[2n + \gamma(1 - (-1)^n)]}{2(1-\gamma)} C(\lambda, n)|b_n| \leq 1 \tag{14}$$

where $C(\lambda, n) = \binom{\lambda + n - 1}{n - 1}$

Corollary 4 *For* $a_1 = 1, 0 \leq \gamma < 1,\ f = h + \overline{g} \in \overline{\mathcal{H}}S_s(\mu, \gamma)$ *if and only if*

$$\sum_{n=2}^{\infty} \frac{[2n - \gamma - (1 - (-1)^n)]}{2(1-\gamma)} \psi(n)|a_n| + \sum_{n=1}^{\infty} \frac{[2n + \gamma(1 - (-1)^n)]}{2(1-\gamma)} \psi(n)|b_n| \leq 1. \tag{15}$$

where $\psi(n) = \frac{\Gamma(n+1)\Gamma(2-\mu)}{\Gamma(n+1-\mu)}$

3. Distortion Bounds and extreme points

Now we obtain the growth result for functions in $\overline{\mathcal{H}}S_s([\alpha_1], \gamma)$.

Theorem 3 *Let* $f \in \overline{\mathcal{H}}S_s([\alpha_1], \gamma)$, *then*

$$|f(z)| \leq (1 + b_1)r + \frac{1}{\Gamma(\alpha_1, n)} \left(\frac{1-\gamma}{2} - \frac{1+\gamma}{2}|b_1| \right) r^2, \quad |z| = r < 1,$$

and

$$|f(z)| \geq (1 - b_1)r - \frac{1}{\Gamma(\alpha_1, n)} \left(\frac{1-\gamma}{2} - \frac{1+\gamma}{2}|b_1| \right) r^2, \quad |z| = r < 1.$$

Proof. We prove only the left hand inequality, let $f(z) \in \overline{\mathcal{H}}S_s([\alpha_1], \gamma)$. Taking the absolute value of $f(z)$, we have

$$\begin{aligned}
|f(z)| &\geq (1-|b_1|)r - \sum_{n=2}^{\infty}(|a_n|+|b_n|)r^n \\
&\geq (1-|b_1|)r - \sum_{n=2}^{\infty}(|a_n|+|b_n|)r^2 \\
&= (1-|b_1|)r - \frac{1-\gamma}{2\Gamma(\alpha_1,2)}\sum_{n=2}^{\infty}\left(\frac{2\Gamma(\alpha_1,n)}{1-\gamma}|a_n| + \frac{2\Gamma(\alpha_1,n)}{1-\gamma}|b_n|\right)r^2 \\
&\geq (1-|b_1|)r - \\
&\quad \frac{(1-\gamma)r^2}{2\Gamma(\alpha_1,2)}\sum_{n=2}^{\infty}\left(\frac{2n-\gamma(1-(-1)^n)}{2(1-\gamma)}|a_n| + \frac{2n+\gamma(1-(-1)^n)}{2(1-\gamma)}|b_n|\right)\Gamma(\alpha_1,n) \\
&\geq (1-|b_1|)r - \frac{1}{\Gamma(\alpha_1,2)}\left(\frac{1-\gamma}{2} - \frac{1+\gamma}{2}|b_1|\right)r^2.
\end{aligned}$$

The proof of the right hand inequality follows on lines similar to that of the left hand inequality.Which completes the proof of Theorem 3. ■

Now we determine the extreme points of closed convex hulls of $\overline{\mathcal{H}}S_s([\alpha_1], \gamma)$ denoted $clco\overline{\mathcal{H}}S_s([\alpha_1], \gamma)$.

Theorem 4 *A function* $f = h + \overline{g} \in clco\overline{\mathcal{H}}S_s([\alpha_1], \gamma)$ *if and only if* $f(z)$ *can be expressed in the form* $f(z) = \sum_{n=1}^{\infty}(X_n h_n(z) + Y_n g_n(z))$ *where*

$$h_1(z) = z, h_n(z) = z - \frac{2(1-\gamma)}{\Gamma(\alpha_1,n)[2n-\gamma(1-(-1)^n)]}z^n, \ \ (n = 2, 3, \dots);$$

$$g_n(z) = z + \frac{(1-\gamma)}{\Gamma(\alpha_1,n)[2n+\gamma(1-(-1)^n)]}\overline{z}^n, \ (n = 1, 2, \dots);$$

$$\sum_{n=1}^{\infty}(X_n + Y_n) = 1, \quad X_n \geq 0 \ \ \textit{and} \ \ Y_n \geq 0.$$

Proof. For functions $f(z)$ as in Theorem 4, we have

$$\begin{aligned}
f(z) &= \sum_{n=1}^{\infty}(X_n h_n(z) + Y_n g_n(z)) \\
&= z - \sum_{n=2}^{\infty}\frac{2(1-\gamma)}{\Gamma(\alpha_1,n)[2n-\gamma(1-(-1)^n)]}X_n z^n \\
&\quad + \sum_{n=1}^{\infty}\frac{2(1-\gamma)}{\Gamma(\alpha_1,n)[2n+\gamma(1-(-1)^n)]}Y_n \overline{z}^n
\end{aligned}$$

Then by Theorem 2

$$\begin{aligned}
&\sum_{n=2}^{\infty}\frac{[2n-\gamma(1-(-1)^n)]}{2(1-\gamma)}\Gamma(\alpha_1,n)|a_n|+\sum_{n=1}^{\infty}\frac{[2n+\gamma(1-(-1)^n)]}{2(1-\gamma)}\Gamma(\alpha_1,n)|b_n|\\
=\ &\sum_{n=2}^{\infty}\frac{\Gamma(\alpha_1,n)[2n-\gamma(1-(-1)^n)]}{2(1-\gamma)}\left(\frac{2(1-\gamma)}{\Gamma(\alpha_1,n)[2n-\gamma(1-(-1)^n)]}X_n\right)\\
&+\sum_{n=1}^{\infty}\frac{\Gamma(\alpha_1,n)[2n+\gamma(1-(-1)^n)]}{2(1-\gamma)}\left(\frac{2(1-\gamma)}{\Gamma(\alpha_1,n)[2n+\gamma(1-(-1)^n)]}Y_n\right)\\
&=\sum_{n=2}^{\infty}X_n+\sum_{n=1}^{\infty}Y_n=1-X_1\leq 1.
\end{aligned}$$

Therefore, $f(z)\in clco\,\overline{\mathcal{H}}S_s([\alpha_1],\gamma)$. Conversely, suppose that $f(z)\in clco\,\overline{\mathcal{H}}S_s([\alpha_1],\gamma)$. Set

$$X_n=\frac{[2n-\gamma(1-(-1)^n)]}{2(1-\gamma)}\Gamma(\alpha_1,n)|a_n|, n=2,3,\ldots,$$

and

$$Y_n=\frac{[2n+\gamma(1-(-1)^n)]}{2(1-\gamma)}\Gamma(\alpha_1,n)|b_n|, n=1,2,\ldots,$$

where $\sum\limits_{n=1}^{\infty}(X_n+Y_n)=1$. Then

$$\begin{aligned}
f(z) &= z-\sum_{n=2}^{\infty}a_nz^n+\sum_{n=1}^{\infty}b_n\overline{z}^n\\
&= z-\sum_{n=2}^{\infty}\frac{2(1-\gamma)}{\Gamma(\alpha_1,n)[2n-\gamma(1-(-1)^n)]}X_nz^n\\
&\quad+\sum_{n=1}^{\infty}\frac{2(1-\gamma)}{\Gamma(\alpha_1,n)[2n+\gamma(1-(-1)^n)]}Y_n\overline{z}^n\\
&= z-\sum_{n=2}^{\infty}[X_n(h_n(z)-z)]+\sum_{n=1}^{\infty}[Y_n(g_n(z)-z)]\\
&= \sum_{n=1}^{\infty}(X_nh_n(z)+Y_ng_n(z))
\end{aligned}$$

as required. ■

4. Inclusion results

Now we show that $\overline{\mathcal{H}}S_s([\alpha_1],\gamma)$ is closed under convex combinations of its member and also closed under the convolution product .

Theorem 5 *For $0 \leq \nu \leq \gamma < 1$, let $f(z) \in \overline{\mathcal{H}}S_s([\alpha_1], \gamma)$ and $F(z) \in \overline{\mathcal{H}}S_s([\alpha_1], \nu)$. Then $(f * F) \in \overline{\mathcal{H}}S_s([\alpha_1], \gamma) \subset \overline{\mathcal{H}}S_s([\alpha_1], \nu)$.*

Proof. Let

$$f(z) = z - \sum_{n=2}^{\infty} |a_n| z^n + \sum_{n=1}^{\infty} |b_n| \overline{z}^n \in \overline{\mathcal{H}}S_s([\alpha_1], \gamma)$$

and

$$F(z) = z - \sum_{n=2}^{\infty} |A_n| z^n + \sum_{n=1}^{\infty} |B_n| \overline{z}^n \in \overline{\mathcal{H}}S_s([\alpha_1], \nu).$$

Then the convolution of $f(z)$ and $F(z)$ is given by

$$f(z) * F(z) = z - \sum_{n=2}^{\infty} |a_n A_n| z^n + \sum_{n=1}^{\infty} |b_n B_n| \overline{z}^n.$$

Note that $|A_n| \leq 1$ and $|B_n| \leq 1$, since $F \in \overline{\mathcal{H}}S_s([\alpha_1], \nu)$. Then we have

$$\sum_{n=2}^{\infty} [2n - \gamma(1 - (-1)^n)] \Gamma(\alpha_1, n) |a_n| \ \ |A_n| + \sum_{n=1}^{\infty} [2n + \gamma(1 - (-1)^n)] \Gamma(\alpha_1, n) |b_n| \ \ |B_n|$$

$$\leq \sum_{n=2}^{\infty} [2n - \gamma(1 - (-1)^n)] \Gamma(\alpha_1, n) |a_n| \ + \sum_{n=1}^{\infty} [2n + \gamma(1 - (-1)^n)] \Gamma(\alpha_1, n) |b_n| \ .$$

Therefore $f(z) * F(z) \in \overline{\mathcal{H}}S_s([\alpha_1], \gamma) \subset \overline{\mathcal{H}}S_s([\alpha_1], \nu)$, since the above inequality bounded by $2(1-\gamma)$ while $2(1-\gamma) \leq 2(1-\nu)$. ■

Theorem 6 *The class $\overline{\mathcal{H}}S_s([\alpha_1], \gamma)$ is closed under convex combination.*

Proof. For $i = 1, 2, \ldots$, suppose that $f_i(z) \in \overline{\mathcal{H}}S_s([\alpha_1], \gamma)$ where $f_i(z)$ is given by

$$f_i(z) = z - \sum_{n=2}^{\infty} |a_{n,i}| z^n + \sum_{n=2}^{\infty} |b_{n,i}| \overline{z}^n.$$

For $\sum\limits_{i=1}^{\infty} t_i = 1$, $0 \leq t_i \leq 1$, the convex combination of $f_i(z)$ may be written as

$$\begin{aligned} \sum_{i=1}^{\infty} t_i f_i(z) &= z \sum_{i=1}^{\infty} t_i - \sum_{n=2}^{\infty} \left(\sum_{i=1}^{\infty} t_i |a_{n,i}| \right) z^n + \sum_{n=1}^{\infty} \left(\sum_{i=1}^{\infty} t_i |b_{n,i}| \right) \overline{z}^n \\ &= z - \sum_{n=2}^{\infty} \left(\sum_{i=1}^{\infty} t_i |a_{n,i}| \right) z^n + \sum_{n=1}^{\infty} \left(\sum_{i=1}^{\infty} t_i |b_{n,i}| \right) \overline{z}^n. \end{aligned}$$

By Theorem 2,

$$\sum_{n=2}^{\infty}[2n-\gamma(1-(-1)^n)]\Gamma(\alpha_1,n)\left(\sum_{i=1}^{\infty}t_i|a_{n,i}|\right)$$
$$+\sum_{n=1}^{\infty}[2n+\gamma(1-(-1)^n)]\Gamma(\alpha_1,n)\left(\sum_{i=1}^{\infty}t_i|b_{n,i}|\right)$$
$$=\sum_{i=1}^{\infty}t_i\left(\sum_{n=2}^{\infty}[2n-\gamma(1-(-1)^n)]\Gamma(\alpha_1,n)|a_{n,i}|+\sum_{n=1}^{\infty}[2n+\gamma(1-(-1)^n)]\Gamma(\alpha_1,n)|b_{n,i}|\right).$$

Hence,

$$\leq 2(1-\gamma)\sum_{i=1}^{\infty}t_i=2(1-\gamma).$$

Hence $\sum\limits_{i=1}^{\infty}t_if_i\in\overline{\mathcal{H}}S_s([\alpha_1],\gamma)$. ■

Now, we will examine the closure properties of the class $\overline{\mathcal{H}}S_s([\alpha_1],\gamma)$ under the generalized Bernardi-Libera -Livingston integral operator$L_c(f)$ which is defined by

$$L_c(f)=\frac{c+1}{z^c}\int_0^z t^{c-1}f(t)\ dt,\quad c>-1.$$

Theorem 7 *Let* $f(z)\in\overline{\mathcal{H}}S_s([\alpha_1],\gamma)$ *Then* $L_c(f(z))\in\overline{\mathcal{H}}S_s([\alpha_1],\gamma)$

Proof. From the representation of $L_c(f(z))$, it follows that

$$\begin{aligned}L_c(f)&=\frac{c+1}{z^c}\int_0^z t^{c-1}\left[h(t)+\overline{g(t)}\right]dt\\&=\frac{c+1}{z^c}\left(\int_0^z t^{c-1}\left(t-\sum_{n=2}^{\infty}a_nt^n\right)dt+\overline{\int_0^z t^{c-1}\left(\sum_{n=1}^{\infty}b_nt^n\right)dt}\right)\\&=z-\sum_{n=2}^{\infty}A_nz^n+\sum_{n=1}^{\infty}B_nz^n\end{aligned}$$

where

$$A_n=\frac{c+1}{c+n}\ a_n;B_n=\frac{c+1}{c+n}\ b_n.$$

Therefore,

$$\sum_{n=1}^{\infty}\left(\frac{2n-\gamma(1-(-1)^n)}{2(1-\gamma)}[\frac{c+1}{c+n}|a_n|\ ]+\frac{2n+\gamma(1-(-1)^n)}{2(1-\gamma)}[\frac{c+1}{c+n}|b_n|\ ]\right)\Gamma(\alpha_1,n)$$
$$\leq\quad\sum_{n=1}^{\infty}\left(\frac{2n-\gamma(1-(-1)^n)}{2(1-\gamma)}|a_n|+\frac{2n+\gamma(1-(-1)^n)]}{2(1-\gamma)}|b_n|\right)\Gamma(\alpha_1,n)\leq\ 1,$$

since $f(z)\in\overline{\mathcal{H}}S_s([\alpha_1],\gamma)$. Hence by Theorem 2, $L_c(f(z))\in\overline{\mathcal{H}}S_s([\alpha_1],\gamma)$ ■

References

[1] O.P.Ahuja and J.M. Jahangiri, *Sakaguchi-type harmonic univalent functions,* Sci.Math.Japonica., 59 No.1 (2004), 163- 168.

[2] J. Clunie and T. Sheil-Small, *Harmonic Univalent Functions,* Ann. Acad. Aci. Fenn. Ser. A.I. Math., **9** (1984) 3-25.

[3] B.C.Carlson *Special Functions of Applied Mathematics,* Academic Press, New York 1977.

[4] B.C.Carlson and S.B.Shaffer, *Starlike and prestarlike hypergeometric functions,* SIAM, J. Math. Anal., **15** (2002), 737 - 745.

[5] J.Dziok and H.M.Srivastava, *Certain subclasses of analytic functions associated with the generalized hypergeometric function,* Intergral Transform and Spec. Funct., **14** (2003), 7 - 18.

[6] J.M.Jahangiri, *Harmonic Functions Starlike in the Unit disc.,* J.Math.Anal.Appl., 235 (1999), 470-477.

[7] J.M.Jahangiri, G.Murugusundaramoorthy and K.Vijaya, *Starlikeness of Rucheweyh type harmonic univalent functions.,* J. Indian. Academy. Math., 26 (1) (2004), 191 - 200.

[8] S.Ponnusamy and S.Sabapathy, *Geometric properties of generalized hypergeometric functions,* Ramanujan Journal, 1(1997), 187 - 210.

[9] E. D. Rainville, *Special Functions,* Chelsea Publishing Company, New York 1960.

[10] S. Ruscheweyh, *New criteria for Univalent Functions,* Proc. Amer. Math. Soc., 49 (1975), 109 -115.

[11] H. Silverman, *Harmonic univalent functions with negative coefficients ,*

[12] H.M.Srivastava and S.Owa, *Some characterization and distortion theorems involving fractional calculus, generalized hypergeometric functions, Hadamard products, linear operators and certain subclasses of analytic functions,* Nagoya Math. J., 106 (1987), 1 - 28.

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Remarks on the certain subclass of univalent functions

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ABSTRACT: We investigate the family $\mathcal{LP}_\alpha$ ($\alpha \in (-\pi, \pi]$) of functions $f(z) = z + \sum_{n=2}^{\infty} a_n z^n$ that are analytic in the unit disk with the property that the domain of values $f'(z) + \frac{1+e^{i\alpha}}{2} z f''(z)$ is the parabolic region $(\mathrm{Im} w)^2 < 2\mathrm{Re} w - 1$. We give inclusion theorems and bounds of $\mathrm{Re} f'(z)$ for this class

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1. Introduction and definitions

Let $\mathcal{A}$ be the class of functions of the form $f(z) = z + \sum_{n=2}^{\infty} a_n z^n$ analytic in the unit disk $\Delta = \{z \in C : |z| < 1\}$ and let S, K be the subclasses of $\mathcal{A}$ consisting of functions which are univalent and convex in Δ respectively. Let $\mathcal{R} = \{f \in S : \mathrm{Re} f'(z) > 0,\ z \in \Delta\}$. In 1988 St. Ruscheweyh [6] introduced the class

$$\mathcal{D} = \{f \in \mathcal{A} : |zf''(z)| < \mathrm{Re} f'(z),\ z \in \Delta\}$$

which is convex subset of $\mathcal{K}$. The alternative definition of $\mathcal{D}$ is the following

$$f \in \mathcal{D} \Leftrightarrow \mathrm{Re}\left\{f'(z) + e^{i\alpha} z f''(z)\right\} > 0 \text{ for } z \in \Delta \text{ and for all } \alpha \in (-\pi, \pi].$$

In 1998 Silverman and Silvia [7] introduced and investigated the class

$$\mathcal{L}_\alpha = \left\{f \in \mathcal{A} : \mathrm{Re}\left(f'(z) + \frac{1+e^{i\alpha}}{2} z f''(z)\right) > 0,\ z \in \Delta\right\}$$

where $\alpha \in (-\pi, \pi]$ is fixed. Let $\mathcal{L} = \bigcap_{-\pi<\alpha\leq\pi} \mathcal{L}_\alpha$.

Let

$$Q(z) = 1 + \frac{2}{\pi^2}\left(\log\frac{1+\sqrt{z}}{1-\sqrt{z}}\right)^2, \; z \in \Delta,$$

where the branch of square root is chosen such that $\mathrm{Im}\sqrt{z} \geq 0$. The function Q is analytic and univalent in Δ with the following power series expansion [3]

$$Q(z) = 1 + \frac{8}{\pi^2}\sum_{n=1}^{\infty}\left(\frac{1}{n}\sum_{k=0}^{n-1}\frac{1}{2k+1}\right)z^n = 1 + \sum_{n=1}^{\infty} B_n z^n$$

and it maps Δ onto the set

$$Q(\Delta) = \{w \in \mathrm{C} : |w-1| < \mathrm{Re}w\} = \left\{w \in \mathrm{C} : (\mathrm{Im}w)^2 < 2\mathrm{Re}w - 1\right\}.$$

For functions g and h, analytic in Δ, a function g is called subordinate to h, written $g \prec h$ (or $g(z) \prec h(z)$) if h is univalent in Δ, $g(0) = h(0)$ and $g(\Delta) \subset h(\Delta)$.

In [8] author investigated the class $\mathcal{LP}_\alpha$ defined as follows:

$$\mathcal{LP}_\alpha = \left\{f \in \mathcal{A} : f'(z) + \frac{1+e^{i\alpha}}{2} z f''(z) \prec Q(z), \; z \in \Delta\right\}, \tag{1}$$

where $\alpha \in (-\pi, \pi]$ is fixed.

2. Inclusion relations

First, we recall the following

Lemma 1 (Noshiro [5]) *If the function $f(z)$ is analytic in $|z| < R$ and $\mathrm{Re}f'(z) > 0$ for $|z| < R$, then $f(z)$ is univalent in $|z| < R$.*

Notice that the parabola $\partial Q(\Delta)$ is symmetric w.r.t. the real axis and its vertex is in the point $w = \frac{1}{2}$. Therefore for $f \in \mathcal{LP}_\pi$ we have $\mathrm{Re}f'(z) > \frac{1}{2}$. Consequently, by Lemma 1, the class $\mathcal{LP}_\pi$ consists of univalent functions.

Now, we will show that for each $\alpha \in (-\pi, \pi)$ the inclusion $\mathcal{LP}_\alpha \subset \mathcal{LP}_\pi$ holds. We need the following result

Lemma 2 [4] *Let β and γ be complex constants, and let h be convex (univalent) in Δ, with $h(0) = 1$ and $\mathrm{Re}(\beta h(z) + \gamma) > 0$. If $p(z) = 1 + p_1 z + \ldots$ is analytic in Δ, then*

$$p(z) + \frac{zp'(z)}{\beta p(z) + \gamma} \prec h(z) \Rightarrow p(z) \prec h(z).$$

Theorem 1 *For each $\alpha \in (-\pi, \pi)$ we have*

$$\mathcal{LP}_\alpha \subset \mathcal{LP}_\pi.$$

Proof. Observe that for all $\alpha \in (-\pi, \pi)$

$$\mathrm{Re}\frac{2}{1+e^{i\alpha}} = \frac{2(1+\cos\alpha)}{|1+e^{i\alpha}|^2} > 0.$$

Thus for $f \in \mathcal{LP}_\alpha$ it is sufficient to take $p = f'$, $\beta = 0$ and $\gamma = \frac{2}{1+e^{i\alpha}}$ in Lemma 2. This completes the proof. ■

Basing on that result we can conclude that each $\mathcal{LP}_\alpha$ consists of univalent functions and that $\bigcup_{-\pi<\alpha\leq\pi} \mathcal{LP}_\alpha = \mathcal{LP}_\pi$.

Let $\mathcal{LP} = \bigcap_{-\pi<\alpha\leq\pi} \mathcal{LP}_\alpha$.

Theorem 2 *The class $\mathcal{LP}$ is nonempty.*

Proof. It is easy to check that the function $f_{\frac{1}{4}}(z) := -4\log(1-\frac{z}{4})$ belongs to $\mathcal{LP}$. Let

$$g_{\frac{1}{4}}(z,\alpha) = \mathrm{Re}\left\{f'_{\frac{1}{4}}(z) + \frac{1+e^{i\alpha}}{2} z f''_{\frac{1}{4}}(z)\right\} - \left|f'_{\frac{1}{4}}(z) + \frac{1+e^{i\alpha}}{2} z f''_{\frac{1}{4}}(z) - 1\right|.$$

It is enough to show that the condition $g_{\frac{1}{4}}(z,\alpha) > 0$ holds for all $|z| = 1$ and for all $\alpha \in (-\pi, \pi]$. From

$$g_{\frac{1}{4}}(z,\alpha) = \mathrm{Re}\left\{\frac{4}{4-z} + \frac{2z}{(4-z)^2} + \frac{2ze^{i\alpha}}{(4-z)^2}\right\} - \left|\frac{z(6-z)+2ze^{i\alpha}}{(4-z)^2}\right| \geq$$

$$\geq 2\mathrm{Re}\frac{8-z}{(4-z)^2} - \frac{|z|[4+|6-z|]}{|4-z|^2}$$

it follows that

$$g_{\frac{1}{4}}(e^{i\theta},\alpha) \geq 2\mathrm{Re}\frac{8-e^{i\theta}}{(4-e^{i\theta})^2} - \frac{4+|6-e^{i\theta}|}{|4-e^{i\theta}|^2} =$$

$$= \frac{32\cos^2\theta - 130\cos\theta + 188 - (17-8\cos\theta)\sqrt{37-12\cos\theta}}{(17-8\cos\theta)^2}.$$

Direct computation leads to the conclusion that for all $\theta \in [0, 2\pi)$ the function $\phi(\theta) = 32\cos^2\theta - 130\cos\theta + 188 - (17-8\cos\theta)\sqrt{37-12\cos\theta}$ has the positive values. Therefore $g_{\frac{1}{4}}(e^{i\theta},\alpha) > 0$ for all $\theta \in [0, 2\pi)$ and for all $\alpha \in (-\pi, \pi]$. Consequently $f_{\frac{1}{4}}$ is in $\mathcal{LP}$. The proof is completed. ■

Silverman and Silvia proved in 1999 [7] that the inclusion $\mathcal{D} \subset \mathcal{L} \subset \mathcal{K}$ holds. Note that for each $\alpha \in (-\pi, \pi]$ we have $\mathcal{LP}_\alpha \subset \mathcal{L}_\alpha$, therefore $\mathcal{LP} \subset \mathcal{L}$. Consequently, $\mathcal{LP}$ consists of convex functions. We have $\mathcal{LP} \subset \mathcal{L}$ and $\mathcal{D} \subset \mathcal{L}$. It will be interesting to answering to the question what are the inclusion relationships between $\mathcal{LP}$ and $\mathcal{D}$. The next theorem presents the partial solution of this problem.

Theorem 3 *We have*

(i) *$\mathcal{LP} \cap \mathcal{D}$ is nonempty,*

(ii) *$\mathcal{D} \setminus \mathcal{LP}$ is nonempty.*

Proof. It is known [1] that the function $f_r(z) = \frac{-\log(1-rz)}{r}$ belongs to the class $\mathcal{D}$ if and only if $0 < r \le \frac{1}{2}$. On the other hand, as we showed in the proof of Theorem 2, $f_{\frac{1}{4}}(z) \in \mathcal{LP}$. Consequently, $\mathcal{LP} \cap \mathcal{D}$ is nonempty.

To show that (ii) holds it is sufficient to prove that $f_{\frac{1}{2}}(z) \notin \mathcal{LP}_0$. We observe that $f_{\frac{1}{2}}(z) \in \mathcal{LP}_0$ if and only if

$$g_{\frac{1}{2}}(z) := \mathrm{Re}\left\{f'_{\frac{1}{2}}(z) + zf''_{\frac{1}{2}}(z)\right\} - \left|f'_{\frac{1}{2}}(z) + zf''_{\frac{1}{2}}(z) - 1\right| > 0$$

for all $z \in \Delta$. Note that

$$g_{\frac{1}{2}}(z) = \mathrm{Re}\frac{4}{(2-z)^2} - \frac{|4z - z^2|}{|2-z|^2}.$$

It is sufficient to look at $z = e^{i\theta},\ \theta \in (0, 2\pi]$. A straightforward computation leads to the observation that

$$g_{\frac{1}{2}}(e^{i\theta}) = \frac{8\cos^2\theta - 16\cos\theta + 12 - (5 - 4\cos\theta)\sqrt{17 - 8\cos\theta}}{|2 - e^{i\theta}|^4} =: \frac{\zeta(\theta)}{|2 - e^{i\theta}|^4}.$$

It is easy to check that $\zeta(\frac{\pi}{2}) = 12 - 5\sqrt{17} < 0$. Therefore $f_{\frac{1}{2}}(z) \notin \mathcal{LP}_0$ and consequently $f_{\frac{1}{2}}(z) \notin \mathcal{LP}$. The proof is completed. ■

3. Bounds of the real part of derivative

For $f \in \mathcal{LP}_\pi = \bigcup_{-\pi<\alpha\le\pi} \mathcal{LP}_\alpha$ we have $\mathrm{Re}f'(z) > \frac{1}{2}$. In this section we give an answer to the question how large is $\mathrm{Re}f'(z)$ for $f \in \mathcal{LP}_\alpha,\ \alpha \in (-\pi, \pi)$ fixed. We will use the following result of Hallenbeck and Ruscheweyh.

Lemma 3 [2] *Let $h(z)$ be convex in Δ with $h(0) = a$, $\gamma \ne 0$ and $\mathrm{Re}\gamma \ge 0$. If $p(z) = a + a_n z^n + a_{n+1}z^{n+1} + \dots$ is analytic in Δ and*

$$p(z) + \frac{zp'(z)}{\gamma} \prec h(z) \tag{2}$$

then

$$p(z) \prec q(z) = \frac{\gamma}{nz^{\frac{\gamma}{n}}}\int_0^z h(t)t^{\frac{\gamma}{n}-1}dt$$

and q is convex and this is the best dominant of (2).

Theorem 4 *Let $\alpha \in (-\pi, \pi)$ and let $\gamma := \frac{2}{1+e^{i\alpha}}$. If $f \in \mathcal{LP}_\alpha$, then*

$$f'(z) \prec q_{\gamma(z)} = 1 + \frac{2\gamma}{\pi^2}z^{-\gamma}\int_0^{\log\frac{1+\sqrt{z}}{1-\sqrt{z}}} u^2\frac{(\tanh\frac{u}{2})^{2\gamma-1}}{(\cosh\frac{u}{2})^2}du,\ z \in \Delta$$

and q is the best dominant. Furthermore

$$\mathrm{Re}f'(z) > q_\gamma(-1).$$

Proof. Let $\alpha \in (-\pi, \pi)$ and $f \in \mathcal{LP}_\alpha$. Note that $\mathrm{Re}\frac{1+e^{i\alpha}}{2} = \frac{1}{2}(1+\cos\alpha) \geq 0$ for all $\alpha \in (-\pi, \pi)$. Setting $n = 1,\ \gamma = \frac{2}{1+e^{i\alpha}},\ p = f'$and $h = Q$ in Lemma 3 we obtain

$$f'(z) \prec \gamma z^{-\gamma} \int_0^z t^{\gamma-1} \left[1 + \frac{2}{\pi^2}\left(\log\frac{1+\sqrt{t}}{1-\sqrt{t}}\right)^2\right] dt =$$

$$= 1 + \frac{2\gamma}{\pi^2} z^{-\gamma} \int_0^z t^{\gamma-1} \left(\log\frac{1+\sqrt{t}}{1-\sqrt{t}}\right)^2 dt.$$

Substituting $u = \log\frac{1+\sqrt{t}}{1-\sqrt{t}}$ and $dt = \frac{4e^u(e^u-1)}{(e^u+1)^3} du$ we obtain

$$f'(z) \prec 1 + \frac{8\gamma}{\pi^2} z^{-\gamma} \int_0^{\log\frac{1+\sqrt{z}}{1-\sqrt{z}}} u^2 e^u \frac{(e^u-1)^{2\gamma-1}}{(e^u+1)^{2\gamma+1}} du =$$

$$= 1 + \frac{2\gamma}{\pi^2} z^{-\gamma} \int_0^{\frac{1+\sqrt{z}}{1-\sqrt{z}}} u^2 \frac{(\tanh\frac{u}{2})^{2\gamma-1}}{(\cosh\frac{u}{2})^2} du =: q_\gamma(z).$$

Since f' is subordinate to the convex function, hence

$$\mathrm{Re} f'(z) > \min_{|z|=1} q_\gamma(z) = q_\gamma(-1).$$

This completes the proof. ■

Setting $\alpha = 0$ in Theorem 3 we obtain the following result:

Corollary 1 *If $f \in \mathcal{LP}_0$ then*

$$\mathrm{Re} f'(z) > \frac{4}{\pi}\left(1 - \frac{2}{\pi}\ln 2\right) \approx 0.711395603.$$

Proof. For $\alpha = 0$ we have $\gamma = \frac{2}{1+e^{i\alpha}} = 1$, so making use of Theorem 4 we immediately obtain

$$\mathrm{Re} f'(z) > q_1(-1) = 1 - \frac{2}{\pi^2} \int_0^{i\frac{\pi}{2}} u^2 \frac{\sinh\frac{u}{2}}{(\cosh\frac{u}{2})^3} du.$$

Integrating by parts we get

$$\int u^2 \frac{\sinh\frac{u}{2}}{\cosh^3\frac{u}{2}} du = -\frac{u^2}{\cosh^2\frac{u}{2}} + 4u\tanh\frac{u}{2} - 8\log(\cosh\frac{u}{2}).$$

Therefore

$$\mathrm{Re} f'(z) > \frac{4}{\pi}\left(1 - \frac{2}{\pi}\ln 2\right).$$

The proof has been completed. ■

References

[1] V. Gruenberg, F. Rønning and St. Ruscheweyh, *On a multiplier conjecture for univalent functions*, Trans. Amer. Math. Soc., **322(1990)**, 377-393.

[2] D. J. Hallenbeck, St. Ruscheweyh, *Subordination by convex functions*, Proc. Amer. Math. Soc. **52 (1975)** , 191-195.

[3] W. Ma, D. Minda, *Uniformly convex functions II*, Ann. Polon. Math. **58(3)**(1993), 275-285.

[4] P. Eenigenburg, S.S. Miller, P.T.Mocanu, M.O. Reade, *On a Briot-Bouquet differential subordination*, Rev. Roumaine Math. Pures Appl. **29(1984)**, 567-573.

[5] K. Noshiro, *On the univalency of certain analytic functions*, J. Fac. Sci. Hokkaida Imp. Univ. **2 (1934),** 89-101.

[6] St. Ruscheweyh, *Extension of Szëgo's theorem on the sections of univalent functions*, Siam J, Math. Anal. **19(1988)**, 1442-1449.

[7] H. Silverman, E.M. Silvia, *Characterizations for subclasses of univalent functions*, Math. Japonica **50** , No. 1(1999), 103-109.

[8] L. Trojnar-Spelina, *Characterizations of subclasses of univalent functions*, Demonstratio Mathematica No 1, **38(2005)**,35-41.

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Strict pseud-contraction strong convergence theorems for strict pseud-contractions

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Abstract: In this paper, we prove two strong convergence theorems for strict pseudo-contractions in Hilbert spaces by hybrid methods. Our results extend and improve the recent ones announced by Nakajo and Takahashi [K. Nakajo, W. Takahashi, Strong convergence theorems for nonexpansive mappings and nonexpansive semigroups, J. Math. Anal. Appl. 279 (2003), 372-379], Marino and Xu [G. Marino, H.K. Xu, Weak and strong convergence theorems for strict pseudo-contractions in Hilbert spaces, J. Math. Anal. Appl. 329 (2007), 336-346], Martinez-Yanes and Xu [C. Martinez-Yanes, H.K. Xu, Strong convergence of the CQ method for fixed point iteration processes, Nonlinear Anal. 64 (2006), 2400-2411] and some others

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1. Introduction and Preliminaries

Let H be a real Hilbert space, C a nonempty closed convex subset of E, and $T : C \to C$ a mapping. Recall that T is nonexpansive if

$$\|Tx - Ty\| \leq \|x - y\| \quad \forall x, y \in C.$$

A point $x \in C$ is a fixed point of T provided $Tx = x$. Denote by $F(T)$ the set of fixed points of T; that is, $F(T) = \{x \in C : Tx = x\}$.

Some iteration processes are often used to approximate a fixed point of a nonexpansive mapping T. The first iteration process is now known as Mann's iteration process [7] which is defined as

$$x_{n+1} = \alpha_n x_n + (1 - \alpha_n)Tx_n, \quad n \geq 0, \tag{1.1}$$

where the initial guess x_0 is taken in C arbitrarily and the sequence $\{\alpha_n\}_{n=0}^{\infty}$ is in the interval [0, 1].

The second iteration process is referred to as Ishikawa's [4] iteration process which is defined recursively by

$$\begin{cases} y_n = \beta_n x_n + (1-\beta_n)Tx_n, \\ x_{n+1} = \alpha_n x_n + (1-\alpha_n)Ty_n, \quad n \geq 0, \end{cases} \tag{1.2}$$

where the initial guess x_0 is taken in C arbitrarily, $\{\alpha_n\}$ and $\{\beta_n\}$ are sequences in the interval [0, 1].

But both (1.1) and (1.2) have only weak convergence, in general (see [2] for an example). For example, Reich [15], shows that if E is a uniformly convex and has a Frehét differentiable norm and if the sequence $\{\alpha_n\}$ is such that $\alpha_n(1-\alpha_n) = \infty$, then the sequence $\{x_n\}$ generated by processes (1.1) converges weakly to a point in $F(T)$. (An extension of this result to processes (1.2) can be found in [21].) On the other hand, process (1.1) may fail to converge while process (1.2) can still converge for a Lipschitz pseudo-contractive mapping in a Hilbert space [1]. Therefore, many authors attempt to modify (1.1) and (1.2) to have strong convergence in Hilbert spaces and Banach spaces, respectively, see [10,12-14,18] for more details.

Attempts to modify the Mann iteration method (1.1) so that strong convergence is guaranteed have recently been made. Nakajo and Takahashi [11] proposed the following modification of the Mann iteration (1.1) for a single nonexpansive mapping T in a Hilbert space. To be more precise, They proved the following result.

Theorem NT. *Let C be a closed convex subset of a Hilbert space H and let $T : C \to C$ be a nonexpansive mapping such that $F(T) \neq \emptyset$. Assume that $\{\alpha_n\}_{n=0}^{\infty}$ is a sequence in [0, 1] such that $\alpha_n \leq 1-\delta$ for some $\delta \in (0, 1]$. Define a sequence $\{x_n\}_{n=0}^{\infty}$ in C by the algorithm:*

$$\begin{cases} x_0 \in C \quad \textit{chosen arbitrarily}, \\ y_n = \alpha_n x_n + (1-\alpha_n)Tx_n, \\ C_n = \{z \in C : \|y_n - z\| \leq \|x_n - z\|\}, \\ Q_n = \{z \in C : \langle x_0 - x_n, x_n - z\rangle \geq 0\}, \\ x_{n+1} = P_{C_n \cap Q_n} x_0. \end{cases} \tag{1.3}$$

Then $\{x_n\}$ converges in norm to $P_{F(T)}x_0$.

Recently, Kim and Xu [5] has adapted the iteration (1.1) in Hilbert spaces. They extended the recent one of Nakajo an Takahashi [11] from nonexpansive mappings to asymptotically nonexpansive mappings. To be more precise, they gave the following results.

Theorem KX. *Let C be a nonempty bounded closed convex subset of a Hilbert space H and let $T : C \to C$ be an asymptotically nonexpansive mapping with a sequence $\{k_n\}$ such that $k_n \to 1$ as $n \to \infty$. Assume that $\{\alpha_n\}_{n=0}^{\infty}$ is a sequence in [0,1] such* *that* $\limsup_{n\to\infty}\alpha_n < 1$. *Define a sequence* $\{x_n\}$ *in* C *by the following algorithm:*

$$\begin{cases} x_0 \in C \quad \textit{chosen arbitrarily}, \\ y_n = \alpha_n x_n + (1-\alpha_n)T^n z_n, \\ C_n = \{z \in C : \|y_n - z\|^2 \le \|x_n - z\|^2 + \theta_n\}, \\ Q_n = \{z \in C : \langle x_0 - x_n, x_n - z\rangle \ge 0\}, \\ x_{n+1} = P_{C_n \cap Q_n} x_0, \end{cases} \tag{1.4}$$

where

$$\theta_n = (1-\alpha_n)(k_n^2 - 1)(diamC)^2 \to 0, \quad \text{as } n \to \infty.$$

Then $\{x_n\}$ *defined by (1.4) converges strongly to* $P_{F(T)}x_0$.

Very recently, Marino and Xu [9] adapted the iteration (1.1) in Hilbert spaces. They extended the recent one of Nakajo an Takahashi [11] from nonexpansive mappings to strict pseudo-contractions. To be more precise, they proved the following results.

Theorem MX. *Let* C *be a closed convex subset of a Hilbert space* H *and let* $T : C \to C$ *be a k-strict pseudo-contraction for some* $0 \le k < 1$ *and assume that the fixed point set* $F(T)$ *of* T *is nonempty. Define a sequence* $\{x_n\}_{n=0}^{\infty}$ *in* C *by the algorithm:*

$$\begin{cases} x_0 \in C \quad \textit{chosen arbitrarily}, \\ y_n = \alpha_n x_n + (1-\alpha_n)T x_n, \\ C_n = \{z \in C : \|y_n - z\|^2 \le \|x_n - z\|^2 \\ \qquad -(k-\alpha_n)(1-\alpha_n)\|x_n - Tx_n\|^2\}, \\ Q_n = \{z \in C : \langle x_0 - x_n, x_n - z\rangle \ge 0\}, \\ x_{n+1} = P_{C_n \cap Q_n} x_0. \end{cases} \tag{1.5}$$

Assume that the control sequence $\{\alpha_n\}_{n=0}^{\infty}$ *is such that* $0 \le \alpha_n < 1$ *for all* n. *Then* $\{x_n\}$ *converges in norm to* $P_{F(T)}x_0$.

On the other hand, Attempts to modify the Ishikawa iteration method (1.2) so that strong convergence is guaranteed have recently been made. Martinez-Yanes and Xu [8] adapted the iteration (1.2) in Hilbert space to have strong convergence. To be more precise, they obtained the following convergence theorem.

Theorem MYX1. *Let* C *be a nonempty closed convex subset of a Hilbert space* H *and let* $T : C \to C$ *be a nonexpansive mapping such that* $F(T) \ne \emptyset$. *Assume that* $\{\alpha_n\}_{n=0}^{\infty}$ *and* $\{\beta_n\}_{n=0}^{\infty}$ *are sequences in (0,1) such that* $\lim_{n\to\infty}\alpha_n \le 1-\delta$ *for some* $\delta \in (0,1]$ *and* $\lim_{n\to\infty}\beta_n = 1$. *Define a sequence* $\{x_n\}$ *in* C *by the following* *algorithm*

$$\begin{cases} x_0 \in C \quad \textit{chosen arbitrarily}, \\ z_n = \beta_n x_n + (1-\beta_n)Tx_n, \\ y_n = \alpha_n x_n + (1-\alpha_n)Tz_n, \\ C_n = \{v \in C : \|y_n - v\|^2 \le \|x_n - v\|^2 + (1-\alpha_n)(\|z_n\|^2 \\ \qquad -\|x_n\|^2 + 2\langle x_n - z_n, v\rangle)\}, \\ Q_n = \{v \in C : \langle x_0 - x_n, x_n - v\rangle \ge 0\}, \\ x_{n+1} = P_{C_n \cap Q_n} x_0, \end{cases} \tag{1.6}$$

then $\{x_n\}$ *converges in norm to* $q = P_{F(T)}x_0$.

It is well know that Halpern iterations process [3] which is defined as

$$x_{n+1} = \alpha_n x_0 + (1-\alpha_n)Tx_n, \quad n \ge 0, \tag{1.7}$$

where $\{\alpha_n\}_{n=0}^{\infty}$ is a sequence in the interval [0,1] is also usually used to approximate a fixed point of nonexpansive. The iteration process (1.7) has been proved to be strongly convergent in both Hilbert spaces [3,6,19] and uniformly smooth Banach spaces [16,17,20] unless the sequence $\{\alpha_n\}$ satisfies the conditions

(C1) $\lim_{n\to\infty} \alpha_n = 0$;

(C2) $\sum_{n=1}^{\infty} \alpha_n = \infty$;

(C3) either $\sum_{n=0}^{\infty} |\alpha_n - \alpha_{n+1}| < \infty$ or $\lim_{n\to\infty} \frac{\alpha_n}{\alpha_{n+1}} = 1$.

It is well know that process (1.7) is widely believed to have slow convergence because the restriction of condition C_2. Moreover, Halpern [3] proved that condition (C_1) and (C_2) are indeed necessary in the sense that if the iterative process (1.7) is strongly convergent for all closed convex subsets C of a Hilbert space H and all nonexpansive mappings T on C, then the sequence $\{\alpha_n\}$ must satisfy conditions (C_1) and (C_2). (However, It is unknown whether these two conditions are also sufficient; see [20] for more detail.) Thus to improve the rate of convergence of the iterative process (1.7), one cannot rely only on the process itself. In [8], Martinez-Yanes and Xu studied the following iteration process:

Theorem MYX2. *Let H be a real Hilbert space, C a closed convex subset of H and $T : C \to C$ a nonexpansive mapping such that $F(T) \ne \emptyset$. Assume that $\alpha_n \subset (0,1)$ is chosen such that* $\lim_{n\to\infty} \alpha_n = 0$. *Then the sequence $\{x_n\}_{n=0}^{\infty}$ generated by*

$$\begin{cases} x_0 \in C \quad \textit{arbitrarily}, \\ y_n = \alpha_n x_0 + (1-\alpha_n)Tx_n, \\ C_n = \{z \in C : \|y_n - z\|^2 \le \|x_n - z\|^2 + \alpha_n(\|x_0\|^2 + 2\langle x_n - x_0, z\rangle)\}, \\ Q_n = \{z \in C : \langle x_0 - x_n, x_n - z\rangle \ge 0\}, \\ x_{n+1} = P_{C_n \cap Q_n} x_0. \end{cases} \tag{1.8}$$

converges strongly in norm to $P_{F(T)}x_0$.

The purpose of this paper is to employ Nakajo and Takahashi's [11] idea to modify process (1.2) and (1.7) to have strong convergence for strict pseudo-contractions. Our results improve and extend the ones announced by Martinez-Yanes and Xu [8] from nonexpansive mappings to strict pseudo-contractions.

Let C be a nonempty subset of a Hilbert space H. Recall that A mapping $T : C \to C$ is said to be k-strictly pseudo-contractive if

$$\|Tx - Ty\|^2 \leq \|x - y\|^2 + k\|(I - T)x - (I - T)y\|^2, \tag{1.9}$$

for some $k \in [0, 1)$, for all x, $y \in C$.

Note that the class of k-strict pseudo-contractions strictly includes the class of nonexpansive mappings. That is, T is nonexpansive if and only if T is 0-strictly pseudo-contractive.

In order to prove our main results, we shall make use of the following lemmas, [8,9].

Lemma 1.1. *Let H be a real Hilbert space. there hold the following identities:*

(i) $\|x - y\|^2 = \|x\|^2 - \|y\|^2 - 2\langle x - y, y\rangle, \quad \forall x, y \in H;$

(ii) $\|tx + (1-t)y\|^2 = t\|x\|^2 + (1-t)\|y\|^2 - t(1-t)\|x - y\|^2, \quad \forall t \in [0, 1], \forall x, y \in H.$

Lemma 1.2. *Let C be a closed convex subset of real Hilbert space H and let P_C be the metric projection from H onto C(i.e., for $x \in H$, P_C is the only point in C such that $\|x - P_C x\| = inf\{\|x - z\| : z \in C\}$). Given $x \in H$ and $z \in C$. Then $z = P_C x$ if and only if there holds the relations:*

$$\langle x - z, y - z\rangle \leq 0 \quad \forall y \in C. \tag{1.9}$$

Lemma 1.3. *Let H be a real Hilbert space. Let C be a nonempty closed convex subset of E and $T : C \to C$ a k-strict pseudo-contraction with a nonempty fixed point set. Then $(I - T)$ is demi-closed at zero.*

Lemma 1.4. *Let E be a real Banach space, C a nonempty subset of E and $T : C \to C$ a k-strict pseudo-contraction. Then T is L-Lipschitzian.*

Lemma 1.5. *Let H be a real Hilbert space, C a nonempty subset of H and $T : C \to C$ a k-strict pseudo-contraction. Then the fixed points set $F(T)$ of T is closed and convex so that the projection $P_{F(T)}$ is well defined.*

Lemma 1.6. *Let H be a real Hilbert space. Given a closed convex subset $C \subset H$ and points $x, y, z \in H$. Given also a real number $a \in R$. The set*

$$D = \{v \in C : \|y - v\|^2 \leq \|x - v\|^2 + \langle w, v\rangle + a\}$$

is closed (and convex).

2. Main Results

2.1 The hybrid projection method for Ishikawa's iteration process

Theorem 2.1. *Let C be a closed convex subset of a Hilbert space H, $T : C \to C$ a k-strict pseudo-contraction. Assume that the fixed point set $F(T)$ of T is nonempty and $\{\alpha_n\}$, $\{\beta_n\}$ are sequences in (0,1) such that $\alpha_n < 1$ for all $n \geq 0$ and $\lim_{n\to\infty} \beta_n = 1$. Define a sequence $\{x_n\}$ in C by the following algorithm:*

$$\begin{cases} x_0 \in C \quad \textit{choen arbitrarily}, \\ z_n = \beta_n x_n + (1-\beta_n)Tx_n, \\ y_n = \alpha_n x_n + (1-\alpha_n)Tz_n, \\ C_n = \{v \in C : \|y_n - v\|^2 \leq \|x_n - v\|^2 \\ \qquad +(1-\alpha_n)(1-\beta_n)(k-\beta_n)\|Tx_n - x_n\|^2 \\ \qquad +(1-\alpha_n)(k\|z_n - Tz_n\|^2 - \alpha_n\|Tz_n - x_n\|^2)\}, \\ Q_n = \{v \in C : \langle x_0 - x_n, x_n - v\rangle \geq 0\}, \\ x_{n+1} = P_{C_n \cap Q_n} x_0, \end{cases}$$

then $\{x_n\}$ converges strongly to $P_{F(T)}x_0$.

Proof. First observe that C_n is convex by Lemma 1.6. Next, we show that $F(T) \subset C_n$ for all n. Indeed, we have, for all $p \in F(T)$,

$$\begin{aligned} \|y_n - p\|^2 &= \|\alpha_n(x_n - p) + (1-\alpha_n)(Tz_n - p)\|^2 \\ &\leq \alpha_n\|x_n - p\|^2 + (1-\alpha_n)\|Tz_n - p\|^2 - \alpha_n(1-\alpha_n)\|Tz_n - x_n\|^2 \\ &\leq \alpha_n\|x_n - p\|^2 + (1-\alpha_n)(\|z_n - p\|^2 + k\|z_n - Tz_n\|^2) \\ &\quad - \alpha_n(1-\alpha_n)\|Tz_n - x_n\|^2 \\ &\leq \alpha_n\|x_n - p\|^2 + (1-\alpha_n)\|z_n - p\|^2 + (1-\alpha_n)(k\|z_n - Tz_n\|^2 \\ &\quad - \alpha_n\|Tz_n - x_n\|^2). \end{aligned} \tag{2.1}$$

On the other hand, we also have

$$\begin{aligned} \|z_n - p\|^2 &= \|\beta_n(x_n - p) + (1-\beta_n)(Tx_n - p)\|^2 \\ &\leq \beta_n\|x_n - p\|^2 + (1-\beta_n)\|Tx_n - p\|^2 - \beta_n(1-\beta_n)\|Tx_n - x_n\|^2 \\ &\leq \beta_n\|x_n - p\|^2 + (1-\beta_n)(\|x_n - p\|^2 + k\|Tx_n - x_n\|^2) \\ &\quad - \beta_n(1-\beta_n)\|Tx_n - x_n\|^2 \\ &= \|x_n - p\|^2 + (1-\beta_n)(k-\beta_n)\|Tx_n - x_n\|^2 \end{aligned} \tag{2.2}$$

Substitute (2.2) into (2.1) yields that

$$\begin{aligned} \|y_n - p\|^2 \leq \|x_n - p\|^2 &+ (1-\alpha_n)(1-\beta_n)(k-\beta_n)\|Tx_n - x_n\|^2 \\ &+ (1-\alpha_n)(k\|z_n - Tz_n\|^2 - \alpha_n\|Tz_n - x_n\|^2). \end{aligned}$$

So $p \in C_n$ for all n. Next we show that

$$F(T) \subset Q_n, \quad \forall n \geq 0. \tag{2.3}$$

We prove this by induction. For $n = 0$, we have $F(T) \subset C = Q_0$. Assume that $F(T) \subset Q_n$. Since x_{n+1} is the projection of x_0 onto $C_n \cap Q_n$, by Lemma 1.2 we have

$$\langle x_0 - x_{n+1}, x_{n+1} - z\rangle \geq 0, \quad \forall\, z \in C_n \cap Q_n.$$

As $F(T) \subset C_n \cap Q_n$ by the induction assumptions, the last inequality holds, in particular, for all $z \in F(T)$. This together with the definition of Q_{n+1} implies that $F(T) \subset Q_{n+1}$. Hence (2.3) holds for all $n \geq 0$. In order to prove

$$\lim_{n\to\infty} \|x_{n+1} - x_n\| = 0,$$

from the definition of Q_n we have $x_n = P_{Q_n} x_0$ which together with the fact that $x_{n+1} \in C_n \cap Q_n \subset Q_n$ implies that

$$\|x_0 - x_n\| \leq \|x_0 - x_{n+1}\|.$$

This shows that the sequence $\{\|x_n - x_0\|\}$ is nondecreasing. On the other hand, since $x_n = P_{Q_n} x_0$ (by the definition of Q_n) and since $F(T) \subset Q_n$, we have

$$\|x_n - x_0\| \leq \|p - x_0\|, \quad \forall p \in F(T).$$

In particular, $\{x_n\}$ is bounded and

$$\|x_n - x_0\| \leq \|P_{F(T)} x_0 - x_0\|, \tag{2.4}$$

We obtain that $\lim_{n\to\infty} \|x_n - x_0\|$ exists. Noticing again that $x_n = P_{Q_n} x_0$ and $x_{n+1} \in Q_n$ which give that $\langle x_{n+1} - x_n, x_n - x_0\rangle \geq 0$. Therefore, we have

$$\begin{aligned}\|x_{n+1} - x_n\|^2 &= \|(x_{n+1} - x_0) - (x_n - x_0)\|^2\\ &= \|x_{n+1} - x_0\|^2 - \|x_n - x_0\|^2 - 2\langle x_{n+1} - x_n, x_n - x_0\rangle\\ &\leq \|x_{n+1} - x_0\|^2 - \|x_n - x_0\|^2.\end{aligned}$$

It follows that

$$\lim_{n\to\infty} \|x_n - x_{n+1}\| = 0. \tag{2.5}$$

On the other hand, It follows from the definition of C_n that $x_{n+1} \in C_n$, Therefore, we have

$$\begin{aligned}\|y_n - x_{n+1}\|^2 \leq{}& \|x_n - x_{n+1}\|^2 + (1-\alpha_n)(1-\beta_n)(k-\beta_n)\|Tx_n - x_n\|^2\\ &+ (1-\alpha_n)(k\|z_n - Tz_n\|^2 - \alpha_n\|Tz_n - x_n\|^2).\end{aligned} \tag{2.6}$$

Moreover, since $y_n = \alpha_n x_n + (1-\alpha_n)Tz_n$, we obtain

$$\begin{aligned}&\|y_n - x_{n+1}\|^2\\ &= \|\alpha_n(x_n - x_{n+1}) + (1-\alpha_n)(Tz_n - x_{n+1})\|^2\\ &= \alpha_n\|x_n - x_{n+1}\|^2 + (1-\alpha_n)\|Tz_n - x_{n+1}\|^2\\ &\quad - \alpha_n(1-\alpha_n)\|Tz_n - x_n\|^2.\end{aligned} \tag{2.7}$$

Substituting (2.7) into (2.6), we arrive at

$$\begin{aligned}&(1-\alpha_n)\|x_{n+1}-Tz_n\|^2\\&\le(1-\alpha_n)\|x_n-x_{n+1}\|^2+(1-\alpha_n)k\|Tz_n-z_n\|^2\\&\quad+(1-\alpha_n)(1-\beta_n)(k-\beta_n)\|Tx_n-x_n\|^2.\end{aligned}$$

Since $\alpha_n<1$ for all $n\ge 0$, the last inequality becomes

$$\begin{aligned}\|x_{n+1}-Tz_n\|^2\le&\|x_n-x_{n+1}\|^2+k\|Tz_n-z_n\|^2\\&+(1-\beta_n)(k-\beta_n)\|Tx_n-x_n\|^2.\end{aligned}\tag{2.8}$$

On the other hand, we have

$$\begin{aligned}&\|x_{n+1}-Tz_n\|^2\\&=\|x_{n+1}-x_n+x_n-Tz_n\|^2\\&=\|x_{n+1}-x_n\|^2+\|x_n-Tz_n\|^2+2\langle x_{n+1}-x_n,x_n-Tz_n\rangle.\end{aligned}\tag{2.9}$$

Combining (2.8) with (2.9), we obtain

$$\begin{aligned}&\|x_n-Tz_n\|^2+2\langle x_{n+1}-x_n,x_n-Tz_n\rangle\\&\le k\|Tz_n-z_n\|^2+(1-\beta_n)(k-\beta_n)\|Tx_n-x_n\|^2.\end{aligned}$$

That is,

$$\begin{aligned}&\|x_n-Tx_n\|^2+\|Tx_n-Tz_n\|^2+2\langle x_n-Tx_n,Tx_n-Tz_n\rangle\\&\quad+2\langle x_{n+1}-x_n,x_n-Tz_n\rangle\\&\le k\|Tz_n-Tx_n\|^2+k\|Tx_n-x_n\|^2+k\|x_n-z_n\|^2\\&\quad+2k\langle Tx_n-x_n,x_n-z_n\rangle+2k\langle Tz_n-Tx_n,Tx_n-z_n\rangle+\delta_n,\end{aligned}$$

where $\delta_n=(1-\beta_n)(k-\beta_n)\|Tx_n-x_n\|^2$. It follows from $\lim_{n\to\infty}\beta_n=1$ and the boundness of $\{x_n\}$ that $\delta_n\to 0$, as $n\to\infty$. Therefore, we obtain

$$\begin{aligned}(1-k)\|x_n-Tx_n\|^2\le&\, k\|x_n-z_n\|^2+2k\|Tx_n-x_n\|\|x_n-z_n\|\\&+2\|Tz_n-Tx_n\|(\|Tx_n-z_n\|+\|x_n-Tx_n\|)\\&+2\|x_{n+1}-x_n\|\|x_n-Tz_n\|+\delta_n.\end{aligned}$$

On the hand, we have

$$\|x_n-z_n\|=(1-\beta_n)\|x_n-Tx_n\|.$$

It follows from $\lim_{n\to\infty}\beta_n=1$ and the boundness of $\{x_n\}$ that

$$\lim_{n\to\infty}\|x_n-z_n\|=0.\tag{2.10}$$

Noticing that T is L-Lipschitzian, (2.5) and (2.10), we obtain

$$\lim_{n\to\infty}\|x_n-Tx_n\|=0.$$

Assume that $\{x_{n_i}\}$ is a subsequence of $\{x_n\}$ such that $x_{n_i} \rightharpoonup \tilde{x}$. by Lemma 1.3 we have $\tilde{x} \in F(T)$. Next we show that $\tilde{x} = P_{F(T)}x_0$ and convergence is strong. Put $\bar{x} = P_{F(T)}x_0$ and consider the sequence $\{x_0 - x_{n_i}\}$. Then we have $x_0 - x_{n_i} \rightharpoonup x_0 - \tilde{x}$ and by the weak lower semicontinuity of the norm and by the fact that $\|x_0 - x_{n+1}\| \leq \|x_0 - \bar{x}\|$ for all $n \geq 0$ which is implied by the fact that $x_{n+1} = P_{C_n \cap Q_n}x_0$, we have

$$\|x_0 - \bar{x}\| \leq \|x_0 - \tilde{x}\| \leq \liminf_{i\to\infty} \|x_0 - x_{n_i}\| \leq \limsup_{i\to\infty} \|x_0 - x_{n_i}\| \leq \|x_0 - \bar{x}\|.$$

This gives that

$$\|x_0 - \bar{x}\| = \|x_0 - \tilde{x}\| \quad \text{and} \quad \|x_0 - x_{n_i}\| \to \|x_0 - \bar{x}\|$$

It follows that $x_0 - x_{n_i} \to x_0 - \bar{x}$; hence, $x_{n_i} \to \bar{x}$. Since $\{x_{n_i}\}$ is an arbitrary subsequence of $\{x_n\}$, we conclude that $x_n \to \bar{x}$ as $n \to \infty$. The proof is completed.

2.2 The hybrid method for Halpern's iteration process

Theorem 2.2. *Let C be a closed convex subset of a Hilbert space H and let $T : C \to C$ be a k-strict pseudo-contraction and assume that the fixed point set $F(T)$ of T is nonempty. Define a sequence $\{x_n\}_{n=0}^{\infty}$ in C by the algorithm:*

$$\begin{cases} x_0 \in C \quad \textit{choesn arbitrarily}, \\ y_n = \alpha_n x_0 + (1-\alpha_n)Tx_n, \\ C_n = \{z \in C : \|y_n - z\|^2 \leq \|x_n - z\|^2 \\ \qquad\qquad +\alpha_n(\|x_0\|^2 - \|x_n\|^2 + 2\langle x_n - x_0, z\rangle) \\ \qquad\qquad +(1-\alpha_n)[k\|Tx_n - x_n\|^2 - \alpha_n\|Tx_n - x_0\|^2]\}, \\ Q_n = \{z \in C : \langle x_0 - x_n, x_n - z\rangle \geq 0\}, \\ x_{n+1} = P_{C_n \cap Q_n}x_0. \end{cases}$$

Assume that the control sequence $\{\alpha_n\}_{n=0}^{\infty}$ is chosen such that $\lim_{n\to\infty} \alpha_n = 0$. Then $\{x_n\}$ converges in norm to $P_{F(T)}x_0$.

Proof. We first show that C_n is convex. Since

$$\begin{aligned} \|y_n - z\|^2 \leq &\|x_n - z\|^2 + \alpha_n(\|x_0\|^2 - \|x_n\|^2 + 2\langle x_n - x_0, z\rangle) \\ &+ (1-\alpha_n)[k\|Tx_n - x_n\|^2 - \alpha_n\|Tx_n - x_0\|^2] \end{aligned}$$

is equivalent to

$$\begin{aligned} 2\langle x_0 - y_n, z\rangle \leq (1-\alpha_n)\|x_n\|^2 - \|y_n\|^2 + \alpha_n\|x_0\|^2 \\ + (1-\alpha_n)[k\|Tx_n - x_n\|^2 - \alpha_n\|Tx_n - x_0\|^2]. \end{aligned} \tag{2.11}$$

It is easy to get C_n is convex. Next, we show that $F(T) \subset C_n$ for all n. Indeed, we have, for all $p \in F(T)$

$$
\begin{aligned}
\|y_n - p\|^2 &= \|\alpha(x_0 - p) + (1-\alpha_n)(Tx_n - p)\|^2 \\
&\le \alpha\|x_0 - p\|^2 + (1-\alpha_n)\|Tx_n - p\|^2 - \alpha_n(1-\alpha_n)\|Tx_n - x_0\| \\
&\le \alpha\|x_0 - p\|^2 + (1-\alpha_n)(\|x_n - p\|^2 + k\|Tx_n - x_n\|^2) \\
&\quad - \alpha_n(1-\alpha_n)\|Tx_n - x_0\| \\
&\le \|x_n - p\|^2 + \alpha_n(\|x_0\|^2 - \|x_n\|^2 + 2\langle x_n - x_0, p\rangle) \\
&\quad + (1-\alpha_n)[k\|Tx_n - x_n\|^2 - \alpha_n\|Tx_n - x_0\|^2].
\end{aligned}
$$

So $p \in C_n$ for all n. It follows from the methods of Theorem 2.1 that

$$F(T) \subset Q_n \quad \text{for all } n \ge 0. \tag{2.12}$$

In order to prove $\lim_{n\to\infty}\|x_{n+1} - x_n\| = 0$, from the definition of Q_n we have $x_n = P_{Q_n}x_0$ which together with the fact that $x_{n+1} \in C_n \cap Q_n \subset Q_n$ implies that

$$\|x_0 - x_n\| \le \|x_0 - x_{n+1}\|.$$

This shows that the sequence $\{\|x_n - x_0\|\}$ is nondecreasing. On the other hand, we have $\{x_n\}$ is bounded. Indeed, the definition of Q_n and Lemma 1.2 imply that $x_n = P_{Q_n}x_0$ which in turn implies that $\|x_n - x_0\| \le \|p - x_0\|$ for all $p \in F(T)$. In particular, one has

$$\|x_n - x_0\| \le \|P_{F(T)}x_0 - x_0\|.$$

This shows that $\{x_n\}$ is bounded. Therefore, we obtain that $\lim_{n\to\infty}\|x_n - x_0\|$ exists. Noticing again that $x_n = P_{Q_n}x_0$ and $x_{n+1} \in Q_n$ which give that $\langle x_{n+1} - x_n, x_n - x_0\rangle \ge 0$. Therefore, we have

$$
\begin{aligned}
\|x_{n+1} - x_n\|^2 &= \|(x_{n+1} - x_0) - (x_n - x_0)\|^2 \\
&\le \|x_{n+1} - x_0\|^2 - \|x_n - x_0\|^2 - 2\langle x_{n+1} - x_n, x_n - x_0\rangle \\
&\le \|x_{n+1} - x_0\|^2 - \|x_n - x_0\|^2.
\end{aligned}
$$

It follows that

$$\lim_{n\to\infty}\|x_n - x_{n+1}\| = 0. \tag{2.13}$$

On the other hand, it follows from $x_{n+1} \in C_n$ that

$$
\begin{aligned}
\|y_n - x_{n+1}\|^2 \le{}& \|x_n - x_{n+1}\|^2 + \alpha_n(\|x_0\|^2 - \|x_n\|^2 + 2\langle x_n - x_0, x_{n+1}\rangle) \\
&+ (1-\alpha_n)[k\|Tx_n - x_n\|^2 - \alpha_n\|Tx_n - x_0\|^2].
\end{aligned} \tag{2.14}
$$

Moreover, since $y_n = \alpha_n x_0 + (1-\alpha_n)Tx_n$, we obtain

$$
\begin{aligned}
&\|y_n - x_{n+1}\|^2 \\
&= \|\alpha_n(x_0 - x_{n+1}) + (1-\alpha_n)(Tx_n - x_{n+1})\|^2 \\
&= \alpha_n\|x_0 - x_{n+1}\|^2 + (1-\alpha_n)\|Tx_n - x_{n+1}\|^2 - \alpha_n(1-\alpha_n)\|Tx_n - x_0\|^2.
\end{aligned} \tag{2.15}
$$

On the other hand, we have

$$\begin{aligned}&\|x_{n+1}-Tx_n\|^2\\&=\|x_{n+1}-x_n+x_n-Tx_n\|^2\\&=\|x_{n+1}-x_n\|^2+\|x_n-Tx_n\|^2+2\langle x_{n+1}-x_n,x_n-Tx_n\rangle.\end{aligned}\tag{2.16}$$

Combine (2.14), (2.15) with (2.16) yields that

$$\begin{aligned}&(1-\alpha_n)(1-k)\|Tx_n-x_n\|^2\\&\le 2(1-\alpha_n)\|x_{n+1}-x_n\|\|x_n-Tx_n\|\\&+\|x_n-x_{n+1}\|+\alpha_n(\|x_0\|^2-\|x_n\|^2+2\langle x_n-x_0,x_{n+1}\rangle).\end{aligned}$$

Since (2.13) and $\lim_{n\to\infty}\alpha_n=0$, we obtain

$$\lim_{n\to\infty}\|Tx_n-x_n\|=0.$$

Next, we can obtain the desired conclusion easily by following the method of Theorem 2.1. The proof is completed.

As some applications of our main results, we have the following results.

If $\beta_n=1$ for all $n\ge 0$ in Theorem 2.1, then Theorem 2.1 includes the corresponding result of Marino and Xu [9] as a special case.

Note that the class of k-strict pseudo-contractions strictly includes the class of nonexpansive mappings. That is, T is nonexpansive if and only if T is 0-strict pseudo-contraction. by using Theorem 2.1 and Theorem 2.2, we can obtain the following desired conclusions easily.

Corollary 2.3 (Martinez-Yanes and Xu [8]). *Let C be a nonempty closed convex subset of a Hilbert space H and let $T: C\to C$ be a nonexpansive mapping such that $F(T)\ne\emptyset$. Assume that $\{\alpha_n\}_{n=0}^\infty$ and $\{\beta_n\}_{n=0}^\infty$ are sequences in (0,1) such that* $\lim_{n\to\infty}\alpha_n\le 1-\delta$ *for some $\delta\in(0,1]$ and* $\lim_{n\to\infty}\beta_n=1$. *Define a sequence $\{x_n\}$ in C by the following algorithm:*

$$\begin{cases}x_0\in C\quad \textit{choesn arbitrarily},\\ z_n=\beta_n x_n+(1-\beta_n)Tx_n,\\ y_n=\alpha_n x_n+(1-\alpha_n)Tz_n,\\ C_n=\{v\in C:\|y_n-v\|^2\le\|x_n-v\|^2+(1-\alpha_n)(\|z_n\|^2\\ \qquad -\|x_n\|^2+2\langle x_n-z_n,v\rangle)\},\\ Q_n=\{v\in C:\langle x_0-x_n,x_n-v\rangle\ge 0\},\\ x_{n+1}=P_{C_n\cap Q_n}x_0,\end{cases}$$

then $\{x_n\}$ converges strongly to $q=P_{F(T)}x_0$.

Corollary 2.4 (Martinez-Yanes and Xu [8]). *Let H be a real Hilbert space, C a closed convex subset of H and $T: C\to C$ a nonexpansive mapping such that $F(T)\ne\emptyset$.*

Assume that $\alpha_n \subset (0,1)$ *is chosen such that* $\lim_{n\to\infty} = 0$. *Then the sequence* $\{x_n\}_{n=0}^{\infty}$ *generated by*

$$\begin{cases} x_0 \in C \quad \textit{chosen arbitrarily}, \\ y_n = \alpha_n x_0 + (1-\alpha_n)Tx_n, \\ C_n = \{z \in C : \|y_n - z\|^2 \le \|x_n - z\|^2 + \alpha_n(\|x_0\|^2 + 2\langle x_n - x_0, z\rangle)\}, \\ Q_n = \{z \in C : \langle x_0 - x_n, x_n - z\rangle \ge 0\}, \\ x_{n+1} = P_{C_n \cap Q_n} x_0. \end{cases}$$

converges strongly to $P_{F(T)}x_0$.

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References

[1] C.E. Chidume, S.A. Mutangadura, An example on the Mann iteration method for Lipschitz pseudocontractions, Proc. Am. Math. Soc. **129** (2001), 2359-2363.

[2] A. Genel, J. Lindenstrass, An example concerning fixed points, Israel J. Math. **22** (1975), 81-86.

[3] B. Halpern, fixed points of nonexpanding maps, Bull. Am. Math. Soc. **73** (1967), 957-961.

[4] S. Ishikawa, Fixed points by a new iteration medthod, Proc. Am. Math. Soc. **44** (1974), 147-150.

[5] T.H. Kim, H.K. Xu, Strong convergence of modified mann iterations for asymptotically nonexpansive mappings and semigroups, Nonlinear Anal. **64** (2006), 1140-1152.

[6] P.L. Lions, Approximation de points fixes de contractions, C.R. Acad. Sci. Sèr. A-B Paris **284** (1977), 1357-1359.

[7] W.R. Mann, Mean value methods in iteration, Proc. Amer. Math. Soc. **4** (1953), 506-510.

[8] C. Martinez-Yanes, H.K. Xu, Strong convergence of the CQ method for fixed point iteration processes Nonlinear Anal. **64** (2006), 2400-2411.

[9] G. Marino, H.K. Xu, Weak and strong convergence theorems for strict pseudo-contractions in Hilbert spaces, J. Math. Anal. Appl. **329** (2007), 336-346.

[10] S. Matsushita, W. Takahashi, A strong convergence theorem for relatively nonexpansive mappings in a Banach space. J. Approx. Theory **134** (2005), 257-266.

[11] K. Nakajo, W. Takahashi, Strong convergence theorems for nonexpansive mappings and nonexpansive semigroups, J. Math. Anal. Appl. **279** (2003), 372-379.

[12] X. Qin, Y. Su, Strong convergence theorems for relatively nonexpansive mappings in a Banach space, Nonlinear Anal. **67** (2007), 1958-1965.

[13] X. Qin, Y. Su, M. Shang, Strong convergence theorems for asymptotically nonexpansive mappings by hybrid methods, Kyungpook Math. J. **48**, (2008), 133-142.

[14] X. Qin, Y. Su, M. Shang, Strong convergence of the composite Halpern iteration J. Math. Anal. Appl. **339** (2008), 996-1002.

[15] S. Reich, Weak convergence theorems for nonexpansive mappings in Banach spaces, J. Math. Anal. Appl. **67** (1979), 274-276.

[16] S. Reich, Strong convergence theorems for resolvents of accretive operators in Banach spaces, J. Math. Anal. Appl. **75** (1980), 287-292.

[17] N. Shioji, W. Takahashi, Strong convergence of approximated sequences for nonexpansive mappings in Banach spaces, Proc. Am. Math. Soc. **125** (1997), 3641-3645.

[18] Y. Su, X. Qin, Strong convergence of modified Ishikawa iterations for nonlinear mappings, Proc. Indian Acad. Sci. Math. Sci. **117** (2007), 97-107.

[19] R. Wittmann, Approximation of fixed points of nonexpansive mappings, Arch. Math. **58** (1992), 486-491.

[20] H.K. Xu, Iterative algorithms for nonlinear operator, J. London Math. Soc. **66** (2002), 240-256.

[21] K.K. Tan, K.K. Xu, Approximating fixed points of nonexpansive mappings by the Ishikawa iteration process, J. Math. Anal. Appl. **178** (1993), 301-308.

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On an application of certain sufficient condition for starlikeness

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Abstract: In this paper we consider a sufficient condition for function to be α-starlike function, when $\alpha \in [0, 1/2]$. We use it for certain subclass of strongly starlike functions defined by a geometric condition. We take advantage of the techniques of differential subordinations

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1. Introduction

Let $\mathcal{H}$ denote the class of analytic functions in the unit disc $U = \{z : \ |z| < 1\}$ on the complex plane $\mathbb{C}$. Let $\mathcal{A}$ denote the subclass of $\mathcal{H}$ consisting of functions normalized by $f(0) = 0,\ f'(0) = 1$. We say that $f \in \mathcal{H}$ is subordinate to $g \in \mathcal{H}$ in U, written $f \prec g$, if and only if there exists a function $\omega \in \mathcal{H}$ with $\omega(0) = 0$ and $|\omega(z)| < 1$ in U such that $f(z) = g(\omega(z))$ for $z \in U$. If $f \prec g$ in U, then $f(U) \subseteq g(U)$. Many classes of functions studied in geometric function theory can be described in terms of subordination. Let us denote $p_\alpha(z) = (1 + (1 - 2\alpha)z)/(1 - z)$, $z \in U$, and let

$$\mathcal{S}^*(\alpha) := \left\{ f \in \mathcal{A} : \ \frac{zf'(z)}{f(z)} \prec p_\alpha(z) \text{ in } U \right\} = \left\{ f \in \mathcal{A} : \ \mathrm{Re}\left[\frac{zf'(z)}{f(z)}\right] > \alpha \text{ for } z \in \ U \right\}$$

be the class of α-starlike functions, $\alpha \in [0, 1)$. $\mathcal{S}^*(0)$ is the class of starlike functions which map U onto a starlike domain with respect to the origin. We say that the function $f \in \mathcal{H}$ is convex when $f(U)$ is a convex set. It is easy to see that p_α is a convex univalent function.

Robertson [4] obtained the following theorem.

Theorem A([4]). *If $f \in \mathcal{A}$, with $f(z)/z \neq 0$ and if there exists a $k \in (0, 2]$, then*

$$\left|\frac{zf''(z)}{f'(z)}\right| \le k \left|\frac{zf'(z)}{f(z)}\right| \quad \Rightarrow \quad \frac{zf'(z)}{f(z)} \prec \frac{2}{2 + kz}. \tag{1}$$

In particular $f \in \mathcal{S}^*(\alpha)$, *with* $\alpha = 2/(2+k)$.

In this paper we consider a condition similar to (1). We shall use the Jack's Lemma given below.

Lemma A(see [2]). *If a function* ω *is analytic for* $|z| \le |z_0| < 1$, $\omega(0) = 0$ *and* $|\omega(z_0)| = \max\{|\omega(z)| : |z| \le |z_0|\}$, *then*

$$\frac{z_0\omega'(z_0)}{\omega(z_0)} \ge 1.$$

2. Main results

Theorem 1. *If* $f \in \mathcal{A}$, *then*

$$\mathrm{Re}\left[\frac{zf''(z)}{f'(z)}\right] < \mathrm{Re}\left[\frac{zf'(z)}{f(z)}\right] - \frac{3}{4} \quad \Rightarrow \quad \frac{zf'(z)}{f(z)} \prec q_0(z) := \sqrt{1+z},$$

where the branch of the square root is chosen in order to $q_0(0) = 1$.

Proof. Let us denote $Q(f,z) = zf'(z)/f(z)$. Suppose that $Q(f,z) \not\prec q_0(z)$. The function q_0 is univalent in U so there exist z_0, ζ_0 such that $|z_0| = r_0 < 1$, $|\zeta_0| = 1$, $Q(f,z)(\{|z| < r_0\}) \subset q_0(U)$ and $Q(f,z_0) = q_0(\zeta_0)$. Then the function $\omega(z) = q_0^{-1}(Q(f,z))$ is analytic in $|z| < r_0$ and $\omega(0) = 0$, $\omega(z_0) = \zeta_0$. Thus $|\omega(z)|$ assumes at z_0 its maximum in $|z| \le |z_0|$ and by Lemma A $z_0\omega'(z_0) = m\omega(z_0)$, $m \ge 1$. Logarithmic differentiating $q_0(\omega(z)) = Q(f,z)$ we obtain

$$\frac{z\omega'(z)}{\omega(z)}\frac{\omega(z)}{2(1+\omega(z))} = 1 + \frac{zf''(z)}{f'(z)} - \frac{zf'(z)}{f(z)}.$$

Then we have

$$\mathrm{Re}\left[1 + \frac{z_0f''(z_0)}{f'(z_0)} - \frac{z_0f'(z_0)}{f(z_0)}\right] = \mathrm{Re}\left[\frac{z_0\omega'(z_0)}{\omega(z_0)}\frac{\omega(z_0)}{2(1+\omega(z_0))}\right] = \frac{m}{4} \ge \frac{1}{4},$$

which contradicts the hypothesis of the theorem so $Q(f,z) \prec q_0(z) = \sqrt{1+z}$. □

For the function

$$f_0(z) := \frac{4z\exp(2\sqrt{1+z}-2)}{(1+\sqrt{1+z})^2} = z + \frac{1}{2}z^2 + \frac{1}{16}z^3 + \frac{1}{96}z^4 - \frac{1}{128}z^5 + \cdots \qquad (2)$$

we have $zf_0'(z)/f_0(z) = q_0(z)$ and $1 + zf_0''(z)/f_0'(z) = q_0(z) + \frac{z}{2(1+z)}$, hence

$$\frac{zf_0''(z)}{f_0(z)} - \frac{zf_0'(z)}{f_0(z)} = -\frac{z+2}{2(z+1)}.$$

Note that the function $g(z) = -\frac{z+2}{2(z+1)}$ maps U onto the half-plane $\{w : \mathrm{Re}\ w < -3/4\}$.

Let us still denote $Q(f,z) = zf'(z)/f(z)$. Kanas and Wiśniowska introduced in [3] the concept of a k–starlike functions

$$k - \mathcal{ST} := \{f \in \mathcal{A} : \mathrm{Re}[Q(f,z)] > k|Q(f,z) - 1|\}, \quad k \geq 0.$$

In this way they obtained a continuous passage from starlike functions ($k = 0$) to the class $\mathcal{S}_p^*$ considered by Rønning [5], where $\mathcal{S}_p^* = (1 - \mathcal{ST})$. Moreover for $0 < k < 1$ the quantity $Q(f,z)$ takes its values in a convex domain on the right of a hyperbola while for $k > 1$ inside an ellipse. Now, let us consider the class $\mathcal{SL}^*$:

$$\mathcal{SL}^* = \{f \in \mathcal{A} : |Q^2(f,z) - 1| < 1\}. \tag{3}$$

It is easy to see that $f \in \mathcal{SL}^*$ if and only if $Q(f,z) \prec q_0(z) = \sqrt{1+z}$, $q_0(0) = 1$. Therefore by Theorem 1 we obtain the following corollary.

Corollary 1. *If $f \in \mathcal{A}$ and*

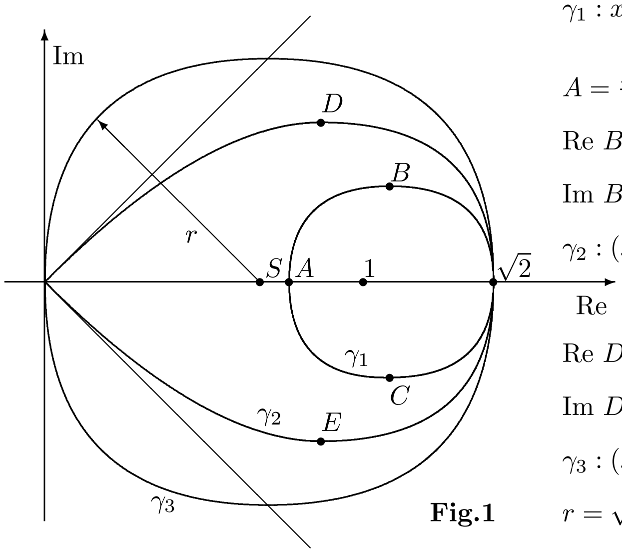
$$\mathrm{Re}\left[\frac{zf''(z)}{f'(z)}\right] < \mathrm{Re}\left[\frac{zf'(z)}{f(z)}\right] - \frac{3}{4}, \tag{4}$$

then $f \in \mathcal{SL}^*$.

Notice that $\mathcal{L} := \{w \in \mathbb{C} : \mathrm{Re}\, w > 0, \ \ |w^2 - 1| < 1\}$ is the interior of the right half of the lemniscate of Bernoulli $\gamma_2 : (x^2 + y^2)^2 - 2(x^2 - y^2) = 0$. It can be verified that $\mathcal{L} \subset \{w : |w - \sqrt{2}/2| < \sqrt{2}/2\}$ (see Fig. 1). Moreover $\mathcal{L} \subset \{w : |\mathrm{Arg}\, w| < \pi/4\}$, thus $\mathcal{SL}^* \subset \mathcal{SS}^*(1/2) \subset \mathcal{S}^*$, where $\mathcal{SS}^*(\beta)$ denotes the class of strongly starlike functions of order β

$$\mathcal{SS}^*(\beta) := \{f \in \mathcal{A} : |\mathrm{Arg}\ Q(f,z)| < \beta\pi/2\}, \quad 0 < \beta \leq 1$$

which was introduced in [6] and [1]. Let us consider the conic region $P(k) = \{w \in \mathbb{C} : \mathrm{Re}\ w > k|w - 1|\}$ connected with the class $k - \mathcal{ST}$ described above. For $k > 1$ the curve $\partial P(k)$ is the ellipse $\gamma_1 : x^2 = k^2(x-1)^2 + k^2y^2$. For $k \geq 2 + \sqrt{2}$ this ellipse lies entirely inside $\overline{\mathcal{L}}$. Therefore $k - \mathcal{ST} \subset \mathcal{SL}^*$, for $k \geq 2 + \sqrt{2}$.



$\gamma_1 : x^2 = (6+4\sqrt{2})(x-1)^2 + (6+4\sqrt{2})y^2$

$A = \frac{4+\sqrt{2}}{7}$

$\mathrm{Re}\ B = \mathrm{Re}\ C = \frac{4\sqrt{2}-2}{7}$

$\mathrm{Im}\ B = -\mathrm{Im}\ C = \frac{1}{\sqrt{5+4\sqrt{2}}}$

$\gamma_2 : (x^2+y^2)^2 - 2(x^2-y^2) = 0$

$\mathrm{Re}\ D = \mathrm{Re}\ E = \frac{\sqrt{3}}{2}$

$\mathrm{Im}\ D = -\mathrm{Im}\ E = \frac{1}{2}$

$\gamma_3 : (x-\sqrt{2}/2)^2 + y^2 = 1/2,$

$r = \sqrt{2}/2$

Fig.1

A simple calculation shows that the function $g(z) = z\exp(az)$ satisfy (4) when $|a| < 1/3$. Thus $g \in \mathcal{SL}^*$ for $|a| < 1/3$. The condition (3) gives after much more intricate calculation the sharp bound $|a| < \sqrt{2}-1$. Moreover by (3) we obtain

$$\frac{z}{(1-az)^2} \in \mathcal{SL}^* \iff |a| < 3-2\sqrt{2} = 0.17\ldots.$$

References

[1] D. A. Brannan, W. E. Kirwan, On some classes of bounded univalent functions, J. London Math. Soc. 1(2)(1969) 431–443.

[2] I. S. Jack, Functions starlike and convex of order α, J. London Math. Soc. 3(1971) 469–474.

[3] S. Kanas, A. Wiśniowska, Conic regions and k–uniform convexity, J. Comput Appl. Math. 105(1999) 327–336.

[4] M. S. Robertson, Certain classes of starlike functions, Michigan Math. J. 76, no.1, (1954) 755–758.

[5] F. Rønning, Uniformly convex functions and a corresponding class of starlike functions, Proc. Amer. Math. Soc. 118(1993) 189–196.

[6] J. Stankiewicz, Quelques problèmes extrèmaux dans les classes des fonctions α -angulairement ètoilèes, Ann. Univ. Mariae Curie–Skłodowska, Sect. A 20(1966) 59–75.

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Stability of solutions for a class of convex minimization problems on reflexive Banach spaces

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ABSTRACT: In this paper we study stability of solutions of minimization problems $f(x) \to \min$, $x \in C$, where f is a convex lower semicontinuous function and a set C is the countable intersection of a decreasing sequence of closed sets C_i in a reflexive Banach space X

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Key Words and Phrases: *Complete metric space, convex function, lower semicontinuous function, minimization problem, open everywhere dense set*

1. Introduction

In this paper we study the minimization problem

$$f(x) \to \min,\ x \in C \qquad (P^f)$$

and the convergence of solutions of the problems

$$f(x) \to \min,\ x \in C_i,\ i = 1, 2, \dots$$

to a solution of the problem (P^f), where

$$C = \cap_{i=1}^{\infty} C_i,$$

C_i, $i = 1, 2, \dots$ is a decreasing sequence of convex closed subsets of a reflexive Banach space X, and f is a convex lower semicontinuous function defined on X. Such convergence properties for minimization problems on reflexive Banach spaces and Hilbert spaces were studied in [2-5].

In the present paper we will prove two main results. The first of them stated in Section 2 and proved in Section 3 establishes that if a function f satisfies a strict convexity condition, then for all sufficiently large natural numbers i all approximate solutions of the problem

$$f(x) \to \min, \ x \in C_i$$

are close to a unique solution of the problem (P^f).

Our second main result stated in Section 4 and proved in Section 5 establishes the existence of an open everywhere dense subset $\mathcal{F}$ of a space of convex lower semicontinuous functions on X equipped with a natural complete metric such that for each $f \in \mathcal{F}$ the following property holds:

If a function g belongs to a small neighborhood of f and a natural number i is large enough, then approximate solutions of the problem

$$g(x) \to \min, \ x \in C_i$$

are close to a unique solution of the problem (P^f).

2. The first main result

We use the convention that $\infty - \infty = 0$ and $\infty/\infty = 1$. Let X be a reflexive Banach space with the norm $||\cdot||$ and let

$$C_\infty = \cap_{i=1}^\infty C_i \neq \emptyset, \tag{2.1}$$

$$C_{i+1} \subset C_i, \ i = 1, 2, \dots,$$

where for all natural numbers i, C_i is a closed convex subset of X.

Let $f : C_1 \to R^1 \cup \{\infty\}$ be a convex lower semicontinuous function which is not identically infinity on C_∞ and satisfy

$$\lim_{||x|| \to \infty} f(x) = \infty. \tag{2.2}$$

For each nonempy set $C \subset C_1$ put

$$\inf(f; C) = \inf\{f(x) : \ x \in C\}. \tag{2.3}$$

Since the space X is reflexive and the convex lower semicontinuous function f satisfies (2.2) for each $i \in \{1, 2, \dots\} \cup \{\infty\}$ the following minimization problem

$$f(x) \to \min, \ x \in C_i$$

has a solution.

In this section we assume that f possesses the following property:

(P1) For each natural number $n \geq 1$ there is a number $\delta > 0$ such that for each $x, y \in C_1$ satisfying $||x||, ||y|| \leq n$ and $||x-y|| \geq 1/n$ and each $\alpha \in [(2n)^{-1}, 1-(2n)^{-1}]$,

$$f(\alpha x + (1-\alpha)y) + \delta \leq \alpha f(x) + (1-\alpha) f(y).$$

Property (P1) implies that for each $i \in \{1, 2, \dots\} \cup \{\infty\}$ there is a unique $x_i \in C_i$ such that

$$f(x_i) = \inf(f; C_i). \tag{2.4}$$

The following theorem is our first main result.

Theorem 2.1.
1. $\lim_{i\to\infty} \inf(f; C_i) = \inf(f; C_\infty)$.
2. Let $\epsilon > 0$. *Then there exist* $\delta > 0$ *and a natural number* i_0 *such that for each integer* $i \geq i_0$ *and each* $x \in C_i$ *satisfying* $f(x) \leq \inf(f; C_i) + \delta$,

$$||x - x_\infty|| \leq \epsilon.$$

Theorem 2.1 will be proved in Section 3.

Note that if $C_1 = X$ where X is a Hilbert space with the inner product $< \cdot, \cdot >$ and if $f(x) =< x, x >$, $x \in X$, then the function f is convex and the property (P1) holds. The minimization problem (P_f) with the function $f(x) =< x, x >$ was studied in [5].

It was shown in [1] that if in a complete metric space of convex lower semicontinuous functions there is a function which possesses the property (P1) then most functions of the space (in the sense of Baire category) have this property.

3. Proof of Theorem 2.1

It is not difficult to see that assertion 1 holds (see also Lemma 2.1 of [2] or Lemma 3.3 of [3]).

Let us prove assertion 2. Let $\epsilon > 0$. Choose $c_0 > 0$ such that

$$c_0 > |\inf(f; C_\infty)| + |\inf(f; C_1)| + 4. \tag{3.1}$$

Let γ be an arbitrary positive number such that

$$\gamma < 4^{-1}\epsilon. \tag{3.2}$$

By (2.2) there is a natural number k such that

$$\text{if } z \in C_1 \text{ and } f(z) \leq c_0, \text{ then } ||z|| \leq k, \tag{3.3}$$

$$k^{-1} < \gamma/8, \; k \geq 4. \tag{3.4}$$

By property (P1) there is $\Delta \in (0, 2^{-1})$ such that the following property holds:

(P2) For each $x, y \in C_1$ satisfying

$$||x||, |y|| \leq k, \; ||x - y|| \geq 1/k$$

we have

$$f(2^{-1}x + 2^{-1}y) + 8\Delta \leq 2^{-1}f(x) + 2^{-1}f(y). \tag{3.5}$$

By assertion 1 there is a natural number i_0 such that for each integer $i \geq i_0$

$$|\inf(f;C_\infty) - \inf(f;C_i)| \leq \Delta/2. \tag{3.6}$$

Let integers i,j satisfy

$$i,j \geq i_0,$$
$$x \in C_i,\ f(x) \leq \inf(f;C_i) + \Delta,\ y \in C_j,\ f(y) \leq \inf(f;C_j) + \Delta. \tag{3.7}$$

We show that $||x-y|| \leq \gamma$. Assume the contrary. Then by (3.4)

$$||x-y|| > \gamma > 8/k. \tag{3.8}$$

We may assume that $j \geq i$. In view of (3.7), the inequality $\Delta < 1/2$, (2.1) and (3.1)

$$f(y), f(x) \leq \inf(f;C_\infty) + 1 < c_0. \tag{3.9}$$

It follows from (3.3) and (3.9) that

$$||x||, ||y|| \leq k. \tag{3.10}$$

Clearly,

$$2^{-1}(x+y) \in C_i. \tag{3.11}$$

By (P2), (3.7), (3.8) and (3.10)

$$f(2^{-1}(x+y)) \leq 2^{-1}f(x) + 2^{-1}f(y) - 8\Delta. \tag{3.12}$$

In view (3.6), (3.7) and (3.12),

$$\begin{aligned} f(2^{-1}(x+y)) &\leq 2^{-1}(\inf(f;C_i)+\Delta) + 2^{-1}(\inf(f;C_j)+\Delta) - 8\Delta \\ &\leq \ 2^{-1}(\inf(f;C_i)+\Delta) + 2^{-1}(\inf(f;C_i)+\Delta) - 8\delta = \inf(f;C_i) + 2\Delta - 8\Delta. \end{aligned}$$

This contradicts (3.11). The contradiction we have reached shows that $||x-y|| \leq \gamma$.

We have shown that the following property holds:

(P3) For each pair of integers $i,j \geq i_0$ and each $x,y \in X$ satisfying (3.7) the inequality $||x-y|| \leq \gamma$ holds.

Since γ is an arbitrary positive number satisfying (3.2) it follows from (P3) that $\{x_i\}_{i=1}^\infty$ is a Cauchy sequence (see (2.4)). Since f is lower semicontinuous we obtain that

$$f(\lim_{i\to\infty} x_i) \leq \lim_{i\to\infty} f(x_i). \tag{3.13}$$

Clearly,

$$\lim_{i\to\infty} x_i \in C_\infty. \tag{3.14}$$

In view of (3.13), assertion 1, (2.4) and (3.14),

$$f(\lim_{i\to\infty} x_i) \leq f(x_\infty).$$

Since x_∞ is a unique minimizer of f on C_∞ the relation above and (3.14) imply that

$$x_\infty = \lim_{i\to\infty} x_i \text{ in the norm topology.} \tag{3.15}$$

By (P3) and (2.4) for each pair of integers $i, j \geq i_0$

$$||x_i - x_j|| \leq \gamma.$$

Together with (3.15) this implies that

$$||x_\infty - x_i|| \leq \gamma \text{ for all integers } i \geq i_0. \tag{3.16}$$

Let an integer $i \geq i_0$ and let $x \in C_i$ satisfy

$$f(x) \leq \inf(f; C_i) = \Delta.$$

By (P3) $||x - x_i|| \leq \gamma$. Combined with (3.2) and (3.16) this inequality implies that

$$||x - x_\infty|| \leq 2\gamma < \epsilon.$$

Assertion 2 is proved. This completes the proof of Theorem 2.1.

4. The second main result

We again use the convention that $\infty - \infty = 0$ and $\infty/\infty = 1$. Let X be a reflexive Banach space with the norm $||\cdot||$ and let

$$C_\infty = \cap_{i=1}^{\infty} C_i \neq \emptyset,$$

where for all natural numbers i, C_i is a closed convex subset of X such that $C_{i+1} \subset C_i$, $i = 1, 2, \dots$.

For each function $g : C_1 \to R^1 \cup \{\infty\}$ put

$$\mathrm{dom}(g) = \{z \in C_1 : \ g(z) < \infty\}.$$

Let $\phi : C_1 \to R^1$ be such that

$$\phi(x) \to \infty \text{ as } ||x|| \to \infty. \tag{4.1}$$

Denote by $\mathcal{M}$ the set of all convex lower semicontinuous functions $f : C_1 \to R^1 \cup \{\infty\}$ which are not identically ∞ such that

$$f(x) \geq \phi(x) \text{ for all } x \in C_1. \tag{4.2}$$

Denote by $\mathcal{M}_v$ the set of all finite valued functions $f \in \mathcal{M}$, by $\mathcal{M}_c$ the set of all continuous functions $f \in \mathcal{M}_v$, by $\mathcal{M}_{lL}$ the set of all locally Lipschitz functions $f \in \mathcal{M}_v$ and by $\mathcal{M}_{\mathcal{L}}$ the set of all Lipschitz on bounded subsets of C_1 functions $f \in \mathcal{M}_v$.

For each integer $n \geq 1$ set

$$\mathcal{E}(n) = \{(f,g) \in \mathcal{M} \times \mathcal{M} : \ |f(x) - g(x)| \leq 1/n \text{ for all } x \in C_1 \text{ such that } ||x|| \leq n\}$$
$$\cap\{(f,g) \in \mathcal{M} \times \mathcal{M} : \ |(f-g)(x) - (f-g)(y)| \leq n^{-1}||x-y||$$
$$\text{for all } x, y \in C_1 \cap \ \text{dom}(f) \text{ such that } ||x||, ||y|| \leq n\}. \quad (4.3)$$

We equip the space $\mathcal{M}$ with the uniformity determined by the base $\mathcal{E}(n)$, $n = 1, 2, \dots$. It is clear that the uniform space $\mathcal{M}$ is metrizable and complete. We equip the space $\mathcal{M}$ with the topology generated by this uniformity. Clearly, $\mathcal{M}_v$, $\mathcal{M}_c$, $\mathcal{M}_{lL}$ and $\mathcal{M}_L$ are closed subsets of $\mathcal{M}$.

Note that for each $i \in \{1, 2, \dots\} \cup \{\infty\}$, $\inf(f; C_i)$ is finite.

In the following theorem which is our second main result we assume that $\mathcal{A}$ is one of the following subspaces of $\mathcal{M}$ with the relative topology:

$\mathcal{M}$; $\mathcal{M}_v$; $\mathcal{M}_c$; $\mathcal{M}_{lL}$; $\mathcal{M}_L$.

Theorem 4.1. *There exists an open everywhere dense set $\mathcal{F} \subset \mathcal{A}$ such that for each $f \in \mathcal{F}$ there exist $x_f \in C_\infty$ and an open neighborhood $V \subset \mathcal{F}$ of f such that:*

$$f(x_f) = \inf(f; C_\infty);$$

For each $\epsilon > 0$ there exist $\delta > 0$ and an integer $i_0 \geq 1$ such that for each integer $i \geq i_0$, each $g \in V$ and each $x \in C_i$ satisfying $g(x) \leq \inf(g; C_i) + \delta$,

$$||x - x_f|| \leq \epsilon.$$

5. Proof of Theorem 4.1

Lemma 5.1. *Let $f \in \mathcal{M}$. Then* $\inf(f; C_\infty) = \lim_{i\to\infty} \inf(f; C_i)$.

Proof Clearly,

$$\inf(f; C) \geq \lim_{i\to\infty} \inf(f; C_i)$$

and for each $i \in \{1, 2, \dots\} \cup \{\infty\}$, $\inf(f; C_i)$ is finite.

For each integer $i \geq 1$ set

$$D_i = \{z \in C_i : \ f_i(z) \leq \lim_{j\to\infty} \inf(f; C_j)\}.$$

Clearly for any integer $i \geq 1$ $D_i \neq \emptyset$ and the set D_i is closed convex and bounded and therefore it is weakly compact. Hence

$$\cap_{i=1}^\infty D_i \neq \emptyset.$$

Let

$$z \in \cap_{i=1}^\infty D_i.$$

Then

$$f(z) \le \lim_{i\to\infty} \inf(f; C_i),\ z \in C_\infty$$

and

$$\inf(f; C) \le \lim_{i\to\infty} \inf(f; C_i).$$

Lemma 5.1 is proved.

Let $\mathcal{A}$ be $\mathcal{M}$ or $\mathcal{M}_v$ or $\mathcal{M}_c$ or $\mathcal{M}_{lL}$ or $\mathcal{M}_L$,

$$f \in \mathcal{A},\ \gamma \in (0,1),\ x_f \in C_\infty,\ f(x_f) = \inf(f; C_\infty). \tag{5.1}$$

Set

$$f_\gamma(x) = f(x) + \gamma||x - x_f||,\ x \in C_1. \tag{5.2}$$

Clearly, $f_\gamma \in \mathcal{A}$ and

$$f_\gamma \to f \text{ as } \gamma \to 0_+ \text{ in } \mathcal{A}. \tag{5.3}$$

Lemma 5.2. *Let*

$$f \in \mathcal{A},\ \gamma \in (0,1),\ x_f \in C_\infty,\ f(x_f) = \inf(f; C_\infty) \tag{5.4}$$

and let $\epsilon > 0$. Then there exist an integer i_0 and $\delta > 0$ such that if an integer $i \ge i_0$ and if $x \in C_i$ satisfies $f_\gamma(x) \le \inf(f_\gamma; C_i) + \delta$, then $||x - x_f|| \le \epsilon$.

Proof By (5.1) and (5.2),

$$f_\gamma(x_f) = f(x_f) = \inf(f; C_\infty) = \inf(f_\gamma; C_\infty). \tag{5.5}$$

Choose $\delta \in (0,1)$ such that

$$4\delta/\gamma < \epsilon. \tag{5.6}$$

By Lemma 5.1 there is an integer $i_0 \ge 1$ such that

$$|\inf(f; C_\infty) - \inf(f; C_i)| \le \delta \text{ for all integers } i \ge i_0. \tag{5.7}$$

Let an integer i satisfy

$$i \ge i_0,\ x \in C_i,\ f_\gamma(x) \le \inf(f_\gamma; C_i) + \delta. \tag{5.8}$$

In view of (2.1), (5.3), (5.5), (5.7) and (5.8)

$$\begin{aligned} f(x) + \gamma||x - x_f|| &= f_\gamma(x) \le \inf(f_\gamma; C_i) + \delta \le \inf(f_\gamma; C_\infty) + \delta = \inf(f; C_\infty) + \delta \\ &\le \inf(f; C_i) + 2\delta \le f(x_f) + 2\delta. \end{aligned}$$

These relations imply that

$$\gamma||x - x_f|| \le 2\delta.$$

Together with (5.6) this inequality implies that

$$||x - x_f|| \le 2\delta\gamma^{-1} < \epsilon.$$

Lemma 5.2 is proved.

Lemma 5.3. *Let*

$$f \in \mathcal{A},\ \gamma \in (0,1),\ x_f \in C_\infty \text{ and } f(x_f) = \inf(f; C_\infty). \tag{5.9}$$

Then there exists a natural number n such that the following assertion holds.

For each $\epsilon > 0$ there is $\delta > 0$ and an integer $i_0 \geq 1$ such that if an integer $i \geq i_0$, if $g \in \mathcal{A}$ satisfies $(g, f_\gamma) \in \mathcal{E}(n)$ and if $x \in C_i$ satisfies

$$g(x) \leq \inf(g; C_i) + \delta,$$

then $||x - x_f|| \leq \epsilon$.

Proof By (5.1), (5.2) and (5.9),

$$f_\gamma(x_f) = f(x_f) = \inf(f; C_\infty) = \inf(f_\gamma; C_\infty). \tag{5.10}$$

In view of (4.1) there is a natural number n such that

$$n > ||x_f|| + 4 \text{ and } 1/n < \gamma/4 \tag{5.11}$$

and

$$\text{if } x \in C_1 \text{ satisfies } \phi(x) \leq |\inf(f; C_1)| + |\inf(f; C_\infty)| + 8, \text{ then } ||x|| \leq n. \tag{5.12}$$

Let $\epsilon > 0$. Choose $\delta \in (0,1)$ such that

$$8\delta\gamma^{-1} < \epsilon. \tag{5.13}$$

By Lemma 5.1 there is a natural number i_0 such that for each integer $i \geq i_0$

$$|\inf(f; C_i) - \inf(f; C_\infty)| \leq \delta/4. \tag{5.14}$$

Assume that

$$g \in \mathcal{A},\ (g, f_\gamma) \in \mathcal{E}(n), \text{ an integer } i \geq i_0,\ x \in C_i,\ g(x) \leq \inf(g; C_i) + \delta. \tag{5.15}$$

In view of (5.11), (5.15), (2.1) and (5.9)

$$|g(x_f) - f_\gamma(x_f)| \leq 1/n. \tag{5.16}$$

It follows from (5.15) that

$$g(x) \leq \inf(g; C_i) + \delta \leq \inf(g; C_\infty) + \delta \leq g(x_f) + \delta. \tag{5.17}$$

By (4.2), (5.16) and (5.10),

$$\phi(x) \leq g(x) \leq f(x_f) + 1. \tag{5.18}$$

By (5.18), (5.12) and (5.15),

$$||x|| \le n, \ |g(x) - f_\gamma(x)| \le 1/n. \tag{5.19}$$

By (5.15), (4.3), (5.19), (5.11) and (5.3),

$$\begin{aligned} g(x) &= f_\gamma(x) + g(x) - f_\gamma(x) \\ &= f_\gamma(x) + ((g - f_\gamma)(x) - (g - f_\gamma)(x_f)) + (g - f_\gamma)(x_f) \\ &\ge f_\gamma(x) - n^{-1}||x - x_f|| + (g - f_\gamma)(x_f) \\ &= f(x) + \gamma||x - x_f|| - n^{-1}||x - x_f|| + (g - f_\gamma)(x_f) \\ &\ge f(x) + (\gamma/2)||x - x_f|| + g(x_f) - f_\gamma(x_f). \end{aligned}$$

Combined with (5.10) and (5.17) this implies that

$$\delta \ge f(x) + (\gamma/2)||x - x_f|| - f(x_f). \tag{5.20}$$

By (5.20), (5.15), (5.14) and (5.9),

$$\begin{aligned} f(x_f) + \delta &\ge f(x) + (\gamma/2)||x - x_f|| \ge \inf(f; C_i) + (\gamma/2)||x - x_f|| \\ &\ge \inf(f; C_\infty) - \delta/4 + (\gamma/2)||x - x_f|| \\ &= f(x_f) - \delta/4 + (\gamma/2)||x - x_f|| \end{aligned}$$

and

$$4\delta\gamma^{-1} \ge ||x - x_f||.$$

Combined with (5.13) this inequality implies that

$$||x - x_f|| < \epsilon.$$

Lemma 5.3 is proved.

Completion of the proof of Theorem 4.1.

Let $f \in \mathcal{A}$, $x_f \in C_\infty$, $f(x_f) = \inf(f; C_\infty)$, $\gamma \in (0,1)$. By Lemma 5.3 there exists an open neighborhood $V(f,\gamma)$ of f_γ in $\mathcal{A}$ such that the following property holds:

For each $\epsilon > 0$ there exist $\delta > 0$ and an integer $i_0 \ge 1$ such that if an integer $i \ge i_0$, $g \in V(f,\gamma)$ and if $x \in C_i$ satisfies

$$g(x) \le \inf(g; C_i) + \delta,$$

then $||x - x_f|| \le \epsilon$.

Put

$$\mathcal{F} = \cup\{V(f,\gamma) : \ f \in \mathcal{A}, \ \gamma \in (0,1)\}.$$

By (5.3), $\mathcal{F}$ is an open everywhere dense subset of $\mathcal{A}$. It is not difficult to see that Theorem 4.1 holds by the definition of $V(f,\gamma)$ ($f \in \mathcal{A}$, $\gamma \in (0,1)$).

References

[1] D. Butnariu, S. Reich and A. J. Zaslavski, *There are many totally convex functions*, J. Convex Analysis **13**, 623-632 (2006)

[2] P.G. Howlett and A. J. Zaslavski, *On the minimization of convex functions in reflexive Banach spaces*, Communications in Applied Analysis, **5**, 535-545, (2001)

[3] P.G. Howlett and A. J. Zaslavski, *A porosity result in convex minimization*, Abstract and Applied Analysis, (2005), 319-320.

[4] M.M. Israel and S. Reich, *Extension and selection problems for nonlinear semigroups in Banach spaces*, Math. Japonica, **28**, 1-8, (1983)

[5] J. Semple, *Infinite positive-definite quadratic programming in a Hilbert space*, J. Optim. Theory Appl. **88**, 743-749, (1996)

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Certain subclass of multivalent functions involving the Cho-Kwon-Srivastava operator

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Abstract: In the present paper, we introduce a new subclass $\mathcal{S}_{p,\lambda}^{(j)}(\alpha; a, c; \phi)$ of multivalent functions involving the Cho-Kwon-Srivastava operator. Such results as inclusion relationships, coefficient estimates and convolution properties for this class are proved. The results presented here would provide extensions of those given in earlier works

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1. Introduction

Let $\mathcal{A}_p$ denote the class of functions of the following form:

$$f(z) = z^p + \sum_{n=1}^{\infty} a_{n+p} z^{n+p} \qquad (p \in \mathbb{N} := \{1, 2, 3, \ldots\}), \tag{1}$$

which are *analytic* in the *open* unit disk

$$\mathbb{U} := \{z : z \in \mathbb{C} \quad \text{and} \quad |z| < 1\}.$$

For simplicity, we write

$$\mathcal{A}_1 =: \mathcal{A}.$$

Also let $\mathcal{P}$ denote the class of functions of the form:

$$\mathfrak{p}(z) = 1 + \sum_{n=1}^{\infty} \mathfrak{p}_n z^n \qquad (z \in \mathbb{U}),$$

which are analytic and convex in $\mathbb{U}$ and satisfy the following inequality:

$$\Re(\mathfrak{p}(z)) > 0.$$

Let $f,\ g \in \mathcal{A}_p$, where f is given by (1.1) and g is defined by

$$g(z) = z^p + \sum_{n=1}^{\infty} b_{n+p} z^{n+p}.$$

Then the Hadamard product (or convolution) $f * g$ of the functions f and g is defined by

$$(f * g)(z) := z^p + \sum_{n=1}^{\infty} a_{n+p} b_{n+p} z^{n+p} =: (g * f)(z).$$

For parameters

$$a \in \mathbb{R}, \quad c \in \mathbb{R} \setminus \mathbb{Z}_0^- \quad (\mathbb{Z}_0^- := \{0, -1, -2, \ldots\}),$$

Saitoh [5] introduced a linear operator:

$$\mathcal{L}_p(a, c) : \mathcal{A}_p \longrightarrow \mathcal{A}_p$$

defined by

$$\mathcal{L}_p(a, c) f(z) = \phi_p(a, c; z) * f(z) \qquad (z \in \mathbb{U};\ f \in \mathcal{A}_p)$$

where

$$\phi_p(a, c; z) = \sum_{n=0}^{\infty} \frac{(a)_n}{(c)_n} z^{n+p} \tag{2}$$

and $(\lambda)_n$ is the Pochhammer symbol defined by

$$(\lambda)_n := \begin{cases} 1, & (n = 0), \\ \lambda(\lambda + 1) \cdots (\lambda + n - 1), & (n \in \mathbb{N}). \end{cases}$$

In a recent paper, Cho *et al.* [2] introduced the following family of linear operators $\mathcal{I}_p^\lambda(a, c)$ analogous to $\mathcal{L}_p(a, c)$:

$$\mathcal{I}_p^\lambda(a, c) : \mathcal{A}_p \longrightarrow \mathcal{A}_p,$$

which is defined as

$$\mathcal{I}_p^\lambda(a, c) f(z) := \phi_p^\dagger(a, c; z) * f(z) \qquad (a, c \in \mathbb{R} \setminus \mathbb{Z}_0^-;\ \lambda > -p;\ z \in \mathbb{U};\ f \in \mathcal{A}_p), \tag{3}$$

where $\phi_p^\dagger(a, c; z)$ is the function defined in terms of the Hadamard product (or convolution) by the following condition:

$$\phi_p(a, c; z) * \phi_p^\dagger(a, c; z) = \frac{z^p}{(1 - z)^{\lambda + p}}. \tag{4}$$

We can easily find from (2), (3) and (4) that

$$\mathcal{I}_p^{\lambda}(a,c)f(z) = \sum_{n=0}^{\infty} \frac{(\lambda+p)_n (c)_n}{n!(a)_n} a_{n+p} z^{n+p} \qquad (z \in \mathbb{U};\ \lambda > -p). \tag{5}$$

It is also readily verified from (5) that

$$z\left(\mathcal{I}_p^{\lambda}(a,c)f\right)^{(j+1)}(z) = (p-j-c)\left(\mathcal{I}_p^{\lambda}(a,c)f\right)^{(j)}(z) + c\left(\mathcal{I}_p^{\lambda}(a,c+1)f\right)^{(j)}(z) \tag{6}$$

$$(z \in \mathbb{U};\ j \in \{0,1,\ldots,p-1\}).$$

For two functions f and g, analytic in $\mathbb{U}$, we say that the function f is subordinate to g in $\mathbb{U}$, and write

$$f(z) \prec g(z) \qquad (z \in \mathbb{U}),$$

if there exists a Schwarz function $\omega(z)$, which is analytic in $\mathbb{U}$ with

$$\omega(0) = 0 \quad \text{and} \quad |\omega(z)| < 1 \quad (z \in \mathbb{U})$$

such that

$$f(z) = g\big(\omega(z)\big) \qquad (z \in \mathbb{U}).$$

Indeed it is known that

$$f(z) \prec g(z) \quad (z \in \mathbb{U}) \Longrightarrow f(0) = g(0) \quad \text{and} \quad f(\mathbb{U}) \subset g(\mathbb{U}).$$

Furthermore, if the function g is univalent in $\mathbb{U}$, then we have the following equivalence:

$$f(z) \prec g(z) \quad (z \in \mathbb{U}) \Longleftrightarrow f(0) = g(0) \quad \text{and} \quad f(\mathbb{U}) \subset g(\mathbb{U}).$$

In recent years, several authors obtained many interesting results involving the Cho-Kwon-Srivastava operator (see, for details, [1, 4, 6]). In the present paper, by making use of the operator $\mathcal{I}_p^{\lambda}(a,c)$ and the above-mentioned principle of subordination between analytic functions, we introduce and investigate the following subclass of the class $\mathcal{A}_p$ of p-valent analytic functions.

Definition 1 *A function $f \in \mathcal{A}_p$ is said to be in the class $\mathcal{S}_{p,\lambda}^{(j)}(\alpha;a,c;\phi)$ if it satisfies the following subordination condition:*

$$\frac{z\left[(1-\alpha)\left(\mathcal{I}_p^{\lambda}(a,c)f\right)^{(j+1)}(z) + \alpha\left(\mathcal{I}_p^{\lambda}(a,c+1)f\right)^{(j+1)}(z)\right]}{(1-\alpha)\left(\mathcal{I}_p^{\lambda}(a,c)f\right)^{(j)}(z) + \alpha\left(\mathcal{I}_p^{\lambda}(a,c+1)f\right)^{(j)}(z)} \prec (p-j)\phi(z) \qquad (z \in \mathbb{U}) \tag{7}$$

for some α ($\alpha \geqq 0$) and j ($j \in \{0,1,\ldots,p-1\}$), where $\phi \in \mathcal{P}$.

For simplicity, we write

$$\mathcal{S}_{p,\lambda}^{(j)}(0;a,c;\phi) =: \mathcal{S}_{p,\lambda}^{(j)}(a,c;\phi).$$

Remark 1 If we set

$$\alpha = j = 0 \quad \text{and} \quad \phi(z) = \frac{1 + Az}{1 + Bz} \qquad (-1 \leqq B < A \leqq 1)$$

in the class $\mathcal{S}_{p,\lambda}^{(j)}(\alpha; a, c; \phi)$, then it reduces to the class $\mathcal{S}_{p,\lambda}(a, c; A, B)$ which was studied recently by Aghalary [1].

In order to establish our main results, we shall also make use of the following lemma.

Lemma 1 (see [3]) *Let $\beta, \gamma \in \mathbb{C}$. Suppose also that $\phi(z)$ is convex and univalent in $\mathbb{U}$ with*

$$\phi(0) = 1 \quad \text{and} \quad \Re(\beta\phi(z) + \gamma) > 0 \qquad (z \in \mathbb{U}).$$

If $\mathfrak{p}(z)$ is analytic in $\mathbb{U}$ with $\mathfrak{p}(0) = 1$, then the following subordination:

$$\mathfrak{p}(z) + \frac{z\mathfrak{p}'(z)}{\beta\mathfrak{p}(z) + \gamma} \prec \phi(z) \qquad (z \in \mathbb{U})$$

implies that

$$\mathfrak{p}(z) \prec \phi(z) \qquad (z \in \mathbb{U}).$$

In the present paper, we aim at proving such results as inclusion relationships, coefficient estimates and convolution properties for the class $\mathcal{S}_{p,\lambda}^{(j)}(\alpha; a, c; \phi)$. The results presented here would provide extensions of those given in earlier works.

2. A set of inclusion relationships

At first, we prove some inclusion relationships for the class $\mathcal{S}_{p,\lambda}^{(j)}(\alpha; a, c; \phi)$, which was defined in the preceding section.

Theorem 1 *Let $\phi \in \mathcal{P}$ with*

$$\Re\left((p - j)\phi(z) + \frac{c}{\alpha} - p + j\right) > 0 \qquad (\alpha > 0;\ j \in \{0, 1, \ldots, p - 1\};\ z \in \mathbb{U}).$$

Then

$$\mathcal{S}_{p,\lambda}^{(j)}(\alpha; a, c; \phi) \subset \mathcal{S}_{p,\lambda}^{(j)}(a, c; \phi).$$

Proof. Let $f \in \mathcal{S}_{p,\lambda}^{(j)}(\alpha; a, c; \phi)$ and suppose that

$$\psi(z) = \frac{z\left(\mathcal{I}_p^\lambda(a, c)f\right)^{(j+1)}(z)}{(p - j)\left(\mathcal{I}_p^\lambda(a, c)f\right)^{(j)}(z)} \qquad (z \in \mathbb{U}). \tag{8}$$

Then ψ is analytic in $\mathbb{U}$ and $\psi(0) = 1$. It follows from (6) and (8) that

$$c - p + j + (p - j)\psi(z) = \frac{c\left(\mathcal{I}_p^\lambda(a, c + 1)f\right)^{(j)}(z)}{\left(\mathcal{I}_p^\lambda(a, c)f\right)^{(j)}(z)}. \tag{9}$$

We can easily find from (8) and (9) that

$$z\left(\mathcal{I}_p^\lambda(a,c+1)f\right)^{(j+1)}(z) = \frac{p-j}{c}\left\{z\psi'(z)+[c-p+j+(p-j)\psi(z)]\,\psi(z)\right\}\left(\mathcal{I}_p^\lambda(a,c)f\right)^{(j)}(z). \tag{10}$$

It now follows from (6), (8), (9) and (10) that

$$\begin{aligned}
&\frac{z\left[(1-\alpha)\left(\mathcal{I}_p^\lambda(a,c)f\right)^{(j+1)}(z)+\alpha\left(\mathcal{I}_p^\lambda(a,c+1)f\right)^{(j+1)}(z)\right]}{(p-j)\left[(1-\alpha)\left(\mathcal{I}_p^\lambda(a,c)f\right)^{(j)}(z)+\alpha\left(\mathcal{I}_p^\lambda(a,c+1)f\right)^{(j)}(z)\right]}\\
&\quad=\frac{(1-\alpha)\psi(z)+\frac{\alpha}{c}\left\{z\psi'(z)+[c-p+j+(p-j)\psi(z)]\,\psi(z)\right\}}{(1-\alpha)+\frac{\alpha}{c}[c-p+j+(p-j)\psi(z)]}\\
&\quad=\frac{\frac{\alpha}{c}z\psi'(z)+\left\{(1-\alpha)+\frac{\alpha}{c}[c-p+j+(p-j)\psi(z)]\right\}\psi(z)}{(1-\alpha)+\frac{\alpha}{c}[c-p+j+(p-j)\psi(z)]}\\
&\quad=\psi(z)+\frac{z\psi'(z)}{\frac{c}{\alpha}-p+j+(p-j)\psi(z)}\prec\phi(z)\qquad(z\in\mathbb{U}).
\end{aligned}\tag{11}$$

Moreover, since

$$\Re\left((p-j)\phi(z)+\frac{c}{\alpha}-p+j\right)>0\qquad(\alpha>0;\ z\in\mathbb{U}),$$

by Lemma 1 and (11), we know that

$$\psi(z)=\frac{z\left(\mathcal{I}_p^\lambda(a,c)f\right)^{(j+1)}(z)}{(p-j)\left(\mathcal{I}_p^\lambda(a,c)f\right)^{(j)}(z)}\prec\phi(z)\qquad(z\in\mathbb{U}),$$

that is, that $f\in\mathcal{S}_{p,\lambda}^{(j)}(a,c;\phi)$. This implies that

$$\mathcal{S}_{p,\lambda}^{(j)}(\alpha;a,c;\phi)\subset\mathcal{S}_{p,\lambda}^{(j)}(a,c;\phi).$$

Hence the proof of Theorem 1 is complete. ■

Theorem 2 *Let* $\phi\in\mathcal{P}$ *with*

$$\Re\left((p-j)\phi(z)+c-p+j\right)>0\qquad(j\in\{0,1,\ldots,p-1\};\ z\in\mathbb{U}).$$

Then

$$\mathcal{S}_{p,\lambda}^{(j)}(a,c+1;\phi)\subset\mathcal{S}_{p,\lambda}^{(j)}(a,c;\phi).$$

Proof. Suppose that $f\in\mathcal{S}_{p,\lambda}^{(j)}(a,c+1;\phi)$. Then we have

$$\frac{z\left(\mathcal{I}_p^\lambda(a,c+1)f\right)^{(j+1)}(z)}{(p-j)\left(\mathcal{I}_p^\lambda(a,c+1)f\right)^{(j)}(z)}\prec\phi(z)\qquad(z\in\mathbb{U}). \tag{12}$$

Differentiating both sides of (9) with respect to z logarithmically and using (8), we have

$$\psi(z)+\frac{z\psi'(z)}{c-p+j+(p-j)\psi(z)}=\frac{z\left(\mathcal{I}_p^{\lambda}(a,c+1)f\right)^{(j+1)}(z)}{(p-j)\left(\mathcal{I}_p^{\lambda}(a,c+1)f\right)^{(j)}(z)}\qquad (z\in\mathbb{U}).\quad (13)$$

It now follows from (12) and (13) that

$$\psi(z)+\frac{z\psi'(z)}{c-p+j+(p-j)\psi(z)}\prec\phi(z)\qquad (z\in\mathbb{U}).\quad (14)$$

Moreover, since

$$\Re\left((p-j)\phi(z)+c-p+j\right)>0\qquad (z\in\mathbb{U}),$$

by (14) and Lemma 1, we know that

$$\psi(z)=\frac{z\left(\mathcal{I}_p^{\lambda}(a,c)f\right)^{(j+1)}(z)}{(p-j)\left(\mathcal{I}_p^{\lambda}(a,c)f\right)^{(j)}(z)}\prec\phi(z)\qquad (z\in\mathbb{U}),$$

that is, that $f\in\mathcal{S}_{p,\lambda}^{(j)}(a,c;\phi)$. This implies that

$$\mathcal{S}_{p,\lambda}^{(j)}(a,c+1;\phi)\subset\mathcal{S}_{p,\lambda}^{(j)}(a,c;\phi).$$

The proof of Theorem 2 is thus completed. ■

3. Coefficient estimates

In this section, we give the coefficient estimates of functions belonging to the class $\mathcal{S}_{p,\lambda}^{(j)}(\alpha;a,c;\phi)$.

Theorem 3 *If $f\in\mathcal{S}_{p,\lambda}^{(j)}\left(\alpha;a,c;\frac{1+z}{1-z}\right)$, then*

$$|a_{n+p}|\leqq\frac{2n!(p-j)(p-j+1)_n(2p-3j+1)_{n-1}(a)_n}{(1-j)_n(\lambda+p)_n(p+1)_n(c+1)_{n-1}(c+n\alpha)}\quad (15)$$

$(j\in\{0,1,\ldots,p-1\};\ n,p\in\mathbb{N})$.

Proof. Suppose that $f\in\mathcal{S}_{p,\lambda}^{(j)}\left(\alpha;a,c;\frac{1+z}{1-z}\right)$. It follows that

$$\frac{z\left[(1-\alpha)\left(\mathcal{I}_p^{\lambda}(a,c)f\right)^{(j+1)}(z)+\alpha\left(\mathcal{I}_p^{\lambda}(a,c+1)f\right)^{(j+1)}(z)\right]}{(p-j)\left[(1-\alpha)\left(\mathcal{I}_p^{\lambda}(a,c)f\right)^{(j)}(z)+\alpha\left(\mathcal{I}_p^{\lambda}(a,c+1)f\right)^{(j)}(z)\right]}=:p(z),\quad (16)$$

where

$$p(z)=1+p_1z+p_2z^2+\cdots\prec\frac{1+z}{1-z}\qquad (z\in\mathbb{U}).$$

Upon substituting the series expansion of $f(z)$ and $p(z)$ in (16) and equating the coefficients of z^{n+p-j} on both sides of the resulting equation, we obtain

$$\begin{aligned}(n-j)&\left[(1-\alpha)k_{c,n}+\alpha k_{c+1,n}\right]\\ &=p_1(p-j)\left[(1-\alpha)k_{c,n-1}+\alpha k_{c+1,n-1}\right]+p_2(p-j)\left[(1-\alpha)k_{c,n-2}+\alpha k_{c+1,n-2}\right]\\ &\quad+\cdots+p_n(p-j)\left[(1-\alpha)k_{c,0}+\alpha k_{c+1,0}\right]\qquad (n\in\mathbb{N}),\end{aligned} \tag{17}$$

where

$$k_0:=p_0:=1,$$

and

$$k_{c,n}:=\frac{(\lambda+p)_n(c)_n}{n!(a)_n}(n+p)\cdots(n+p-j+1)a_{n+p}.$$

Using the well known coefficient estimates:

$$|p_n|\leqq 2\qquad (n\in\mathbb{N})$$

in (17), we get the required result (15) asserted by Theorem 3. ∎

Remark 2 If we set $\alpha=j=0$ in Theorem 3, we can get the corresponding result obtained by Aghalary [1].

4. Convolution properties

In this section, we provide some convolution properties for the class $\mathcal{S}_{p,\lambda}^{(j)}(a,c;\phi)$.

Theorem 4 *Let* $f\in\mathcal{S}_{p,\lambda}^{(j)}(a,c;\phi)$. *Then*

$$f^{(j)}(z)=\left[z^{p-j}\exp\left((p-j)\int_0^z\frac{\phi(\omega(z))-1}{\xi}d\xi\right)\right]*\left(\sum_{n=0}^{\infty}\frac{n!(a)_n}{(\lambda+p)_n(c)_n}z^{n+p-j}\right),\tag{18}$$

$(j\in\{0,1,\ldots,p-1\})$ *where* ω *is analytic in* $\mathbb{U}$ *with*

$$\omega(0)=0\quad\text{and}\quad|\omega(z)|<1\quad(z\in\mathbb{U}).$$

Proof. Suppose that $f\in\mathcal{S}_{p,\lambda}^{(j)}(a,c;\phi)$. We know from (7) (with $\alpha=0$) that

$$\frac{z\left(\mathcal{I}_p^{\lambda}(a,c)f\right)^{(j+1)}(z)}{(p-j)\left(\mathcal{I}_p^{\lambda}(a,c)f\right)^{(j)}(z)}=\phi(\omega(z))\qquad(z\in\mathbb{U}),\tag{19}$$

where ω is analytic in $\mathbb{U}$ with

$$\omega(0)=0\quad\text{and}\quad|\omega(z)|<1\quad(z\in\mathbb{U}).$$

We next find from (19) that

$$\frac{\left(\mathcal{I}_p^\lambda(a,c)f\right)^{(j+1)}(z)}{\left(\mathcal{I}_p^\lambda(a,c)f\right)^{(j)}(z)} - \frac{p-j}{z} = (p-j)\frac{\phi(\omega(z))-1}{z} \qquad (z \in \mathbb{U}). \tag{20}$$

Upon integrating (20), we have

$$\log\left(\frac{\left(\mathcal{I}_p^\lambda(a,c)f\right)^{(j)}(z)}{z^{p-j}}\right) = (p-j)\int_0^z \frac{\phi(\omega(\xi))-1}{\xi}d\xi,$$

or equivalently,

$$\left(\mathcal{I}_p^\lambda(a,c)f\right)^{(j)}(z) = z^{p-j}\cdot\exp\left((p-j)\int_0^z \frac{\phi(\omega(\xi))-1}{\xi}d\xi\right). \tag{21}$$

On the other hand, we know from (5) that

$$\left(\mathcal{I}_p^\lambda(a,c)f\right)^{(j)}(z) = \left(\sum_{n=0}^{\infty}\frac{(\lambda+p)_n(c)_n}{n!(a)_n}z^{n+p-j}\right) * f^{(j)}(z). \tag{22}$$

The assertion (18) of Theorem 4 can now easily be derived from (21) and (22). ■

Theorem 5 *Let*

$$f \in \mathcal{A}_p \quad \textit{and} \quad \phi \in \mathcal{P}.$$

Then $f \in \mathcal{S}_{p,\lambda}^{(j)}(a,c;\phi)$ *if and only if*

$$\frac{1}{z}\left[f^{(j)}(z) * \left(\sum_{n=0}^{\infty}\frac{(\lambda+p)_n(c)_n}{n!(a)_n}\left(n+p-j-(p-j)\phi(e^{i\theta})\right)z^{n+p-j}\right)\right] \neq 0 \tag{23}$$

$$(z \in \mathbb{U};\ 0 \leqq \theta < 2\pi).$$

Proof. Suppose that $f \in \mathcal{S}_{p,\lambda}^{(j)}(a,c;\phi)$. Since the following subordination condition:

$$\frac{z\left(\mathcal{I}_p^\lambda(a,c)f\right)^{(j+1)}(z)}{(p-j)\left(\mathcal{I}_p^\lambda(a,c)f\right)^{(j)}(z)} \prec \phi(z)$$

is equivalent to

$$\frac{z\left(\mathcal{I}_p^\lambda(a,c)f\right)^{(j+1)}(z)}{(p-j)\left(\mathcal{I}_p^\lambda(a,c)f\right)^{(j)}(z)} \neq \phi(e^{i\theta}) \qquad (z \in \mathbb{U};\ 0 \leqq \theta < 2\pi). \tag{24}$$

It is easy to see that the condition (24) can be written as follows:

$$\frac{1}{z}\left[z\left(\mathcal{I}_p^\lambda(a,c)f\right)^{(j+1)}(z)-(p-j)\left(\mathcal{I}_p^\lambda(a,c)f\right)^{(j)}(z)\phi(e^{i\theta})\right]\neq 0 \tag{25}$$

$(z\in\mathbb{U};\ \ 0\leqq\theta<2\pi)$.

On the other hand, we know from (5) that

$$z\left(\mathcal{I}_p^\lambda(a,c)f\right)^{(j+1)}(z)=\left(\sum_{n=0}^{\infty}\frac{(\lambda+p)_n(c)_n}{n!(a)_n}(n+p-j)z^{n+p-j}\right)*f^{(j)}(z). \tag{26}$$

Upon substituting (22) and (26) into (25), we can easily get the convolution property (23) asserted by Theorem 5. ■

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References

[1] R. Aghalary, On subclasses of p-valent analytic functions defined by integral operators, *Kyungpook Math. J.* **47** (2007), 393–401.

[2] N. E. Cho, O. S. Kwon, H. M. Srivastava, Inclusion relationships and argument properties for certain subclasses of multivalent functions associated with a family of linear operators, *J. Math. Anal. Appl.* **292** (2004), 470–483.

[3] P. Eenigenburg, S. S. Miller, P. T. Mocanu and M. O. Reade, On a Briot-Bouquet differential subordination, in *General Mathematics* **3**, International Series of Numerical Mathematics, Vol. **64**, Birkhäuser Verlag, Basel, 1983, pp. 339–348; see also *Rev. Roumaine Math. Pures Appl.* **29** (1984), 567–573.

[4] J. Patel, On certain subclasses of multivalent functions involving Cho-Kwon-Srivastava operator, *Ann. Univ. Mariae Curie-Skłodowska Sect. A* **60** (2006), 75–86.

[5] H. Saitoh, A linear operator and its applications of first order differential subordinations, *Math. Japon.* **44** (1996), 31–38.

[6] J. Sokól and L. Trojnar-Spelina, Convolution properties for certain classes of multivalent functions, *J. Math. Anal. Appl.* **337** (2008), 1190–1197.

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