Instability to differential equations of fourth order with a variable deviating argument

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ABSTRACT: The main purpose of this paper is to give two instability theorems to fourth order nonlinear differential equations with a variable deviating argument.

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1. Introduction

In 2000, Ezeilo [5] proved two instability theorems for the fourth order nonlinear differential equations without delay

$$x^{(4)} + a_1 x''' + g(x, x', x'', x''')x'' + h(x)x' + f(x, x', x'', x''') = 0 \qquad (1.1)$$

and

$$x^{(4)} + p(x''', x'') + q(x, x', x'', x''')x'' + a_3 x' + a_4 x = 0. \qquad (1.2)$$

In this paper, instead of Eq. (1.1) and Eq. (1.2), we consider the fourth order nonlinear differential equations with a variable deviating argument, $\tau(t)$:

$$
\begin{aligned}
x^{(4)}(t) \quad + \quad & a_1 x'''(t) + g(x(t - \tau(t)), ..., x'''(t - \tau(t)))x'' \\
+ \quad & h(x(t))x'(t) + f(x(t - \tau(t)), ..., x'''(t - \tau(t)))x(t) = 0 \qquad (1.3)
\end{aligned}
$$

and

$$
\begin{aligned}
x^{(4)}(t) \quad + \quad & p(x'''(t), x''(t)) + q(x(t - \tau(t)), ..., x'''(t - \tau(t)))x'' \\
+ \quad & a_3 x'(t) + a_4 x(t) = 0. \qquad (1.4)
\end{aligned}
$$

We write Eq. (1.3) and Eq. (1.4) in system form as

$$\begin{aligned}
x' &= y, \\
y' &= z, \\
z' &= u, \\
u' &= -a_1 u - g(x(t - \tau(t)), ..., u(t - \tau(t)))z - h(x)y \\
&\quad - f(x(t - \tau(t)), ..., u(t - \tau(t)))x
\end{aligned} \tag{1.5}$$

and

$$\begin{aligned}
x' &= y, \\
y' &= z, \\
z' &= u, \\
u' &= -p(u, z) - q(x(t - \tau(t)), ..., u(t - \tau(t)))z \\
&\quad - a_3 y - a_4 x,
\end{aligned} \tag{1.6}$$

respectively, where $\tau(t)$ is fixed delay, $t - \tau(t)$ is strictly increasing, $\lim\limits_{t \to \infty} (t - \tau(t)) = \infty$, $t \in \Re_+ = [0, \infty)$; a_1, a_3 and a_4 are constants; g, h, f, p and q are continuous functions in their respective arguments on $\Re^4$, $\Re$, $\Re^4$, $\Re^2$ and $\Re^4$, respectively, with $p(0, z) = 0$ and satisfy a Lipschitz condition in their respective arguments; the derivative $\frac{\partial p}{\partial z}(u, z)$ exists and is also continuous. Hence, the existence and uniqueness of the solutions of Eq. (1.3) and Eq. (1.4) are guaranteed (see [[2], pp.14]). We assume in what follows that $x(t)$, $y(t)$, $z(t)$ and $u(t)$ are abbreviated as x, y, z and u, respectively.

So far, the instability of solutions to certain fourth order nonlinear scalar and vector differential equations without delay has been investigated by many authors (see Dong and Zhang [1], Ezeilo ([3]-[5]), Li and Duan [8], Li and Yu [9], Lu and Liao [10], Sadek [11], Skrapek [12], Sun and Hou [13], Tiryaki [14], Tunç ([15]-[18]), C. Tunç and E. Tunç [20] and the references cited thereof). However, by now, the instability of solutions to fourth order nonlinear differential equations with deviating arguments has only been studied by Tunç [19]. This paper is the second attempt on the topic in the literature. It is worth mentioning that throughout all of the papers, based on Krasovskii's properties (see Krasovskii [6]), the Lyapunov's second (or direct) method has been used as a basic tool to prove the results established therein. The motivation for this paper comes from the above mentioned papers. Our aim is to carry out the results established in Ezeilo [5] to nonlinear differential equations of fourth order, Eq. (1.3) and Eq. (1.4), with a deviating argument for the instability of zero solution of these equations.

Note that the instability criteria of Krasovskii [6] can be summarized as the following: According to these criteria, it is necessary to show here that there exists a Lyapunov function $V(.) \equiv V(x, y, z, u)$ which has Krasovskii properties, say (K_1), (K_2) and (K_3) :

(K_1) In every neighborhood of $(0, 0, 0, 0)$, there exists a point (ξ, η, ζ, μ) such that $V(\xi, \eta, \zeta, \mu) > 0$;

(K_2) the time derivative $\dot{V} = \frac{d}{dt}V(x, y, z, u)$ along solution paths of the system (1.5) is positive semi-definite;

(K_3) the only solution $(x, y, z, u) = (x(t), y(t), z(t), u(t))$ of the system (1.5) which satisfies $\dot{V} = 0$, $(t \geq 0)$, is the trivial solution $(0, 0, 0, 0)$.

Let $r \geq 0$ be given, and let $C = C([-r, 0], \; \Re^n)$ with

$$\|\phi\| = \max_{-r \leq s \leq 0} |\phi(s)|, \;\; \phi \in C.$$

For $H > 0$ define $C_H \subset C$ by

$$C_H = \{\phi \in C : \|\phi\| < H\}.$$

If $x : [-r, \; A) \to \Re^n$ is continuous, $0 < A \leq \infty$, then, for each t in $[0, A)$, x_t in C is defined by

$$x_t(s) = x(t + s), -r \leq s \leq 0, t \geq 0.$$

Let G be an open subset of C and consider the general autonomous delay differential system with finite delay

$$\dot{x} = F(x_t), x_t = x(t + \theta), -r \leq \theta \leq 0, t \geq 0,$$

where $F(0) = 0$ and $F : G \to \Re^n$ is continuous and maps closed and bounded sets into bounded sets. It follows from these conditions on F that each initial value problem

$$\dot{x} = F(x_t), x_0 = \phi \in G$$

has a unique solution defined on some interval $[0, A)$, $0 < A \leq \infty$. This solution will be denoted by $x(\phi)(.)$ so that $x_0(\phi) = \phi$.

Definition 1.1. Let $F(0) = 0$. The zero solution, $x = 0$, of $\dot{x} = F(x_t)$ is stable if for each $\varepsilon > 0$ there exists $\delta = \delta(\varepsilon) > 0$ such that $\|\phi\| < \delta$ implies that $|x(\phi)(t)| < \varepsilon$ for all $t \geq 0$. The zero solution is said to be unstable if it is not stable.

Theorem 1.1. *(Instability Theorem of Cetaev's). Let Ω be a neighborhood of the origin. Let there be given a function $V(x)$ and region Ω_1 in Ω with the following properties:*

(i) $V(x)$ has continuous first partial derivatives in Ω_1.

(ii) $V(x)$ and $\dot{V}(x)$ are positive in Ω_1.

(iii) At the boundary points of Ω_1 inside Ω, $V(x) = 0$.

(iv) The origin is a boundary point of Ω_1.

Under these conditions the origin is unstable (see LaSalle and Lefschetz [7]).

2. Main results

The first main result is the following theorem.

Theorem 2.1. *Suppose that*

$$f(x(t-\tau(t)), ..., u(t-\tau(t))) - \frac{1}{4}g^2(x(t-\tau(t)), ..., u(t-\tau(t))) > 0$$

for arbitrary $x(t-\tau(t)), ..., \ u(t-\tau(t))$. *Then the zero solution of Eq. (1.3) is unstable.*

Proof. Consider the Lyapunov function $V = V(x, y, z, u)$ defined by

$$V = yz + \frac{1}{2}a_1 y^2 - xu - a_1 xz - \int_0^x h(s)sds, \quad \text{(where } a_1 \text{ is a constant)},$$

so that

$$V(0, \varepsilon^2, \varepsilon, 0) = \varepsilon^3 + \frac{1}{2}a_1\varepsilon^4 > 0$$

for sufficiently small ε. In fact, if ε is an arbitrary positive constant, then

$$V(0, \varepsilon^2, \varepsilon, 0) > 0$$

for sufficiently small ε. Thus V satisfies the property (K_1), (see [6]).

By an elementary differentiation the time derivative of V along the solutions of (1.5) can be estimated as follows

$$
\begin{aligned}
\dot{V} &= z^2 + xzg(x(t-\tau(t)), ..., u(t-\tau(t))) + x^2 f(x(t-\tau(t)), ..., u(t-\tau(t))) \\
&= [z + 2^{-1}xg(x(t-\tau(t)), ..., u(t-\tau(t)))]^2 \\
&\quad + \left[f(x(t-\tau(t)), ..., u(t-\tau(t))) - \frac{1}{4}g^2(x(t-\tau(t)), ..., u(t-\tau(t))) \right]x^2 \\
&\geq \left[f(x(t-\tau(t)), ..., u(t-\tau(t))) - \frac{1}{4}g^2(x(t-\tau(t)), ..., u(t-\tau(t))) \right]x^2 > 0.
\end{aligned}
$$

Thus V satisfies the property (K_2), (see [6]).

Further, it follows that $\dot{V} = 0 \Leftrightarrow x = 0$. In turn, this implies that

$$x = y = z = u = 0.$$

Thus V satisfies the property (K_3), (see [6]). This completes the proof of Theorem 2.1. □

Example 2.1. Consider nonlinear differential equation of fourth order with a variable deviating argument, $\tau(t) = t/2$:

$$
\begin{aligned}
x^{(4)} \ + \ & x''' + \{2 + \frac{2}{1 + x^2(t/2) + ... + x'''^2(t/2)}\}x'' \\
+ \ & 4xx' + (9 + x^2(t/2) + ... + x'''^2(t/2))x = 0
\end{aligned}
$$

so that

$$
\begin{aligned}
x' &= y, \\
y' &= z, \\
z' &= u, \\
u' &= -u - \{2 + \frac{2}{1 + x^2(t/2) + ... + u^2(t/2)}\}z - 4xy \\
&\quad - \{9 + x^2(t/2) + ... + u^2(t/2)\}x(t) = 0.
\end{aligned}
$$

We have the following estimates:

$$
\begin{aligned}
a_1 &= 1, \\
\tau(t) &= t/2, \\
g(x(t - \tau(t)), ..., u(t - \tau(t))) &= 2 + \frac{2}{1 + x^2(t/2) + ... + u^2(t/2)}, \\
h(x) &= 4x
\end{aligned}
$$

and

$$
f(x(t - \tau(t)), ..., u(t - \tau(t))) = 9 + x^2(t/2) + ... + u^2(t/2)
$$

so that

$$
\begin{aligned}
f(.) - \frac{1}{4}g^2(.) &= 9 + x^2(t/2) + ... + u^2(t/2) \\
&\quad - \left[1 + \frac{1}{1 + x^2(t/2) + ... + u^2(t/2)}\right]^2 > 0.
\end{aligned}
$$

This shows that the zero solution of the above equation is unstable.

The second main result is the following theorem.

Theorem 2.2. *Suppose that*

$$
p(0, z) = 0, \quad a_4 > 0 \quad \text{and } a_4 - \frac{1}{4}q^2(x(t - \tau(t)), ..., u(t - \tau(t))) > 0
$$

for arbitrary $x(t - \tau(t)), ..., u(t - \tau(t))$, *and* $\frac{\partial p}{\partial z}(u, z)sgnu \leq 0$ *for arbitrary* u, z.
Then the zero solution of Eq. (1.4) is unstable for arbitrary a_3.

Proof. Consider the Lyapunov function $V_1 = V_1(x, y, z, u)$ defined by

$$
V_1 = -\int_0^u p(s, z)ds - a_3yu + \frac{1}{2}a_3z^2 - a_4xu + a_4yz
$$

so that

$$
V_1(0, \varepsilon^2, \varepsilon, 0) = a_4\varepsilon^3 + \frac{1}{2}a_3\varepsilon^4 > 0, (a_4 > 0), (a_3 \in \Re),
$$

for sufficiently small ε. Indeed, if ε is an arbitrary positive constant, then

$$V_1(0, \varepsilon^2, \varepsilon, 0) > 0$$

for sufficiently small ε. Thus V_1 satisfies the property (K_1), (see [6]).

The time derivativeof V_1 along the solutions of (1.6) can be calculated as follows:

$$\begin{aligned} \dot{V}_1 &= -u' \{p(u,z) + a_3 y + a_4 x\} + a_4 z^2 - u \int_0^u \frac{\partial p}{\partial z}(s,z)ds \\ &= u' \{u' + q(x(t-\tau(t)),...,u(t-\tau(t)))z\} \\ &\quad + a_4 z^2 - u \int_0^u \frac{\partial p}{\partial z}(s,z)ds. \end{aligned}$$

The last estimate leads

$$\begin{aligned} \dot{V}_1 &= \{u' + 2^{-1} q(x(t-\tau(t)),...,u(t-\tau(t)))z\}^2 \\ &\quad + \{a_4 - 4^{-1} q^2(x(t-\tau(t)),...,u(t-\tau(t)))\}z^2 \\ &\quad - u \int_0^u \frac{\partial p}{\partial z}(s,z)ds \end{aligned}$$

so that

$$\begin{aligned} \dot{V}_1 &\geq \{u' + 2^{-1} q(x(t-\tau(t)),...,u(t-\tau(t)))z\}^2 \\ &\quad + \{a_4 - 4^{-1} q^2(x(t-\tau(t)),...,u(t-\tau(t)))\}z^2 > 0. \end{aligned}$$

Thus V_1 satisfies the property (K_2), (see [6]).

On the other hand, $\dot{V}_1 = 0 \Leftrightarrow z = 0$, this implies that $z = u = 0$. System (1.6) and $\dot{V}_1 = 0$ leads that

$$a_3 y + a_4 x = 0 \Rightarrow a_3 x' + a_4 x = 0.$$

Because of $x'' = 0$, it follows that $x' =$ constant so that $a_3 x' + a_4 x = 0 \Rightarrow x =$ constant. However, this implies $x' = 0$ since $a_4 \neq 0$. Hence $a_4 > 0$ implies $x = 0$. Thus V_1 satisfies the property (K_3), (see [6]). This completes the proof of Theorem 2.2. $\qquad \square$

Example 2.2. Consider nonlinear differential equation of fourth order with a variable deviating argument, $\tau(t) = t/2$:

$$x^{(4)} - (arctgx'')x''' + 2\cos(x(t/2) + ... + x'''(t/2))x'' + 3x' + 4x = 0.$$

so that

$$\begin{aligned} x' &= y, \\ y' &= z, \\ z' &= u, \\ u' &= (arctgz)u - 2\cos(x(t/2) + ... + u(t/2))z - 3y - 4x. \end{aligned}$$

We have the following estimates:

$$\tau(t) = t/2, \qquad a_3 = 3, \qquad a_4 = 4,$$
$$p(u, z) = -(arctgz)u,$$
$$\frac{\partial p}{\partial z}(u, z)sgnu = -\frac{u}{1 + z^2}sgnu \leq 0,$$
$$q(x(t - \tau(t)), ..., u(t - \tau(t))) = 2\cos(x(t/2) + ... + u(t/2)),$$

so that

$$a_4 - \frac{1}{4}q^2(.) = 4 - \cos^2(x(t/2) + ... + u(t/2)) > 0.$$

This shows that the zero solution of the above equation is unstable.

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