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The Joint Laplace-Hankel Transforms for Fractional Diffusion Equation

Arman Aghili

Abstract: Operational methods are used to accomplish the solution of certain problems with less effort and in a simple routine way. Laplace transforms can be used to solve certain types of fractional singular integral equation not considered in the literature. In this study, the author implemented an analytical technique the joint Laplace-Hankel transforms to provide the exact solution for a time fractional non-homogeneous diffusion equation with non-constant coefficients in cylindrical coordinates. The obtained results reveal that the joint transform method is very convenient and effective. Certain non trivial integral identities involving Airy functions and modified Bessel functions of the second kind are also provided.

AMS Subject Classification: 26A33, 44A10, 44A20, 35A22.
Keywords and Phrases: Laplace transforms; Hankel transform; Modified Bessel function; Airy function; Gross Levi.

1. Introduction and Preliminaries

In recent years, a growing number of research works done by many researchers from various fields of engineering and science deal with dynamical systems described by equations of fractional order which means equations involving derivatives and integrals of fractional order.
In this work, the author studied analytically distribution functions during ion cyclotron resonance heating (ICRH) by using the one-dimensional Fokker-Planck equation incorporating ion-electron and ion-ion collisions and quasi-linear diffusion. In the equation, we include source and loss terms and we find the steady-state and time-dependent solutions which are regular in the origin and vanish at high energies. The

main purpose of the current study is to develop a method for evaluation of certain integrals and finding analytic solutions of fractional PDEs. An analytical technique approaches, the joint Laplace-Hankel transforms to provide the exact solution for a time fractional non-homogeneous diffusion equation with non-constant coefficients in cylindrical coordinates.

1.1. Definitions and Notations

Definition 1.1. With $\mathcal{D}_t^{c,\alpha}$ we denote the time fractional derivative of order α $(0<\alpha<1)$ regularized in the Dzhrbashyan-Caputo sense defined for a sufficiently regular function $\phi(t)$, as

$$D_t^{c,\alpha}\phi(t)=\frac{1}{\Gamma(1-\alpha)}\int_a^t \frac{1}{(t-\xi)^\alpha}\phi'(\xi)d\xi. \tag{1.1}$$

Remark. In this work, we prefer Caputo fractional derivative to Riemann-Liouville one since the former is more popular in real applications. When we adopt the Caputo fractional derivative of order-α, the initial values of $y(0), y'(0), ..., y^m(0)$, where $m=[\alpha]$, are enough. Obviously, these initial values are prone to measure since they have all physical meaning. On the other hand, we choose Caputo fractional derivative due to another fact that the non-homogeneous initial conditions are permitted if such conditions are necessary.

Definition 1.2. The Laplace transform of the function $f(t)$ is given by[1-3]

$$\mathcal{L}\{f(t)\}=\int_0^\infty e^{-st}f(t)dt:=F(s). \tag{1.2}$$

If $\mathcal{L}\{f(t)\}=F(s)$, then $\mathcal{L}^{-1}\{F(s)\}$ is as follows

$$f(t)=\frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty}e^{st}F(s)ds, \tag{1.3}$$

where $F(s)$ is analytic in the region $\mathrm{R}e(s)>c$.
The expression in equation (1.3) is the inverse Laplace transform for the function $F(s)$, and is often called the Bromwich integral.

Lemma 1.1. *Let $L\{f(t)\}=F(s)$ then, the following identities hold*

1. $\mathcal{L}^{-1}(\frac{1}{\sqrt{s}(\sqrt{s}+\lambda)})=e^{\lambda^2 t}Erfc(\lambda\sqrt{t})$,
2. $e^{-\omega s^\beta}=\frac{1}{\pi}\int_0^\infty e^{-r^\beta(\omega\cos\beta\pi)}\frac{\sin(\omega r^\beta\sin\beta\pi)}{s+r}dr$,
3. $\mathcal{L}^{-1}(F(s^\alpha))=\frac{1}{\pi}\int_0^\infty f(u)\int_0^\infty e^{-tr-ur^\alpha\cos\alpha\pi}\sin(ur^\alpha\sin\alpha\pi)drdu$,
4. $\mathcal{L}^{-1}(F(\sqrt[3]{s}))=\frac{1}{3\pi}\int_0^\infty(\frac{u}{t})^{\frac{3}{2}}K_{\frac{1}{3}}(\frac{2u\sqrt{u}}{3\sqrt{3t}})f(u)du$.

Proof. See [1].

Example 1.1. The fractional integral of order α of the function $\phi(t)$, with $0 < \alpha < 1$ is defined as follows

$$\mathcal{J}^{\alpha}\phi(t) = \frac{1}{\Gamma(\alpha)}\int_0^t (t-\xi)^{\alpha-1}\phi(\xi)d\xi,$$

then the Laplace transform of the fractional integral of order α is as below

$$\mathcal{L}[\mathcal{J}^{\alpha}\phi(t)] = \int_0^{+\infty} e^{-st}[\frac{1}{\Gamma(\alpha)}\int_0^t (t-\xi)^{\alpha-1}\phi(\xi)d\xi]dt = \frac{\Phi(s)}{s^{\alpha}}.$$

Lemma 1.2. *The following integral relation holds*

$$\mathcal{L}^{-1}[\frac{e^{-k\sqrt{s}}}{s^{\nu}+\lambda}; s \to t] = f(t) = \frac{1}{\pi}\int_0^{+\infty} e^{-t\xi}[\frac{\xi^{\nu}\sin(\pi\nu - k\sqrt{\xi}) - \lambda\sin(k\sqrt{\xi})}{\xi^{2\nu} + 2\lambda\xi^{\nu}\cos(\pi\nu) + \lambda^2}]d\xi.$$

Proof. In view of the Titchmarch theorem or Gross-Levi lemma [3], we have the following

$$f(t) = \frac{1}{\pi}\int_0^{+\infty} e^{-t\xi} Im[\frac{e^{-k\sqrt{\xi e^{-i\pi}}}}{(\xi e^{-i\pi})^{\nu} + \lambda}]d\xi,$$

or

$$f(t) = \frac{1}{\pi}\int_0^{+\infty} e^{-t\xi} Im[\frac{e^{-ik\sqrt{\xi}}}{\xi^{\nu}(\cos(\pi\nu) - i\sin(\pi\nu)) + \lambda}]d\xi,$$

after simplifying we have

$$f(t) = \frac{1}{\pi}\int_0^{+\infty} e^{-t\xi} Im\frac{[\cos(k\sqrt{\xi}) - i\sin(k\sqrt{\xi})][\xi^{\nu}\cos(\pi\nu) + \lambda + i\xi^{\nu}\sin(\pi\nu)]}{\xi^{2\nu} + 2\lambda\xi^{\nu}\cos(\pi\nu) + \lambda^2}d\xi,$$

or

$$f(t) = \frac{1}{\pi}\int_0^{+\infty} e^{-t\xi}[\frac{\xi^{\nu}\sin(\pi\nu - k\sqrt{\xi}) - \lambda\sin(k\sqrt{\xi})}{\xi^{2\nu} + 2\lambda\xi^{\nu}\cos(\pi\nu) + \lambda^2}]d\xi.$$

Let us consider the special cases
1. $\lambda = k = 0,\ 0 < \nu < 1$ we have

$$\mathcal{L}^{-1}[\frac{1}{s^{\nu}}; s \to t] = f(t) = \frac{\sin(\pi\nu)}{\pi}\int_0^{+\infty} e^{-t\xi}\xi^{-\nu}d\xi = \frac{t^{\nu-1}}{\Gamma(\nu)}.$$

2. $k = 0$, we have

$$\mathcal{L}^{-1}[\frac{1}{s^{\nu}+\lambda}; s \to t] = f(t) = \frac{\sin(\pi\nu)}{\pi}\int_0^{+\infty}[\frac{\xi^{\nu}e^{-t\xi}}{\xi^{2\nu} + 2\lambda\xi^{\nu}\cos(\pi\nu) + \lambda^2}]d\xi.$$

□

Lemma 1.3. *The following integral relation holds*

$$\mathcal{L}^{-1}[\frac{1}{s^{\beta}(s^{\nu}+\lambda)}; s\to t] = \int_0^t \frac{(t-\eta)^{\beta-1}}{\Gamma(\beta)}\left[\frac{\sin(\pi\nu)}{\pi}\int_0^{+\infty}[\frac{\xi^{\nu}e^{-\eta r}}{\xi^{2\nu}+2\lambda r^{\nu}\cos(\pi\nu)+\lambda^2}]d\xi\right]d\eta.$$

Proof. Making use of the convolution theorem for the Laplace transform.
Corollary 1.1. *Let us show that*

$$\mathcal{L}^{-1}[\frac{\pi}{\sqrt{3}}e^{-3\sqrt[3]{s}}; s\to t] = t^{-\frac{3}{2}}K_{\frac{1}{3}}(\frac{2}{\sqrt{t}}).$$

Note. In the above relation $K_{\nu}(.)$ stands for the modified Bessel function of the second kind of order ν.
Proof. Let us choose $f(u)=\delta(u-\lambda)$ then we have $F(s)=e^{-\lambda s}$, in view of part four of the Lemma 1.1. we get

$$\mathcal{L}^{-1}[e^{-3\sqrt[3]{s}}; s\to t] = \frac{1}{3\pi}\int_0^{\infty}(\frac{u}{t})^{\frac{3}{2}}K_{\frac{1}{3}}(\frac{2u\sqrt{u}}{3\sqrt{3t}})\delta(u-\lambda)du = \frac{1}{3\pi}(\frac{\lambda}{t})^{\frac{3}{2}}K_{\frac{1}{3}}(\frac{2\lambda\sqrt{\lambda}}{3\sqrt{3t}}).$$

If we choose $\lambda=3$, after simplifying we arrive at

$$\mathcal{L}^{-1}[\frac{\pi}{\sqrt{3}}e^{-3\sqrt[3]{s}}; s\to t] = t^{-\frac{3}{2}}K_{\frac{1}{3}}\left(\frac{2}{\sqrt{t}}\right).$$

In the above relation if we set $s=0$ we have

$$\int_0^{+\infty} t^{-\frac{3}{2}}K_{\frac{1}{3}}(\frac{2}{\sqrt{t}})dt = \int_0^{+\infty}K_{\frac{1}{3}}(\xi)d\xi = \frac{\pi}{\sqrt{3}}.$$

□

Theorem 1.1. *Let us consider fractional singular integro-differential equation*

$$D_{0,t}^{c,\alpha}\phi(t) = f(t) + \lambda\int_t^{+\infty}\phi(\xi)d\xi, \quad 0<t<+\infty$$

$$\phi(0)=u_0, \quad \int_0^{+\infty}\phi(\xi)d\xi = k, \quad 0<\alpha<1,$$

then, the above fractional singular integro-differential equation has the following formal solution

$$\phi(t) = u_0\sum_{n=0}^{+\infty}\frac{(-\lambda)^n t^{(\alpha+1)n}}{\Gamma(1+(\alpha+1)n)} + \sum_{n=0}^{+\infty}(-\lambda)^n\int_0^t f(t-\eta)\frac{\eta^{(\alpha+1)n}}{\Gamma(1+(\alpha+1)n)}d\eta$$

$$-\lambda k\sum_{n=0}^{+\infty}\frac{(-\lambda)^n t^{(\alpha+1)(1+n)-1}}{\Gamma(1+(\alpha+1)n)}.$$

Note. To the best of the author's knowledge this kind of singular integral equation is not considered in the literature.
Solution. Taking the Laplace transform of the above fractional singular integral equation term wise, leads to

$$s^{\alpha}\Phi(s) - s^{\alpha-1}u_0 = F(s) + \lambda\frac{\Phi(s)-\Phi(0)}{s} = F(s) + \lambda\frac{\Phi(s)-k}{s}.$$

After solving the above equation, we obtain

$$\Phi(s) = \frac{sF(s)}{\lambda + s^{\alpha+1}} + \frac{u_0 s^{\alpha} - \lambda k}{\lambda + s^{\alpha+1}},$$

or

$$\Phi(s) = \sum_{n=0}^{+\infty}(-\lambda)^n\left[\frac{F(s)}{s^{n(\alpha+1)+\alpha}} + \frac{u_0}{s^{(\alpha+1)n+1}} - \frac{\lambda k}{s^{(\alpha+1)(n+1)}}\right].$$

At this point, taking the inverse Laplace transform term-wise, we arrive at

$$\phi(t) = \sum_{n=0}^{+\infty}(-\lambda)^n[\int_0^t f(t-\xi)\frac{\xi^{n(\alpha+1)+\alpha-1}}{\Gamma(n(\alpha+1)+\alpha)}d\xi$$

$$+\frac{u_0 t^{(\alpha+1)n}}{\Gamma(n(\alpha+1)+1)} - \frac{\lambda k t^{(\alpha+1)(n+1)-1}}{\Gamma((\alpha+1)(n+1))}], \quad 0 < t < +\infty.$$

It is easy to verify that $\phi(0) = u_0$.
Let us consider the special case $\alpha = 0.5$, we have

$$\phi(t) = \sum_{n=0}^{+\infty}(-\lambda)^n[\int_0^t f(t-\xi)\frac{\xi^{\frac{3n-1}{2}}}{\Gamma(\frac{3n+1}{2})}d\xi$$

$$+\frac{u_0 t^{\frac{3n}{2}}}{\Gamma(\frac{3n}{2}+1)} - \frac{\lambda k t^{\frac{3n+1}{2}}}{\Gamma((\frac{3}{2}(n+1))}], \quad 0 < t < +\infty.$$

Example 1.2. Let us assume that

$$\Psi_n(s) = \int_0^{+\infty}(\xi^2+1)^{\frac{n-1}{2}}e^{-s\sqrt{\xi^2+1}}d\xi$$

then we have

$$\mathcal{L}^{-1}[\Psi_n(s); s \to t] = \frac{t^{\frac{n+1}{2}}}{\sqrt{t^2-1}}.$$

Proof. Let us start with the integral representation of $\Psi_1(s)$, $\Psi_1(s)=\int_0^{+\infty} e^{-s\sqrt{\xi^2+1}}d\xi$, then taking n-times derivative with respect to parameter s leads to

$$\Psi_n(s) = \int_0^{+\infty} (\xi^2+1)^{\frac{n-1}{2}} e^{-s\sqrt{\xi^2+1}}d\xi.$$

By taking inverse Laplace transform followed by the complex inversion formula, we have

$$\psi_n(t) = \mathcal{L}^{-1}[\Psi_n(s); s \to t] = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} e^{st}[\int_0^{+\infty} (\xi^2+1)^{\frac{n-1}{2}} e^{-s\sqrt{\xi^2+1}}d\xi]ds.$$

At this stage changing the order of integration leads to

$$\phi_n(t) = \mathcal{L}^{-1}[\Psi_n(s); s \to t] = \int_0^{+\infty} (\xi^2+1)^{\frac{n-1}{2}}[\frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} e^{(t-\sqrt{\xi^2+1})s}ds]d\xi.$$

The value of the inner integral is $\delta(t-\sqrt{\xi^2+1})$, we arrive at

$$\psi_n(t) = \mathcal{L}^{-1}[\Psi_n(s); s \to t] = \int_0^{+\infty} (\xi^2+1)^{\frac{n-1}{2}}\delta(t-\sqrt{\xi^2+1})d\xi.$$

In order to evaluate the above integral, we make a change of variable

$$t-\sqrt{\xi^2+1} = \eta$$

$$\psi_n(t) = \mathcal{L}^{-1}[\Psi_n(s); s \to t] = \int_{-\infty}^{t-1} (t-\eta)^{\frac{n-1}{2}}.\frac{t-\eta}{\sqrt{(t-\eta)^2-1}}\delta(\eta)d\eta = \frac{t^{\frac{n+1}{2}}}{\sqrt{t^2-1}}.$$

Finally using convolution theorem for the Laplace transform, we have the following relation

$$\psi(t) = \mathcal{L}^{-1}[\Psi_n(s)\Psi_m(s); s \to t] = \int_0^t \frac{(t-\xi)^{\frac{m+1}{2}}}{\sqrt{(t-\xi)^2-1}}\frac{\xi^{\frac{n+1}{2}}}{\sqrt{\xi^2-1}}d\xi.$$

□

2. Generalized Bessel's Equation, Bessel Functions, Hankel Transform

Let us consider the following second order differential equation with non-constant coefficients

$$x^2y'' + (1-2\alpha)xy' + [(kcx^c)^2 + \alpha^2 - \nu^2c^2]y = 0, \tag{2.1}$$

the above equation has the following solution

$$y(x) = x^{\alpha}[C_1 J_\nu(kx^c) + C_2 Y_\nu(kx^c)]. \tag{2.2}$$

We note that if $\alpha = 0, c = 1$ we obtain the Bessel equation

$$x^2 y'' + xy' + [(kx)^2 - \nu^2 c^2]y = 0, \tag{2.3}$$

with the solution as follows

$$y(x) = C_1 J_\nu(kx) + C_2 Y_\nu(kx). \tag{2.4}$$

In Eq.(2.1), if we set $\alpha = 0.5,\ c = \frac{3}{2},\ \nu = \frac{1}{3},\ k = \frac{2i}{3}$ we get

$$x^2 y'' + [(ix^{\frac{3}{2}})^2 + \frac{1}{4} - \frac{1}{4}]y = 0, \tag{2.5}$$

after simplifying we obtain

$$y'' - xy = 0, \tag{2.6}$$

the above equation is known as an Airy differential equation with the solution as below

$$y(x) = \sqrt{x}[C_1 J_{\frac{1}{3}}(\frac{2i}{3}x^{\frac{3}{2}}) + C_2 J_{-\frac{1}{3}}(\frac{2i}{3}x^{\frac{3}{2}})]. \tag{2.7}$$

At this stage using the fact that

$$J_\nu(ix) = e^{\frac{-i\pi\nu}{2}} I_\nu(x), \qquad K_\nu(x) = \frac{2}{\sin(\pi\nu)}[I_{-\nu}(x) - I_\nu(x)].$$

Where $I_\nu(x),\ K_\nu(x)$ are the modifed Bessel functions of the first and second kind respectively. Therefore, we get

$$y(x) = \sqrt{x}[C_1' I_{\frac{1}{3}}(\frac{2}{3}x^{\frac{3}{2}}) + C_2' I_{-\frac{1}{3}}(\frac{2}{3}x^{\frac{3}{2}})]. \tag{2.8}$$

In the special case $\nu = \frac{1}{3}$, we have the following relations [8,11]

$$Ai(x) = \frac{1}{\pi}\sqrt{\frac{x}{3}} K_{\frac{1}{3}}(\frac{2x^{\frac{3}{2}}}{3}) = \frac{\sqrt{x}}{3}[I_{-\frac{1}{3}}(\frac{2x^{\frac{3}{2}}}{3}) - I_{\frac{1}{3}}(\frac{2x^{\frac{3}{2}}}{3})]$$

and

$$Bi(x) = \sqrt{\frac{x}{3}}[I_{-\frac{1}{3}}(\frac{2x^{\frac{3}{2}}}{3}) + I_{\frac{1}{3}}(\frac{2x^{\frac{3}{2}}}{3})].$$

Finally, equation (2.6) has the following solution in terms of the Airy functions $Ai(x),\ Bi(x)$

$$y(x) = C_1'' Ai(x) + C_2'' Bi(x).$$

Remark. It is worth mentioning that the Airy function $Ai(x)$ is used in physics to model of the diffraction of light.

Theorem 2.1. *We have the following integral representation of the square of the Airy function*

$$Ai^2(\phi) = \frac{1}{\pi\sqrt{3}} \int_0^{+\infty} \eta J_0(2\phi\eta + \frac{2\eta^3}{3}) d\eta. \tag{2.9}$$

Note. In the literature the integral representation of the square of the Airy function is given [8,11].

Proof. Let us start with an integral representation of the product of the modified Bessel functions of order ν as follows

$$K_\nu(x)K_\nu(y) = \frac{\pi}{2\sin(\pi\nu)} \int_{\ln(\frac{y}{x})}^{+\infty} J_0(\sqrt{2xy\cosh\xi - (x^2+y^2)}) \sinh(\nu\xi) d\xi,$$

by taking $\nu = \frac{1}{3},\ x = y$, we have the following relation [8]

$$K^2_{\frac{1}{3}}(x) = \frac{\pi}{2\sin(\frac{\pi}{3})} \int_0^{+\infty} J_0(x\sqrt{2\cosh\xi - 2}) \sinh(\frac{\xi}{3}) d\xi.$$

At this stage using the well-known identity $K_{\frac{1}{3}}(x) = \frac{\pi\sqrt{3}}{\sqrt{\phi}} Ai(\phi)$, where $x = \frac{2}{3}\phi^{\frac{3}{2}}$, therefore, we have

$$[\frac{\pi\sqrt{3}}{\sqrt{\phi}} Ai(\phi)]^2 = \frac{\pi}{\sqrt{3}} \int_0^{+\infty} J_0(\frac{2}{3}\phi^{\frac{3}{2}}\sqrt{2\cosh\xi - 2}) 2\sinh(\frac{\xi}{6})\cosh(\frac{\xi}{6}) d\xi,$$

after simplifying we obtain

$$Ai^2(\phi) = \frac{\phi}{3\pi\sqrt{3}} \int_0^{+\infty} J_0[\frac{4\phi\sqrt{\phi}}{3}(3\sinh(\frac{\xi}{6}) + 4\sinh^3(\frac{\xi}{6}))] 2\sinh(\frac{\xi}{6})\cosh(\frac{\xi}{6}) d\xi.$$

Let us introduce a change of variable $\sinh(\frac{\xi}{6}) = \frac{\eta}{2\sqrt{\phi}}$, then we have $\frac{1}{6}\cosh(\frac{\xi}{6})d\xi = \frac{d\eta}{2\sqrt{\phi}}$, from which we deduce that

$$Ai^2(\phi) = \frac{\phi}{3\pi\sqrt{3}} \int_0^{+\infty} J_0[\frac{4\phi\sqrt{\phi}}{3}[(\frac{3\eta}{2\sqrt{\phi}}) + 4(\frac{\eta}{2\sqrt{\phi}})^3]] 12 \frac{\eta}{2\sqrt{\phi}} \frac{d\eta}{2\sqrt{\phi}}.$$

Finally, we obtain

$$Ai^2(\phi) = \frac{1}{\pi\sqrt{3}} \int_0^{+\infty} \eta J_0(2\phi\eta + \frac{2\eta^3}{3}) d\eta. \tag{2.10}$$

Let us consider the following special cases

1. $\phi = 0$, we get

$$Ai^2(0) = \frac{1}{3^{\frac{4}{3}}\Gamma^2(\frac{2}{3})} = \frac{1}{\pi\sqrt{3}} \int_0^{+\infty} \eta J_0(\frac{2\eta^3}{3}) d\eta.$$

2. In Eq.(2.10), taking derivitive with respect to ϕ and setting $\phi = 0$, we have

$$2Ai(0)Ai'(0) = \frac{-2}{2\pi\sqrt{3}} = \frac{-2}{\pi\sqrt{3}} \int_0^{+\infty} \eta^2 J_1(\frac{2\eta^3}{3}) d\eta,$$

or

$$\int_0^{+\infty} 2\eta^2 J_1(\frac{2\eta^3}{3})d\eta = \int_0^{+\infty} J_1(\tau)d\tau = 1.$$

□

Theorem 2.2. *We have the following integral identity for the modified Bessel function of the second kind or Macdonald function*

$$\int_0^{+\infty} K_\nu(\lambda\sqrt{x^2+z^2})\frac{x^{2\beta+1}}{(x^2+z^2)^{\frac{\nu}{2}}}dx = \frac{2^\beta\Gamma(\beta+1)}{\lambda^{\beta+1}z^{\nu-(\beta+1)}}K_{\nu-(\beta+1)}(\lambda z).$$

Proof. Let us start with the left hand side, by using an integral representation for the modified Bessel function, we have

$$\int_0^{+\infty} K_\nu(\lambda\sqrt{x^2+z^2})\frac{x^{2\beta+1}}{(x^2+z^2)^{\frac{\nu}{2}}}dx =$$

$$= \int_0^{+\infty} \frac{x^{2\beta+1}}{(x^2+z^2)^{\frac{\nu}{2}}}[(\frac{\lambda(\sqrt{x^2+z^2})}{2})^\nu \int_0^{+\infty} e^{-\xi-\frac{\lambda^2(x^2+z^2)}{4\xi}}\frac{d\xi}{2\xi^{\nu+1}}]dx,$$

changing the order of integration in the double integral and after simplifying, we obtain

$$L.H.S = (\frac{\lambda}{2})^\nu \int_0^{+\infty} e^{-\xi-\frac{\lambda^2 z^2}{4\xi}}[\int_0^{+\infty} x^{2\beta+1}e^{-\frac{\lambda^2x^2}{4\xi}}dx]\frac{d\xi}{2\xi^{\nu+1}}.$$

At this point let us make a change of variable $u = \frac{\lambda^2x^2}{4\xi}$ in the inner integral after simplification we obtain

$$L.H.S = \frac{1}{2}(\frac{\lambda}{2})^\nu\Gamma(\beta+1)(\frac{2}{\lambda})^{2(\beta+1)} \int_0^{+\infty} e^{-\xi-\frac{\lambda^2z^2}{4\xi}}\frac{d\xi}{2\xi^{\nu+1}}.$$

Let us rewrite the above relation as follows

$$L.H.S = \frac{1}{2}(\frac{\lambda}{2})^\nu\Gamma(\beta+1)(\frac{2}{\lambda})^{2(\beta+1)}(\frac{\lambda z}{2})^{-\nu+(\beta+1)})[(\frac{\lambda z}{2})^{\nu-(\beta+1)}(\int_0^{+\infty} e^{-\xi-\frac{\lambda^2z^2}{4\xi}}\frac{d\xi}{2\xi^{\nu+1}}].$$

But the expression in the brackets is the integral representation for the modified Bessel function $K_{\nu-(\beta+1)}(\lambda z)$, therefore we get

$$L.H.S = \frac{2^\beta\Gamma(\beta+1)}{z^{\nu-(\beta+1)}\lambda^{\beta+1}}K_{\nu-(\beta+1)}(\lambda z).$$

Let us consider the special case $\nu = 0$ then we get

$$\int_0^{+\infty} K_0(\lambda\sqrt{x^2+z^2})x^{2\beta+1}dx = \frac{2^\beta\Gamma(\beta+1)}{\lambda^{\beta+1}z^{-(\beta+1)}}K_{-(\beta+1)}(\lambda z) = \frac{2^\beta z^{(\beta+1)}\Gamma(\beta+1)}{\lambda^{\beta+1}}K_{(\beta+1)}(\lambda z).$$

Also, considering the special case $\beta = -\frac{1}{2}$ we obtain

$$\int_0^{+\infty} K_0(\lambda\sqrt{x^2+z^2})dx = \sqrt{\frac{\lambda z}{2\pi}} K_{\frac{1}{2}}(\lambda z).$$

In the above theorem, we used the fact that $K_\nu(.) = K_{-\nu}(.)$ and the well-known integral representation $K_\nu(az) = (\frac{az}{2})^\nu \int_0^{+\infty} e^{-\xi - \frac{a^2z^2}{4\xi}} \frac{d\xi}{2\xi^{\nu+1}}$, [5,8]. $\square$

Hankel Transforms

Hankel transforms arise naturally in solving boundary-value problems formulated in cylindrical coordinates. They also occur in other applications such as determining oscillations of the suspended heavy chain from one end.
We define the general Hankel transforms of order ν by

$$\mathcal{H}_\nu[\phi(r);\rho] = \int_0^{+\infty} rJ_\nu(\rho r)\phi(r)dr = \Phi(\rho). \tag{2.11}$$

The corresponding inversion formula of which takes the form

$$\mathcal{H}_\nu^{-1}[\Phi(\rho);r] = \int_0^{+\infty} \rho J_\nu(r\rho)\Phi(\rho)d\rho = \phi(r). \tag{2.12}$$

The basic requirement for the existence of the Hankel transform is that the function $\sqrt{r}f(r)$ be absolutely integrable and piecewise continuous on the positive real line. In this section we will determine the Hankel transform of certain functions and develop some of the fundamental operational properties of the Hankel transform.

Lemma 2.1. *Let us assume that $\mathcal{H}_\nu[\phi(r);\rho] = \Phi(\rho)$, then we have*

$$1. \qquad \mathcal{H}_\nu[\frac{1}{r^{\nu+1}}\frac{d}{dr}[r^{2\nu+1}\frac{d}{dr}(\frac{1}{r^\nu}\phi(r))];\rho] = -\rho^2\Phi(\rho). \tag{2.13}$$

$$2. \qquad \mathcal{H}_0[\frac{1}{r}\frac{d}{dr}[r\frac{d}{dr}(\phi(r))];\rho] = -\rho^2\Phi(\rho). \tag{2.14}$$

Proof. See [3,4,9].

Example 2.1. Show that

$$\mathcal{H}_0[\frac{1}{\sqrt{r^2+a^2}};\rho] = \frac{1}{\rho}e^{-a\rho}.$$

Proof. Let us start with the Laplace transform of the function $J_0(r\rho)$, we have

$$\mathcal{L}[J_0(r\rho);\rho \to a] = \int_0^{+\infty} e^{-a\rho}J_0(r\rho)d\rho = \frac{1}{\sqrt{a^2+r^2}}.$$

In terms of the Hankel transform of order zero we have

$$\mathcal{H}_0[\frac{e^{-a\rho}}{\rho}; \rho \to r] = \frac{1}{\sqrt{r^2 + a^2}}.$$

Inverting the above relation leads to

$$\mathcal{H}_0^{-1}[\frac{1}{\sqrt{r^2 + a^2}}; r \to \rho] = \int_0^{+\infty} \rho J_0(\rho r) \frac{1}{\sqrt{r^2 + a^2}} dr = \frac{e^{-a\rho}}{\rho}.$$

□

Lemma 2.2. *Parseval identity for the Hankel transform.*
If $\Phi(\rho)$ and $\Psi(\rho)$ are the Hankel transforms of the functions $\phi(r)$ and $\psi(r)$, respectively, then

$$\int_0^{+\infty} r\phi(r)\psi(r)dr = \int_0^{+\infty} \rho\Phi(\rho)\Psi(\rho)d\rho. \tag{2.15}$$

Proof. The integral on the right side can be rewritten as follows

$$\int_0^{+\infty} \rho\Phi(\rho)\Psi(\rho)d\rho = \int_0^{+\infty} \rho\Phi(rho)[\int_0^{+\infty} rJ_\nu(\rho r)\psi(r)dr]d\rho.$$

Changing the order of integration, we get

$$\int_0^{+\infty} \rho\Phi(\rho)\Psi(\rho)d\rho = \int_0^{+\infty} r\psi(r)[\int_0^{+\infty} \rho J_\nu(r\rho)\Phi(\rho)d\rho]dr = \int_0^{+\infty} r\psi(r)\phi(r)dr.$$

□

Lemma 2.3. *The following integral identity holds*

$$\frac{1}{2}\delta(\frac{a^2 - b^2}{4}) = \int_0^{+\infty} \rho J_\nu(a\rho)J_\nu(b\rho)d\rho. \tag{2.16}$$

Proof. Let us take $\phi(r) = \frac{1}{2}\delta(\frac{r^2-a^2}{4})$ and $\psi(r) = \frac{1}{2}\delta(\frac{r^2-b^2}{4})$. In view of the Parseval identity and using Lemma 2.4. we have

$$\int_0^{+\infty} \frac{1}{2}\delta(\frac{r^2 - a^2}{4})\frac{1}{2}\delta(\frac{r^2 - b^2}{4})rdr = \frac{1}{2}\delta(\frac{a^2 - b^2}{4}) = \int_0^{+\infty} \rho J_\nu(a\rho)J_\nu(b\rho)d\rho. \tag{2.17}$$

□

Lemma 2.4. *We have the following relations for the Hankel transform*

$$\mathcal{H}_\nu[\frac{1}{2}\delta(\frac{r^2 - a^2}{4}); \rho] = \int_0^{+\infty} rJ_\nu(\rho r)\delta(\frac{r^2 - a^2}{4})dr = J_\nu(a\rho). \tag{2.18}$$

Proof. Let us make a change of variable $\xi = \frac{r^2-a^2}{4}$ in the above integral, we get

$$\mathcal{H}_\nu[\frac{1}{2}\delta(\frac{r^2-a^2}{4});\rho] = \int_{-\frac{a^2}{4}}^{+\infty} \sqrt{4\xi+a^2}J_\nu(\rho\sqrt{4\xi+a^2})\delta(\xi)\frac{2d\xi}{\sqrt{4\xi+a^2}} = J_\nu(a\rho). \tag{2.19}$$

□

3. Solution for the Time Fractional Heat Equation in Cylindrical Coordinates Via the Joint Laplace-Hankel Transform

Fractional calculus deals with the fractional integrals and derivatives of arbitrary order. It provides better models for systems having long range memory and non-local effects and it has important applications in several fields of engineering and sciences. Fractional differential equations are widely used for modeling anomalous diffusion phenomena. In this section, the author implemented the joint Laplace-Hankel transforms to construct the exact solution for the time fractional heat conduction equation. In the past three decades, considerable research work has been invested in the study of the anomalous diffusion using the time fractional equation.

Problem 3.1 Let us solve the following impulsive time fractional heat conduction equation in cylindrical coordinates

$$D_t^{c,\alpha}u = \frac{a^2}{r}\frac{\partial}{\partial r}(r\frac{\partial u}{\partial r}) + \delta(t)\delta(r-r_0), \qquad \alpha = 0.5, \quad t > 0, \quad 0 < r < +\infty.$$

with the boundary conditions as follows

$$1. \quad u(r,0) = f(r), \quad 2. \quad \lim_{r\to 0}|u(r,t)| < +\infty, \quad 3. \quad \lim_{r\to+\infty} u(r,t) = 0.$$

Solution. Let us define the joint Laplace-Hankel transform of order zero as follows

$$U(\rho,s) = \int_0^{+\infty} rJ_0(\rho r)[\int_0^{+\infty} e^{-st}u(r,t)dt]dr. \tag{3.1}$$

Application of the joint Laplace-Hankel transform the above equation leads to the following transformed equation with the boundary conditions as follows

$$s^\alpha U(\rho,s) + a^2\rho^2 U(\rho,s) = s^{\alpha-1}F(\rho) + r_0J_0(r_0\rho), \qquad \mathcal{H}_0[f(r);\rho] = F(\rho). \tag{3.2}$$

Solving the above equation (3.2) yields

$$U(\rho,s) = \frac{s^{\alpha-1}F(\rho) + r_0J_0(r_0\rho)}{s^\alpha + a^2\rho^2} = F(\rho)[\frac{1}{s^{1-\alpha}(s^\alpha+a^2\rho^2)}] + J_0(r_0\rho)\frac{r_0}{s^{1-\alpha}(s^\alpha+a^2\rho^2)}. \tag{3.3}$$

At this point, taking the joint inverse Laplace-Hankel transform of order zero to obtain

$$\begin{aligned} u(r,t) &= \int_0^{+\infty} \rho J_0(r\rho)F(\rho)[\mathcal{L}^{-1}[\frac{1}{s^{1-\alpha}(s^\alpha + a^2\rho^2)}]d\rho \\ &+ r_0 \int_0^{+\infty} \rho J_0(r_0\rho)J_0(r\rho)[\mathcal{L}^{-1}[\frac{1}{s^{1-\alpha}(s^\alpha + a^2\rho^2)}]d\rho. \end{aligned} \tag{3.4}$$

At this stage let us take $\alpha = 0.5$, then we have

$$\mathcal{L}^{-1}[\frac{1}{s^{1-\alpha}(s^\alpha + a^2\rho^2)}] = \mathcal{L}^{-1}[\frac{1}{\sqrt{s}(\sqrt{s} + a^2\rho^2)}] = e^{a^4\rho^4 t} Erfc(a^2\rho^2\sqrt{t}). \tag{3.5}$$

In relation (3.6), let us replace $F(\rho) = \mathcal{H}_0[f(r);\rho]$, $r_0J_0(r_0\rho) = \mathcal{H}_0[\delta(r - r_0);\rho]$ by the following integrals

$$F(\rho) = \int_0^{+\infty} \xi J_0(\rho\xi)f(\xi)d\xi, \qquad r_0J_0(r_0\rho) = \int_0^{+\infty} \tau J_0(\rho\tau)\delta(\tau - r_0)d\tau \tag{3.6}$$

we arrive at

$$u(r,t) = \int_0^{+\infty} \rho J_0(r\rho)e^{a^4\rho^4 t} Erfc(a^2\rho^2\sqrt{t})[\int_0^{+\infty} \xi J_0(\rho\xi)f(\xi)d\xi]d\rho +$$

$$+ \int_0^{+\infty} \rho J_0(r\rho)e^{a^4\rho^4 t} Erfc(a^2\rho^2\sqrt{t})[\int_0^{+\infty} \tau J_0(\rho\tau)\delta(\tau - r_0)d\tau]d\rho \tag{3.7}$$

By changing the order of integration we obtain the formal solution to boundary-value problem

$$u(r,t) = \int_0^{+\infty} \xi f(\xi)[\int_0^{+\infty} \rho J_0(r\rho)e^{a^4\rho^4 t} Erfc(a^2\rho^2\sqrt{t})J_0(\xi\rho)d\rho]d\xi +$$

$$+ \int_0^{+\infty} \tau\delta(\tau - r_0)[\int_0^{+\infty} \rho J_0(r\rho)e^{a^4\rho^4 t} Erfc(a^2\rho^2\sqrt{t})J_0(\tau\rho)d\rho]d\tau. \tag{3.8}$$

Note. In the above relation $Erfc(\xi) = \frac{2}{\sqrt{\pi}} \int_\xi^{+\infty} e^{-t^2} dt$.
The last step is to evaluate $u(r,0)$ as below

$$u(r,0) = \int_0^{+\infty} \xi f(\xi)[\int_0^{+\infty} \rho J_0(r\rho)J_0(\xi\rho)d\rho]d\xi + \int_0^{+\infty} \tau\delta(\tau - r_0)[\int_0^{+\infty} \rho J_0(r\rho)J_0(\tau\rho)d\rho]d\tau. \tag{3.9}$$

In view of the Lemma 2.4. the value of the inner integrals are $\frac{1}{2}\delta(\frac{r^2-\xi^2}{4})$ and $\frac{1}{2}\delta(\frac{r^2-\tau^2}{4})$ respectively, therefore

$$u(r,0) = \int_0^{+\infty} \xi f(\xi)[\frac{1}{2}\delta(\frac{r^2 - \xi^2}{4})]d\xi + \int_0^{+\infty} \tau\delta(\tau - r_0)[\frac{1}{2}\delta(\frac{r^2 - \tau^2}{4})]d\tau = f(r). \tag{3.10}$$

Note. In the last step we have made a change of variable $\frac{r^2-\xi^2}{4} = \eta$ in the above integral.

4. Main Result. Solution for The Time Fractional Non-Homogeneous Heat Equation in Cylindrical Coordinates via the Joint Laplace-Hankel Transform

Let us consider the following time fractional heat conduction equation a fractional generalization of the problem Ion distribution function during ion cyclotron resonance heating at the fundamental frequency [6]

$$D_t^{c,\alpha} u = \frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r} - \lambda u + \phi(r) + \mathcal{J}^{\alpha} h(t), \qquad 0 < \alpha < 1, \quad t > 0, \quad 0 < r < +\infty$$

with the boundary conditions as below

$$1. \quad u(r,0) = \psi(r), \quad 2. \quad \lim_{r\to 0} |u(r,t)| < +\infty, \quad 3. \quad \lim_{r\to +\infty} u(r,t) = 0.$$

Note. Analytic solutions are more important than numerical solutions, because these are valid in the whole domain of definition whereas the numerical solutions are only valid at chosen points in the domain of definition.
Solution. Let us define the joint Laplace-Hankel transforms of order zero as follows

$$U(\rho, s) = \int_0^{+\infty} r J_0(\rho r)[\int_0^{+\infty} e^{-st} u(r,t)dt]dr. \tag{4.1}$$

By applying the joint Laplace-Hankel transforms of order zero the above equation, we arrive at the following transformed equation with the boundary conditions

$$(s^\alpha + \rho^2 + \lambda)U(\rho, s) = s^{\alpha-1}\Psi(\rho) + \frac{\Phi(\rho)}{s} + \frac{H(s)}{s^\alpha}. \tag{4.2}$$

Solution of the above equation (4.2) leads to

$$U(\rho, s) = \frac{\Psi(\rho)}{s^{1-\alpha}(s^\alpha + \rho^2 + \lambda)} + \frac{\Phi(\rho)}{s(s^\alpha + \rho^2 + \lambda)} + \frac{H(s)}{s^\alpha(s^\alpha + \rho^2 + \lambda)}. \tag{4.3}$$

By taking the inverse joint Laplace-Hankel transform of order zero, we have

$$u(r,t) = \int_0^{+\infty} \rho J_0(r\rho)\Psi(\rho)[\mathcal{L}^{-1}[\frac{1}{s^{1-\alpha}(s^\alpha + \rho^2 + \lambda)}]d\rho +$$

$$+ \int_0^{+\infty} \rho J_0(r\rho)\Phi(\rho)[\mathcal{L}^{-1}[\frac{1}{s(s^\alpha + \rho^2 + \lambda)}]d\rho + \int_0^{+\infty} \rho J_0(r\rho)[\mathcal{L}^{-1}[\frac{H(s)}{s^\alpha(s^\alpha + \rho^2 + \lambda)}]d\rho. \tag{4.4}$$

In view of the Corollary 1.2. we have the following formal solution

$u(r,t) =$

$$= \frac{\sin(\pi\alpha)}{\pi\Gamma(1-\alpha)}\int_0^{+\infty}\rho J_0(r\rho)\Psi(\rho)[\int_0^t\frac{1}{(t-\eta)^\alpha}[\int_0^{+\infty}\frac{\xi^\alpha e^{-\eta\xi}d\xi}{\xi^{2\alpha}+2(\sqrt{\rho^2+\lambda})\xi^\alpha\cos(\pi\alpha)+\lambda+\rho^2}]d\eta]d\rho+$$

$$+\frac{\sin(\pi\alpha)}{\pi}\int_0^{+\infty}\rho J_0(r\rho)\Phi(\rho)[\int_0^t[\int_0^{+\infty}[\frac{\xi^\alpha e^{-\eta\xi}}{\xi^{2\alpha}+2(\sqrt{\rho^2+\lambda})\xi^\alpha\cos(\pi\alpha)+\lambda+\rho^2}]d\xi]d\eta]d\rho+$$

$$+\frac{\sin(\pi\alpha)}{\pi}\int_0^{+\infty}\rho J_0(r\rho)[\int_0^t\mathcal{J}^\alpha h(t-\eta)[\int_0^{+\infty}[\frac{\xi^\alpha e^{-\eta\xi}}{\xi^{2\alpha}+2(\sqrt{\rho^2+\lambda})\xi^\alpha\cos(\pi\alpha)+\lambda+\rho^2}]d\xi]d\eta]d\rho. \tag{4.5}$$

At this stage let us take $\alpha = 0.5$, then we obtain the solution as follows

$$u(r,t) = \frac{1}{\pi\Gamma(\frac{1}{2})}\int_0^{+\infty}\rho J_0(r\rho)\Psi(\rho)[\int_0^t\frac{1}{(t-\eta)^{\frac{1}{2}}}[\int_0^{+\infty}\frac{\sqrt{\xi}e^{-\eta\xi}d\xi}{\xi+\lambda+\rho^2}]d\eta]d\rho+$$

$$+\frac{1}{\pi}\int_0^{+\infty}\rho J_0(r\rho)\Phi(\rho)[\int_0^t[\int_0^{+\infty}[\frac{\sqrt{\xi}e^{-\eta\xi}}{\xi+\lambda+\rho^2}]d\xi]d\eta]d\rho$$

$$+\frac{1}{\pi}\int_0^{+\infty}\rho J_0(r\rho)[\int_0^t\mathcal{J}^\alpha h(t-\eta)[\int_0^{+\infty}[\frac{\sqrt{\xi}e^{-\eta\xi}}{\xi+\lambda+\rho^2}]d\xi]d\eta]d\rho. \tag{4.6}$$

At this point, we may use the following integral identity in order to evaluate the inner most integral [5]

$$\int_0^{+\infty}\frac{\sqrt{\xi}e^{-\eta\xi}}{\xi+(\lambda+\rho^2)}d\xi = \sqrt{\lambda+\rho^2}e^{\eta(\lambda+\rho^2)}\Gamma(-\frac{1}{2},\eta(\lambda+\rho^2)),$$

therefore we get

$$u(r,t)=\frac{1}{\pi\Gamma(\frac{1}{2})}\int_0^{+\infty}\rho J_0(r\rho)\Psi(\rho)[\int_0^t\frac{1}{(t-\eta)^{\frac{1}{2}}}[\sqrt{\lambda+\rho^2}e^{\eta(\lambda+\rho^2)}\Gamma(-\frac{1}{2},\eta(\lambda+\rho^2))]d\eta]d\rho+$$

$$+\frac{1}{\pi}\int_0^{+\infty}\rho J_0(r\rho)\Phi(\rho)[\int_0^t[\sqrt{\lambda+\rho^2}e^{\eta(\lambda+\rho^2)}\Gamma(-\frac{1}{2},\eta(\lambda+\rho^2))]d\eta]d\rho+$$

$$+\frac{1}{\pi}\int_0^{+\infty}\rho J_0(r\rho)[\int_0^t\mathcal{J}^\alpha h(t-\eta)[\sqrt{\lambda+\rho^2}e^{\eta(\lambda+\rho^2)}\Gamma(-\frac{1}{2},\eta(\lambda+\rho^2))]d\eta]d\rho. \tag{4.7}$$

Note. In the above relation $\Gamma(a,\xi) = \int_\xi^{+\infty} t^{s-1}e^{-t}dt$ stands for the incomplete gamma function.

5. Conclusion

The paper is devoted to studying and application of the joint Laplace-Hankel transform for solving time fractional heat equation in cylindrical coordinates. The main purpose of this work is to develop a method for finding analytic solutions of fractional PDEs, evaluation of certain integrals. These results should be applicable to obtaining solutions of a wide class of problems in applied mathematics, engineering and mathematical physics. The methods and techniques discussed in this article can also be applied to solve other types of the fractional partial differential equations.
Compliance with Ethical Standards: The author declares that he has no conflict of interest.
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References

[1] A. Aghili, *Solution to time fractional non-homogeneous first order PDE with non-constant coefficients*, Tbilisi Math. J. 12 (4) (2019) 149–155.

[2] A. Aghili, *Complete solution for the time fractional diffusion problem with mixed boundary conditions by operational method*, Appl. Math. Nonlinear Sci. April (2020) (aop) 1–12.

[3] A. Apelblat, *Laplace Transforms and Their Applications*, Nova Science Publishers, Inc., New York, 2012.

[4] H.J. Glaeske, A.P. Prudnikov, K.A. Skornik, *Operational Calculus and Related Topics*, Chapman & Hall / CRC, 2006.

[5] I.S. Gradshteyn, I.M. Ryzhik, *Table of Integrals, Series, and Products*, Seventh Edition, 2007.

[6] M.H.A. Hassan, *Ion distribution function during ion cyclotron resonance heating at the fundamental frequency*, Phys. Fluids 31 (1988) 596–601.

[7] A.A. Kilbas, J.J. Trujillo, *Differential equation of fractional order: Methods, results and problems. II*, Appl. Anal. 81 (2) (2002) 435–493.

[8] N.N. Lebedev, *Special Functions and Their Applications*, Prentice-Hall, Inc., 1972.

[9] B. Patra, *An Introduction to Integral Transforms*, CRC Press, 2016.

[10] I. Podlubny, *Fractional Differential Equations*, Academic Press, San Diego, CA, 1999.

[11] O. Vallee, M. Suares, *Airy Functions and Applications to Physics*, World Scientific, Imperial College Press, 2003.

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Asymptotic Properties of General Nonlinear Differential Equations Containing Nonconformable Fractional Derivatives

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ABSTRACT: In this paper the author employs the nonconformable fractional derivative developed by J. E. Nápoles Valdes and his coauthors to study the asymptotic properties of solutions of a broad class of nonlinear fractional differential equations containing such a type of derivative. Sufficient conditions for the boundedness and convergence to zero of all solutions are presented.

AMS Subject Classification: 26A33, 34A08, 34C10, 34C11.
Keywords and Phrases: Nonconformable fractional derivative; Nonlinear differential equations; Asymptotic properties of solutions.

1. Introduction

In this paper we utilize the nonconformable fractional derivative introduced in [1] and [4] to study the asymptotic behavior of solutions to very general nonlinear fractional differential equations that are generalizations of Emden-Fowler and other types of ordinary (integer order) equations. One advantage of using this type of fractional derivative, which we will denote by N, is that if a function is α-order, $\alpha \in (0, 1]$, differentiable at a point $t_0 \in (0, \infty)$, then it is continuous at that point (see [1, Theorem 2.2]). Also, this fractional derivative obeys product and quotient rules that mimic those for ordinary (integer order) derivatives (see [1, Theorem 2.3]). But probably its most important feature is that it satitfies a chain rule like the one for integer order derivatives (see Lemma 2.5 below). This type of fractional derivative is well described in the paper [1]. We also obtain a Gronwall type inequality for this kind of fractional derivative.

2. Preliminaries and Basic Concepts

We begin with the notion of the nonconformable fractional derivative.

Definition 2.1. ([1, Definition 2.1], [5, Definition 1]) Let $f : [0, \infty) \to \mathbb{R}$. The nonconformable fractional derivative of f of order $\alpha \in (0, 1)$ is defined by

$$(N^\alpha f)(t) = \lim_{\epsilon \to 0} \frac{f(t + \epsilon e^{t^{-\alpha}}) - f(t)}{\epsilon}$$

for all $t > 0$.

Remark. If $(N^\alpha f)(t)$ exists in some $(0, a)$ and $\lim_{t \to 0^+}(N^\alpha f)(t)$ exists, then we define $(N^\alpha f)(0) = \lim_{t \to 0^+}(N^\alpha f)(t)$.

Corresponding to the nonconformable fractional derivative, we have the nonconformable fractional integral.

Definition 2.2. ([5, Definition 2]) Let $f : [0, \infty) \to \mathbb{R}$. The nonconformable fractional integral of f of order $\alpha \in (0, 1)$ is defined by

$$({}_N J^\alpha_{t_0} f)(t) = \int_{t_0}^{t} \frac{f(s)}{e^{s^{-\alpha}}} ds.$$

In view of Definitions 2.1 and 2.2 it is obvious that the following lemma is needed.

Lemma 2.3. ([5, Theorem 3]) *If f is N^α–differentiable on (t_0, ∞) with $\alpha \in (0, 1]$, then for $t > t_0$:*

(a) If f is differentiable, ${}_N J^\alpha_{t_0}(N^\alpha f)(t) = f(t) - f(t_0)$.

(b) $N^\alpha({}_N J^\alpha_{t_0} f)(t) = f(t)$.

For convenience, we next give some properties of the nonconformable fractional derivative.

Lemma 2.4. *Let f and g be N^α differentiable, $\alpha \in (0, 1]$, at a point $t > 0$; then:*

(1) $N^\alpha(c) = 0$ for any constant $c \in \mathbb{R}$.

(2) $N^\alpha(fg)(t) = f(t)(N^\alpha g)(t) + g(t)(N^\alpha f)(t)$.

(3) $N^\alpha\left(\frac{f}{g}\right) = \dfrac{g(t)(N^\alpha f)(t) - f(t)(N^\alpha g)(t)}{g^2(t)}$.

(4) If f is differentiable (in the ordinary sense), then $(N^\alpha f)(t) = e^{t^{-\alpha}} f'(t)$.

Proof. This is parts (c)–(f) of Theorem 2.3 in [1]. □

Remark. ([1, p. 91]) If $(N^\alpha f)(t)$ exists for $t > 0$, then f is differentiable (in the ordinary sense) at t, and

$$f'(t) = e^{-t^{-\alpha}}(N^\alpha f)(t).$$

As mentioned earlier, a very important advantage that the nonconformable fractional derivative has over other fractional derivatives is the existence of a chain rule that mimics the one for ordinary (integer valued) derivatives. We state it here as the following lemma; its proof can be found in [1, Theorem 3.1].

Lemma 2.5. *Let $\alpha \in (0, 1]$, g be N^α differentiable at $t > 0$, and f be differentiable at $g(t)$. Then*

$$N^\alpha(f \circ g)(t) = f'(g(t))(N^\alpha g)(t).$$

In the study of continuability, boundedness, stability, and other asymptotic properties of solutions of nonlinear differential equations, the kinetic energy of the system often appears as an integral such as $F(x) = \int_0^x f(s)ds$. It then becomes necessary to differentiate this quantity. By applying the above chain rule, we obtain,

$$N^\alpha F(x) = f(x(t))(N^\alpha x)(t).$$

Due to its importance, we formulate this as the following corollary.

Corollary 2.6. *Let $f : \mathbb{R} \to \mathbb{R}$ and define $F(x) = \int_0^x f(s)ds$. Then*

$$(N^\alpha F)(x) = f(x(t))(N^\alpha x)(t).$$

Remark. An intermediate value theorem for nonconformable derivatives can be found in [3, Theorem 4] framed in a multivariate setting, as can a multivariate chain rule [3, Theorem 8]. Similarly, there is an implicit function theorem [3, Theorem 12].

We conclude this section with a Gronwall type inequality for nonconformable fractional derivatives. Here, we let $\mathbb{R} = (-\infty, \infty)$ and $\mathbb{R}_+ = [0, \infty)$.

Lemma 2.7. *Let $c \in \mathbb{R}_+$ and a, $u : \mathbb{R} \to \mathbb{R}_+$. If*

$$u(t) \leq c + ({}_N J^\alpha_{t_0} au)(t), \tag{2.1}$$

then

$$u(t) \leq c \exp\{({}_N J^\alpha_{t_0} a)(t)\}. \tag{2.2}$$

Proof. If we let $K(t)$ denote the right hand side of (2.1), then it is easy to see that (2.1) can be rewritten as

$$\frac{N^\alpha K(t)}{K(t)} \leq a(t).$$

This implies

$$\frac{K'(t)}{K(t)} \leq e^{-t^{-\alpha}} a(t)$$

by Remark 2. Integrating, we have

$$\ln K(t) \leq \ln K(t_0) + \int_{t_0}^t e^{-s^{-\alpha}} a(s)ds,$$

so

$$K(t) \le K(t_0) \exp \int_{t_0}^{t} e^{-s^{-\alpha}} a(s) ds.$$

Hence,

$$u(t) \le K(t) \le c \exp\{({}_N J_{t_0}^{\alpha} a)(t)\},$$

which proves (2.2). □

3. Main Results

Consider the perturbed nonlinear differential equation with nonconformable fractional derivatives

$$N^{\alpha}(a(t)N^{\alpha}x) + b(t, x, N^{\alpha}x) + q(t)f(x)g(N^{\alpha}x) = e(t, x, N^{\alpha}x), \tag{E}$$

where a, $q : \mathbb{R}_+ \to \mathbb{R}_+$, b, $e : \mathbb{R}_+ \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$, and f, $g : \mathbb{R} \to \mathbb{R}$ are continuous functions with $g(v) > 0$ for $v \in \mathbb{R}$.

Special cases of the left hand side of this equation include the Emden–Fowler equation ($b \equiv 0$ and $g \equiv 1$), the Liénard equation ($a \equiv 1 \equiv q$, $b(t, u, v) = b(u)v$, $g \equiv 1$), and the Rayleigh equation ($a \equiv 1 \equiv q$, $b(t, u, v) = b(v)$, $g \equiv 1$). We will make use of a variety of different conditions on the coefficient functions including:

$$|e(t, u, v)| \le r(t), \tag{3.1}$$

where $r : \mathbb{R}_+ \to \mathbb{R}$ is a continuous function;

$$b(t, u, v)v \ge 0, \tag{3.2}$$

$$F(x) = \int_0^x f(s) ds \to \infty \text{ as } |x| \to \infty, \tag{3.3}$$

$$\frac{|v|}{g(v)} \le m + nG(v), \tag{3.4}$$

where m and n are nonnegative constants and $G(v) = \displaystyle\int_0^v \frac{s ds}{g(s)}$,

$$\frac{v^2}{g(v)} \le MG(v) \quad \text{for all } v, \tag{3.5}$$

where M is a positive constant;

$$N^{\alpha}a(t) \ge 0, \tag{3.6}$$

and

$$a(t) \le A, \tag{3.7}$$

where $A > 0$ is a constant.

For any continuous function $d : [0, \infty) \to \mathbb{R}$, we set $(N^\alpha d)(t)_+ = \max\{(N^\alpha d)(t), 0\}$ and $N^\alpha d(t)_- = \max\{-(N^\alpha d)(t), 0\}$ which means that $(N^\alpha d)(t) = (N^\alpha d)(t)_+ - (N^\alpha d)(t)_-$. Also, if we let

$$b(t) = \exp\left\{-\left({}_N J_{t_0}^\alpha \frac{N^\alpha d(t)_-}{d(t)}\right)(t)\right\} \quad \text{and} \quad c(t) = \exp\left\{\left({}_N J_{t_0}^\alpha \frac{N^\alpha d(t)_+}{d(t)}\right)(t)\right\}$$

then $d(t) = d(t_0)b(t)c(t)$. Moreover, it is not hard to show that if

$$ {}_N J_{t_0}^\alpha \left(\frac{N^\alpha d(t)_-}{d(t)}\right)(\infty) < \infty, \tag{3.8}$$

then $d(t)$ is bounded from below away from 0, and if

$$ {}_N J_{t_0}^\alpha \left(\frac{N^\alpha d(t)_+}{d(t)}\right)(\infty) < \infty,$$

then then $d(t)$ is bounded from above.

In view of the above discussion, we list the following possible assumptions to be used in this paper:

$$ {}_N J_{t_0}^\alpha \left(\frac{N^\alpha a(s)_+}{a(s)}\right)(\infty) < \infty, \tag{3.9}$$

$$ {}_N J_{t_0}^\alpha \left(\frac{N^\alpha a(s)_-}{a(s)}\right)(\infty) < \infty, \tag{3.10}$$

$$ {}_N J_{t_0}^\alpha \left(\frac{N^\alpha q(s)_+}{q(s)}\right)(\infty) < \infty, \tag{3.11}$$

$$ {}_N J_{t_0}^\alpha \left(\frac{N^\alpha q(s)_-}{q(s)}\right)(\infty) < \infty. \tag{3.12}$$

For convenience, we will write equation (E) as the system

$$\begin{cases} N^\alpha x = y, \\ N^\alpha y = [-(N^\alpha(a(t))y - b(t, x, y) - q(t)f(x)g(y) + e(t, x, y)]/a(t). \end{cases} \tag{S_1}$$

Note: As long as there is no ambiguity to the meaning, in what follows we will write

$$ {}_N J_{t_0}^\alpha M(t) \quad \text{to mean} \quad ({}_N J_{t_0}^\alpha M)(t).$$

It is important to know that solutions to our problem can be defined for all time in the future, i.e., they are continuable. One such result is given in the following theorem. By interchanging some of the conditions, it is possible to obtain some variations of it.

Theorem 3.1. *Assume that $F(x)$ is bounded from below and conditions* (3.1), (3.2), (3.4) *and* (3.6) *hold. If $G(v) \to \infty$ as $|v| \to \infty$, then all solutions of system* (S_1) *and hence equation* (E) *are defined for all $t > 0$.*

Proof. Let $x(t)$ be a solution of equation (E) and $(x(t), y(t))$ be the corresponding solution of system (S_1), and assume that the solution is not continuable, i.e.,

$$\limsup_{t\to T^-}[|x(t)| + |y(t)|] = +\infty$$

for some $0 < T < \infty$ (that is, the solution has finite escape time).

Now $F(x(t)) \geq -K$ for some constant $K \geq 0$, so we define

$$V(t) = V(t, x(t), y(t)) = [F(x) + K]/a(t) + G(y)/q(t), \tag{3.13}$$

where we have suppressed some of the dependence on t.

Then, by Lemmas 2.4 and 2.5 and Corollary 2.6,

$$\begin{aligned} N^\alpha V(t) &= -[F(x) + K]N^\alpha a(t)/a^2(t) + f(x)N^\alpha x/a(t) - G(y)N^\alpha q(t)/q^2(t) \\ &\quad + \frac{y}{g(y)q(t)}N^\alpha y \\ &\leq -G(y)N^\alpha q(t)/q^2(t) + \frac{e(t,x,y)y}{g(y)q(t)a(t)} \\ &\leq -G(y)N^\alpha q(t)/q^2(t) + \frac{r(t)}{q(t)a(t)}\Big(m + nG(y)\Big). \end{aligned}$$

If we now integrate $N^\alpha V(t)$ from t_0 to T, we see that

$$\frac{G(y(t))}{q(t)} \leq V(t) \leq {}_N J^\alpha_{t_0}\left(\frac{G(y(t))}{q(t)}\left[N^\alpha q(t)_-/q(t) + \frac{nr(t)}{a(t)}\right]\right) + {}_N J^\alpha_{t_0}\left(\frac{mr(t)}{a(t)q(t)}\right) + V(t_0), \tag{3.14}$$

or

$$G(y(t))/q(t) \leq C + {}_N J^\alpha_{t_0}\left\{\frac{G(y(t))}{q(t)}\left[N^\alpha q(t)_-/q(t) + \frac{nr(t)}{a(t)}\right]\right\}$$

for some constant $C > 0$. By Lemma 2.7 we see that $G(y(t))/q(t)$ and hence $G(y(t))$ is bounded on $(0, T)$. This implies $y(t)$ is bounded on $(0, T)$ and an integration shows that $x(t)$ is bounded there as well. Therefore, the solution $(x(t), y(t))$ of (S_1) does not have finite escape time, and this proves the theorem. □

It is possible to formulate alternate versions of Theorem 3.1, for example, if $b(t, u, v) \equiv 0$, then obviously condition (3.2) is not needed; if $e(t, u, v) \equiv 0$, then (3.1) and (3.4) are not needed; if $a(t) \equiv 1$, (3.6) is not needed; and (3.6) can be dropped if we add condition (3.5). We leave the formulation and proofs of such results to the reader.

Based on Theorem 3.1 and its proof, we can formulate a number of different boundedness results. As an example, we have the following one. We will need the condition

$${}_N J^\alpha_{t_0}\left(\frac{r}{a}\right)(\infty) < \infty. \tag{3.15}$$

Theorem 3.2. *If conditions* (3.1)–(3.4), (3.6), (3.7), (3.12), *and* (3.15) *hold, then all solutions of equation* (E) *are bounded. If in addition, $q(t) \leq q_2 < \infty$ and*

$$G(v) \to \infty \quad as \quad |v| \to \infty, \tag{3.16}$$

then all solutions of system (S_1) are bounded.

Proof. First observe that condition (3.3) ensures that $F(x)$ is bounded from below. Then proceeding as in the proof of Theorem 3.1, we obtain (3.14) and so

$$V(t) \leq {}_N J_{t_0}^{\alpha} \left\{ V(t) \left[N^{\alpha} q(t)_- / q(t) + \frac{nr(t)}{a(t)} \right] \right\} + {}_N J_{t_0}^{\alpha} \left\{ \frac{mr(t)}{a(t)q(t)} \right\} + V(t_0).$$

An application of Gronwall's inequality (Lemma 2.7) and conditions (3.8) and (3.15) show that $V(t)$ is bounded. Hence, $[F(x)+K]/a(t)$ is bounded, and so $x(t)$ is bounded by (3.3) and (3.7).

Now $V(t)$ bounded implies $\frac{G(y(t))}{q(t)}$ is bounded, and the additional hypotheses imply that $y(t)$ is bounded. This completes the proof of the theorem. $\square$

In order to show the versatility of the nonconformable fractional derivative, let us consider the special case of equation (E)

$$N^{\alpha}(N^{\alpha}x) + b(x)N^{\alpha}x + f(x) = 0, \tag{L}$$

i.e., the fractional Liènard equation, which we will write as the system

$$\begin{cases} N^{\alpha}x = y - B(x) \\ N^{\alpha}y = -f(x) \end{cases} \tag{S_2}$$

where $B(x) = {}_N J_{t_0}^{\alpha} b(x)$. Define

$$W(t) = W(t, x(t), y(t)) = \frac{y^2(t)}{2} + F(x).$$

Then along solutions of system (S_2), we have

$$N^{\alpha}W(t) = yN^{\alpha}y + f(x)N^{\alpha}x = -yf(x) + f(x)(y - B(x)) = -f(x)B(x).$$

Condition (3.2) implies $xB(x) \geq 0$, so if $xf(x) \geq 0$, we have $N^{\alpha}W(t) \leq 0$. Thus, $W(t)$ is decreasing along solutions of (S_2). Standard Lyapunov stability theorems imply that the zero solution of (S_2) is stable. In addition, if $F(x) \to \infty$ as $|x| \to \infty$, then all solutions of (S_2) are bounded.

We indicated earlier that variations of Theorem 3.1 can be obtained by swapping some of the hypotheses. This is also the case for the boundedness result in Theorem 3.2. One such result is contained in the following theorem.

Theorem 3.3. *In addition to conditions* (3.2), (3.3), (3.5), (3.7), (3.10), *and* (3.12), *assume that*

$$|e(t,x,y)| \le \frac{r(t)a(t)}{q(t)} \tag{3.17}$$

and

$$_N J^{\alpha}_{t_0}\left(\frac{r}{q}\right)(\infty) < \infty. \tag{3.18}$$

Then all solutions of equation (E) *are bounded. If, in addition,* $q(t) \le q_2 < \infty$ *and* (3.16) *holds, then all solutions of system* (S_1) *are bounded.*

Proof. Since the proof will proceed along the same lines as that of Theorem 3.2, let us consider the terms arising from the differentiation of (3.13). First, we see that

$$-[F(x)+K]N^{\alpha}a(t)/a^2(t) \le V(t)N^{\alpha}a(t)_-/a(t)$$

and

$$-G(y)N^{\alpha}q(t)/q^2(t) \le V(t)N^{\alpha}q(t)_-/q(t).$$

Also,

$$\frac{y}{g(y)q(t)}[-yN^{\alpha}a(t)] \le +\frac{MG(y)}{q(t)}\frac{N^{\alpha}a(t)_-}{a(t)} \le MV(t)\frac{N^{\alpha}a(t)_-}{a(t)}.$$

Now if $|y| \le 1$, then $\frac{|y|}{g(y)} \le M_1$ for some $M_1 > 0$, and if $|y| \ge 1$, then $|y|/g(y) \le |y|^2/g(y)$, so $\frac{|y|}{g(y)} \le M_1 + |y|^2/g(y)$ for all y. In view of condition (3.5), it is easy to see that $\frac{|y|}{g(y)} \le M_1 + MG(y)$ for all y. Also, (3.12) implies that $q(t) \ge q_1 > 0$. Hence, by (3.17),

$$\frac{ye(t,x,y)}{g(y)q(t)a(t)} \le (M_1 + MG(y))\frac{r(t)}{q(t)} \le \left(\frac{M_1}{q_1} + \frac{MG(y)}{q(t)}\right)\frac{r(t)}{q(t)}.$$

We then have

$$N^{\alpha}V(t) \le V(t)\left\{(1+M)N^{\alpha}a(t)_-/a(t) + N^{\alpha}q(t)_-/q(t) + M\frac{r(t)}{q(t)}\right\} + \frac{M_1}{q_1}\frac{r(t)}{q(t)}.$$

Applying our Gronwall type inequality and the hypotheses easily completes the proof. □

Let us consider another Lyapunov (energy) type function,

$$W_1(t) = W_1(t,x(t),y(t)) = q(t)[F(x)+K]/a(t) + G(y). \tag{3.19}$$

Then,

$$
\begin{aligned}
N^\alpha W_1(t) &\le W_1(t)\frac{N^\alpha\left(\frac{q(t)}{a(t)}\right)}{\frac{q(t)}{a(t)}} + \frac{N^\alpha a(t)_-}{a(t)}\left(\frac{y^2}{g(y)}\right) + \frac{e(t,x,y)y}{a(t)g(y)} \\
&\le W_1(t)\left[\frac{N^\alpha\left(\frac{q(t)}{a(t)}\right)}{\frac{q(t)}{a(t)}} + M\frac{N^\alpha a(t)_-}{a(t)}\right] + (M_1 + MG(y))\frac{r(t)}{a(t)} \\
&= W_1(t)\left[\frac{N^\alpha\left(\frac{q(t)}{a(t)}\right)}{\frac{q(t)}{a(t)}} + M\frac{N^\alpha a(t)_-}{a(t)} + M\frac{r(t)}{a(t)}\right] + M_1\frac{r(t)}{a(t)}.
\end{aligned}
$$

Based on the above calculations, we can formulate the following result.

Theorem 3.4. *In addition to conditions* (3.1)–(3.3), (3.5), *and* (3.15), *assume that*

$$
{}_N J_{t_0}^\alpha\left(\frac{N^\alpha\left(\frac{q(t)}{a(t)}\right)}{\frac{q(t)}{a(t)}}\right)(\infty) < \infty \tag{3.20}
$$

and

$$
\frac{q(t)}{a(t)} \ge B_1 > 0 \tag{3.21}
$$

for some constant B_1. Then all solutions of equation (E) *are bounded. If, in addition, $q(t) \le q_2 < \infty$ and* (3.16) *holds, then all solutions of system* (S_1) *are bounded.*

For our next boundedness theorem, we modify the Lyapunov (energy) functions we have been using and see that this leads to a different set of conditions to be satisfied. We begin by defining

$$
v(t) = \exp\left\{{}_N J_{t_0}^\alpha\left(\frac{N^\alpha q(t)_-}{q(t)}\right)(t)\right\} \quad \text{and} \quad w(t) = \exp\left\{{}_N J_{t_0}^\alpha\left(\frac{N^\alpha a(t)_-}{a(t)}\right)(t)\right\}
$$

and note that $v(t) \le 1$ and $w(t) \le 1$.

Theorem 3.5. *In addition to conditions* (3.1)–(3.3), (3.5), (3.7), (3.10), (3.12), *and* (3.15), *assume that*

$$
y^2/g(y) \le N_1 \quad \textit{for all } y \tag{3.22}
$$

and

$$
{}_N J_{t_0}^\alpha\left(\frac{r}{aq}\right)(\infty) < \infty. \tag{3.23}
$$

Then all solutions of equation (E) *are bounded. If, in addition, If, in addition, $q(t) \le q_2 < \infty$ and* (3.16) *holds, then all solutions of system* (S_1) *are bounded.*

Proof. Define

$$W_2(t) = W_2(t, x(t), y(t)) = v(t)w(t)\left\{[F(x) + K]/a(t) + G(y)/q(t)\right\}. \tag{3.24}$$

Then,

$$N^\alpha W_2(t) \le v(t)w(t)\left\{[F(x)+K]\frac{N^\alpha a(t)_-}{a^2(t)} + f(x)y/a(t) + \frac{y}{q(t)g(y)}N^\alpha y \right.$$
$$\left. -G(y)\frac{N^\alpha q(t)_-}{q^2(t)} + \big([F(x)+K]/a(t) + G(y)/q(t)\big)\left(\frac{N^\alpha q(t)_-}{q(t)} + \frac{N^\alpha a(t)_-}{a(t)}\right)\right\}.$$

Condition (3.12) implies $q(t) \ge q_1 > 0$ and $v(t) \ge v_1 > 0$, and (3.10) implies $a(t) \ge a_1 > 0$ and $w(t) \ge w_1 > 0$. We also see that $|y|/g(y)$ is bounded for $|y| \le 1$ and $|y|/g(y) \le |y|^2/g(y)$ for $|y| > 1$, so from condition (3.22), $|y|/g(y) \le N_2$ for all y and some $N_2 > 0$. Hence,

$$N^\alpha W_2(t) \le W_2(t)\left[\frac{N^\alpha a(t)_-}{a(t)} + \frac{N^\alpha q(t)_-}{q(t)} + \frac{1}{v_1 w_1}\left(\frac{N^\alpha a(t)_-}{a(t)} + \frac{N^\alpha q(t)_-}{q(t)} + \frac{N_2}{q_1}\frac{N^\alpha a(t)_-}{a(t)}\right)\right]$$
$$+ v(t)w(t)\frac{yr(t)}{g(y)a(t)q(t)}.$$

Therefore,

$$N^\alpha W_2(t) \le W_2(t)\left[\frac{N^\alpha a(t)_-}{a(t)} + \frac{N^\alpha q(t)_-}{q(t)} + \frac{1}{v_1 w_1}\left(\frac{N^\alpha a(t)_-}{a(t)} + \frac{N^\alpha q(t)_-}{q(t)} + \frac{N_2}{q_1}\frac{N^\alpha a(t)_-}{a(t)}\right)\right]$$
$$+ \frac{N_2 r(t)}{a(t)q(t)}.$$

The remainder of the proof follows as before with an application of the Gronwall inequality and the conditions of the theorem. □

We conclude this section with the following observation. Notice that conditions (3.15), (3.18), and (3.23) do not require that the perturbation term e be small, even in the case where (3.1) holds. Many existing results on boundedness in the literature, even for those not involving fractional derivatives, require

$${}_N J_{t_0}^\alpha(r)(\infty) < \infty.$$

This is not the case with Theorems 3.2–3.5 in this paper.

4. Asymptotic Properties of Solutions

The publication of the paper by Hammett [2] in 1971 generated a great deal of interest in obtaining sufficient conditions for ensuring that nonoscillatory solutions $x(t)$ of various differential equations satisfy $\liminf_{t\to\infty} |x(t)| = 0$, and this interest continues to the present day. For the purposes of our discussion here, we classify solutions of equation (E) as follows. A solution of equation (E) is said to be *nonoscillatory* if for any $t_0 > 0$ there exists $t_1 > t_0$ such that $x(t) \neq 0$ for $t \geq t_1$. A solution of equation (E) is said to be *oscillatory* if for any $t_0 > 0$ there exist $t_1 > t_0$ and $t_2 > t_0$, with $x(t_1) > 0$ and $x(t_2) < 0$. A solution will be said to be a *Z-type* solution if it has arbitrarily large zeros but is eventually nonnegative or nonpositive. It turns out that asymptotic properties of nonoscillatory solutions often hold for the Z-type solutions as well.

We begin with two results that give sufficient conditions for bounded nonoscillatory and Z-type solutions to satisfy $\liminf_{t\to\infty} |x(t)| = 0$. This is followed by four theorems ensuring that all solutions of equation (E) converge to zero.

In what follows we will assume that

$$xf(x) > 0 \quad \text{if} \quad x \neq 0 \tag{4.1}$$

and that $f(x)$ is bounded away from 0 if x is bounded away from 0.

This means that the constant K appearing in the Lyapunov type functions (3.13), (3.19), and (3.24) can be chosen to be 0. In addition, we will use the conditions:

if u is bounded, there exists a continuous function $k_1 : \mathbb{R}_+ \to \mathbb{R}_+$ such that

$$|b(t,u,v)| \leq k_1(t)g(v), \tag{4.2}$$

$$g(v) \geq C \quad \text{for some constant } C > 0, \tag{4.3}$$

$$_N J^\alpha_{t_0}(q)(\infty) = \infty, \tag{4.4}$$

$$\frac{k_1(t)}{q(t)} \to 0 \quad \text{and} \quad \frac{r(t)}{q(t)} \to 0 \text{ as } t \to \infty, \tag{4.5}$$

$$_N J^\alpha_{t_0}\left(\frac{1}{a}\right)(\infty) = \infty, \tag{4.6}$$

$$a(t)k_1(t) \to 0 \quad \text{and} \quad a(t)r(t) \to 0 \quad \text{as} \quad t \to \infty. \tag{4.7}$$

Theorem 4.1. *Assume conditions* (3.1) *and* (4.1)–(4.6) *hold. If $x(t)$ is a bounded nonoscillatory or Z-type solution of* (E)*, then* $\liminf_{t\to\infty} |x(t)| = 0$.

Proof. If $x(t)$ is a Z-type solution, the conclusion obviously holds, so let $x(t)$ is a bounded nonoscillatory solution of (E), say $0 < x(t) < c_1$ for $t \geq t_0 > 0$ and some $c_1 > 0$. The proof in case $x(t)$ is eventually negative is similar. If $\liminf_{t\to\infty} x(t) \neq 0$, then there exists $t_1 \geq t_0$ and $c_2 > 0$ so that $x(t) \geq c_2$ for $t \geq t_1$. Thus, $f(x(t)) > c_3 > 0$ for $t \geq t_1$ for some c_3 by (4.1).

From equation (E) we have

$$\begin{aligned} N^\alpha(a(t)N^\alpha x)/g(N^\alpha x) &\leq -b(t,x,N^\alpha x)/g(N^\alpha x) - q(t)f(x) + e(t,x,N^\alpha x)g(N^\alpha x) \\ &\leq k_1(t) - q(t)c_3 + r(t)/C \\ &\leq q(t)[k_1(t)/q(t) - c_3 + r(t)/q(t)]. \end{aligned}$$

Since $k_1(t)/q(t) \to 0$ and $r(t)/q(t) \to 0$ as $t \to \infty$, we can choose $t_2 > t_1$ such that

$$N^\alpha(a(t)N^\alpha x)/g(N^\alpha x) \leq -(c_3/3)q(t)$$

for $t \geq t_2$.

Integrating and applying condition (4.4) shows that $a(t)N^\alpha x$ is eventually negative, and this fact together with condition (4.6) shows that $x(t)$ is eventually negative, which is a contradiction. Therefore, $\liminf_{t\to\infty} x(t) = 0$. □

We also have the companion result.

Theorem 4.2. *Assume that conditions* (3.1), (4.1)–(4.3), (4.6), *and* (4.7) *hold, and* $a(t)q(t) \geq B_2$ *for some* $B_2 > 0$. *If* $x(t)$ *is a bounded nonoscillatory or* Z*-type solution of* (E), *then* $\liminf_{t\to\infty} |x(t)| = 0$.

Proof. Proceeding as in the proof of Lemma 4.1, we arrive at

$$\begin{aligned} N^\alpha(a(t)N^\alpha x)/g(N^\alpha x) &\leq k_1(t) - q(t)c_3 + r(t)/C \\ &\leq \frac{1}{a(t)}[a(t)k_1(t) - a(t)q(t)c_3 + a(t)r(t)/C]. \end{aligned}$$

Condition (4.7) implies there exits $T > 0$ such that

$$N^\alpha(a(t)N^\alpha x)/g(N^\alpha x) \leq \frac{B_2 c_3}{2}$$

for $t \geq T$. The remainder of the proof is similar to that of Theorem 4.1 □

Our first theorem guaranteeing that all solutions converge to zero is built upon Theorem 3.2.

Theorem 4.3. *If conditions* (3.1)–(3.4), (3.6), (3.7), (3.12), *and* (3.15) *hold, then every solution of* (E) *converges to zero as* $t \to \infty$.

Proof. Let $x(t)$ be solution of (E). By Theorem 3.2, $x(t)$ is bounded. Define $V(t)$ as in the proof of Theorem 3.2 (see (3.13) in the proof of Theorem 3.1) taking (4.1) into account. Differentiating, we obtain

$$N^\alpha V(t) \leq \left\{V(t)\left[N^\alpha q(t)_-/q(t) + \frac{nr(t)}{a(t)}\right]\right\} + \frac{mr(t)}{a(t)q(t)}. \tag{4.8}$$

From the proof of Theorem 3.2, we have that $V(t)$ is bounded, say $V(t) \leq K_1$ for some $K_1 > 0$. Let $\epsilon > 0$ be given. By conditions (3.12) and (3.15), we can choose $T_\epsilon > t_0$ such that

$${}_N J^\alpha_{T_\epsilon}\left(\frac{N^\alpha q(s)_-}{q(s)}\right)(t) < \frac{\epsilon}{4K_1} \quad \text{and} \quad {}_N J^\alpha_{T_\epsilon}\left(\frac{r}{a}\right)(t) < \min\left\{\frac{q_1\epsilon}{4m}, \frac{\epsilon}{4nK_1}\right\}$$

for $t \geq T_\epsilon$. Then, an integration of (4.8) shows that $V(t) \leq \epsilon$ for $t \geq T_\epsilon$, that is,

$$\frac{F(x(t))}{A} \leq \frac{F(x(t))}{a(t)} \leq V(t) \to 0$$

as $t \to \infty$, which implies $x(t) \to 0$ as $t \to \infty$. □

Our next theorem is based on Theorem 3.3.

Theorem 4.4. *Let conditions* (3.2), (3.3), (3.5), (3.7), (3.10), *and* (3.12), (3.17) *and* (3.18) *hold. Then then every solution of* (E) *converges to zero as $t \to \infty$.*

Proof. Let $x(t)$ be a solution of (E); it is bounded by Theorem 3.3. Define $V(t)$ as used in the proof of Theorem 4.3. Differentiating, we obtain

$$N^\alpha V(t) \le V(t) \left\{ (1+M)N^\alpha a(t)_-/a(t) + N^\alpha q(t)_-/q(t) + M\frac{r(t)}{q(t)} \right\} + \frac{M_1}{q_1}\frac{r(t)}{q(t)}.$$

Again $V(t)$ is bounded, say $V(t) \le K_2$ for some $K_2 > 0$. Let $\epsilon > 0$. We then find $T_1 > t_0$ so that

$${}_N J^\alpha_{T_1}\left(\frac{N^\alpha a(s)_-}{a(s)}\right)(t) < \frac{\epsilon}{4(1+M)K_1}, \quad {}_N J^\alpha_{T_1}\left(\frac{N^\alpha q(s)_-}{q(s)}\right)(t) < \frac{\epsilon}{4K_1}$$

and

$${}_N J^\alpha_{T_1}\left(\frac{r}{q}\right)(t) < \min\left\{\frac{\epsilon}{4MK_1}, \frac{q_1\epsilon}{K_1M_1}\right\}$$

for $t \ge T_1$. The remainder of the proof follows as before. □

Corresponding to the boundedness result in Theorem 3.4 we have the following theorem.

Theorem 4.5. *Let conditions* (3.1)–(3.3), (3.5), (3.15), (3.20) *and* (3.21) *hold. Then any solution $x(t)$ of equation* (E) *satisfies $x(t) \to 0$ as $t \to \infty$.*

Proof. Let $x(t)$ be a solution of (E) and define $W_1(t)$ by

$$W_1(t) = W_1(t, x(t), y(t)) = q(t)F(x)/a(t) + G(y).$$

We then have

$$N^\alpha W_1(t) = W_1(t)\left[\frac{N^\alpha\left(\frac{q(t)}{a(t)}\right)}{\frac{q(t)}{a(t)}} + M\frac{N^\alpha a(t)_-}{a(t)} + M\frac{r(t)}{a(t)}\right] + M_1\frac{r(t)}{a(t)}.$$

The boundedness of W_1 follows from the conditions in the theorem. Denote this fact by $W_1(t) \le K_3$ for all $t > t_0$ and let $\epsilon > 0$ be given. Our conditions allow us to choose $T_2 > t_0$ such that

$${}_N J^\alpha_{T_2}\left(\frac{N^\alpha\left(\frac{q(t)}{a(t)}\right)}{\frac{q(t)}{a(t)}}\right)(t) < \frac{\epsilon}{4K_3} \quad {}_N J^\alpha_{T_2}\left(\frac{N^\alpha a(t)_-}{a(t)}\right)(t) < \frac{\epsilon}{4MK_3}$$

and

$${}_N J^\alpha_{T_2}\left(\frac{r(t)}{a(t)}\right)(t) < \frac{\epsilon}{4K_3(M+M_1)}$$

for $t \ge T_2$. The remainder of the proof proceeds as before. □

Based on Theorem 3.5 we have our last result in this paper.

Theorem 4.6. *Let conditions* (3.1)–(3.3)*,* (3.5)*,* (3.7)*,* (3.10)*,* (3.12)*, and* (3.15) (3.22) *and* (3.23) *hold. Then every solution $x(t)$ of equation* (E) *converges to* 0 *as $t \to \infty$.*

Proof. With $W_2(t)$ defined as in the proof of Theorem 3.5, we find that

$$N^\alpha W_2(t) \le W_2(t) \left[\frac{N^\alpha a(t)_-}{a(t)} + \frac{N^\alpha q(t)_-}{q(t)} + \frac{1}{v_1 w_1} \left(\frac{N^\alpha a(t)_-}{a(t)} + \frac{N^\alpha q(t)_-}{q(t)} \right) \right] + \frac{N_2 r(t)}{a(t)q(t)}$$

and $W_2(t) \le K_4$ for $t \ge t_0$.

For a given $\epsilon > 0$, we choose $T_3 > t_0$ with

$${}_N J^\alpha_{T_2} \left(\frac{N^\alpha a(t)_-}{a(t)} \right)(t) < \frac{\epsilon}{K_4(1 + \frac{1}{v_1 w_1})}, \quad {}_N J^\alpha_{T_2} \left(\frac{N^\alpha q(t)_-}{q(t)} \right)(t) < \frac{\epsilon}{K_4(1 + \frac{1}{v_1 w_1})}$$

and

$${}_N J^\alpha_{T_2} \left(\frac{r(t)_-}{a(t)q(t)} \right)(t) < \frac{\epsilon}{4}$$

for all $t \ge T_2$. The remainder of the proof is straightforward and is left to the reader. $\square$

In conclusion, we wish to point out that all the results in this section are new for fractional differential equations of any type. Also, we remark that it would be interesting to apply this definition of a nonconformable fractional derivative to equations on time scales.

References

[1] P.M. Guzmán, G. Langton, L.M.L.M. Bittencurt, J. Medina, J.E. Nápoles Valdes, *A new definition of a fractional derivative of local type*, J. Math. Anal. 9 (2018) 88–98.

[2] M.E. Hammett, *Nonoscillation properties of a nonlinear differential equation*, Proc. Amer. Math. Soc. 20 (1971) 92–96.

[3] F. Martínez, J.E. Nápoles Valdes, *Towards a non-conformable fractional calculus of n-variables*, J. Math. Appl. 43 (2020) 87–98.

[4] J.E. Nápoles Valdes, P.M. Guzman, L.M. Lugo, *On the stability of solutions of nonconformable differential equations*, Stud. Univ. Babeş-Bolyai Math., to appear.

[5] J.E. Nápoles Valdes, C. Tunç, *On the boundedness and oscillation of non-conformable Liénard equation*, J. Fract. Calc. Appl. 11 (2020) 92–101.

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On Some Metric in the Family of Compact Convex Sets

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ABSTRACT: We define the Demyanov metric and new metric and compare with the Hausdorff and Vitale metrics. Vitale compared the Hausdorff metric ρ_H and Vitale metric ρ_V. We proved that main metric ρ_{LV} is equivalence with ρ_H metric and that the family of nonempty, convex, compact sets and the ρ_{LV} metric is the complete space.

AMS Subject Classification: 00A69, 97E60.
Keywords and Phrases: Demyanov metric; Hausdorff metric; Convex sets; Support function.

1. Introduction

In the convex sets space metrics has a crucial role which we use for approximation of convex sets, optimization, multifunction theory, control theory etc.

The well-known a Hausdorff metric is widely applied. In some situations the Hausdorff metric is not fine enough to capture some changes in sets which may be crucial. If we rotate a polytope then the Hausdorff distance is small but their faces will not be parallel. Diamond et al in [2] reformulated the Demyanov metric and showed that the Demyanov metric majorizes the Hausdorff metric but this metrics are not equivalent for the of compact, convex sets. In 1985 R.A. Vitale give a metric similar to metric use in the L^p function space and give relation from the Hausdorff metric.

2. Basic notations and preliminaries

We introduce some notation. By $\mathcal{K}^d$ will stand for the space of nonempty, compact, convex subsets of $\mathbb{R}^d$. To each $A \in \mathcal{K}^d$ we assign a support function in the direction v: $p_A(v) = sup_{a \in A} < a, v >$ where $< \cdot, \cdot >$ is a scalar product and v is a vector the unit sphere S^{d-1}.For bounded A this is a convex, positively homogenous functional on $\mathbb{R}^d$.

By $A(v) = \{a \in A :< a, v >= p_A(v)\}$ we define the face of set $A \in \mathbb{K}^d$ in the direction v.

Definition 2.1. The point $e \in A$ is exposed point of a set A if exist a vector $v \in S^{d-1}$ such that $A(v) = e$.

Let $T_A = \{v \in \mathbb{R}^d : A(v) \ \textit{is singleton}\}$. The set T_A is a set of full Lebesque measure in $\mathbb{R}^d$. The set $\mathbb{R}^d \setminus T_A$ has always measure 0 and so for any two $A, B \in \mathbb{K}^d$ the complement of $T_A \cap T_B$ has also measure 0.

As aritmetic operations in $\mathcal{K}^d$ we use the classical ones, namely the Minkowski sum and scalar multiplication:

$$A + b = \{a + b : a \in A, \ b \in B\} \text{ for } A, B \in \mathcal{K}^d,$$
$$\lambda A = \{\lambda a : a \in A\} \text{ for } \lambda \in \mathbb{R}^d, \ A \in \mathcal{K}^d.$$

3. The Hausdorff and Demyanov metric

Definition 3.1. Let $A, B \in \mathcal{K}^d$. The Hausdorff metric is defined by

$$\rho_H(A, B) = \max\{e(A, B), e(B, A)\}$$

where $e(A, B) = \sup_{a \in A} \inf_{b \in B} \|a - b\|$, ($\| \cdot \|$ is Euclidean norm in $\mathbb{R}^d$).

We use this paper the following definition the Hausdorff distance

$$\rho_H(A, B) = \sup_{v \in S^{d-1}} |p_A(v) - p_B(v)|.$$

We recall a definition the stricly convex set.

Definition 3.2. The set $A \in \mathcal{K}^d$ we call stricly convex if for all $v \in \mathbb{R}^d$, $A(v)$ is singleton.
By $\bar{\mathcal{K}^d}$ be shall denote the family of nonempty, compact and stricly convex sets.

The following examples a shows that the space $(\bar{\mathcal{K}^d}, \rho_H)$ is not complete.

Example 3.3. We consider the following sequence sets in $\bar{\mathcal{K}^2}$

$$A_n = \{(x_1, x_2) : x_1^2 + 2^n x_2^2 \leq 1\}.$$

Then $\rho_H(A_n, A_m) = |\frac{1}{2^n} - \frac{1}{2^m}|$ and is so the Cauchy sequence converges to a set $A = \{(x_1, x_2) : -1 \leq x_1 \leq 1, \ x_2 = 0\}$ which is not stricly convex.

Now we define the Demyanov metric has been introduce earlier by Pliś [4].

Definition 3.4. Let $A, B \in \mathcal{K}^d$. The Demyanov distance we define

$$\rho_D(A,B) = \sup\{\|A(v) - B(v)\| : v \in T_A \cap T_B\}.$$

The triangle inequality and the symmetricity are obvious. To prove that defines a metric we remark that $A = clco\{A(v) : v \in T_A\}$,clco stands here for the closed, convex hull of a set.This equality is a consequence that every compact, convex set is the closed, convex hull of the set of its extreme points and with the Straszewicz theorem give that the set of extreme points of a set in $\mathcal{K}^d$ is contained in the closure of the set of its exposed points. So if $\rho_D(A,B) = 0$ then the boundaries of A and B coincide and $A = B$.

Use the inequality $|p_A(v) - p_B(v)| \leq \|A(v) - B(v)\|$ for all $v \in S^{d-1}$ we have that $\rho_H(A,B) \leq \rho_D(A,B)$.

The following example ilustrate that the Hausdorff metric not respond on of rotation the sets.

Example 3.5. Let be the family sets from $\mathcal{K}^2$

$$A_x = clco\{(0,0),(cosx, sinx)\}$$

where $x \in< 0, 2\pi >$. We find the Hausdorff distance

$$\rho_H(A_x, A_y) = \sqrt{(cosx - cosy)^2 + (sinx - siny)^2} = \sqrt{2(1 - cos(x-y))} = \\ sin|x-y| \leq |x-y|.$$

So we have that if $x \to y$ then $\rho_H(A_x, A_y) \to 0$. Now we find the Demyanov distance for the sets A_x and A_y. Fix z such that $\frac{\pi}{2} + x < z < \frac{\pi}{2} + y$. Then $A_x(z) = (0,0)$ and $A_y(z) = (cosy, siny)$ so $\rho_D(A_x, A_y) = 1$.

T.Rzeżuchowski in([5] Theorem 2.2) prove that in $\bar{\mathcal{K}}^d$ the Hausdorff metric and the Demyanov metric are equivalent. T.Rzeżuchowski prove the following theorem.

Theorem 3.6. *The metrics ρ_H and ρ_D are equivalent in $\bar{\mathcal{K}}^d$, the metric space $(\bar{\mathcal{K}}^d, \rho_H)$ is not complete and the space $(\bar{\mathcal{K}}^d, \rho_D)$ is complete.*

4. The Vitale metric

In 1985 R.Vitale in [7] defined a new metric in $\mathcal{K}^d$ as follows

$$\rho_V(A,B) = (\int_{S^{d-1}} |p_A(v) - p_B(v)|^p) d\mu(v))^{\frac{1}{p}} \text{ for } 1 \leq p < \infty$$

where $\mu(\cdot)$ is Lebesque unit measure on S^{d-1}.

The inequality $|p_A(v) - p_B(v)| \leq \sup_{v \in S^{d-1}} |p_A(v) - p_B(v)|$ is true for all $v \in S^{d-1}$ which implies immediately that $\rho_V(A,B) \leq \rho_H(A,B)$.

In [7] Vitale showing that if we have the sequence sets A_n from $\mathcal{K}^d$ and $A \in \mathcal{K}^d$ then $\rho_V(A_n, A) \to 0 \Leftrightarrow \rho_H(A_n, A) \to 0$ for all $1 \leq p < \infty$.
This fact imply the following theorem (Theorem 3 in [7]).

Theorem 4.1. *All of the ρ_V metrics, $1 \leq p < \infty$, induce the same topology on $\mathcal{K}^d$ and yield complete metric spaces in which closed, bounded sets are compact.*

5. Main result

We introduce now the new metric

$$\rho_{LV}(A, B) = (\int_{T_A \cap T_B} \|A(v) - B(v)\|^p d\mu(v))^{\frac{1}{p}}.$$

Obviously we have the inequality $\rho_{LV}(A, B) \leq \rho_D(A, B)$ and

$$\rho_V(A, B) \leq \rho_{LV}(A, B).$$

Now we consider the example which a showed that ρ_D and ρ_{LV} metrics are not equivalent.

Example 5.1. Let be the sequence sets from $\mathcal{K}^2$:

$$A_n = clco\{(-1, 1), (1, -1), (1, 1 + \frac{1}{n}), (-1, 1)\} \text{ and } A \in \mathcal{K}^2$$

where

$$A = clco\{(-1, 1), (1, -1), (1, 1), (-1, 1)\}.$$

Then the metrics are:

$$\rho_H(A_n, A) = \frac{1}{n}, \qquad \rho_V(A_n, A) = \frac{1}{n}(\frac{\pi}{2} + arctg\frac{1}{n}), \qquad \rho_D(A_n, A) = \sqrt{4 + \frac{1}{n^2}}$$

and main metric

$$\rho_{LV}(A_n, A) = \sqrt{4 + \frac{1}{n^2}} arctg\frac{1}{n}.$$

We have that $\rho_H(A_n, A) \to 0$ and $\rho_V(A_n, A) \to 0$ and $\rho_{LV}(A_n, A) \to 0$.
The Vitale result in [7] and the inequality $\rho_V(A, B) \leq \rho_{LV}(A, B)$ give that

$$\rho_{LV}(A_n, A) \to 0 => \rho_V(A_n, A) \to 0 \implies \rho_H(A_n, A) \to 0.$$

Now we showing the inverse implication.

Theorem 5.2. *Let $A \in \mathcal{K}^d$ and the sequence sets $A_n \in \mathcal{K}^d$ be such that $\rho_H(A_n, A) \to 0$. Then for all $v \in T = T_A \cap \bigcap_{n=1}^{\infty} T_{A_n}$, $\|A_n(v) - A(v)\| \to 0$.*

Proof. Fix the $v \in T$ where $T = T_A \cap \bigcap_{n=1}^{\infty} A_n$ and we assume that

$$\|A_n(v) - A(v)\|$$

not converges to 0. Exist $\alpha > 0$ and the subsequence of the sequence A_n, denoted again by A_n for which $\|y_n - A(v)\| > \alpha > 0$ for $y_n \in A_n(v)$.
Let $y_n \to y_0$, then $\|y_0 - A(v)\| \geq \alpha$. Because $A(v)$ is the exposed point then

$$< y_0, v > - < A(v), v >> 2\alpha.$$

The following condition with assume the theorem $\rho_H(A_n, A) \to 0$ imply that

$$\sup_{v \in S^{d-1}} |p_{A_n}(v) - p_A(v)| \to 0.$$

For sufficiently large n and for $\epsilon = \alpha$ this the condition give that

$$< y_n, v > - < A(v), v > \leq \epsilon.$$

Because $y_n \to y_0$ we have that

$$2\epsilon < < y_0, v > - < A(v), v > \leq \epsilon.$$

This contradiction shows that $\|A_n(v) - A(v)\| \to 0$. □

The Theorem 5.2 implies:

Corollary 5.3. *Let be $A \in \mathcal{K}^d$ and the sequence sets A_n be such that*

$$T = T_A \cap \bigcap_{n=1}^{\infty} T_{A_n}.$$

Then the metrics ρ_H and ρ_{LV} are equivalent.

This result showed the following corollary:

Corollary 5.4. *The space $(\mathcal{K}^d, \rho_{LV})$ is complete.*

6. Summary

We can use the metrics to characterization of a set-valued Lipschitz map by uniformly the Lipschitz selections in the cases for ρ_V the Lipschitz maps with the convex, compact images or the ρ_{LV} Lipschitz maps with the convex, compact images.

References

[1] J.P. Aubin, A. Cellina, *Differential Inclusions*, Grundlehren Math. Wiss. 264, Springer, Berlin, 1984.

[2] P. Diamond, P. Kloeden, A. Rubinov, A. Vladimirov, *Comparative properties of three metrics in the space of compact convex sets*, Set-Valued Anal. 5 (3) (1997) 267–289.

[3] A. Leśniewski, T. Rzeżuchowski, *The Demyanov metric for convex, bounded sets and existence of Lipschitzian selectors*, J. Convex Anal. 18 (3) (2011) 737–747.

[4] A. Pliś, *Uniqueness of optimal trajectories for non-linear control systems*, Ann. Polon. Math. 29 (4) (1975) 397–401.

[5] T. Rzeżuchowski, *The Demyanov metric and some other metrics in the family of convex sets*, Cent. Eur. J. Math. 10 (6) (2012) 2229–2239.

[6] R. Schneider, *Convex Bodies: The Brunn-Minkowski Theory*, Encyclopedia Math. Appl. 44, Cambridge University Press, Cambridge, 1993.

[7] R.A. Vitale, L_p *metrics for compact, convex sets*, J. Approximation Theory 45 (1985) 280–287.

[8] J. Sadowski, *Young inequality for convolution and its application in convex and set-valued analysis*, J. Math. Anal. Appl. (2015) 1274–1294.

[9] L. Coutin, N. Marie, P.R. de Fitte, *On a set-valued Young integral with applications to differential inclusions*, hal-03252856v1 (2021).

[10] A. Leśniewski, *The Demyanov continuous and Cesari's property*, Trans. J. Math. Anal. Appl. 2 (1) (2014).

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Nano b-I-Continuous Functions and Nano b-I-Open Functions

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Abstract: The aim of this paper is to define and study certain new classes of continuous, irresolute and open functions namely nano b-I-continuous, nano b-I-irresolute and nano b-I-open functions in nano ideal topological spaces. Some characterizations and properties regarding these concepts are discussed. All these concepts will be helpful for further generalizations of nano continuous mappings in nano ideal topological spaces.

AMS Subject Classification: 54A05, 54D10.
Keywords and Phrases: Nano b-I-continuous function; Nano b-I-irresolute function; Nano b-I-open function.

1. Introduction

Thivagar and Richard [23] established the field of nano topological spaces. In 2016, Thivagar and Devi [21] introduced the notion of nano local functions and explore the field of nano topological spaces. In 2018, Parimala and Jafari [18] introduced the notion of nano I-continuous functions in nano ideal topological spaces. Jamal M. Mustafa [13 - 16] studied weakly nano semi-I-open sets and weakly nano semi-I-continuous functions and some covering properties using the b-open sets. In this paper we introduce and study the new classes of continuous, irresolute and open functions namely $nano\ b-I-continuous$, $nano\ b-I-irresolute$ and $nano\ b-I-open$ functions in nano ideal topological spaces and we discuss some of their properties.

Let (D, ζ) be a topological space and $A \subseteq D$. The complement of A in D, the closure of A, the interior of A and the power set of A will be denoted by $D - A = A^c$, $Cl\,(A)$, $Int\,(A)$ and $\mathcal{P}\,(A)$, respectively. The subject of ideals in topological spaces has been studied by Kuratowski [10] and Vaidyanathaswamy [25]. An ideal on a topological space (D, ζ) is defined as a non-empty collection I of subsets of D satisfying the following two conditions: (1) If $A \in I$ and $B \subseteq A$, then $B \in I$; (2) If $A \in I$ and $B \in I$, then $A \cup B \in I$. An ideal topological space is a topological space (D, ζ) with an ideal I on D and is denoted by (D, ζ, I). For a subset $A \subseteq D$, $A^*(I) = \{x \in D : U \cap A \notin I$ for every $U \in \zeta$ with $x \in U\}$ is called the local function of A with respect to I and ζ [10]. We simply write A^* instead of $A^*(I)$ in case there is no chance of confusion. It is well known that $Cl^*(A) = A \cup A^*$ defines a Kuratowski closure operator for $\tau^*(I)$.

First we shall recall some definitions used in the sequel.

Definition 1.1. Let A be a subset of a topological space (D, ζ). Then

a) A is called $semi-open$ [9] if $A \subseteq Cl(Int(A))$.

b) A is called $pre-open$ [10] if $A \subseteq Int(Cl(A))$.

c) A is called $\alpha-open$ [10] if $A \subseteq Int(Cl(Int(A)))$.

c) A is called $b-open$ [1] if $A \subseteq Cl(Int(A)) \cup Int(Cl(A))$.

d) A is called $semi-closed$ [4] if it is the complement of a semi-open set.

e) The $semi-closure$ of A [4], denoted by $sCl\,(A)$, is the smallest semi-closed set that contains A.

Definition 1.2. A subset A of an ideal topological space (D, ζ, I) is said to be

a) $I-open$ [9] if $A \subseteq Int(A^*)$.

b) $semi-I-open$ [7] if $A \subseteq Cl^*(Int(A))$.

c) $pre-I-open$ [5] if $A \subseteq Int(Cl^*(A))$.

d) $b-I-open$ [6] if $A \subseteq Cl^*(Int(A)) \cup Int(Cl^*(A))$.

2. Preliminaries

Definition 2.1. [23] Let U be a non-empty finite set of all objects called the universe and R be an equivalence relation on U named as indiscernibility relation. Then U is divided into disjoint equivalence classes. Elements belonging to the same equivalence class are said to be indiscernible with one another. The pair (U, R) is said to be the approximation space. Let $D \subseteq U$. Then,

(1) The lower approximation of D with respect to R is the set of all objects which can be for certain classified as D with respect to R and is denoted by $L_R(D)$. $L_R(D) = \cup\{R(x) : R(x) \subseteq D, x \in U\}$ where $R(x)$ denotes the equivalence class determined by $x \in U$.

(2) The upper approximation of D with respect to R is the set of all objects which can be possibly classified as D with respect to R and is denoted by $U_R(D) = \cup\{R(x) : R(x) \cap D \neq \phi, x \in U\}$.

(3) The boundary region of X with respect to R is the set of all objects which can be classified neither as X nor as not$-D$ with respect to R and is denoted by $B_R(D)$. $B_R(D) = U_R(D) - L_R(D)$.

Remark. [23] If (U, R) is an approximation space and $D, E \subseteq U$, then

(1) $L_R(D) \subseteq D \subseteq U_R(D)$.

(2) $L_R(\phi) = U_R(\phi) = \phi$.

(3) $L_R(U) = U_R(U) = U$.

(4) $U_R(D \cup E) = U_R(D) \cup U_R(E)$.

(5) $U_R(D \cap E) \subseteq U_R(D) \cap U_R(E)$.

(6) $L_R(D \cup E) \supseteq L_R(D) \cup L_R(E)$.

(7) $L_R(D \cap E) = L_R(D) \cap L_R(E)$.

(8) $L_R(D) \subseteq L_R(E)$ and $U_R(D) \subseteq U_R(E)$ whenever $D \subseteq E$.

(9) $U_R(D^c) = [L_R(D)]^c$ and $L_R(D^c) = [U_R(D)]^c$.

(10) $U_R(U_R(D)) = L_R(U_R(D)) = U_R(D)$.

(11) $L_R(L_R(D)) = U_R(L_R(D)) = L_R(D)$.

Definition 2.2. [23] Let U be the universe, R be an equivalence relation on U and $\zeta_R(D) = \{U, \phi, L_R(D), U_R(D), B_R(D)\}$ where $D \subseteq U$. Then by the last remark, $\zeta_R(D)$ satisfies the following axioms:

(1) U and $\phi \in \zeta_R(D)$.

(2) The union of the elements of any subcollection of $\zeta_R(D)$ is in $\zeta_R(D)$.

(3) The intersection of the elements of any finite subcollection of $\zeta_R(D)$ is in $\zeta_R(D)$.

Then $\zeta_R(D)$ is a topology on U called the nano topology on U with respect to X. $(U, \zeta_R(D))$ is called the nano topological space. Elements of the nano topology are known as nano open sets in U and the complement of a nano open set is called nano closed.

Definition 2.3. [23] If $\zeta_R(D)$ is the nano topology on U with respect to D, then the set $B = \{U, L_R(D), B_R(D)\}$ is the basis for $\zeta_R(D)$.

Definition 2.4. [23] If $(U, \zeta_R(D))$ is a nano topological space with respect to D where $D \subseteq U$ and if $A \subseteq U$, then

(1) The nano interior of the set A is defined as the union of all nano open subsets contained in A and is denoted by $nInt(A)$. $nInt(A)$ is the largest nano open subset of A.

(2) The nano closure of the set A is defined as the intersection of all nano closed sets containing A and is denoted by $nCl(A)$. $nCl(A)$ is the smallest nano closed set containing A.

Definition 2.5. [23] Let $(U, \zeta_R(D))$ be a nano topological space and $A \subseteq U$. Then A is said to be:

(1) *nano semi − open* if $A \subseteq nCl(nInt(A))$.

(2) *nano pre open* if $A \subseteq nInt(nCl(A))$.

Definition 2.6. Let $(U, \zeta_R(D))$ and $(V, \zeta_{R'}(E))$ be two nano topological spaces. A function $f : (U, \zeta_R(D)) \to (V, \zeta_{R'}(E))$ is called:

(1) nano continuous [24] if $f^{-1}(B)$ is nano open in U for every nano open set B in V .

(2) nano semi-continuous [20] if $f^{-1}(B)$ is nano semi-open in U for every nano open set B in V .

(3) nano precontinuous [22] if $f^{-1}(B)$ is nano preopen in U for every nano open set B in V .

(4) nano open if $f(A)$ is nano open in V for every nano open set A in U.

(5) nano closed if $f(C)$ is nano closed in V for every nano closed set C in U.

3. Nano ideal topological spaces

In 2016, Thivagar and Devi [21] considered the nano local function in nano ideal topological space and they obtained a new topology. A nano ideal topological space is a nano topological space $(U, \zeta_R(D))$ with an ideal I on U and is denoted by $(U, \zeta_R(D), I)$. For a subset $A \subseteq U$, $nA^*(I) = \{x \in U : W \cap A \notin I$ for every $W \in \zeta_R(D)$ with $x \in W\}$ is called the nano local function of A with respect to I and $\zeta_R(D)$ [21]. We simply write nA^* instead of $nA^*(I)$ in case there is no chance of confusion. It is well known that $nCl^*(A) = A \cup nA^*$ defines a nano closure operator for $(\zeta_R(D))^*(I)$.

Theorem 3.1. *[21] Let $(U, \zeta_R(D))$ be a nano topological space with ideals I, J on U and A, B be subsets of U. Then the following statements are true:*

(i) if $A \subseteq B$, then $nA^* \subseteq nB^*$

(ii) if $I \subseteq J$, then $nA^*(I) \subseteq nA^*(J)$.

(iii) $nA^* = nCl(nA^*) \subseteq nCl(A)$.

(iv) $n(nA^*)^* = nA^*$.

(v) $nA^* \cup nB^* = n(A \cup B)^*$.

(vi) $nA^* - nB^* = n(A - B)^* - nB^* \subseteq n(A - B)^*$.

(vii) *if* $V \in \tau_R(D)$, *then* $V \cap nA^* = V \cap n(V \cap A)^* \subseteq n(V \cap A)^*$.

(viii) *if* $E \in I$, *then* $n(A \cup E)^* = nA^* = n(A - E)^*$.

Theorem 3.2. *[21] The nano closure operator* nCl^* *satisfies the following conditions:*

(i) $A \subseteq nCl^*(A)$.

(ii) $nCl^*(\phi) = \phi$ *and* $nCl^*(U) = U$.

(iii) *if* $A \subseteq B$ *then* $nCl^*(A) \subseteq nCl^*(B)$.

(iv) $nCl^*(A) \cup nCl^*(B) = nCl^*(A \cup B)$.

(v) $nCl^*(nCl^*(A)) = nCl^*(A)$.

Definition 3.3. *A subset A of a nano ideal topological space* $(U, \zeta_R(D), I)$ *is said to be*

(1) *nano semi* $I - open$ *[21] if* $A \subseteq nCl^*(nInt(A))$.

(2) *nano pre* $I - open$ *[8] if* $A \subseteq nInt(nCl^*(A))$.

(3) *nano* $\alpha - I - open$ *[21] if* $A \subseteq nInt(nCl^*(nInt(A)))$.

Definition 3.4. A subset $A \subseteq U$ in a nano ideal topological space $(U, \zeta_R(D), I)$ is said to be *nano* $b - I - open$ [19] if $A \subseteq nCl^*(nInt(A)) \cup nInt(nCl^*(A))$.

The family of all nano b-I-open sets of the space $(U, \zeta_R(D), I)$ will be denoted by $NbIO(U, \zeta_R(D))$.

A subset $A \subseteq U$ in a nano ideal topological space $(U, \zeta_R(D), I)$ is said to be *nano* $b - I - closed$ if its complement is *nano* $b - I - open$.

Theorem 3.5. *For a subset of a nano ideal topological space, the following properties hold:*

(a) *Every nano semi* $I - open$ *set is nano b-I-open.*

(b) *Every nano pre* $I - open$ *set is nano b-I-open.*

(c) *Every nano* $\alpha - I - open$ *set nano b-I-open.*

The converse of each part in the above theorem need not be true as shown in the following example.

Example 3.6. Let $U = \{a,b,c,d\}$ be the universe, $D = \{b,d\} \subseteq U$, $U/R = \{\{a\},\{b\},\{c,d\}\}$, $\zeta_R(D) = \{\phi, U, \{b\}, \{c,d\}, \{b,c,d\}\}$ and the ideal $I = \{\phi, \{a\}\}$. Then

(1) The set $\{a,b,d\}$ is a *nano b-I-open set* but it is not nano *semi-I-open.*

(2) The set $\{a,b\}$ is a *nano b-I-open set* but it is not nano *pre-I-open and not nano* $\alpha - I - open$.

Lemma 3.7. *[9] Let A and B be subsets of U in a nano ideal topological space $(U, \zeta_R(D), I)$.*

a) If $A \subseteq B$, *then* $A^* \subseteq B^*$.

b) If $V \in \zeta_R(D)$, *then* $V \cap A^* \subseteq (V \cap A)^*$.

c) A^* *is nano closed in* $(U, \zeta_R(D))$.

Theorem 3.8. *Let $(U, \zeta_R(D), I)$ be an ideal topological space and A, B subsets of U.*

1) If $A_\alpha \in NbIO(U, \zeta_R(D))$ *for each* $\alpha \in \Delta$, *then* $\bigcup\{A_\alpha : \alpha \in \Delta\} \in NbIO(U, \zeta_R(D))$.

2) If $A \in NbIO(U, \zeta_R(D))$ *and* $B \in \zeta_R(D)$, *then* $A \cap B \in NbIO(U, \zeta_R(D))$.

Proof. 1) Since $A_\alpha \in NbIO(U, \zeta_R(D))$, we have

$$\begin{aligned}
\bigcup_{\alpha\in\Delta} A_\alpha &\subseteq \bigcup_{\alpha\in\Delta} [nCl^*(nInt(A_\alpha)) \cup nInt(nCl^*(A_\alpha))] \\
&\subseteq \bigcup_{\alpha\in\Delta} \{[(nInt(A_\alpha)) \cup (nInt(A_\alpha))^*] \cup [nInt(A_\alpha \cup A_\alpha^*)]\} \\
&\subseteq [nInt(\bigcup_{\alpha\in\Delta} A_\alpha) \cup (nInt(\bigcup_{\alpha\in\Delta} A_\alpha))^*] \cup [nInt((\bigcup_{\alpha\in\Delta} A_\alpha) \cup (\bigcup_{\alpha\in\Delta} A_\alpha)^*)] \\
&= nCl^*(nInt(\bigcup_{\alpha\in\Delta} A_\alpha)) \cup nInt(nCl^*(\bigcup_{\alpha\in\Delta} A_\alpha)).
\end{aligned}$$

Hence $\bigcup_{\alpha\in\Delta} A_\alpha \in NbIO(U, \zeta_R(D))$.

2) Let $A \in NbIO(U, \zeta_R(D))$ and $B \in \zeta_R(D)$. Then $A \subseteq nCl^*(nInt(A)) \cup nInt(nCl^*(A))$ and so

$$\begin{aligned}
A \cap B &\subseteq [nCl^*(nInt(A)) \cup nInt(nCl^*(A))] \cap B \\
&= [nCl^*(nInt(A)) \cap B] \cup [nInt(nCl^*(A)) \cap B] \\
&= [[nInt(A) \cup (nInt(A))^*] \cap B] \cup [nInt(A \cup A^*) \cap B] \\
&\subseteq [(nInt(A) \cap B) \cup ((nInt(A)) \cap B)^*] \cup [nInt[(A \cap B) \cup (A^* \cap B)]] \\
&\subseteq [(nInt(A) \cap B) \cup (nInt(A) \cap B)^*] \cup [nInt[(A \cap B) \cup (A \cap B)^*]] \\
&\subseteq [(nInt(A \cap B)) \cup (nInt(A \cap B))^*] \cup [nInt[(A \cap B) \cup (A \cap B)^*]] \\
&= nCl^*(nInt(A \cap B)) \cup nInt(nCl^*(A \cap B)).
\end{aligned}$$

This shows that $A \cap B \in NbIO(U, \zeta_R(D))$. $\square$

The following example shows that the finite intersection of nano b-I-open sets need not be nano b-I-open.

Example 3.9. Let $U = \{a, b, c, d\}$ be the universe, $D = \{b, d\} \subseteq U$, $U/R = \{\{a\}, \{b\}, \{c, d\}\}$, $\zeta_R(D) = \{\phi, U, \{b\}, \{c, d\}, \{b, c, d\}\}$ and the ideal $I = \{\phi, \{a\}\}$. Then the sets $A = \{a, b, d\}$ and $B = \{a, c, d\}$ are *nano b-I-open sets* but $A \cap B = \{a, d\}$ is not *nano b-I-open.*

4. Nano b-I-continuous functions

Definition 4.1. [8] A function $f : (U, \zeta_R(D), I) \to (V, \zeta_{R'}(E))$ is said to be *nano* $semi - I - continuous$ (resp. $nano\ pre - I - continuous$, $nano\ \alpha - I - continuous$) if $f^{-1}(B)$ is nano semi I-open (resp. nano pre I -open, nano α-I-open) set in $(U, \zeta_R(D), I)$ for every nano open set B in $(V, \zeta_{R'}(E))$.

Definition 4.2. A function $f : (U, \zeta_R(D), I) \to (V, \zeta_{R'}(E))$ is called $nano\ b - I - continuous$ if the inverse image of each nano open set in $(V, \zeta_{R'}(E))$ is a nano b-I-open set in $(U, \zeta_R(D), I)$.

Remark.

1) *Every nano continuous function is nano b-I-continuous.*

2) *Every nano semi-I-continuous function is nano b-I-continuous.*

3) *Every nano pre-I-continuous function is nano b-I-continuous.*

4) *Every nano α-I-continuous function is nano b-I-continuous.*

The converse in each part of the above remark need not be true as shown in the following three examples.

Example 4.3. Let $U = \{a, b, c, d\}$ be the universe, $D = \{a, d\} \subseteq U$, $U/R = \{\{a, d\}, \{b\}, \{c\}\}$, $\zeta_R(D) = \{\phi, U, \{a, d\}\}$ and the ideal $I = \{\phi, \{a\}\}$ and let $V = \{a, b, c\}$, $Y = \{a, c, d\} \subseteq V$, $V/R' = \{\{b\}, \{a, c\}, \{d\}\}$, $\zeta_{R'}(E)) = \{\phi, V, \{a, c, d\}\}$. Define $f : (U, \zeta_R(D), I) \to (V, \zeta_{R'}(E))$ by $f(a) = a$, $f(b) = b$, $f(c) = c$, $f(d) = d$. We note that, $f^{-1}(\{a, c, d\}) = \{a, c, d\}$ is a $nano\ b - I - open\ set\ but\ not\ nano\ open.$ Hence, f is nano b-I-continuous but not nano continuous.

Example 4.4. Let $U = \{a, b, c, d\}$ be the universe, $D = \{b, d\} \subseteq U$, $U/R = \{\{a, d\}, \{b\}, \{c\}\}$, $\zeta_R(D) = \{\phi, U, \{a, d\}, \{b\}, \{a, b, d\}\}$ and the ideal $I = \{\phi, \{a\}\}$ and let $V = \{a, b, c, d\}$, $E = \{a, b, d\} \subseteq V$, $V/R' = \{\{a\}, \{b, d\}, \{c\}\}$, $\zeta_{R'}(E)) = \{\phi, V, \{a, b, d\}\}$.

(1) Define $f : (U, \zeta_R(D), I) \to (V, \zeta_{R'}(E))$ by $f(a) = c$, $f(b) = b$, $f(c) = a$, $f(d) = d$. Then f is nano b-I-continuous but not nano semi I-continuous.

(2) Define $g : (U, \zeta_R(D), I) \to (V, \zeta_{R'}(E))$ by $g(a) = a$, $g(b) = c$, $g(c) = b$, $g(d) = d$. Then g is nano b-I-continuous but not nano pre I-continuous.

Example 4.5. Let $U = \{a, b, c, d\}$ be the universe, $D = \{a, b\} \subseteq U$, $U/R = \{\{a\}, \{b, d\}, \{c\}\}$, $\zeta_R(D) = \{\phi, U, \{a\}, \{b, d\}, \{a, b, d\}\}$ and the ideal $I = \{\phi, \{a\}\}$ and let $V = \{a, b, c\}$, $E = \{a\} \subseteq V$, $V/R' = \{\{a\}, \{b, c\}\}$, $\zeta_{R'}(E)) = \{\phi, V, \{a\}\}$. Define $f : (U, \zeta_R(D), I) \to (V, \zeta_{R'}(E))$ by $f(a) = f(b) = f(c) = a$, $f(d) = c$. Then f is *nano* $b - I - continuous$ but not *nano* $\alpha - I - continuous$.

Theorem 4.6. *For a function $f : (U, \zeta_R(D), I) \to (V, \zeta_{R'}(E))$ the following statements are equivalent:*

1) *f is nano b-I-continuous.*

2) *For each $x \in D$ and each $B \in \zeta_{R'}(E)$ with $f(x) \in B$, there exists $A \in NbIO(U, \zeta_R(D))$ with $x \in A$ such that $f(A) \subseteq B$.*

3) *The inverse image of each nano closed set in $(V, \zeta_{R'}(E))$ is nano b-I-closed in $(U, \zeta_R(D), I)$.*

Proof. Straightforward. □

Definition 4.7. Let $A \subseteq U$ in a nano ideal topological space $(U, \zeta_R(D), I)$ and $x \in U$. Then A is called a *nano* $b - I - neighborhood$ of x, if there exists a nano b-I-open set B containing x such that $B \subseteq A$.

Theorem 4.8. *For a function $f : (U, \zeta_R(D), I) \to (V, \zeta_{R'}(E))$, the following statements are equivalent:*

1) f is nano b-I-continuous.

2) For each $x \in U$ and each nano open set B in $(V, \zeta_{R'}(E))$ with $f(x) \in B$, $f^{-1}(B)$ is nano b-I-neighborhood of x.

Proof. (1) $\Rightarrow$ (2). Let $x \in U$ and let B be a nano open set in $(V, \zeta_{R'}(E))$ such that $f(x) \in B$. By Theorem 4.7, there exists a nano b-I-open set A in $(U, \zeta_R(D), I)$ with $x \in A$ such that $f(A) \subseteq B$. So $x \in A \subseteq f^{-1}(B)$. Hence $f^{-1}(B)$ is a nano b-I-neighborhood of x.

(2) $\Rightarrow$ (1). Let B be a nano open set in $(V, \zeta_{R'}(E))$ and let $f(x) \in B$. Then by assumption, $f^{-1}(B)$ is a nano b-I-neighborhood of x. Thus for each $x \in f^{-1}(B)$ there exists a nano b-I-open set A_x containing x such that $x \in A_x \subseteq f^{-1}(B)$. Hence $f^{-1}(B) = \cup\{A_x : x \in f^{-1}(B)\}$ and so $f^{-1}(B) \in NbIO(U, \zeta_R(D))$. □

Definition 4.9. A function $f : (U, \zeta_R(D), I) \to (V, \zeta_{R'}(E))$ is called nano $b - I - irresolute$ if $f^{-1}(B)$ is nano b-I-open in $(U, \zeta_R(D), I)$ for every nano b-I-open set B in $(V, \zeta_{R'}(E))$.

Theorem 4.10. *Let $f : (U, \zeta_R(D), I) \to (V, \zeta_{R'}(E), J)$ and $g : (V, \zeta_{R'}(E), J) \to (W, \zeta_{R''}(Z), K)$ then*

1) gof is nano b-I-continuous if f is nano b-I-continuous and g is nano continuous.

2) gof is nano b-I-continuous if f is nano b-I-irresolute and g is nano b-I-continuous.

If $(U, \zeta_R(D), I)$ is a nano ideal topological space and A is a subset of U, we denote by $\zeta_R(D)_{|A}$ the relative nano topology on A and $I_{|A} = \{A \cap B : B \in I\}$ is obviously an ideal on A.

The proofs of the following two lemmas is similar to the proofs of Lemma 3.14 and Lemma 3.15 in [18].

Lemma 4.11. *Let $(U, \zeta_R(D), I)$ be a nano ideal topological space and A, B be subsets of U such that $B \subseteq A$. Then $nB^*(\zeta_R(D)_{|A}, I_{|A}) = nB^*(\zeta_R(D), I) \cap A$.*

Lemma 4.12. *Let $(U, \zeta_R(D), I)$ be a nano ideal topological space, $A \subseteq U$ and $W \in \zeta_R(D)$. Then $nCl^*(A) \cap W = nCl^*_W(A \cap W)$.*

Theorem 4.13. *Let $(U, \zeta_R(D), I)$ be a nano ideal topological space, $A \subseteq W \in \zeta_R(D)$. If $A \in NbIO(U, \zeta_R(D))$ then $A \in NbIO(W, \zeta_R(D)_{|W}, I_{|W})$.*

Proof. Since $W \in \zeta_R(D)$ and $A \in NbIO(U, \zeta_R(D))$, we have

$$\begin{aligned} A = W \cap A &\subseteq W \cap [nCl^*(nInt(A)) \cup nInt(nCl^*(A))] \\ &= [W \cap (nCl^*(nInt(A)))] \cup [W \cap (nInt(nCl^*(A)))] \\ &\subseteq nCl^*(W \cap nInt(A)) \cup (W \cap nInt(nCl^*(A))) \\ &= nCl^*(Int(W \cap A)) \cup nInt(W \cap nCl^*(A)) \\ &= nCl^*(nInt_W(W \cap A)) \cup nInt_W(W \cap nCl^*(A)). \end{aligned}$$

Since $W \in \zeta_R(X) \subseteq \zeta_R(X)^*$, we obtain

$$\begin{aligned} A = W \cap A &\subseteq W \cap [nCl^*(nInt_W(W \cap A)) \cup nInt_W(W \cap nCl^*(A))] \\ &= [W \cap (nCl^*(nInt_W(W \cap A)))] \cup [W \cap (nInt_W(W \cap nCl^*(A)))] \\ &= nCl^*_W(nInt_W(W \cap A)) \cup nInt_W(W \cap nCl^*(A)) \\ &= nCl^*_W(nInt_W(A)) \cup nInt_W(nCl^*_W(A)). \end{aligned}$$

Then $A \in NbIO(W, \zeta_R(X)_{|W}, I_{|W})$. □

Corollary 4.14. *Let $(U, \zeta_R(D), I)$ be a nano ideal topological space, $W \in \zeta_R(D)$ and $A \in NbIO(U, \zeta_R(D))$, then $W \cap A \in NbIO(U, \zeta_{|U}, I_{|U})$.*

Proof. Since $W \in \zeta_R(D)$ and $A \in NbIO(U, \zeta_R(D))$, by Theorem 3.8, $W \cap A \in NbIO(U, \zeta_R(D))$. Since $W \in \zeta_R(D)$, by Theorem 4.14, $W \cap A \in NbIO(W, \zeta_R(D)_{|W}, I_{|W})$. □

Theorem 4.15. *Let $f : (U, \zeta_R(D), I) \to (V, \zeta_{R'}(E))$ be a nano $b-I-$continuous function and $W \in \zeta_R(D)$. Then the restriction $f_{|W} : (W, \zeta_R(D)_{|W}, I_{|W}) \to (V, \zeta_{R'}(E))$ is nano $b - I -$ continuous.*

Proof. Let G be any nano open set in $(V, \zeta_{R'}(E))$. Since f is nano $b-I-$ *continuous*, we have $f^{-1}(G) \in NbIO(U, \zeta_R(D))$. Since $W \in \zeta_R(D)$, by Theorem 4.14, we have $W \cap f^{-1}(G) \in NbIO(W, \tau_R(D)_{|W}, I_{|W})$. On the other hand, $(f_{|W})^{-1}(G) = W \cap f^{-1}(G)$ and $(f_{|W})^{-1}(G) \in NbIO(W, \tau_R(D)_{|W}, I_{|W})$. This shows that $f_{|W} : NbIO(W, \zeta_R(D)_{|W}, I_{|W}) \to (V, \zeta_{R'}(E))$ is nano $b-I-$ *continuous*. □

Definition 4.16. A nano ideal topological space $(U, \zeta_R(D), I)$ is said to be *nano* $b-I-$ *normal* if for each pair of non-empty disjoint nano closed subsets A and B of U, there exist two nano b-I-open subsets G and W of U such that $A \subseteq G$, $B \subseteq W$ and $G \cap W = \phi$.

Theorem 4.17. *If $f : (U, \zeta_R(D), I) \to (V, \zeta_{R'}(E))$ is nano b-I-continuous, nano closed injection and V is nano normal, then $(U, \zeta_R(D), I)$ is nano b-I-normal.*

Proof. Let A and B be two disjoint nano closed subsets of U. Since f is nano closed and injective, $f(A)$ and $f(B)$ are disjoint nano closed subsets of $(V, \zeta_{R'}(E))$. Since V is nano normal, there exist two nano open subsets G and W of V such that $f(A) \subseteq G$, $f(B) \subseteq W$ and $G \cap W = \phi$. Now $f^{-1}(G)$ and $f^{-1}(W)$ are nano b-I-open in U with $A \subseteq f^{-1}(G)$, $B \subseteq f^{-1}(W)$ and $f^{-1}(G) \cap f^{-1}(W) = \phi$. Thus $(U, \zeta_R(D), I)$ is nano b-I-normal. □

Definition 4.18. A nano ideal topological space $(U, \zeta_R(D), I)$ is said to be *nano* $b-I-$ *connected* if U can't be written as a union of two disjoint nano b-I-open subsets of D.

Theorem 4.19. *The nano b-I-continuous image of a nano b-I-connected space is nano connected.*

Proof. Let $f : (U, \zeta_R(D), I) \to (V, \zeta_{R'}(E))$ be a nano b-I-continuous function of a nano b-I-connected space $(U, \zeta_R(D), I)$ onto a nano topological space $(V, \zeta_{R'}(E))$. Assume that V is not nano connected, then $V = A \cup B$ where A and B are non-empty nano clopen with $A \cap B = \phi$. Since f is nano b-I-continuous, $f^{-1}(A)$ and $f^{-1}(B)$ are non-empty nano b-I-open in U. Also, $U = f^{-1}(V) = f^{-1}(A \cup B) = f^{-1}(A) \cup f^{-1}(B)$ and $f^{-1}(A) \cap f^{-1}(B) = \phi$. Hence U is not nano b-I-connected which is a contradiction. Therefore, V is nano connected. □

5. Nano b-I-open functions

Recall that a subset F of a nano ideal topological space $(U, \zeta_R(D), I)$ is said to be *nano* $semi-I-closed$ [*26*] if its complement is nano semi-I-open.

Definition 5.1. A function $f : (U, \zeta_R(D)) \to (V, \zeta_{R'}(E), I)$ is called *nano* $semi-I-open$ (resp.,*nano* $semi-I-closed$) if the image of every nano open (resp., nano closed) set in $(U, \zeta_R(D))$ is nano semi-I-open (resp., nano semi-I-closed) in $(V, \tau_{R'}(E), I)$.

Definition 5.2. A function $f : (U, \zeta_R(D)) \to (V, \zeta_{R'}(E), I)$ is called *nano* $b-I-open$ (resp., *nano* $b-I-closed$) if the image of every nano open (resp., nano closed) set in $(U, \zeta_R(D))$ is nano b-I-open (resp., nano b-I-closed) in $(V, \zeta_{R'}(E), I)$.

Remark. *Every nano semi-I-open (resp., nano semi-I-closed) function is nano b-I-open (resp., nano b-I-closed).*

The converse of the above remark need not be true as shown in the following example.

Example 5.3. Let $U = \{a, b, c, d\}$ be the universe, $D = \{a, b, d\} \subseteq U$, $U/R = \{\{a\}, \{b, d\}, \{c\}\}$, $\zeta_R(D) = \{\phi, U, \{a, b, d\}\}$ and let $V = \{a, b, c, d\}$, $E = \{b, d\} \subseteq V$, $V/R' = \{\{a, d\}, \{b\}, \{c\}\}$, $\zeta_{R'}(E)) = \{\phi, V, \{a, d\}, \{b\}, \{a, b, d\}\}$ and the ideal $I = \{\phi, \{a\}\}$. Define $f : (U, \zeta_R(D)) \to (V, \zeta_{R'}(E), I)$ by $f(a) = c$, $f(b) = b$, $f(c) = a$, $f(d) = d$. Then f is nano b-I-open but not nano semi I-open.

Theorem 5.4. *A function* $f : (U, \zeta_R(D)) \to (V, \zeta_{R'}(E), I)$ *is nano b-I-open if and only if for each* $x \in U$ *and each nano neighborhood* W *of* x *there exists* $G \in NbIO(V, \zeta_{R'}(E))$ *containing* $f(x)$ *such that* $G \subseteq f(W)$.

Proof. $\Rightarrow$) Suppose that f is a nano $b-I-open$ function. For each $x \in U$ and each nano neighborhood W of x, there exists $W_x \in \zeta_R(D)$ such that $x \in W_x \subseteq W$. Let $G = f(W_x)$. Since f is nano b-I-open, $G \in NbIO(V, \zeta_{R'}(E))$ and $f(x) \in G \subseteq f(W)$.

$\Leftarrow$) Let W be a nano open set in $(U, \zeta_R(D))$. For each $x \in W$, there exists $G_x \in NbIO(V, \zeta_{R'}(E))$ such that $f(x) \in G_x \subseteq f(W)$. Now $f(W) = \cup\{G_x : x \in W\}$ and so $f(W) \in NbIO(V, \zeta_{R'}(E))$. This shows that f is nano b-I-open. □

Theorem 5.5. *Let* $f : (U, \zeta_R(D)) \to (V, \zeta_{R'}(E), I)$ *be a nano b-I-open function. If* G *is any subset of* V *and* C *is a nano closed subset of* U *with* $f^{-1}(G) \subseteq C$*, then there exists a nano b-I-closed subset* H *of* V *with* $G \subseteq H$ *such that* $f^{-1}(H) \subseteq C$.

Proof. Suppose that f is a nano b-I-open function. Let G be any subset of V and C a nano closed subset of U with $f^{-1}(G) \subseteq C$. Then $U - C$ is nano open. Since f is nano b-I-open, $f(U - C)$ is nano b-I-open in V. Let $H = V - f(U - C)$. Then H is nano b-I-closed in V. Since $f^{-1}(G) \subseteq C$, $G \subseteq H$. Also, we obtain $f^{-1}(H) \subseteq C$. □

Theorem 5.6. *Let* $f : (U, \zeta_R(D)) \to (V, \zeta_{R'}(E), I)$ *be nano b-I-closed. If* G *is any subset of* V *and* W *is a nano open subset of* U *with* $f^{-1}(G) \subseteq W$*, then there exists a nano b-I-open subset* H *of* V *with* $G \subseteq H$ *such that* $f^{-1}(H) \subseteq W$.

Proof. Similar to that used in Theorem 5.6. □

Theorem 5.7. *For any bijective function* $f : (U, \zeta_R(D)) \to (V, \zeta_{R'}(E), I)$*, the following are equivalent:*

1) $f^{-1} : (V, \zeta_{R'}(E), I) \to (U, \zeta_R(D))$ *is nano b-I-continuous.*

2) f *is nano b-I-open.*

3) f *is nano b-I-closed.*

Proof. It is straightforward. $\square$

Theorem 5.8. *Let* $f : (U, \zeta_R(D)) \rightarrow (V, \zeta_{R'}(E), I)$ *and* $g : (V, \zeta_{R'}(E), I) \rightarrow (W, \zeta_{R"}(Z), K)$.

1) gof is nano b-I-open if f is nano open and g is a nano b-I-open.

2) f is nano b-I-open if gof is nano open and g is a nano b-I-continuous injection.

References

[1] D. Andrijević, *On b-open sets*, Mat. Vesnik. 48 (1996) 59–64.

[2] A.A. Atoom, F. Bani-Ahmad, *Between pairwise -$\alpha-$ perfect functions and pairwise -t–$\alpha-$ perfect functions*, Journal of Applied Mathematics and Informatics 42 (1) (2024) 15–29.

[3] A.A. Atoom, H. Qoqazeh, R. Alrababah, E. Almuhur, N. Abu-Alkishik, *Significant modification of pairwise omega continuous functions with associated concepts*, Wseas Transcations on Mathematics (2023). DOI: 10.37394/23206.2023.22.105.

[4] S.G. Crossley, S.K. Hildebrand, *Semi-closure*, Texas J. Sci. 22 (1971) 99–112.

[5] J. Dontchev, *On pre-I-open sets and a decomposition of I-continuity*, Banyan Math. J. 2 (1996).

[6] A. Caksu Guler, G. Aslim, *b-I-open sets and decomposition of continuity via idealization*, Proceedings of Institute of Mathematics and Mechanics. National Academy of Sciences of Azerbaijan 22 (2005) 27–32.

[7] E. Hatir, T. Noiri, *On decomposition of continuity via idealization*, Acta Math. Hungar. 96 (2002) 341–349.

[8] V. Inthumathi, R. Abinprakash, M. Parveen Banu, *Some weaker forms of continuous and irresolute mappings in nano ideal topological spaces*, Journal of New Results in Science (JNRS) 8 (1) (2019) 14–25.

[9] D.S. Janković, T.R. Hamlett, *New topologies from old via ideals*, Amer. Math. Monthly 97 (4) (1990) 295–310.

[10] K. Kuratowski, *Topology*, Vol. I, Academic Press, New York, 1966.

[11] N. Levine, *Semi-open sets and semi-continuity in topological spaces*, Amer. Math. Monthly 70 (1963) 36–41.

[12] A.S. Mashhour, M.E. Abd El-Monssef, S.N. El-Deeb, *On precontinuous and weak precontinuous mappings*, Proc. Math. Phys. Soc. Egypt 53 (1982) 47–53.

[13] J.M. Mustafa, *Totally supra b-continuous and slightly supra b-continuous functions*, Studia Universitaties Babes-Bolyai Mathematica 57 (1) (2012) 135–144.

[14] J.M. Mustafa, *Supra b-compact and supra b-Lindelof spaces*, Journal of Mathematics and Applications 36 (2013) 79–83.

[15] J.M. Mustafa, *Supra soft b-compact and supra soft b-Lindelof spaces*, Journal of Advances in Mathematics 16 (2019) 8376–8383.

[16] J.M. Mustafa, *Weakly nano semi-I-open sets and weakly namo semi-I-continuous functions*, Poincare Journal of Analysis and Applications 10 (1) (2023) 75–85.

[17] R.L. Newcomb, *Topologies which are Compact Modulo Anideal, Ph.D. dissertation*, University of California, Santa Barbara, Calif., USA, 1967.

[18] M. Parimala, S. Jafari, *On some new notions in nano ideal topological spaces*, International Balkan Journal of Mathematics 1 (3) (2018) 85–92.

[19] I. Rajasekaran, O. Nethaji, *Simple forms of nano open sets in an ideal nano topological spaces*, Journal of New Theory 24 (2018) 35–43.

[20] P. Sathishmohan, V. Rajendran, A. Devika, R. Vani, *On nano semi continuity and nano pre continuity*, International Journal of Applied Research 3 (2) (2017) 76–79.

[21] M.L. Thivagar, V.S. Devi, *New sort of operator in nano ideal topology*, Ultra Scientist 28 (1)A (2016) 51–64.

[22] M.L. Thivagar, S. Jafari, V.S. Devi, *On new class of contra continuity in nano topology*, Italian Journal of Pure and Appl. Math. 41 (2017) 1–12.

[23] M.L. Thivagar, C. Richard, *On nano forms of weakly open sets*, International Journal of Mathematics and Statistics Invention 1 (1) (2013) 31–37.

[24] M.L. Thivagar, C. Richard, *On nano continuity*, Mathematical Theory and Modelling 3 (7) (2013) 32–37.

[25] R. Vaidyanathaswamy, *The localization theory in set topology*, Proc. Indian Acad. Sci. 20 (1945) 51–61.

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On a Unified Form of Fractional Volterra-type Integro-Differential Equations and its applications

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ABSTRACT: In the present article, an extension for the family of Volterra-type integro-differential equations, involving a generalization of Hilfer fractional derivative with the Lorenzo-Hartley's G-function (LHGF) in the kernel, is proposed. A compact and computable solution of the considered family of integro-differential equations is established in terms of an infinite series of LHGF. Further, certain known and new special cases of the proposed family are also established. Furthermore, some examples of the integro-differential equation are also discussed. Moreover, from the application point of view, generalized fractional free-electron laser equations involving the Caputo and the Riemann-Liouville fractional derivatives are also determined. Finally, the graphical illustrations for the solutions of the studied generalized fractional free-electron laser equations are demonstrated.

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1. Introduction

The study of fractional calculus (FC) is gaining popularity in the scientific community. It is applied to analyze several complicated phenomena in applied sciences. Several fractional-order models are explored in the recent past that characterize the multi-faceted behavior of a number of systems with complex dynamics. As an emerging area the subject has a wide variety of applications in different fields such as Medical Sciences, Space Sciences, Statistical mechanics, Control systems, Nuclear-physics, Thermal power, Finance and Material sciences, etcetera. Substantial development for the profound understanding of fractional calculus is noticed in the last few decades. For some recent and new real-world applications of fractional calculus, we refer to Sun at al. [62]. To review and insightful study of different concepts of fractional calculus we refer to standard monographs such as [6, 7, 15, 24, 30, 32, 37, 39, 41, 42, 44, 48, 54].

Special functions (SFs) are widely used in mathematics [8]. Mathematicians and applied scientists have made lots of efforts for the development of SFs while seeking an exhaustive and unified theory for the subject matter. In most of the cases, classical SFs are emerged as solutions of ordinary differential equations/partial differential equations and represented in terms of series, or integrals, or in both [51, 52]. Several classical SFs are useful in applied analysis and may be represented as particular cases of generalized special functions (GSFs), such as Fox's H-functions, Meijer's G-functions and generalized hypergeometric functions ${}_pF_q$, etcetera (we may call them generalized classical functions). For a more detailed description of classical SFs, we refer to classical monographs [1, 51, 52].

From the available corpus of classical SFs some may be referred as Special functions for fractional calculus (SFs for FC) [32]. Most often, Special functions for fractional calculus appear in the solution of arbitrary order differential equation or may arise during modelling of complex physical systems, see [22, 29, 34, 38, 50] etcetera. FC, in general, consists of differentiation and integration of arbitrary order and involves differential and integral operator of fractional order. The development of the theory of fractional calculus is largely dependent on the development of functions for the fractional calculus [35]. Thus, one may expect that exploration about generalized functions for fractional calculus may contribute towards the establishment of a unified theory of fractional calculus. We believe that such generalizations of fractional calculus may also provide a coherent methodology for analysis and applications. Generalized fractional integral and derivative operators are generally introduced by the suitable choice of functions that appeared in the kernel by more generalized functions, particularly for more details we refer [16, 18, 19, 20, 21, 25, 26, 27, 28, 29, 49, 50, 56, 60, 61, 63]. One can believe that the future growth in the theory of fractional calculus as the generalized fractional calculus would be an outcome of the manifestation of generalized special functions in different branches of science.

Fractional-order integro-differential equations are observed frequently in modelling and analysis of physical systems, see [3, 13, 14, 55, 56]. For more background, we refer to [43] and references therein. The present paper is about the applications of generalized fractional operators and generalized functions for fractional calculus to determine a unification of several fractional-order integro-differential equations that arise in applied sciences. The work presented in this paper is inspired by the remarkable contributions of other researchers (see [3, 4, 9, 10, 13, 14, 28, 43, 55]).

We present a brief description of different classical and novel fractional calculus operators and introduce the Lorenzo-Hartley's G-function (say LHGF) in the current section. In Section 2, we propose a unification for family of fractional-ordered integro-differential equations including a generalized fractional function in the kernel and a generalized fractional derivative operator (i.e., the Hilfer-Prabhakar derivative). Next, we investigate the convergence of the obtained solution for further computational requirements. Further, some of the corollaries of Theorem 2.1 are derived in the next Section 3. For the applications of the derived unification, two examples are discussed in Section 4. Furthermore, in Section 5, solutions for two generalized fractional free-electron laser equations, involving Caputo and Riemann-Liouville derivatives respectively, are determined. Moreover, some graphical illustrations for the considered generalized fractional free-electron laser equations are demonstrated in the same Section 5. Finally, in Section 6 we present some concluding remarks.

1.1. Riemann-Liouville fractional-order derivative

If $h(t)$, where $-\infty \leq a < t < b \leq \infty$, is locally integrable real-valued function in $\mathcal{L}^1[a,b]$, then the $\mu^{th}(\mu \in \mathbb{C})$ order right-sided Riemann-Liouville fractional integral of $h(t)$ is denoted by ${}^{\mathsf{RL}}\mathbf{I}^{\mu}_{a+}h$ and defined as [42, 48, 54]:

$$({}^{\mathsf{RL}}\mathbf{I}^{\mu}_{a+}h)(t) = \frac{1}{\Gamma(\mu)}\int_a^t \frac{h(u)}{(t-u)^{1-\mu}}du = (h * \mathcal{F}_\mu)(t), \tag{1}$$

with the condition that $(t > 0; \mathfrak{Re}(\mu) > 0)$. The expression $\mathcal{F}_\mu(t)$ is given by $\mathcal{F}_\mu(t) = \dfrac{t^{\mu-1}}{\Gamma(\mu)}$.

If $h(t) \in \mathcal{L}^1[a,b]$, where $-\infty < a < t < b < \infty$ and $h * \mathcal{F}_{m-\mu} \in \mathcal{W}^{m,1}[a,b]$, $m = \lceil \mu \rceil$, $\mu > 0$, where $\lceil \cdot \rceil$ is the least integer function. Also, $\mathcal{W}^{m,1}[a,b]$ is used to denote the Sobolev space defined as:

$$\mathcal{W}^{m,1}[a,b] = \Big\{ h(t) \in \mathcal{L}^1[a,b] : \frac{d^m}{dt^m}h(t) \in \mathcal{L}^1[a,b] \Big\}. \tag{2}$$

The classical Riemann-Liouville right-sided fractional derivative of order μ ($\mu \in \mathbb{C}$, $\mathfrak{Re}(\mu) > 0$) is defined as:

$$({}^{\mathsf{RL}}\mathbf{D}^{\mu}_{a+}h)(t) = \Big(\frac{d}{dt}\Big)^m \Big(({}^{\mathsf{RL}}\mathbf{I}^{m-\mu}_{a+}h)(t)\Big) = \frac{1}{\Gamma(m-\mu)}\frac{d^m}{dt^m}\int_a^t (t-u)^{m-\mu-1}h(u)du, \tag{3}$$

with $m = -[-\mathfrak{Re}(\mu)]$, where $[\cdot]$ denotes the integral part of the argument, i.e.

$$m = \begin{cases} [\mathfrak{Re}(\mu)] + 1 & \text{for } \mu \notin \mathbb{N}_0, \\ \mu & \text{for } \mu \in \mathbb{N}_0. \end{cases} \tag{4}$$

Particularly for $\mu = m \in \mathbb{N}_0$, we write

$$({}^{\mathsf{RL}}\mathbf{D}^{\mu}_{a+}h)(t) = h^{(m)}(t), \tag{5}$$

where $h^{(m)}(t)$ is the standard m^{th} order derivative of the function $h(t)$.

If $\mathcal{AC}[a,b]$ is the space of absolutely continuous functions and $h(t)$ be the real-valued functions with continuous derivative up to order $(m-1)$ on the interval $[a,b]$ such that $h^{m-1}(t) \in \mathcal{AC}[a,b]$, we say that function $h(t) \in \mathcal{AC}^m[a,b] (m \in \mathbb{N})$. The space $\mathcal{AC}^m[a,b]$ of real-valued function is given as:

$$\mathcal{AC}^m[a,b] = \Big\{ h : [a,b] \to \mathbb{R} : \frac{d^{m-1}}{dt^{m-1}} h(t) \in \mathcal{AC}[a,b] \Big\}. \tag{6}$$

1.2. Caputo fractional-order derivative

The Caputo fractional derivative of a function $h(t)$, denoted by ${}^{\mathsf{C}}\mathbf{D}^{\mu}_{a+}h(t)$, has a close connection with Riemann-Liouville fractional derivative ${}^{\mathsf{RL}}\mathbf{D}^{\mu}_{a+}h(t)$ (see [15, 39, 41, 42]).

If $h(t) \in \mathcal{AC}^m[a,b]$, $\mu \in \mathbb{C}$ $(\mathfrak{Re}(\mu) > 0), m = \lceil \mu \rceil$ then the right-sided μ^{th} order Caputo fractional derivative of $h(t)$ is defined as:

$${}^{\mathsf{C}}\mathbf{D}^{\mu}_{a+}h(t) = \Big({}^{\mathsf{RL}}\mathbf{I}^{m-\mu}_{a+} \frac{d^m}{dt^m} h \Big)(t) = \frac{1}{\Gamma(m-\mu)} \int_a^t (t-u)^{m-\mu-1} \frac{d^m}{du^m} h(u) du. \tag{7}$$

The study of generalized fractional-order derivatives, being part of the investigation of several researchers [19, 24, 27, 28, 29, 31, 34, 40, 55, 56, 59, 60, 61, 63], are of great need as such generalized fractional derivatives play a vital role in the justification of various phenomena in different complex systems. Now we consider some of the popular generalizations of the above-defined classical fractional derivatives.

1.3. Hilfer derivative

If $h(t) \in \mathcal{L}^1[a,b], h * \mathcal{F}_{(1-\mu)(1-\nu)}(\cdot) \in \mathcal{AC}^1[a,b]$ with the restrictions $-\infty \le a < t < b \le \infty, \mu \in (0,1)$ and $\nu \in [0,1]$, then the right-sided Hilfer fractional-order derivative of $h(t)$, symbolically denoted by $({}^{\mathsf{H}}\mathbf{D}^{\mu,\nu}_{a+}h)(t)$, is defined as [24, 25, 26, 27, 30, 63]:

$$({}^{\mathsf{H}}\mathbf{D}^{\mu,\nu}_{a+}h)(t) = \Big({}^{\mathsf{RL}}\mathbf{I}^{\nu(1-\mu)}_{a+} \frac{d}{dt} {}^{\mathsf{RL}}\mathbf{I}^{(1-\nu)(1-\mu)}_{a+} h \Big)(t). \tag{8}$$

For $\nu = 0$, the derivative ${}^{\mathsf{H}}\mathbf{D}^{\mu,\nu}_{a+}$ reduces into the classical Riemann-Liouville fractional-order derivative ${}^{\mathsf{RL}}\mathbf{D}^{\mu}_{a+}$. Also on taking $\nu = 1$ the derivative ${}^{\mathsf{H}}\mathbf{D}^{\mu,\nu}_{a+}$ becomes Caputo fractional-order derivative [33].

1.4. Prabhakar integral

If $h \in \mathcal{L}^1(a,b), 0 \le a < t \le b \le \infty$, then the right-sided Prabhakar integral ${}^{\mathsf{P}}\mathbf{E}^{\gamma}_{\rho,\mu,\omega,a^+}$ of the function $h(t)$ is given as [29, 49, 50]:

$$({}^{\mathsf{P}}\mathbf{E}^{\gamma}_{\rho,\mu,\omega,a^+}h)(t) = h * e^{\gamma}_{\rho,\mu,\omega}(t) = \int_a^t (t-u)^{\mu-1} E^{\gamma}_{\rho,\mu}(\omega(t-u)^{\rho}) h(u) du, \tag{9}$$

with $\gamma, \rho, \mu, \omega \in \mathbb{C}$ and $\mathfrak{Re}(\rho) > 0, \mathfrak{Re}(\mu) > 0$. The symbol $e^{\gamma}_{\rho,\mu,\omega}(t)$ in above Eq. (9) is $t^{\mu-1}E^{\gamma}_{\rho,\mu}(\omega t^{\rho})$, and $E^{\gamma}_{\rho,\mu}(\cdot)$ (where $(\cdot)$ denotes argument of the function) is the generalized Mittag-Leffler function, for more details, see [50]. If we take $\gamma = 0$, the integral operator ${}^{\mathsf{P}}\mathbf{E}^{\gamma}_{\rho,\mu,\omega,a^+}$ reduces into the Riemann-Liouville fractional-order integral operator (see Eq. (1)).

1.5. Prabhakar derivative

The Prabhakar derivative is defined as the inverse operator of Prabhakar integral. It is a generalization of the classical Riemann-Liouville derivative. If $h \in \mathcal{L}^1(a,b)$, $0 \le a < t \le b \le \infty$, and $h * e^{-\gamma}_{\rho,m-\mu,\omega}(\cdot) \in \mathcal{W}^{m,1}(a,b), m = \lceil \mu \rceil$, the right-sided Prabhakar derivative ${}^{\mathsf{P}}\mathbf{D}^{\gamma}_{\rho,\mu,\omega,a^+}$ of a function $h(t)$ is given as [29, 49, 50]:

$$({}^{\mathsf{P}}\mathbf{D}^{\gamma}_{\rho,\mu,\omega,a^+}h)(t) = \Big(\frac{d^m}{dt^m}({}^{\mathsf{P}}\mathbf{E}^{-\gamma}_{\rho,m-\mu,\omega,a^+}h)\Big)(t), \tag{10}$$

where $\gamma, \rho, \mu, \omega \in \mathbb{C}$ and $\mathfrak{Re}(\rho) > 0, \mathfrak{Re}(\mu) > 0$.

1.6. Regularized Prabhakar derivative

The regularized Prabhakar derivative operator can be considered as a generalization of Caputo fractional derivative operator. If $h(t) \in \mathcal{AC}^m(a,b), 0 \le a < t \le b \le \infty$, the regularized Prabhakar derivative is defined as [49]:

$$({}^{\mathsf{C}}\mathbf{D}^{\gamma}_{\rho,\mu,\omega,a^+}h)(t) = \Big({}^{\mathsf{P}}\mathbf{E}^{-\gamma}_{\rho,m-\mu,\omega,a^+}\frac{d^m}{dt^m}h\Big)(t). \tag{11}$$

On substituting $\gamma = 0$ the derivative $({}^{\mathsf{C}}\mathbf{D}^{\gamma}_{\rho,\mu,\omega,a^+}h)(t)$ reduces into Caputo derivative ${}^{\mathsf{C}}\mathbf{D}^{\mu}_{a+}h(t)$, defined by Eq. (7) in the subsection 1.2.

1.7. Hilfer-Prabhakar derivative

The Hilfer-Prabhakar derivatives (also known as the generalized Hilfer derivative) is emerging as a useful differential operator, particularly in mathematical physics and other branches of applied mathematics. Garra et al. [19] have described the dynamics of the generalized renewal stochastic process and some other classical equations of mathematical physics in terms of Hilfer-Prabhakar derivatives.
If $h \in \mathcal{L}^1(a,b), h * e^{-\gamma(1-\nu)}_{\rho,(1-\nu)(1-\mu),\omega}(\cdot) \in \mathcal{AC}^1(a,b)$ with the restrictions $0 \le a < t \le b \le \infty$, $\mu \in (0,1)$, and $\nu \in [0,1]$, then the Hilfer-Prabhakar derivative is defined as [19]:

$$({}^{\mathsf{HP}}\mathbf{D}^{\gamma,\mu,\nu}_{\rho,\omega,a^+}h)(t) = \Big({}^{\mathsf{P}}\mathbf{E}^{-\gamma\nu}_{\rho,\nu(1-\mu),\omega,a^+}\frac{d}{dt}({}^{\mathsf{P}}\mathbf{E}^{-\gamma(1-\nu)}_{\rho,(1-\mu)(1-\nu),\omega,a^+}h)\Big)(t), \tag{12}$$

where $\omega, \rho, \gamma \in \mathbb{C}$ with $\mathfrak{Re}(\rho) > 0$. Particularly, If we put $\gamma = 0$, the Hilfer-Prabhakar derivative becomes the Hilfer derivative given in above Eq. (8). The remarkable property of the Hilfer-Prabhakar derivative is that it interpolates between the Prabhakar derivative and its regularized version, given in Eq. (10) and Eq. (11), respectively.

1.8. The Lorenzo-Hartley's function

Special functions arise ubiquitously in solutions of fractional differential equations. The Agarwal's function, Mittag-Leffler functions (with one, two & three parameters), Erdélyi's function, Robotnov & Hartley's function, Miller & Ross's function are some of the appear naturally in the solution of various differential equations of integer and non-integer orders. Lorenzo and Hartley [34] investigated a multivalued generalization of standard exponential function known as Lorenzo-Hartley's G-function (LHGF), denoted as $G_{\{\rho,\beta,\delta\}}(\omega, v, t)$. Being an eigenfunction, all the order fractional differintegrals of LHGF appear in terms of LHGF (with suitably modified parameters). In a recent monograph [35], it is shown that such generalized functions have a great potential in investigations of scientific applications pertaining to Galactic classification, Shell morphology, Weather prediction, etcetera. The infinite series representation of LHGF given as:

$$G_{\{\rho,\beta,\delta\}}(\omega, v, t) = \sum_{k=0}^{\infty} \frac{(\delta)_k \omega^k (t-v)^{(k+\delta)\rho-\beta-1}}{k!\Gamma((k+\delta)\rho-\beta)}, \text{ with } \mathfrak{Re}(\rho\delta-\beta) > 0, \tag{13}$$

where $(\delta)_k$ is the generalization of factorial (also known as rising factorial or Pochhammer's symbol), is defined as:

$$(\delta)_0 = 1, (\delta)_1 = \delta, (\delta)_2 = (\delta)(\delta+1), ..., (\delta)_n = (\delta)(\delta+1)...(\delta+n-1). \tag{14}$$

On substituting $v = 0$ Eq. (13) reduces in to following convenient form:

$$G_{\{\rho,\beta,\delta\}}(\omega, 0, t) = G_{\{\rho,\beta,\delta\}}(\omega, t) = \sum_{k=0}^{\infty} \frac{(\delta)_k \omega^k t^{(k+\delta)\rho-\beta-1}}{k!\Gamma((k+\delta)\rho-\beta)}, \text{ with } \mathfrak{Re}(\rho\delta-\beta) > 0. \tag{15}$$

A number of functions have direct and elegant relationships with the LHGF $G_{\{\rho,\beta,\delta\}}(\omega, b, t)$, for more details one can refer recent investigation [46].

A fractional function LHGF is gaining importance in applications and analysis as it can handle increased time-domain complexity. In [64] Yang has discussed generalized fractional derivatives and integrals involving LHGFs (of one and two parameters) in the kernel and illustrates some applications in applied sciences. In a most recent monograph, Yang et al. [65] have demonstrated applications of such fractional operators for the investigations of models pertaining to viscoelasticity. Chaurasia and Pandey [11, 12] have extended the work of Haubold and Mathai [23] and studied computable closed-forms of some generalized fractional kinetic equations in terms of LHGF. Saxena et al. [57] have used LHGF in the investigation of generalized fractional kinetic equations. Goufo [17] have applied this function in the study related to bio-mathematical analysis associated with cellulose degradation dynamics. For some more details about LHGF one can also refer to Mahmood et al. [36], Saha et al. [53], Shakeel et al. [58] and recent investigations by Pandey [45, 46] .

2. A unification of fractional Volterra-type integro-differential equations

In this section, after presenting a chronological review pertaining to the development of a Volterra-type fractional integro-differential equation (FIDE), we propose a unified family of Volterra-type fractional integro-differential equations involving LHGF in the kernel and a generalized Hilfer derivative. It is emphasized that the solution obtained is also represented in a closed-form of LHGF. For the sake of simplified computations via LHGF we have assumed that the order of fractional derivative lies between 0 and 1. Moreover, we discuss the convergence of the solution by the method recently used by Giusti and Colombaro in [20] during the investigation of a generalized Viscoelastic model.

Dattoli et al. [14] studied the following first-order integro-differential equation of Volterra-type involving exponential function in the kernel:

$$\frac{d}{dt}(h(t)) = -i\pi g_0 \int_0^t \zeta h(t-\zeta)e^{i\omega\zeta}d\zeta, \ \ 0 \le t \le 1, \ \text{ with } \ h(0) = h_0 \text{ and } g_0, \omega \in \mathbb{R}, \tag{16}$$

and discussed analytical treatment that describes the unsaturated behaviour of the free-electron laser equation (for other details, see [13]).

In this direction, Boyadjiev et al. [9] proposed following fractional analogue form of the Volterra-type integro-differential Eq. (16) and examined analytic and numerical behaviour of the solution:

$$\left({}^{\mathsf{RL}}\mathbf{D}_t^{\mu} h\right)(t) = \ell \int_0^t \zeta h(t-\zeta)e^{i\omega\zeta}d\zeta, \quad 0 \le t \le 1, \tag{17}$$

with ${}^{\mathsf{RL}}\mathbf{D}_t^{\mu-j} h(t)|_{t=0} = h_j \in \mathbb{R} \ \ (j = 1, 2, 3, \dots, n)$, and where $\mu, \ell \in \mathbb{C}$, $(n-1) < \Re e(\mu) \le n$, $n = -[-\Re e(\mu)]$, $\omega \in \mathbb{R}$. The symbol ${}^{\mathsf{RL}}\mathbf{D}_t^{\mu}$ in above Eq. (17) denotes is the well-known Riemann-Liouville fractional derivative of order μ.
On substituting $\vartheta = (t-\zeta)$, the Eq. (17) reduces into the following alternative form:

$$\left({}^{\mathsf{RL}}\mathbf{D}_t^{\mu} h\right)(t) = \ell \int_0^t (t-\vartheta)h(\vartheta)e^{i\omega(t-\vartheta)}d\vartheta. \tag{18}$$

Concurrently, Boyadjiev et al. [10] studied and investigated non-homogeneous FIDE of the form:

$$\left({}^{\mathsf{RL}}\mathbf{D}_t^{\mu} h\right)(t) = \ell \int_0^t \zeta h(t-\zeta)e^{i\omega\zeta}d\zeta + \beta' e^{i\omega t}, \quad 0 \le t \le 1, \tag{19}$$

with ${}^{\mathsf{RL}}\mathbf{D}_t^{\mu-j} h(t)|_{t=0} = h_j \in \mathbb{R} \ \ (j = 1, 2, 3, \dots n)$, where $\mu, \beta', \ell \in \mathbb{C}; \omega \in \mathbb{R}; (n-1) < \Re e(\mu) \le n,$ and $n = -[-\Re e(\mu)]$. An alternative form of the above FIDE can also be obtained as:

$$\left({}^{\mathsf{RL}}\mathbf{D}_t^{\mu}h\right)(t) = \ell\int_0^t (t-\vartheta)h(\vartheta)e^{i\omega(t-\vartheta)}d\vartheta + \beta' e^{i\omega t}. \tag{20}$$

Al-Shammery et al. [4] discussed following generalized form of FIDE and extended the idea of Boyadjiev et al. [10]:

$$\left({}^{\mathsf{RL}}\mathbf{D}_t^{\mu}h\right)(t) = \ell\int_0^t \zeta^{\kappa}h(t-\zeta)e^{i\omega\zeta}d\zeta + \beta' e^{i\omega t}, \quad 0 \leq t \leq 1, \tag{21}$$

with $\mu, \beta', \ell \in \mathbb{C}$, and $\omega \in \mathbb{R}$, and $\kappa > -1$. The above FIDE can be alternatively put in following form:

$$\left({}^{\mathsf{RL}}\mathbf{D}_t^{\mu}h\right)(t) = \ell\int_0^t (t-\vartheta)^{\kappa}h(\vartheta)e^{i\omega(t-\vartheta)}d\vartheta + \beta' e^{i\omega t}, \quad 0 \leq t \leq 1, \tag{22}$$

with $\mu, \beta', \ell \in \mathbb{C}, \omega \in \mathbb{R}$ and $\kappa > -1$.

In continuation Saxena and Kalla [55] considered following extension of FIDE involving Kummer's hypergeometric function [38, 39]:

$$\left({}^{\mathsf{RL}}\mathbf{D}_t^{\mu}h\right)(t) = \ell\int_0^t \zeta^{\kappa}h(t-\zeta)\Phi(b,\kappa+1;i\omega\zeta)d\zeta + \rho' t^{\gamma}\Phi(\beta',\gamma+1;i\omega t),\ 0 \leq t \leq 1, \tag{23}$$

with $\mu, b, \beta', \rho', \ell \in \mathbb{C}, \omega \in \mathbb{R}$, and $\kappa > -1$. The above FIDE (23) alternatively can be put in following form:

$$\left({}^{\mathsf{RL}}\mathbf{D}_t^{\mu}h\right)(t) = \ell\int_0^t (t-\vartheta)^{\kappa}h(\vartheta)\Phi(b,\kappa+1;i\omega(t-\vartheta))d\vartheta + \rho' t^{\gamma}\Phi(\beta',\gamma+1;i\omega t), \tag{24}$$

$0 \leq t \leq 1$, with $\mu, b, \beta', \rho', \ell \in \mathbb{C}$ and $\omega \in \mathbb{R}$ and $\kappa > -1$.

At the same time Kilbas et al. [28] have proposed and studied following interesting and generalized form of the of above Eq. (24) which involves the well-known Mittag-Leffler function [22] in the kernel and a general function $\psi(t)$:

$$\left({}^{\mathsf{RL}}\mathbf{D}_{a+}^{\mu}h\right)(t) = \ell\int_a^t (t-\vartheta)^{(\kappa-1)}E_{\rho,\kappa}^{\delta}(\omega(t-\vartheta)^{\rho})h(\vartheta)d\vartheta + \psi(t), \tag{25}$$

where $\mu, \rho, \kappa, \delta$ and $\omega \in \mathbb{C}$ (with $\mathfrak{Re}(\kappa) > 0,\ \mathfrak{Re}(\mu) > 0,\ \mathfrak{Re}(\rho) > 0$).

Now, we propose a unified family of fractional integro-differential equations of Volterra-type. Such unifications may deduce several interesting forms of well-known (maybe also new) fractional integro-differential equations and provide a common framework for computation of numerous problems in applied sciences.

Theorem 2.1. *If $\psi(t)$ is a general function with $h(t) \in \mathcal{L}^1(0,\infty); \gamma, \delta, \rho, \omega, \alpha, \beta \in \mathbb{C}$; $0 < \mu < 1, 0 \leq \nu \leq 1$; $\mathfrak{Re}(\gamma) \geq 0,\ \mathfrak{Re}(\delta) \geq 0,\ \mathfrak{Re}(\rho\delta - \beta) > 0,\ \mathfrak{Re}(\rho) > 0,\ \mathfrak{Re}(\omega) > 0$, then for FIDE:*

$$\left({}^{\mathsf{HP}}\boldsymbol{D}^{\gamma,\mu,\nu}_{\rho,\omega,0+}h\right)(t) + \alpha\int_0^t G_{\{\rho,\beta,\delta\}}(\omega,x,t)h(x)dx = \psi(t), \tag{26}$$

with $\left({}^{\mathsf{P}}\boldsymbol{E}^{-\gamma(1-\nu)}_{\rho,(1-\nu)(1-\mu),\omega,0+}h(t)\right)_{t=0+} = c$, *the following solution holds:*

$$h(t) = \psi(t) * \sum_{k=0}^{\infty}(-\alpha)^k\left[G_{\{\rho,[(\beta-\mu-\rho\delta)k-\mu+\rho\{(\gamma+\delta)k+\gamma\}],[(\gamma+\delta)k+\gamma]\}}(\omega,0,t)\right]$$

$$+c\sum_{k=0}^{\infty}(-\alpha)^k\left[G_{\{\rho,[(\beta-\mu-\rho\delta)k-\mu-\nu(1-\mu)+\rho\{(\gamma+\delta)k+\gamma-\nu\gamma\}],[(\gamma+\delta)k+\gamma-\nu\gamma]\}}(\omega,0,t)\right], \tag{27}$$

provided the sums on the RHS of above Eq. (27) converges.

Proof. The proof of the theorem is based on the Laplace transform method [47]. Applying the Laplace transform both the sides of the above integro-differential Eq. (26), and using the following well-known result pertaining to the Laplace transform of Hilfer-Prabhakar derivative operator (see for more details [19, 46, 49]):

$$L\left({}^{\mathsf{HP}}\mathbf{D}^{\gamma,\mu,\nu}_{\rho,\omega,a^+}h\right)(s) = L\left(\left({}^{\mathsf{P}}\mathbf{E}^{-\gamma\nu}_{\rho,\nu(1-\mu),\omega,a^+}\frac{d}{dt}\left({}^{\mathsf{P}}\mathbf{E}^{-\gamma(1-\nu)}_{\rho,(1-\mu)(1-\nu),\omega,a^+}h\right)\right)(s)$$

$$= s^{\mu}[1-\omega s^{-\rho}]^{\gamma}L[h](s) - s^{-\nu(1-\mu)}[1-\omega s^{-\rho}]^{\gamma\nu}\left({}^{\mathsf{P}}\mathbf{E}^{-\gamma(1-\nu)}_{\rho,(1-\mu)(1-\nu),\omega,a^+}h(t)\right)_{t=a+}, \tag{28}$$

we get

$$s^{\mu}(1-\omega s^{-\rho})^{\gamma}\bar{h}(s) - cs^{-\nu(1-\mu)}(1-\omega s^{-\rho})^{\gamma\nu} + \alpha\frac{s^{\beta-\rho\delta}}{(1-\omega s^{-\rho})^{\delta}}\bar{h}(s) = \bar{g}(s), \tag{29}$$

where $\bar{h}(s)$ and $\bar{g}(s)$ are the Laplace transforms of $h(t)$ and $\psi(t)$, respectively. Also, Eq. (29) can be rewritten as:

$$\bar{h}(s)\left[s^{\mu}(1-\omega s^{-\rho})^{\gamma} + \alpha\frac{s^{\beta-\rho\delta}}{(1-\omega s^{-\rho})^{\delta}}\right] = \bar{g}(s) + cs^{-\nu(1-\mu)}(1-\omega s^{-\rho})^{\nu\gamma}, \tag{30}$$

or alternatively

$$\bar{h}(s) = \frac{\bar{g}(s)}{s^{\mu}(1-\omega s^{-\rho})^{\gamma}}\left[1 + \frac{\alpha}{s^{\mu+\rho\delta-\beta}(1-\omega s^{-\rho})^{(\gamma+\delta)}}\right]^{-1}$$

$$+\frac{s^{-\nu(1-\mu)}(1-\omega s^{-\rho})^{\nu\gamma}c}{s^{\mu}(1-\omega s^{-\rho})^{\gamma}}\left[1 + \frac{\alpha}{s^{\mu+\rho\delta-\beta}(1-\omega s^{-\rho})^{(\gamma+\delta)}}\right]^{-1}. \tag{31}$$

By the use of binomial series expansion the above Eq. (31) reduces into following computable series:

$$\bar{h}(s) = \bar{g}(s)\sum_{k=0}^{\infty}(-\alpha)^k \frac{s^{(\beta-\mu-\rho\delta)k-\mu}}{(1-\omega s^{-\rho})^{(\gamma+\delta)k+\gamma}} + c\sum_{k=0}^{\infty}(-\alpha)^k \frac{s^{(\beta-\mu-\rho\delta)k-\mu-\nu(1-\mu)}}{(1-\omega s^{-\rho})^{(\gamma+\delta)k+\gamma-\nu\gamma}}. \quad (32)$$

It is easy to see that the expressions involved in Eq. (31) will be there in the existence provided both the infinite series are absolutely convergent power series, i.e., we must have following condition:

$$\left|\frac{\alpha}{s^{\mu+\rho\delta-\beta}(1-\omega s^{-\rho})^{(\gamma+\delta)}}\right| < 1. \quad (33)$$

By the application of the well-known convolution theorem of the Laplace transform and taking inverse Laplace transform on both the sides of above Eq. (32) , we arrive on the desired solution of Eq. (26), given in Eq. (27).

Computation of the solution obtained in Eq. (27) is less trivial and based on the convergence of each term involved therein. The first expression involves convolution of the function $\psi(t)$ with each term of infinite series of LHGF, i.e.

$$\psi(t) * \sum_{k=0}^{\infty}(-\alpha)^k \Big[G_{\{\rho,[(\beta-\mu-\rho\delta)k-\mu+\rho\{(\gamma+\delta)k+\gamma\}],[(\gamma+\delta)k+\gamma]\}}(\omega,0,t)\Big]. \quad (34)$$

The convergence of the above expression, Eq. (34), is based on the convergence of following infinite series consisting repeated series of LHGF:

$$\sum_{k=0}^{\infty}(-\alpha)^k \Big[G_{\{\rho,[(\beta-\mu-\rho\delta)k-\mu+\rho\{(\gamma+\delta)k+\gamma\}],[(\gamma+\delta)k+\gamma]\}}(\omega,0,t)\Big],$$

which can be rewritten by the use of series form of LHGF as:

$$\sum_{k=0}^{\infty}(-\alpha)^k \sum_{n=0}^{\infty} \frac{((\gamma+\delta)k+\gamma)_n \omega^n t^{\{(n+(\gamma+\delta)k+\gamma)\rho-(\beta-\mu-\rho\delta)k+\mu-\rho((\gamma+\delta)k+\gamma)-1\}}}{n!\Gamma\{(n+(\gamma+\delta)k+\gamma)\rho-(\beta-\mu-\rho\delta)k+\mu-\rho((\gamma+\delta)k+\gamma)\}}, \quad (35)$$

or equivalently

$$\sum_{k=0}^{\infty}(-\alpha)^k \sum_{n=0}^{\infty} \frac{((\gamma+\delta)k+\gamma)_n \omega^n t^{\{\rho n+(\rho\delta+\mu-\beta)k+\mu-1\}}}{n!\Gamma\{\rho n+(\rho\delta+\mu-\beta)k+\mu\}}. \quad (36)$$

In order to prove the absolute convergence of the series labelled by k, we need to show that the series

$$\sum_{k=0}^{\infty}(-\alpha)^k \frac{((\gamma+\delta)k+\gamma)_n \omega^n t^{\{\rho n+(\rho\delta+\mu-\beta)k+\mu-1\}}}{n!\Gamma\{\rho n+(\rho\delta+\mu-\beta)k+\mu\}}, \quad (37)$$

is absolutely convergent for each (fixed) $n \in \mathbb{N} \cup \{0\}$ (for more details, see [5]).
Let us define

$$u_k(n,t) = (-\alpha)^k \frac{((\gamma+\delta)k+\gamma)_n \omega^n t^{\{\rho n+(\rho\delta+\mu-\beta)k+\mu-1\}}}{n!\Gamma\{\rho n+(\rho\delta+\mu-\beta)k+\mu\}}$$

$$= (-\alpha)^k \frac{\Gamma\{(\gamma+\delta)k+\gamma+n\}}{\Gamma\{(\gamma+\delta)k+\gamma\}} \frac{(\omega t^\rho)^n t^{\{(\rho\delta+\mu-\beta)k+\mu-1\}}}{n!\Gamma\{\rho n+(\rho\delta+\mu-\beta)k+\mu\}}, \tag{38}$$

using the asymptotic behaviour of the ratio of gamma functions we get

$$\Big|\frac{u_{k+1}(n,t)}{u_k(n,t)}\Big| \sim \Big|\frac{(-\alpha)t^{(\rho\delta+\mu-\beta)}}{k(\rho\delta+\mu-\beta)}\Big| \quad \forall\ t>0,\ \forall\ n \in \mathbb{N}\cup\{0\}, \tag{39}$$

hence for $k \to \infty$ we get

$$\lim_{k\to\infty} \Big|\frac{u_{k+1}(n,t)}{u_k(n,t)}\Big| = 0,\ \forall\ t>0, \forall\ n \in \mathbb{N}\cup\{0\}, \tag{40}$$

which indicates that the absolute convergence of the series involved in the first term of Eq. (27). Also, if the function $\psi(t)$ is continuous and suitably defined in $\mathcal{L}^1(0,\infty)$ then the convolution must be convergent and the first term of Eq. (27) must be convergent. The convergence of the second term of Eq. (27) can also be investigated in a similar manner, thus we omit the details here. $\square$

3. Certain Volterra-type fractional integro-differential equations based on the family of unified fractional Volterra-type integro-differential equations

The above-discussed family of Volterra-type FIDE is general in nature and unifies several elegant and interesting results proposed by eminent scholars. In the present section, based on Theorem 2.1 we deduce some of the corollaries which may be directly applicable in different fields of sciences, such as laser, nuclear, astrophysics, thermal analysis, heat transfer etcetera.
For $\delta = 0; \beta = -\eta$, Theorem 2.1 reduces into the following result recently investigated by Pandey [46]:

Corollary 3.1. *If* $h(t) \in \mathcal{L}^1(0,\infty)$; $\gamma,\ \rho,\ \omega,\ \alpha,\ \eta \in \mathbb{C}$; $0 < \mu < 1, 0 \leq \nu \leq 1; \mathfrak{Re}(\gamma) \geq 0, \mathfrak{Re}(\delta) \geq 0,\ \mathfrak{Re}(\eta) > 0, \mathfrak{Re}(\rho) > 0, \mathfrak{Re}(\omega) > 0,$ *then for the FIDE:*

$$\left({}^{\mathsf{HP}}\boldsymbol{D}^{\gamma,\mu,\nu}_{\rho,\omega,0+}h\right)(t) + \alpha \int_0^t G_{\{\rho,-\eta,0\}}(\omega,x,t)h(x)dx = \psi(t), \tag{41}$$

or, equivalently

$$\left({}^{\mathsf{HP}}\boldsymbol{D}^{\gamma,\mu,\nu}_{\rho,\omega,0+}h\right)(t) + \frac{\alpha}{\Gamma(\eta)} \int_0^t (t-x)^{\eta-1}h(x)dx = \psi(t), \tag{42}$$

with $\left({}^{\mathsf{P}}\boldsymbol{E}^{-\gamma(1-\nu)}_{\rho,(1-\nu)(1-\mu),\omega,0+}h(t)\right)_{t=0+} = c$, *following solution holds:*

$$h(t) = \psi(t) * \sum_{k=0}^{\infty}(-\alpha)^k \left[G_{\{\rho,[\rho\gamma(k+1)-(\eta+\mu)k-\mu],[\gamma(k+1)]\}}(\omega,0,t)\right]$$

$$+c\sum_{k=0}^{\infty}(-\alpha)^k \left[G_{\{\rho,[\nu(\mu-1)-\mu-(\eta+\mu)k+\rho\{\gamma(k+1)-\nu\gamma\}],[\gamma(k+1)-\nu\gamma]\}}(\omega,0,t)\right], \tag{43}$$

provided the sums on the RHS of above Eq. (43) converges.

Remark. The results presented as Corollaries 7, 8 and 9 in [46] can also be deduced as the particular cases of the theorem 2.1. For more details, see [2] and [63].

On taking $\gamma = 0$ in Theorem 2.1, we arrive on the following corollary pertaining to certain family of Volterra-type FIDE based on the Hilfer derivative that involves LHGF in the kernel.

Corollary 3.2. *If* $h(t) \in \mathcal{L}^1(0,\infty)$; $0 < \mu < 1, 0 \leq \nu \leq 1$; $\alpha, \rho, \beta, \delta \in \mathbb{C}$; $\mathfrak{Re}(\rho\delta - \beta) > 0$, $\mathfrak{Re}(\omega) > 0$ *then for FIDE of the form:*

$$\left({}^{\mathsf{H}}\mathcal{D}^{\mu,\nu}_{0+}h\right)(t) + \alpha\int_0^t G_{\{\rho,\beta,\delta\}}(\omega,x,t)h(x)dx = \psi(t), \tag{44}$$

with $\left({}^{\mathsf{RL}}I^{(1-\mu)(1-\nu)}_{0+}h(t)\right)_{t=0+} = c$, *following solution holds:*

$$h(t) = \psi(t) * \sum_{k=0}^{\infty}(-\alpha)^k \left[G_{\{\rho,[(\beta-\mu)k-\mu],\delta k\}}(\omega,0,t)\right]$$

$$+c\sum_{k=0}^{\infty}(-\alpha)^k \left[G_{\{\rho,[(\beta-\mu)k-\mu-\nu(1-\mu)],[\delta k]\}}(\omega,0,t)\right], \tag{45}$$

provided the sums on the RHS of above Eq. (45) converges.

On setting $\nu = 0$ in Corollary 3.2, we get the following form of Volterra-type FIDE:

Corollary 3.3. *If* $h(t) \in \mathcal{L}^1(0,\infty)$; $0 < \mu < 1$; $\alpha, \rho, \beta, \delta \in \mathbb{C}$; $\mathfrak{Re}(\rho\delta - \beta) > 0$, $\mathfrak{Re}(\omega) > 0$, *then for FIDE:*

$$\left({}^{\mathsf{RL}}\boldsymbol{D}^{\mu}_{0+}h\right)(t) + \alpha\int_0^t G_{\{\rho,\beta,\delta\}}(\omega,x,t)h(x)dx = \psi(t), \tag{46}$$

with $\left({}^{\mathsf{RL}}\boldsymbol{I}^{(1-\mu)}_{0+}h(t)\right)_{t=0+} = c$, *following solution holds:*

$$h(t) = \psi(t) * \sum_{k=0}^{\infty}(-\alpha)^k \left[G_{\{\rho,[(\beta-\mu)k-\mu],\delta k\}}(\omega,0,t)\right]$$

$$+c\sum_{k=0}^{\infty}(-\alpha)^{k}\Big[G_{\{\rho,[(\beta-\mu)k-\mu],\delta k\}}(\omega,0,t)\Big], \tag{47}$$

provided the sums on the RHS of above Eq. (47) converges.

Remark. If we substitute $\alpha=-\ell(\Gamma(\kappa+1))$ with $\rho\to1$; $\beta\to(\varpi-\kappa+1)$; $\delta\to\varpi$; $\omega\to i\omega$ and $\psi(t)=\rho'\Gamma(\gamma+1)G_{\{1,(\beta'-\gamma+1),\beta'\}}(i\omega,0,t)$ in the above Eq. (46) of Corollary 3.3, we arrive on the following Volterra-type FIDE:

$$\left({}^{\mathsf{RL}}\mathbf{D}^{\mu}_{0+}h\right)(t)=\ell(\Gamma(\kappa+1))\int_0^t G_{\{1,(\varpi-\kappa+1),\varpi\}}(i\omega,x,t)h(x)dx$$
$$+\rho'\Gamma(\gamma+1)G_{\{1,(\beta'-\gamma+1),\beta'\}}(i\omega,0,t), \tag{48}$$

with $\left({}^{\mathsf{RL}}\mathbf{I}^{(1-\mu)}_{0+}h(t)\right)_{t=0+}=c$. The FIDE in Eq. (48) is equivalent to the result studied by Saxena and Kalla [55], discussed in Eq. (24).

Remark. Using the relation given in [46], Eq. (25), in above Eq. (46), we arrive on following FIDE:

$$\left({}^{\mathsf{RL}}\mathbf{D}^{\mu}_{0+}h\right)(t)+\alpha\int_0^t(t-x)^{\rho\delta-\beta-1}E^{\delta}{}_{\rho,(\rho\delta-\beta)}(\omega(t-x)^{\rho})(h(x)dx=\psi(t), \tag{49}$$

with $\left({}^{\mathsf{RL}}\mathbf{I}^{(1-\mu)}_{0+}h(t)\right)_{t=0+}=c$, which on substituting $\alpha=-\ell$; $\rho\delta-\beta=\kappa$ yields the well-known Volterra-type integro-differential equation investigated by Kilbas et al. [28], given in above Eq. (25).

On taking $\nu=1$ in Corollary 3.2, we get following family of Volterra-type FIDE in terms of the Caputo derivative involving LHGF in the kernel.

Corollary 3.4. *If* $h(t)\in\mathcal{L}^1(0,\infty)$; $0<\mu<1$; $\alpha,\rho,\beta,\delta\in\mathbb{C}$; $\mathfrak{Re}(\rho\delta-\beta)>0$, $\mathfrak{Re}(\omega)>0$, *then for FIDE:*

$$\left({}^{\mathsf{C}}\boldsymbol{D}^{\mu}_{0+}h\right)(t)+\alpha\int_0^t G_{\{\rho,\beta,\delta\}}(\omega,x,t)h(x)dx=\psi(t), \tag{50}$$

with $h(t)_{t=0+}=c$, *following solution holds:*

$$h(t)=\psi(t)*\sum_{k=0}^{\infty}(-\alpha)^{k}\Big[G_{\{\rho,[(\beta-\mu)k-\mu],\delta k\}}(\omega,0,t)\Big]$$
$$+c\sum_{k=0}^{\infty}(-\alpha)^{k}\Big[G_{\{\rho,[(\beta-\mu)k-\mu-(1-\mu)],\delta k\}}(\omega,0,t)\Big], \tag{51}$$

provided the sum on the RHS of above Eq. (51) converges.

Remark. The detailed analysis of the corollaries concerning convergence discussed in this section can be done exactly in the same way as we have proposed in Theorem 2.1.

4. Certain examples pertaining to the unified family of Volterra-type fractional integro-differential equations

In this section, we investigate some applications of Theorem 2.1 by considering some particular forms of the function $\psi(t)$. Let's consider the case when $\psi(t) = G_{\{\rho,\eta,\xi\}}(\omega,0,t)$, we arrive on the following result:

Example 4.1. If $h(t)\in\mathcal{L}^1(0,\infty)$; $0<\mu<1$, $0\le\nu\le 1$; $\gamma\ge 0$; $\rho,\omega,\alpha,\eta,\beta,\delta,\xi,\lambda\in\mathbb{C}$; $\mathfrak{Re}(\rho\delta-\beta)>0$, $\mathfrak{Re}(\rho\xi-\eta)>0$, $\mathfrak{Re}(\rho)>0$, $\mathfrak{Re}(\omega)>0$, then for FIDE:

$$\left({}^{\mathsf{HP}}\mathbf{D}^{\gamma,\mu,\nu}_{\rho,\omega,0+}h\right)(t)+\alpha\int_0^t G_{\{\rho,\beta,\delta\}}(\omega,x,t)h(x)dx=\psi(t)=\lambda G_{\{\rho,\eta,\xi\}}(\omega,0,t), \tag{52}$$

with $\left({}^{\mathsf{P}}\mathbf{E}^{-\gamma(1-\nu)}_{\rho,(1-\nu)(1-\mu),\omega,0+}h(t)\right)_{t=0+}=c,$ following solution holds:

$$h(t)=c\sum_{k=0}^{\infty}(-\alpha)^k\Big[G_{\{\rho,[(\beta-\mu+\rho\gamma)k-\nu(1-\mu)-\mu+\rho\gamma(1-\nu)],[(\gamma+\delta)k+\gamma(1-\nu)]\}}(\omega,0,t)\Big]$$

$$+\lambda\sum_{k=0}^{\infty}(-\alpha)^k\Big[G_{\{\rho,[(\beta-\mu+\rho\gamma)k+(\eta-\mu+\rho\gamma)],[(\gamma+\delta)k+\gamma+\xi]\}}(\omega,0,t)\Big], \tag{53}$$

provided the sum on RHS of above Eq. (53) converges.

Particularly, if we substitute $\lambda=0$ in the above example we arrive on the following homogeneous FIDE:

$$\left({}^{\mathsf{HP}}\mathbf{D}^{\gamma,\mu,\nu}_{\rho,\omega,0+}h\right)(t)+\alpha\int_0^t G_{\{\rho,\beta,\delta\}}(\omega,x,t)h(x)dx=0, \tag{54}$$

with $\left({}^{\mathsf{P}}\mathbf{E}^{-\gamma(1-\nu)}_{\rho,(1-\nu)(1-\mu),\omega,0+}h(t)\right)_{t=0+}=c$, then following solution holds:

$$h(t)=c\sum_{k=0}^{\infty}(-\alpha)^k\Big[G_{\{\rho,[(\beta-\mu+\rho\gamma)k-\nu(1-\mu)-\mu+\rho\gamma(1-\nu)],[(\gamma+\delta)k+\gamma(1-\nu)]\}}(\omega,0,t)\Big], \tag{55}$$

provided the sum on RHS of above Eq. (55) converges.

Furthermore, if we substitute $\delta=0$; $\beta=-\tau$ with the conditions $\mathfrak{Re}(\rho)>0$, $\mathfrak{Re}(\tau)>0, \mathfrak{Re}(\rho\xi-\eta)>0$, $\mathfrak{Re}(\omega)>0$ in Example 4.1, we arrive on following FIDE:

$$\left({}^{\mathsf{HP}}\mathbf{D}^{\gamma,\mu,\nu}_{\rho,\omega,0+}h\right)(t)+\alpha\int_0^t G_{\{\rho,-\tau,0\}}(\omega,x,t)h(x)dx=\lambda G_{\{\rho,\eta,\xi\}}(\omega,0,t), \tag{56}$$

or equivalently (applying the relation [46], Eq. (16))

$$\left({}^{\mathsf{HP}}\mathbf{D}^{\gamma,\mu,\nu}_{\rho,\omega,0+}h\right)(t) + \frac{\alpha}{\Gamma(\tau)}\int_0^t (t-x)^{\tau-1}h(x)dx = \lambda G_{\{\rho,\eta,\xi\}}(\omega,0,t), \tag{57}$$

with $\left({}^{\mathsf{P}}\mathbf{E}^{-\gamma(1-\nu)}_{\rho,(1-\nu)(1-\mu),\omega,0+}h(t)\right)_{t=0+} = c,$ which has its solution as:

$$h(t) = c\sum_{k=0}^{\infty}(-\alpha)^k\Big[G_{\{\rho,[(\rho\gamma-\tau-\mu)k-\nu(1-\mu)-\mu+\rho\gamma(1-\nu)],[\gamma k+\gamma(1-\nu)]\}}(\omega,0,t)\Big]$$

$$+\lambda\sum_{k=0}^{\infty}(-\alpha)^k\Big[G_{\{\rho,[(\rho\gamma-\tau-\mu)k+(\eta-\mu+\rho\gamma)],[\gamma k+\gamma+\xi]\}}(\omega,0,t)\Big], \tag{58}$$

provided the sum on the RHS of above Eq. (58) converges.

Let us consider the case when the function $\psi(t) = \lambda G_{\{\rho,-\eta,0\}}(\omega,0,t)$, then by the Theorem 2.1 we can deduce following particular example:

Example 4.2. If $h(t) \in \mathcal{L}^1(0,\infty)$; $0 < \mu < 1$, $0 \le \nu \le 1$; $\gamma \ge 0$; $\rho,\omega,\alpha,\eta,\beta,\delta \in \mathbb{C}$; $\mathfrak{Re}(\rho\delta-\beta) > 0$, $\mathfrak{Re}(\eta) > 0$, $\mathfrak{Re}(\rho) > 0$, $\mathfrak{Re}(\omega) > 0$ then for FIED:

$$\left({}^{\mathsf{HP}}\mathbf{D}^{\gamma,\mu,\nu}_{\rho,\omega,0+}h\right)(t) + \alpha\int_0^t G_{\{\rho,\beta,\delta\}}(\omega,x,t)h(x)dx = \lambda G_{\{\rho,-\eta,0\}}(\omega,0,t), \tag{59}$$

or equivalently (applying the relation [46], Eq. (21))

$$\left({}^{\mathsf{HP}}\mathbf{D}^{\gamma,\mu,\nu}_{\rho,\omega,0+}h\right)(t) + \alpha\int_0^t G_{\{\rho,\beta,\delta\}}(\omega,x,t)h(x)dx = \lambda\frac{t^{\eta-1}}{\Gamma(\eta)}, \tag{60}$$

with $\left({}^{\mathsf{P}}\mathbf{E}^{-\gamma(1-\nu)}_{\rho,(1-\nu)(1-\mu),\omega,0+}h(t)\right)_{t=0+} = c,$ following solution holds:

$$h(t) = c\sum_{k=0}^{\infty}(-\alpha)^k\Big[G_{\{\rho,[(\beta-\mu+\rho\gamma)k-\nu(1-\mu)-\mu+\rho\gamma(1-\nu)],[(\gamma+\delta)k+\gamma(1-\nu)]\}}(\omega,0,t)\Big]$$

$$+\lambda\sum_{k=0}^{\infty}(-\alpha)^k\Big[G_{\{\rho,[(\beta-\mu+\rho\gamma)k+(\rho\gamma-\eta-\mu)],[(\gamma+\delta)k+\gamma]\}}(\omega,0,t)\Big], \tag{61}$$

provided the sum on RHS of above Eq. (61) converges.

Moreover, for $\delta = 0; \beta = -\sigma$ with $\mathfrak{Re}(\sigma) > 0, \mathfrak{Re}(\eta) > 0$ FIDE, presented as Eq. (59), reduces into following form:

$$\left({}^{\mathsf{HP}}\mathbf{D}^{\gamma,\mu,\nu}_{\rho,\omega,0+}h\right)(t) + \alpha\int_0^t G_{\{\rho,-\sigma,0\}}(\omega,x,t)h(x)dx = \lambda G_{\{\rho,-\eta,0\}}(\omega,0,t), \tag{62}$$

which equivalently can be rewritten as

$$\left({}^{\mathsf{HP}}\mathbf{D}^{\gamma,\mu,\nu}_{\rho,\omega,0+}h\right)(t)+\frac{\alpha}{\Gamma(\sigma)}\int_0^t(t-x)^{\sigma-1}h(x)dx=\lambda\frac{t^{\eta-1}}{\Gamma(\eta)}, \tag{63}$$

with $\left({}^{\mathsf{P}}\mathbf{E}^{-\gamma(1-\nu)}_{\rho,(1-\nu)(1-\mu),\omega,0+}h\right)_{t=0+}=c$ has its solution as:

$$\begin{aligned}h(t)&=c\sum_{k=0}^{\infty}(-\alpha)^k\Big[G_{\{\rho,[(\rho\gamma-\sigma-\mu)k-\nu(1-\mu)-\mu+\rho\gamma(1-\nu)],[\gamma k+\gamma(1-\nu)]\}}(\omega,0,t)\Big]\\&\quad+\lambda\sum_{k=0}^{\infty}(-\alpha)^k\Big[G_{\{\rho,[(\rho\gamma-\sigma-\mu)k+(\rho\gamma-\eta-\mu)],[\gamma k+\gamma]\}}(\omega,0,t)\Big],\end{aligned} \tag{64}$$

provided the sum on RHS of above Eq. (64) converges.

The solutions of Theorem 2.1, its corollaries, and associated examples are obtained in terms of LHGF where we tactically assumed that the range of different parameters are taken in such a way that the obtained solutions must be convergent.

5. Applications in Free-electron laser (FEL) equations

To demonstrate applications of the presented unified family of fractional Volterra-type integro-differential equation, we deduce two fractional-order generalizations of free-electron laser equations involving LHGF in the kernel as the special cases of Example 4.1.

5.1. Fractional free-electron laser equation based on Caputo derivative

On setting $\lambda=0,\ \gamma=0,\ \nu=1$, above Example 4.1 reduces into following generalization of fractional free-electron laser equation based on Caputo derivative:

If $0<\mu<1$, $\mathfrak{Re}(\rho\delta-\beta)>0$, then FIDE that represents generalized FEL :

$$\left({}^{\mathsf{C}}\mathbf{D}^{\mu}_{0+}h\right)(t)+\alpha\int_0^t G_{\{\rho,\beta,\delta\}}(\omega,x,t)h(x)dx=0, \tag{65}$$

with $[h(t)]_{t=0+}=c$, has its solution in terms of LHGF as:

$$h(t)=c\sum_{k=0}^{\infty}(-\alpha)^k\Big[G_{\{\rho,[(\beta-\mu)k-1],\delta k\}}(\omega,0,t)\Big], \tag{66}$$

provided the sum on RHS of Eq. (66) converges.

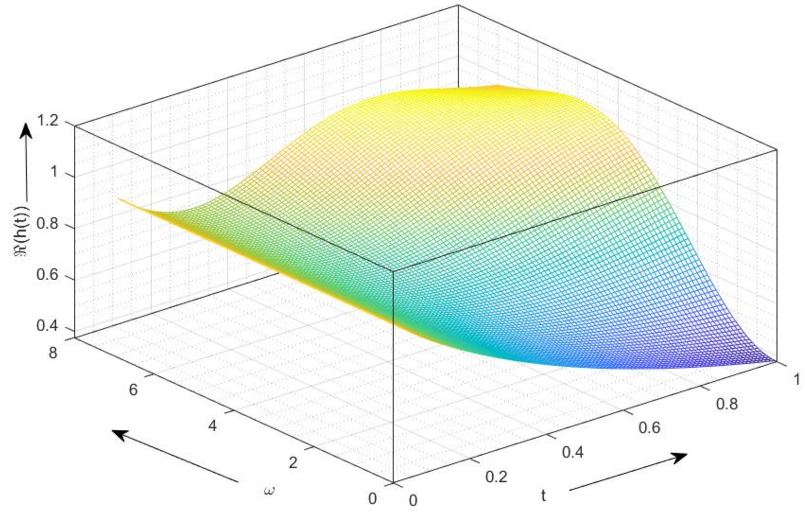

Figure 1: Graph demonstrates the real part of the solution for Caputo derivative of order $\mu = 1/9$

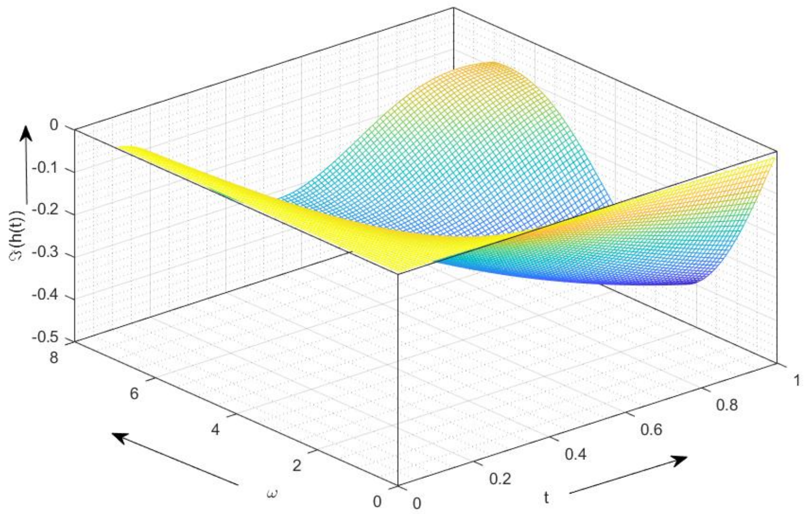

Figure 2: Graph exhibits the imaginary part of the solution for Caputo derivative of order $\mu = 1/9$

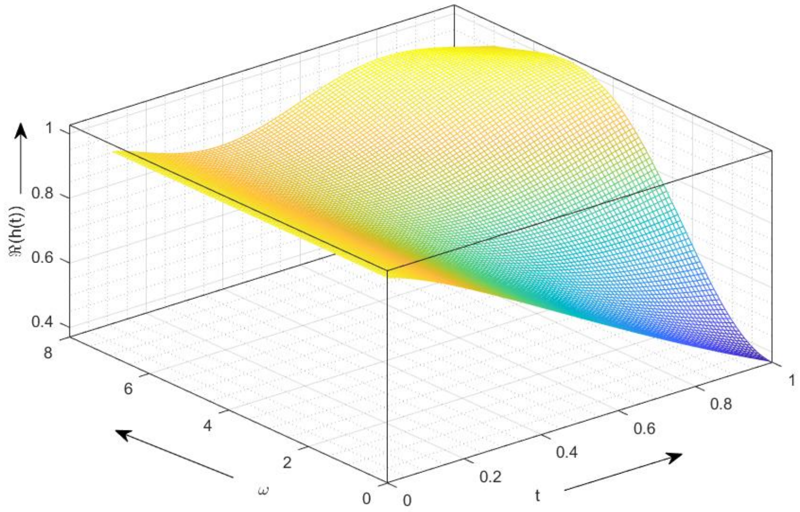

Figure 3: Graph displays the real part of the solution for Caputo derivative of order $\mu = 1/2$

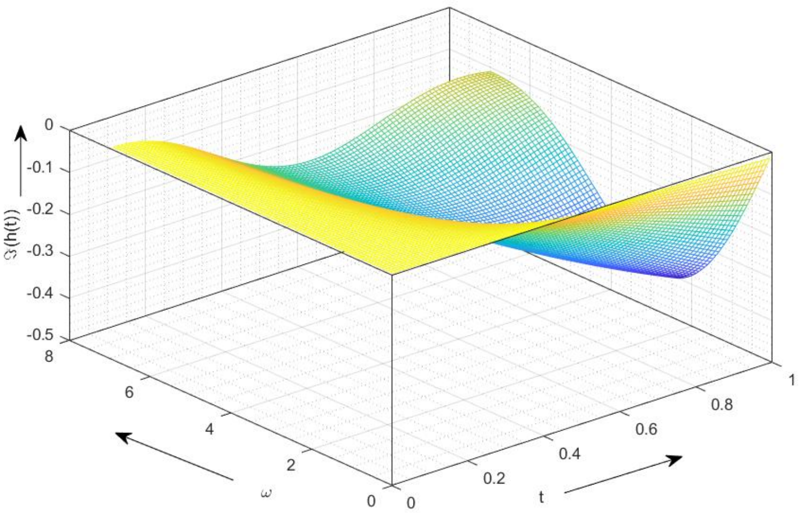

Figure 4: Graph describes the imaginary part of the solution for Caputo derivative of order $\mu = 1/2$

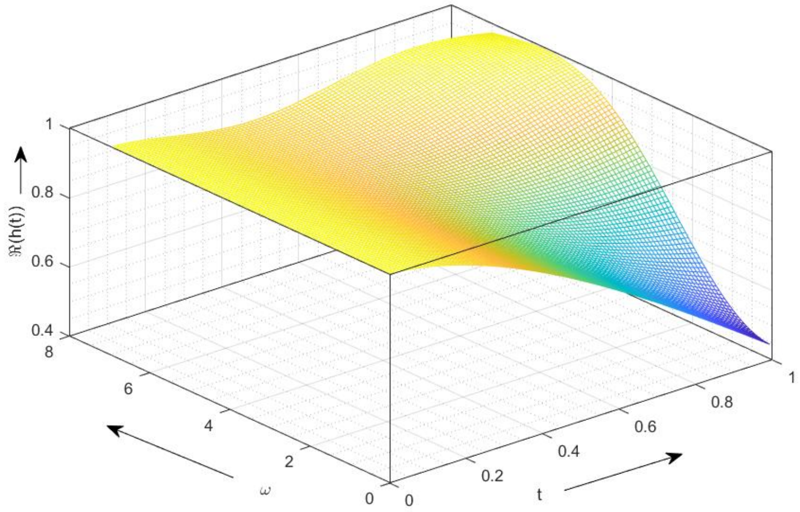

Figure 5: Graph indicates the real part of the solution for Caputo derivative of order $\mu = 8/9$

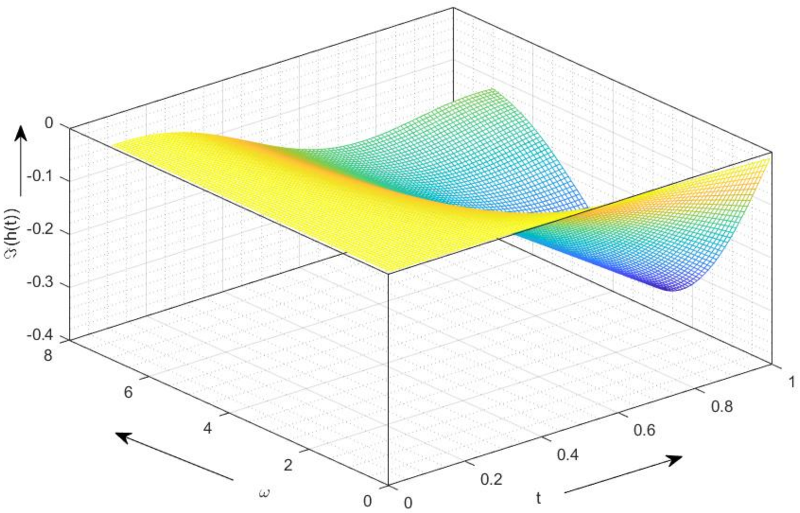

Figure 6: Graph reflects the imaginary part of the solution for Caputo derivative of order $\mu = 8/9$

5.2. Fractional free-electron laser equation based on Riemann-Liouville derivative

On substituting $\lambda = 0,\ \gamma = 0,\ \nu = 0$, above Example 4.1 give rise to the following fractional homogeneous fractional free-electron laser equation based on Riemann-Liouville derivative:

If $0 < \mu < 1$, $\mathfrak{Re}(\rho\delta - \beta) > 0$, then FIDE that represents generalized FEL:

$$\left({}^{\mathsf{RL}}\mathcal{D}^{\mu}_{0+}h\right)(t) + \alpha \int_0^t G_{\{\rho,\beta,\delta\}}(\omega, x, t)h(x)dx = 0, \tag{67}$$

with $\left({}^{\mathsf{RL}}\mathcal{I}^{(1-\mu)}_{0+}h(t)\right)_{t=0+} = c,$ has it solution in terms of LHGF as:

$$h(t) = c\sum_{k=0}^{\infty}(-\alpha)^k\Big[G_{\{\rho,[(\beta-\mu)k-\mu],\delta k\}}(\omega, 0, t)\Big], \tag{68}$$

provided the sum on RHS of Eq. (68) converges.

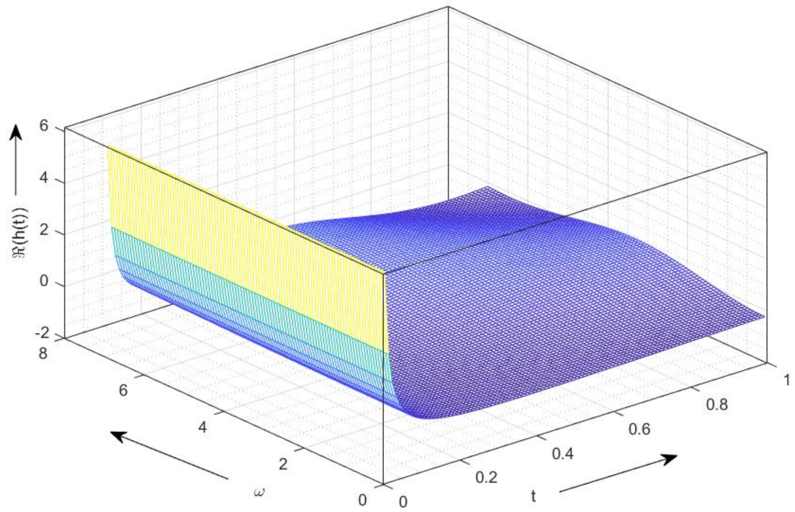

Figure 7: Graph demonstrates the real part of the solution for Riemann-Liouville derivative of order $\mu = 1/9$

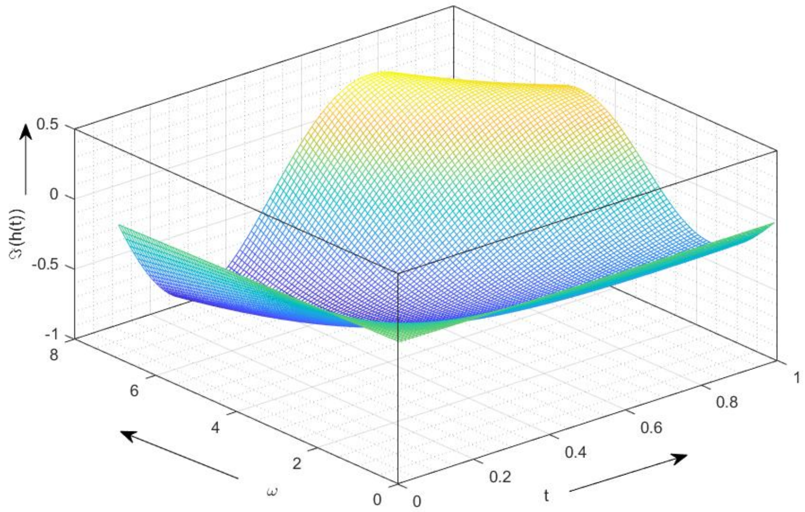

Figure 8: Graph exhibits the imaginary part of the solution for Riemann-Liouville derivative of order $\mu = 1/9$

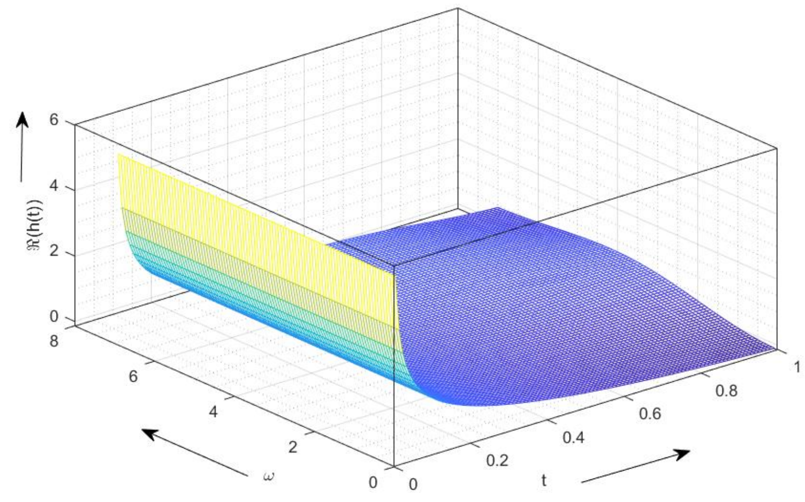

Figure 9: Graph displays the real part the of the solution for Riemann-Liouville derivative of order $\mu = 1/2$

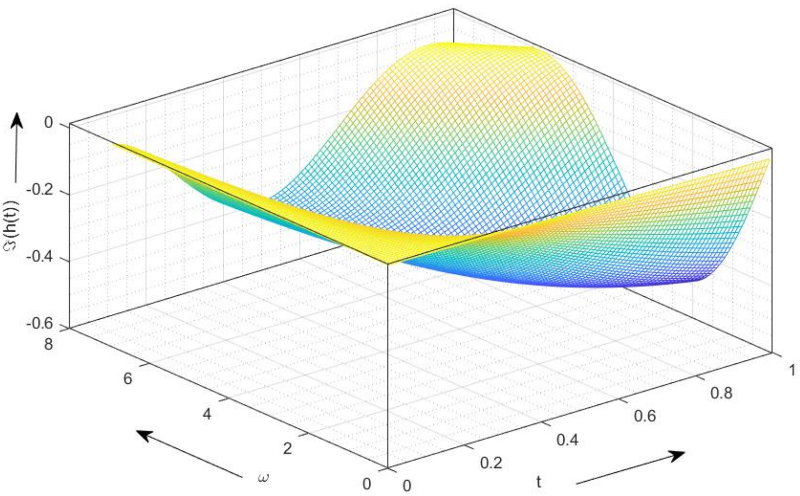

Figure 10: Graph describes the imaginary part of the solution for Riemann-Liouville derivative of order $\mu = 1/2$

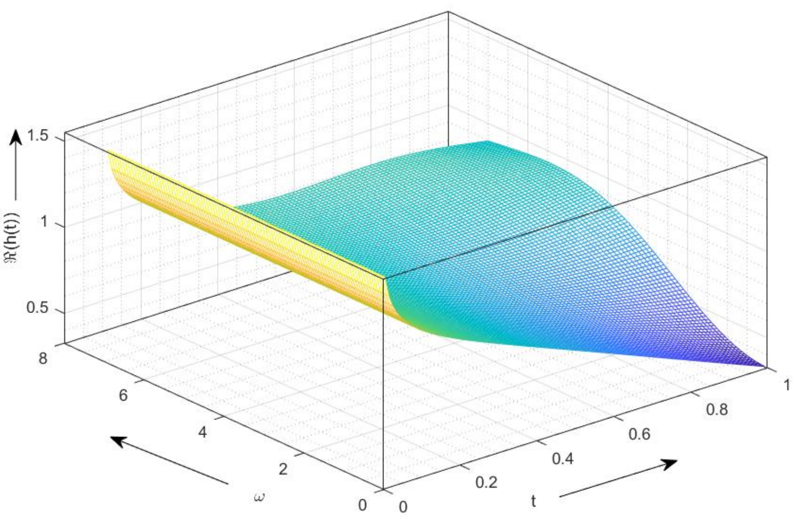

Figure 11: Graph indicates the real part the of the solution for Riemann-Liouville derivative of order $\mu = 8/9$

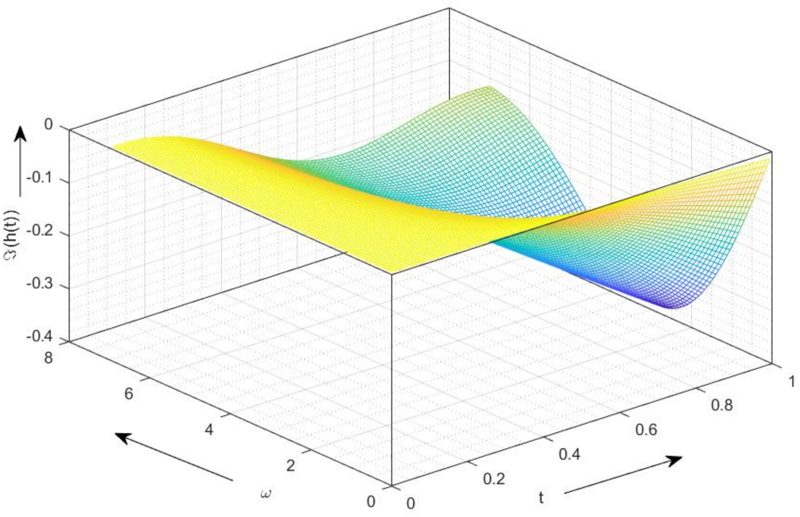

Figure 12: Graph reflects the imaginary part of the solution for Riemann-Liouville derivative of order $\mu = 8/9$

To illustrate the behaviour of the solutions of the above-mentioned generalized free-electron laser Equations (65) and (67) based on Caputo derivative and Riemann-Liouville derivative, respectively, the computation for the solutions are performed on MATLAB using series representation of LHGF. Particularly, the parameters for computations are taken as: $\alpha = 1, c = 1,\ \rho = 1, \delta = 1, (0 + i0.07) < \omega < (0 + i7)$, (with the difference of $(0 + i0.07)$) $0.01 < t < 1.0$, and $\beta = 0.2$ (with the difference of 0.01). The behavior of the obtained solutions are shown in figures. Figure 1 through Figure 6 exhibit the behaviour of the real and imaginary parts of the solutions for homogeneous fractional free-electron laser Eq. (65) with Caputo derivative. Figure 7 through Figure 12 demonstrate the behaviour of the real and imaginary parts of the solutions for homogeneous fractional free-electron laser Eq. (67) with Riemann-Liouville derivative.

6. Concluding remarks

In this paper, we have investigated a unified family of Volterra-type fractional integro-differential equations. The solution of the considered family is determined in the closed-forms of LHGF, which works well in case of increased time-domain complexity. To investigate the computational nature of the solution of the considered unified family, we have discussed the convergence of the solution. Several known and new fractional integro-differential equations involving different functions for fractional calculus in the kernel (obtained by proper choice of parameters in LHGF) accompanied by various forms of fractional derivatives, can be derived by specializing the parameters involved therein. Notably, the results can also be used to obtain closed-form solutions for several other Volterra-type fractional integro-differential equations that arise in different engineering sciences fields. From the application point of view, we have illustrated two forms of fractional free-electron laser and obtained their solutions in the closed and computable form of LHGF. Several graphical illustrations are presented, which demonstrate the behaviour of the solutions.

Remarkably, Hilfer-Prabhakar derivative interpolates between the Prabhakar derivative and its regularized version, given in Eq. (10) and Eq. (11), respectively. It can also be reduced into Hilfer derivative which may reduced into Riemann-Liouville and Caputo fractional derivatives by proper choice of parameters. Thus, the results established in the paper may be used to derive closed-form solutions for several Volterra-type fractional integro-differential equations, hitherto scattered in the literature.

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References

[1] M. Abramowitz, I.A. Stegun, D. Miller, *Handbook of Mathematical Functions with Formulas, Graphs, and Mathematical Tables*, National Bureau of Standards Applied Mathematics Series No. 55 (1965).

[2] B.N. Al-Saqabi, V.K. Tuan, *Solution of a fractional differential equation*, Integral Transform. Spec. Funct. 4 (1996) 321–326.

[3] A.H. Al-Shammery, S.L. Kalla, H.G. Khajah, *A fractional generalization of the free electron laser equation*, Fract. Calc. Appl. Anal. 2 (4) (1999) 501–508.

[4] A.H. Al-Shammery, S.L. Kalla, H.G. Khajah, *On a generalized fractional integro-differential equation of Volterra-type*, Integral Transform. Spec. Funct. 9 (2) (2000) 81–90.

[5] T.M. Apostol, *Mathematical Analysis*, Addison-Wesley Publ. Co., Reading Mass and London, 1960.

[6] D. Baleanu, K. Diethelm, E. Scalas, J.J. Trujillo, *Fractional Calculus: Models and Numerical Methods*, World Scientific Publishing, Singapore, 2017.

[7] D. Baleanu, J.A.T. Machado, A.C.J. Luo, *Fractional Dynamics and Control*, Springer, New York, 2012.

[8] M. Berry, *Why are special functions special?*, Phys. Today 54 (4) (2001) 11–12.

[9] L. Boyadjiev, H.J. Dobner, S.L. Kalla, *A fractional integro-differential equation of Volterra type*, Math. Comput. Modelling 28 (10) (1998) 103–113.

[10] L. Boyadjiev, S.L. Kalla, H.G. Khajah, *Analytical and numerical treatment of a fractional integro-differential equation of Volterra-type*, Math. Comput. Modelling 25 (12) (1998) 1–9.

[11] V.B.L. Chaurasia, S.C. Pandey, *On a new computable solution of generalized fractional kinetic equations involving the generalized function for the fractional calculus and related functions*, Astrophys. Space Sci. 317 (2008) 213–219.

[12] V.B.L. Chaurasia, S.C. Pandey, *Computable extensions of generalized fractional kinetic equations in astrophysics*, Res. Astron. Astrophys. 10 (1) (2010) 22–32.

[13] G. Dattoli, A. Renieri, A. Torre, *Lectures on the Free Electron Laser Theory and Related Topics*, World Scientific, Singapore, 1993.

[14] G. Dattoli, S. Lorenzutta, G. Maino, A. Torre, *Analytical treatment of the high-gain free electron laser equation*, Radiat. Phys. Chem. 48 (1) (1996) 29–40.

[15] K. Diethelm, *The Analysis of Fractional Differential Equations: An Application-Oriented Exposition Using Differential Operators of Caputo Type*, Springer Science & Business Media, 2010.

[16] M.A.F. Dos Santos, *Fractional Prabhakar derivative in diffusion equation with non-static stochastic resetting*, Phys. 1 (2019) 40–58.

[17] E.F. Doungmo Goufo, *A biomathematical view on the fractional dynamics of cellulose degradation*, Fract. Calc. Appl. Anal. 18 (3) (2015) 554–564.

[18] M. D'Ovidio, F. Polito, *Fractional diffusion-telegraph equations and their associated stochastic solutions*, arXiv:1307.1696v3 [math.PR], 22 pages (2013).

[19] R. Garra, R. Gorenflo, F. Polito, Ž. Tomovski, *Hilfer-Prabhakar derivatives and some applications*, Appl. Math. Comput. 242 (2014) 576–589.

[20] A. Giusti, I. Colombaro, *Prabhakar-like fractional viscoelasticity*, Commun. Nonlinear Sci. Numer. Simul. 56 (2018) 138–143.

[21] A. Giusti, *General fractional calculus and Prabhakar's theory*, Commun. Nonlinear Sci. Numer. Simul. 83 (2020) 105114.

[22] R. Gorenflo, A.A. Kilbas, F. Mainardi, S.V. Rogosin, *Mittag-Leffler Functions, Related Topics and Applications*, Springer-Verlag Berlin, Heidelberg, 2014.

[23] H.J. Haubold, A.M. Mathai, *The fractional kinetic equation and thermonuclear functions*, Astrophys. Space Sci. 273 (2000) 53–63.

[24] R. Hilfer, *Applications of Fractional Calculus in Physics*, World Scientific Publishing Company, Singapore, 2000.

[25] R. Hilfer, *Experimental evidence for fractional time evolution in glass forming materials*, J. Chem. Phys. 284 (2002) 399–408.

[26] R. Hilfer, *Threefold introduction to fractional derivatives*, Anomalous Transport: Foundations and Applications, Wiley-VCH Verlag, Weinheim (2008) 17–73.

[27] R. Hilfer, Y. Luchko, Ž. Tomovski, *Operational method for solution of the fractional differential equations with the generalized Riemann-Liouville fractional derivatives*, Fract. Calc. Appl. Anal. 12 (2009) 299–318.

[28] A.A. Kilbas, M. Saigo, R.K. Saxena, *Solution of Volterra integro-differential equations with generalized Mittag-Leffler functions in the kernels*, J. Integral Equations Appl. 14 (2002) 377–396.

[29] A.A. Kilbas, M. Saigo, R.K. Saxena, *Generalized Mittag-Leffler function and generalized fractional calculus operators*, Integral Transform. Spec. Funct. 15 (2004) 31–49.

[30] A.A. Kilbas, H.M. Srivastava, J.J. Trujillo, *Theory and Applications of Fractional Differential Equations*, North-Holland Mathematical Studies Vol. 204, Elsevier, Amsterdam, 2006.

[31] V. Kiryakova, *Generalized Fractional Calculus and Applications*, CRC Press, 1993.

[32] V. Kiryakova, *The special functions of fractional calculus as generalized fractional calculus operators of some basic functions*, Comput. Math. Appl. 59 (2010) 1128–1141.

[33] J. Liouville, *Mémoire sur quelques questions de géométrie et de mécanique et sur un nouveau genre de calcul pour résoudre ces questions*, J. École Polytech. 13 (21) (1832) 1–69.

[34] C.F. Lorenzo, T.T. Hartley, *Generalized functions for the fractional calculus*, NASA TP 209424 (1999).

[35] C.F. Lorenzo, T.T. Hartley, *The Fractional Trigonometry: With Application to Fractional Differential Equations and Science*, John Wiley & Sons, Hoboken, New Jersey, 2017.

[36] A. Mahmood, S. Parveen, A. Ara, N.A. Khan, *Exact analytic solution for the unsteady flow of a non-Newtonian fluid between two cylinders with fractional derivative model*, Commun. Nonlinear Sci. Numer. Simul. 14 (2009) 3309–3319.

[37] F. Mainardi, *Fractional Calculus and Waves in Linear Viscoelasticity*, Imperial College Press, London, 2010.

[38] A.M. Mathai, H.J. Haubold, *Special Functions for Applied Scientists*, Springer, New York, 2008.

[39] A.M. Mathai, H.J. Haubold, *Introduction to Fractional Calculus*, Nova Science Publishers, 2017.

[40] D. Matignon, *Stability properties for generalized fractional differential systems*, ESAIM Proc. Systèmes Différentiels Fractionnaires – Modèles, Methods et Applications 5 (1998).

[41] C. Milici, G. Drăgănescu, J.A.T. Machado, *Introduction to Fractional Differential Equations*, Nonlinear Syst. Complexity Vol. 25, Springer, Switzerland, 2019.

[42] K.S. Miller, B. Ross, *An Introduction to the Fractional Calculus and Fractional Differential Equations*, Wiley, New York, 1993.

[43] R. Nawaz, A. Khattak, M. Akbar, S. Ahsan, Z. Shah, A. Khan, *Solution of fractional-order integro-differential equations using optimal homotopy asymptotic method*, J. Thermal Anal. Calorimetry (2020) 1–13.

[44] K.B. Oldham, J. Spanier, *The Fractional Calculus: Theory and Applications of Differentiation and Integration to Arbitrary Order*, Academic Press, San Diego, 1974.

[45] S.C. Pandey, *The Lorenzo-Hartley function for fractional calculus and its applications pertaining to fractional order modelling of anomalous relaxation in dielectrics*, Comput. Appl. Math. 37 (3) (2018) 2648–2666.

[46] S.C. Pandey, *On some computable solutions of unified families of fractional differential equations*, São Paulo J. Math. Sci. (2022) 1280–1308.

[47] I. Podlubny, *The Laplace transform method for linear differential equations of the fractional order*, Inst. Exper. Phys. Slovak Acad. Sci. 2 (1994) 1–35.

[48] I. Podlubny, *Fractional Differential Equations*, Academic Press, San Diego, 1999.

[49] F. Polito, Ž. Tomovski, *Some properties of Prabhakar-type fractional calculus operators*, Fract. Differ. Calculus 6 (1) (2016) 73–94.

[50] T.R. Prabhakar, *A singular integral equation with a generalized Mittag-Leffler function in the kernel*, Yokohama Math. J. 19 (1971) 7–15.

[51] A.P. Prudnikov, I.A. Brychkov, O.I. Marichev, *Integrals and Series: Special Functions* Vol. 2, CRC Press, 1986.

[52] I.M. Ryzhik, I.S. Gradshteĭn, *Table of Integrals, Series, and Products*, Academic Press, New York, 1965.

[53] U.K. Saha, L.K. Arora, A.K. Arora, *On the relationships of the R-function of Lorenzo and Hartley with other special functions of fractional calculus*, Fract. Calc. Appl. Anal. 12 (4) (2009) 453–458.

[54] S.G. Samko, A.A. Kilbas, O.I. Marichev, *Fractional Integrals and Derivatives: Theory and Applications*, Gordon and Breach, Yverdon, Switzerland, 1993.

[55] R.K. Saxena, S.L. Kalla, *On a fractional generalization of the free electron laser equation*, Appl. Math. Comput. 143 (1) (2003) 89–97.

[56] R. Saxena, S.L. Kalla, L. Galue, *On a fractional generalization of the free electron laser equations associated with generalized Mittag-Leffler function and a confluent hypergeometric function of two variables*, J. Inequalities Spec. Funct. 8 (2) (2017) 157–165.

[57] R.K. Saxena, J. Ram, D. Kumar, *Alternative derivation of generalized fractional kinetic equations*, J. Fract. Calc. Appl. 4 (2) (2013) 322–334.

[58] A. Shakeel, S. Ahmad, H. Khan, H. Vieru, *Solutions with Wright functions for time fractional convection flow near a heated vertical plate*, Adv. Differ. Equations 51 (2016) 1–11.

[59] H.M. Srivastava, *Remarks on some families of fractional-order differential equations*, Integral Transform. Spec. Funct. 28 (7) (2017) 560–564.

[60] H.M. Srivastava, R.K. Saxena, *Some Volterra-type fractional integro-differential equations with a multivariable confluent hypergeometric function as their kernel*, J. Integral Equations Appl. 17 (2005) 199–217.

[61] H.M. Srivastava, Ž. Tomovski, *Fractional calculus with an integral operator containing a generalized Mittag-Leffler function in the kernel*, Appl. Math. Comput. 211 (2009) 198–210.

[62] H. Sun, Y. Zhang, D. Baleanu, W. Chen, Y. Chen, *A new collection of real world applications of fractional calculus in science and engineering*, Commun. Nonlinear Sci. Numer. Simul. 64 (2018) 213–231.

[63] Ž. Tomovski, R. Hilfer, H.M. Srivastava, *Fractional and operational calculus with generalized fractional derivative operators and Mittag-Leffler type functions*, Integral Transform. Spec. Funct. 21 (11) (2010) 797–814.

[64] X.J. Yang, *General Fractional Derivatives: Theory, Methods and Applications*, CRC Press, 2019.

[65] X.J. Yang, F. Gao, Y. Ju, *General Fractional Derivatives with Applications in Viscoelasticity*, Academic Press, 2020.

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Pure Strategy Solutions in the Progressive Discrete Silent Duel with Identical Linear Accuracy Functions and Shooting Uniform Jitter

Vadim Romanuke

Abstract: The progressive discrete silent duel is studied modeling limited observability within a system in order to make the best discrete-time decision. The moments to make a decision (to take an action, to shoot a bullet) are scheduled beforehand. The kernel of the duel is skew-symmetric, and the duelists (players) have identical linear accuracy functions. The duel is a finite zero-sum game defined on a subset of the unit square. As the duel starts, time moments of possible shooting become denser by a geometric progression. Apart from the duel beginning and end time moments, every following time moment is the partial sum of the respective geometric series, to which a value of the jitter is added. Regardless of the jitter, both the duelists have the same optimal strategies and the game optimal value is 0 due to the skew-symmetry. The only optimal behavior of the duelist at any positive jitter is to shoot at the positively jittered middle of the duel time span. The only optimal behavior of the duelist in the 3×3 duel at any negative jitter is to shoot at the very end of the duel. In the 4×4 and bigger duels, there is an open interval of the negative jitter, between $\frac{\sqrt{17}-5}{8}$ and 0, at which the duel does not have a pure strategy solution. Value $-\frac{1}{4}$ is the boundary case of the negative jitter, at which the 4×4 duel has four versions of the solution. At any other negative jitter, the only optimal behavior of the duelist in the 4×4 duel is to shoot at the very end of the duel. Bigger duels are more affected by negative jitter. There are two intervals of the pure strategy solution nonexistence in 5×5 and bigger duels, one of which is mentioned above, and the other one approaches to interval $\left(-\frac{1}{2};\ -\frac{1}{4}\right)$ on the left endpoint from the right.

AMS Subject Classification: 91A05, 91A50, 91A55.
Keywords and Phrases: Game of timing; Silent duel; Linear accuracy function; Matrix game; Pure strategy solution; Jittered possible shooting moments.

1. Introduction and motivation

Games of timing that represent a wide class of competitive interaction models are intended to develop rational time decisions for participants under limited system observability [2, 3, 6, 14]. The player (participant) must make a decision of innovation, adoption, response, etc., during a time span on which the game exists [16, 17, 15, 7]. If a decision is made in a two-person game commonly referred to as a duel, the other player either learns it or does not learn it until the duel ends. The latter is the case of the silent duel [19, 7, 9, 1], whose solution heavily depends on whether the game is finite or not [4, 7, 12], apart from the game symmetry [6, 3, 11].

A common pattern of the symmetric silent duel is a zero-sum game

$$\langle X,\ Y,\ K(x,\ y)\rangle \tag{1}$$

defined on unit square

$$X \times Y = [0;\ 1] \times [0;\ 1] \tag{2}$$

with kernel

$$K(x,\ y) = x - y + xy\,\mathrm{sign}\,(y - x)\,, \tag{3}$$

where $X = [0;\ 1]$ and $Y = [0;\ 1]$ are the sets of pure strategies of the duelists, in which the pure strategy is a time moment of possible shooting (i. e., making a decision). Obviously, kernel (3) is skew-symmetric:

$$K(y,\ x) = y - x + yx\,\mathrm{sign}\,(x - y) = -K(x,\ y)\,. \tag{4}$$

Game (1) by (2) and (3) is a silent duel with identical linear accuracy functions of the duelists, which are allowed to shoot at any moment during the duel time span $[0;\ 1]$. Owing to property (4), both the duelists in this duel have the same optimal strategies and the game optimal value is 0 [19, 7, 5].

To get rid of infinite supports in the duelists' optimal strategies [4, 19, 7, 18, 10], a discrete version of duel (1) is considered, where the sets of pure strategies of the duelists are

$$X = \{x_i\}_{i=1}^{N} = Y = \{y_j\}_{j=1}^{N} = T = \{t_q\}_{q=1}^{N} \subset [0;\ 1] \tag{5}$$

by

$$t_q < t_{q+1}\ \ \forall\, q = \overline{1,\ N-1}\ \text{ and }\ t_1 = 0,\ \ t_N = 1\ \text{ for }\ N \in \mathbb{N} \backslash \{1\}\,. \tag{6}$$

The discrete silent duel includes the moments of the duel beginning $x = y = 0$ and duel end $x = y = 1$. By (5), finite symmetric game (1) is a matrix game whose solution is of finite supports only [7, 19]. Moreover, the solution is computed far easier than that in the case of infinite game (1) by $X = [0;\ 1]$ and $Y = [0;\ 1]$.

A specific case of possible shooting moments $\{t_q\}_{q=1}^{N}$ is when they, still obeying (6), are assigned according to a geometrical progression:

$$t_q = \sum_{l=1}^{q-1} 2^{-l} = \frac{2^{q-1} - 1}{2^{q-1}} \quad \text{for} \quad q = \overline{2,\ N-1}. \tag{7}$$

In this case, the density of pure strategies of the duelist grows in the geometrical progression as the duelist approaches to the duel end [18, 8]. Apart from the duel beginning and end moments, every following moment is the partial sum of the respective geometric series. However, the precise assignment is not always realizable in practice (e. g., due to finite accuracy in measuring the distance between neighboring moments of possible shooting), so

$$t_q = \xi + \sum_{l=1}^{q-1} 2^{-l} = \xi + \frac{2^{q-1} - 1}{2^{q-1}} \quad \text{for} \quad q = \overline{2,\ N-1} \quad \text{and} \quad \xi \in \left(-\frac{1}{2};\ \frac{1}{2^{N-2}}\right) \tag{8}$$

instead of (7). The possible shooting moments $\{t_q\}_{q=2}^{N-1}$ specified by (8) is a shooting uniform jitter, which slightly moves points $\{t_q\}_{q=2}^{N-1}$ by (7) within the duel time span $[0;\ 1]$ not violating their relative order (topology) within $[t_2;\ t_{N-1}]$.

The case of $\xi = 0$ is the known progressive discrete silent duel (PDSD) with identical linear accuracy functions whose solutions are studied in [13]: the pure strategy solution is situation

$$\{x_2,\ y_2\} = \left\{\frac{1}{2},\ \frac{1}{2}\right\} \tag{9}$$

in 3×3 PDSDs and bigger. In PDSDs bigger than the 3×3 PDSD, optimal pure strategy situation (9) is the single one. For a trivial 3×3 PDSD, in which the duelist possesses just one moment of possible shooting between the duel beginning and end moments, any pure strategy situation, not containing the duel beginning moment, is optimal.

2. Objective and tasks to be fulfilled

The objective is to study pure strategy solutions of the PDSD

$$\left\langle \{x_i\}_{i=1}^{N},\ \{y_j\}_{j=1}^{N},\ \mathbf{K}_N \right\rangle \tag{10}$$

with identical linear accuracy functions and shooting uniform jitter by (8), where payoff matrix

$$\begin{gathered} \mathbf{K}_N = [k_{ij}]_{N \times N} \quad \text{by} \quad k_{ij} = K(x_i,\ y_j) = \\ = x_i - y_j + x_i y_j \operatorname{sign}(y_j - x_i) \quad \text{and} \quad N \in \mathbb{N} \setminus \{1\} \end{gathered} \tag{11}$$

and it is skew-symmetric, i. e. $\mathbf{K}_N = -\mathbf{K}_N^{\mathrm{T}}$ or

$$k_{ij} = -k_{ji} \ \ \forall\, i = \overline{1,\ N} \ \text{ and } \ \forall\, j = \overline{1,\ N}.$$

The primary task is to encompass all existing pure strategy solutions for

$$\xi \in \left(-\frac{1}{2};\ \frac{1}{2^{N-2}} \right) \setminus \{0\} \,. \tag{12}$$

The secondary task is to determine all ξ by (12) such that no pure strategy solution exists. Finally, the solution results are to be summarized along with recapitulating their peculiarities, whereupon the study is discussed and concluded in the last section.

3. Trivial cases

If the duelist is allowed to shoot at either the very beginning or end of the PDSD, this is the most trivial case, where $N = 2$ and the respective payoff matrix (11)

$$\mathbf{K}_2 = [k_{ij}]_{2\times 2} = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \tag{13}$$

does not depend on ξ. The single optimal solution here is pure strategy situation

$$\{x_2,\ y_2\} = \{1,\ 1\} \,. \tag{14}$$

The case with payoff matrix (13) is not referenced further.

It is worth mentioning that

$$K\left(x_1,\ y_j\right) = K\left(0,\ y_j\right) = -y_j < 0 \ \ \forall\, j = \overline{2,\ N}$$

and therefore the minimum of the first row of matrix (11) does not exceed $-1 < 0$ and thus the game optimal value $v_{\mathrm{opt}} = 0$ cannot be reached in this row, whichever number N is. So, the first row of matrix (11) cannot be an optimal pure strategy of the first duelist (the first row does not contain saddle points). Due to the skew-symmetry of matrix (11), the stated inference is immediately followed by that the first column does not contain saddle points either (the first column cannot be an optimal pure strategy of the second duelist). In the further consideration, only the inferences on saddle points in definite rows of matrix (11), which imply the same inferences on saddle points in respective columns, will be stated. As only a zero entry of matrix (11) can be a saddle point, then a row containing a negative entry does not contain saddle points; neither does the respective column containing the positive entry. Meanwhile, a nonnegative row contains a saddle point on the main diagonal of the payoff matrix. A row whose entries are positive, except for the main diagonal entry, contains a single saddle point which is the single one in such a duel (all the other $N - 1$ rows of the respective column contain negative entries).

In the next case of triviality, when the shooting, apart from the very beginning and end moments $t_1 = 0$, $t_3 = 1$, is also allowed at moment $t_2 = \dfrac{1}{2}$, the solution depends on the sign of ξ. The following assertion supplements the abovementioned case of $\xi = 0$ [13].

Theorem 1. *In a PDSD (10) by (8) and (11) for $N = 3$, pure strategy situation*

$$\{x_2,\ y_2\} = \left\{\frac{1}{2} + \xi,\ \frac{1}{2} + \xi\right\} \tag{15}$$

is solely optimal by $\xi > 0$, whereas pure strategy situation

$$\{x_3,\ y_3\} = \{1,\ 1\} \tag{16}$$

is solely optimal by $\xi < 0$.

Proof. Due to $k_{13} = -1$, situation

$$\{x_1,\ y_1\} = \{0,\ 0\} \tag{17}$$

is never optimal in the PDSD. The respective payoff matrix is

$$\mathbf{K}_3 = [k_{ij}]_{3\times 3} = \begin{bmatrix} 0 & -\frac{1}{2} - \xi & -1 \\ \frac{1}{2} + \xi & 0 & 2\xi \\ 1 & -2\xi & 0 \end{bmatrix}. \tag{18}$$

If $\xi > 0$ then the second row of matrix (18) is nonnegative and the third row contains a negative entry. The only zero entry in the second row is k_{22}, whence situation (15) is optimal and it is the single saddle point for the 3×3 PDSD with kernel (18) by $\xi > 0$.

If $\xi < 0$ then the second row of matrix (18) contains a negative entry, and thus the second row does not contain saddle points. The third row is nonnegative and its single zero entry is k_{33}, whence situation (16) is optimal and it is the single saddle point for the 3×3 PDSD with kernel (18) by $\xi < 0$. □

In the further consideration, the case with $\xi > 0$ will be called a positive jitter, and the case with $\xi < 0$ will be called a negative jitter. Time moment

$$t_q = \xi + \frac{2^{q-1} - 1}{2^{q-1}} \ \text{ at } \ q \in \left\{\overline{2,\ N-1}\right\}$$

will be called positively ξ-jittered moment and negatively $|\xi|$-jittered moment by $\xi > 0$ and $\xi < 0$, respectively.

4. The positive jitter duel solution

In fact, Theorem 1 determines the single solution of the 3×3 PDSD with a positive jitter, according to which the best decision is made right after the duel passes its start. The question of whether this property remains for bigger PDSDs is answered by the following assertion.

Theorem 2. *In a PDSD (10) by (8) and (11) for $N \in \mathbb{N} \backslash \{1,\ 2\}$, pure strategy situation (15) is solely optimal by $\xi > 0$.*

Proof. Due to Theorem 1, situation (15) is the single saddle point for $N = 3$. For $N \in \mathbb{N} \backslash \{1,\ 2,\ 3\}$ consider entry k_{22} that is the result of when both the duelists simultaneously shoot at the positively ξ-jittered middle of the duel time span corresponding to situation (15). This entry is in the second row of matrix (11), where

$$K\left(x_2,\ y_1\right) = K\left(\frac{1}{2}+\xi,\ 0\right) = \frac{1}{2}+\xi > 0 \tag{19}$$

and

$$K\left(x_2,\ y_j\right) = K\left(\frac{1}{2}+\xi,\ y_j\right) = \frac{1}{2}+\xi - y_j + \left(\frac{1}{2}+\xi\right)\cdot y_j =$$
$$= \frac{1}{2}+\xi-\frac{1}{2}y_j+\xi y_j > 0 \ \text{ by } \ j=\overline{3,\ N} \ \text{ and } \ \xi>0 \tag{20}$$

inasmuch as $\frac{1}{2}-\frac{1}{2}y_j \geqslant 0$ for $0 \leqslant y_j \leqslant 1$. So, the second row of matrix (11), apart from the main diagonal entry k_{22}, is positive and therefore situation (15) is optimal; the second row does not contain any other saddle points. Inequalities (19) and (20) also imply that entries $k_{i2} < 0\ \forall\, i=\overline{3,\ N}$ in the second column, so saddle point (15) is the single one by $\xi > 0$. □

5. Negative jitter duel solutions

In the 3×3 PDSD with a negative jitter, according to Theorem 1, the best decision is made at the very end of the duel. This rule is generally broken in bigger PDSDs.

Theorem 3. *In a PDSD (10) by (8) and (11) for $N \in \mathbb{N}\backslash\{1,\ 2,\ 3\}$, pure strategy situation*

$$\{x_3,\ y_3\} = \left\{\frac{3}{4}+\xi,\ \frac{3}{4}+\xi\right\} \tag{21}$$

is solely optimal by

$$\xi \in \left(-\frac{1}{4};\ \frac{\sqrt{17}-5}{8}\right]. \tag{22}$$

Proof. Inasmuch as

$$K\left(x_2,\ y_N\right) = K\left(\frac{1}{2}+\xi,\ 1\right) =$$
$$= \frac{1}{2}+\xi-1+\frac{1}{2}+\xi = 2\xi < 0 \ \text{ by } \ \xi<0, \tag{23}$$

the second row of matrix (11) does not contain saddle points, whichever the negative jitter is. In the third row, the first entry is

$$k_{31} = K\left(x_3,\ y_1\right) = K\left(\frac{3}{4}+\xi,\ 0\right) = \frac{3}{4}+\xi > 0 \ \text{ by } \ \xi \in \left(-\frac{1}{2};\ 0\right), \tag{24}$$

the second entry is

$$k_{32} = K\left(x_3,\ y_2\right) = K\left(\frac{3}{4}+\xi,\ \frac{1}{2}+\xi\right) =$$

$$= \frac{3}{4} + \xi - \left(\frac{1}{2} + \xi\right) - \left(\frac{3}{4} + \xi\right)\left(\frac{1}{2} + \xi\right) =$$

$$= -\frac{1}{8} - \frac{5}{4}\xi - \xi^2 \geqslant 0 \text{ by } \xi \in \left[\frac{-\sqrt{17}-5}{8};\ \frac{\sqrt{17}-5}{8}\right], \tag{25}$$

where

$$\frac{-\sqrt{17}-5}{8} < -1 < -\frac{1}{2} < -\frac{1}{4} < \frac{\sqrt{17}-5}{8} < 0. \tag{26}$$

The remaining entries of the third row, apart from (24), (25), $k_{33} = 0$, are

$$k_{3j} = K(x_3,\ y_j) = K\left(\frac{3}{4} + \xi,\ y_j\right) = \frac{3}{4} + \xi - y_j + \left(\frac{3}{4} + \xi\right) \cdot y_j =$$

$$= \frac{3}{4} + \xi - \frac{1}{4}y_j + \xi y_j \ \ \forall\, y_j > \frac{3}{4} + \xi \text{ by } j = \overline{4,\ N}, \tag{27}$$

whence

$$\frac{3}{4} + \xi - \frac{1}{4}y_j + \xi y_j \geqslant \frac{1}{2} + 2\xi > 0 \text{ by } \xi > -\frac{1}{4}. \tag{28}$$

So,

$$k_{3j} = K(x_3,\ y_j) > 0 \ \ \forall\, y_j > \frac{3}{4} + \xi \text{ by } j = \overline{4,\ N} \text{ and } \xi > -\frac{1}{4}. \tag{29}$$

With using (24)—(29), the third row of matrix (11), apart from the main diagonal entry k_{33}, is positive by

$$\xi \in \left(-\frac{1}{4};\ \frac{\sqrt{17}-5}{8}\right). \tag{30}$$

Therefore, situation (21) is solely optimal by (30). If $\xi = \frac{\sqrt{17}-5}{8}$ then, according to (25), $k_{32} = 0$, while still entries (27) are positive by (28); but as the second row does not contain saddle points, situation (21) remains solely optimal by (22). □

Inequality (23) means that any negative jitter precludes optimality of situation (15). By a negative jitter, shooting straight after the duel begins (at the time moment following the very beginning) is not optimal. The optimality jumps one moment farther by (22), still being achieved without mixing pure strategies. As it will turn out below, the PDSD is not solved in pure strategies at shallower negative jitter compared to (22).

Theorem 4. *No pure strategy solutions exist in a PDSD (10) by (8) and (11) for* $N \in \mathbb{N} \backslash \{1,\ 2,\ 3\}$ *by*

$$\xi \in \left(\frac{\sqrt{17}-5}{8};\ 0\right). \tag{31}$$

Proof. If (31) holds, then, using (25), $k_{32} < 0$, i.e. the third row does not contain saddle points. As (29) is true (the third row entries above the main diagonal are positive),

$$k_{j3} = -k_{3j} = -K\left(x_3,\ y_j\right) < 0\ \ \forall\, y_j > \frac{3}{4} + \xi$$
$$\text{by}\ \ j = \overline{4,\ N}\ \ \text{and}\ \ \xi > \frac{\sqrt{17} - 5}{8} > -\frac{1}{4}, \tag{32}$$

i. e. the third column entries below the main diagonal are negative and rows whose number $i = \overline{4,\ N}$ do not contain saddle points. Consequently, the PDSD is not solved in pure strategies by (31). □

Another interesting aspect is when the negative jitter equals to the left endpoint of the half-open interval in (22). This boundary case is treated differently for the 4×4 PDSD and bigger ones.

Theorem 5. *At* $\xi = -\frac{1}{4}$ *a PDSD (10) by (8) and (11) for* $N \in \mathbb{N} \backslash \{1,\ 2,\ 3,\ 4\}$ *has a single optimal situation*

$$\{x_3,\ y_3\} = \left\{\frac{1}{2},\ \frac{1}{2}\right\}. \tag{33}$$

Proof. If $\xi = -\frac{1}{4}$ then $k_{31} = \frac{1}{2}$, $k_{32} = \frac{1}{8}$,

$$k_{3N} = K\left(x_3,\ y_N\right) = K\left(\frac{1}{2},\ 1\right) = 0 \tag{34}$$

while

$$k_{3j} = K\left(x_3,\ y_j\right) = K\left(\frac{1}{2},\ y_j\right) =$$
$$= \frac{1}{2} - y_j + \frac{1}{2}y_j = \frac{1}{2} - \frac{1}{2}y_j > 0\ \ \forall\, y_j > \frac{3}{4} + \xi = \frac{1}{2}\ \ \text{by}\ \ j = \overline{4,\ N-1}. \tag{35}$$

So, the third row is nonnegative containing a saddle point on the main diagonal, which is situation (33). Besides, inequalities (35) imply that columns whose number $j = \overline{4,\ N-1}$ (or rows whose number $i = \overline{4,\ N-1}$) do not contain saddle points. Despite (34), the N-th column (row) for $N \geqslant 5$ does not contain saddle points either because

$$k_{N4} = K\left(x_N,\ y_4\right) = K\left(1,\ -\frac{1}{4} + \frac{2^3 - 1}{2^3}\right) =$$
$$= K\left(1,\ \frac{5}{8}\right) = 1 - 2 \cdot \frac{5}{8} = -\frac{1}{4} < 0, \tag{36}$$

and thus saddle point (33) is the single one at $\xi = -\frac{1}{4}$ and $N \geqslant 5$. □

Theorem 6. *The* 4×4 *PDSD (10) by (8) and (11) at* $\xi = -\frac{1}{4}$ *has four optimal situations: (33),*

$$\{x_4,\ y_4\} = \{1,\ 1\}, \tag{37}$$

$$\{x_3,\ y_4\} = \left\{\frac{1}{2},\ 1\right\}, \tag{38}$$

$$\{x_4,\ y_3\} = \left\{1,\ \frac{1}{2}\right\}. \tag{39}$$

Proof. It is easy to see that the payoff matrix of the 4×4 PDSD at $\xi = -\dfrac{1}{4}$ is

$$\mathbf{K}_4 = [k_{ij}]_{4\times 4} = \begin{bmatrix} 0 & -\frac{1}{4} & -\frac{1}{2} & -1 \\ \frac{1}{4} & 0 & -\frac{1}{8} & -\frac{1}{2} \\ \frac{1}{2} & \frac{1}{8} & 0 & 0 \\ 1 & \frac{1}{2} & 0 & 0 \end{bmatrix}. \tag{40}$$

Payoff matrix (40) has four saddle points (33), (37) — (39). □

Finally, the case when

$$\xi \in \left(-\frac{1}{2};\ -\frac{1}{4}\right) \tag{41}$$

is to be considered. Once again, 4×4 PDSDs are recognized differently from bigger PDSDs, which will be shown in the following two assertions. Besides, a subinterval within interval (41) will be determined, by which the PDSDs bigger than the 4×4 PDSD are not solved in pure strategies.

Theorem 7. *In a PDSD (10) by (8) and (11) for $N \in \mathbb{N} \backslash \{1,\ 2,\ 3,\ 4\}$, pure strategy situation*

$$\{x_N,\ y_N\} = \{1,\ 1\} \tag{42}$$

is solely optimal by

$$\xi \in \left(-\frac{1}{2};\ -\frac{1}{2} + \frac{1}{2^{N-2}}\right]. \tag{43}$$

Proof. Then N-th entry in the third row of matrix (11) is

$$k_{3N} = K(x_3,\ y_N) = K\left(\frac{3}{4} + \xi,\ 1\right) = \frac{1}{2} + 2\xi < 0 \ \text{ by } \ \xi < -\frac{1}{4}, \tag{44}$$

so situation (21) is not optimal. The last row of matrix (11) contains saddle point (42) if

$$k_{Nj} = K(x_N,\ y_j) = K(1,\ y_j) = 1 - 2y_j \geqslant 0 \ \ \forall\, j = \overline{1,\ N} \tag{45}$$

or, briefly,

$$y_j \leqslant \frac{1}{2} \ \ \forall\, j = \overline{2,\ N-1} \tag{46}$$

owing to $y_1 = 0 \leqslant \frac{1}{2}$ and $k_{NN} = 0$ regardless of $y_N = 1 > \frac{1}{2}$. Using (8), inequality (46) is re-written as

$$y_j = \xi + \frac{2^{j-1} - 1}{2^{j-1}} \leqslant \frac{1}{2},$$

whence

$$\xi \leqslant \frac{1}{2^{j-1}} - \frac{1}{2} < -\frac{1}{4} \text{ by } j = \overline{2, N-1}. \tag{47}$$

The strict inequality in (47) is $\frac{1}{2^{j-1}} < \frac{1}{4}$ or $2^{j-1} > 4$, which holds $\forall\, j = \overline{4, N-1}$ for $N \in \mathbb{N} \setminus \{1, 2, 3, 4\}$ by the least possible value $-\frac{1}{2} + \frac{1}{2^{N-2}}$ of the negative jitter. Therefore, situation (42) is optimal if (43) is true.

If

$$\xi \in \left(-\frac{1}{2}; -\frac{1}{2} + \frac{1}{2^{N-2}}\right) \text{ for } N \in \mathbb{N} \setminus \{1, 2, 3, 4\} \tag{48}$$

then inequality (46) holds strictly, that is

$$k_{Nj} = K(x_N, y_j) = K(1, y_j) = 1 - 2y_j > 0 \;\; \forall\, j = \overline{1, N-1}, \tag{49}$$

whence the N-th row of matrix (11), apart from the main diagonal entry k_{NN}, is positive and therefore optimal situation (42) is the single one. If

$$\xi = -\frac{1}{2} + \frac{1}{2^{N-2}} \text{ for } N \in \mathbb{N} \setminus \{1, 2, 3, 4\} \tag{50}$$

then

$$y_{N-1} = \xi + \frac{2^{N-2} - 1}{2^{N-2}} = -\frac{1}{2} + \frac{1}{2^{N-2}} + \frac{2^{N-2} - 1}{2^{N-2}} = \frac{1}{2} \tag{51}$$

and

$$y_j < y_{N-1} = \frac{1}{2} \;\; \forall\, j = \overline{2, N-2}, \tag{52}$$

where (51) and (52) imply that

$$k_{Nj} = K(x_N, y_j) = K(1, y_j) = 1 - 2y_j > 0 \;\; \forall\, j = \overline{1, N-2}, \tag{53}$$

that is the first $N-2$ entries of the N-th row are positive. Next,

$$k_{N,N-1} = 1 - 2y_{N-1} = 0 = k_{N-1,N},$$

but

$$x_{N-1} = \frac{1}{2},$$

$$y_{N-2} = \xi + \frac{2^{N-3} - 1}{2^{N-3}} = -\frac{1}{2} + \frac{1}{2^{N-2}} + 1 - \frac{1}{2^{N-3}} = \frac{1}{2} - \frac{1}{2^{N-2}},$$

and

$$k_{N-1,N-2} = K\left(x_{N-1},\ y_{N-2}\right) = K\left(\frac{1}{2},\ \frac{1}{2} - \frac{1}{2^{N-2}}\right) =$$
$$= \frac{1}{2} - \frac{1}{2} + \frac{1}{2^{N-2}} - \frac{1}{2} \cdot \left(\frac{1}{2} - \frac{1}{2^{N-2}}\right) =$$
$$= \frac{1}{2^{N-2}} - \frac{1}{4} + \frac{1}{2^{N-1}} = \frac{3}{2^{N-1}} - \frac{1}{4} < 0 \ \text{ for } \ N \in \mathbb{N} \backslash \left\{1,\ 2,\ 3,\ 4\right\},$$

which implies that the $(N-1)$-th row of matrix (11) does not contain saddle points by (50). □

Theorem 8. *In the* 4×4 *PDSD (10) by (8) and (11), situation (42) is solely optimal by (41).*

Proof. At $N = 4$
$$\left(-\frac{1}{2};\ -\frac{1}{2} + \frac{1}{2^{N-2}}\right) = \left(-\frac{1}{2};\ -\frac{1}{4}\right).$$
In the respective 4×4 PDSD inequality (46) holds as
$$y_{N-1} = y_3 = \xi + \frac{3}{4} < \frac{1}{2} \ \text{ by } \ \xi < -\frac{1}{4}, \tag{54}$$
so situation (42) is optimal as well. Besides, it is solely optimal due to inequality (49) holds after (54). □

Theorem 9. *No pure strategy solutions exist in a PDSD (10) by (8) and (11) for* $N \in \mathbb{N} \backslash \left\{1,\ 2,\ 3,\ 4\right\}$ *by*
$$\xi \in \left(-\frac{1}{2} + \frac{1}{2^{N-2}};\ -\frac{1}{4}\right). \tag{55}$$

Proof. The first row of matrix (11) does not contain saddle points; the second row does not contain saddle points due to (23) holds; the third row does not contain saddle points due to (44) holds. Consider entry k_{nn} in matrix (11) for $n \in \left\{\overline{4,\ N-1}\right\}$ and $N \in \mathbb{N} \backslash \left\{1,\ 2,\ 3,\ 4\right\}$. This entry is the result of when both the duelists shoot at moment
$$t_n = \xi + \frac{2^{n-1} - 1}{2^{n-1}} \tag{56}$$
corresponding to situation
$$\left\{x_n,\ y_n\right\} = \left\{\xi + \frac{2^{n-1} - 1}{2^{n-1}},\ \xi + \frac{2^{n-1} - 1}{2^{n-1}}\right\}. \tag{57}$$
If situation (57) is optimal, then, in the n-th row of matrix (11), inequality
$$k_{nj} = K\left(x_n,\ y_j\right) = x_n - y_j - x_n y_j \geqslant 0 \ \ \forall\, y_j < x_n \ \text{ by } \ j = \overline{1,\ n-1} \tag{58}$$
must hold. From inequality (58) it follows that
$$\frac{x_n}{1 + x_n} \geqslant y_j \ \ \forall\, y_j < x_n \ \text{ by } \ j = \overline{1,\ n-1}. \tag{59}$$

As

$$y_j \leqslant \xi + \frac{2^{n-2}-1}{2^{n-2}} = y_{n-1} < \xi + \frac{2^{n-1}-1}{2^{n-1}} = x_n, \tag{60}$$

then inequality (59) is transformed into

$$\left(\xi + \frac{2^{n-1}-1}{2^{n-1}}\right) \cdot \frac{1}{1 + \xi + \dfrac{2^{n-1}-1}{2^{n-1}}} \geqslant \xi + \frac{2^{n-2}-1}{2^{n-2}},$$

$$\frac{\xi \cdot 2^{n-1} + 2^{n-1} - 1}{2^n + \xi \cdot 2^{n-1} - 1} \geqslant \frac{\xi \cdot 2^{n-2} + 2^{n-2} - 1}{2^{n-2}},$$

$$\frac{\left(\xi \cdot 2^{n-1} + 2^{n-1} - 1\right) \cdot 2^{n-2} - \left(2^n + \xi \cdot 2^{n-1} - 1\right) \cdot \left(\xi \cdot 2^{n-2} + 2^{n-2} - 1\right)}{\left(2^n + \xi \cdot 2^{n-1} - 1\right) \cdot 2^{n-2}} \geqslant 0. \tag{61}$$

It is clear that $2^n + \xi \cdot 2^{n-1} - 1 > 0$ in the denominator of the fraction in (61), so inequality (61) holds as the numerator of the fraction in (61)

$$\begin{gathered} \xi \cdot 2^{2n-3} + 2^{2n-3} - 2^{n-2} - \\ -\left(\xi \cdot 2^{2n-2} + \xi^2 \cdot 2^{2n-3} - \xi \cdot 2^{n-2} + 2^{2n-2} + \xi \cdot 2^{2n-3} - \right. \\ \left. -2^{n-2} - 2^n - \xi \cdot 2^{n-1} + 1\right) = \\ = -\xi^2 \cdot 2^{2n-3} + \xi \cdot \left(2^{2n-3} - 2^{2n-2} + 2^{n-2} - 2^{2n-3} + 2^{n-1}\right) + \\ +2^{2n-3} - 2^{n-2} - 2^{2n-2} + 2^{n-2} + 2^n - 1 = \\ = -\xi^2 \cdot 2^{2n-3} + \xi \cdot 2^{n-2} \cdot (3 - 2^n) - 2^{2n-3} + 2^n - 1 \geqslant 0. \end{gathered} \tag{62}$$

The discriminant of the respective quadratic equation

$$-\xi^2 \cdot 2^{2n-3} + \xi \cdot 2^{n-2} \cdot (3 - 2^n) - 2^{2n-3} + 2^n - 1 = 0 \tag{63}$$

is

$$\begin{gathered} D = 2^{2n-4} \cdot (3 - 2^n)^2 + 4 \cdot 2^{2n-3} \cdot \left(-2^{2n-3} + 2^n - 1\right) = \\ = 2^{2n-4} \cdot \left(9 - 6 \cdot 2^n + 2^{2n} - 8 \cdot 2^{2n-3} + 8 \cdot 2^n - 8\right) = 2^{2n-4} \cdot \left(1 + 2^{n+1}\right), \end{gathered}$$

whence (63) holds by

$$\begin{gathered} \xi = \frac{-2^{n-2} \cdot (3 - 2^n) - \sqrt{2^{2n-4} \cdot (1 + 2^{n+1})}}{-2^{2n-2}} = \\ = \frac{2^{n-2} \cdot (3 - 2^n) + 2^{n-2} \cdot \sqrt{(1 + 2^{n+1})}}{2^{2n-2}} = \frac{3 - 2^n + \sqrt{1 + 2^{n+1}}}{2^n} \end{gathered}$$

and

$$\xi = \frac{3 - 2^n - \sqrt{1 + 2^{n+1}}}{2^n}.$$

So, (62) is true by

$$\xi \in \left[\frac{3 - 2^n - \sqrt{1 + 2^{n+1}}}{2^n};\ \frac{3 - 2^n + \sqrt{1 + 2^{n+1}}}{2^n}\right]. \tag{64}$$

At $n = 4$,

$$\frac{3 - 2^4 + \sqrt{1 + 2^5}}{2^4} = \frac{\sqrt{33} - 13}{16} \in (-0.5;\ -0.45)\,,$$

but

$$k_{43} = K\,(x_4,\ y_3) = K\left(\xi + \frac{7}{8},\ \xi + \frac{3}{4}\right) =$$
$$= \xi + \frac{7}{8} - \left(\xi + \frac{3}{4}\right) - \left(\xi + \frac{7}{8}\right)\left(\xi + \frac{3}{4}\right) =$$
$$= \frac{1}{8} - \xi^2 - \xi \cdot \frac{13}{8} - \frac{21}{32} = -\xi^2 - \xi \cdot \frac{13}{8} - \frac{17}{32} \geqslant 0$$
$$\text{by}\ \ \xi \in \left[-\frac{\sqrt{33} + 13}{16};\ \frac{\sqrt{33} - 13}{16}\right] \tag{65}$$

and

$$k_{4N} = K\,(x_4,\ y_N) = K\left(\xi + \frac{7}{8},\ 1\right) =$$
$$= \xi + \frac{7}{8} - 1 + \xi + \frac{7}{8} = 2\xi + \frac{3}{4} \geqslant 0\ \text{ by }\ \xi \in \left[-\frac{3}{8};\ -\frac{1}{4}\right), \tag{66}$$

where

$$\frac{\sqrt{33} - 13}{16} < -\frac{3}{8}$$

and thus

$$\left[-\frac{\sqrt{33} + 13}{16};\ \frac{\sqrt{33} - 13}{16}\right] \cap \left[-\frac{3}{8};\ -\frac{1}{4}\right) = \emptyset.$$

The latter means that inequalities (65) and (66) are impossible together, and so the fourth row of matrix (11) does not contain saddle points.

At $n = 5$,

$$\frac{3 - 2^5 - \sqrt{1 + 2^6}}{2^5} = -\frac{\sqrt{65} + 29}{32} < -1.15,$$

$$\frac{3 - 2^5 + \sqrt{1 + 2^6}}{2^5} = \frac{\sqrt{65} - 29}{32} < -0.65,$$

i. e. inequality (62) holds only at ξ for which (55) is false, or, in other words, inequality (62) is impossible for $n = 5$ and (55). Denote $b = 2^n$ for $n \geqslant 5$ and consider the right endpoint of the interval in (64) as a function of 2^n:

$$\frac{3 - 2^n + \sqrt{1 + 2^{n+1}}}{2^n} = f\,(b) = \frac{3 - b + \sqrt{1 + 2b}}{b}. \tag{67}$$

The first derivative of function (67) is

$$\frac{df(b)}{db}=\frac{-b+\dfrac{2b}{2\sqrt{1+2b}}-3+b-\sqrt{1+2b}}{b^2}=$$
$$=\frac{b-3\sqrt{1+2b}-1-2b}{b^2\sqrt{1+2b}}=-\frac{3\sqrt{1+2b}+1+b}{b^2\sqrt{1+2b}}<0,$$

so function (67) is decreasing. Therefore,

$$\max_{b\geqslant 32} f(b)=\max_{b\geqslant 32}\frac{3-b+\sqrt{1+2b}}{b}=$$
$$=\max_{n\geqslant 5}\frac{3-2^n+\sqrt{1+2^{n+1}}}{2^n}=f\left(2^5\right)=$$
$$=\frac{\sqrt{65}-29}{32}<-0.65<-\frac{1}{2}+\frac{1}{2^{N-2}},\tag{68}$$

whence inequality (62) is impossible for $n\geqslant 5$ and (55). The latter means that situation (57) is not optimal also for $n\in\left\{\overline{5,\ N-1}\right\}$ and $N\in\mathbb{N}\backslash\{1,\ 2,\ 3,\ 4\}$.

In the last row of matrix (11),

$$k_{N,N-1}=K(x_N,\ y_{N-1})=K\left(1,\ \xi+\frac{2^{N-2}-1}{2^{N-2}}\right)=$$
$$=1-\left(\xi+\frac{2^{N-2}-1}{2^{N-2}}\right)-\left(\xi+\frac{2^{N-2}-1}{2^{N-2}}\right)=1-2\xi-\frac{2^{N-2}-1}{2^{N-3}}=$$
$$=-1-2\xi+\frac{1}{2^{N-3}}=2\cdot\left(-\frac{1}{2}+\frac{1}{2^{N-2}}\right)-2\xi<0$$

due to (55), and thus the payoff matrix of any PDSD (10) by (8) and (11) for $N\in\mathbb{N}\backslash\{1,\ 2,\ 3,\ 4\}$ does not contain saddle points by (55). □

6. Recapitulation

In the PDSD with a positive jitter, the only optimal behavior of the duelist is to shoot at the positively ξ-jittered middle of the duel time span. This is ascertained by Theorem 1 and Theorem 2. When a negative jitter exists, it is reasonable to consider 3×3 and 4×4 PDSDs separately from bigger PDSDs. The only optimal behavior of the duelist is to shoot at the very end in the 3×3 PDSD with a negative jitter (Theorem 1). The 4×4 PDSD with a negative jitter higher than $\dfrac{\sqrt{17}-5}{8}$ does not have a pure strategy solution (Theorem 4). Neither do bigger PDSDs by such a negative jitter (Theorem 4). The only optimal behavior of the duelist in the 4×4 PDSD with a negative jitter higher than $-\dfrac{1}{4}$ and not higher than $\dfrac{\sqrt{17}-5}{8}$ is to shoot at the negatively $|\xi|$-jittered moment following the negatively $|\xi|$-jittered middle of the duel time span (Theorem 3). Such a behavior remains optimal for bigger PDSDs as well (Theorem 3).

Value $-\frac{1}{4}$ is the boundary case of the negative jitter, at which, as Theorem 6 asserts, the 4×4 PDSD has four optimal situations whose strategies include only the duel end moment and middle of the duel time span (the latter is not the moment following the duel beginning moment, but it is the moment following the negatively $\frac{1}{4}$-jittered middle of the duel time span). Bigger PDSDs, however, have the single optimal situation at the middle of the duel time span (Theorem 5). Here, the assertion of Theorem 3 might have been modified in order to consider the closed interval between $-\frac{1}{4}$ and $\frac{\sqrt{17}-5}{8}$, by considering only 5×5 PDSDs and bigger, and thus to merge with Theorem 5.

The only optimal behavior of the duelist is to shoot at the very end in the 4×4 PDSD with a negative jitter higher than $-\frac{1}{2}$ and lower than $-\frac{1}{4}$ (Theorem 8). Such a behavior remains optimal for bigger $N \times N$ PDSDs with a negative jitter higher than $-\frac{1}{2}$ and not higher than $-\frac{1}{2} + \frac{1}{2^{N-2}}$ (Theorem 7). Such PDSDs do not have pure strategy solutions when a negative jitter falls between $-\frac{1}{2} + \frac{1}{2^{N-2}}$ and $-\frac{1}{4}$ (Theorem 9).

So, the positive jitter does not affect the possibility of implementing the best decision in a single action (or, in terms of the duel, in a single shot). In this case, all the possible shooting moments followed by the duel beginning moment are shifted towards the duel end moment. The negative jitter does not effect the 3×3 PDSD at all, but it affects the 4×4 PDSD at a lesser negative jitter, when its magnitude is below $\frac{5-\sqrt{17}}{8}$ (Figure 1). Nevertheless, the relative interval of the pure strategy solution nonexistence in the 4×4 PDSD with a negative jitter is narrower than the half-open interval between $-\frac{1}{2}$ and $\frac{\sqrt{17}-5}{8}$, at which the 4×4 PDSD is solved in pure strategies.

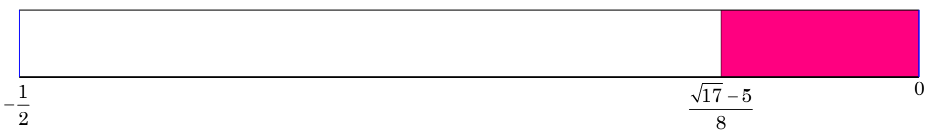

Figure 1: The relative interval of the pure strategy solution nonexistence in the 4×4 PDSD with a negative jitter

Bigger PDSDs, in which the duelist, apart from the duel beginning and end moments, possesses no fewer than three possible shooting moments, are affected to a more considerable extent. The negative jitter splits the open interval between $-\frac{1}{2}$

and 0 into four subintervals (Figure 2), at two of which an $N \times N$ PDSD has a single

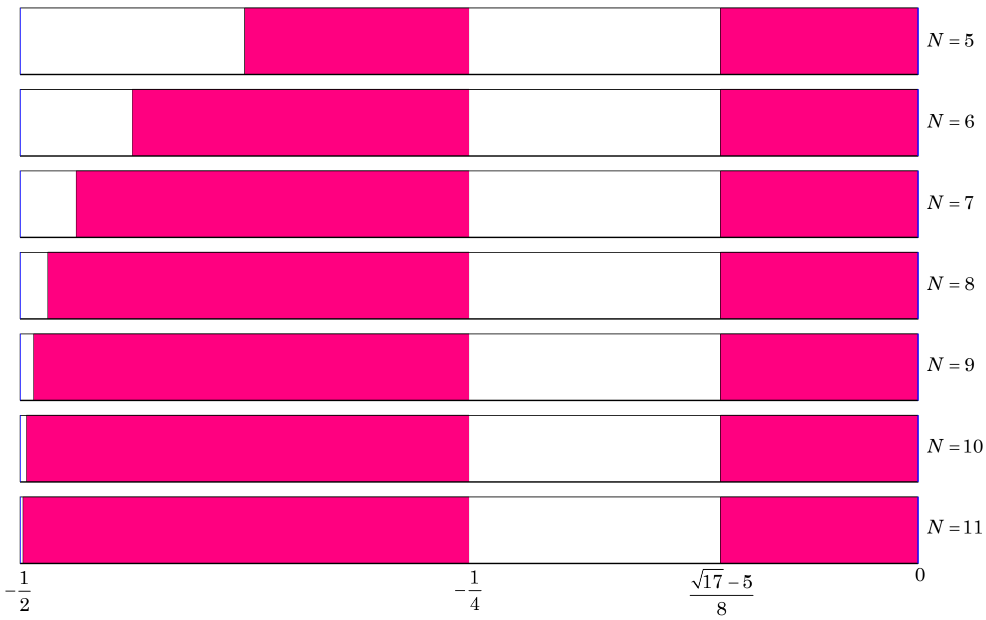

Figure 2: The change of the two relative intervals of the pure strategy solution nonexistence in $N \times N$ PDSDs with a negative jitter by $N = \overline{5,\ 11}$ (as the PDSD gets bigger)

optimal situation. Namely, if

$$\xi \in \left(-\frac{1}{2};\ -\frac{1}{2} + \frac{1}{2^{N-2}}\right] \cup \left[-\frac{1}{4};\ \frac{\sqrt{17}-5}{8}\right] \quad \text{for} \quad N \in \mathbb{N} \backslash \{1,\ 2,\ 3,\ 4\}\,, \tag{69}$$

then the $N \times N$ PDSD has a single optimal situation. As the PDSD gets bigger, the leftmost interval in (69) fades away.

7. Discussion and conclusion

The jitter is a substantially important component of a game-of-timing model that reflects imperfection of time setting and measurements. The duelists' accuracy functions presumed to be linear are hardly identical in practical applications as well, but their identity is attained on average. However, the considered shooting uniform jitter is just the first step in studying game-of-timing models with imperfection, where the symmetry is still maintained. Subsequently, the jitter may be considered non-uniform, with probably known statistical properties.

The importance of possessing an optimal pure strategy is hard to overestimate.

In real-world applications, it allows almost instantly implementing or starting to implement the best decision, unlike a mixed strategy requiring long-run repetitions of the game conditions without deviations. Pure strategy solutions in the PDSD with identical linear accuracy functions are guaranteed only for positive jitter. An exception from the rule exists for the trivial case, where any 3×3 PDSD is solved in pure strategies, whichever sign and magnitude of the jitter are.

The 4×4 PDSD does not have a pure strategy solution only if a negative jitter is higher than $\dfrac{\sqrt{17}-5}{8}$. Bigger PDSDs, in addition to this rule, have another open interval of the pure strategy solution nonexistence — it is (55) for $N \times N$ PDSDs, $N \in \mathbb{N}\backslash\{1, 2, 3, 4\}$. The latter interval gets wider as N increases, approaching to interval $\left(-\dfrac{1}{2}; -\dfrac{1}{4}\right)$ on the left endpoint from the right. The narrowest interval of pure strategy solution nonexistence at $\xi < -\dfrac{1}{4}$ has the length of $\dfrac{1}{8}$ (it is when $N = 5$), which is more than 1.14 times longer than the low-negative-jitter interval of pure strategy solution nonexistence (31). As N increases (i. e., the PDSD gets bigger), the ultimately-high-negative-jitter interval (43) gets narrower, making the only optimal behavior of the duelist to shoot at the very end of the duel less probable compared to other intervals of negative jitter.

References

[1] E.N. Barron, *Game Theory : An Introduction* (2nd ed.), Wiley, Hoboken, New Jersey, 2013.

[2] R.A. Epstein, *The Theory of Gambling and Statistical Logic* (2nd ed.), Academic Press, Burlington, Massachusetts, 2013.

[3] S. Karlin, *The Theory of Infinite Games. Mathematical Methods and Theory in Games, Programming, and Economics*, Pergamon, London — Paris, 1959.

[4] J.P. Lang, G. Kimeldorf, *Duels with continuous firing*, Management Science 2 (4) (1975) 470 — 476.

[5] M.J. Osborne, *An Introduction to Game Theory*, Oxford University Press, Oxford, UK, 2003.

[6] J.F. Reinganum, *Chapter 14 — The Timing of Innovation: Research, Development, and Diffusion*, in: R. Willig, R. Schmalensee (Eds.), Handbook of Industrial Organization, Elsevier, North-Holland, Volume 1, 1989, pp. 849 — 908.

[7] V.V. Romanuke, *Theory of Antagonistic Games*, New World — 2000, Lviv, 2010.

[8] V.V. Romanuke, *Fast solution of the discrete noiseless duel with the nonlinear scale on the linear accuracy functions*, Herald of Khmelnytskyi National University. Economical Sciences 5 (4) (2010) 61 — 66.

[9] V.V. Romanuke, *Discrete noiseless duel with a skewsymmetric payoff function on the unit square for models of socioeconomic competitive processes with a finite number of pure strategies*, Cybernetics and Systems Analysis 47 (5) (2011) 818 — 826.

[10] V.V. Romanuke, *Convergence and estimation of the process of computer implementation of the optimality principle in matrix games with apparent play horizon*, Journal of Automation and Information Sciences 45 (10) (2013) 49 — 56.

[11] V.V. Romanuke, *Zero-sum games on a product of staircase-function finite spaces*, Journal of Mathematics and Applications 44 (2021) 75 — 91.

[12] V.V. Romanuke, *Finite uniform approximation of two-person games defined on a product of staircase-function infinite spaces*, International Journal of Approximate Reasoning 145 (2022) 139 — 162.

[13] V.V. Romanuke, *Pure strategy saddle points in the generalized progressive discrete silent duel with identical linear accuracy functions*, Journal of Information and Organizational Sciences 48 (1) (2024).

[14] S. Siddharta, M. Shubik, *A model of a sudden death field-goal football game as a sequential duel*, Mathematical Social Sciences 15 (3) (1988) 205 — 215.

[15] J.-H. Steg, *On identifying subgame-perfect equilibrium outcomes for timing games*, Games and Economic Behavior 135 (2022) 74 — 78.

[16] Y. Teraoka, *A two-person game of timing with random arrival time of the object*, Mathematica Japonica 24 (1979) 427 — 438.

[17] S. Trybuła, *Solution of a silent duel with arbitrary motion and with arbitrary accuracy functions*, Optimization 27 (1993) 151 — 172.

[18] T. Vincent, J. Brown, *Evolutionary Game Theory, Natural Selection, and Darwinian Dynamics*, Cambridge University Press, Cambridge, UK, 2005.

[19] N.N. Vorob'yov, *Game Theory for Economists-Cyberneticists*, Nauka, Moscow, 1985.

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