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# Fixed Point Theorems for Monotone Mappings in Ordered Banach Spaces Under Weak Topology Features

*Abdullah Alahmari, Mohamed Mabrouk and Mohamed-Aziz Taoudi*

Abstract: We present several fixed point theorems for monotone nonlinear operators in ordered Banach spaces. The main assumptions of our results are formulated in terms of the weak topology. As an application, we study the existence of solutions to a class of first-order vector-valued ordinary differential equations. Our conclusions generalize many well-known results.

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*Keywords and Phrases:* Fixed point theorem; Order cone; Increasing operator; Decreasing operator; Weakly condensing; Measure of weak noncompactness.

## 1. Introduction

Fixed point theory furnishes an effective and important tool for proving theoretical as well as constructive existence for a variety of nonlinear problems arising from the mathematical modelling of real world phenomena. The usual topological fixed point methods (Schauder, Darbo, Sadovskii,...) are generally only suited to nonlinear problems with continuity and compactness. However, many problems in theory and applications have no compactness. Some attempts have been made to overcome this difficulty by using the weak topology, see [2, 3, 6, 7, 8, 9, 10, 11, 14, 34]. The interest of the weak topology is mainly due to the vital role played by weak compactness in the theory of infinite dimensional linear spaces. In particular, a Banach space X is reflexive if and only if the closed unit ball is weakly compact. Equally, fixed point theorems using the weak topology (Schauder-Tychonov, Arino-Gautier-Penot,...) are

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generally only suited to nonlinear problems with weak (sequential) continuity and weak compactness. In several situations, the weak (sequential) continuity could rise several difficulties. For example, in $L^1$-spaces, which are the most natural functional settings of many real world problems in physics and population dynamics (notably when the unknown is a density), only linear superposition (Nemytskii) operators are weakly (sequentially) continuous [4]. To our knowledge, the first paper where the weak topology was successfully applied to fixed point theorems without requiring the weak continuity of the involved operators, was [29]. In the quoted paper, the authors used the concepts of ws-compactness and ww-compactness instead of the (sequential) weak continuity. Such concepts proved to be more effective in many practical situations especially when we work in nonreflexive Banach spaces. This fact was illustrated by proving the existence of an integrable solution for a stationary nonlinear problem arising in transport theory and kinetic of gas and in many other situations [12, 13, 16, 20, 21, 22, 29, 30].

In the present paper, we provide a new general treatment of fixed point theory of monotone mappings in ordered vector spaces. Specifically, we will show how weak topology is successfully used in conjunction with the order in fixed point problems. As the functional setting of many nonlinear problems arising from the mathematical modeling of real world phenomena is usually an ordered vector space, our approach gives an extremely powerful and direct tool to investigate the solvability of a large class of evolution equations with lack of compactness. To illustrate our results, we investigate the solvability of a class of first-order vector-valued ordinary differential equations. Before proceeding to the detailed discussion, we recall some related definitions and auxiliary results. Let $X$ be a Banach space and let $P$ be a subset of $X$. The set $P$ is called an *order cone* if and only if:

(i) $P$ is closed, nonempty and $P \neq \{0\}$,

(ii) $a, b \in \mathbb{R}, a, b \geq 0, x, y \in P \Rightarrow ax + by \in P$,

(iii) $x \in P$ and $-x \in P \Rightarrow x = 0$.

An order cone permits to define a partial order in $X$ by

$$x \leq y \text{ iff } y - x \in P.$$

Conversely, let $X$ be a real Banach space with a partial order compatible with the algebraic operations in $X$, that is,

$$x \geq 0 \text{ and } \lambda \geq 0 \text{ implies } \lambda x \geq 0$$

$$x_1 \leq y_1 \text{ and } x_2 \leq y_2 \text{ implies } x_1 + x_2 \leq y_1 + y_2.$$

The *positive cone* of $X$ is defined by

$$X^+ = \{x \in X : \ 0 \leq x\}.$$

**Definition 1.1.**

(i) A subset $M \subset X$ is said *order bounded* if there exist $u, v \in X$ such that $u \leq x \leq v$, for all $x \in M$.

(ii) The order cone $P$ is called *normal* if and only if there is a number $c > 0$ such that for all $x, y \in X$ we have

$$0 \leq x \leq y \Rightarrow \|x\| \leq c\|y\|. \tag{1.1}$$

The least positive number $c$ (if it exists) satisfying (1.1) is called a *normal constant*.

**Remark 1.2.** If the cone $P$ is normal, then every order interval is norm bounded (see e.g. [23, Theorem 2.1.1]).

**Remark 1.3.** Let $K$ be a compact Hausdorff space and $E$ be an ordered Banach space with normal positive cone. We denote by $C(K, E)$ the Banach space of all continuous $E$ -valued functions on $K$ endowed with the usual maximum norm. Plainly $C(K, E)$ is an ordered Banach space with the natural ordering whose positive cone is given by

$$C^+(K, E) = \{f \in C(K, E) : \, f(x) \in E^+, \, \forall x \in K\}.$$

Since $E^+$ is normal so is $C^+(K, E)$.

The following definitions are frequently used in the sequel.

**Definition 1.4.** Let $M \subset X$. The operator $T \colon M \to X$ is said to be an increasing operator if $x, y \in M$, $x \leq y$ implies $Tx \leq Ty$. The operator $T \colon M \to X$ is said to be a decreasing operator if $x, y \in M$, $x \leq y$ implies $Ty \leq Tx$.

**Definition 1.5.** Let $M$ be a nonempty closed subset of $X$. The operator $T \colon M \to X$ is said to be monotone-subcontinuous if for any monotone sequence (increasing or decreasing) $(x_n)$ in $M$ that converges strongly to $x$ the sequence $(Tx_n)$ converges weakly to $Tx$.

The following elementary result serves as the key tool in the proof of more sophisticated results.

**Lemma 1.6.** *[26] Let $X$ be an ordered real Banach space with a normal order cone. Suppose that $\{x_n\}$ is a monotone sequence which has a subsequence $\{x_{n_k}\}$ converging weakly to $x_\infty$. Then $\{x_n\}$ converges strongly to $x_\infty$. Moreover, if $\{x_n\}$ is an increasing sequence, then $x_n \leq x_\infty (n = 1, 2, 3, \ldots)$; if $\{x_n\}$ is a decreasing sequence, then $x_\infty \leq x_n$ $(n = 1, 2, 3, \ldots)$.*

By a poset $F = (F, \leq)$ we mean a nonempty set $F$ equipped with a partial ordering relation $\leq$.

**Lemma 1.7.** *[25, Lemma 1.1.5] Let $\{x_n\}$ be a sequence in a poset $F$.*

(a) *If $\{x_n\}$ is totally ordered, then it has a monotone subsequence.*

(b) *If $\{x_n\}$ is nondecreasing (resp. nonincreasing), then it has the supremum (resp. the infimum) $x$ if and only if $x$ is the supremum (resp. the infimum) of some of its subsequences.*

Combining Lemma 1.6 and Lemma 1.7 we obtain the following interesting result.

**Lemma 1.8.** *Let $X$ be an ordered real Banach space with a normal order cone. Suppose that $\{x_n\}$ is a totally ordered sequence which is contained in a relatively weakly compact set. Then $\{x_n\}$ converges strongly in $X$.*

In what follows, $\psi$ will always denote a measure of weak noncompactness (MWNC) on the Banach space $X$. We refer the reader to [5] for the axiomatic definition of a measure of weak noncompactness. One of the most frequently exploited measure of weak noncompactness was defined by De Blasi [15] as follows:

$$w(M) = \inf\{r > 0 : \text{ there exists } W \text{ weakly compact such that } M \subseteq W + B_r\},$$

for each bounded subset $M$ of $X$; Here, $B_r$ stands for the closed ball of $X$ centered at origin with radius $r$.
The following results are crucial for our purposes. We first state a theorem of Ambrosetti type (see [31] for a proof).

**Theorem 1.9.** *Let $E$ be a Banach space and let $H \subseteq C([0,T],E)$ be bounded and equicontinuous. Then the map $t \to w(H(t))$ is continuous on $[0,T]$ and*

$$w(H) = \sup_{t\in[0,T]} w(H(t)) = w(H[0,T]),$$

*where $H(t) = \{h(t)\colon h \in H\}$ and $H[0,T] = \bigcup_{t\in[0,T]}\{h(t)\colon h \in H\}$.*

The following Lemma is well-known (see for example [32]).

**Lemma 1.10.** *If $H \subseteq C([0,T],E)$ is equicontinuous and $x_0 \in C([0,T],E)$, then $\overline{co}(H \cup \{x_0\})$ is also equicontinuous in $C([0,T],E)$.*

## 2. Fixed point results

In this section, we prove some fixed point theorems for monotone mappings in ordered Banach spaces. Our results combine the advantages of the strong topology (i.e. the involved mappings will be continuous (or subcontinuous) with respect to the strong topology) with the advantages of the weak topology (i.e. the maps will satisfy some compactness conditions relative to the weak topology) to draw new conclusions about fixed points for a given monotone map.

**Theorem 2.1.** *Let $X$ be an ordered Banach space with a normal cone $P$. Let $u_0, v_0 \in X$ with $u_0 < v_0$ and $A\colon [u_0, v_0] \to X$ be a monotone-subcontinuous increasing operator satisfying the following:*

$$u_0 \le Au_0,\ Av_0 \le v_0. \tag{2.1}$$

*If, in addition, $A$ verifies*

($\mathcal{P}(n_0)$): *There exists an integer $n_0 \ge 1$ such that: for any monotone sequence $V = \{x_n\}$ of $[u_0, v_0]$ and any finite subset $F$ of $[u_0, v_0]$ of cardinal $n_0$, we have:*

$$V = F \cup A^{n_0}(V) \text{ implies } V \text{ is relatively weakly compact.}$$

*Then, $A$ has a minimal fixed point $u_*$ and a maximal fixed point $u^*$ in $[u_0, v_0]$ and*

$$u_* = \lim_{n\to\infty} u_n \text{ and } u^* = \lim_{n\to\infty} v_n, \tag{2.2}$$

*where $u_n = Au_{n-1}$ and $v_n = Av_{n-1},\ n = 1, 2, \ldots$*

$$u_0 \le u_1 \le \cdots u_* \le u^* \le \cdots \le v_n \le \cdots \le v_1 \le v_0. \tag{2.3}$$

**Proof.** Let $u_n = Au_{n-1}$ and $v_n = Av_{n-1}$ for $n \ge 1$. Since $A$ is increasing, then

$$u_0 \le u_1 \le \cdots \le u_n \le \cdots v_n \le \cdots v_1 \le v_0. \tag{2.4}$$

Let $S = \{u_0, u_1, \ldots, u_n, \ldots\}$. Clearly, for any integer $k \ge 1$ we have

$$A^k(S) \cup \{u_0, u_1, \ldots, u_{k-1}\} = S.$$

From our hypotheses we know that $S$ is relatively weakly compact. Referring to Lemma 1.8, we see that $\{u_n\}$ is convergent. Let $u_*$ be its limit. The monotone-subcontinuity of $A$ yields $Au_* = u_*$. Similarly, we can prove that $\{v_n\}$ converges to some $u^*$ and $Au^* = u^*$. Finally, we prove that $u^*$ and $u_*$ are the maximal and minimal fixed points of $A$ in $[u_0, v_0]$. Let $x \in [u_0, v_0]$ and $Ax = x$. Since $A$ is increasing, it follows from $u_0 \le x \le v_0$ that $Au_0 \le Ax \le Av_0$, i.e. $u_1 \le x \le v_1$. Using the same argument, we get $u_2 \le x \le v_2$ and, in general, $u_n \le x \le v_n$ $(n = 1, 2, 3, \ldots)$. Now, letting $n$ go to infinity we get $u_* \le x \le u^*$. □

As a convenient specialization of Theorem 2.1, we state the following.

**Corollary 2.2.** *Let $X$ be an ordered Banach space with a normal cone $P$. Let $u_0, v_0 \in X$ with $u_0 < v_0$ and $A\colon [u_0, v_0] \to X$ be a monotone-subcontinuous increasing operator satisfying the following:*

$$u_0 \le Au_0,\ Av_0 \le v_0. \tag{2.5}$$

*If, in addition, $A$ verifies*

($\mathcal{P}(1)$): *if $V = \{x_n\}$ is a monotone sequence of $[u_0, v_0]$ and $a \in [u_0, v_0]$, then $V = \{a\} \cup A(V)$ implies $V$ is relatively weakly compact.*
*Then $A$ has a minimal fixed point $u_*$ and a maximal fixed point $u^*$ in $[u_0, v_0]$ satisfying (2.2) and (2.3).*

**Proof.** Apply Theorem 2.1 with $n_0 = 1$. $\square$

Another consequence of Theorem 2.1 is the following. Recall that a measure of weak noncompactness $\psi$ on a Banach space $X$ is said to be *nonsingular* if $\psi(M \cup \{a\}) = \psi(M)$ for every $a \in X$ and every nonempty bounded subset $M$ of $X$.

**Corollary 2.3.** *Let $X$ be an ordered Banach space with a normal cone $P$ and $\psi$ be a nonsingular measure of weak noncompactness on $X$. Let $u_0, v_0 \in X$ with $u_0 < v_0$ and $A\colon [u_0, v_0] \to X$ be a monotone-subcontinuous increasing operator satisfying the following:*

$$u_0 \leq Au_0,\ Av_0 \leq v_0. \tag{2.6}$$

*In addition, if for any $\Omega = \{u_n\} \subset [u_0, v_0]$ countable and monotone with $\psi(\Omega) \neq 0$ we have*

$$\psi(A^{n_0}(\Omega)) < \psi(\Omega),$$

*for some integer $n_0 \geq 1$. Then, $A$ has a minimal fixed point $u_*$ and a maximal fixed point $u^*$ in $[u_0, v_0]$ satisfying (2.2) and (2.3).*

**Proof.** By virtue of Theorem 2.1, it suffices to show that $(\mathcal{P}(n_0))$ holds true. To do this, let $V = \{x_n\}$ be a monotone sequence of $[u_0, v_0]$ and $F$ be a finite subset of $[u_0, v_0]$ of cardinal $n_0$ such that $V = F \cup A^{n_0}(V)$. Since $P$ is normal then, according to Remark 1.2, the order interval $[u_0, v_0]$ is bounded. This implies that $V$ and $A^{n_0}(V)$ are bounded and we have $\psi(V) = \psi(F \cup A^{n_0}(V)) = \psi(A^{n_0}(V))$. Consequently, it follows from our hypotheses that $\psi(V) = 0$, which means that $V$ is relatively weakly compact. This achieves the proof.

$\square$

**Remark 2.4.** Corollary 2.3 extends [23, Theorem 3.1.1].

**Corollary 2.5.** *Let $u_0, v_0 \in X$ with $u_0 < v_0$ and $A\colon [u_0, v_0] \to X$ be a monotone-subcontinuous increasing operator satisfying (2.6). If $P$ is normal and $A^{n_0}([u_0, v_0])$ is relatively weakly compact for some integer $n_0 \geq 1$, then $A$ has a minimal fixed point $u_*$ and a maximal fixed point $u^*$ in $[u_0, v_0]$ satisfying (2.2) and (2.3).*

For later use, we consider the following condition.

$$(\mathcal{C})\ \begin{cases} A\colon P \to P \text{ satisfies } A^2\theta \geq \epsilon A\theta \text{ where } 0 < \epsilon < 1, \text{and for any} \\ \epsilon A\theta \leq x \leq A\theta \text{ and } \epsilon \leq t < 1,\ \text{there exists } \eta = \eta(x,t) > 0,\ \text{such that} \\ A(tx) \leq (t(1+\eta))^{-1} Ax. \end{cases}$$

We will need the following lemmas from [23].

**Lemma 2.6.** *[23, Lemma 3.2.1] Let $A\colon P \to P$ be a decreasing operator satisfying the condition $(\mathcal{C})$. If $u, v \in P$ with $Au = v$ and $Av = u$, then $u = v$.*

**Lemma 2.7.** *[23, Lemma 3.2.2] Let $A\colon P \to P$ be a decreasing operator satisfying the condition $(\mathcal{C})$. If $u, v \in P$ with $Au = u$ and $Av = v$, then $u = v$.*

**Theorem 2.8.** *Let $X$ be an ordered Banach space with a normal cone $P$. Let $A\colon P \to P$ be a monotone-subcontinuous decreasing operator satisfying the conditions $(\mathcal{C})$ and $(\mathcal{P}(n_0))$ for some integer $n_0 \geq 1$. Then $A$ has a unique fixed point $u^*$ in $P$ and*

$$u^* = \lim_{n\to\infty} u_n, \tag{2.7}$$

*where* $u_n = Au_{n-1},\ n = 1, 2, \ldots$

**Proof.** Keeping in mind that $A\colon P \to P$ is decreasing we easily deduce that

$$\theta = u_0 \leq u_2 \leq \cdots \leq u_{2n} \leq \cdots \leq u_{2n+1} \leq \cdots \leq u_1 = A\theta. \tag{2.8}$$

Let $S = \{u_0, u_1, \ldots, u_n, \ldots\}$. From (2.8) and the normality of $P$ we infer that $S$ is bounded. Clearly, for any integer $k \geq 1$ we have

$$A^k(S) \cup \{u_0, u_1, \ldots, u_{k-1}\} = S.$$

From our hypotheses we know that $S$ is relatively weakly compact. This implies that the increasing sequence $\{u_{2n}\}$ has a weakly convergent subsequence. Referring to Lemma 1.6, we see that $\{u_{2n}\}$ is convergent. Let $u_*$ be its limit. Similarly we can prove that the sequence $\{u_{2n+1}\}$ converges to some $u^*$. Taking the limit at the both sides of $u_{2n+1} = Au_{2n}$ and $u_{2n+2} = Au_{2n+1}$ and using the monotone-subcontinuity of $A$ we get $u_* \leq u^*$, $u^* = Au_*$ and $u^* = Au_*$. Invoking Lemma 2.6 we infer that $u^* = u_*$ is a fixed point of $A$. The uniqueness follows from Lemma 2.7. □

As a convenient specialization of Theorem 2.8 we obtain the following result.

**Corollary 2.9.** *Let $X$ be an ordered Banach space with a normal cone $P$ and $\psi$ be a nonsingular measure of weak noncompactness on $X$. Let $A\colon P \to P$ be a monotone-subcontinuous decreasing operator satisfying the condition $(\mathcal{C})$. In addition, if for any $\Omega = \{u_n\} \subset P$ countable and monotone with $\psi(\Omega) \neq 0$ we have*

$$\psi(A^{n_0}(\Omega)) < \psi(\Omega),$$

*for some integer $n_0 \geq 1$, then $A$ has a unique fixed point $u^*$ in $P$ and*

$$u^* = \lim_{n\to\infty} u_n, \tag{2.9}$$

*where* $u_n = Au_{n-1},\ n = 1, 2, \ldots$

**Proof.** In view of Theorem 2.8, it suffices to show that $A$ verifies $(\mathcal{P}(n_0))$. The reasoning in Corollary 2.3 yields the result. □

**Remark 2.10.** Theorem 2.8 and Corollary 2.9 extend [23, Theorem 3.2.1].

# 3. Application to differential equations

We shall use the results in previous sections to get an existence theorem for a nonlinear ODE in a Banach space. The nonlinear term satisfies an appropriate condition expressed in terms of the De Blasi measure of weak noncompactness. Let $E$ be an ordered Banach space with a normal cone $P$. We consider the following initial value problem

$$u' = f(t,u) \text{ on } \ I, \ u(0) = u_0, \tag{3.1}$$

where $I = [0,1], \ u \in C^1(I,E), \ f \in C(I \times E, E)$. A vector-valued function $u\colon I \to E$ is said to be a solution of (3.1) on $I$ if $u(t)$ is continuously differentiable and satisfies (3.1) on $I$.
In [18], Du and Lakshmikantham proved that if the problem (3.1) has a lower solution $v_0$ and an upper solution $w_0$ with $v_0 \leq w_0$, and the nonlinear term satisfies the monotonicity condition

$$f(t,y) - f(t,x) \geq -M(y-x) \text{ whenever } v_0(t) \leq x \leq y \leq w_0(t) \tag{3.2}$$

for some $M > 0$, and the compactness measure condition

$$\alpha(f(t,V)) \leq \tau\alpha(V) \tag{3.3}$$

for any $t \in I$ and any bounded subset $V$ of $E$, where $\tau$ is a positive constant and $\alpha(.)$ denotes the Kuratowski measure of noncompactness in $E$, then the problem (3.1) has a minimal and a maximal solution between $v_0$ and $w_0$, which can be obtained by a monotone iterative procedure starting from $v_0$ and $w_0$ respectively. When $E$ is weakly sequentially complete, Y. Du [17] improved the result of [18] and removed the condition (3.3).
Our aim in this section is to improve and extend the aforementioned results. We will replace the noncompactness measure condition (3.3) by a weaker condition expressed in terms of the De Blasi measure of weak noncompactness. From now on, we assume the following:

(i) There exist $v_0, w_0 \in C^1(I,E)$ with $v_0(t) \leq w_0(t)$ on $I$ such that:

$$v_0'(t) \leq f(t, v_0(t)),\ v_0(0) \leq u_0$$

$$w_0'(t) \geq f(t, w_0(t)),\ w_0(0) \geq u_0.$$

(ii) For some $M > 0$,

$$f(t,y) - f(t,x) \geq -M(y-x) \text{ whenever } v_0(t) \leq x \leq y \leq w_0(t).$$

(iii) There is a constant $\tau \geq 0$ such that for any equicontinuous monotone sequence $V = \{u_n\}$ of $[v_0, w_0]$ and for any $a, b \in [0,1]$ with $a < b$ we have

$$w(f([a,b] \times V)) \leq \tau w(V[a,b]),$$

where $f([a,b] \times V) := \{f(s, x(s)),\, a \leq s \leq b,\, x \in V\}$.

**Remark 3.1.** Let $g(s,x) = f(s,x) + Mx$. Then, for any monotone sequence $V = \{u_n\}$ of $[v_0, w_0]$ and for any $a, b \in [0,1]$ with $a < b$ we have

$$w(g([a,b] \times V)) \leq \mu w(V[a,b]), \tag{3.4}$$

where $\mu = \tau + M$.
Now, let $t \in [0,1]$ be fixed and let $h(s,x) = e^{-M(t-s)} g(s,x)$, for $s \in [0,t]$ and $x \in E$. It is readily verified that

$$h([0,t] \times V) \subset co\left(g([0,t] \times V) \cup \{0\}\right). \tag{3.5}$$

Combining (3.4) and (3.5) we arrive at

$$w(h([0,t] \times V)) \leq \mu w(V[0,t]), \tag{3.6}$$

where $h([0,t] \times V) := \{h(s, x(s)),\, 0 \leq s \leq t,\, x \in V\}$.

Now, we are in a position to state our main result.

**Theorem 3.2.** *Let assumptions* (i)–(iii) *be satisfied. Then the problem (3.1) has a maximal and a minimal solution between $v_0$ and $w_0$, which can be obtained by a monotone iterative procedure starting from $v_0$ and $w_0$ respectively.*

**Proof.** We consider the equivalent modified problem

$$u' + Mu = f(t,u) + Mu \text{ on } I,\ u(0) = u_0, \tag{3.7}$$

which is equivalent to the problem

$$\left(e^{Mt}u\right)' = e^{Mt}\left(f(t,u) + Mu\right) \text{ on } I,\ u(0) = u_0. \tag{3.8}$$

Let us write (3.8) as an integral equation

$$u(t) = e^{-Mt}u_0 + \int_0^t e^{-M(t-s)}\left(f(s,u(s)) + Mu(s)\right) ds. \tag{3.9}$$

□

Define the operator $A$ on $C(I,E)$ by

$$(Au)(t) = e^{-Mt}u_0 + \int_0^t e^{-M(t-s)}\left(f(s,u(s)) + Mu(s)\right) ds,\ t \in I. \tag{3.10}$$

It is easy to check that a fixed point of $A$ is a solution of (3.1). We will demonstrate that $A$ satisfies all the hypotheses of Theorem 2.1. It is apparent that $A$ is continuous. From Hypothesis (ii) we know that $A$ is increasing on $[v_0, w_0]$. To illustrate that $v_0 \leq Av_0$, let $k(t) = v_0'(t) + Mv_0(t)$. Clearly, $k \in C(I, E)$ and $k(t) \leq f(t, v_0(t)) + Mv_0(t),\ t \in I$. Keeping in mind the fact that $\left(e^{Mt}v_0(t)\right)' = e^{Mt}k(t)$, we deduce that for all $t \in I$ we have:

$$\begin{aligned} e^{Mt}v_0(t) &= v_0(0) + \int_0^t e^{Ms}k(s)ds \\ &\leq u_0 + \int_0^t e^{Ms}(f(s, v_0(s)) + Mv_0(s))ds. \end{aligned}$$

Accordingly, $v_0 \leq Av_0$. Similarly, we can prove that $Aw_0 \leq w_0$. We claim that for any integer $k \geq 1$ and any $V \subset [u_0, v_0]$ the set $A^k(V)$ is equicontinuous. Indeed, let $t, t_0 \in I$ with $t < t_0$ and $u \in [v_0, w_0]$. Then,

$$\begin{aligned} \|Au(t) - Au(t_0)\| \leq\ & (e^{-Mt} - e^{-Mt_0})\|u_0\| + \int_0^t (e^{-M(t-s)} - e^{-M(t_0-s)})\|g(s, u(s))\|ds \\ & + \int_t^{t_0} \|g(s, u(s))\|ds. \end{aligned}$$

For any $u \in [v_0, w_0]$, by Assumption (ii), we have

$$g(s, v_0(s)) \leq g(s, u(s)) \leq g(s, w_0(s)).$$

By the normality of the cone $P$, there exists $C_g > 0$ such that

$$\|g(t, u(t))\| \leq C_g,\ \ u \in [v_0, w_0].$$

Accordingly,

$$\begin{aligned} \|Au(t) - Au(t_0)\| &\leq (e^{-Mt} - e^{-Mt_0})\|u_0\| + C_g \int_0^t (e^{-M(t-s)} - e^{-M(t_0-s)})ds \\ & \quad + C_g(t_0 - t). \end{aligned}$$

Consequently,

$$\|Au(t) - Au(t_0)\| \to 0 \ \text{ as } t \to t_0^-,$$

uniformly with respect to $u$. Similarly, we get

$$\|Au(t) - Au(t_0)\| \to 0 \ \text{ as } t \to t_0^+,$$

uniformly with respect to $u$. This proves that $A(V)$ is equicontinuous. Therefore, for any integer $k \geq 1$ the set $A^k(V)$ is equicontinuous.

Now, let $V \subset [v_0, w_0]$ and $F$ be a finite subset of $[v_0, w_0]$ such that $V = A^k(V) \cup F$, for some integer $k \geq 1$. Since $A^k(V)$ is equicontinuous, then by invoking Lemma 1.10

we conclude that $V$ is equicontinuous. Let $h$ be as described in Remark 3.1, then for each $t \in I$, we have

$$\begin{aligned} w(A(V)(t)) &= w\left(\left\{e^{-Mt}u_0 + \int_0^t h(s,u(s))ds\colon u \in V\right\}\right) \\ &\leq w(t\overline{co}\{h(s,u(s))\colon u \in V,\ s \in [0,t]\}) \\ &= tw(\overline{co}\{h(s,u(s))\colon u \in V,\ s \in [0,t]\}) \\ &\leq tw(h([0,t] \times V) \\ &\leq t\mu w(V[0,t]). \end{aligned}$$

Theorem 1.9 implies (since $V$ is equicontinuous) that

$$w(A(V)(t)) \leq t\mu w(V). \tag{3.11}$$

Using (3.11) we get

$$\begin{aligned} w(A^2(V)(t)) &= w\left(\left\{e^{-Mt}u_0 + \int_0^t h(s,u(s))ds\colon u \in A(V)\right\}\right) \\ &= w\left(\left\{\int_0^t h(s,u(s))ds\colon u \in A(V)\right\}\right). \end{aligned} \tag{3.12}$$

Fix $t \in [0,1]$. We divide the interval $[0,t]$ into $m$ parts $0 = t_0 < t_1 < \cdots < t_m = t$ in such a way that $\Delta t_i = t_i - t_{i-1} = \frac{t}{m}$, $i = 1,\dots,m$. For each $u \in A(V)$ we have

$$\begin{aligned} \int_0^t h(s,u(s))ds &= \sum_{i=1}^m \int_{t_{i-1}}^{t_i} h(s,u(s))ds \\ &\in \sum_{i=1}^m \Delta t_i \overline{co}\{h(s,u(s))\colon u \in A(V),\ s \in [t_{i-1},t_i]\} \\ &\subseteq \sum_{i=1}^m \Delta t_i \overline{co}(h([t_{i-1},t_i] \times A(V)). \end{aligned}$$

Using again Theorem 1.9 we infer that for each $i = 2,\dots,m$ there is a $s_i \in [t_{i-1},t_i]$ such that

$$\sup_{s\in[t_{i-1},t_i]} w(A(V)(s)) = w(A(V)[t_{i-1},t_i]) = w(A(V)(s_i)). \tag{3.13}$$

Consequently

$$\begin{aligned} w(\{\int_0^t h(s,x(s))ds\colon u \in A(V)\} &\leq \sum_{i=1}^m \Delta t_i w(\overline{co}(h([t_{i-1},t_i] \times A(V))) \\ &\leq \mu \sum_{i=1}^m \Delta t_i w((A(V)([t_{i-1},t_i])) \\ &\leq \mu \sum_{i=1}^m \Delta t_i w(A(V)((s_i)). \end{aligned}$$

On the other hand, if $m \to \infty$ then

$$\sum_{i=1}^{m} \Delta t_i w(A(V)((s_i)) \to \int_0^t w(A(V)(s))ds. \tag{3.14}$$

As a result,

$$w(A^2(V)(t)) \leq \frac{(\mu t)^2}{2} w(V). \tag{3.15}$$

By induction we get

$$w(A^n(V)(t)) \leq \frac{(\mu t)^n}{n!} w(V). \tag{3.16}$$

Invoking Theorem 1.9 we obtain

$$w(A^n(V)) \leq \frac{\mu^n}{n!} w(V). \tag{3.17}$$

Since $\lim_{n\to\infty} \frac{\mu^n}{n!} = 0$, we may choose $n_0$ as large as we please such that $\frac{\mu^{n_0}}{n_0!} < 1$.

Now, let $V \subset [v_0, w_0]$ and $F$ be a finite subset of $[v_0, w_0]$ such that $V = A^{n_0}(V) \cup F$. Then, $w(V) = w(A^{n_0}(V) \cup F) = w(A^{n_0}(V)) \leq \frac{\mu^{n_0}}{n_0!} w(V)$. Thus, $w(V) = 0$ and therefore $V$ is relatively weakly compact. By applying Theorem 2.1 we infer that $A$ has a maximal and a minimal fixed points between $v_0$ and $w_0$, which can be obtained by a monotone iterative procedure starting from $v_0$ and $w_0$ respectively. This completes the proof.

**Remark 3.3.** If $E$ is weakly sequentially complete (reflexive, in particular), then the condition (iii) in Theorem 3.2 holds automatically. In fact, according to [17, Theorem 2.2] any monotone order-bounded sequence is relatively compact. Thus, Theorem 3.2 greatly improves [17, Theorem 4.1] and [18, Theorem 3.1].

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**Abdullah Alahmari**
email: `aaahmari@uqu.edu.sa`
ORCID: 0000-0002-9596-910X
Department of Mathematics
College of Applied Sciences
P. O. Box 715, Makkah 21955
KSA

**Mohamed Mabrouk**
email: `Mohamed.Mabrouk@fsg.rnu.tn`
Department of Mathematics
College of Applied Sciences
P. O. Box 715, Makkah 21955
KSA

Department of Mathematics
Faculty of Sciences of Gabès
University of Gabès
Cité Erriadh, 6072 Zrig, Gabès
TUNISIA

**Mohamed-Aziz Taoudi**
email: `a.taoudi@uca.ma`
ORCID: 0000-0002-8851-8714
National School of Applied Sciences
Cadi Ayyad University
Marrakech
MOROCCO

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# On Some Fixed Point Theorems for Expansive Mappings in Dislocated Cone Metric Spaces with Banach Algebras

*Abba Auwalu, Evren Hinçal and Lakshmi Narayan Mishra**

ABSTRACT: In this paper, we introduced the notion of generalized expansive mappings in dislocated cone metric spaces with Banach algebras. Furthermore, we prove some fixed point theorems for generalized expansive mappings in dislocated cone metric spaces with Banach algebras without the assumption of normality of cones. Moreover, we give an example to elucidate our result. Our results are significant extension and generalizations of many recent results in the literature.

*AMS Subject Classification:* 47H10, 54H25.
*Keywords and Phrases:* Dislocated cone metric space over Banach algebras; Expansive mapping; Fixed point; $c$-sequence.

## 1. Introduction

The concept of cone metric space was introduced by Huang and Zhang [9]. They supplanted the set of real numbers in metric space by a complete normed space and proved some fixed point results for different contractive conditions in such a space. Later on, Liu and Xu [13] introduced the notion of cone metric space over Banach algebras by supplanting the complete normed space in cone metric space with Banach algebras and proved that cone metric space over Banach algebras are not equivalent to metric space in terms of existence of the fixed points of mappings. Subsequently, many authors established interesting and significant results in a cone metric space over Banach algebras (see [20], [7], [8]). In 2017, George et al. [6] introduced the notion of dislocated cone metric space over Banach algebras as a generalization of cone metric space over Banach algebras and proved some fixed point results for Banach, Kannan

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and Perov type contractive conditions in such a space. Very recently, Jiang et al. [11] introduced the concept of expansive mapping defined on cone metric space over Banach algebras and proved some fixed point results for such mapping. In this work, we use the concept of expansive mapping defined on dislocated cone metric space over Banach algebras and prove some fixed point theorems. Our results unify, complement and/or generalized the recent results of [11, 2, 10, 1, 3, 19], and many others, that will be useful in dynamic programming and integral equation, (see; [4] - [15] and references therein).

# 2. Preliminaries

In this section, we recall some definitions and results needed in the sequel.

**Definition 2.1.** ([18]) A Banach algebra $\mathcal{A}$ is a real Banach space in which an operation of multiplication is defined subject to the following properties for all $p, q, r \in \mathcal{A}$, $\lambda \in \mathbb{R}$

1. $(pq)r = p(qr)$,
2. $p(q + r) = pq + pr$ and $(p + q)r = pr + qr$,
3. $\lambda(pq) = (\lambda p)q = p(\lambda q)$,
4. $\|pq\| \leq \|p\| \, \|q\|$.

A subset $\mathcal{K}$ of a Banach algebra $\mathcal{A}$ is called a cone (see [13]) if

1. $\mathcal{K}$ is nonempty closed and $\{\theta, e\} \subset \mathcal{K}$;
2. $\alpha\mathcal{K} + \beta\mathcal{K} \subset \mathcal{K}$ for all nonnegative real numbers $\alpha, \beta$;
3. $\mathcal{K}^2 = \mathcal{K}\mathcal{K} \subset \mathcal{K}$;
4. $\mathcal{K} \cap (-\mathcal{K}) = \{\theta\}$,

where $\theta$ and $e$ denote the zero and unit elements of the Banach algebra $\mathcal{A}$, respectively. For a given cone $\mathcal{K} \subset \mathcal{A}$, we write $z \preccurlyeq y$ if and only if $y - z \in \mathcal{K}$, where $\preccurlyeq$ is a partial order relation defined on $\mathcal{K}$. Also, $x \ll y$ will stand for $y - x \in int\mathcal{K}$, where $int\mathcal{K}$ denotes the interior of $\mathcal{K}$. If $int\mathcal{K} \neq \varnothing$ then $\mathcal{K}$ is called a solid cone.

**Definition 2.2.** ([6]) Let $\mathcal{Z}$ be a nonempty set. Suppose that $\rho : \mathcal{Z} \times \mathcal{Z} \to \mathcal{A}$ be a mapping satisfying the following conditions:
$(D_1)$ $\theta \preccurlyeq \rho(z, y)$ for all $z, y \in \mathcal{Z}$ and $\rho(z, y) = \theta \Longrightarrow z = y$;
$(D_2)$ $\rho(z, y) = \rho(y, z)$ for all $z, y \in \mathcal{Z}$;
$(D_3)$ $\rho(z, y) \preccurlyeq \rho(z, x) + \rho(x, y)$ for all $z, y, x \in \mathcal{Z}$.
Then $\rho$ is called a dislocated cone metric on $\mathcal{Z}$, and $(\mathcal{Z}, \rho)$ is called a dislocated cone metric space over Banach algebra $\mathcal{A}$.

**Remark 2.3.** In a dislocated cone metric space $(\mathcal{Z},\rho)$, $\rho(z,z)$ need not be zero for $z \in \mathcal{Z}$. Hence every cone metric space over Banach algebras is a dislocated cone metric space over Banach algebras, but the converse is not necessarily true. (see [6]).

**Example 2.4.** ([6]) Let $\mathcal{A} = \{b = (b_{i,j})_{3\times 3} : b_{i,j} \in \mathbb{R}, 1 \leq i,j \leq 3\}$, $\|b\| = \sum_{1\leq i,j\leq 3}|b_{i,j}|$, $\mathcal{K} = \{b \in \mathcal{A} : b_{i,j} \geq 0, 1 \leq i,j \leq 3\}$ be a cone in $\mathcal{A}$. Let $\mathcal{Z} = \mathbb{R}^+ \cup \{0\}$. Let a mapping $\rho : \mathcal{Z}\times\mathcal{Z} \to \mathcal{A}$ be define by

$$\rho(z,y) = \begin{pmatrix} z+y & z+y & z+y \\ 2z+2y & 2z+2y & 2z+2y \\ 3z+3y & 3z+3y & 3z+3y \end{pmatrix}, \text{ for all } z,y \in \mathcal{Z}.$$

Then $(\mathcal{Z},\rho)$ is a dislocated cone metric space over a Banach algebra $\mathcal{A}$ but not a cone metric space over a Banach algebra $\mathcal{A}$ since

$$\rho\left(\frac{1}{2},\frac{1}{2}\right) = \begin{pmatrix} 1 & 1 & 1 \\ 2 & 2 & 2 \\ 3 & 3 & 3 \end{pmatrix} \neq \theta.$$

**Definition 2.5.** ([6]) Let $(\mathcal{Z},\rho)$ be a dislocated cone metric space over Banach algebra $\mathcal{A}$, $z \in \mathcal{Z}$ and $\{z_i\}$ be a sequence in $(\mathcal{Z},\rho)$. Then

1. $\{z_i\}$ converges to $z$ whenever for each $c \in \mathcal{A}$ with $\theta \ll c$, there is a natural number $N$ such that $\rho(z_i,z) \ll c$ for all $i \geq N$. We denote this by $z_i \to z$ $(i \to \infty)$.

2. $\{z_i\}$ is a Cauchy sequence whenever for each $c \in \mathcal{A}$ with $\theta \ll c$, there is a natural number $N$ such that $\rho(z_i,z_j) \ll c$ for all $i,j \geq N$.

3. $(\mathcal{Z},\rho)$ is said to be complete if every Cauchy sequence in $\mathcal{Z}$ is convergent.

**Definition 2.6.** ([12]) Let $\mathcal{K}$ be a solid cone in a Banach algebra $\mathcal{A}$. A sequence $\{z_i\} \subset \mathcal{K}$ is said to be a $c$-sequence if for each $\theta \ll c$, there exists $N \in \mathbb{N}$ such that $z_i \ll c$ for all $i > N$.

**Lemma 2.7.** ([18]) *Let $\mathcal{A}$ be a Banach algebra with a unit $e$ and $\tau \in \mathcal{A}$, then $\lim_{n\to\infty}\|\tau^n\|^{\frac{1}{n}}$ exists and the spectral radius $\delta(\tau)$ satisfies*

$$\delta(\tau) = \lim_{n\to\infty}\|\tau^n\|^{\frac{1}{n}} = inf\,\|\tau^n\|^{\frac{1}{n}}.$$

*If $\delta(\tau) < 1$, then $(e-\tau)$ is invertible in $\mathcal{A}$. Moreover,*

$$(e-\tau)^{-1} = \sum_{k=0}^{\infty}\tau^k,$$

*and*

$$\delta\left[(e-\tau)^{-1}\right] \leq \frac{1}{1-\delta(\tau)}.$$

**Remark 2.8.** ([20]). If $\delta(\tau) < 1$ then $\left\|\tau^i\right\| \to 0$ $(i \to \infty)$.

**Lemma 2.9.** ([7]) *If $E$ is a real Banach space with a solid cone $\mathcal{K}$ and $\{z_i\} \subset \mathcal{K}$ be a sequence with $\|z_i\| \to 0$ $(i \to \infty)$, then for each $\theta \ll c$, there exists $N \in \mathbb{N}$ such that for any $i > N$, we have $z_i \ll c$.*

**Lemma 2.10.** ([6]) *Let $(\mathcal{Z}, \rho)$ be a complete dislocated cone metric space over Banach algebra $\mathcal{A}$ and $\mathcal{K}$ be the underlying solid cone. Let $\{z_i\}$ be a sequence in $(\mathcal{Z}, \rho)$. If $\{z_i\}$ converges to $z \in \mathcal{Z}$, then*

1. *$\{\rho(z_i, z)\}$ is a c-sequence.*
2. *For any $j \in \mathbb{N}$, $\{\rho(z_i, z_{i+j})\}$ is a c-sequence.*

**Lemma 2.11.** ([12]) *Let $\mathcal{A}$ be a real Banach algebra with a solid cone $\mathcal{K}$ and let $\{\alpha_n\}$ and $\{\beta_n\}$ be sequences in $\mathcal{K}$. If $\{\alpha_n\}$ and $\{\beta_n\}$ are c-sequences and $k_1, k_2 \in \mathcal{K}$ then $\{k_1\alpha_n + k_2\beta_n\}$ is also a c-sequence.*

**Lemma 2.12.** ([12]) *If $E$ is a real Banach space with a solid cone $\mathcal{K}$*

1. *If $a, b, c \in E$ and $a \preccurlyeq b \ll c$, then $a \ll c$.*
2. *If $a \in \mathcal{K}$ and $\theta \preccurlyeq a \ll c$ for each $\theta \ll c$, then $a = \theta$.*
3. *If $a \preccurlyeq \tau a$, where $a, \tau \in \mathcal{K}$ and $\delta(\tau) < 1$, then $a = \theta$.*

# 3. Main results

First, we introduce the notion of expansive mapping in the setting of dislocated cone metric space over Banach algebra $\mathcal{A}$.

**Definition 3.1.** Let $(\mathcal{Z}, \rho)$ be a dislocated cone metric space over Banach algebra $\mathcal{A}$, $\mathcal{K}$ be the underlying solid cone. Then $\mathfrak{F} : \mathcal{Z} \to \mathcal{Z}$ is called an expansive mapping if there exist $\vartheta, \vartheta^{-1} \in \mathcal{K}$ such that $\delta(\vartheta^{-1}) < 1$ and

$$\rho(\mathfrak{F}z, \mathfrak{F}y) \succcurlyeq \vartheta\rho(z, y), \text{ for all } z, y \in \mathcal{Z}. \tag{3.1}$$

**Example 3.2.** Let $\mathcal{A} = C^1_{\mathbb{R}}[0, 1]$ and define a norm on $\mathcal{A}$ by $\|z\| = \|z\|_\infty + \|z'\|_\infty$ for $z \in \mathcal{A}$, where multiplication in $\mathcal{A}$ is defined in the usual way. Then $\mathcal{A}$ is a Banach algebra with unit element $e = 1$ and the set $\mathcal{K} = \{z \in \mathcal{A} : z(t) \geq 0, t \in [0, 1]\}$ is a cone in $\mathcal{A}$. Let $\mathcal{Z} = [0, \infty)$. Consider a mapping $\rho : \mathcal{Z} \times \mathcal{Z} \to \mathcal{A}$ define by

$$\rho(z, y)(t) = (z + y)e^t, \text{ for all } z, y \in \mathcal{Z}.$$

Then $(\mathcal{Z}, \rho)$ is a dislocated cone metric space over Banach algebra $\mathcal{A}$. Define a mapping $\mathfrak{F} : \mathcal{Z} \to \mathcal{Z}$ by $\mathfrak{F}z = 2z$, for all $z \in \mathcal{Z}$. Take $\vartheta = 2$. Hence, $\mathfrak{F}$ is expansive mapping.

Next, we prove the existence of fixed point for generalized expansive mapping in dislocated cone metric space over Banach algebra $\mathcal{A}$ without the assumption of normality of cone.

**Theorem 3.3.** *Let $(\mathcal{Z}, \rho)$ be a complete dislocated cone metric space over Banach algebra $\mathcal{A}$ with a unit $e$, $\mathcal{K}$ be the underlying solid cone. Let the mapping $\mathfrak{F} : \mathcal{Z} \to \mathcal{Z}$ be a surjection and satisfy the generalized expansive condition:*

$$\rho(\mathfrak{F}z, \mathfrak{F}y) + \vartheta_1[\rho(z, \mathfrak{F}y) + \rho(y, \mathfrak{F}z)] \succcurlyeq \vartheta_2\rho(z, y) + \vartheta_3\rho(z, \mathfrak{F}z) + \vartheta_4\rho(y, \mathfrak{F}y), \tag{3.2}$$

*for all $z, y \in \mathcal{Z}$, where $\vartheta_j \in \mathcal{K}$ $(j = 1, 2, 3, 4)$ such that $(\vartheta_2 + \vartheta_3 - 3\vartheta_1)^{-1}, (\vartheta_2 - \vartheta_1 + \vartheta_4)^{-1} \in \mathcal{K}$ and spectral radius $\delta[(\vartheta_2 + \vartheta_3 - 3\vartheta_1)^{-1}(e + \vartheta_1 - \vartheta_4)] < 1$. Then $\mathfrak{F}$ has a fixed point $z_*$ in $\mathcal{Z}$.*

**Proof.** Let $z_0 \in \mathcal{Z}$. Since $\mathfrak{F}$ is surjective, there exists $z_1 \in \mathcal{Z}$ such that $\mathfrak{F}z_1 = z_0$. Again, we can choose $z_2 \in \mathcal{Z}$ such that $\mathfrak{F}z_2 = z_1$. Continuing this process, we can construct a sequence $\{z_i\}$ in $(\mathcal{Z}, \rho)$ by

$$z_i = \mathfrak{F}z_{i+1}, \text{ for } i = 0, 1, 2, \ldots. \tag{3.3}$$

Suppose $z_k = z_{k+1}$ for some $k \in \mathbb{N}$, then $z_* = z_k$ is a fixed point of $\mathfrak{F}$ and the result is proved. Hence, we assume that $z_{i+1} \neq z_i$, $\forall i \in \mathbb{N}$. Using (3.2) and (3.3), we get

$$\begin{aligned}
\rho(\mathfrak{F}z_{i+1}, \mathfrak{F}z_i) + \vartheta_1[\rho(z_{i+1}, \mathfrak{F}z_i) + \rho(z_i, \mathfrak{F}z_{i+1})] &\succcurlyeq \vartheta_2\rho(z_{i+1}, z_i) \\
&\quad + \vartheta_3\rho(z_{i+1}, \mathfrak{F}z_{i+1}) + \vartheta_4\rho(z_i, \mathfrak{F}z_i) \\
\rho(z_i, z_{i-1}) + \vartheta_1[\rho(z_{i+1}, z_{i-1}) + \rho(z_i, z_i)] &\succcurlyeq \vartheta_2\rho(z_{i+1}, z_i) + \vartheta_3\rho(z_{i+1}, z_i) + \vartheta_4\rho(z_i, z_{i-1}) \\
\rho(z_i, z_{i-1}) + \vartheta_1[3\rho(z_{i+1}, z_i) + \rho(z_i, z_{i-1})] &\succcurlyeq (\vartheta_2 + \vartheta_3)\rho(z_{i+1}, z_i) + \vartheta_4\rho(z_i, z_{i-1}) \\
(e + \vartheta_1 - \vartheta_4)\rho(z_i, z_{i-1}) &\succcurlyeq (\vartheta_2 + \vartheta_3 - 3\vartheta_1)\rho(z_{i+1}, z_i) \\
(\vartheta_2 + \vartheta_3 - 3\vartheta_1)\rho(z_{i+1}, z_i) &\preccurlyeq (e + \vartheta_1 - \vartheta_4)\rho(z_i, z_{i-1}) \\
\rho(z_i, z_{i+1}) &\preccurlyeq \tau\rho(z_{i-1}, z_i),
\end{aligned} \tag{3.4}$$

where $\tau = (\vartheta_2 + \vartheta_3 - 3\vartheta_1)^{-1}(e + \vartheta_1 - \vartheta_4)$.
Hence, from (3.4), we get

$$\begin{aligned}
\rho(z_i, z_{i+1}) &\preccurlyeq \tau\rho(z_{i-1}, z_i) \\
&\preccurlyeq \tau^2\rho(z_{i-2}, z_{i-1}) \\
&\ \ \vdots \\
\rho(z_i, z_{i+1}) &\preccurlyeq \tau^i\rho(z_0, z_1), \text{ for all } i \in \mathbb{N}.
\end{aligned} \tag{3.5}$$

Since $\delta(\tau) < 1$, it follows, by Lemma 2.7, that $(e - \tau)$ is invertible in $\mathcal{A}$. Moreover,

$$(e - \tau)^{-1} = \sum_{k=0}^{\infty} \tau^k. \tag{3.6}$$

Also, by Remark 2.8, we get

$$\|\tau^i\| \to 0 \ (i \to \infty). \tag{3.7}$$

Hence, for $i, j \in \mathbb{N}$ with $i < j$, using (3.5) and (3.6), we have

$$\begin{aligned}
\rho(z_i, z_j) &\preccurlyeq \rho(z_i, z_{i+1}) + \rho(z_{i+1}, z_j) \\
&\preccurlyeq \rho(z_i, z_{i+1}) + \rho(z_{i+1}, z_{i+2}) + \rho(z_{i+2}, z_j) \\
&\preccurlyeq \rho(z_i, z_{i+1}) + \rho(z_{i+1}, z_{i+2}) + \rho(z_{i+2}, z_{i+3}) \\
&\quad + \cdots + \rho(z_{j-2}, z_{j-1}) + \rho(z_{j-1}, z_j) \\
&\preccurlyeq \tau^i \rho(z_0, z_1) + \tau^{i+1}\rho(z_0, z_1) + \tau^{i+2}\rho(z_0, z_1) \\
&\quad + \cdots + \tau^{j-2}\rho(z_0, z_1) + \tau^{j-1}\rho(z_0, z_1) \\
&= \tau^i(e + \tau + \tau^2 + \cdots + \tau^{j-i-2} + \tau^{j-i-1})\rho(z_0, z_1) \\
&\preccurlyeq \tau^i \left( \sum_{k=0}^{\infty} \tau^k \right) \rho(z_0, z_1) \\
&= \tau^i (e - \tau)^{-1} \rho(z_0, z_1).
\end{aligned}$$

Therefore, using (3.7), we have that $\left\|\tau^i(e-\tau)^{-1}\rho(z_0, z_1)\right\| \to 0$ $(i \to \infty)$, and it follows, by Lemma 2.9, that for any $c \in \mathcal{A}$ with $\theta \ll c$, there exists $N \in \mathbb{N}$ such that

$$\rho(z_i, z_j) \preccurlyeq \tau^i (e-\tau)^{-1} \rho(z_0, z_1) \ll c, \text{ for all } j > i > N,$$

which implies, by Lemma 2.12 and Definition 2.5, that $\{z_i\}$ is a Cauchy sequence. Since $(\mathcal{Z}, \rho)$ is complete, there exists $z_* \in \mathcal{Z}$ such that $z_i \to z_*$ $(i \to \infty)$. Since $\mathfrak{F}$ is a surjection mapping, there exists a point $y_*$ in $\mathcal{Z}$ such that $\mathfrak{F}y_* = z_*$. Next, we show that $y_* = z_*$. Using (3.2) and (3.3), we have that

$$\begin{aligned}
\rho(z_i, z_*) &= \rho(\mathfrak{F}z_{i+1}, \mathfrak{F}y_*) \\
&\succcurlyeq -\vartheta_1[\rho(z_{i+1}, \mathfrak{F}y_*) + \rho(y_*, \mathfrak{F}z_{i+1})] + \vartheta_2\rho(z_{i+1}, y_*) \\
&\quad + \vartheta_3\rho(z_{i+1}, \mathfrak{F}z_{i+1}) + \vartheta_4\rho(y_*, \mathfrak{F}y_*) \\
&\succcurlyeq -\vartheta_1[\rho(z_{i+1}, z_*) + \rho(y_*, z_i)] + \vartheta_2\rho(z_{i+1}, y_*) \\
&\quad + \vartheta_3\rho(z_{i+1}, z_i) + \vartheta_4\rho(y_*, z_*) \\
\rho(z_i, z_{i+1}) + \rho(z_{i+1}, z_*) &\succcurlyeq -\vartheta_1\rho(z_{i+1}, z_*) - \vartheta_1[\rho(y_*, z_{i+1}) - \rho(z_i, z_{i+1})] \\
&\quad + \vartheta_2\rho(z_{i+1}, y_*) + \vartheta_3\rho(z_{i+1}, z_i) \\
&\quad + \vartheta_4[\rho(y_*, z_{i+1}) - \rho(z_*, z_{i+1})] \\
(\vartheta_2 - \vartheta_1 + \vartheta_4)\rho(z_{i+1}, y_*) &\preccurlyeq (e + \vartheta_1 + \vartheta_4)\rho(z_{i+1}, z_*) + (e - \vartheta_1 - \vartheta_3)\rho(z_i, z_{i+1}) \\
\rho(z_{i+1}, y_*) &\preccurlyeq (\vartheta_2 - \vartheta_1 + \vartheta_4)^{-1}[(e + \vartheta_1 + \vartheta_4)\rho(z_{i+1}, z_*) \\
&\quad + (e - \vartheta_1 - \vartheta_3)\rho(z_i, z_{i+1})].
\end{aligned}$$

This implies that

$$\rho(z_{i+1}, y_*) \preccurlyeq \alpha_1\rho(z_{i+1}, z_*) + \alpha_2\rho(z_i, z_{i+1}),$$

where $\alpha_1 = (\vartheta_2 - \vartheta_1 + \vartheta_4)^{-1}(e + \vartheta_1 + \vartheta_4), \alpha_2 = (\vartheta_2 - \vartheta_1 + \vartheta_4)^{-1}(e - \vartheta_1 - \vartheta_3) \in \mathcal{K}$. Now, by Lemma 2.10, Lemma 2.11; $\{\rho(z_{i+1}, z_*)\}$, $\{\rho(z_i, z_{i+1})\}$ and $\{\alpha_1\rho(z_{i+1}, z_*) + \alpha_2\rho(z_i, z_{i+1})\}$ are $c$-sequences. Hence, for any $c \in \mathcal{A}$ with $\theta \ll c$, there exists $N \in \mathbb{N}$ such that

$$\rho(z_{i+1}, y_*) \preccurlyeq \alpha_1\rho(z_{i+1}, z_*) + \alpha_2\rho(z_i, z_{i+1}) \ll c, \text{ for all } i > N,$$

which implies that $z_{i+1} \to y_*$. Since the limit of a convergent sequence in cone metric space is unique, we have that $y_* = z_*$. Hence, $z_*$ is a fixed point of $\mathfrak{F}$. □

**Remark 3.4.** Note that $\mathfrak{F}$ may have more than one fixed point (e.g. see [11, 1]).

**Theorem 3.5.** *Let $(\mathcal{Z}, \rho)$ be a complete dislocated cone metric space over Banach algebra $\mathcal{A}$ with a unit $e$, $\mathcal{K}$ be the underlying solid cone. Let the mapping $\mathfrak{F} : \mathcal{Z} \to \mathcal{Z}$ be a surjection and satisfy the following condition:*

$$\rho(\mathfrak{F}z, \mathfrak{F}y) \succcurlyeq \vartheta_1\rho(z, y) + \vartheta_2\rho(z, \mathfrak{F}y), \text{ for all } z, y \in \mathcal{Z}, \tag{3.8}$$

*where $\vartheta_1, \vartheta_2 \in \mathcal{K}$ such that $(\vartheta_1+\vartheta_2)^{-1} \in \mathcal{K}$ and spectral radius $\delta[(\vartheta_1+\vartheta_2)^{-1}(e+\vartheta_2)] < 1$. Then $\mathfrak{F}$ has a fixed point $z_*$ in $\mathcal{Z}$.*

**Proof.** Let $z_0$ be an arbitrary point in $\mathcal{Z}$. Since $\mathfrak{F}$ is surjective, there exists $z_1 \in \mathcal{Z}$ such that $\mathfrak{F}z_1 = z_0$. Again, we can choose $z_2 \in \mathcal{Z}$ such that $\mathfrak{F}z_2 = z_1$. Continuing this process, we can construct a sequence $\{z_i\}$ in $(\mathcal{Z}, \rho)$ by

$$z_i = \mathfrak{F}z_{i+1}, \text{ for } i = 0, 1, 2, \ldots. \tag{3.9}$$

Suppose $z_{j-1} = z_j$ for some $j \in \mathbb{N}$, then $z_* = z_j$ is a fixed point of $\mathfrak{F}$ and the result is proved. Hence, we assume that $z_i \neq z_{i-1}$ for all $i \in \mathbb{N}$. Now, using (3.8) and (3.9), we have

$$\begin{aligned}
\rho(z_i, z_{i-1}) &= \rho(\mathfrak{F}z_{i+1}, \mathfrak{F}z_i)\\
&\succcurlyeq \vartheta_1\rho(z_{i+1}, z_i) + \vartheta_2\rho(z_{i+1}, z_{i-1})\\
&\succcurlyeq \vartheta_1\rho(z_{i+1}, z_i) + \vartheta_2[\rho(z_{i+1}, z_i) - \rho(z_{i-1}, z_i)]\\
(e + \vartheta_2)\rho(z_i, z_{i-1}) &\succcurlyeq (\vartheta_1 + \vartheta_2)\rho(z_{i+1}, z_i)\\
\rho(z_i, z_{i+1}) &\preccurlyeq (\vartheta_1 + \vartheta_2)^{-1}(e + \vartheta_2)\rho(z_{i-1}, z_i)\\
\rho(z_i, z_{i+1}) &\preccurlyeq \tau\rho(z_{i-1}, z_i),
\end{aligned} \tag{3.10}$$

where $\tau = (\vartheta_1 + \vartheta_2)^{-1}(e + \vartheta_2)$.
Hence, from (3.10), we have

$$\begin{aligned}
\rho(z_i, z_{i+1}) &\preccurlyeq \tau\rho(z_{i-1}, z_i)\\
&\preccurlyeq \tau^2\rho(z_{i-2}, z_{i-1})\\
&\vdots\\
\rho(z_i, z_{i+1}) &\preccurlyeq \tau^i\rho(z_0, z_1), \text{ for all } i \in \mathbb{N}.
\end{aligned} \tag{3.11}$$

Using the same argument to the proof in Theorem 3.3, we get that $\{z_i\}$ is a Cauchy sequence. Since $(\mathcal{Z}, \rho)$ is complete, there exists $z_* \in \mathcal{Z}$ such that $z_i \to z_*$ $(i \to \infty)$. Since $\mathfrak{F}$ is a surjection mapping, there exists a point $z_{**}$ in $\mathcal{Z}$ such that $\mathfrak{F}z_{**} = z_*$. Now, we show that $z_{**} = z_*$. Using (3.8) and (3.9), we have that

$$\begin{aligned}
\rho(z_*, z_i) &= \rho(\mathfrak{F}z_{**}, \mathfrak{F}z_{i+1}) \\
&\succcurlyeq \vartheta_1\rho(z_{**}, z_{i+1}) + \vartheta_2\rho(z_{**}, \mathfrak{F}z_{i+1}) \\
&= \vartheta_1\rho(z_{**}, z_{i+1}) + \vartheta_2\rho(z_{**}, z_i) \\
\rho(z_*, z_{i+1}) + \rho(z_{i+1}, z_i) &\succcurlyeq \vartheta_1\rho(z_{**}, z_{i+1}) + \vartheta_2[\rho(z_{**}, z_{i+1}) - \rho(z_i, z_{i+1})] \\
(\vartheta_1 + \vartheta_2)\rho(z_{i+1}, z_{**}) &\preccurlyeq \rho(z_{i+1}, z_*) + (e + \vartheta_2)\rho(z_i, z_{i+1}) \\
\rho(z_{i+1}, z_{**}) &\preccurlyeq (\vartheta_1 + \vartheta_2)^{-1}[\rho(z_{i+1}, z_*) + (e + \vartheta_2)\rho(z_i, z_{i+1})].
\end{aligned}$$

This implies that

$$\rho(z_{i+1}, z_{**}) \preccurlyeq \beta_1\rho(z_{i+1}, z_*) + \beta_2\rho(z_i, z_{i+1}),$$

where $\beta_1 = (\vartheta_1 + \vartheta_2)^{-1}, \beta_2 = (\vartheta_1 + \vartheta_2)^{-1}(e + \vartheta_2) \in \mathcal{K}$. Now, by Lemma 2.10, Lemma 2.11; $\{\rho(z_{i+1}, z_*)\}$, $\{\rho(z_i, z_{i+1})\}$ and $\{\beta_1\rho(z_{i+1}, z_*) + \beta_2\rho(z_i, z_{i+1})\}$ are $c$-sequences. Hence, for any $c \in \mathcal{A}$ with $\theta \ll c$, there exists $N \in \mathbb{N}$ such that

$$\rho(z_{i+1}, z_{**}) \preccurlyeq \beta_1\rho(z_{i+1}, z_*) + \beta_2\rho(z_i, z_{i+1}) \ll c, \text{ for all } i > N,$$

which implies that $z_{i+1} \to z_{**}$. Since the limit of a convergent sequence in a cone metric space is unique, we have that $z_{**} = z_*$. Hence, $z_*$ is a fixed point of $\mathfrak{F}$. □

**Corollary 3.6.** *Let $(\mathcal{Z}, \rho)$ be a complete dislocated cone metric space over Banach algebra $\mathcal{A}$ with a unit $e$, $\mathcal{K}$ be the underlying solid cone. Let the mapping $\mathfrak{F} : \mathcal{Z} \to \mathcal{Z}$ be a surjection and satisfy the following condition:*

$$\rho(\mathfrak{F}z, \mathfrak{F}y) \succcurlyeq \vartheta_1\rho(z, y) + \vartheta_2\rho(z, \mathfrak{F}z) + \vartheta_3\rho(y, \mathfrak{F}y), \tag{3.12}$$

*for all $z, y \in \mathcal{Z}$. where $\vartheta_k \in \mathcal{K}$ $(k = 1, 2, 3)$ such that $(\vartheta_1 + \vartheta_2)^{-1}, (\vartheta_1 + \vartheta_3)^{-1} \in \mathcal{K}$ and spectral radius $\delta[(\vartheta_1 + \vartheta_2)^{-1}(e - \vartheta_3)] < 1$. Then $\mathfrak{F}$ has a fixed point $z_*$ in $\mathcal{Z}$.*

**Proof.** Putting $\vartheta_1 = \theta$ in Theorem 3.3, the result follows. □

**Corollary 3.7.** *Let $(\mathcal{Z}, \rho)$ be a complete dislocated cone metric space over Banach algebra $\mathcal{A}$ with a unit $e$, $\mathcal{K}$ be the underlying solid cone. Let the mapping $\mathfrak{F} : \mathcal{Z} \to \mathcal{Z}$ be a surjection and satisfy the following condition:*

$$\rho(\mathfrak{F}z, \mathfrak{F}y) \succcurlyeq \vartheta_1\rho(z, \mathfrak{F}z) + \vartheta_2\rho(y, \mathfrak{F}y), \tag{3.13}$$

*for all $z, y \in \mathcal{Z}$. where $\vartheta_k \in \mathcal{K}$ $(k = 1, 2)$ such that ${\vartheta_1}^{-1}, {\vartheta_2}^{-1} \in \mathcal{K}$ and spectral radius $\delta[{\vartheta_1}^{-1}(e - \vartheta_2)] < 1$. Then $\mathfrak{F}$ has a fixed point $z_*$ in $\mathcal{Z}$.*

**Proof.** Putting $\vartheta_1 = \vartheta_2 = \theta$ in Theorem 3.3, the result follows. □

**Theorem 3.8.** *Let $(\mathcal{Z}, \rho)$ be a complete dislocated cone metric space over Banach algebra $\mathcal{A}$ with a unit $e$, $\mathcal{K}$ be the underlying solid cone. Let the mapping $\mathfrak{F} : \mathcal{Z} \to \mathcal{Z}$ be a surjection and satisfy the following condition:*

$$\rho(\mathfrak{F}z, \mathfrak{F}y) \succcurlyeq \vartheta\rho(z, y), \tag{3.14}$$

*for all $z, y \in \mathcal{Z}$. where $\vartheta, \vartheta^{-1} \in \mathcal{K}$ such that spectral radius $\delta(\vartheta^{-1}) < 1$. Then $\mathfrak{F}$ has a unique fixed point $z_*$ in $\mathcal{Z}$.*

**Proof.** Using Theorem 3.3, Theorem 3.5, we only need to show that the fixed point is unique. Suppose that $y_*$ is another fixed point of $\mathfrak{F}$, then using (3.14), we have that

$$\begin{aligned}
\rho(z_*, y_*) &= \rho(\mathfrak{F}z_*, \mathfrak{F}y_*) \\
&\succcurlyeq \vartheta\rho(z_*, y_*) \\
\rho(z_*, y_*) &\preccurlyeq \vartheta^{-1}\rho(z_*, y_*) = \tau\rho(z_*, y_*),
\end{aligned}$$

where $\tau = \vartheta^{-1}$.
Hence, we have

$$\begin{aligned}
\rho(z_*, y_*) &\preccurlyeq \tau\rho(z_*, y_*) \\
&\preccurlyeq \tau^2\rho(z_*, y_*) \\
&\vdots \\
\rho(z_*, y_*) &\preccurlyeq \tau^i\rho(z_*, y_*), \text{ for all } i \in \mathbb{N}.
\end{aligned}$$

Since $\delta(\tau) < 1$, then, by Remark 2.8, it follows that

$$\|\tau^i\| \to 0 \ (i \to \infty).$$

Hence, we have that $\left\|\tau^i\rho(z_*, y_*)\right\| \to 0 \ (i \to \infty)$ and by Lemma 2.9, it follows that for any $c \in \mathcal{A}$ with $\theta \ll c$, there exists $N \in \mathbb{N}$ such that

$$\rho(z_*, y_*) \preccurlyeq \tau^i\rho(z_*, y_*) \ll c, \text{ for all } i > N,$$

which implies that $\rho(z_*, y_*) = \theta$. Therefore $z_* = y_*$. This completes the proof. □

**Corollary 3.9.** *Let $(\mathcal{Z}, \rho)$ be a complete dislocated cone metric space over Banach algebra $\mathcal{A}$ with a unit $e$, $\mathcal{K}$ be the underlying solid cone. Let the mapping $\mathfrak{F} : \mathcal{Z} \to \mathcal{Z}$ be a surjection and satisfy the following condition:*

$$\rho(\mathfrak{F}^m z, \mathfrak{F}^m y) \succcurlyeq \vartheta\rho(z, y), \quad m \in \mathbb{Z}^+ \tag{3.15}$$

*for all $z, y \in \mathcal{Z}$. where $\vartheta, \vartheta^{-1} \in \mathcal{K}$ such that $\delta(\vartheta^{-1}) < 1$. Then $\mathfrak{F}$ has a unique fixed point $z_*$ in $\mathcal{Z}$.*

**Proof.** Using Theorem 3.8, we get that $\mathfrak{F}^m$ a has a fixed point $z$ in $\mathcal{Z}$. Since $\mathfrak{F}^m(\mathfrak{F}z) = \mathfrak{F}(\mathfrak{F}^m z) = \mathfrak{F}z$, then $\mathfrak{F}z$ is also a fixed point of $\mathfrak{F}^m$. Thus $\mathfrak{F}z = z$, $z$ is a fixed of $\mathfrak{F}$. Since the fixed in Theorem 3.8 is unique, the result follows. □

**Theorem 3.10.** *Let $(\mathcal{Z}, \rho)$ be a complete dislocated cone metric space over Banach algebra $\mathcal{A}$ with a unit $e$, $\mathcal{K}$ be the underlying solid cone. Let the mapping $\mathfrak{F} : \mathcal{Z} \to \mathcal{Z}$ be a continuous, surjection and satisfy the following condition:*

$$\rho(\mathfrak{F}z, \mathfrak{F}y) \succcurlyeq \vartheta\big\{\rho(z, y), \rho(z, \mathfrak{F}z), \rho(y, \mathfrak{F}y)\big\}, \tag{3.16}$$

*for all $z, y \in \mathcal{Z}$, where $\vartheta_j \in \mathcal{K}$ $(j = 1, 2, 3, 4)$ such that $\vartheta, \vartheta^{-1} \in \mathcal{K}$ and spectral radius $\delta(\vartheta^{-1}) < 1$. Then $\mathfrak{F}$ has a fixed point $z_*$ in $\mathcal{Z}$.*

**Proof.** Let $z_0$ be an arbitrary point in $\mathcal{Z}$. Since $\mathfrak{F}$ is surjective, there exists $z_1 \in \mathcal{Z}$ such that $\mathfrak{F}z_1 = z_0$. Again, we can choose $z_2 \in \mathcal{Z}$ such that $\mathfrak{F}z_2 = z_1$. Continuing this process, we can construct a sequence $\{z_i\}$ in $\mathcal{Z}$ by

$$z_i = \mathfrak{F}z_{i+1}, \text{ for } i = 0, 1, 2, \ldots. \tag{3.17}$$

Suppose $z_{j-1} = z_j$ for some $j \in \mathbb{N}$, then $z_* = z_{j-1}$ is a fixed point of $\mathfrak{F}$ and the result is proved. Hence, we assume that $z_{i-1} \neq z_i$ for all $i \in \mathbb{N}$. Now, using (3.16) and (3.17), we have

$$\begin{aligned} \rho(z_{i-1}, z_i) &= \rho(\mathfrak{F}z_i, \mathfrak{F}z_{i+1}) \\ &\succcurlyeq \vartheta\big\{\rho(z_i, z_{i+1}), \rho(z_i, \mathfrak{F}z_i), \rho(z_{i+1}, \mathfrak{F}z_{i+1})\big\} \\ &= \vartheta\big\{\rho(z_i, z_{i+1}), \rho(z_i, z_{i-1})\big\}. \end{aligned} \tag{3.18}$$

We consider the following two cases:

1. If $\rho(z_{i-1}, z_i) \succcurlyeq \vartheta\rho(z_i, z_{i-1})$ then $\rho(z_{i-1}, z_i) \preccurlyeq \vartheta^{-1}\rho(z_{i-1}, z_i)$. Hence, by Lemma 2.12, $\rho(z_{i-1}, z_i) = \theta$, that is $z_{i-1} = z_i$. This is a contradiction.
2. If $\rho(z_{i-1}, z_i) \succcurlyeq \vartheta\rho(z_i, z_{i+1})$ then $\rho(z_i, z_{i+1}) \preccurlyeq \vartheta^{-1}\rho(z_{i-1}, z_i) = \tau\rho(z_{i-1}, z_i)$.

Hence, we have

$$\begin{aligned} \rho(z_i, z_{i+1}) &\preccurlyeq \tau\rho(z_{i-1}, z_i) \\ &\preccurlyeq \tau^2\rho(z_{i-2}, z_{i-1}) \\ &\ \ \vdots \\ \rho(z_i, z_{i+1}) &\preccurlyeq \tau^i\rho(z_0, z_1), \text{ for all } i \in \mathbb{N}. \end{aligned} \tag{3.19}$$

Using the same argument to the proof in Theorem 3.3, we get that $\{z_i\}$ is a Cauchy sequence. Since $(\mathcal{Z}, \rho)$ is complete, there exists $z_* \in \mathcal{Z}$ such that $z_i \to z_*$ $(i \to \infty)$. To show that $z_*$ is a fixed point of $\mathfrak{F}$, since $\mathfrak{F}$ is continuous, so $\mathfrak{F}z_i \to \mathfrak{F}z_*$ $(i \to \infty)$, which implies that $z_{i-1} \to \mathfrak{F}z_*$ $(i \to \infty)$. Hence, $\mathfrak{F}z_* = z_*$. This completes the proof. □

**Example 3.11.** Let $\mathcal{A} = C^1_{\mathbb{R}}[0, 1/5]$ and define a norm on $\mathcal{A}$ by $\|z\| = \|z\|_\infty + \|z'\|_\infty$ for $z \in \mathcal{A}$, where multiplication in $\mathcal{A}$ is defined in the usual way. Then $\mathcal{A}$ is a Banach algebra with unit element $e = 1$ and the set $\mathcal{K} = \{z \in \mathcal{A} : z(t) \geq 0, t \in [0, 1/5]\}$ is a non normal cone in $\mathcal{A}$. Let $\mathcal{Z} = [0, \infty)$. Consider a mapping $\rho : \mathcal{Z} \times \mathcal{Z} \to \mathcal{A}$ define by

$$\rho(z, y)(t) = (z + y)e^t, \text{ for all } z, y \in \mathcal{Z}.$$

Then $(\mathcal{Z}, \rho)$ is a dislocated cone metric space over Banach algebra $\mathcal{A}$. Define a mapping $\mathfrak{F} : \mathcal{Z} \to \mathcal{Z}$ by $\mathfrak{F}z = 2z$, for all $z \in \mathcal{Z}$. Let $\vartheta \in \mathcal{K}$ be defined by $\vartheta(t) = \frac{5}{3t+4}$. Simple calculations show that all the conditions of Theorem 3.8 are satisfied and $z_* = \theta$ is the unique fixed point of $\mathfrak{F}$.

# 4. Conclusion

The aim of this paper is to introduce the notion of generalized expansive mappings on dislocated cone metric space over Banach algebras and prove some fixed point theorems for such mappings. Our results are actual generalization of the recent results in [11, 2, 10, 1, 3, 19] and others in the literature. We hope the results will be useful in fixed point theory and may be generalized in further spaces with efficient conditions.

**Competing interests**
The authors declare that they have no competing interests.

**Authors contributions**
All authors have equally contributed towards writing this article. All authors read and approved the final manuscript.

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**Abba Auwalu**
email: abba.auwalu@neu.edu.tr, abbaauwalu@yahoo.com
ORCID: 0000-0002-6859-5923
Department of Mathematics
Near East University
Nicosia-TRNC, Mersin 10
TURKEY

**Evren Hinçal**
email: evren.hincal@neu.edu.tr, evrenhincal@yahoo.com
Graduate School of Applied Sciences
Near East University
Nicosia-TRNC, Mersin 10
TURKEY

**Lakshmi Narayan Mishra***
email: lakshminarayanmishra04@gmail.com, lakshminarayan.mishra@vit.ac.in
ORCID: 0000-0001-7774-7290
School of Advanced Sciences
Vellore Institute of Technology (VIT) University
Vellore 632 014, Tamil Nadu
INDIA
*Corresponding author

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# Existence and Uniqueness of Solutions for Nonlinear Katugampola Fractional Differential Equations

*Bilal Basti, Yacine Arioua* and Nouredine Benhamidouche*

ABSTRACT: The present paper deals with the existence and uniqueness of solutions for a boundary value problem of nonlinear fractional differential equations with Katugampola fractional derivative. The main results are proved by means of Guo-Krasnoselskii and Banach fixed point theorems. For applications purposes, some examples are provided to demonstrate the usefulness of our main results.

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*Keywords and Phrases:* Fractional equation; Fixed point theorems; Boundary value problem; Existence; Uniqueness.

## 1. Introduction

The differential equations of fractional order are generalizations of classical differential equations of integer order. They are increasingly used in a variety of fields such as fluid flow, control theory of dynamical systems, signal and image processing, aerodynamics, electromagnetics, probability and statistics, (Samko et al. 1993 [18], Podlubny 1999 [17], Kilbas et al. 2006 [9], Diethelm 2010 [3]) books can be checked as a reference.

Boundary value problem of fractional differential equations is recently approached by various researchers ([1], [8], [19], [20]).

In [20], Bai and L used some fixed point theorems on cone to show the existence and multiplicity of positive solutions for a Dirichlet-type problem of the nonlinear fractional differential equation:

$$\begin{cases} \mathcal{D}_{0^+}^{\alpha} u(t) + f(t, u(t)) = 0, \quad 0 < t < 1, \\ u(0) = u(1) = 0, \end{cases}$$

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where $\mathcal{D}_{0+}^{\alpha}u$ is the standard Riemann Liouville fractional derivative of order $1 < \alpha \leq 2$ and $f : [0,1] \times [0,\infty) \to [0,\infty)$ is continuous function.

In a recent work [8], Katugampola studied the existence and uniqueness of solutions for the following initial value problem:

$$\begin{cases} {}_{c}^{\rho}\mathcal{D}_{0+}^{\alpha}u(t) = f(t,u(t)), \ \alpha > 0, \\ D^{k}u(0) = u_{0}^{(k)}, \ k = 1,2,...,m-1, \end{cases}$$

where $m = [\alpha]$, ${}_{c}^{\rho}\mathcal{D}_{0+}^{\alpha}$ is the Caputo-type generalized fractional derivative, of order $\alpha$, and $f : G \to \mathbb{R}$ is a given continuous function with:

$$G = \left\{ (t,u) : t \in [0,h^{*}], \ \left| u - \sum_{k=0}^{m-1} \frac{t^{k}u_{0}^{(k)}}{k!} \right| \leq K, \ K, h^{*} > 0 \right\}.$$

This paper focuses on the existence and uniqueness of solutions for a nonlinear fractional differential equation involving Katugampola fractional derivative:

$$ {}^{\rho}\mathcal{D}_{0+}^{\alpha}u(t) + \beta f(t,u(t)) = 0, \ 0 < t < T, \tag{1.1}$$

supplemented with the boundary conditions:

$$u(0) = 0, \ u(T) = 0, \tag{1.2}$$

where $\beta \in \mathbb{R}$, and ${}^{\rho}\mathcal{D}_{0+}^{\alpha}$ for $\rho > 0$, presents Katugampola fractional derivative of order $1 < \alpha \leq 2$, $f : [0,T] \times [0,\infty) \to [h,\infty)$ is a continuous function, with finite positive constants $h, T$.

## 2. Background materials and preliminaries

In this section, some necessary definitions from fractional calculus theory are presented. Let $\Omega = [0,T] \subset \mathbb{R}$ be a finite interval.

As in [9], let us denote by $X_{c}^{p}[0,T]$, $(c \in \mathbb{R}, \ 1 \leq p \leq \infty)$ the space of those complex-valued Lebesgue measurable functions $y$ on $[0,T]$ for which $\|y\|_{X_{c}^{p}} < \infty$ is defined by

$$\|y\|_{X_{c}^{p}} = \left( \int_{0}^{T} |s^{c}y(s)|^{p} \frac{ds}{s} \right)^{\frac{1}{p}} < \infty,$$

for $1 \leq p < \infty$, $c \in \mathbb{R}$, and

$$\|y\|_{X_{c}^{\infty}} = \operatorname{ess} \sup_{0 \leq t \leq T} [t^{c}|y(t)|], \ (c \in \mathbb{R}).$$

**Definition 2.1** (Riemann-Liouville fractional integral [9])**.** The left-sided Riemann-Liouville fractional integral of order $\alpha > 0$ of a continuous function $y : [0,T] \to \mathbb{R}$ is given by:

$$ {}^{RL}\mathcal{I}_{0+}^{\alpha}y(t) = \frac{1}{\Gamma(\alpha)} \int_{0}^{t} (t-s)^{\alpha-1} y(s)\,ds, \ t \in [0,T],$$

where $\Gamma(\alpha) = \int_{0}^{+\infty} e^{-s}s^{\alpha-1}ds$, is the Euler gamma function.

**Definition 2.2** (Riemann-Liouville fractional derivative [9])**.** The left-sided Riemann Liouville fractional derivative of order $\alpha > 0$ of a continuous function $y : [0, T] \to \mathbb{R}$ is given by:

$$^{RL}\mathcal{D}^{\alpha}_{0^+} y(t) = \frac{1}{\Gamma(n-\alpha)} \left(\frac{d}{dt}\right)^n \int_0^t (t-s)^{n-\alpha-1} y(s)\, ds,\ t \in [0, T],\ n = [\alpha] + 1,$$

**Definition 2.3** (Hadamard fractional integral [9])**.** The left-sided Hadamard fractional integral of order $\alpha > 0$ of a continuous function $y : [0, T] \to \mathbb{R}$ is given by:

$$^{H}\mathcal{I}^{\alpha}_{0^+} y(t) = \frac{1}{\Gamma(\alpha)} \int_0^t \left(\log \frac{t}{s}\right)^{\alpha-1} y(s) \frac{ds}{s},\quad t \in [0, T].$$

**Definition 2.4** (Hadamard fractional derivative [9])**.** The left-sided Hadamard fractional derivative of order $\alpha > 0$ of a continuous function $y : [0, T] \to \mathbb{R}$ is given by:

$$^{H}\mathcal{D}^{\alpha}_{0^+} y(t) = \frac{1}{\Gamma(n-\alpha)} \left(t\frac{d}{dt}\right)^n \int_0^t \left(\log \frac{t}{s}\right)^{n-\alpha-1} y(s) \frac{ds}{s},\quad t \in [0, T],\ n = [\alpha] + 1,$$

if the integral exist.

A recent generalization in 2011, introduced by Udita Katugampola [6], combines the Riemann-Liouville fractional integral and the Hadamard fractional integral into a single form (see [9]) , the integral is now known as Katugampola fractional integral, it is given in the following definition:

**Definition 2.5** (Katugampola fractional integral [6])**.**
The left-sided Katugampola fractional integral of order $\alpha > 0$ of a function $y \in X^p_c[0, T]$ is defined by:

$$\left({}^{\rho}\mathcal{I}^{\alpha}_{0^+} y\right)(t) = \frac{\rho^{1-\alpha}}{\Gamma(\alpha)} \int_0^t \frac{s^{\rho-1} y(s)}{(t^\rho - s^\rho)^{1-\alpha}} ds,\ \rho > 0,\ t \in [0, T]. \tag{2.1}$$

Similarly, we can define right-sided integrals [6]-[7], [9].

**Definition 2.6** (Katugampola fractional derivatives [7])**.**
Let $\alpha, \rho \in \mathbb{R}^+$, and $n = [\alpha] + 1$. The Katugampola fractional derivative corresponding to the Katugampola fractional integral (2.1) are defined for $0 \leq t \leq T \leq \infty$ by:

$$^{\rho}\mathcal{D}^{\alpha}_{0^+} y(t) = \left(t^{1-\rho}\frac{d}{dt}\right)^n \left({}^{\rho}\mathcal{I}^{n-\alpha}_{0^+} y\right)(t) = \frac{\rho^{\alpha-n+1}}{\Gamma(n-\alpha)} \left(t^{1-\rho}\frac{d}{dt}\right)^n \int_0^t \frac{s^{\rho-1} y(s)}{(t^\rho - s^\rho)^{\alpha-n+1}} ds. \tag{2.2}$$

**Theorem 2.7** ([7]). *Let $\alpha, \rho \in \mathbb{R}^{+}$, then*

$$\begin{aligned}
\lim_{\rho\to 1}\left({}^{\rho}\mathcal{I}_{0^+}^{\alpha}y\right)(t) &= {}^{RL}\mathcal{I}_{0^+}^{\alpha}y(t) = \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}y(s)\,ds,\\
\lim_{\rho\to 0^+}\left({}^{\rho}\mathcal{I}_{0^+}^{\alpha}y\right)(t) &= {}^{H}\mathcal{I}_{0^+}^{\alpha}y(t) = \frac{1}{\Gamma(\alpha)}\int_0^t \left(\log\frac{t}{s}\right)^{\alpha-1}\frac{y(s)}{s}ds,\\
\lim_{\rho\to 1}\left({}^{\rho}\mathcal{D}_{0^+}^{\alpha}y\right)(t) &= {}^{RL}\mathcal{D}_{0^+}^{\alpha}y(t) = \frac{1}{\Gamma(n-\alpha)}\left(\frac{d}{dt}\right)^n\int_0^t (t-s)^{n-\alpha-1}y(s)\,ds,\\
\lim_{\rho\to 0^+}\left({}^{\rho}\mathcal{D}_{0^+}^{\alpha}y\right)(t) &= {}^{H}\mathcal{D}_{0^+}^{\alpha}y(t) = \frac{1}{\Gamma(n-\alpha)}\left(t\frac{d}{dt}\right)^n\int_0^t \left(\log\frac{t}{s}\right)^{n-\alpha-1}\frac{y(s)}{s}ds.
\end{aligned}$$

**Remark.** As an example, for $\alpha, \rho > 0$, and $\mu > -\rho$, we have

$$ {}^{\rho}\mathcal{D}_{0^+}^{\alpha}t^{\mu} = \frac{\rho^{\alpha-1}\Gamma\left(1+\frac{\mu}{\rho}\right)}{\Gamma\left(1-\alpha+\frac{\mu}{\rho}\right)}t^{\mu-\alpha\rho}. \tag{2.3}$$

In particular

$$ {}^{\rho}\mathcal{D}_{0^+}^{\alpha}t^{\rho(\alpha-m)} = 0,\ \text{for each } m = 1,2,\ldots,n.$$

For $\mu > -\rho$, we have

$$\begin{aligned}
{}^{\rho}\mathcal{D}_{0^+}^{\alpha}t^{\mu} &= \frac{\rho^{\alpha-n+1}}{\Gamma(n-\alpha)}\left(t^{1-\rho}\frac{d}{dt}\right)^n\int_0^t s^{\rho+\mu-1}\left(t^{\rho}-s^{\rho}\right)^{n-\alpha-1}ds\\
&= \frac{\rho^{\alpha-n}}{\Gamma(n-\alpha)}\left(t^{1-\rho}\frac{d}{dt}\right)^n t^{\rho(n-\alpha)+\mu}\int_0^1 \tau^{\frac{\mu}{\rho}}(1-\tau)^{n-\alpha-1}d\tau\\
&= \frac{\rho^{\alpha-n}}{\Gamma(n-\alpha)}B\left(n-\alpha,1+\frac{\mu}{\rho}\right)\left(t^{1-\rho}\frac{d}{dt}\right)^n t^{\rho(n-\alpha)+\mu}\\
&= \frac{\rho^{\alpha-n}\Gamma\left(1+\frac{\mu}{\rho}\right)}{\Gamma\left(1+n-\alpha+\frac{\mu}{\rho}\right)}\left(t^{1-\rho}\frac{d}{dt}\right)^n t^{\rho(n-\alpha)+\mu}.
\end{aligned}$$

Then

$$ {}^{\rho}\mathcal{D}_{0^+}^{\alpha}t^{\mu} = \frac{\rho^{\alpha-1}\Gamma\left(1+\frac{\mu}{\rho}\right)}{\Gamma\left(1+n-\alpha+\frac{\mu}{\rho}\right)}\left[n-\alpha+\frac{\mu}{\rho}\right]\left[n-\alpha-1+\frac{\mu}{\rho}\right]\cdots\left[1-\alpha+\frac{\mu}{\rho}\right]t^{\mu-\alpha\rho}. \tag{2.4}$$

As

$$\Gamma\left(1+n-\alpha+\frac{\mu}{\rho}\right) = \left[n-\alpha+\frac{\mu}{\rho}\right]\left[n-\alpha-1+\frac{\mu}{\rho}\right]\cdots\left[1-\alpha+\frac{\mu}{\rho}\right]\Gamma\left(1-\alpha+\frac{\mu}{\rho}\right),$$

we get

$$ {}^{\rho}\mathcal{D}_{0^+}^{\alpha}t^{\mu} = \frac{\rho^{\alpha-1}\Gamma\left(1+\frac{\mu}{\rho}\right)}{\Gamma\left(1-\alpha+\frac{\mu}{\rho}\right)}t^{\mu-\alpha\rho}.$$

In case $m = \alpha - \frac{\mu}{\rho}$, it follows from (2.4) , that

$$^{\rho}\mathcal{D}_{0+}^{\alpha} t^{\rho(\alpha-m)} = \rho^{\alpha-1}\frac{\Gamma(\alpha-m+1)}{\Gamma(n-m+1)}(n-m)(n-m-1)\cdots(1-m)\,t^{-\rho m}.$$

So, for $m = 1, 2, \ldots, n$, we get

$$^{\rho}\mathcal{D}_{0+}^{\alpha} t^{\rho(\alpha-m)} = 0.$$

Similarly, for all $\alpha, \rho > 0$, we have:

$$^{\rho}\mathcal{I}_{0+}^{\alpha} t^{\mu} = \frac{\rho^{-\alpha}\Gamma\left(1+\frac{\mu}{\rho}\right)}{\Gamma\left(1+\alpha+\frac{\mu}{\rho}\right)} t^{\mu+\alpha\rho},\ \forall\mu > -\rho. \tag{2.5}$$

By $C[0,T]$, we denote the Banach space of all continuous functions from $[0,T]$ into $\mathbb{R}$ with the norm:

$$\|y\| = \max_{0\le t\le T} |y(t)|.$$

**Remark.** Let $p \geq 1$, $c > 0$ and $T \leq (pc)^{\frac{1}{pc}}$. Far all $y \in C[0,T]$, note that

$$\|y\|_{X_c^p} = \left(\int_0^T |s^c y(s)|^p \frac{ds}{s}\right)^{\frac{1}{p}} \leq \left(\|y\|^p \int_0^T s^{pc-1} ds\right)^{\frac{1}{p}} = \frac{T^c}{(pc)^{\frac{1}{p}}}\|y\|,$$

and

$$\|y\|_{X_c^\infty} = \operatorname{ess}\sup_{0\le t\le T}[t^c |y(t)|] \leq T^c \|y\|,$$

which imply that $C[0,T] \hookrightarrow X_c^p[0,T]$, and

$$\|y\|_{X_c^p} \leq \|y\|_\infty, \text{ for all } T \leq (pc)^{\frac{1}{pc}}.$$

We express some properties of Katugampola fractional integral and derivative in the following result.

**Theorem 2.8** ([6]-[7]-[8])**.**
*Let $\alpha, \beta, \rho, c \in \mathbb{R}$, be such that $\alpha, \beta, \rho > 0$. Then, for any $y \in X_c^p[0,T]$, where $1 \leq p \leq \infty$, we have:*
*- Index property:*

$$\begin{aligned} ^{\rho}\mathcal{I}_{0+}^{\alpha}\ ^{\rho}\mathcal{I}_{0+}^{\beta} y(t) &= {}^{\rho}\mathcal{I}_{0+}^{\alpha+\beta} y(t), \quad \textit{for all } \alpha, \beta > 0, \\ ^{\rho}\mathcal{D}_{0+}^{\alpha}\ ^{\rho}\mathcal{D}_{0+}^{\beta} y(t) &= {}^{\rho}\mathcal{D}_{0+}^{\alpha+\beta} y(t), \quad \textit{for all } 0 < \alpha, \beta < 1. \end{aligned}$$

*- Inverse property*

$$^{\rho}\mathcal{D}_{0+}^{\alpha}\ ^{\rho}\mathcal{I}_{0+}^{\alpha} y(t) = y(t), \quad \textit{for all } \alpha \in (0,1).$$

From Definitions 2.5 and 2.6, and Theorem 2.8, we deduce that

$$
\begin{aligned}
{}^{\rho}\mathcal{I}_{0^+}^{1}\left(t^{1-\rho}\frac{d}{dt}\right){}^{\rho}\mathcal{I}_{0^+}^{\alpha+1}y\left(t\right) &= \int_0^t s^{\rho-1}\left(s^{1-\rho}\frac{d}{ds}\right){}^{\rho}\mathcal{I}_{0^+}^{\alpha+1}y\left(s\right)\,ds \\
&= \int_0^t \frac{d}{ds}\,{}^{\rho}\mathcal{I}_{0^+}^{\alpha+1}y\left(s\right)\,ds \\
&= \left[\frac{1}{\rho^{\alpha}\Gamma\left(\alpha+1\right)}\int_0^s \tau^{\rho-1}\left(t^{\rho}-\tau^{\rho}\right)^{\alpha}y\left(\tau\right)d\tau\right]_0^t \\
&= {}^{\rho}\mathcal{I}_{0^+}^{\alpha+1}y\left(t\right).
\end{aligned}
$$

Consequently

$$\left(t^{1-\rho}\frac{d}{dt}\right){}^{\rho}\mathcal{I}_{0^+}^{\alpha+1}y\left(t\right) = {}^{\rho}\mathcal{I}_{0+}^{\alpha}y\left(t\right),\ \forall\alpha>0. \tag{2.6}$$

**Definition 2.9** ([4]). Let $E$ be a real Banach space, a nonempty closed convex set $P\subset E$ is called a cone of $E$ if it satisfies the following conditions:
(i) $u\in P$, $\lambda\geq 0$, implies $\lambda u\in P$.
(ii) $u\in P$, $-u\in P$, implies $u=0$.

**Definition 2.10** ([2]). Let $E$ be a Banach space, $P\in C\left(E\right)$ is called an equicontinuous part if and only if

$$\forall\varepsilon>0,\ \exists\delta>0,\ \forall u,v\in E,\ \forall\mathcal{A}\in P,\ \left\|u-v\right\|<\delta\Rightarrow\left\|\mathcal{A}\left(u\right)-\mathcal{A}\left(v\right)\right\|<\varepsilon.$$

**Theorem 2.11** (Ascoli-Arzel [2]). *Let $E$ be a compact space. If $\mathcal{A}$ is an equicontinuous, bounded subset of $C\left(E\right)$, then $\mathcal{A}$ is relatively compact.*

**Definition 2.12** (Completely continuous [4]). We say $\mathcal{A}:E\rightarrow E$ is completely continuous if for any bounded subset $P\subset E$, the set $\mathcal{A}\left(P\right)$ is relatively compact.

The following fixed-point theorems are fundamental in the proofs of our main results.

**Lemma 2.13** (Guo-Krasnosel'skii fixed point theorems [12]).
*Let $E$ be a Banach space, $P\subseteq E$ a cone, and $\Omega_1$, $\Omega_2$ two bounded open balls of $E$ centered at the origin with $\bar{\Omega}_1\subset\Omega_2$. Suppose that $\mathcal{A}:P\cap\left(\bar{\Omega}_2\backslash\Omega_1\right)\rightarrow P$ is a completely continuous operator such that either*

(i) $\left\|\mathcal{A}x\right\|\leq\left\|x\right\|,\ x\in P\cap\partial\Omega_1$ *and* $\left\|\mathcal{A}x\right\|\geq\left\|x\right\|,\ x\in P\cap\partial\Omega_2$, *or*

(ii) $\left\|\mathcal{A}x\right\|\geq\left\|x\right\|,\ x\in P\cap\partial\Omega_1$ *and* $\left\|\mathcal{A}x\right\|\leq\left\|x\right\|,\ x\in P\cap\partial\Omega_2$,

*holds. Then $\mathcal{A}$ has a fixed point in $P\cap\left(\bar{\Omega}_2\backslash\Omega_1\right)$.*

**Theorem 2.14** (Banach's fixed point [5]). *Let $E$ be a Banach space, $P\subseteq E$ a nonempty closed subset. If $\mathcal{A}:P\rightarrow P$ is a contraction mapping, then $\mathcal{A}$ has a unique fixed point in $P$.*

# 3. Main results

In the sequel, $T$, $p$ and $c$ are real constants such that

$$p \geq 1,\ \ c > 0,\ \text{ and } T \leq (pc)^{\frac{1}{pc}}.$$

Now, we present some important lemmas which play a key role in the proofs of the main results.

**Lemma 3.1.** *Let $\alpha, \rho \in \mathbb{R}^+$. If $u \in C[0,T]$, then:*

(i) *The fractional equation ${}^{\rho}\mathcal{D}_{0+}^{\alpha} u(t) = 0$, has a solution as follows:*

$$u(t) = C_1 t^{\rho(\alpha-1)} + C_2 t^{\rho(\alpha-2)} + \cdots + C_n t^{\rho(\alpha-n)},\ \text{ where } C_m \in \mathbb{R},\ \text{ with } m = 1,2,\ldots,n.$$

(ii) *If ${}^{\rho}\mathcal{D}_{0+}^{\alpha} u \in C[0,T]$ and $1 < \alpha \leq 2$, then:*

$$ {}^{\rho}\mathcal{I}_{0+}^{\alpha}\ {}^{\rho}\mathcal{D}_{0+}^{\alpha}\ u(t) = u(t) + C_1 t^{\rho(\alpha-1)} + C_2 t^{\rho(\alpha-2)},\quad \text{for some } C_1, C_2 \in \mathbb{R}. \tag{3.1}$$

**Proof.** (i) Let $\alpha, \rho \in \mathbb{R}^+$. From remark 2, we have:

$${}^{\rho}\mathcal{D}_{0+}^{\alpha} t^{\rho(\alpha-m)} = 0,\quad \text{for each } m = 1,2,\ldots,n.$$

Then, the fractional differential equation ${}^{\rho}\mathcal{D}_{0+}^{\alpha} u(t) = 0$, admits a solution as follows:

$$u(t) = C_1 t^{\rho(\alpha-1)} + C_2 t^{\rho(\alpha-2)} + \cdots + C_n t^{\rho(\alpha-n)},\ C_m \in \mathbb{R},\ m = 1,2,\ldots,n.$$

(ii) Let ${}^{\rho}\mathcal{D}_{0+}^{\alpha} u \in C[0,T]$ be the fractional derivative (2.2) of order $1 < \alpha \leq 2$. If we apply the operator ${}^{\rho}\mathcal{I}_{0+}^{\alpha}$ to ${}^{\rho}\mathcal{D}_{0+}^{\alpha} u(t)$ and use Definitions 2.5, 2.6, Theorem 2.8 and property (2.6), we get

$$\begin{aligned}
{}^{\rho}\mathcal{I}_{0+}^{\alpha}\ {}^{\rho}\mathcal{D}_{0+}^{\alpha} u(t) &= \left(t^{1-\rho}\frac{d}{dt}\right) {}^{\rho}\mathcal{I}_{0+}^{\alpha+1}\ {}^{\rho}\mathcal{D}_{0+}^{\alpha} u(t) \\
&= \left(t^{1-\rho}\frac{d}{dt}\right)\left[\frac{\rho^{-\alpha}}{\Gamma(\alpha+1)}\int_0^t (t^{\rho}-s^{\rho})^{\alpha} s^{\rho-1}\ {}^{\rho}\mathcal{D}_{0+}^{\alpha} u(s)\, ds\right] \\
&= \left(t^{1-\rho}\frac{d}{dt}\right)\left[\frac{\rho^{-\alpha}}{\Gamma(\alpha+1)}\int_0^t (t^{\rho}-s^{\rho})^{\alpha} s^{\rho-1}\left[\left(s^{1-\rho}\frac{d}{ds}\right)^2 {}^{\rho}\mathcal{I}_{0+}^{2-\alpha} u(s)\right] ds\right] \\
&= \left(t^{1-\rho}\frac{d}{dt}\right)\left[\frac{\rho^{-\alpha}}{\Gamma(\alpha+1)}\int_0^t (t^{\rho}-s^{\rho})^{\alpha}\frac{d}{ds}\left[\left(s^{1-\rho}\frac{d}{ds}\right) {}^{\rho}\mathcal{I}_{0+}^{2-\alpha} u(s)\right] ds\right] \\
&= \left(t^{1-\rho}\frac{d}{dt}\right)\left[\frac{\rho^{-\alpha}}{\Gamma(\alpha+1)}\left(\left[(t^{\rho}-s^{\rho})^{\alpha}\left(s^{1-\rho}\frac{d}{ds}\right) {}^{\rho}\mathcal{I}_{0+}^{2-\alpha} u(s)\right]_0^t\right.\right. \\
&\quad \left.\left. + \alpha\rho\int_0^t s^{\rho-1}(t^{\rho}-s^{\rho})^{\alpha-1}\left(s^{1-\rho}\frac{d}{ds}\right) {}^{\rho}\mathcal{I}_{0+}^{2-\alpha} u(s)\, ds\right)\right].
\end{aligned}$$

From (2.6) , we have

$$\left(s^{1-\rho}\frac{d}{ds}\right){}^{\rho}\mathcal{I}_{0^+}^{2-\alpha}u\left(s\right)={}^{\rho}\mathcal{I}_{0^+}^{1-\alpha}u\left(s\right). \tag{3.2}$$

On the other hand, from (2.2) , we have

$$\left(s^{1-\rho}\frac{d}{ds}\right){}^{\rho}\mathcal{I}_{0^+}^{2-\alpha}u\left(s\right)=\left(s^{1-\rho}\frac{d}{ds}\right)^{1}{}^{\rho}\mathcal{I}_{0^+}^{1-(\alpha-1)}u\left(s\right)={}^{\rho}\mathcal{D}_{0^+}^{\alpha-1}u\left(s\right). \tag{3.3}$$

Then

$$\begin{aligned}{}^{\rho}\mathcal{I}_{0^+}^{\alpha}\ {}^{\rho}\mathcal{D}_{0^+}^{\alpha}u\left(t\right) &= \underbrace{t^{1-\rho}\frac{d}{dt}\left(\frac{\rho^{1-\alpha}}{\Gamma\left(\alpha\right)}\int_0^t\left(t^{\rho}-s^{\rho}\right)^{\alpha-1}\frac{d}{ds}\ {}^{\rho}\mathcal{I}_{0^+}^{2-\alpha}u\left(s\right)ds\right)}_{\psi}\\ &\quad -\frac{\rho^{1-\alpha}\ {}^{\rho}\mathcal{I}_{0^+}^{1-\alpha}u\left(0^+\right)}{\Gamma\left(\alpha\right)}t^{\rho(\alpha-1)},\end{aligned}$$

where

$$\begin{aligned}\psi &= t^{1-\rho}\frac{d}{dt}\frac{\rho^{1-\alpha}}{\Gamma\left(\alpha\right)}\left(\left[\left(t^{\rho}-s^{\rho}\right)^{\alpha-1}\ {}^{\rho}\mathcal{I}_{0^+}^{2-\alpha}u\left(s\right)\right]_0^t\right.\\ &\quad \left.+\rho\left(\alpha-1\right)\int_0^t s^{\rho-1}\left(t^{\rho}-s^{\rho}\right)^{\alpha-2}\ {}^{\rho}\mathcal{I}_{0^+}^{2-\alpha}u\left(s\right)ds\right)\\ &= t^{1-\rho}\frac{d}{dt}\left(\frac{\rho^{2-\alpha}}{\Gamma\left(\alpha-1\right)}\int_0^t s^{\rho-1}\left(t^{\rho}-s^{\rho}\right)^{\alpha-2}\ {}^{\rho}\mathcal{I}_{0^+}^{2-\alpha}u\left(s\right)ds\right.\\ &\quad \left.-\frac{\rho^{1-\alpha}\ {}^{\rho}\mathcal{I}_{0^+}^{2-\alpha}u\left(0^+\right)}{\Gamma\left(\alpha\right)}t^{\rho(\alpha-1)}\right)\\ &= t^{1-\rho}\frac{d}{dt}\left({}^{\rho}\mathcal{I}_{0^+}^{\alpha-1}\ {}^{\rho}\mathcal{I}_{0^+}^{2-\alpha}u\left(t\right)-\frac{\rho^{1-\alpha}\ {}^{\rho}\mathcal{I}_{0^+}^{2-\alpha}u\left(0^+\right)}{\Gamma\left(\alpha\right)}t^{\rho(\alpha-1)}\right)\\ &= t^{1-\rho}\frac{d}{dt}\left({}^{\rho}\mathcal{I}_{0^+}^{1}u\left(t\right)-\frac{\rho^{1-\alpha}\ {}^{\rho}\mathcal{I}_{0^+}^{2-\alpha}u\left(0^+\right)}{\Gamma\left(\alpha\right)}t^{\rho(\alpha-1)}\right)\\ &= u\left(t\right)-\frac{\rho^{2-\alpha}\ {}^{\rho}\mathcal{I}_{0^+}^{2-\alpha}u\left(0^+\right)}{\Gamma\left(\alpha-1\right)}t^{\rho(\alpha-2)}.\end{aligned}$$

Finally, for $1<\alpha\leq 2$, we have:

$${}^{\rho}\mathcal{I}_{0^+}^{\alpha}\ {}^{\rho}\mathcal{D}_{0^+}^{\alpha}u\left(t\right)=u\left(t\right)-\frac{\rho^{1-\alpha}\ {}^{\rho}\mathcal{I}_{0^+}^{1-\alpha}u\left(0^+\right)}{\Gamma\left(\alpha\right)}t^{\rho(\alpha-1)}-\frac{\rho^{2-\alpha}\ {}^{\rho}\mathcal{I}_{0^+}^{2-\alpha}u\left(0^+\right)}{\Gamma\left(\alpha-1\right)}t^{\rho(\alpha-2)}. \tag{3.4}$$

As

$${}^{\rho}\mathcal{I}_{0^+}^{\alpha}t^{\mu}=\frac{\rho^{-\alpha}\Gamma\left(1+\frac{\mu}{\rho}\right)}{\Gamma\left(1+\alpha+\frac{\mu}{\rho}\right)}t^{\mu+\alpha\rho},\ \forall\mu>-\rho,$$

we use $(3.2)$, $(3.3)$, to prove that

$$ {}^{\rho}\mathcal{I}_{0^+}^{1-\alpha}\left[C_1 t^{\rho(\alpha-1)}\right] = C_1 \frac{\rho^{-(1-\alpha)}\Gamma\left(1+\frac{\rho(\alpha-1)}{\rho}\right)}{\Gamma\left(1+(1-\alpha)+\frac{\rho(\alpha-1)}{\rho}\right)} t^{\rho(\alpha-1)+(1-\alpha)\rho} = C_1\rho^{\alpha-1}\Gamma(\alpha), \quad (3.5) $$

$$ {}^{\rho}\mathcal{I}_{0^+}^{1-\alpha}\left[C_2 t^{\rho(\alpha-2)}\right] = C_2\ {}^{\rho}\mathcal{D}_{0^+}^{\alpha-1} t^{\rho(\alpha-2)} = C_2\ {}^{\rho}\mathcal{D}_{0^+}^{\alpha-1} t^{\rho((\alpha-1)-1)} = 0, \quad (3.6) $$

for some $C_1, C_2 \in \mathbb{R}$, and

$$ {}^{\rho}\mathcal{I}_{0^+}^{2-\alpha}\left[C_1 t^{\rho(\alpha-1)}\right] = C_1 \frac{\rho^{-(2-\alpha)}\Gamma\left(1+\frac{\rho(\alpha-1)}{\rho}\right)}{\Gamma\left(1+(2-\alpha)+\frac{\rho(\alpha-1)}{\rho}\right)} t^{\rho(\alpha-1)+(2-\alpha)\rho} = C_1\rho^{\alpha-2}\Gamma(\alpha)\, t^{\rho} \quad (3.7) $$

$$ {}^{\rho}\mathcal{I}_{0^+}^{2-\alpha}\left[C_2 t^{\rho(\alpha-2)}\right] = C_2 \frac{\rho^{-(2-\alpha)}\Gamma\left(1+\frac{\rho(\alpha-2)}{\rho}\right)}{\Gamma\left(1+(2-\alpha)+\frac{\rho(\alpha-2)}{\rho}\right)} t^{\rho(\alpha-2)+(2-\alpha)\rho} = C_2\rho^{\alpha-2}\Gamma(\alpha-1). \quad (3.8) $$

Then, for $u(t) = C_1 t^{\rho(\alpha-1)} + C_2 t^{\rho(\alpha-2)}$, we have respectively:

$$ {}^{\rho}\mathcal{I}_{0^+}^{1-\alpha}u\left(0^+\right) = {}^{\rho}\mathcal{I}_{0^+}^{1-\alpha}\left[C_1 t^{\rho(\alpha-1)}\right]\left(0^+\right) + {}^{\rho}\mathcal{I}_{0^+}^{1-\alpha}\left[C_2 t^{\rho(\alpha-2)}\right]\left(0^+\right) = C_1\rho^{\alpha-1}\Gamma(\alpha), \quad (3.9) $$

$$ {}^{\rho}\mathcal{I}_{0^+}^{2-\alpha}u\left(0^+\right) = {}^{\rho}\mathcal{I}_{0^+}^{2-\alpha}\left[C_1 t^{\rho(\alpha-1)}\right]\left(0^+\right) + {}^{\rho}\mathcal{I}_{0^+}^{2-\alpha}\left[C_2 t^{\rho(\alpha-2)}\right]\left(0^+\right) = C_2\rho^{\alpha-2}\Gamma(\alpha-1). \quad (3.10) $$

From $(3.4), (3.5), (3.6), (3.7), (3.8), (3.9)$ and $(3.10)$ we get $(3.1)$. □

In the following lemma, we define the integral solution of the boundary value problem (1.1)-(1.2).

**Lemma 3.2.** *Let $\alpha, \rho \in \mathbb{R}^+$, be such that $1 < \alpha \le 2$. We give ${}^{\rho}\mathcal{D}_{0^+}^{\alpha}u \in C[0,T]$, and $f(t,u)$ is a continuous function. Then the boundary value problem* (1.1)-(1.2), *is equivalent to the fractional integral equation*

$$ u(t) = \beta \int_0^T G(t,s)\, f(s,u(s))\, ds, \ t \in [0,T], $$

*where*

$$ G(t,s) = \begin{cases} \frac{\rho^{1-\alpha}s^{\rho-1}}{\Gamma(\alpha)}\left[\left[\frac{t^{\rho}}{T^{\rho}}\left(T^{\rho}-s^{\rho}\right)\right]^{\alpha-1} - \left(t^{\rho}-s^{\rho}\right)^{\alpha-1}\right], & 0 \le s \le t \le T, \\ \frac{\rho^{1-\alpha}s^{\rho-1}}{\Gamma(\alpha)}\left[\frac{t^{\rho}}{T^{\rho}}\left(T^{\rho}-s^{\rho}\right)\right]^{\alpha-1}, & 0 \le t \le s \le T, \end{cases} \quad (3.11) $$

*is the Green's function associated with the boundary value problem* (1.1)-(1.2).

**Proof.** Let $\alpha, \rho \in \mathbb{R}^+$, be such that $1 < \alpha \le 2$. We apply Lemma 3.1 to reduce the fractional equation (1.1) to an equivalent fractional integral equation. It is easy to prove the operator ${}^{\rho}\mathcal{I}_{0+}^{\alpha}$ has the linearity property for all $\alpha > 0$ after direct integration. Then by applying ${}^{\rho}\mathcal{I}_{0+}^{\alpha}$ to equation (1.1) , we get

$$ {}^{\rho}\mathcal{I}_{0+}^{\alpha}\ {}^{\rho}\mathcal{D}_{0+}^{\alpha}u(t) + \beta\ {}^{\rho}\mathcal{I}_{0+}^{\alpha}f(t,u(t)) = 0. $$

From Lemma 3.1, we find for $1 < \alpha \leq 2$,

$$ {}^{\rho}\mathcal{I}_{0+}^{\alpha}\ {}^{\rho}\mathcal{D}_{0+}^{\alpha}u(t) = u(t) + C_1 t^{\rho(\alpha-1)} + C_2 t^{\rho(\alpha-2)}, $$

for some $C_1, C_2 \in \mathbb{R}$. Then, the integral solution of the equation (1.1) is:

$$ u(t) = -\frac{\beta\rho^{1-\alpha}}{\Gamma(\alpha)}\int_0^t \frac{s^{\rho-1}f(s,u(s))}{(t^{\rho}-s^{\rho})^{1-\alpha}}ds - C_1 t^{\rho(\alpha-1)} - C_2 t^{\rho(\alpha-2)}. \tag{3.12} $$

The conditions (1.2) imply that:

$$ \begin{cases} u(0) = 0 = 0 - 0 - \lim\limits_{t\to 0} C_2 t^{\rho(\alpha-2)} & \Rightarrow \quad C_2 = 0, \\ u(T) = 0 = -\frac{\beta\rho^{1-\alpha}}{\Gamma(\alpha)}\int\limits_0^T \frac{s^{\rho-1}f(s,u(s))}{(T^{\rho}-s^{\rho})^{1-\alpha}}ds - C_1 T^{\rho(\alpha-1)} & \Rightarrow \quad C_1 = -\frac{\beta\rho^{1-\alpha}}{T^{\rho(\alpha-1)}\Gamma(\alpha)}\int\limits_0^T \frac{s^{\rho-1}f(s,u(s))}{(T^{\rho}-s^{\rho})^{1-\alpha}}ds. \end{cases} $$

The integral equation (3.12) is equivalent to:

$$ u(t) = -\frac{\beta\rho^{1-\alpha}}{\Gamma(\alpha)}\int_0^t \frac{s^{\rho-1}f(s,u(s))}{(t^{\rho}-s^{\rho})^{1-\alpha}}ds + \frac{\beta t^{\rho(\alpha-1)}\rho^{1-\alpha}}{T^{\rho(\alpha-1)}\Gamma(\alpha)}\int_0^T \frac{s^{\rho-1}f(s,u(s))}{(T^{\rho}-s^{\rho})^{1-\alpha}}ds. $$

Therefore, the unique solution of problem (1.1)-(1.2) is:

$$ \begin{aligned} u(t) &= \beta\int_0^t \frac{\rho^{1-\alpha}s^{\rho-1}\left[\left[\frac{t^{\rho}}{T^{\rho}}(T^{\rho}-s^{\rho})\right]^{\alpha-1} - (t^{\rho}-s^{\rho})^{\alpha-1}\right]}{\Gamma(\alpha)} f(s,u(s))\,ds \\ &\quad +\beta\int_t^T \frac{\rho^{1-\alpha}s^{\rho-1}\left[\frac{t^{\rho}}{T^{\rho}}(T^{\rho}-s^{\rho})\right]^{\alpha-1}}{\Gamma(\alpha)} f(s,u(s))\,ds \\ &= \beta\int_0^T G(t,s)\,f(s,u(s))\,ds. \end{aligned} $$

The proof is complete. □

## 3.1. Application of Guo-Krasnosel'skii fixed point theorem

In this part, we assume that $\beta > 0$ and $0 < \rho \leq 1$. We impose some conditions on $f$, which allow us to obtain some results on existence of positive solutions for the boundary value problem (1.1)-(1.2) .

We note that $u(t)$ is a solution of (1.1)-(1.2) if and only if:

$$ u(t) = \beta\int_0^T G(t,s)\,f(s,u(s))\,ds,\ t \in [0,T]. $$

Now we prove some properties of the Green's function $G(t,s)$ given by (3.11) .

**Lemma 3.3.** *Let $1 < \alpha \leq 2$ and $0 < \rho \leq 1$, then the Green's function $G(t,s)$ given by (3.11) satisfies:*

(1) $G(t,s) > 0$ *for* $t, s \in (0,T)$.

(2) $\max\limits_{0\leq t\leq T} G(t,s) = G(s,s)$, *for each* $s \in [0,T]$.

(3) *For any* $t \in [0,T]$,

$$G(t,s) \geq b(t)\, G(s,s)\,,\ \text{for any } \frac{T}{8} \leq s \leq T \text{ and some } b \in C[0,T]. \tag{3.13}$$

**Proof.** (1) Let $1 < \alpha \leq 2$ and $0 < \rho \leq 1$. In the case $0 < t \leq s < T$, we have:

$$\frac{\rho^{1-\alpha} s^{\rho-1}}{\Gamma(\alpha)} \left[\frac{t^\rho}{T^\rho}(T^\rho - s^\rho)\right]^{\alpha-1} > 0.$$

Moreover, for $0 < s \leq t < T$, we have $\frac{t^\rho}{T^\rho} < 1$, then $\frac{t^\rho}{T^\rho}s^\rho < s^\rho$ and $t^\rho - \frac{t^\rho}{T^\rho}s^\rho > t^\rho - s^\rho$, thus

$$t^\rho - \frac{t^\rho}{T^\rho}s^\rho = \frac{t^\rho}{T^\rho}(T^\rho - s^\rho) > t^\rho - s^\rho \Rightarrow \left[\frac{t^\rho}{T^\rho}(T^\rho - s^\rho)\right]^{\alpha-1} - (t^\rho - s^\rho)^{\alpha-1} > 0,$$

which imply that $G(t,s) > 0$ for any $t, s \in (0,T)$.

(2) To prove that

$$\max_{0\leq t\leq T} G(t,s) = G(s,s) = \frac{\rho^{1-\alpha} s^{\rho-1}}{\Gamma(\alpha)} \left[\frac{s^\rho}{T^\rho}(T^\rho - s^\rho)\right]^{\alpha-1},\ \forall s \in [0,T], \tag{3.14}$$

we choose

$$g_1(t,s) = \frac{\rho^{1-\alpha} s^{\rho-1}}{\Gamma(\alpha)} \left[\left[\frac{t^\rho}{T^\rho}(T^\rho - s^\rho)\right]^{\alpha-1} - (t^\rho - s^\rho)^{\alpha-1}\right],$$

$$g_2(t,s) = \frac{\rho^{1-\alpha} s^{\rho-1}}{\Gamma(\alpha)} \left[\frac{t^\rho}{T^\rho}(T^\rho - s^\rho)\right]^{\alpha-1}.$$

Indeed, we put $\max\limits_{0\leq t\leq T} G(t,s) = G(t^*,s)$, where $0 \leq t^* \leq T$. Then, we get for some $0 < t_1 < t_2 < T$, that

$$\begin{aligned}\max_{0\leq t\leq T} G(t,s) &= \begin{cases} g_1(t^*,s), & s \in [0,t_1], \\ \max\{g_1(t^*,s), g_2(t^*,s)\}, & s \in [t_1,t_2], \\ g_2(t^*,s), & s \in [t_2,T], \end{cases} \\ &= \begin{cases} g_1(t^*,s), & s \in [0,r], \\ g_2(t^*,s), & s \in [r,T], \end{cases}\end{aligned}$$

where $r \in [t_1,t_2]$, is the unique solution of equation

$$g_1(t^*,s) = g_2(t^*,s) \Leftrightarrow t^* = s,$$

which shows the equality $(3.14)$.
(3) In the following, we divide the proof into two-part, to show the existence $b \in C[0,T]$, such that

$$G(t,s) \geq b(t)\, G(s,s), \text{ for any } \frac{T}{8} \leq s \leq T.$$

(i) Firstly, if $0 \leq t \leq s \leq T$, we see that $\frac{G(t,s)}{G(s,s)}$ is decreasing with respect to $s$. Consequently

$$\frac{G(t,s)}{G(s,s)} = \frac{\left[\frac{t^\rho}{T^\rho}(T^\rho - s^\rho)\right]^{\alpha-1}}{\left[\frac{s^\rho}{T^\rho}(T^\rho - s^\rho)\right]^{\alpha-1}} = \left(\frac{t}{s}\right)^{\rho(\alpha-1)} \geq \left(\frac{t}{T}\right)^{\rho(\alpha-1)} = b_1(t), \ \forall t \in [0,s].$$

(ii) In the same way, if $0 \leq s \leq t \leq T$, we have $\frac{s^\rho}{T^\rho} < \frac{t^\rho}{T^\rho} \leq 1$, $\left(\frac{t^\rho}{T^\rho}\right)^{\alpha-2} \geq 1$, $\forall \alpha \in (1,2]$, and

$$\begin{aligned}
G(t,s) &= \frac{\rho^{1-\alpha}s^{\rho-1}}{\Gamma(\alpha)}\left[\left[\frac{t^\rho}{T^\rho}(T^\rho - s^\rho)\right]^{\alpha-1} - (t^\rho - s^\rho)^{\alpha-1}\right] \\
&= \frac{(\alpha-1)\rho^{1-\alpha}s^{\rho-1}}{\Gamma(\alpha)}\int_{t^\rho - s^\rho}^{\frac{t^\rho}{T^\rho}(T^\rho - s^\rho)} \tau^{\alpha-2}d\tau \\
&\geq \frac{(\alpha-1)\rho^{1-\alpha}s^{\rho-1}}{\Gamma(\alpha)}\left(\frac{t^\rho}{T^\rho}\right)^{\alpha-2}(T^\rho - s^\rho)^{\alpha-2}\left(\frac{t^\rho}{T^\rho}(T^\rho - s^\rho) - (t^\rho - s^\rho)\right) \\
&\geq \frac{(\alpha-1)\rho^{1-\alpha}s^{\rho-1}}{\Gamma(\alpha)}(T^\rho - s^\rho)^{\alpha-1}\frac{s^\rho(T^\rho - t^\rho)}{T^\rho(T^\rho - s^\rho)}.
\end{aligned}$$

As $0 < \rho \leq 1$, we get

$$T^\rho - t^\rho = \rho\int_t^T \tau^{\rho-1}d\tau \geq \rho T^{\rho-1}(T-t), \text{ and } T^\rho - s^\rho = \rho\int_s^T \tau^{\rho-1}d\tau \leq \rho s^{\rho-1}(T-s).$$

Therefore

$$\begin{aligned}
\frac{G(t,s)}{G(s,s)} &\geq \frac{\frac{(\alpha-1)\rho^{1-\alpha}s^{\rho-1}}{\Gamma(\alpha)}(T^\rho - s^\rho)^{\alpha-1}\frac{s^\rho(T^\rho - t^\rho)}{T^\rho(T^\rho - s^\rho)}}{\frac{\rho^{1-\alpha}s^{\rho-1}}{\Gamma(\alpha)}\left[\frac{s^\rho}{T^\rho}(T^\rho - s^\rho)\right]^{\alpha-1}} = (\alpha-1)\frac{s^\rho(T^\rho - t^\rho)}{T^\rho(T^\rho - s^\rho)}\left(\frac{T^\rho}{s^\rho}\right)^{\alpha-1} \\
&\geq (\alpha-1)\frac{s(T-t)}{T(T-s)} \\
&\geq (\alpha-1)\frac{s(T-t)}{T^2}.
\end{aligned}$$

Finally, for $s \in \left[\frac{T}{8}, t\right]$, we have:

$$\frac{G(t,s)}{G(s,s)} \geq \frac{(\alpha-1)(T-t)}{8T} = b_2(t).$$

It is clear that $b_1(t)$ and $b_2(t)$ are positive functions, it is enough to choose:

$$b(t) = \begin{cases} \left(\frac{t}{T}\right)^{\rho(\alpha-1)}, & \text{for } t \in [0, \bar{t}], \\ \frac{(\alpha-1)(T-t)}{8T}, & \text{for } t \in [\bar{t}, T], \end{cases} \tag{3.15}$$

where $\bar{t} \in (0, T)$ is the unique solution of the equation $b_1(t) = b_2(t)$. We see that

$$b(t) \leq \bar{b} = b(\bar{t}) = \left(\frac{\bar{t}}{T}\right)^{\rho(\alpha-1)} = \frac{(\alpha-1)(T-\bar{t})}{8T} < 1 \text{ for all } t \in [0, T].$$

Finally, we have $\forall s \in \left[\frac{T}{8}, T\right]$,

$$G(t,s) \geq b(t)\, G(s,s), \ \forall t \in [0, T].$$

The proof is complete. □

**Lemma 3.4.** *Let $1 < \alpha \leq 2$ and $0 < \rho \leq 1$, then there exists a positive constant*

$$\lambda = 1 + \frac{8^{\rho\alpha} L(\alpha+1)\left[8^{\rho\alpha} - (8^\rho - 1)^\alpha\right]}{h(8^\rho - 1)^\alpha \left[8^\rho(\alpha+1) + 8^{\rho(\alpha-1)}(\alpha-1)(8^\rho - 1)\right]}, \text{ for some } h, L > 0,$$

*such that*

$$\int_0^T G(s,s)\, f(s, u(s))\, ds \leq \lambda \int_{\frac{T}{8}}^T G(s,s)\, f(s, u(s))\, ds. \tag{3.16}$$

**Proof.** As $f(t, u(t)) \geq h$, for any $t \in [0, T]$, we get

$$\begin{aligned} \int_{\frac{T}{8}}^T G(s,s) f(s, u(s))\, ds &\geq h \int_{\frac{T}{8}}^T \frac{\rho^{1-\alpha} s^{\rho-1}}{\Gamma(\alpha)} \left[\frac{s^\rho}{T^\rho}(T^\rho - s^\rho)\right]^{\alpha-1} ds \\ &\geq -\frac{h}{\alpha\rho^\alpha T^{\rho(\alpha-1)}\Gamma(\alpha)} \int_{\frac{T}{8}}^T s^{\rho(\alpha-1)} \left[-\rho\alpha s^{\rho-1}(T^\rho - s^\rho)^{\alpha-1}\right] ds. \end{aligned}$$

The integral by part gives:

$$\begin{aligned} \int_{\frac{T}{8}}^T G(s,s) f(s, u(s))\, ds &\geq \frac{h\left[\frac{T^{\rho(\alpha-1)}}{8^{\rho(\alpha-1)}}\left(T^\rho - \frac{T^\rho}{8^\rho}\right)^\alpha + \rho(\alpha-1)\int_{\frac{T}{8}}^T s^{\rho(\alpha-1)-1}(T^\rho - s^\rho)^\alpha ds\right]}{\rho^\alpha T^{\rho(\alpha-1)}\Gamma(\alpha+1)} \\ &\geq \frac{h\left[\frac{T^\rho}{8^{\rho(\alpha-1)}}\left(T^\rho - \frac{T^\rho}{8^\rho}\right)^\alpha + \rho(\alpha-1)\int_{\frac{T}{8}}^T \frac{s^{\rho(\alpha-2)}}{T^{\rho(\alpha-2)}} s^{\rho-1}(T^\rho - s^\rho)^\alpha ds\right]}{\rho^\alpha T^\rho \Gamma(\alpha+1)} \\ &\geq \frac{h\left[\frac{T^\rho}{8^{\rho(\alpha-1)}}\left(T^\rho - \frac{T^\rho}{8^\rho}\right)^\alpha - \frac{\alpha-1}{\alpha+1}\int_{\frac{T}{8}}^T \left[-\rho(\alpha+1)s^{\rho-1}(T^\rho - s^\rho)^\alpha\right] ds\right]}{\rho^\alpha T^\rho \Gamma(\alpha+1)} \\ &\geq \frac{hT^{\rho\alpha}(8^\rho - 1)^\alpha}{\rho^\alpha 8^{\rho\alpha}\Gamma(\alpha+1)}\left[\frac{8^\rho(\alpha+1) + 8^{\rho(\alpha-1)}(\alpha-1)(8^\rho - 1)}{8^{\rho\alpha}(\alpha+1)}\right]. \end{aligned}$$

Then

$$\frac{\rho^{\alpha}8^{\rho\alpha}\Gamma(\alpha+1)}{hT^{\rho\alpha}(8^{\rho}-1)^{\alpha}}\left[\frac{8^{\rho\alpha}(\alpha+1)}{8^{\rho}(\alpha+1)+8^{\rho(\alpha-1)}(\alpha-1)(8^{\rho}-1)}\right]\int_{\frac{T}{8}}^{T}G(s,s)\,f(s,u(s))\,ds\geq 1. \tag{3.17}$$

On the other hand, if $\max_{0\leq t\leq T} f(t,u)$ is bounded for $u\in[0,\infty)$, then there exists $L_0>0$, such that

$$|f(t,u(t))|\leq L_0,\ \forall t\in[0,T].$$

In the similar way, if $\max_{0\leq t\leq T} f(t,u)$ is unbounded for $u\in[0,\infty)$, then there exists $M_0>0$, such that

$$\sup_{0\leq u\leq M_0}\max_{0\leq t\leq T}|f(t,u(t))|\leq L_1,\ \text{for some } L_1>0.$$

In all cases, for $L=\max\{L_0,L_1\}$, we have:

$$\int_0^{\frac{T}{8}}G(s,s)\,f(s,u(s))\,ds\leq L\int_0^{\frac{T}{8}}G(s,s)\,ds\leq\frac{LT^{\rho\alpha}\left[8^{\rho\alpha}-(8^{\rho}-1)^{\alpha}\right]}{8^{\rho\alpha}\rho^{\alpha}\Gamma(\alpha+1)}.$$

From $(3.17)$, we get

$$\begin{aligned}
\int_0^{T}G(s,s)\,f(s,u(s))\,ds &= \int_{\frac{T}{8}}^{T}G(s,s)\,f(s,u(s))\,ds+\int_0^{\frac{T}{8}}G(s,s)\,f(s,u(s))\,ds\\
&\leq \int_{\frac{T}{8}}^{T}G(s,s)\,f(s,u(s))\,ds+\frac{LT^{\rho\alpha}\left[8^{\rho\alpha}-(8^{\rho}-1)^{\alpha}\right]}{\rho^{\alpha}8^{\rho\alpha}\Gamma(\alpha+1)}\\
&\leq \int_{\frac{T}{8}}^{T}G(s,s)\,f(s,u(s))\,ds\\
&\quad+\frac{LT^{\rho\alpha}\left[8^{\rho\alpha}-(8^{\rho}-1)^{\alpha}\right]}{\rho^{\alpha}8^{\rho\alpha}\Gamma(\alpha+1)}\times\frac{\rho^{\alpha}8^{\rho\alpha}\Gamma(\alpha+1)}{hT^{\rho\alpha}(8^{\rho}-1)^{\alpha}}\\
&\quad\times\left[\frac{8^{\rho\alpha}(\alpha+1)}{8^{\rho}(\alpha+1)+8^{\rho(\alpha-1)}(\alpha-1)(8^{\rho}-1)}\right]\\
&\quad\times\int_{\frac{T}{8}}^{T}G(s,s)\,f(s,u(s))\,ds\\
&\leq \lambda\int_{\frac{T}{8}}^{T}G(s,s)\,f(s,u(s))\,ds.
\end{aligned}$$

□

Let us define the cone $P$ by:

$$P=\left\{u\in C[0,T]\ \mid u(t)\geq\frac{b(t)}{\lambda}\|u\|,\ \forall t\in[0,T]\right\}. \tag{3.18}$$

**Lemma 3.5.** *Let $\mathcal{A} : P \to C[0,T]$ be an integral operator defined by:*

$$\mathcal{A}u(t) = \beta \int_0^T G(t,s) f(s,u(s))\, ds, \tag{3.19}$$

*equipped with standard norm*

$$\|\mathcal{A}u\| = \max_{0 \le t \le T} |\mathcal{A}u(t)|.$$

*Then $\mathcal{A}(P) \subset P$.*

**Proof.** For any $u \in P$, we have from $(3.13)$, $(3.16)$ and $(3.18)$, that

$$\begin{aligned} \mathcal{A}u(t) &= \beta \int_0^T G(t,s) f(s,u(s))\, ds \ge \beta b(t) \int_{\frac{T}{8}}^T G(s,s) f(s,u(s))\, ds \\ &\ge \frac{\beta b(t)}{\lambda} \int_0^T G(s,s) f(s,u(s))\, ds \\ &\ge \frac{b(t)}{\lambda} \max_{0 \le t \le T} \left( \beta \int_0^T G(t,s) f(s,u(s))\, ds \right) \\ &\ge \frac{b(t)}{\lambda} \|\mathcal{A}u\|,\ \forall t \in [0,T]. \end{aligned}$$

Thus $\mathcal{A}(P) \subset P$. The proof is complete. □

**Lemma 3.6.** *$\mathcal{A} : P \to P$ is a completely continuous operator.*

**Proof.** In view of continuity of $G(t,s)$ and $f(t,u)$, the operator $\mathcal{A} : P \to P$ is a continuous.
Let $\Omega \subset P$ be a bounded. Then there exists a positive constant $M > 0$, such that:

$$\|u\| \le M,\ \forall u \in \Omega.$$

By choice

$$L = \sup_{0 \le u \le M} \max_{0 \le t \le T} |f(t,u)| + 1.$$

In this case, we get $\forall u \in \Omega$,

$$\begin{aligned} |\mathcal{A}u(t)| &= \left| \beta \int_0^T G(t,s) f(s,u(s))\, ds \right| \le \beta \int_0^T |G(t,s) f(s,u(s))|\, ds \\ &\le \beta L \int_0^T G(s,s)\, ds \le \frac{\beta L}{\rho^{\alpha-1}\Gamma(\alpha)} \int_0^T s^{\rho-1} (T^\rho - s^\rho)^{\alpha-1}\, ds \\ &\le \frac{\beta L T^{\alpha\rho}}{\rho^\alpha \Gamma(\alpha+1)}. \end{aligned}$$

Consequently, $|\mathcal{A}u(t)| \leq \frac{\beta L T^{\alpha\rho}}{\rho^{\alpha}\Gamma(\alpha+1)}$, $\forall u \in \Omega$. Hence, $\mathcal{A}(\Omega)$ is bounded.
Now, for $1 < \alpha \leq 2$ and $0 < \rho \leq 1$, we give:

$$\delta(\varepsilon) = \left(\frac{\rho^{\alpha}\Gamma(\alpha)}{T^{\rho}\beta L}\varepsilon\right)^{\frac{1}{\rho(\alpha-1)}}, \text{ for some } \varepsilon > 0.$$

Then $\forall u \in \Omega$, and $t_1, t_2 \in [0,T]$, where $t_1 < t_2$, and $t_2 - t_1 < \delta$, we find $|\mathcal{A}u(t_2) - \mathcal{A}u(t_1)| < \varepsilon$.
Consequently, for $0 \leq s \leq t_1 < t_2 \leq T$, we have:

$$\begin{aligned}
G(t_2,s) - G(t_1,s) &= \frac{\rho^{1-\alpha}s^{\rho-1}}{\Gamma(\alpha)}\left[\left[t_2^{\rho(\alpha-1)} - t_1^{\rho(\alpha-1)}\right]\left(\frac{T^{\rho}-s^{\rho}}{T^{\rho}}\right)^{\alpha-1}\right. \\
&\quad \left. - \left[(t_2^{\rho}-s^{\rho})^{\alpha-1} - (t_1^{\rho}-s^{\rho})^{\alpha-1}\right]\right] \\
&< \frac{\rho^{1-\alpha}s^{\rho-1}}{\Gamma(\alpha)}\left[t_2^{\rho(\alpha-1)} - t_1^{\rho(\alpha-1)}\right]\left(\frac{T^{\rho}-s^{\rho}}{T^{\rho}}\right)^{\alpha-1} \\
&< \frac{\rho^{1-\alpha}s^{\rho-1}}{\Gamma(\alpha)}\left[t_2^{\rho(\alpha-1)} - t_1^{\rho(\alpha-1)}\right].
\end{aligned}$$

In the same way, for $0 \leq t_1 \leq s < t_2 \leq T$ or $0 \leq t_1 < t_2 \leq s \leq T$, we have:

$$G(t_2,s) - G(t_1,s) < \frac{\rho^{1-\alpha}s^{\rho-1}}{\Gamma(\alpha)}\left[t_2^{\rho(\alpha-1)} - t_1^{\rho(\alpha-1)}\right].$$

Then

$$\begin{aligned}
|\mathcal{A}u(t_2) - \mathcal{A}u(t_1)| &= \left|\beta\int_0^T [G(t_2,s) - G(t_1,s)]\, f(s,u(s))\, ds\right| \\
&\leq \beta L\int_0^T |G(t_2,s) - G(t_1,s)|\, ds \\
&< \beta L\int_0^T \frac{\rho^{1-\alpha}s^{\rho-1}}{\Gamma(\alpha)}\left[t_2^{\rho(\alpha-1)} - t_1^{\rho(\alpha-1)}\right] ds \\
&< \frac{\beta L\rho^{1-\alpha}}{\Gamma(\alpha)}\left[t_2^{\rho(\alpha-1)} - t_1^{\rho(\alpha-1)}\right]\left[\frac{1}{\rho}s^{\rho}\right]_0^T.
\end{aligned}$$

Finally

$$|\mathcal{A}u(t_2) - \mathcal{A}u(t_1)| < \frac{\beta L T^{\rho}}{\rho^{\alpha}\Gamma(\alpha)}\left[t_2^{\rho(\alpha-1)} - t_1^{\rho(\alpha-1)}\right]. \tag{3.20}$$

In the following, we divide the proof into three cases.
(a) If $\delta \leq t_1 < t_2 \leq T$, we have:

$$\delta \leq t_1 < t_2 \Leftrightarrow t_2^{\rho(\alpha-2)} < t_1^{\rho(\alpha-2)} \leq \delta^{\rho(\alpha-2)}, \text{ and } t_2^{\rho-1} < t_1^{\rho-1} \leq \delta^{\rho-1}.$$

Thus

$$t_2^{\rho} - t_1^{\rho} = t_2 t_2^{\rho-1} - t_1 t_1^{\rho-1} < t_2 t_2^{\rho-1} - t_1 t_2^{\rho-1} = t_2^{\rho-1}(t_2 - t_1) < \delta^{\rho-1}(t_2 - t_1) < \delta^{\rho}.$$

In similar way

$$\begin{aligned}
t_2^{\rho(\alpha-1)} - t_1^{\rho(\alpha-1)} &= t_2^{\rho} t_2^{\rho(\alpha-2)} - t_1^{\rho} t_1^{\rho(\alpha-2)} < t_2^{\rho} t_2^{\rho(\alpha-2)} - t_1^{\rho} t_2^{\rho(\alpha-2)} = t_2^{\rho(\alpha-2)} \left(t_2^{\rho} - t_1^{\rho}\right) \\
&< \delta^{\rho(\alpha-2)} \left(t_2^{\rho} - t_1^{\rho}\right) \\
&< \delta^{\rho(\alpha-1)}.
\end{aligned}$$

Then, the inequality (3.20) gives:

$$\begin{aligned}
\left|\mathcal{A}u(t_2) - \mathcal{A}u(t_1)\right| &< \frac{\beta L T^{\rho}}{\rho^{\alpha}\Gamma(\alpha)} \left[t_2^{\rho(\alpha-1)} - t_1^{\rho(\alpha-1)}\right] < \frac{\beta L T^{\rho}}{\rho^{\alpha}\Gamma(\alpha)} \delta^{\rho(\alpha-1)} \\
&< \frac{\beta L T^{\rho}}{\rho^{\alpha}\Gamma(\alpha)} \left[\left(\frac{\rho^{\alpha}\Gamma(\alpha)}{T^{\rho}\beta L}\varepsilon\right)^{\frac{1}{\rho(\alpha-1)}}\right]^{\rho(\alpha-1)} \\
&< \varepsilon. \qquad (3.21)
\end{aligned}$$

(b) If $t_1 \leq \delta < t_2 < 2\delta$, we have:

$$t_1 \leq \delta < t_2 \Leftrightarrow t_2^{\rho(\alpha-2)} < \delta^{\rho(\alpha-2)} \leq t_1^{\rho(\alpha-2)},$$

and

$$\begin{aligned}
t_2^{\rho(\alpha-1)} - t_1^{\rho(\alpha-1)} &= t_2^{\rho} t_2^{\rho(\alpha-2)} - t_1^{\rho} t_1^{\rho(\alpha-2)} < t_2^{\rho} \delta^{\rho(\alpha-2)} - t_1^{\rho} \delta^{\rho(\alpha-2)} \\
&< \delta^{\rho(\alpha-2)} \left(t_2^{\rho} - t_1^{\rho}\right) < \delta^{\rho(\alpha-1)}.
\end{aligned}$$

Also, we find the same result $(3.21)$.
(c) If $t_1 < t_2 \leq \delta$, we have:

$$\begin{aligned}
\left|\mathcal{A}u(t_2) - \mathcal{A}u(t_1)\right| &< \frac{\beta L T^{\rho}}{\rho^{\alpha}\Gamma(\alpha)} \left[t_2^{\rho(\alpha-1)} - t_1^{\rho(\alpha-1)}\right] < \frac{\beta L T^{\rho}}{\rho^{\alpha}\Gamma(\alpha)} t_2^{\rho(\alpha-1)} \\
&< \frac{\beta L T^{\rho}}{\rho^{\alpha}\Gamma(\alpha)} \delta^{\rho(\alpha-1)} \\
&< \varepsilon.
\end{aligned}$$

By the means of the Ascoli-Arzel Theorem 2.11, we have $\mathcal{A} : P \to P$ is completely continuous. □

We define some important constants

$$\begin{aligned}
&F_0 = \lim_{u\to 0^+} \max_{t\in[0,T]} \tfrac{f(t,u)}{u}, && F_{\infty} = \lim_{u\to +\infty} \max_{t\in[0,T]} \tfrac{f(t,u)}{u}, \\
&f_0 = \lim_{u\to 0^+} \min_{t\in[0,T]} \tfrac{f(t,u)}{u}, && f_{\infty} = \lim_{u\to +\infty} \min_{t\in[0,T]} \tfrac{f(t,u)}{u}, \\
&\omega_1 = \textstyle\int_0^T G(s,s)\,ds, && \omega_2 = \tfrac{\bar{b}}{\lambda^2} \textstyle\int_0^T G(s,s)\,b(s)\,ds.
\end{aligned}$$

Assume that $\frac{1}{\omega_2 f_{\infty}} = 0$ if $f_{\infty} \to \infty$, $\frac{1}{\omega_1 F_0} = \infty$ if $F_0 \to 0$, $\frac{1}{\omega_2 f_0} = 0$ if $f_0 \to \infty$, and $\frac{1}{\omega_1 F_{\infty}} = \infty$ if $F_{\infty} \to 0$.

**Theorem 3.7.** *If $\omega_2 f_\infty > \omega_1 F_0$ holds, then for each:*

$$\beta \in \left( (\omega_2 f_\infty)^{-1}, (\omega_1 F_0)^{-1} \right), \tag{3.22}$$

*the boundary value problem* (1.1)-(1.2) *has at least one positive solution.*

**Proof.** Let $\beta$ satisfies (3.22) and $\varepsilon > 0$, be such that

$$((f_\infty - \varepsilon)\,\omega_2)^{-1} \leq \beta \leq ((F_0 + \varepsilon)\,\omega_1)^{-1}. \tag{3.23}$$

From the definition of $F_0$, we see that there exists $r_1 > 0$, such that

$$f(t,u) \leq (F_0 + \varepsilon)\,u,\ \forall t \in [0,T],\ 0 < u \leq r_1. \tag{3.24}$$

Consequently, for $u \in P$ with $\|u\| = r_1$, we have from $(3.23), (3.24)$, that

$$\begin{aligned} \|\mathcal{A}u\| &= \max_{0<t<T} \left| \beta \int_0^T G(t,s)\, f(s,u(s))\, ds \right| \\ &\leq \beta \int_0^T G(s,s)\,(F_0+\varepsilon)\, u(s)\, ds \\ &\leq \beta\,(F_0+\varepsilon)\,\|u\| \int_0^T G(s,s)\, ds \\ &\leq \beta\,(F_0+\varepsilon)\,\|u\|\,\omega_1 \\ &\leq \|u\|. \end{aligned}$$

Hence, if we choose $\Omega_1 = \{u \in C[0,T] : \ \|u\| < r_1\}$, then

$$\|\mathcal{A}u\| \leq \|u\|,\ \text{for } u \in P \cap \partial\Omega_1. \tag{3.25}$$

By definition of $f_\infty$, there exists $r_3 > 0$, such that

$$f(t,u) \geq (f_\infty - \varepsilon)\,u,\ \forall t \in [0,T],\ u \geq r_3. \tag{3.26}$$

Therefore, for $u \in P$ with $\|u\| = r_2 = \max\{2r_1, r_3\}$, we have from $(3.23), (3.26)$, that

$$\begin{aligned} \|\mathcal{A}u\| \ &\geq\ \mathcal{A}u(\bar{t}) = \beta \int_0^T G(\bar{t},s)\, f(s,u(s))\, ds \geq \beta \int_{\frac{T}{8}}^T b(\bar{t})\, G(s,s)\, f(s,u(s))\, ds \\ &\geq \frac{\beta\bar{b}}{\lambda} \int_0^T G(s,s)\, f(s,u(s))\, ds \geq \frac{\beta\bar{b}}{\lambda} \int_0^T G(s,s)\,[(f_\infty-\varepsilon)\,u(s)]\, ds, \quad \forall t \in [0,T]. \end{aligned}$$

By definition of $P$ in $(3.18)$, we have:

$$\begin{aligned} \|\mathcal{A}u\| &\geq \frac{\beta\bar{b}\,(f_\infty-\varepsilon)}{\lambda^2}\,\|u\| \int_0^T G(s,s)\, b(s)\, ds \\ &\geq \beta\,(f_\infty-\varepsilon)\,\|u\|\,\omega_2 \\ &\geq \|u\|. \end{aligned}$$

If we set $\Omega_2 = \{u \in C[0,T] : \|u\| < r_2\}$, then

$$\|\mathcal{A}u\| \geq \|u\|, \text{ for } u \in P \cap \partial\Omega_2. \tag{3.27}$$

Now, from $(3.25)$, $(3.27)$, and Lemma 2.13, we guarantee that $\mathcal{A}$ has a fix point $u \in P \cap \left(\bar{\Omega}_2 \backslash \Omega_1\right)$ with $r_1 \leq \|u\| \leq r_2$. It is clear that $u$ is a positive solution of (1.1)-(1.2). The proof is complete. □

**Theorem 3.8.** *If $\omega_2 f_0 > \omega_1 F_\infty$ holds, then for each:*

$$\beta \in \left((\omega_2 f_0)^{-1}, (\omega_1 F_\infty)^{-1}\right), \tag{3.28}$$

*the boundary value problem* (1.1)-(1.2) *has at least one positive solution.*

**Proof.** Let $\beta$ satisfies (3.28) and $\varepsilon > 0$, be such that

$$\left((f_0 - \varepsilon)\omega_2\right)^{-1} \leq \beta \leq \left((F_\infty + \varepsilon)\omega_1\right)^{-1}. \tag{3.29}$$

From definition of $f_0$, we see that there exists $r_1 > 0$, such that

$$f(t,u) \geq (f_0 - \varepsilon)u, \ \forall t \in [0,T], \ 0 < u \leq r_1.$$

Further, if $u \in P$ with $\|u\| = r_1$, then similar to the proof's second part of Theorem 3.7, we can get that $\|\mathcal{A}u\| \geq \|u\|$. Then, if we choose $\Omega_1 = \{u \in C[0,T] : \ \|u\| < r_1\}$, thus

$$\|\mathcal{A}u\| \geq \|u\|, \text{ for } u \in P \cap \partial\Omega_1. \tag{3.30}$$

Next, and by definition of $F_\infty$, we may choose $R_1 > 0$, such that

$$f(t,u) \leq (F_\infty + \varepsilon)u, \text{ for } u \geq R_1. \tag{3.31}$$

We consider two cases:
1) If $\max\limits_{0\leq t\leq T} f(t,u)$ is bounded for $u \in [0,\infty)$. Then, there exists some $L > 0$, such that

$$f(t,u) \leq L, \text{ for all } t \in [0,T], \ u \in P.$$

Let us denote by $r_3 = \max\{2r_1, \beta L\omega_1\}$, if $u \in P$ with $\|u\| = r_3$, then

$$\|\mathcal{A}u\| = \max_{0\leq t\leq T}\left|\beta\int_0^T G(t,s) f(s,u(s))\,ds\right| \leq \beta L\int_0^T G(s,s)\,ds = \beta L\omega_1 \leq r_3 = \|u\|.$$

Hence,

$$\|\mathcal{A}u\| \leq \|u\|, \text{ for } u \in \partial P_{r_3} = \{u \in P : \|u\| \leq r_3\}. \tag{3.32}$$

2) If $\max\limits_{0\leq t\leq T} f(t,u)$ is unbounded for $u \in [0,\infty)$, then there exists some $r_4 = \max\{2r_1, R_1\}$, such that

$$f(t,u) \leq \max_{0\leq t\leq T} f(t,r_4), \text{ for all } 0 < u \leq r_4, \ t \in [0,T].$$

Let $u \in P$ with $\|u\| = r_4$. Then, from $(3.29)$, $(3.31)$, we have:

$$\begin{aligned} \|\mathcal{A}u\| &= \max_{0<t<T} \left| \beta \int_0^T G(t,s) f(s,u(s))\, ds \right| \le \beta \int_0^T G(s,s)(F_\infty + \varepsilon) u(s)\, ds \\ &\le \beta (F_\infty + \varepsilon) \|u\| \int_0^T G(s,s)\, ds = \beta (F_\infty + \varepsilon) \|u\| \omega_1 \\ &\le \|u\|. \end{aligned}$$

Thus, $(3.32)$ is also true for $u \in \partial P_{r_4}$.
In both cases 1 and 2, if we set $\Omega_2 = \{u \in C[0,T] : \ \|u\| < r_2 = \max\{r_3, r_4\}\}$, then

$$\|\mathcal{A}u\| \le \|u\|, \text{ for } u \in P \cap \partial \Omega_2. \tag{3.33}$$

Now, from $(3.30)$, $(3.33)$, and Lemma 2.13, we guarantee that $\mathcal{A}$ has a fix point $u \in P \cap \left(\bar{\Omega}_2 \backslash \Omega_1\right)$ with $r_1 \le \|u\| \le r_2$. It is clear that $u$ is a positive solution of $(1.1)$-$(1.2)$. The proof is complete. □

**Theorem 3.9.** *Suppose there exists* $r_2 > r_1 > 0$, *such that*

$$\sup_{0 \le u \le r_2} \max_{0 \le t \le T} f(t,u) \le \frac{r_2}{\beta \omega_1}, \text{ and } \inf_{0 \le u \le r_1} f(t,u) \ge \frac{r_1}{\beta \lambda \omega_2} b(t), \ \forall t \in [0,T]. \tag{3.34}$$

*Then, the boundary value problem* $(1.1)$-$(1.2)$ *has a positive solution* $u \in P$, *with* $r_1 \le \|u\| \le r_2$.

**Proof.** Choose $\Omega_1 = \{u \in C[0,T] : \ \|u\| < r_1\}$. Then, for $u \in P \cap \partial \Omega_1$, we get

$$\begin{aligned} \|\mathcal{A}u\| &\ge \mathcal{A}u(\bar{t}) = \beta \int_0^T G(\bar{t},s) f(s,u(s))\, ds \ge \beta \int_{\frac{T}{8}}^T b(\bar{t}) G(s,s) f(s,u(s))\, ds \\ &\ge \frac{\beta \bar{b}}{\lambda} \int_0^T G(s,s) \inf_{0 \le u \le r_1} f(s,u(s))\, ds \ge \frac{\beta \bar{b}}{\lambda} \int_0^T G(s,s) \frac{r_1}{\beta \lambda \omega_2} b(s)\, ds \\ &\ge r_1 = \|u\|. \end{aligned}$$

On the other hand, choose $\Omega_2 = \{u \in C[0,T] : \ \|u\| < r_2\}$. Then, for $u \in P \cap \partial \Omega_2$, we get

$$\begin{aligned} \|\mathcal{A}u\| &= \max_{0<t<T} \left| \beta \int_0^T G(t,s) f(s,u(s))\, ds \right| \le \beta \int_0^T G(s,s) \sup_{0 \le u \le r_2} \max_{0 \le t \le T} f(s,u(s))\, ds \\ &\le \beta \int_0^T G(s,s) \frac{r_2}{\beta \omega_1} ds = r_2 = \|u\|. \end{aligned}$$

Now, from Lemma 2.13, we guarantee that $\mathcal{A}$ has a fix point $u \in P \cap \left(\bar{\Omega}_2 \backslash \Omega_1\right)$ with $r_1 \le \|u\| \le r_2$. It is clear that $u$ is a positive solution of $(1.1)$-$(1.2)$. The proof is complete. □

## 3.2. Application of Banach fixed point theorem

In this part, we assume that $\beta \in \mathbb{R}$ and $\rho > 0$, and $f : [0, T] \times [0, \infty) \to [0, \infty)$ satisfies the conditions:

(H1) $f(t, u)$ is Lebesgue measurable function with respect to $t$ on $[0, T]$,

(H2) $f(t, u)$ is continuous function with respect to $u$ on $\mathbb{R}$.

**Theorem 3.10.** *Assume* (H1), (H2) *hold, and there exists a constant* $\sigma > 0$, *such that*

$$|f(t, u) - f(t, v)| \leq \sigma |u - v|, \text{ for almost every } t \in [0, T], \text{ and all } u, v \in C[0, T]. \tag{3.35}$$

*If*

$$|\beta| < \frac{\rho^{\alpha}\Gamma(\alpha + 1)}{\sigma T^{\alpha\rho}}. \tag{3.36}$$

*Then, there exists a unique solution of the boundary value problem* (1.1)-(1.2) *on* $[0, T]$.

**Proof.** Assume that $|\beta| < \frac{\rho^{\alpha}\Gamma(\alpha+1)}{\sigma T^{\alpha\rho}}$, and consider the operator $\mathcal{A} : C[0, T] \to C[0, T]$ defined by (3.19) as follows

$$\mathcal{A}u(t) = \beta \int_0^T G(t, s) f(s, u(s)) \, ds.$$

We shall show that $\mathcal{A}$ is a contraction mapping. In fact, for any $u, v \in C[0, T]$, we have

$$\begin{aligned}
|\mathcal{A}u(t) - \mathcal{A}v(t)| &= \left| \beta \int_0^T G(t, s) [f(s, u(s)) - f(s, v(s))] \, ds \right| \\
&\leq |\beta| \int_0^T G(t, s) |f(s, u(s)) - f(s, v(s))| \, ds \\
&\leq |\beta| \sigma \int_0^T G(s, s) |u(s) - v(s)| \, ds,
\end{aligned}$$

then

$$\begin{aligned}
\|\mathcal{A}u - \mathcal{A}v\| &\leq |\beta| \sigma \|u - v\| \int_0^T G(s, s) \, ds \\
&\leq \frac{|\beta| \sigma T^{\alpha\rho}}{\rho^{\alpha}\Gamma(\alpha + 1)} \|u - v\|. 
\end{aligned} \tag{3.37}$$

This imply from (3.37) that $\mathcal{A}$ is a contraction operator. As a consequence of Theorem 2.14, by Banach's contraction principle [5], we deduce that $\mathcal{A}$ has a unique fixed point which is the unique solution of the problem (1.1)-(1.2) on $[0, T]$. □

# 4. Examples

In this section, we present some examples to illustrate the usefulness of our main results.

**Example 1.** Consider the following boundary value problem

$$\begin{cases} {}^1\mathcal{D}_{0^+}^{\frac{3}{2}} u(t) + \beta(1+t)\, u(t) \ln(1+u(t)) = 0, \quad t \in [0,1]. \\ u(0) = u(1) = 0. \end{cases} \tag{4.1}$$

Set $\beta > 0$ any finite positive real number, and

$$f(t,u) = (1+t)\, u \ln(1+u).$$

In this case, the function $f$ is jointly continuous for any $t \in [0,1]$, and any $u > 0$.

We get

$$F_0 = \lim_{u \to 0^+} \max_{t \in [0,T]} \tfrac{f(t,u)}{u} = 0^+, \quad f_\infty = \lim_{u \to +\infty} \min_{t \in [0,T]} \tfrac{f(t,u)}{u} = \infty.$$

On the other hand, we get

$$\omega_1 = \int_0^1 G(s,s)\, ds = \frac{1}{\Gamma\left(\frac{3}{2}\right)} \int_0^1 \sqrt{s(1-s)} ds = \frac{1}{\frac{1}{2}\sqrt{\pi}} \frac{\pi}{8} = \frac{\sqrt{\pi}}{4}, \tag{4.2}$$

and

$$b(t) = \begin{cases} \sqrt{t} & \text{for } t \in [0, \bar{t}], \\ \frac{1-t}{16} & \text{for } t \in [\bar{t}, 1]. \end{cases} \tag{4.3}$$

Then

$$\omega_2 = \frac{\bar{b}}{\lambda^2 \Gamma\left(\frac{3}{2}\right)} \left[ \int_0^{\bar{t}} s\sqrt{(1-s)} ds + \frac{1}{16} \int_{\bar{t}}^1 \sqrt{s}\, (1-s)^{\frac{3}{2}}\, ds \right] \simeq \frac{\bar{b}\sqrt{\pi}}{128\lambda^2}. \tag{4.4}$$

Where $\bar{t} \simeq 0,003876\ldots$ and $\bar{b} \simeq 0,062258\ldots$ and the choice of $\lambda$ depends directly by choice of $r_1, r_2$ in $(3.25)$, $(3.27)$.

Because $\omega_1, \omega_2 > 0$, two finite constants for any choice of $0 < r_1 < r_2 < \infty$. We have always:

$$\frac{1}{\omega_2 f_\infty} = 0, \text{ and } \frac{1}{\omega_1 F_0} = \infty.$$

Then, the condition (3.22) is satisfied for any $0 < \beta < \infty$.

It follows from Theorem 3.7 that the problem (4.1) has at least one solution.

**Example 2.** Consider

$$\begin{cases} {}^1\mathcal{D}_{0^+}^{\frac{3}{2}} u(t) + \beta(1+t)\, u(t) \exp\left(\frac{1}{u(t)} - [u(t)]^2\right) = 0, \quad t \in [0,1]. \\ u(0) = u(1) = 0. \end{cases} \tag{4.5}$$

Set $\beta > 0$ any finite positive real number, and

$$f\left(t,u\right) = \left(1+t\right) u \exp\left(\frac{1}{u} - u^2\right).$$

Clearly, for any $t \in [0,1]$ and any $u > 0$, the function $f$ is jointly continuous.

Here, we have:

$$f_0 = \lim_{u \to 0^+} \min_{t\in[0,T]} \tfrac{f(t,u)}{u} = \infty, \quad F_\infty = \lim_{u\to+\infty} \max_{t\in[0,T]} \tfrac{f(t,u)}{u} = 0^+.$$

Also, we find the same function $b\left(t\right)$ in $(4.3)$, and same constant $\omega_1$, $\omega_2$ respectively in $(4.2)$, $(4.4)$.

The choice of $\lambda > 1$ depends directly by choice of $r_1, r_2$ in $(3.30)$, $(3.33)$.

Because $\omega_1, \omega_2 > 0$, two finite constants for any choice of $0 < r_1 < r_2 < \infty$. We have always:

$$\frac{1}{\omega_2 f_0} = 0, \text{ and } \frac{1}{\omega_1 F_\infty} = \infty.$$

Then, the condition $(3.28)$ is satisfied for any $0 < \beta < \infty$.

It follows from Theorem 3.8 that the problem $(4.5)$ has at least one solution.

**Example 3.** Consider the following boundary value problem

$$\begin{cases} {}^1\mathcal{D}_{0^+}^{\frac{3}{2}} u\left(t\right) + \frac{(1+t)(1+u(t))}{\sqrt{\pi}} = 0, \quad t \in [0,1]. \\ u\left(0\right) = u\left(1\right) = 0. \end{cases} \tag{4.6}$$

Set $\beta = \frac{1}{\sqrt{\pi}}$, and

$$f\left(t,u\right) = \left(1+t\right)\left(1+u\right).$$

The function $f$ is jointly continuous for any $t \in [0,1]$ and any $u > 0$.

We find the same function $b\left(t\right)$ in $(4.3)$, such that $0 \leq b\left(t\right) < 1$, and

$$\omega_1 = \int_0^1 G\left(s,s\right) ds = \frac{\sqrt{\pi}}{4}.$$

Choosing $r_1 = \frac{1}{10^4} < r_2 = 2$. Then, for all $t \in [0,1]$, we have:

$$h = 1 \leq f\left(t,u\right) \leq 6 = L.$$

In this case

$$\begin{aligned} \lambda &= 1 + \frac{8^{\rho\alpha} L\left(\alpha+1\right)\left[8^{\rho\alpha} - \left(8^\rho - 1\right)^\alpha\right]}{h\left(8^\rho - 1\right)^\alpha \left[8^\rho\left(\alpha+1\right) + 8^{\rho(\alpha-1)}\left(\alpha-1\right)\left(8^\rho - 1\right)\right]} \\ &= 1 + \frac{8^{\frac{3}{2}} \times 6 \times \frac{5}{2} \times \left(8^{\frac{3}{2}} - 7^{\frac{3}{2}}\right)}{7^{\frac{3}{2}} \times \left(8 \times \frac{5}{2} + \sqrt{8} \times \frac{7}{2}\right)} \\ &\simeq 3,517426\ldots \end{aligned}$$

Then

$$\omega_2 \simeq \frac{\bar{b}\sqrt{\pi}}{128\lambda^2} \simeq \frac{0,062258 \times \sqrt{\pi}}{128 \times 3,517426^2} \simeq \frac{3,9313\sqrt{\pi}}{10^5}.$$

It remains to show that the conditions in (3.34) , which is

$$\sup_{0 \leq u \leq r_2} \max_{0 \leq t \leq T} f(t,u) = 6 \leq \frac{r_2}{\beta\omega_1} \simeq 8,$$

and

$$\inf_{0 \leq u \leq r_1} f_3(t,u) = 1 + t \geq \frac{r_1}{\beta\lambda\omega_2} b(t) \simeq 0,72317 \times b(t), \quad \forall t \in [0,1].$$

Are satisfied. It follows from Theorem 3.9 that the problem (4.6) has at least one solution.

**Example 4.** Let

$$\begin{cases} {}^{\frac{2}{3}}\mathcal{D}_{0+}^{\frac{3}{2}} u(t) + \frac{\cos(t)[2+|u(t)|]}{\pi\left(\sqrt{2}\cos(t)+\sin(t)\right)[1+|u(t)|]} = 0, & t \in \left[0, \frac{\pi}{4}\right], \\ u(0) = u\left(\frac{\pi}{4}\right) = 0. \end{cases} \tag{4.7}$$

Set $\beta = \frac{1}{\pi}$ and

$$f(t,u) = \frac{\cos(t)[2+|u|]}{\left(\sqrt{2}\cos(t)+\sin(t)\right)[1+|u|]}, \quad t \in \left[0, \frac{\pi}{4}\right], \; u, v \in \mathbb{R}.$$

As $\sin(t)$, $\cos(t)$ are continuous positive functions $\forall t \in \left[0, \frac{\pi}{4}\right]$, the function $f$ is jointly continuous. For any $u, v \in \mathbb{R}$ and $t \in \left[0, \frac{\pi}{4}\right]$, we have $\frac{\sqrt{2}}{2} \leq \cos(t) \leq 1$, and $0 \leq \sin(t) \leq \frac{\sqrt{2}}{2}$, then

$$\begin{aligned} |f(t,u) - f(t,v)| &= \left| \frac{\cos(t)[2+|u|]}{\left(\sqrt{2}\cos(t)+\sin(t)\right)[1+|u|]} - \frac{\cos(t)[2+|v|]}{\left(\sqrt{2}\cos(t)+\sin(t)\right)[1+|v|]} \right| \\ &= \left| \frac{\cos(t)}{\sqrt{2}\cos(t)+\sin(t)} \right| \left| \frac{2+|u|}{1+|u|} - \frac{2+|v|}{1+|v|} \right| \\ &\leq ||u| - |v|| \leq |u - v|. \end{aligned}$$

Hence, the condition (3.35) is satisfied with $\sigma = 1$. It remains to show that the condition (3.36)

$$0 < \beta = \frac{1}{\pi} \simeq 0,318309\ldots < \frac{\rho^\alpha \Gamma(\alpha+1)}{\sigma T^{\alpha\rho}} = \frac{\frac{2}{3}^{\frac{3}{2}} \times \Gamma\left(\frac{5}{2}\right)}{\frac{\pi}{4}} \simeq 0,921317\ldots$$

is satisfied. It follows from Theorem 3.10 that the problem (4.7) has a unique solution.

## 5. Conclusion

In this paper we have discussed the existence and the uniqueness of solutions for a class of nonlinear fractional differential equations with a boundary value, by using the properties of Guo-Krasnosel'skii and Banach fixed point theorems. The used differential operator is developed by Katugampola, which generalizes the Riemann-Liouville and the Hadamard fractional derivatives into a single form.

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**Bilal Basti**
email: `bilalbasti@univ-msila.dz`
ORCID: 0000-0001-8216-3812
Laboratory for Pure and Applied Mathematics
University of M'sila
M'sila 28000
ALGERIA

**Yacine Arioua***
email: `yacine.arioua@univ-msila.dz`
ORCID: 0000-0002-9681-9568
Laboratory for Pure and Applied Mathematics
University of M'sila
M'sila 28000
ALGERIA
*Corresponding author

**Nouredine Benhamidouche**
email: `nbenhamidouche@univ-msila.dz`
ORCID: 0000-0002-5740-8504
Laboratory for Pure and Applied Mathematics
University of M'sila
M'sila 28000
ALGERIA

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# Fast Growing Solutions to Linear Differential Equations with Entire Coefficients Having the Same $\rho_\varphi$-order

*Benharrat Belaïdi*

Abstract: This paper deals with the growth of solutions of a class of higher order linear differential equations

$$f^{(k)} + A_{k-1}(z) f^{(k-1)} + \cdots + A_1(z) f' + A_0(z) f = 0,\ k \geq 2$$

when most coefficients $A_j(z)$ $(j = 0, ..., k-1)$ have the same $\rho_\varphi$-order with each other. By using the concept of $\tau_\varphi$-type, we obtain some results which indicate growth estimate of every non-trivial entire solution of the above equations by the growth estimate of the coefficient $A_0(z)$. We improve and generalize some recent results due to Chyzhykov-Semochko and the author.

*AMS Subject Classification:* 34M10, 30D35.
*Keywords and Phrases:* Linear differential equations; Entire function; Meromorphic function; $\rho_\varphi$-order; $\mu_\varphi$-order; $\tau_\varphi$-type.

## 1. Introduction and main results

Throughout this paper, the term "meromorphic" will mean meromorphic in the complex plane $\mathbb{C}$. Also, we shall assume that readers are familiar with the fundamental results and the standard notation of the Nevanlinna value distribution theory of meromorphic functions such as $m(r, f)$, $N(r, f)$, $T(r, f)$ (see, [12, 24]). For all $r \in \mathbb{R}$, we define $\exp_1 r := e^r$ and $\exp_{p+1} r := \exp(\exp_p r)$, $p \in \mathbb{N}$. We also define for all $r$ sufficiently large $\log_1 r := \log r$ and $\log_{p+1} r := \log(\log_p r)$, $p \in \mathbb{N}$. Moreover, we denote by $\exp_0 r := r$, $\log_0 r := r$, $\log_{-1} r := \exp_1 r$ and $\exp_{-1} r := \log_1 r$, see [17, 18].

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**Definition 1.1** ([17]). Let $p \geq 1$ be an integer. The iterated $p$-order of a meromorphic function $f$ is defined by

$$\rho_p(f) = \limsup_{r \longrightarrow +\infty} \frac{\log_p T(r,f)}{\log r},$$

where $T(r,f)$ is the Nevanlinna characteristic function of $f$. If $f$ is entire, then the iterated $p$-order of $f$ is defined by

$$\rho_p(f) = \limsup_{r \longrightarrow +\infty} \frac{\log_p T(r,f)}{\log r} = \limsup_{r \longrightarrow +\infty} \frac{\log_{p+1} M(r,f)}{\log r},$$

where $M(r,f) = \max_{|z|=r} |f(z)|$ is the maximum modulus function.

**Definition 1.2** ([17]). The finiteness degree of the order of a meromorphic function $f$ is defined by

$$i(f) := \begin{cases} 0, & \text{for } f \text{ rational}, \\ \min\{j \in \mathbb{N} : \rho_j(f) < \infty\}, & \text{for } f \text{ transcendental for which} \\ & \text{some } j \in \mathbb{N} \text{ with } \rho_j(f) < \infty \text{ exists}, \\ +\infty, & \text{for } f \text{ with } \rho_j(f) = +\infty,\ \forall j \in \mathbb{N}. \end{cases}$$

**Definition 1.3** Let $f$ be a meromorphic function. Then the iterated $p$-type of $f$, with iterated $p$-order $0 < \rho_p(f) < \infty$ is defined by

$$\tau_p(f) = \limsup_{r \longrightarrow +\infty} \frac{\log_{p-1} T(r,\ f)}{r^{\rho_p(f)}} \quad (p \geq 1 \ \text{ is an integer}).$$

If $f$ is an entire function, then the iterated $p$-type of $f$, with iterated $p$-order $0 < \rho_p(f) < \infty$ is defined by

$$\tau_{M,p}(f) = \limsup_{r \longrightarrow +\infty} \frac{\log_p M(r,\ f)}{r^{\rho_p(f)}} \quad (p \geq 1 \ \text{ is an integer}).$$

**Remark 1.1** Note that for $p = 1$, we can have $\tau_{M,1}(f) \neq \tau_1(f)$. For example if $f(z) = e^z$, then $\tau_{M,1}(f) = 1 \neq \tau_1(f) = \dfrac{1}{\pi}$. However, by Proposition 2.2.2 in [18], we have $\tau_{M,p}(f) = \tau_p(f)$ for $p \geq 2$.

Consider for $k \geq 2$ the linear differential equation

$$f^{(k)} + A_{k-1}(z) f^{(k-1)} + \cdots + A_1(z) f' + A_0(z) f = 0, \tag{1.1}$$

where $A_0(z) \not\equiv 0, \ldots, A_{k-1}(z)$ are entire functions. It is well-known that all solutions of equation (1.1) are entire functions and if some of the coefficients of (1.1) are transcendental, then (1.1) has at least one solution with order $\rho(f) = +\infty$. As far as we known, Bernal [7] firstly introduced the idea of iterated order to express the fast growth of solutions of complex linear differential equations. Since then, many authors obtained further results on iterated order of solutions of (1.1), see e.g. [2, 8, 9, 17].

In [17], Kinnunen have investigated the growth of solutions of equation (1.1) and obtained the following theorem.

**Theorem A** ([17]). *Let* $A_0(z), ..., A_{k-1}(z)$ *be entire functions such that* $i(A_0) = p$ $(0 < p < \infty)$. *If either* $\max\{i(A_j)\colon j = 1, 2, ..., k-1\} < p$ *or* $\max\{\rho_p(A_j)\colon j = 1, 2, ..., k-1\} < \rho_p(A_0)$, *then every solution* $f \not\equiv 0$ *of* (1.1) *satisfies* $i(f) = p+1$ *and* $\rho_{p+1}(f) = \rho_p(A_0)$.

Note that the result of Theorem A occur when there exists only one dominant coefficient. In the case that there are more than one dominant coefficients, the author [2] obtained the following result.

**Theorem B** ([2]). *Let* $A_0(z), ..., A_{k-1}(z)$ *be entire functions, and let* $i(A_0) = p$ $(0 < p < \infty)$. *Assume that either*

$$\max\{i(A_j) : j = 1, 2, ..., k-1\} < p$$

*or*

$$\max\{\rho_p(A_j) : j = 1, 2, ..., k-1\} \leq \rho_p(A_0) = \rho \ \ (0 < \rho < +\infty)$$

*and*

$$\max\{\tau_{M,p}(A_j) : \rho_p(A_j) = \rho_p(A_0)\} < \tau_{M,p}(A_0) = \tau \ \ (0 < \tau < +\infty).$$

*Then every solution* $f \not\equiv 0$ *of* (1.1) *satisfies* $i(f) = p+1$ *and* $\rho_{p+1}(f) = \rho_p(A_0) = \rho$.

In [15, 16], Juneja, Kapoor and Bajpai have investigated some properties of entire functions of $[p, q]$-order and obtained some results about their growth. In [20], in order to maintain accordance with general definitions of the entire function $f$ of iterated $p$-order [17, 18], Liu-Tu-Shi gave a minor modification of the original definition of the $[p, q]$-order given in [15, 16]. With this new concept of $[p, q]$-order, Liu, Tu and Shi [20] have considered equation (1.1) with entire coefficients and obtained different results concerning the growth of their solutions. After that, several authors used this new concept to investigate the growth of solutions in the complex plane and in the unit disc [3, 4, 5, 13, 19, 23, 25]. For the unity of notations, we here introduce the concept of $[p, q]$-order, where $p$, $q$ are positive integers satisfying $p \geq q \geq 1$ (e.g. see, [19, 20]).

**Definition 1.4** ([19, 20]). Let $p \geq q \geq 1$ be integers. If $f$ is a transcendental meromorphic function, then the $[p, q]$-order of $f$ is defined by

$$\rho_{[p,q]}(f) = \limsup_{r \longrightarrow +\infty} \frac{\log_p T(r, f)}{\log_q r}.$$

It is easy to see that $0 \leq \rho_{[p,q]}(f) \leq \infty$. If $f$ is rational, then $T(r, f) = O(\log r)$, and so $\rho_{[p,q]}(f) = 0$ for any $p \geq q \geq 1$. By Definition 1.4, we have that $\rho_{[1,1]}(f) =$

$\rho_1(f) = \rho(f)$ usual order, $\rho_{[2,1]}(f) = \rho_2(f)$ hyper-order and $\rho_{[p,1]}(f) = \rho_p(f)$ iterated $p-$order.

**Remark 1.2** Both definitions of iterated order and of $[p,q]$-order have the disadvantage that they do not cover arbitrary growth, i.e., there exist entire or meromorphic functions of infinite $[p,q]$-order and $p$-th iterated order for arbitrary $p \in \mathbb{N}$, i.e., of infinite degree, see Example 1.4 in [10].

Recently, Chyzhykov and Semochko [10] have given general definition of growth for an entire function $f$ in the complex plane, which does not have this disadvantage (see [22]) as follows.

As is [10] , let $\Phi$ be the class of positive unbounded increasing function on $[1,+\infty)$ such that $\varphi(e^t)$ is slowly growing, i.e.,

$$\forall c > 0: \lim_{t\to+\infty} \frac{\varphi(e^{ct})}{\varphi(e^t)} = 1.$$

We give some properties of functions from the class $\Phi$.

**Proposition 1.1** ([10]). *If* $\varphi \in \Phi$, *then*

$$\forall m > 0,\ \forall k \geq 0: \lim_{x\to+\infty} \frac{\varphi^{-1}(\log x^m)}{x^k} = +\infty, \tag{1.2}$$

$$\forall \delta > 0: \lim_{x\to+\infty} \frac{\log \varphi^{-1}((1+\delta)x)}{\log \varphi^{-1}(x)} = +\infty. \tag{1.3}$$

**Remark 1.3** ([10]). If $\varphi$ is non-decreasing, then (1.3) is equivalent to the definition of the class $\Phi$.

**Definition 1.5** ([10]). Let $\varphi$ be an increasing unbounded function on $[1,+\infty)$. Then, the orders of the growth of an entire function $f$ are defined by

$$\tilde{\rho}^0_\varphi(f) = \limsup_{r\longrightarrow+\infty} \frac{\varphi(M(r,f))}{\log r},\ \tilde{\rho}^1_\varphi(f) = \limsup_{r\longrightarrow+\infty} \frac{\varphi(\log M(r,f))}{\log r}.$$

If $f$ is meromorphic, then the orders are defined by

$$\rho^0_\varphi(f) = \limsup_{r\longrightarrow+\infty} \frac{\varphi\left(e^{T(r,f)}\right)}{\log r},\ \rho^1_\varphi(f) = \limsup_{r\longrightarrow+\infty} \frac{\varphi(T(r,f))}{\log r}.$$

**Remark 1.4** Now, if we suppose that $\varphi(r) = \log\log r$, then it is clear that $\varphi \in \Phi$. In this case, the above definition of orders coincide with definitions of usual order and hyper-order, i.e., if $f$ is entire, then

$$\tilde{\rho}^0_{\log\log}(f) = \limsup_{r\longrightarrow+\infty} \frac{\log\log M(r,f)}{\log r} = \rho(f),$$

$$\tilde{\rho}^{1}_{\log\log}(f) = \limsup_{r\longrightarrow+\infty} \frac{\log\log\log M(r,f)}{\log r} = \rho_2(f).$$

If $f$ is meromorphic, then

$$\rho^{0}_{\log\log}(f) = \limsup_{r\longrightarrow+\infty} \frac{\log\log\left(e^{T(r,f)}\right)}{\log r} = \limsup_{r\longrightarrow+\infty} \frac{\log T(r,f)}{\log r} = \rho(f),$$

$$\rho^{1}_{\log\log}(f) = \limsup_{r\longrightarrow+\infty} \frac{\varphi(T(r,f))}{\log r} = \limsup_{r\longrightarrow+\infty} \frac{\log\log T(r,f)}{\log r} = \rho_2(f).$$

**Proposition 1.2** ([10]). *Let $\varphi \in \Phi$ and $f$ be an entire function. Then*

$$\rho^{j}_{\varphi}(f) = \tilde{\rho}^{j}_{\varphi}(f), \ j = 0, 1.$$

Now, by Definition 1.5, we can introduce the concepts of $\mu_{\varphi}$ lower order.

**Definition 1.6** Let $\varphi$ be an increasing unbounded function on $[1, +\infty)$. Then, the lower orders of the growth of an entire function $f$ are defined by

$$\tilde{\mu}^{0}_{\varphi}(f) = \liminf_{r\longrightarrow+\infty} \frac{\varphi(M(r,f))}{\log r}, \quad \tilde{\mu}^{1}_{\varphi}(f) = \liminf_{r\longrightarrow+\infty} \frac{\varphi(\log M(r,f))}{\log r}.$$

If $f$ is meromorphic, then the orders are defined by

$$\mu^{0}_{\varphi}(f) = \liminf_{r\longrightarrow+\infty} \frac{\varphi\left(e^{T(r,f)}\right)}{\log r}, \quad \mu^{1}_{\varphi}(f) = \liminf_{r\longrightarrow+\infty} \frac{\varphi(T(r,f))}{\log r}.$$

**Proposition 1.3** *Let $\varphi \in \Phi$ and $f$ be an entire function. Then*

$$\mu^{j}_{\varphi}(f) = \widetilde{\mu}^{j}_{\varphi}(f), \ j = 0, 1.$$

*Proof.* By using the same proof of Proposition 3.1 in [10] and replacing $\limsup$ by $\liminf$, we can easily obtain the Proposition 1.3.

**Definition 1.7** Let $\varphi$ be an increasing unbounded function on $[1, +\infty)$. Then, the types of an entire function $f$ with $0 < \tilde{\rho}^{i}_{\varphi}(f) < +\infty$ $(i = 0, 1)$ are defined by

$$\tilde{\tau}^{0}_{M,\varphi}(f) = \limsup_{r\longrightarrow+\infty} \frac{\exp\{\varphi(M(r,f))\}}{r^{\tilde{\rho}^{0}_{\varphi}(f)}}, \quad \tilde{\tau}^{1}_{M,\varphi}(f) = \limsup_{r\longrightarrow+\infty} \frac{\exp\{\varphi(\log M(r,f))\}}{r^{\tilde{\rho}^{1}_{\varphi}(f)}}.$$

If $f$ is meromorphic, then the types of $f$ with $0 < \rho^{i}_{\varphi}(f) < +\infty$ $(i = 0, 1)$ are defined by

$$\tau^{0}_{\varphi}(f) = \limsup_{r\longrightarrow+\infty} \frac{\exp\left\{\varphi\left(e^{T(r,f)}\right)\right\}}{r^{\rho^{0}_{\varphi}(f)}}, \quad \tau^{1}_{\varphi}(f) = \limsup_{r\longrightarrow+\infty} \frac{\exp\{\varphi(T(r,f))\}}{r^{\rho^{1}_{\varphi}(f)}}.$$

**Definition 1.8** Let $\varphi$ be an increasing unbounded function on $[1, +\infty)$. Then, the lower types of an entire function $f$ with $0 < \tilde{\mu}^{i}_{\varphi}(f) < +\infty$ $(i = 0, 1)$ are defined by

$$\underline{\tilde{\tau}}^{0}_{M,\varphi}(f) = \liminf_{r\longrightarrow+\infty} \frac{\exp\{\varphi(M(r,f))\}}{r^{\tilde{\mu}^{0}_{\varphi}(f)}}, \quad \underline{\tilde{\tau}}^{1}_{M,\varphi}(f) = \liminf_{r\longrightarrow+\infty} \frac{\exp\{\varphi(\log M(r,f))\}}{r^{\tilde{\mu}^{1}_{\varphi}(f)}}.$$

If $f$ is meromorphic, then the lower types of $f$ with $0 < \mu_{\varphi}^{i}(f) < +\infty$ $(i = 0, 1)$ are defined by

$$\underline{\tau}_{\varphi}^{0}(f) = \liminf_{r \longrightarrow +\infty} \frac{\exp\left\{\varphi\left(e^{T(r,f)}\right)\right\}}{r^{\mu_{\varphi}^{0}(f)}}, \quad \underline{\tau}_{\varphi}^{1}(f) = \liminf_{r \longrightarrow +\infty} \frac{\exp\left\{\varphi\left(T(r,f)\right)\right\}}{r^{\mu_{\varphi}^{1}(f)}}.$$

Very recently, Bandura, Skaskiv and Filevych in [1, Theorem 7] proved that for an arbitrary entire transcendental function $f$ of infinite order, there exists a strictly increasing positive unbounded and continuously differentiable function $\varphi$ on $[1, +\infty)$ such that $\tilde{\rho}_{\varphi}^{0}(f) \in (0, +\infty)$. On the other hand, Chyzhykov and Semochko [10], Semochko [21], Belaïdi [6] used the concepts of $\rho_{\varphi}$-orders in order to investigate the growth of solutions of linear differential equations in the complex plane and in the unit disc. Examples of such results are the following two theorems.

**Theorem C** ([10]). *Let $\varphi \in \Phi$ and let $A_0(z), \ldots, A_{k-1}(z)$ be entire functions satisfying* $\max\{\rho_{\varphi}^{0}(A_j) : j = 1, \ldots, k-1\} < \rho_{\varphi}^{0}(A_0)$. *Then, every solution $f \not\equiv 0$ of equation* (1.1) *satisfies $\rho_{\varphi}^{1}(f) = \rho_{\varphi}^{0}(A_0)$.*

**Theorem D** ([6]). *Let $A_0(z), \ldots, A_{k-1}(z)$ be entire functions, and let $\varphi \in \Phi$. Assume that* $\max\{\tilde{\rho}_{\varphi}^{0}(A_j) : j = 1, \ldots, k-1\} < \tilde{\mu}_{\varphi}^{0}(A_0) \leq \tilde{\rho}_{\varphi}^{0}(A_0) < +\infty$. *Then every solution $f \not\equiv 0$ of* (1.1) *satisfies $\tilde{\mu}_{\varphi}^{0}(A_0) = \tilde{\mu}_{\varphi}^{1}(f) \leq \tilde{\rho}_{\varphi}^{1}(f) = \tilde{\rho}_{\varphi}^{0}(A_0)$.*

The main purpose of this paper is to consider the growth of solutions of equation (1.1) with entire coefficients of finite $\rho_{\varphi}$-order in the complex plane by using the concept of $\tau_{\varphi}$-type. We obtain the following results which extend Theorems A-B-C-D.

**Theorem 1.1** *Let $A_0(z), \ldots, A_{k-1}(z)$ be entire functions, and let $\varphi \in \Phi$. Assume that*

$$\max\{\tilde{\rho}_{\varphi}^{0}(A_j) : j = 1, \ldots, k-1\} \leq \tilde{\rho}_{\varphi}^{0}(A_0) = \rho < +\infty \ \ (0 < \rho < +\infty)$$

*and*

$$\max\{\tilde{\tau}_{M,\varphi}^{0}(A_j) : \tilde{\rho}_{\varphi}^{0}(A_j) = \tilde{\rho}_{\varphi}^{0}(A_0)\} < \tilde{\tau}_{M,\varphi}^{0}(A_0) = \tau \ \ (0 < \tau < +\infty).$$

*Then every solution $f \not\equiv 0$ of* (1.1) *satisfies $\tilde{\rho}_{\varphi}^{1}(f) = \tilde{\rho}_{\varphi}^{0}(A_0)$.*

By using Proposition 1.2, combining Theorem C and Theorem 1.1, we obtain the following result.

**Corollary 1.1** *Let $A_0(z), \ldots, A_{k-1}(z)$ be entire functions, and let $\varphi \in \Phi$. Assume that either*

$$\max\{\tilde{\rho}_{\varphi}^{0}(A_j) : j = 1, \ldots, k-1\} < \tilde{\rho}_{\varphi}^{0}(A_0)$$

*or*

$$\max\{\tilde{\rho}_{\varphi}^{0}(A_j) : j = 1, \ldots, k-1\} \leq \tilde{\rho}_{\varphi}^{0}(A_0) = \rho < +\infty \ \ (0 < \rho < +\infty)$$

*and*

$$\max\{\tilde{\tau}^0_{M,\varphi}(A_j) : \tilde{\rho}^0_\varphi(A_j) = \tilde{\rho}^0_\varphi(A_0)\} < \tilde{\tau}^0_{M,\varphi}(A_0) = \tau \ \ (0 < \tau < +\infty).$$

*Then every solution* $f \not\equiv 0$ *of* (1.1) *satisfies* $\tilde{\rho}^1_\varphi(f) = \tilde{\rho}^0_\varphi(A_0)$.

**Theorem 1.2** *Let* $A_0(z), ..., A_{k-1}(z)$ *be entire functions, and let* $\varphi \in \Phi$*. Assume that*

$$\max\{\tilde{\rho}^0_\varphi(A_j) : j = 1, \dots, k-1\} \le \tilde{\mu}^0_\varphi(A_0) \le \tilde{\rho}^0_\varphi(A_0) = \rho < +\infty \ \ \left(\tilde{\mu}^0_\varphi(A_0) > 0\right)$$

*and*

$$\tau_1 = \max\{\tilde{\tau}^0_{M,\varphi}(A_j) : \tilde{\rho}^0_\varphi(A_j) = \tilde{\mu}^0_\varphi(A_0)\} < \tilde{\underline{\tau}}^0_{M,\varphi}(A_0) = \tau \ \ (0 < \tau < +\infty).$$

*Then every solution* $f \not\equiv 0$ *of* (1.1) *satisfies* $\tilde{\rho}^1_\varphi(f) = \tilde{\rho}^0_\varphi(A_0) \ge \tilde{\mu}^1_\varphi(f) = \tilde{\mu}^0_\varphi(A_0)$.

By combining Theorem D and Theorem 1.2, we obtain the following result.

**Corollary 1.2** *Let* $A_0(z), ..., A_{k-1}(z)$ *be entire functions, and let* $\varphi \in \Phi$*. Assume that either*

$$\max\{\tilde{\rho}^0_\varphi(A_j) : j = 1, \dots, k-1\} < \tilde{\mu}^0_\varphi(A_0) \le \tilde{\rho}^0_\varphi(A_0) < +\infty$$

*or*

$$\max\{\tilde{\rho}^0_\varphi(A_j) : j = 1, \dots, k-1\} \le \tilde{\mu}^0_\varphi(A_0) \le \tilde{\rho}^0_\varphi(A_0) = \rho < +\infty \ \ \left(\tilde{\mu}^0_\varphi(A_0) > 0\right)$$

*and*

$$\tau_1 = \max\{\tilde{\tau}^0_{M,\varphi}(A_j) : \tilde{\rho}^0_\varphi(A_j) = \tilde{\mu}^0_\varphi(A_0)\} < \tilde{\underline{\tau}}^0_{M,\varphi}(A_0) = \tau \ \ (0 < \tau < +\infty).$$

*Then every solution* $f \not\equiv 0$ *of* (1.1) *satisfies* $\tilde{\rho}^1_\varphi(f) = \tilde{\rho}^0_\varphi(A_0) \ge \tilde{\mu}^1_\varphi(f) = \tilde{\mu}^0_\varphi(A_0)$.

## 2. Some preliminary lemmas

We recall the following definition. The logarithmic measure of a set $F \subset (1, +\infty)$ is defined by $lm(F) = \int_1^{+\infty} \frac{\chi_F(t)}{t} dt$, where $\chi_H(t)$ is the characteristic function of a set $H$. Our proofs depend mainly upon the following lemmas.

**Lemma 2.1** ([11]). *Let* $f$ *be a transcendental meromorphic function, and let* $\alpha > 1$ *be a given constant. Then there exist a set* $E_1 \subset (1, \infty)$ *with finite logarithmic measure and a constant* $B > 0$ *that depends only on* $\alpha$ *and* $i, j$ $(0 \le i < j \le k)$*, such that for all* $z$ *satisfying* $|z| = r \notin [0, 1] \cup E_1$*, we have*

$$\left|\frac{f^{(j)}(z)}{f^{(i)}(z)}\right| \le B \left\{\frac{T(\alpha r, f)}{r} (\log^\alpha r) \log T(\alpha r, f)\right\}^{j-i}.$$

**Lemma 2.2** *Let $\varphi \in \Phi$ and $f$ be an entire function with $0 < \tilde{\rho}_{\varphi}^{0}(f) = \rho < +\infty$ and type $0 < \tilde{\tau}_{M,\varphi}^{0}(f) < \infty$. Then for any given $\beta < \tilde{\tau}_{M,\varphi}^{0}(f)$, there exists a set $E_2 \subset [1,+\infty)$ that has infinite logarithmic measure, such that for all $r \in E_2$, we have*

$$\varphi(M(r,f)) > \log(\beta r^{\rho}).$$

*Proof.* By definitions of $\tilde{\tau}_{M,\varphi}^{0}(f)$ type, there exists an increasing sequence $\{r_n\}$, $r_n \to +\infty$ satisfying $\left(1+\frac{1}{n}\right) r_n < r_{n+1}$ and

$$\lim_{r_n \to +\infty} \frac{\exp\{\varphi(M(r_n,f))\}}{r_n^{\rho}} = \tilde{\tau}_{M,\varphi}^{0}(f).$$

Then, there exists a positive integer $n_0$ such that for all $n \geq n_0$ and for any given $\varepsilon$ with $0 < \varepsilon < \tilde{\tau}_{M,\varphi}^{0}(A_0) - \beta$, we have

$$\exp\{\varphi(M(r_n,f))\} > \left(\tilde{\tau}_{M,\varphi}^{0}(f) - \varepsilon\right) r_n^{\rho}. \tag{2.1}$$

For any given $\beta < \tilde{\tau}_{M,\varphi}^{0}(f)$, there exists a positive integer $n_1$ such that for all $n \geq n_1$, we have

$$\left(\frac{n}{n+1}\right)^{\rho} > \frac{\beta}{\tilde{\tau}_{M,\varphi}^{0}(f) - \varepsilon}. \tag{2.2}$$

Taking $n \geq n_2 = \max\{n_0, n_1\}$. By (2.1) and (2.2) for any $r \in \left[r_n, \left(1+\frac{1}{n}\right) r_n\right]$, we obtain

$$\begin{aligned}\exp\{\varphi(M(r,f))\} &\geq \exp\{\varphi(M(r_n,f))\} > \left(\tilde{\tau}_{M,\varphi}^{0}(f) - \varepsilon\right) r_n^{\rho} \\ &\geq \left(\tilde{\tau}_{M,\varphi}^{0}(f) - \varepsilon\right)\left(\frac{n}{n+1} r\right)^{\rho} > \beta r^{\rho}.\end{aligned}$$

Set $E_2 = \bigcup_{n=n_2}^{+\infty} \left[r_n, \left(1+\frac{1}{n}\right) r_n\right]$, then there holds

$$lm(E_2) = \sum_{n=n_2}^{+\infty} \int_{r_n}^{\left(1+\frac{1}{n}\right) r_n} \frac{dt}{t} = \sum_{n=n_2}^{+\infty} \log\left(1+\frac{1}{n}\right) = +\infty.$$

**Lemma 2.3** ([6]). *Let $\varphi \in \Phi$ and $A_0(z), \ldots, A_{k-1}(z)$ be entire functions. Then, every solution $f \not\equiv 0$ of (1.1) satisfies*

$$\tilde{\rho}_{\varphi}^{1}(f) \leq \max\{\tilde{\rho}_{\varphi}^{0}(A_j) : j = 0, 1, \ldots, k-1\}.$$

**Lemma 2.4** ([6]). *Let $\varphi \in \Phi$ and $f$ be a meromorphic function with $\mu_{\varphi}^{1}(f) < +\infty$. Then there exists a set $E_3 \subset (1,+\infty)$ with infinite logarithmic measure such that for $r \in E_3 \subset (1,+\infty)$, we have for any given $\varepsilon > 0$*

$$T(r,f) < \varphi^{-1}\left(\left(\mu_{\varphi}^{1}(f) + \varepsilon\right) \log r\right).$$

**Lemma 2.5** ([14]). *Let* $f(z) = \sum\limits_{n=0}^{\infty} a_n z^n$ *be an entire function,* $\mu(r)$ *be the maximum term, i.e.,*

$$\mu(r) = \max\{|a_n| r^n : n = 0, 1, 2, ...\},$$

$\nu(r, f) = \nu_f(r)$ *be the central index of* $f$*, i.e.,* $\nu(r, f) = \max\{m : \mu(r) = |a_m| r^m\}$. *Then*
(i)

$$\mu(r) = \log|a_0| + \int\limits_0^r \frac{\nu_f(t)}{t} dt$$

*here we assume that* $|a_0| \neq 0$.
(ii) *For* $r < R$

$$M(r, f) < \mu(r) \left\{ \nu_f(R) + \frac{R}{R - r} \right\}.$$

**Lemma 2.6** ([14, 18]). *Let* $f$ *be a transcendental entire function. Then there exists a set* $E_4 \subset (1, +\infty)$ *with finite logarithmic measure such that for all* $z$ *satisfying* $|z| = r \notin E_4$ *and* $|f(z)| = M(r, f)$*, we have*

$$\frac{f^{(n)}(z)}{f(z)} = \left( \frac{\nu_f(r)}{z} \right)^n (1 + o(1)), \quad (n \in \mathbb{N}),$$

*where* $\nu_f(r)$ *is the central index of* $f$.

**Lemma 2.7** [6]. *Let* $\varphi \in \Phi$ *and* $f$ *be an entire function with* $\tilde{\mu}_\varphi^0(f) < +\infty$. *Then there exists a set* $E_5 \subset (1, +\infty)$ *with infinite logarithmic measure such that for* $r \in E_5 \subset (1, +\infty)$, *we have for any given* $\varepsilon > 0$

$$M(r, f) < \varphi^{-1}\left( \left( \tilde{\mu}_\varphi^0(f) + \varepsilon \right) \log r \right).$$

# 3. Proof of Theorem 1.1

Suppose that $f$ $(\not\equiv 0)$ is a solution of equation (1.1). From (1.1), we can write

$$|A_0(z)| \leq \left| \frac{f^{(k)}}{f} \right| + |A_{k-1}(z)| \left| \frac{f^{(k-1)}}{f} \right| + \cdots + |A_1(z)| \left| \frac{f'}{f} \right|. \tag{3.1}$$

If $\max\{\tilde{\rho}_\varphi^0(A_j) : j = 1, \ldots, k-1\} < \tilde{\rho}_\varphi^0(A_0) = \rho$, then by Theorem C, we obtain $\tilde{\rho}_\varphi^1(f) = \tilde{\rho}_\varphi^0(A_0)$. Suppose that $\max\{\tilde{\rho}_\varphi^0(A_j) : j = 1, 2, ..., k-1\} = \tilde{\rho}_\varphi^0(A_0) = \rho$ $(0 < \rho < +\infty)$ and $\max\{\tilde{\tau}_{M,\varphi}^0(A_j) : \tilde{\rho}_\varphi^0(A_j) = \tilde{\rho}_\varphi^0(A_0)\} < \tilde{\tau}_{M,\varphi}^0(A_0) = \tau$ $(0 < \tau < +\infty)$. First, we prove that $\rho_1 = \tilde{\rho}_\varphi^1(f) \geq \tilde{\rho}_\varphi^0(A_0) = \rho$. Suppose the contrary $\rho_1 = \tilde{\rho}_\varphi^1(f) < \tilde{\rho}_\varphi^0(A_0) = \rho$. Then, there exists a set $I \subseteq \{1, 2, ..., k-1\}$ such that $\tilde{\rho}_\varphi^0(A_j) = \tilde{\rho}_\varphi^0(A_0) = \rho$ $(j \in I)$ and $\tilde{\tau}_{M,\varphi}^0(A_j) < \tilde{\tau}_{M,\varphi}^0(A_0)$ $(j \in I)$. Thus, we choose $\alpha_1$, $\alpha_2$ satisfying

$$\max\{\tilde{\tau}_{M,\varphi}^0(A_j) : (j \in I)\} < \alpha_1 < \alpha_2 < \tilde{\tau}_{M,\varphi}^0(A_0) = \tau,$$

for sufficiently large $r$, we have

$$|A_j(z)| \leq \varphi^{-1}(\log(\alpha_1 r^\rho)) \quad (j \in J) \tag{3.2}$$

and

$$|A_j(z)| \leq \varphi^{-1}\left(\log r^{\beta_1}\right) \leq \varphi^{-1}(\log(\alpha_1 r^\rho)) \quad (j \in \{1, ..., k-1\} \setminus J), \tag{3.3}$$

where $0 < \beta_1 < \rho$. By Lemma 2.2, there exists a set $E_2 \subset [1, +\infty)$ with infinite logarithmic measure such that for all $r \in E_2$, we have

$$|A_0(z)| = M(r, A_0) > \varphi^{-1}(\log(\alpha_2 r^\rho)). \tag{3.4}$$

By Lemma 2.1, there exists a constant $B > 0$ and a set $E_1 \subset (1, +\infty)$ having finite logarithmic measure such that for all $z$ satisfying $|z| = r \notin E_1 \cup [0, 1]$, we have

$$\left|\frac{f^{(j)}(z)}{f(z)}\right| \leq B[T(2r, f)]^{k+1} \quad (j = 1, 2, ..., k).$$

Since $\tilde{\rho}_\varphi^1(f) = \rho_1$, then by Proposition 1.2, for any given $\varepsilon$ with $0 < \varepsilon < \rho - \rho_1$ and sufficiently large $|z| = r \notin E_1 \cup [0, 1]$

$$\left|\frac{f^{(j)}(z)}{f(z)}\right| \leq B[T(2r, f)]^{k+1} \leq B\left[\varphi^{-1}\left(\log(2r)^{\rho_1+\varepsilon}\right)\right]^{k+1} \quad (j = 1, 2, ..., k). \tag{3.5}$$

Hence, by substituting $(3.2)$, $(3.3)$, $(3.4)$ and $(3.5)$ into $(3.1)$, for any given $\varepsilon$ $(0 < \varepsilon < \min\left\{\frac{\alpha_2-\alpha_1}{2}, \rho - \rho_1\right\})$ and for sufficiently large $|z| = r \in E_2 \setminus (E_1 \cup [0, 1])$, we have

$$\begin{aligned}\varphi^{-1}(\log(\alpha_2 r^\rho)) &\leq kB\varphi^{-1}(\log(\alpha_1 r^\rho))\left[\varphi^{-1}\left(\log(2r)^{\rho_1+\varepsilon}\right)\right]^{k+1} \\ &\leq \varphi^{-1}(\log((\alpha_1 + 2\varepsilon) r^\rho)).\end{aligned} \tag{3.6}$$

The latter two estimates follow from the properties of (1.2) and (1.3). Since $E_2 \setminus (E_1 \cup [0, 1])$ is a set of infinite logarithmic measure, then there exists a sequence of points $|z_n| = r_n \in E_2 \setminus (E_1 \cup [0, 1])$ tending to $+\infty$. It follows by (3.6) that

$$\varphi^{-1}(\log(\alpha_2 r_n^\rho)) \leq \varphi^{-1}(\log((\alpha_1 + 2\varepsilon) r_n^\rho))$$

holds for all $z_n$ satisfying $|z_n| = r_n \in E_2 \setminus (E_1 \cup [0, 1])$ as $|z_n| \to +\infty$. By arbitrariness of $\varepsilon > 0$ and the monotonicity of the function $\varphi^{-1}$, we obtain that $\alpha_1 \geq \alpha_2$. This contradiction proves the inequality $\tilde{\rho}_\varphi^1(f) \geq \tilde{\rho}_\varphi^0(A_0)$. On the other hand, by Lemma 2.3, we have

$$\tilde{\rho}_\varphi^1(f) \leq \max\{\tilde{\rho}_\varphi^0(A_j) : j = 0, 1, \ldots, k-1\} = \tilde{\rho}_\varphi^0(A_0).$$

Hence, every solution $f \not\equiv 0$ of equation (1.1) satisfies $\tilde{\rho}_\varphi^1(f) = \tilde{\rho}_\varphi^0(A_0)$.

# 4. Proof of Theorem 1.2

Suppose that $f$ $(\not\equiv 0)$ is a solution of equation (1.1). Then by Theorem 1.1, we obtain $\tilde{\rho}^1_\varphi(f) = \tilde{\rho}^0_\varphi(A_0)$. Now, we prove that $\mu_1 = \tilde{\mu}^1_\varphi(f) \geq \tilde{\mu}^0_\varphi(A_0) = \mu$. Suppose the contrary $\mu_1 = \tilde{\mu}^1_\varphi(f) < \tilde{\mu}^0_\varphi(A_0) = \mu$. We set $b = \max\{\tilde{\rho}^0_\varphi(A_j) : \tilde{\rho}^0_\varphi(A_j) < \tilde{\mu}^0_\varphi(A_0)\}$. If $\tilde{\rho}^0_\varphi(A_j) < \tilde{\mu}^0_\varphi(A_0)$, then for any given $\varepsilon$ with $0 < 3\varepsilon < \min\{\mu - b, \tau - \tau_1\}$ and for sufficiently large $r$, we have

$$|A_j(z)| \leq \varphi^{-1}\left(\log r^{b+\varepsilon}\right) \leq \varphi^{-1}\left(\log r^{\tilde{\mu}^0_\varphi(A_0)-2\varepsilon}\right). \tag{4.1}$$

If $\tilde{\rho}^0_\varphi(A_j) = \tilde{\mu}^0_\varphi(A_0)$, $\tilde{\tau}^0_{M,\varphi}(A_j) \leq \tau_1 < \underline{\tilde{\tau}}^0_{M,\varphi}(A_0) = \tau$, then for sufficiently large $r$, we have

$$|A_j(z)| \leq \varphi^{-1}\left(\log(\tau_1+\varepsilon)\, r^{\tilde{\mu}^0_\varphi(A_0)}\right) \tag{4.2}$$

and

$$|A_0(z)| \geq \varphi^{-1}\left(\log(\tau-\varepsilon)\, r^{\tilde{\mu}^0_\varphi(A_0)}\right). \tag{4.3}$$

From $(1.1)$, we can write

$$|A_0(z)| \leq \left|\frac{f^{(k)}}{f}\right| + |A_{k-1}(z)|\left|\frac{f^{(k-1)}}{f}\right| + \cdots + |A_1(z)|\left|\frac{f'}{f}\right|. \tag{4.4}$$

By Lemma 2.1, there exists a constant $B > 0$ and a set $E_1 \subset (1, +\infty)$ having finite logarithmic measure such that for all $z$ satisfying $|z| = r \notin E_1 \cup [0,1]$, we have

$$\left|\frac{f^{(j)}(z)}{f(z)}\right| \leq B\,[T(2r,f)]^{k+1} \quad (j = 1, 2, ..., k).$$

By Proposition 1.3 and Lemma 2.4, for any given $\varepsilon$ with $0 < \varepsilon < \mu - \mu_1$ and sufficiently large $|z| = r \in E_3 \backslash (E_1 \cup [0,1])$

$$\left|\frac{f^{(j)}(z)}{f(z)}\right| \leq B\,[T(2r,f)]^{k+1} < B\left[\varphi^{-1}\left(\log(2r)^{\mu_1+\varepsilon}\right)\right]^{k+1} \quad (j = 1, 2, ..., k), \tag{4.5}$$

where $E_3$ is a set of infinite logarithmic measure. Hence, by substituting (4.1) – (4.3) and (4.5) into (4.4), for the above $\varepsilon$ with $0 < \varepsilon < \min\left\{\frac{\mu-b}{3}, \frac{\tau-\tau_1}{3}, \mu-\mu_1\right\}$), we obtain for sufficiently large $|z| = r \in E_3 \backslash (E_1 \cup [0,1])$

$$\begin{aligned}\varphi^{-1}\left(\log(\tau-\varepsilon)\, r^{\tilde{\mu}^0_\varphi(A_0)}\right) &\leq Bk\varphi^{-1}\left(\log(\tau_1+\varepsilon)\, r^{\tilde{\mu}^0_\varphi(A_0)}\right)[T(2r,f)]^{k+1} \\ &\leq Bk\varphi^{-1}\left(\log(\tau_1+\varepsilon)\, r^{\tilde{\mu}^0_\varphi(A_0)}\right)\left[\varphi^{-1}\left(\log(2r)^{\mu_1+\varepsilon}\right)\right]^{k+1} \\ &\leq \varphi^{-1}\left(\log(\tau_1+2\varepsilon)\, r_n^{\tilde{\mu}^0_\varphi(A_0)}\right). \end{aligned} \tag{4.6}$$

The latter two estimates follow from the properties of (1.2) and (1.3). Since $E_3 \backslash (E_1 \cup [0,1])$ is a set of infinite logarithmic measure, then there exists a sequence of points $|z_n| = r_n \in E_3 \backslash (E_1 \cup [0,1])$ tending to $+\infty$. It follows by (4.6) that

$$\varphi^{-1}\left(\log(\tau - \varepsilon) r_n^{\tilde{\mu}_\varphi^0(A_0)}\right) \leq \varphi^{-1}\left(\log(\tau_1 + 2\varepsilon) r_n^{\tilde{\mu}_\varphi^0(A_0)}\right)$$

holds for all $z_n$ satisfying $|z_n| = r_n \in E_3 \backslash (E_1 \cup [0,1])$ as $|z_n| \to +\infty$. By arbitrariness of $\varepsilon > 0$ and the monotonicity of the function $\varphi^{-1}$, we obtain that $\tau_1 \geq \tau$. This contradiction proves the inequality $\tilde{\mu}_\varphi^1(f) \geq \tilde{\mu}_\varphi^0(A_0)$.

Now, we prove $\tilde{\mu}_\varphi^1(f) \leq \tilde{\mu}_\varphi^0(A_0)$. By (1.1), we have

$$\left|\frac{f^{(k)}}{f}\right| \leq |A_{k-1}(z)| \left|\frac{f^{(k-1)}}{f}\right| + \cdots + |A_1(z)| \left|\frac{f'}{f}\right| + |A_0(z)|. \tag{4.7}$$

By Lemma 2.6, there exists a set $E_4 \subset (1, +\infty)$ of finite logarithmic measure such that the estimation

$$\frac{f^{(j)}(z)}{f(z)} = \left(\frac{\nu_f(r)}{z}\right)^j (1 + o(1)) \quad (j = 1, ..., k) \tag{4.8}$$

holds for all $z$ satisfying $|z| = r \notin E_4$, $r \to +\infty$ and $|f(z)| = M(r,f)$. By Lemma 2.7, for any given $\varepsilon > 0$, there exists a set $E_5 \subset (1, +\infty)$ that has infinite logarithmic measure, such that for $|z| = r \in E_5$

$$|A_0(z)| < \varphi^{-1}\left(\log r^{\tilde{\mu}_\varphi^0(A_0)+\varepsilon}\right). \tag{4.9}$$

Substituting (4.1), (4.2), (4.8) and (4.9) into (4.7), we obtain

$$\begin{aligned}\nu_f(r) &\leq kr^k |1 + o(1)| \varphi^{-1}\left(\log r^{\tilde{\mu}_\varphi^0(A_0)+\varepsilon}\right)\\ &\leq \varphi^{-1}\left(\log r^{\tilde{\mu}_\varphi^0(A_0)+2\varepsilon}\right)\end{aligned} \tag{4.10}$$

for all $z$ satisfying $|z| = r \in E_5 \backslash E_4$, $r \to +\infty$ and $|f(z)| = M(r,f)$. By (4.10), Lemma 2.5 and Proposition 1.1, we obtain for each $\varepsilon > 0$

$$\begin{aligned}T(r,f) &\leq \log M(r,f) < \log\mu(r,f) + \log(\nu(2r,f) + 2)\\ &< 2\nu(r,f)\log r + \log(2\nu(2r,f))\\ &\leq 2\varphi^{-1}\left(\log r^{\tilde{\mu}_\varphi^0(A_0)+2\varepsilon}\right)\log r + \log\left(2\varphi^{-1}\left(\log(2r)^{\tilde{\mu}_\varphi^0(A_0)+\varepsilon}\right)\right)\\ &= 2\varphi^{-1}\left(\log r^{\tilde{\mu}_\varphi^0(A_0)+2\varepsilon}\right)\log r + \log 2 + \log\varphi^{-1}\left(\log(2r)^{\tilde{\mu}_\varphi^0(A_0)+\varepsilon}\right)\\ &\leq \varphi^{-1}\left(\log r^{\tilde{\mu}_\varphi^0(A_0)+3\varepsilon}\right).\end{aligned}$$

Hence,

$$\frac{\varphi\left(T(r,f)\right)}{\log r} \leq \frac{\log r^{\tilde{\mu}_{\varphi}^{0}(A_0)+3\varepsilon}}{\log r} = \tilde{\mu}_{\varphi}^{0}\left(A_0\right) + 3\varepsilon.$$

It follows

$$\mu_{\varphi}^{1}\left(f\right) = \tilde{\mu}_{\varphi}^{1}\left(f\right) = \liminf_{r\longrightarrow+\infty} \frac{\varphi\left(T(r,f)\right)}{\log r} \leq \tilde{\mu}_{\varphi}^{0}\left(A_0\right) + 3\varepsilon.$$

Since $\varepsilon > 0$ is arbitrary, it follows that $\tilde{\mu}_{\varphi}^{1}\left(f\right) \leq \tilde{\mu}_{\varphi}^{0}\left(A_0\right)$. Hence, every solution $f \not\equiv 0$ of equation (1.1) satisfies $\tilde{\mu}_{\varphi}^{0}\left(A_0\right) = \tilde{\mu}_{\varphi}^{1}\left(f\right) \leq \tilde{\rho}_{\varphi}^{1}\left(f\right) = \tilde{\rho}_{\varphi}^{0}\left(A_0\right)$.

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**Benharrat Belaïdi**
email: `benharrat.belaidi@univ-mosta.dz`
ORCID: 0000-0002-6635-2514
Department of Mathematics
Laboratory of Pure and Applied Mathematics
University of Mostaganem (UMAB)
B. P. 227 Mostaganem
ALGERIA

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# A Generalization of the Hahn-Banach Theorem in Seminormed Quasilinear Spaces

*Sümeyye Çakan and Yılmaz Yılmaz*

ABSTRACT: The concept of normed quasilinear spaces which is a generalization of normed linear spaces gives us a new opportunity to study with a similar approach to classical functional analysis. In this study, we introduce the notion of seminormed quasilinear space as a generalization of normed quasilinear spaces and give various auxiliary results and examples. We present an analog of Hahn-Banach theorem, in seminormed quasilinear spaces.

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## 1. Introduction

Normed quasilinear spaces are introduced by Aseev, [2], in an effort to generalize normed linear spaces. A partial order relation was used to define normed quasilinear spaces. Motivated by [2], and using the framework and the tools given in [2], we developed the analysis in these spaces in [5, 6, 7, 8, 9, 12].

In this paper, we introduce the concept of seminormed quasilinear spaces and mention its some basic properties. Also we state and prove a version of Hahn-Banach theorem, one of the fundamental tools for the application of functional analysis, for seminormed quasilinear spaces.

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## 2. Preliminaries and some results on quasilinear spaces and normed quasilinear spaces

In this section, we present some basic definitions and results that appeared in [2] and [12] and which will be using in the sequel. Let us begin with Aseev's main definition.

**Definition 2.1.** [2] A set $X$ is called quasilinear space (qls, for short), if a partial order relation "$\preceq$", an algebraic sum operation and an operation of multiplication by real numbers are defined in it in such a way that the following conditions hold for any elements $x, y, z, v \in X$ and any $\alpha, \beta \in \mathbb{R}$:

$$x \preceq x, \tag{2.1}$$

$$x \preceq z \text{ if } x \preceq y \text{ and } y \preceq z, \tag{2.2}$$

$$x = y \text{ if } x \preceq y \text{ and } y \preceq x, \tag{2.3}$$

$$x + y = y + x, \tag{2.4}$$

$$x + (y + z) = (x + y) + z, \tag{2.5}$$

$$\text{there exists an element } \theta \in X \text{ such that } x + \theta = x, \tag{2.6}$$

$$\alpha \cdot (\beta \cdot x) = (\alpha\beta) \cdot x, \tag{2.7}$$

$$\alpha \cdot (x + y) = \alpha \cdot x + \alpha \cdot y, \tag{2.8}$$

$$1 \cdot x = x, \tag{2.9}$$

$$0 \cdot x = \theta, \tag{2.10}$$

$$(\alpha + \beta) \cdot x \preceq \alpha \cdot x + \beta \cdot x, \tag{2.11}$$

$$x + z \preceq y + v \text{ if } x \preceq y \text{ and } z \preceq v, \tag{2.12}$$

$$\alpha \cdot x \preceq \alpha \cdot y \text{ if } x \preceq y. \tag{2.13}$$

Generally, a qls $X$ with the partial order relation "$\preceq$" is denoted by $(X, \preceq)$. Here, we prefer denote the zero vector of $X$ by $\theta$ for clarity.

Every linear space is a qls with the partial order relation "$=$".

The most favorite example of qls which is not a linear space is the set of all nonempty, compact and convex subsets of real numbers with the inclusion relation "$\subseteq$", the algebraic sum operation

$$A + B = \{a + b : a \in A,\ b \in B\}$$

and multiplication operation by a real number $\lambda$ defined by

$$\lambda \cdot A = \{\lambda a : a \in A\}.$$

We denote this set by $\Omega_C(\mathbb{R})$.

Another one is $\Omega(\mathbb{R})$ which is the set of all nonempty compact subsets of real numbers.

In general, $\Omega(E)$ and $\Omega_C(E)$ stand for the space of all nonempty closed bounded and nonempty convex and closed bounded subsets of any normed linear space $E$, respectively. Both are nonlinear qls with the inclusion relation and a slight modification of addition operation by

$$A + B = \overline{\{a + b : a \in A,\ b \in B\}}$$

and multiplication operation by a $\lambda \in \mathbb{R}$ defined by $\lambda \cdot A = \{\lambda a : a \in A\}$. Where the closure is taken with respect to the standard topology in $\mathbb{R}$.

**Lemma 2.1.** [2] *In a qls $(X, \preceq)$, the element $\theta$ is minimal, i.e., $x = \theta$ if $x \preceq \theta$.*

Let $X$ be a qls and $Y \subseteq X$. Then $Y$ is called a subspace of $X$ if $Y$ is a qls with the same partial order relation and the restriction of the operations on $X$ to $Y$ .

**Theorem 2.1.** [12] *$Y$ is a subspace of qls $X$ if and only if $\alpha \cdot x + \beta \cdot y \in Y$ for every $x, y \in Y$ and $\alpha, \beta \in \mathbb{R}$.*

An element $x' \in X$ is called inverse of $x \in X$ if $x + x' = \theta$. Further, if an inverse element exists, then it is unique. An element $x$ possessing inverse is called regular, otherwise is called singular. $X_r$ and $X_s$ stand for the sets of all regular and singular elements in $X$, respectively, [12].

It will be assumed throughout the text that $-x = (-1) \cdot x$.

Suppose that every element $x$ in a qls $X$ has inverse element $x' \in X$ . Then the partial order in $X$ is determined by equality, the distributivity condition in (2.11) holds and consequently, $X$ is a linear space, [2]. In a real linear space, "=" is only way to define a partial order such that the conditions (2.1)-(2.13) hold.

On the other hand, an element $x \in X$ is said to be symmetric if $-x = x$, and $X_d$ denotes the set of all symmetric elements.

$X_r$, $X_d$ and $X_s \cup \{\theta\}$ are subspaces of $X$ and called regular, symmetric and singular subspaces of $X$, respectively, [12].

**Definition 2.2.** [2] Let $(X, \preceq)$ be a qls. A real function $\|.\|_X : X \longrightarrow \mathbb{R}$ is called a norm if the following conditions hold:

$$\|x\|_X > 0 \text{ if } x \neq \theta, \tag{2.14}$$

$$\|x + y\|_X \leq \|x\|_X + \|y\|_X\,, \tag{2.15}$$

$$\|\alpha \cdot x\|_X = |\alpha|\,\|x\|_X\,, \tag{2.16}$$

$$\text{if } x \preceq y, \text{ then } \|x\|_X \leq \|y\|_X\,, \tag{2.17}$$

$$\text{if for any } \varepsilon > 0 \text{ there exists an element } x_\varepsilon \in X \text{ such that} \tag{2.18}$$

$$x \preceq y + x_\varepsilon \text{ and } \|x_\varepsilon\|_X \leq \varepsilon \text{ then } x \preceq y.$$

A qls $X$ ,with a norm defined on it, is called normed quasilinear space (normed qls, for short).

Let $(X, \preceq)$ be a normed qls. Hausdorff metric or norm metric on $X$ is defined by the equality

$$h_X(x,y) = \inf\left\{r \geq 0 : x \preceq y + a_1^r,\, y \preceq x + a_2^r \text{ and } \|a_i^r\| \leq r,\ i = 1,2\right\}.$$

Since $x \preceq y + (x-y)$ and $y \preceq x + (y-x)$ for any elements $x, y \in X$, the quantity $h_X(x,y)$ is well defined. Also, it is not hard to see that the function $h_X$ satisfies all of the metric axioms and we should note that $h_X(x,y)$ may not equal to $\|x-y\|_X$ if $X$ is a nonlinear qls; however $h_X(x,y) \leq \|x-y\|_X$ is always true for any elements $x, y \in X$. Therefore, we use the metric instead of the norm to discuss topological properties in normed quasilinear spaces. For example, $x_n \to x$ if and only if $h_X(x_n, x) \to 0$ for the sequence $(x_n)$ in a normed qls. Although, always $\|x_n - x\|_X \to 0$ implies $x_n \to x$ in normed quasilinear spaces, $x_n \to x$ may not imply $\|x_n - x\|_X \to 0$.

Let $E$ be a real normed linear space. Then $\Omega(E)$ and $\Omega_C(E)$ are normed quasilinear spaces with the norm defined by

$$\|A\|_\Omega = \sup_{a \in A} \|a\|_E . \tag{2.19}$$

In this case, the Hausdorff metric is defined as usual:

$$h_\Omega(A,B) = \inf\{r \geq 0 : A \subseteq B + S(\theta, r),\, B \subseteq A + S(\theta, r)\},$$

where $S(\theta, r)$ is the closed ball of radius $r$ about $\theta \in X$, [2].

**Lemma 2.2.** [2] *The operations of algebraic sum and multiplication by real numbers are continuous with respect to the Hausdorff metric. The norm is a continuous function with respect to the Hausdorff metric.*

**Lemma 2.3.** [2] *Let $X$ be a normed qls and $n$ be a positive integer.*

- **a)** *Suppose that $x_n \to x_0$, $y_n \to y_0$ and $x_n \preceq y_n$ for any $n$. Then $x_0 \preceq y_0$.*
- **b)** *Let $x_n \to x_0$ and $z_n \to x_0$. If $x_n \preceq y_n \preceq z_n$ for any $n$, then $y_n \to x_0$.*
- **c)** *If $x_n + y_n \to x_0$ and $y_n \to \theta$, then $x_n \to x_0$.*

**Definition 2.3.** [2] Let $(X, \preceq)$ and $(Y, \preccurlyeq)$ be quasilinear spaces. A mapping $T : X \to Y$ is called a quasilinear operator if it satisfies the following three conditions:

$$T(\alpha \cdot x) = \alpha \cdot T(x) \text{ for any } \alpha \in \mathbb{R}, \tag{2.20}$$

$$T(x_1 + x_2) \preccurlyeq T(x_1) + T(x_2), \tag{2.21}$$

$$\text{if } x_1 \preceq x_2, \text{ then } T(x_1) \preccurlyeq T(x_2). \tag{2.22}$$

If $X$ and $Y$ are linear spaces, then the definition of a quasilinear operator coincides with the usual definiton of linear operator. In this case, condition (2.22) is automatically satisfied.

**Definition 2.4.** [2] Let $X$ and $Y$ be normed linear spaces. Any mapping from $X$ to $\Omega(Y)$ is called a multivalued mapping.

A quasilinear operator $T : X \to \Omega(Y)$ is called a multivalued quasilinear mapping. In this case, conditions (2.20) and (2.21) take the form

$$T(\alpha \cdot x) = \alpha \cdot T(x) \text{ for any } \alpha \in \mathbb{R},$$

$$T(x_1 + x_2) \subset T(x_1) + T(x_2).$$

Also, condition (2.22) is automatically satisfied.

On the other hand, any quasilinear operator from $X$ to $\Omega(\mathbb{R})$ is called a quasilinear functional.

## 3. Seminormed quasilinear spaces

In this section, we propose a generalization of normed quasilinear spaces. Let us start the following definition.

**Definition 3.1.** Let $(X, \preceq)$ be a qls. A real function $p : X \to \mathbb{R}$ is called a seminorm if the following conditions hold:

$$p(x) \geq 0 \text{ if } x \neq \theta, \tag{3.1}$$

$$p(x + y) \leq p(x) + p(y), \tag{3.2}$$

$$p(\alpha \cdot x) = |\alpha|\, p(x), \tag{3.3}$$

$$p(x) \leq p(y) \text{ if } x \preceq y. \tag{3.4}$$

A qls $X$ with a seminorm defined on it, is called seminormed quasilinear space (briefly, seminormed qls).

A seminorm $p$ is called total seminorm (or norm) if the condition

$$\begin{gathered}\text{“if for any } \varepsilon > 0 \text{ there exists an element } x_\varepsilon \in X \text{ such that} \\ x \preceq y + x_\epsilon \text{ and } p(x_\epsilon) \leq \epsilon \text{ then } x \preceq y\text{”}\end{gathered} \tag{3.5}$$

holds.

Note that this definition is inspired from the definition of norm presented by Aseev in [2] and every seminormed (normed) qls is a semimetric (metric) qls.

**Proposition 3.1.** *Let $(X, p)$ be a seminormed qls. Then the equality*

$$h_X(x, y) = \inf \{ r \geq 0 : x \preceq y + a_1^r,\ y \preceq x + a_2^r,\ p(a_i^r) \leq r,\ i = 1, 2 \} \tag{3.6}$$

*defines a semimetric on $X$. If $p$ is total, $h_X$ becomes a metric.*

**Proof.** First of all, we should note that the quantity $h_X$ is well defined since $x \preceq y + (x-y)$ and $y \preceq x + (y-x)$ for any elements $x, y \in X$.

Assume that $x = y$. Then $x \preceq y$ and $y \preceq x$. According to this,

$$x \preceq y + a_1^r \text{ and } y \preceq x + a_2^r$$

for $a_1^r = a_2^r = \theta$. That implies $h_X(x,y) = 0$ since $p(a_1^r) = p(a_2^r) = 0$.

Clearly $h_X$ is symmetric. Further, remembering that

$$h_X(x,z) = \inf\left\{r \geq 0 : x \preceq z + a_1^r,\ z \preceq x + a_2^r \text{ and } p(a_i^r) \leq \frac{r}{2},\ i = 1,2\right\}$$

and

$$h_X(z,y) = \inf\left\{r \geq 0 : y \preceq z + b_1^r,\ z \preceq y + b_2^r \text{ and } p(b_i^r) \leq \frac{r}{2},\ i = 1,2\right\},$$

we write $x \preceq y + a_1^r + b_2^r$ for every elements $a_1^r$ and $b_2^r$ such that $x \preceq z + a_1^r$ and $z \preceq y + b_2^r$.

Similarly, we can say $y \preceq x + a_2^r + b_1^r$ for every elements $a_2^r$ and $b_1^r$ such that $y \preceq z + b_1^r$ and $z \preceq x + a_2^r$. Since

$$p(a_1^r + b_2^r) \leq p(a_1^r) + p(b_2^r) \leq \frac{r}{2} + \frac{r}{2} = r$$

and

$$p(a_2^r + b_1^r) \leq p(a_2^r) + p(b_1^r) \leq \frac{r}{2} + \frac{r}{2} = r,$$

we get $h_X(x,y) \leq h_X(x,z) + h_X(z,y)$. Because

$$\begin{aligned} h_X(x,y) &= \inf\{r \geq 0 : x \preceq y + a_1^r + b_2^r,\ y \preceq x + a_2^r + b_1^r, \\ &\qquad p(a_1^r + b_2^r) \leq r \text{ and } p(a_2^r + b_1^r) \leq r\} \\ &\leq \inf\left\{r \geq 0 : x \preceq z + a_1^r,\ z \preceq x + a_2^r,\ p(a_i^r) \leq \frac{r}{2},\ i = 1,2\right\} \\ &+ \inf\left\{r \geq 0 : y \preceq z + b_1^r,\ z \preceq y + b_2^r,\ p(b_i^r) \leq \frac{r}{2},\ i = 1,2\right\} \\ &= h_X(x,z) + h_X(z,y). \end{aligned}$$

Hence the equality (3.6) defines a semimetric.

Now let us show that $h_X$ becomes a metric whenever that the seminorm $p$ is total:

Let $p$ be total and $h_X(x,y) = 0$. Then for any $\epsilon > 0$ there exist elements $x_1^\epsilon, x_2^\epsilon \in X$ such that $x \preceq y + x_1^\epsilon$, $y \preceq x + x_2^\epsilon$ and $p(x_i^\epsilon) \leq \epsilon$, $i = 1,2$. Hence the totality condition implies that $x \preceq y$ and $y \preceq x$, that is $x = y$. □

The function $h_X$ defined with the equality in (3.6) is called semimetric (metric) derived from the seminorm (total seminorm) $p$.

Let $h_X$ be semimetric (metric) derived from the seminorm (total seminorm) $p$. Then the inequality $h_X(x,y) \leq p(x-y)$ holds for every $x, y \in X$.

**Proposition 3.2.** *Let $(X,p)$ be a seminormed qls and $h_X$ be semimetric (metric) derived from the seminorm (total seminorm) $p$. Then we have*

*i)* $h_X(x+y,z+v) \leq h_X(x,z) + h_X(y,v)$,

*ii)* $h_X(\alpha \cdot x, \alpha \cdot y) = |\alpha| h_X(x,y)$,

*iii)* $p(x) = h_X(x,\theta)$

*for each $\alpha \in \mathbb{R}$ and every $x,y,z,v \in X$.*

**Proof.** Let us show that the inequality *i)* holds. Taking into account the definition of $h_X$ and $\inf A + \inf B \geq \inf A + B$, and using (2.12), we write

$$\begin{aligned}
&h_X(x,z) + h_X(y,v) \\
&= \inf\{r \geq 0 : x \preceq z + a_1^r,\ z \preceq x + a_2^r,\ p(a_i^r) \leq r/2,\ i = 1,2\} \\
&+ \inf\{r \geq 0 : y \preceq v + b_1^r,\ v \preceq y + b_2^r,\ p(b_i^r) \leq r/2,\ i = 1,2\} \\
&\geq \inf\left\{\begin{array}{c} r \geq 0 : x \preceq z + a_1^r,\ y \preceq v + b_1^r,\ z \preceq x + a_2^r,\ v \preceq y + b_2^r, \\ p(a_i^r) \leq r/2,\ p(b_i^r) \leq r/2,\ i = 1,2 \end{array}\right\} \\
&= \inf\left\{\begin{array}{c} r \geq 0 : x + y \preceq z + v + a_1^r + b_1^r,\ z + v \preceq x + y + a_2^r + b_2^r, \\ p(a_i^r + b_i^r) \leq r,\ i = 1,2 \end{array}\right\} \\
&= h_X(x+y, z+v).
\end{aligned}$$

The equalities *ii)* and *iii)* can be also easily obtained. □

**Proposition 3.3.** *Let $(X,p)$ be a seminormed qls, $x,y \in X$ and $h_X$ be semimetric (metric) derived from the seminorm (total seminorm) $p$. Then*

$$h_X(x,\theta) \leq h_X(y,\theta) \text{ if } x \preceq y. \tag{3.7}$$

*Further, quasilinear space operations are continuous with respect to the topology induced by $h_X$.*

**Proof.** Primarily, we say that $x \preceq y$ implies $p(x) \leq p(y)$ since $p$ is seminorm. Considering

$$p(x) = h_X(x,\theta) \text{ and } p(y) = h_X(y,\theta),$$

it is obtained $h_X(x,\theta) \leq h_X(y,\theta)$ whenever $x \preceq y$.

Since the topology derived from the semimetric $h_X$ is first countable topology, to say that addition and scalar multiplication operations are continuous, it will be sufficient to show that these operations are sequentially continuous.

For continuity of addition, let $(x_n)$ and $(y_n)$ be two sequences in $X$ such that $x_n \to x$ and $y_n \to y$. Then for every $\epsilon > 0$ there exists $N \in \mathbb{N}$ such that

$$x_n \preceq x + a_{1,n}^\epsilon,\ x \preceq x_n + a_{2,n}^\epsilon \text{ and } p\left(a_{i,n}^\epsilon\right) \leq \frac{\epsilon}{2},\ i = 1,2$$

and

$$y_n \preceq y + b^{\epsilon}_{1,n},\ y \preceq y_n + b^{\epsilon}_{2,n} \text{ and } p\left(b^{\epsilon}_{i,n}\right) \leq \frac{\epsilon}{2},\ i = 1, 2$$

whenever $n \geq N$. Taking into account that $p$ is seminorm and using (2.12), we can write

$$x_n + y_n \preceq x + y + a^{\epsilon}_{1,n} + b^{\epsilon}_{1,n},$$
$$x + y \preceq x_n + y_n + a^{\epsilon}_{2,n} + b^{\epsilon}_{2,n}$$

and

$$p\left(a^{\epsilon}_{1,n} + b^{\epsilon}_{1,n}\right) \leq p\left(a^{\epsilon}_{1,n}\right) + p\left(b^{\epsilon}_{1,n}\right) \leq \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon$$
$$p\left(a^{\epsilon}_{2,n} + b^{\epsilon}_{2,n}\right) \leq p\left(a^{\epsilon}_{2,n}\right) + p\left(b^{\epsilon}_{2,n}\right) \leq \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon.$$

These imply that $x_n + y_n \to x + y$.

Hence, it remains to show that multiplication operation is continuous. Let $(x_n)$ be a sequence in $X$ such that $x_n \to x$. Then for every $\epsilon > 0$ there exists $N \in \mathbb{N}$ such that

$$x_n \preceq x + a^{\epsilon}_{1,n},\ x \preceq x_n + a^{\epsilon}_{2,n} \text{ and } p\left(a^{\epsilon}_{i,n}\right) \leq \frac{\epsilon}{|\lambda|},\ \lambda \in \mathbb{R}^+,\ i = 1, 2$$

whenever $n \geq N$. Using (2.8), (2.12), (2.13) and the fact that $p$ is seminorm, we can say

$$\lambda \cdot x_n \preceq \lambda \cdot x + \lambda \cdot a^{\epsilon}_{1,n},$$
$$\lambda \cdot x \preceq \lambda \cdot x_n + \lambda \cdot a^{\epsilon}_{2,n}$$

and

$$p\left(\lambda \cdot a^{\epsilon}_{i,n}\right) \leq |\lambda|\, p\left(a^{\epsilon}_{i,n}\right) \leq \epsilon,\ i = 1, 2.$$

This implies that $\lambda \cdot x_n \to \lambda \cdot x$. □

Also, we note that the semimetric (metric) $h_X$ induced by a seminorm (total seminorm) on the qls $X$ is not translation invariant. But this semimetric (metric) satisfies the inequality

$$h_X(x + a, y + a) \leq h_X(x, y),\ a \in X.$$

Indeed,

$$h_X(x + a,\ y + a) \leq h_X(x, y) + h_X(a, a) = h_X(x, y).$$

Now let us present an example of seminorm function which is not a norm.

**Example 3.1.** Consider the qls $\Omega_C(\mathbb{R}^2)$ and the function

$$p(A) = \sup\{|x_2| : (x_1, x_2) \in A\}$$

for any $A \in \Omega_C(\mathbb{R}^2)$.

It is easy to see that $p$ holds seminorm axioms. On the other hand, $p$ is not a norm since $p(A) = 0$, for element $A = \{(t, 0) : -1 \leq t \leq 1\} \in \Omega_C(\mathbb{R}^2) \neq \theta$. Also the condition (2.18) is also not satisfied:

Let $A = \{(t, 0) : 0 \leq t \leq 2\}$, $B = \{(t, 0) : 0 \leq t \leq 1\}$ and $\epsilon > 0$ be arbitrary. Let us define as

$$A_\epsilon = \{(t + \epsilon, 0) : 0 \leq t \leq 1\}.$$

Then $p(A_\epsilon) = 0$ and $A \subset B + A_\epsilon$, but $A \subsetneq B$.

**Example 3.2.** The function $q(A) = \frac{p(A)}{1+p(A)}$ formed by aid of the seminorm $p$ in Example 3.1 is not a seminorm on $\Omega_C(\mathbb{R}^2)$, since

$$q(\lambda \cdot A) = \frac{p(\lambda \cdot A)}{1 + p(\lambda \cdot A)} = \frac{|\lambda|\, p(A)}{1 + |\lambda|\, p(A)} \neq \lambda q(A).$$

In the following, we give an example of a semimetric map that is not a metric.

**Example 3.3.** Let $A, B \in \Omega_C(\mathbb{R}^2)$ and

$$d(A, B) = \sup\left\{ \sqrt{|a_1 - b_1|} : (a_1, a_2) \in A,\ (b_1, b_2) \in B \right\}. \tag{3.8}$$

Firstly let us show that this formula defines a function from $\Omega_C(\mathbb{R}^2)$ to $\mathbb{R}$:

Consider the projection

$$p_1 : \mathbb{R}^2 \to \mathbb{R},\ p_1(a_1, a_2) = a_1$$

and remember that $p_1$ is continuous. $p_1(A)$ and $p_1(B)$ are compact subsets of $\mathbb{R}$ since $A$ and $B$ are compact in $\mathbb{R}^2$. Hence there exist the numbers $M_1, M_2 \geq 0$ such that $|x| \leq M_1$ for every $x \in p_1(A)$ and $|x| \leq M_2$ for every $x \in p_1(B)$. Therefore, since

$$\begin{aligned} &\sup\left\{ \sqrt{|a_1 - b_1|} : (a_1, a_2) \in A,\ (b_1, b_2) \in B \right\} \\ &= \sup\left\{ \sqrt{|a_1 - b_1|} : a_1 \in p_1(A),\ b_1 \in p_1(B) \right\} \end{aligned}$$

and $|x| \leq \sqrt{M_1 + M_2}$ for $x \in \left\{ \sqrt{|a_1 - b_1|} : a_1 \in p_1(A),\ b_1 \in p_1(B) \right\}$, the function $d$ is well defined.

It is easy to verify that $d$ is a semimetric. But $d$ is not a metric on $\Omega_C(\mathbb{R}^2)$. Indeed, for elements $A = \{(2, 3)\}$ and $B = \{(2, 4)\}$ in $\Omega_C(\mathbb{R}^2)$, $d(A, B) = 0$, but $A \neq B$.

On the other hand, we can show that the semimetric $d$ defined with (3.8) holds the condition (3.7) and the algebraic operations on $\Omega_C(\mathbb{R}^2)$ are continuous according to this semimetric.

For continuity of addition, let $(A_n)$ and $(B_n)$ be sequences in $\Omega_C(\mathbb{R}^2)$ such that $A_n \to A$, $B_n \to B$ and we take any $x_n \in A_n + B_n$. Then there exist $a_n \in A_n$ and $b_n \in B_n$ such that $x_n = a_n + b_n$.We can write as $a_n = (a_{1,n}, a_{2,n})$ and $b_n = (b_{1,n}, b_{2,n})$ since $A_n, B_n \in \Omega_C(\mathbb{R}^2)$. Because of $A_n \to A$, $B_n \to B$, we have

$$d(A_n, A) = \sup\left\{ \sqrt{|a_{1,n} - a_1|} : (a_{1,n}, a_{2,n}) \in A_n,\ (a_1, a_2) \in A \right\} \to 0$$

and

$$d(B_n, B) = \sup\left\{\sqrt{|b_{1,n} - b_1|} : (b_{1,n}, b_{2,n}) \in B_n,\ (b_1, b_2) \in B\right\} \to 0,$$

whenever $n \to \infty$. Hence, we obtain that

$$\begin{aligned}
&d(A_n + B_n, A + B) \\
&= \sup\left\{\sqrt{|a_{1,n} + b_{1,n} - (a_1 + b_1)|} : (a_{1,n} + b_{1,n}, a_{2,n} + b_{2,n}) \in A_n + B_n,\right. \\
&\qquad \left.(a_1 + b_1, a_2 + b_2) \in A + B\right\} \\
&\leq \sup\left\{\sqrt{|a_{1,n} - a_1|} : (a_{1,n}, a_{2,n}) \in A_n,\ (a_1, a_2) \in A\right\} \\
&+ \sup\left\{\sqrt{|b_{1,n} - b_1|} : (b_{1,n}, b_{2,n}) \in B_n,\ (b_1, b_2) \in B\right\} \\
&\to 0 + 0 = 0\ (n \to \infty).
\end{aligned}$$

This shows that the addition operation is continuous. Similarly it can be seen that the real-scalar multiplication operation is continuous.

Lastly it remains to show that the condition (3.7) is satisfied. Assume that $A \subseteq B$. Then

$$\begin{aligned}
d(A, \theta) &= \sup\left\{\sqrt{|a_1|}\ :\ (a_1, a_2) \in A\right\} \\
&\leq \sup\left\{\sqrt{|a_1|}\ :\ (a_1, a_2) \in B\right\} \\
&= d(B, \theta).
\end{aligned}$$

**Remark 3.1.** Every semimetric on a qls may not be obtained from a seminorm. In Example 3.3, if the semimetric defined on $\Omega_C(\mathbb{R}^2)$ is obtained from a seminorm, the property *ii)* in Proposition 3.2 should hold. However, we see that

$$\begin{aligned}
d(\lambda \cdot A, \lambda \cdot B) &= \sup\left\{\sqrt{|\lambda \cdot a_1 - \lambda \cdot b_1|} : (\lambda \cdot a_1, \lambda \cdot a_2) \in \lambda \cdot A,\ (\lambda \cdot b_1, \lambda \cdot b_2) \in \lambda \cdot B\right\} \\
&= \sqrt{|\lambda|} \sup\left\{\sqrt{|a_1 - b_1|} : (a_1, a_2) \in A,\ (b_1, b_2) \in B\right\} \\
&= \sqrt{|\lambda|} d(A, B).
\end{aligned}$$

The following proposition is a comment of the condition (2.18).

**Proposition 3.4.** *Let $X$ be a normed qls, $\mathcal{N}_\theta$ is the family of all neighbourhoods of $\theta$ and $x, y \in X$. If for any $V \in \mathcal{N}_\theta$ there exists some $b \in V$ such that $x \preceq y + b$, then $x \preceq y$.*

**Remark 3.2.** In Proposition 3.4, the hypothesis "Let X be a normed qls" is indispensable. Indeed, Let us recall from Example 3.1 that the function

$$p(A) = \sup\left\{|x_2| : (x_1, x_2) \in A\right\},\ A \in \Omega_C(\mathbb{R}^2)$$

is seminorm on $\Omega_C(\mathbb{R}^2)$. We can construct a topology $\tau$ on $\Omega_C(\mathbb{R}^2)$ by aid of $p$ in such a way that

$$U \in \tau \Leftrightarrow \{A : p(A) < \epsilon\} \subseteq U, \text{ for some } \epsilon > 0.$$

We note that $\tau$ is a semimetrizable topology with the semimetric

$$d(A,B) = \inf\{r \geq 0 : A \subseteq B + C_1^r,\ B \subseteq A + C_2^r,\ p(C_i^r) \leq r,\ i = 1,2\}.$$

Now, let $A = \{(t,0) \in \mathbb{R}^2 : 0 \leq t \leq 2\}$, $B = \{(t,0) \in \mathbb{R}^2 : 0 \leq t \leq 1\}$, $\epsilon > 0$ be arbitrary and

$$B_\epsilon = \{(x_1,x_2) \in \mathbb{R}^2 : 0 \leq x_1 \leq 1,\ 0 \leq x_2 < \epsilon\}.$$

Then there exists $B_\epsilon \in V$ for every $V \in \mathcal{N}_\theta$ such that $A \subset B + B_\epsilon$. However $A \nsubseteq B$.

**Remark 3.3.** In Lemma 2.3, the hypothesis "Let X be a normed qls" can not be relaxed. Indeed, let us recall that every linear space is a qls with the partial order relation "=" and consider the element $x = (x_1,x_2)$ and the seminorm

$$p(x) = p\left((x_1,x_2)\right) = \left\{|x_1| : (x_1,x_2) \in \mathbb{R}^2\right\}$$

on the qls $(\mathbb{R}^2, =)$.

Let $(x_n) = \left(\left(\frac{1}{n},0\right)\right)_{n=1}^{\infty}$ and $(y_n) = \left(\left(\frac{1}{n},0\right)\right)_{n=1}^{\infty}$.

We see that $x_n = y_n$ for every $n$. On the other hand, the sequence $\left(\left(\frac{1}{n},0\right)\right)_{n=1}^{\infty}$ converges to different two elements of $\mathbb{R}^2$ according to this seminorm. For example,

$$(x_n) = \left(\left(\frac{1}{n},0\right)\right)_{n=1}^{\infty} \to (0,1) = x$$

and

$$(y_n) = \left(\left(\frac{1}{n},0\right)\right)_{n=1}^{\infty} \to (0,2) = y$$

since

$$p\left(\left(\frac{1}{n},0\right) - (0,1)\right) = p\left(\left(\frac{1}{n},-1\right)\right) = \left|\frac{1}{n}\right| \to 0$$

and

$$p\left(\left(\frac{1}{n},0\right) - (0,2)\right) = p\left(\left(\frac{1}{n},-2\right)\right) = \left|\frac{1}{n}\right| \to 0$$

while $n \to \infty$. However $x \neq y$.

Now, let us denote by $\Omega_C^n(\mathbb{R})$ the family of all $n-$tuples intervals which constitute an important part of interval analysis.

$$\Omega_C^n(\mathbb{R}) = \{X = (X_1, X_2, ..., X_n) : X_i \in \Omega_C(\mathbb{R}) \text{ for } 1 \leq i \leq n\}.$$

We emphasize that $\Omega_C^n(\mathbb{R})$ is different from $\Omega_C(\mathbb{R}^n)$ which is the family of all nonempty closed, bounded and convex subsets of $\mathbb{R}^n$.

$\Omega_C^n(\mathbb{R})$ is a qls with the operations "$\oplus$", "$\odot$" and partial order relation "$\preceq$" defined by

$$\oplus : \Omega_C^n(\mathbb{R}) \times \Omega_C^n(\mathbb{R}) \to \Omega_C^n(\mathbb{R}),$$
$$X \oplus Y = (X_1 + Y_1, X_2 + Y_2, ..., X_n + Y_n)$$

and

$$\odot : \mathbb{R} \times \Omega_C^n(\mathbb{R}) \to \Omega_C^n(\mathbb{R}),$$
$$\alpha \odot X = (\alpha \cdot X_1, \alpha \cdot X_2, ..., \alpha \cdot X_n)$$

and

$$X \preceq Y \Leftrightarrow X_i \subseteq Y_i \text{ for every } i \in \{1, 2, ..., n\}$$

for $X = (X_1, X_2, ..., X_n)$, $Y = (Y_1, Y_2, ..., Y_n) \in \Omega_C^n(\mathbb{R})$ and $\alpha \in \mathbb{R}$.

$\Omega_C^n(\mathbb{R})$ is a seminormed qls with equality defined by

$$\|X\|_{\Omega_C^n(\mathbb{R})} = \|X_i\|_{\Omega_C(\mathbb{R})}$$

for fixed $i \in \{1, 2, ..., n\}$.

For example, on seminormed qls $\Omega_C^3(\mathbb{R})$, the equality

$$\|X\|_{\Omega_C^3(\mathbb{R})} = \|X_1\|_{\Omega_C(\mathbb{R})}$$

defines a seminorm. It is not hard to see that seminorm axioms are hold. The function defined by this way is not a norm since $\|X\| = 0$ for element

$$X = \{[0, 0], [1, 3], [-3, -2]\} \in \Omega_C^3(\mathbb{R}) \neq \theta.$$

**Example 3.4.** Also the condition (2.18) is also not satisfied:

Let $X = \{[1, 2], [3, 5], [-4, -3]\}$, $Y = \{[1, 2], [4, 6], [3, 5]\}$ and $\epsilon > 0$ be arbitrary. Let us define as

$$X_\epsilon = \{[0, 0], [-2, 1], [-8, -7]\}.$$

Then $\|X_\epsilon\| = 0$ and $X \preceq Y + X_\epsilon$, but $X \npreceq Y$.

We note that $T$ will be called as a linear operator between quasilinear spaces, if $T$ satisfies the following conditions:

$$T(\alpha \cdot x) = \alpha \cdot T(x) \text{ for any } \alpha \in \mathbb{R}, \tag{3.9}$$

$$T(x_1 + x_2) = T(x_1) + T(x_2). \tag{3.10}$$

Also, any linear operator from the quasilinear space $X$ to $\mathbb{R}$ is called a linear functional on quasilinear space $X$.

The Hahn-Banach theorem is an important tool in functional analysis and there are several versions of it. Let us note that we are largely inspired by Theorem 2.2 in [4], in stating the Hahn-Banach theorem for seminormed quasilinear spaces. The impact of the Hahn-Banach theorem is the existence of linear functionals having specified properties on a quasilinear space. The following is the main result of our work.

**Theorem 3.1.** *Let $p$ be a seminorm on the quasilinear space $X$ and $Y$ be a subspace of $X$. Suppose that $f$ is a linear functional from $Y$ to $\mathbb{R}$ and $f(y) \leq p(y)$ for all $y \in Y$. Suppose also that $\varphi$ is a quasilinear functional from $X$ to $\Omega_C(\mathbb{R})$ and $f(x) \in \varphi(x)$ for every $x \in Y$. Then there exists a linear functional $g$ from $X$ to $\mathbb{R}$ such that $g(x) = f(x)$ for any $x \in Y$ and $g(x) \in \varphi(x)$ for any $x \in X$.*

**Proof.** Let $Z$ be a subspace of $X$ containing $Y$, $g$ be a linear functional on $Z$ that extends $f$, and $g(z) \leq p(z)$ for all $z \in Z$. Also, let $\mathcal{Z}$ be the set of all pairs $(Z, g)$. First of all, since the pair $(Y, f)$ is obviously an element of $\mathcal{Z}$, the set $\mathcal{Z}$ is not empty.

Define a partial order relation "$\ll$" on $\mathcal{Z}$ as follows:

$$(Z_1, g_1) \ll (Z_2, g_2) \Leftrightarrow \begin{cases} Z_1 \subset Z_2 \\ g_2(z) = g_1(z), \text{ for all } z \in Z_1 \end{cases}.$$

Using Zorn Lemma, $\mathcal{Z}$ posesses a maximal totally ordered subset $\{(Z_\alpha, g_\alpha)\}$. If it is defined as $Z = \bigcup Z_\alpha$, clearly, $Z$ is a subspace of $X$. Also, if $z \in Z$, then $z \in Z_\alpha$ for some $\alpha$.

If $z \in Z_\alpha$ and $z \in Z_\beta$, then, without loss of generality, we may assume that $(Z_\alpha, g_\alpha) \ll (Z_\beta, g_\beta)$. Therefore $g_\alpha(z) = g_\beta(z)$, so that we may uniquely define $g(z) = g_\alpha(z)$ whenever $z \in Z_\alpha$.

Now, let us show that the function $g$ defined by this way is a linear functional on $Z$. To do this, let $z_1$ and $z_2$ be elements of $Z$. Then $z_1 \in Z_\alpha$ and $z_2 \in Z_\beta$ for some $\alpha$ and $\beta$. Since the set $\{(Z_\gamma, g_\gamma)\}$ is totally ordered, we may assume, again without loss of generality, that $Z_\alpha \subset Z_\beta$, hence both $z_1$ and $z_2$ are in $Z_\beta$. So

$$g(\lambda_1 \cdot z_1 + \lambda_2 \cdot z_2) = g_\beta(\lambda_1 \cdot z_1 + \lambda_2 \cdot z_2) = \lambda_1 g_\beta(z_1) + \lambda_2 g_\beta(z_2) = \lambda_1 g(z_1) + \lambda_2 g(z_2).$$

We note that if $y \in Y$, then $g(y) = f(y)$, so that $g$ is an extension of $f$. So, $g$ is a linear functional on the subspace $Z$, that extends $f$, for which $g(z) \leq p(z)$ and $g(z) \in \varphi(z)$ for all $z \in Z$, so that the proof will be complete if we show that $Z = X$.

Assume that $Z \neq X$, and $v$ be an element in $X$ which is not in $Z$. Also, $Z'$ denotes the set of all elements in the form $z + \lambda \cdot v$ for $\lambda \in \mathbb{R}$ and $z \in Z$.

On the other hand, since

$$\theta = (1-1) \cdot v \preceq v - v$$

and

$$z + z' \preceq z + z' \text{ for any } z, z' \in Z,$$

we write $z + z' \preceq z + z' + v - v$ from (2.12). Also $p(z + z') \leq p(z + z' + v - v)$ by the fact that $p$ is a seminorm. Therefore, we observe

$$\begin{aligned} g(z) + g(z') &= g(z + z') \\ &\leq p(z + z') \\ &\leq p(z + z' + v - v) \\ &\leq p(z + v) + p(z' - v) \end{aligned}$$

or

$$g(z') - p(z' - v) \leq p(z + v) - g(z)$$

for any $z, z' \in Z$.

Consider the sets

$$W_1 = \{g(z') - p(z' - v) : z' \in Z\} \subset \mathbb{R},$$

$$W_2 = \{p(z + v) - g(z) : z \in Z\} \subset \mathbb{R}$$

and say

$$\sup W_1 = w_1 \text{ and } \inf W_2 = w_2.$$

It is clear that $w_1 \leq w_2$. Take $w_0$ to be any number for which $w_1 \leq w_0 \leq w_2$ and define $g'$ on $Z'$ by

$$g'(z + \lambda \cdot v) = g(z) + \lambda \cdot w_0.$$

It is easy to see that $g'$ is linear and extends $f$.

If $\lambda > 0$, then

$$\begin{aligned}
g'(z + \lambda \cdot v) &= \lambda \left( g\left(\frac{z}{\lambda}\right) + w_0 \right) \\
&\leq \lambda \left( g\left(\frac{z}{\lambda}\right) + w_2 \right) \\
&\leq \lambda \left( g\left(\frac{z}{\lambda}\right) + p\left(\frac{z}{\lambda} + v\right) - g\left(\frac{z}{\lambda}\right) \right) \\
&= \lambda p\left(\frac{z}{\lambda} + v\right) \\
&= p(z + \lambda \cdot v).
\end{aligned}$$

On the other hand, if $\lambda < 0$, then

$$\begin{aligned}
g'(z + \lambda \cdot v) &= |\lambda| \left( g\left(\frac{z}{|\lambda|}\right) - w_0 \right) \\
&\leq |\lambda| \left( g\left(\frac{z}{|\lambda|}\right) - w_1 \right) \\
&\leq |\lambda| \left( g\left(\frac{z}{|\lambda|}\right) - g\left(\frac{z}{|\lambda|}\right) + p\left(\frac{z}{|\lambda|} - v\right) \right) \\
&= |\lambda| \, p\left(\frac{z}{|\lambda|} - v\right) \\
&= p(z + \lambda \cdot v).
\end{aligned}$$

This proves $g'(z + \lambda \cdot v) \leq p(z + \lambda \cdot v)$ for all $z + \lambda \cdot v \in Z'$. Hence $(Z', g') \in \mathcal{Z}$ and $(Z, g) \ll (Z', g')$. But then the element $(Z', g') \in \mathcal{Z}$ will contradicts with the maximality of $(Z, g)$ by the fact that $\{(Z_\alpha, g_\alpha)\}$ is a maximal totally ordered set. This completes the proof. $\square$

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**Sümeyye Çakan**
email: `sumeyye.tay@gmail.com`
ORCID: 0000-0001-8761-8564
Department of Mathematics
İnönü University
Malatya 44280
TURKEY

**Yılmaz Yılmaz**
email: `yyilmaz44@gmail.com`
ORCID: 0000-0002-2197-3579
Department of Mathematics
İnönü University
Malatya 44280
TURKEY

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# Nonincreasing Solutions for Quadratic Integral Equations of Convolution Type

*W.G. El-Sayed and A.A.H. Abd El-Mowla*

ABSTRACT: We study a nonlinear quadratic integral equation of Convolution type in the Banach space of real functions defined and continuous on a bounded and closed interval. By using a suitable measure of noncompactness, we show that the integral equation has monotonic solutions.

*AMS Subject Classification:* 45G10, 47H30, 47N20.
*Keywords and Phrases:* Quadratic integral equation; Measure of noncompactness; Fixed-point theorem; Monotonic solutions.

## 1. Introduction and preliminaries

There are many results in nonlinear functional analysis which contain conditions with the measure of noncompactness. Integral equations are one of the most useful mathematical tools in both pure and applied analysis. This is particulary true of problems in mechanical vibrations and the related fields of engineering and mathematical physics. The theory of integral equations is rapidly developing with the help of several tools of functional analysis, topology and fixed point theory. For details, we refer to ([1]-[23]) and the references therein.
The goal of this paper is to study the solvability of the following nonlinear quadratic integral equation

$$x(t) = g(t) + (Tx)(t) \int_0^t k(t,s) f(s, x(\varphi(s)))\, ds,\ t \in [0, M], \tag{1.1}$$

in the Banach space of real functions being defined and continuous on a bounded and closed interval. The main tool used to study the existence solutions of that equation in the class of monotonic functions is a special measure of noncompactness.

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Now, we collect some facts, basic concepts and sketch some useful theorems which will be needed further on. Let $(E, \| \, . \, \|)$ be an infinite dimensional Banach space with zero element $\theta$. Denote by $B(x, r)$ the closed ball in $E$ centered at $x$ with radius $r$. The symbol $B_r$ stands for the ball $B(\theta, r)$. If $X$ is a nonempty subset of $E$, then $\overline{X}$ and $ConvX$ denote the closure and the convex closure of $X$, respectively. Moreover, the symbol $m_E$ denotes the family of all nonempty and bounded subsets of $E$ while $n_E$ stands for its subfamily consisting of all relatively compact sets.
We will accept the following definition of the concept of a measure of noncompactness [4].

**Definition 1.** A mapping $\mu : m_E \to [0, \infty)$ is said to be a measure of noncompactness in $E$ if the following conditions are satisfied:

1. the family $\ker \mu = \{X \in m_E : \mu(X) = 0\}$ is nonempty and $\ker \mu \subset n_E$.
2. $X \subset Y \Rightarrow \mu(X) \leq \mu(Y)$.
3. $\mu(X) = \mu(\overline{X}) = \mu(ConvX)$.
4. $\mu(\lambda X + (1 - \lambda)Y) \leq \lambda \mu(X) + (1 - \lambda)\mu(Y)$ for $\lambda \in [0, 1]$.
5. If $(X_n), n \in N$ is sequence of closed sets from $m_E$ such that $X_{n+1} \subset X_n$ and if $\lim_{n \to \infty} \mu(X_n) = 0$, then the set $X_\infty = \bigcap_{n=1}^{\infty} X_n$ is nonempty.

The family $\ker(\mu)$ describe in 1 is referred to as the kernel of the measure of noncompactness $\mu$.
A measure $\mu$ is called sublinear if it satisfies the following two conditions:

6. $\mu(\lambda X) = |\lambda| \; \mu(X)$, for $\lambda \in R$
7. $\mu(X + Y) \leq \mu(X) + \mu(Y)$.

Moreover, a measure $\mu$ is called a measure with maximum property if

8. $\mu(X \bigcup Y) = \max \; [ \; \mu(X), \mu(Y) \; ]$.

Other facts concerning measures of noncompactness and their properties may be found in [4].

**Definition 2.** {Darbo condition} Let $M$ be a nonempty subset of a Banach space $E$ and the operator $F : M \to E$ is continuous and transforms bounded sets onto bounded ones, then $F$ satisfies the Darbo condition with constant $k \geq 0$ with respect to a measure of noncompactness $\mu$ if for any bounded subset $X$ of $M$ we have

$$\mu(FX) \leq k\mu(X).$$

If $F$ satisfies the Darbo condition with $k < 1$, then it is called a contraction with respect to $\mu$. Next, we need the following fixed point theorem ([4], [16]).

**Theorem 1.** *Let $Q$ be nonempty bounded closed convex subset of the space $E$ and let $F : Q \to Q$ be continuous and such that $\mu(FX) \leq k\mu(X)$ for any nonempty subset $X$ of $Q$, where $k$ is a constant, $k \in [0, 1)$. Then $F$ has a fixed point in the set $Q$.*

**Remark 1.** Under the assumptions of the above theorem, it can be shown that the set $fixF$ of fixed points of $F$ belonging to $Q$ is a member of the family $\ker \mu$ [4]. This fact permits us to characterize solutions of considered operator equations.

We will work in the classical Banach space $C[0, M]$ consisting of all real functions defined and continuous on the interval $[0, M]$. For convenience, we write $I = [0, M]$ and $C(I) = C[0, M]$. The space $C(I)$ is furnished by the standard norm

$$\| x \| = \max\{| x(t) | : t \in I\}.$$

Now, we will display the definition of a measure of noncompactness in $C(I)$. That measure was introduced and studied in [5].
To do this, let us fix a nonempty and bounded subset $X$ of $C(I)$. For $x \in X$ and $\epsilon \geq 0$ denoted by $\omega(x, \epsilon)$, the modulus of continuity of the function $x$, i.e.,

$$\omega(x, \epsilon) = \sup\{| x(t) - x(s) | : t, s \in I, | t - s | \leq \epsilon\}.$$

Further, let us put

$$\omega(X, \epsilon) = \sup\{\omega(x, \epsilon) : x \in X\},$$

$$\omega_0(X) = \lim_{\epsilon \to 0} \omega(X, \epsilon).$$

Now, let us define the following quantities:

$$d(x) = \sup\{| x(s) - x(t) | - [x(s) - x(t)] : t, s \in I, t \leq s\},$$

$$i(x) = \sup\{| x(t) - x(s) | - [x(t) - x(s)] : t, s \in I, t \leq s\},$$

$$d(X) = \sup\{d(x) : x \in X\},$$

$$i(X) = \sup\{i(x) : x \in X\}.$$

Observe that $d(X) = 0$ if and only if all functions belonging to $X$ are nondecreasing on $I$. In a similar way, we can characterize the set $X$ with $i(X) = 0$.
Finally, we define the function $\mu$ on the family $m_{C(I)}$ by putting

$$\mu(X) = \omega_0(X) + d(X). \tag{1.2}$$

It can be shown [5] that the function $\mu$ is a measure of noncompactness in the space $C(I)$. The kernel $\ker \mu$ of this measure contains nonempty and bounded sets $X$ such that functions from $X$ are equicontinuous and nondecreasing on the interval $I$.

**Remark 2.** By properties of the kernel $\ker \mu$ of the measure of noncompactness $\mu$ together with Remark 1 allow us to characterize solutions of the nonlinear integral equation considered in the next section.

**Remark 3.** Observe that, in a similar way, we can define the measure of noncompactness associated with the set quantity $i(X)$ define above.

## 2. Main result

In this section, we will study the nonlinear quadratic integral equation of Volterra type having the form

$$x(t) = g(t) + \ (Tx)(t) \int_0^t k(t,s) f(s, x(\varphi(s)))\ ds, \quad t \in I.$$

Assume that the following conditions are satisfied:

(i) $g \in C(I)$ is nonincreasing and nonnegative on the interval $I$;

(ii) $f : I \times R_+ \to R_+$ is continuous and there exists a nondecreasing function $m : R_+ \to R_+$ such that the inequality

$$| f(s,x) | \leq m(| x |)$$

holds for all $s \in I$ and $x \in R$;

(iii) The operator $T : C(I) \to C(I)$ is continuous and satisfies the Darbo condition for the measure of noncompactness $\mu$ with a constant $a \geq 0$. Moreover, $T$ is a positive operator, i.e. $Tx \geq 0$ if $x \geq 0$;

(iv) There exists a nonnegative constant $q$ such that

$$| (Tx)(t) | \leq q \parallel x \parallel$$

for each $x \in C(I)$ and $t \in I$;

(v) $k : I \times I \to R_+$ is integrable and nonincreasing in the first argument and

$$K = \sup\{ \int_0^t | k(t,s) | \ \ ds : t, s \in I\};$$

(vi) $\varphi : I \to I$ is increasing and continuous function;

(vii) There exists $r_o > 0$ with $\parallel g \parallel + Kqm(r_o)r_o < r_o$ and $Km(r_0)a < 1$.

Now, we are ready to state the existence theorem.

**Theorem 2.** *Let the assumptions* (i)-(vii) *be satisfied, then equation* (1.1) *has at least one positive and nonincreasing solution* $x \in C(I)$.

**Proof.** Let us consider the operators $V, G$ defined on the space $C(I)$ in the following way:

$$(Vx)(t) = g(t) + (Tx)(t) \int_0^t k(t,s) f(s, x(\varphi(s)))\ ds, \quad t \in [0, M]$$

and

$$(Gx)(t) \ = \ \int_0^t k(t,s) f(s, x(\varphi(s)))\ ds.$$

Firstly, we prove that if $x \in C(I)$ then $Vx \in C(I)$. To do this it is sufficient to show that if $x \in C(I)$ then $Gx \in C(I)$. Fix $\epsilon > 0$, let $x \in C(I)$ and $t, s \in I$ such that $t \leq s$ and $\mid t - s \mid \leq \epsilon$. Then

$$\begin{aligned}
(Gx)(s) - (Gx)(t) \ &= \ \int_0^s k(s,\tau) f(\tau, x(\varphi(\tau)))\ d\tau \ - \ \int_0^t k(t,\tau) f(\tau, x(\varphi(\tau)))\ d\tau \\
&= \ \int_0^t k(s,\tau) f(\tau, x(\varphi(\tau)))\ d\tau \ + \ \int_t^s k(s,\tau) f(\tau, x(\varphi(\tau)))\ d\tau \\
&- \ \int_0^t k(t,\tau) f(\tau, x(\varphi(\tau)))\ d\tau \ = \ \int_0^t k(s,\tau) f(\tau, x(\varphi(\tau)))\ d\tau \\
&- \ \int_0^t k(t,\tau) f(\tau, x(\varphi(\tau)))\ d\tau \ + \ \int_t^s k(s,\tau) f(\tau, x(\varphi(\tau)))\ d\tau \\
&\leq \ \int_0^t k(s,\tau) f(\tau, x(\varphi(\tau)))\ d\tau \ - \ \int_0^t k(s,\tau) f(\tau, x(\varphi(\tau)))\ d\tau \\
&\quad + \ \int_t^s k(s,\tau) f(\tau, x(\varphi(\tau)))\ d\tau
\end{aligned}$$

Now

$$\begin{aligned}
\mid (Gx)(s) - (Gx)(t) \mid \ &\leq \ \int_t^s \mid k(s,\tau) f(\tau, x(\varphi(\tau))) \mid \ d\tau \\
&\leq \ \int_t^s k(s,\tau) m(\mid x(\varphi(\tau)) \mid)\ d\tau \\
&\leq \ \int_t^s k(s,\tau) m(\parallel x \parallel)\ d\tau
\end{aligned}$$

we obtain that

$$\mid (Gx)(s) - (Gx)(t) \mid \ \leq \ m(\parallel x \parallel) \int_t^s k(s,\tau)\ d\tau$$

Now, in virtue of the Lebesgue dominated Theorem we have that $\int_t^s k(s,\tau)\, d\tau \to 0$ as $\epsilon \to 0$. Thus $Gx \in C(I)$ and consequentially, $Vx \in C(I)$. Moreover, for each $t \in I$ we have

$$\begin{aligned}
\mid (Vx)(t) \mid &\leq \mid g(t) \mid + \mid (Tx)(t) \mid \ \mid \int_0^t k(t,s) f(s, x(\varphi(s)))\, ds \mid \\
&\leq \parallel g \parallel + q \parallel x \parallel \int_0^t k(t,s) m(\mid x(\varphi(s)) \mid)\, ds \\
&\leq \parallel g \parallel + q \parallel x \parallel m(\parallel x \parallel) \int_0^t k(t,s)\, ds \\
&\leq \parallel g \parallel + Kqm(\parallel x \parallel) \parallel x \parallel .
\end{aligned}$$

Hence,

$$\parallel Vx \parallel \ \leq \ \parallel g \parallel + Kqm(\parallel x \parallel) \parallel x \parallel .$$

Thus, if $\parallel x \parallel \leq r_o$ we obtain from assumption (vii) that

$$\parallel Vx \parallel \ \leq \ \parallel g \parallel + K\ q\ m(r_o)\ r_o \ \leq \ r_o.$$

As a result the operator $V$ transforms the ball $B_{r_0}$ into itself.
In what follows, we will consider the operator $V$ on the subset $B_{r_0}^+$ of the ball $B_{r_0}$ defined in the following way :

$$B_{r_0}^+ \ = \ \{x \in B_{r_0} : x(t) \geq 0, \text{for} \ \ t \in I\}.$$

Obviously, the set $B_{r_0}^+$ is nonempty, bounded, closed and convex. Let $x \in B_{r_0}^+$. Notice that in view of our assumptions (i)-(iv) if $x(t) \geq 0$ then $(Vx)(t) \geq 0$ for $t \in I$. Thus $V$ transforms the set $B_{r_0}^+$ into itself.

Now, we show that $V$ is continuous on the set $B_{r_0}^+$. To do this, let us fix $\epsilon > 0$ and take arbitrary $x, y \in B_{r_0}^+$ such that $\parallel x - y \parallel \leq \epsilon$. Then, for $t \in I$, we derive the following estimates:

$$\begin{aligned}
&\mid (Vx)(t) - (Vy)(t) \mid \\
&= \mid (Tx)(t) \int_0^t k(t,s) f(s, x(\varphi(s))) ds \ - \ (Ty)(t) \int_0^t k(t,s) f(s, y(\varphi(s)))\, ds \mid \\
&\leq \mid (Tx)(t) \int_0^t k(t,s) f(s, x(\varphi(s)))\, ds \ - \ (Ty)(t) \int_0^t k(t,s) f(s, x(\varphi(s)))\, ds \mid \\
&+ \mid (Ty)(t) \int_0^t k(t,s) f(s, x(\varphi(s)))\, ds \ - \ (Ty)(t) \int_0^t k(t,s) f(s, y(\varphi(s)))\, ds \mid \\
&\leq \mid (Tx)(t) - (Ty)(t) \mid \int_0^t k(t,s) f(s, x(\varphi(s)))\, ds \\
&+ \mid (Ty)(t) \mid \int_0^t k(t,s) \ \mid f(s, x(\varphi(s))) \ - \ f(s, y(\varphi(s))) \mid \ ds
\end{aligned}$$

$$\begin{aligned}
&\leq \ \| Tx - Ty \| \int_0^t k(t,s) m(| x(\varphi(s)) |) \, ds \\
&+ \ q \ \| y \| \int_0^t k(t,s) \beta_{r_0}(\epsilon) \, ds \\
&\leq \ \| Tx - Ty \| \int_0^t k(t,s) m(\| x \|) \, ds \ + \ q \| y \| \beta_{r_0}(\epsilon) \int_0^t k(t,s) \, ds \\
&\leq \ Km(\| x \|) \| Tx - Ty \| \ + \ Kq \| y \| \beta_{r_0}(\epsilon), \\
&\leq \ Km(r_o) \| Tx - Ty \| \ + \ Kqr_0 \beta_{r_0}(\epsilon),
\end{aligned}$$

where we denoted

$$\beta_{r_0}(\epsilon) \ = \ \sup\{| f(u, x(u)) - f(u, y(u)) | : u \in I, x, y \in [0, r_0], \| x - y \| \leq \epsilon\}.$$

Obviously, $\beta_{r_0}(\epsilon) \to 0$ as $\epsilon \to 0$ which is a simple consequence of the uniform continuity of the function $f$ on $I \times [0, r_0]$.
From the above estimate, we can write the following inequality:

$$\| Vx - Vy \| \ \leq \ Km(r_o) \| Tx - Ty \| + Kqr_0 \beta_{r_0}(\epsilon),$$

which implies the continuity of the operator $V$ on the set $B^+_{r_0}$.
In what follows, let us take a nonempty set $X \subset B^+_{r_0}$. Further, fix arbitrary a number $\epsilon > 0$ and choose $x \in X$ and $t, s \in I$ such that $| t - s | \leq \epsilon$ and $t \leq s$. Then by our assumptions we have

$$\begin{aligned}
| (Vx)(s) - (Vx)(t) | \ &\leq \ | g(s) - g(t) | \\
&+ \ | (Tx)(s) \int_0^s k(s,\tau) f(\tau, x(\varphi(\tau))) \, d\tau - (Tx)(t) \int_0^t k(t,\tau) f(\tau, x(\varphi(\tau))) \, d\tau | \\
&\leq \ | g(s) - g(t) | + | (Tx)(s) \int_0^s k(s,\tau) f(\tau, x(\varphi(\tau))) \, d\tau \\
&- \ (Tx)(t) \int_0^s k(s,\tau) f(\tau, x(\varphi(\tau))) \, d\tau | \\
&+ \ | (Tx)(t) \int_0^s k(s,\tau) f(\tau, x(\varphi(\tau))) \, d\tau - (Tx)(t) \int_0^s k(t,\tau) f(\tau, x(\varphi(\tau))) \, d\tau | \\
&+ \ | (Tx)(t) \int_0^s k(t,\tau) f(\tau, x(\varphi(\tau))) \, d\tau - (Tx)(t) \int_0^t k(t,\tau) f(\tau, x(\varphi(\tau))) \, d\tau |
\end{aligned}$$

$$
\begin{aligned}
&\leq \quad | g(s) - g(t) | + | (Tx)(s) - (Tx)(t) | \int_0^s k(s,\tau)m(| x(\varphi(\tau)) |) \ d\tau \\
&+ \quad | (Tx)(t) | \int_0^s | k(s,\tau) - k(t,\tau) | \ m(| x(\varphi(\tau)) |) \ d\tau \\
&+ \quad | (Tx)(t) | \int_t^s k(t,\tau)m(| x(\varphi(\tau)) |) \ d\tau \\
&\leq \quad \omega(g,\epsilon) \ + \ m(\| x \|)\omega(Tx,\epsilon) \int_0^s k(s,\tau) \ d\tau \\
&+q \quad \| x \| \ m(\| x \|) \int_0^s | k(s,\tau) - k(s,\tau) | \ \ d\tau \\
&+ \quad q \ \ \| x \| \ m(\| x \|) \int_t^s k(t,\tau) \ d\tau \\
&\leq \quad \omega(g,\epsilon) + Km(\| x \|)\omega(Tx,\epsilon) + q \quad \| x \| \ m(\| x \|) \int_t^s k(t,\tau) \ d\tau \\
&\leq \quad \omega(g,\epsilon) \ + Km(r_0)\omega(Tx,\epsilon) \ + \ qr_o m(r_0) \int_t^s k(t,\tau) \ d\tau.
\end{aligned}
$$

Hence

$$
\omega_o(VX) \ \leq \ Km(r_o) \ \omega_o(TX) \tag{2.1}
$$

Now, fix arbitrarily $x \in X$ and $t, s \in I$ such that $t \leq s$. Then we have the following chain of estimates:

$$
\begin{aligned}
&| (Vx)(t) - (Vx)(s) | - [(Vx)(t) - (Vx)(s)] \\
&= \quad | g(t) + (Tx)(t) \int_0^t k(t,\tau)f(\tau, x(\varphi(\tau))) \ d\tau \\
&- \quad g(s) - (Tx)(s) \int_0^s k(s,\tau)f(\tau, x(\varphi(\tau))) \ d\tau \ | \\
&- \quad [g(t) + (Tx)(t) \int_0^t k(t,\tau)f(\tau, x(\varphi(\tau))) \ d\tau - g(s) \\
&- \quad (Tx)(s) \int_0^s k(s,\tau)f(\tau, x(\varphi(\tau))) \ d\tau] \leq \{| \ g(t) - g(s) \ | \ -[g(t) - g(s)]\} \\
&+ \quad | \ (Tx)(t) \int_0^t k(t,\tau)f(\tau, x(\varphi(\tau))) \ d\tau - (Tx)(s) \int_0^s k(s,\tau)f(\tau, x(\varphi(\tau))) \ d\tau \ | \\
&- \quad [(Tx)(t) \int_0^t k(t,\tau)f(\tau, x(\varphi(\tau))) \ d\tau - (Tx)(s) \int_0^s k(s,\tau)f(\tau, x(\varphi(\tau))) \ d\tau] \\
&\leq \quad | \ (Tx)(t) \int_0^t k(t,\tau)f(\tau, x(\varphi(\tau))) \ d\tau - (Tx)(s) \int_0^t k(t,\tau)f(\tau, x(\varphi(\tau))) \ d\tau \ | \\
&+ \quad | \ (Tx)(s) \int_0^t k(t,\tau)f(\tau, x(\varphi(\tau))) \ d\tau - (Tx)(s) \int_0^t k(s,\tau)f(\tau, x(\varphi(\tau))) \ d\tau \ |
\end{aligned}
$$

$$
\begin{aligned}
&+ \quad |\,(Tx)(s)\int_0^t k(s,\tau)f(\tau,x(\varphi(\tau)))\,d\tau-(Tx)(s)\int_0^s k(s,\tau)f(\tau,x(\varphi(\tau)))\,d\tau\,|\\
&- \quad \{[(Tx)(t)\int_0^t k(t,\tau)f(\tau,x(\varphi(\tau)))\,d\tau-(Tx)(s)\int_0^t k(t,\tau)f(\tau,x(\varphi(\tau)))\,d\tau]\\
&+ \quad [(Tx)(s)\int_0^t k(t,\tau)f(\tau,x(\varphi(\tau)))\,d\tau-(Tx)(s)\int_0^t k(s,\tau)f(\tau,x(\varphi(\tau)))\,d\tau]\\
&+ \quad [(Tx)(s)\int_0^t k(s,\tau)f(\tau,x(\varphi(\tau)))\,d\tau \;-\; (Tx)(s)\int_0^s k(s,\tau)f(\tau,x(\varphi(\tau)))\,d\tau]\}\\
&\leq \quad |\,(Tx)(t)-(Tx)(s)\,|\int_0^t k(t,\tau)f(\tau,x(\varphi(\tau)))\,d\tau\\
&+ \quad |\,(Tx)(s)\,|\int_0^t |\,k(t,\tau) \;-\; k(s,\tau)\,|\,f(\tau,x(\varphi(\tau)))\,d\tau\\
&+ \quad |\,(Tx)(s)\,|\int_s^t k(s,\tau)f(\tau,x(\varphi(\tau)))\,d\tau\\
&- \quad [(Tx)(t)-(Tx)(s)]\int_0^t k(t,\tau)f(\tau,x(\varphi(\tau)))\,d\tau\\
&- \quad (Tx)(s)\int_0^t (k(t,\tau) \;-\; k(s,\tau))f(\tau,x(\varphi(\tau)))\,d\tau\\
&- \quad (Tx)(s)\int_s^t k(s,\tau)f(\tau,x(\varphi(\tau)))\,d\tau.
\end{aligned}
$$

Since $f \geq 0$, $k \geq 0$ and $t \to k(t,s)$ is nonincreasing then we have

$$\int_0^t (k(t,\tau) \;-\; k(s,\tau))f(\tau,x(\varphi(\tau)))\,d\tau \;\geq\; 0 \tag{2.2}$$

and

$$\int_s^t k(s,\tau)f(\tau,x(\varphi(\tau)))\,d\tau \to 0 \text{ as } \epsilon \to 0. \tag{2.3}$$

Finally, (2.2)-(2.3) imply

$$
\begin{aligned}
& \quad |(Vx)(t)-(Vx)(s)| \;-\; [(Vx)(t)-(Vx)(s)]\\
&\leq \quad \{|\,(Tx)(t)-(Tx)(s)\,| - [(Tx)(t)-(Tx)(s)]\}\int_0^t k(t,\tau)f(\tau,x(\varphi(\tau)))\,d\tau\\
&\leq \quad \{|\,(Tx)(t)-(Tx)(s)\,| - [(Tx)(t)-(Tx)(s)]\}\int_0^t k(t,\tau)m(|\,x(\varphi(\tau))\,|)\,d\tau\\
&\leq \quad \{|\,(Tx)(t)-(Tx)(s)\,| \;-\; [(Tx)(t)-(Tx)(s)]\}\int_0^t k(t,\tau)m(\|\,x\,\|)\,d\tau\\
&\leq \quad m(r_0)\{|\,(Tx)(t)-(Tx)(s)\,| \;-\; [(Tx)(t)-(Tx)(s)]\}\int_0^t k(t,\tau)\,d\tau\\
&= \quad Km(r_0)\; i(Tx).
\end{aligned}
$$

Hence, we get

$$i(Vx) \ \leq \ Km(r_0)\ i(Tx),$$

and consequently,

$$i(VX) \ \leq \ Km(r_0)\ i(TX). \tag{2.4}$$

Finally, by the equations (2.1)-(2.4) we obtain

$$\mu(VX) \ \leq \ Km(r_0)\ \mu(TX) \ \leq \ Km(r_0)a\ \mu(X).$$

Now, since $Km(r_0)a < 1$ and applying Theorem 1, we complete the proof. □

**Remark 4.** By Remarks 1 and 2, we have that solutions of the integral equation (1.1) belonging to the set $B^+_{r_0}$ are positive, nonincreasing and continuous on the interval $I = [0, M]$.

Now, we provide an example illustrating the applicability of Theorem 2.
For example, taking in the assumption (iv) $q = 1$ and putting $Tx = x$ for $x \in C(I)$ we obtain the Volterra integral equation of the form

$$x(t) = g(t) + \ x(t) \int_0^t k(t,s) f(s, x(\varphi(s)))\ ds, \quad t \in I.$$

Obviously that equation is a particular of equation (1.1).

## 2.1. Convolution type

Consider the quadratic integral equation of convolution type of the form

$$x(t) = g(t) + \ (Tx)(t) \int_0^t k(t-s) f(s, x(\varphi(s)))\ ds, \ t \in [0, M]. \tag{2.5}$$

Now, the following Corollary deals with the integral equation of convolution type (2.5) .

**Corollary 1.** *Let $k : I \to R_+$ be nonincreasing function and let the assumptions of Theorem 2 be satisfied, then equation* (2.5) *has at least one positive and nonincreasing solution $x \in C(I)$.*

## 2.2. Fractional order equation

Now, taking $k(t-s) \ = \ \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}$, then we have the following Corollary.

**Corollary 2.** *Let $M^\alpha m(r_o)a < \Gamma(\alpha+1)$. Then under the assumptions* (i)-(iv) *and* (vi) *of Theorem 2, the nonlinear quadratic functional integral equation of fractional order*

$$x(t) \ = \ g(t) + (Tx)(t) \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} f(s, x(\varphi(s)))\ ds, \ t \in [0, M], \ 0 < \alpha < 1.$$

*has at least one positive and nonincreasing solution $x \in C(I)$.*

**Corollary 3.** *Let* $(Tx)(t) = p(t, x(\psi(t)))$ *and* $I = [0, 1]$ *in Corollary 2, we obtain the same result as was proved in* [17].

**Corollary 4.** *Under the same assumptions* (i), (ii) *and* (vi) *of Theorem 2* (*with* $q = 1$ *and* $(Tx)(t) = x(t)$), *then the fractional-order integral equation*

$$x(t) \;=\; g(t) + x(t) \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} f(s, x(\varphi(s)))\, ds, \;\; t \in [0, M], \;\; 0 < \alpha < 1.$$

*has at least one positive and nonincreasing solution* $x \in C(I)$ *if* $M^\alpha m(r_o) < \Gamma(\alpha+1)$.

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**W.G. El-Sayed**
email: wagdygoma@alexu.edu.eg
ORCID: 0000-0003-1271-2681
Faculty of Science
Alexandria University
Alexandria
EGYPT

**A.A.H. Abd El-Mowla**
email: aziza.abdelmwla@yahoo.com
ORCID: 0000-0002-7160-4448
Faculty of Science
Omar Al-Mukhtar University
Derna
LIBYA

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# On the Maximum Modulus of a Polynomial

*V.K. Jain*

Abstract: For a polynomial $p(z)$ of degree $n$, having no zeros in $|z| < 1$ Ankeny and Rivlin had shown that for $R \geq 1$

$$\max_{|z|=R} |p(z)| \leq \frac{R^n + 1}{2} \max_{|z|=1} |p(z)|.$$

Using Govil, Rahman and Schmeisser's refinement of the generalization of Schwarz's lemma we have obtained a refinement of Ankeny and Rivlin's result. Our refinement is also a refinement of Dewan and Pukhta's refinement of Ankeny and Rivlin's result.

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*Keywords and Phrases:*Maximum modulus; Polynomial; Refinement; Refinement of the generalization of Schwarz's lemma; No zeros in $|z| < 1$.

## 1. Introduction and statement of results

For an arbitrary polynomial $f(z)$ let $M(f,r) = \max_{|z|=r} |f(z)|$. Further let $p(z) = \sum_{j=0}^{n} a_j z^j$ be a polynomial of degree $n$. Concerning the estimate of $|p(z)|$ on $|z| \leq r$ we have the following well known result (see [7, Problem III 269, p. 158]).

**Theorem 1.1.** *If $p(z)$ is a polynomial of degree $n$ then*

$$M(p,R) \leq R^n M(p,1), R \geq 1,$$

*with equality only for $p(z) = \lambda z^n$.*

For polynomial not vanishing in $|z| < 1$ Ankeny and Rivlin [1] proved

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**Theorem 1.2.** *Let $p(z)$ be a polynomial of degree $n$, having no zeros in $|z| < 1$. Then*

$$M(p,R) \le \frac{R^n+1}{2} M(p,1), R \ge 1.$$

*The result is the best possible with equality only for the polynomial $p(z) = \lambda + \mu z^n, |\lambda| = |\mu|$.*

Dewan and Pukhta [2] used the generalization of Schwarz's lemma [8, p. 212] to obtain the following refinement of Theorem 1.2.

**Theorem 1.3.** *Let $p(z) = a_n \prod_{t=1}^{n}(z - z_t)$ be a polynomial of degree $n$ and let $|z_t| \ge K_t \ge 1, 1 \le t \le n$. Then for $R \ge 1$*

$$\begin{aligned} M(p,R) &\le \frac{R^n+1}{2} M(p,1) - \frac{R^n-1}{2} BM(p,1) \\ &- \frac{n}{2}\left\{\frac{(1-B)^2(M(p,1))^2 - 4|a_n|^2}{(1-B)M(p,1)}\right\} \times \Big[\frac{(R-1)(1-B)M(p,1)}{(1-B)M(p,1)+2|a_n|} \\ &- \ln\left\{1 + \frac{(R-1)(1-B)M(p,1)}{(1-B)M(p,1)+2|a_n|}\right\}\Big], \end{aligned}$$

*where*

$$B = \frac{1}{1 + \frac{2}{n}\sum_{t=1}^{n} \frac{1}{K_t - 1}}.$$

In this paper we have used Govil, Rahman and Schmeisser's refinement of the generalization of Schwarz's lemma [4, Lemma] to obtain a new refinement of Theorem 1.2. Our refinement is a refinement of Theorem 1.3 also. More precisely we prove

**Theorem 1.4.** *Let*

$$p(z) = \sum_{j=0}^{n} a_j z^j = a_n \prod_{t=1}^{n}(z - z_t)$$

*be a polynomial of degree $n$ such that*

$$|z_t| \ge K_t \ge 1, 1 \le t \le n.$$

*Further let*

$$M = \begin{cases} \dfrac{n}{2}\left\{1 - \dfrac{1}{1 + \frac{2}{n}\sum\limits_{t=1}^{n} \frac{1}{K_t - 1}}\right\} M(p,1), & K_t \ne 1 \text{ for all } t \quad (1.1) \\ \dfrac{n}{2} M(p,1), & K_t = 1 \text{ for certain } t \ (1 \le t \le n) \quad (1.2) \end{cases}$$

$$\begin{aligned} a &= n\overline{a_n}, && (1.3) \\ b &= (n-1)\overline{a_{n-1}} && (1.4) \\ R &\ge 1 && (1.5) \end{aligned}$$

*and*

$$D=\begin{cases}\dfrac{1}{\sqrt{\frac{|a|}{M}-\frac{|b|^2}{4(M-|a|)^2}}}\left(\tan^{-1}\dfrac{R+\frac{|b|}{2(M-|a|)}}{\sqrt{\frac{|a|}{M}-\frac{|b|^2}{4(M-|a|)^2}}}\right.\\ \left.-\tan^{-1}\dfrac{1+\frac{|b|}{2(M-|a|)}}{\sqrt{\frac{|a|}{M}-\frac{|b|^2}{4(M-|a|)^2}}}\right),\ \dfrac{|a|}{M}-\dfrac{|b|^2}{4(M-|a|)^2}>0, & (1.6)\\ \dfrac{1}{2\sqrt{\frac{|b|^2}{4(M-|a|)^2}-\frac{|a|}{M}}}\left(\ln\left|\dfrac{R+\frac{|b|}{2(M-|a|)}-\sqrt{\frac{|b|^2}{4(M-|a|)^2}-\frac{|a|}{M}}}{R+\frac{|b|}{2(M-|a|)}+\sqrt{\frac{|b|^2}{4(M-|a|)^2}-\frac{|a|}{M}}}\right|-\right.\\ \left.\ln\left|\dfrac{1+\frac{|b|}{2(M-|a|)}-\sqrt{\frac{|b|^2}{4(M-|a|)^2}-\frac{|a|}{M}}}{1+\frac{|b|}{2(M-|a|)}+\sqrt{\frac{|b|^2}{4(M-|a|)^2}-\frac{|a|}{M}}}\right|\right),\ \dfrac{|a|}{M}-\dfrac{|b|^2}{4(M-|a|)^2}<0, & (1.7)\\ -\left(R+\dfrac{|b|}{2(M-|a|)}\right)^{-1}+\left(1+\dfrac{|b|}{2(M-|a|)}\right)^{-1},\\ \dfrac{|a|}{M}-\dfrac{|b|^2}{4(M-|a|)^2}=0. & (1.8)\end{cases}$$

*Then*

$$M(p,R)\leq$$

$$\begin{cases}\dfrac{R^n+1}{2}M(p,1)-\dfrac{R^n-1}{2}\Big(\dfrac{1}{1+\frac{2}{n}\sum_{t=1}^{n}\frac{1}{K_t-1}}\Big)M(p,1)\\ -(M-|a|)(R-1)+\dfrac{|b|}{2}\ln\dfrac{(M-|a|)MR^2+M|b|R+|a|(M-|a|)}{(M^2-|a|^2)+M|b|}\\ +\dfrac{2(M-|a|)(M^2-|a|^2)-M|b|^2}{2M(M-|a|)}D, M>|a| \text{ and } K_t\neq 1 \text{ for all } t, & (1.9)\\ \dfrac{R^n+1}{2}M(p,1)-(M-|a|)(R-1)\\ +\dfrac{|b|}{2}\ln\dfrac{(M-|a|)MR^2+M|b|R+|a|(M-|a|)}{M^2-|a|^2+M|b|}\\ +\dfrac{2(M-|a|)(M^2-|a|^2)-M|b|^2}{2M(M-|a|)}D,\\ M>|a| \text{ and } K_t=1 \text{ for certain } t\ (1\leq t\leq n), & (1.10)\\ \dfrac{R^n+1}{2}M(p,1)-\dfrac{R^n-1}{2}\Big(\dfrac{1}{1+\frac{2}{n}\sum_{t=1}^{n}\frac{1}{K_t-1}}\Big)M(p,1),\\ M=|a| \text{ and } K_t\neq 1\forall t, & (1.11)\\ \dfrac{R^n+1}{2}M(p,1), M=|a| \text{ and } K_t=1 \text{ for certain } t, (1\leq t\leq n). & (1.12)\end{cases}$$

The result is the best possible if $K_t = 1$ for certain $t, (1 \leq t \leq n)$ and the equality holds for the polynomial $p(z) = \lambda + \mu z^n, |\lambda| = |\mu|$.

**Remark 1.5.** That Theorem 1.4 is a refinement of Theorem 1.3 can be seen from the fact that a refinement of the generalization of Schwarz's lemma is used to obtain Theorem 1.4.

Further by taking $K_t = K, (K \geq 1), \forall t$, in Theorem 1.4 we get

**Corollary 1.6.** *Let $p(z) = \sum_{j=0}^{n} a_j z^j$ be a polynomial of degree $n$, having no zeros in $|z| < K, (K \geq 1)$. Further let*

$$\begin{aligned} M &= \frac{n}{1+K} M(p,1), \\ a &= n\overline{a_n}, \\ b &= (n-1)\overline{a_{n-1}}, \\ R &\geq 1 \end{aligned}$$

*and*

$$D = \begin{cases} \dfrac{1}{\sqrt{\frac{|a|}{M} - \frac{|b|^2}{4(M-|a|)^2}}} \left( \tan^{-1} \dfrac{R + \frac{|b|}{2(M-|a|)}}{\sqrt{\frac{|a|}{M} - \frac{|b|^2}{4(M-|a|)^2}}} - tan^{-1} \dfrac{1 + \frac{|b|}{2(M-|a|)}}{\sqrt{\frac{|a|}{M} - \frac{|b|^2}{4(M-|a|)^2}}} \right), \\ \dfrac{|a|}{M} - \dfrac{|b|^2}{4(M-|a|)^2} > 0, \\ \dfrac{1}{2\sqrt{\frac{|b|^2}{4(M-|a|)^2} - \frac{|a|}{M}}} \left( \ln \left| \dfrac{R + \frac{|b|}{2(M-|a|)} - \sqrt{\frac{|b|^2}{4(M-|a|)^2} - \frac{|a|}{M}}}{R + \frac{|b|}{2(M-|a|)} + \sqrt{\frac{|b|^2}{4(M-|a|)^2} - \frac{|a|}{M}}} \right| - \right. \\ \left. \ln \left| \dfrac{1 + \frac{|b|}{2(M-|a|)} - \sqrt{\frac{|b|^2}{4(M-|a|)^2} - \frac{|a|}{M}}}{1 + \frac{|b|}{2(M-|a|)} + \sqrt{\frac{|b|^2}{4(M-|a|)^2} - \frac{|a|}{M}}} \right| \right), \dfrac{|a|}{M} - \dfrac{|b|^2}{4(M-|a|)^2} < 0, \\ -\left( R + \dfrac{|b|}{2(M-|a|)} \right)^{-1} + \left( 1 + \dfrac{|b|}{2(M-|a|)} \right)^{-1}, \dfrac{|a|}{M} - \dfrac{|b|^2}{4(M-|a|)^2} = 0. \end{cases}$$

*Then*

$$M(p,R) \leq \begin{cases} \dfrac{R^n + K}{1+K} M(p,1) - (M - |a|)(R-1) \\ + \dfrac{|b|}{2} \ln \dfrac{(M-|a|)MR^2 + M|b|R + |a|(M-|a|)}{(M^2 - |a|^2) + M|b|} \\ + \dfrac{2(M-|a|)(M^2-|a|^2) - M|b|^2}{2M(M-|a|)} D, & M > |a|, \\ \dfrac{R^n + K}{1+K} M(p,1), & M = |a|. \end{cases}$$

*The result is the best possible if $K = 1$ and the equality holds for the polynomial $p(z) = \lambda + \mu z^n$, $|\lambda| = |\mu|$.*

**Remark 1.7.** Corollary 1.6 is a refinement of Dewan and Pukhta's result [2, Corollary].

# 2. Lemmas

For the proof of Theorem 1.4 we require the following lemmas.

**Lemma 2.1.** *If $p(z)$ is a polynomial of degree $n$, having no zeros in $|z| < 1$ then*

$$M(p', 1) \leq \frac{n}{2} M(p, 1).$$

This lemma is due to Lax [5].

**Lemma 2.2.** *Let $p(z) = a_n \prod_{t=1}^{n}(z - z_t)$, be a polynomial of degree $n$. If $|z_t| \geq K_t \geq 1$, $1 \leq t \leq n$, then*

$$M(p', 1) \leq n\Big(\Big(\sum\nolimits_{t=1}^{n} \frac{1}{K_t - 1}\Big)/\Big(\sum\nolimits_{t=1}^{n} \frac{K_t + 1}{K_t - 1}\Big)\Big) M(p, 1).$$

*The result is the best possible with the equality for the polynomial $p(z) = (z + k)^n$, $k \geq 1$.*

This lemma is due to Govil and Labelle [3].

**Lemma 2.3.** *If $f(z)$ is analytic and $|f(z)| \leq 1$ in $|z| < 1$ then*

$$|f(z)| \leq \frac{(1 - |a'|)|z|^2 + |b'||z| + |a'|(1 - |a'|)}{|a'|(1 - |a'|)|z|^2 + |b'||z| + (1 - |a'|)}, (|z| < 1),$$

*where $a' = f(0)$, $b' = f'(0)$. The example*

$$f(z) = \Big(a' + \frac{b'}{1 + a'}z - z^2\Big)/\Big(1 - \frac{b'}{1 + a'}z - a'z^2\Big)$$

*shows that the estimate is sharp.*

This lemma is due to Govil et al. [4].

**Remark 2.4.** By using the result [6, p. 172, exercise # 9] one can show that Lemma 2.3 is a refinement of the generalization of Schwarz's lemma.

**Lemma 2.5.** *If $g(z)$ is analytic in $|z| \leq 1$, with*

$$\begin{aligned} |g(z)| &\leq M_1, |z| \leq 1, \\ g(0) &= a_1, \\ g'(0) &= b_1 \end{aligned}$$

*then*

$$|g(z)| \le \begin{cases} M_1 \dfrac{M_1(M_1-|a_1|)|z|^2 + M_1|b_1||z| + |a_1|(M_1-|a_1|)}{|a_1|(M_1-|a_1|)|z|^2 + M_1|b||z| + M_1(M_1-|a_1|)}, M_1 > |a_1| \text{ and } |z| \le 1, \\ M_1, \quad M_1 = |a_1| \text{ and } |z| \le 1. \end{cases}$$

**Proof.** It follows easily by applying Lemma 2.3 to the function $g(z)/M_1$. □

## 3. Proof of Theorem 1.4

For the polynomial

$$T(z) = z^{n-1}\overline{p'(1/\overline{z})} \tag{3.1}$$

we have

$$|T(z)| = |p'(z)|, |z| = 1,$$

which by Lemma 2.1, Lemma 2.2, (1.1) and (1.2) implies that

$$|T(z)| \le M, |z| \le 1.$$

Therefore on applying Lemma 2.5 to $T(z)$ we get for $|z| \le 1$

$$|T(z)| \le \begin{cases} M \dfrac{M(M-|a|)|z|^2 + M|b||z| + |a|(M-|a|)}{|a|(M-|a|)|z|^2 + M|b||z| + M(M-|a|)}, \ M > |a|, (\text{by } (1.3) \text{ and } (1.4)), \\ M, \quad M = |a|, \ (\text{by } (1.3)), \end{cases}$$

which on using (3.1) and

$$z = \frac{1}{R}e^{i\theta}, \qquad 0 \le \theta \le 2\pi,$$

implies for $0 \le \theta \le 2\pi$

$$|p'(Re^{i\theta})| \le \begin{cases} MR^{n-1}\left\{1 - \dfrac{(M-|a|)^2(R^2-1)}{|a|(M-|a|) + M|b|R + M(M-|a|)R^2}\right\}, \\ M > |a|, (\text{by } (1.5)) & (3.2) \\ MR^{n-1}, \quad M = |a|, \ (\text{by } (1.5)). & (3.3) \end{cases}$$

Now we consider the case $M > |a|$. For $0 \le \theta \le 2\pi$ we have

$$
\begin{aligned}
|p(Re^{i\theta}) - p(e^{i\theta})| &= |\int_1^R p'(re^{i\theta})e^{i\theta}dr| \text{ (by (1.5))} \\
&\leq \int_1^R |p'(re^{i\theta})|dr \leq M\int_1^R r^{n-1}dr \\
&- M(M-|a|)^2\int_1^R \frac{r^{n-1}(r^2-1)}{|a|(M-|a|)+M|b|r+M(M-|a|)r^2}dr \text{ (by (3.2)))} \\
&\leq M\frac{R^n-1}{n} - M(M-|a|)^2\int_1^R \frac{r^2-1}{|a|(M-|a|)+M|b|r+M(M-|a|)r^2}dr \\
&= M\frac{R^n-1}{n} - (M-|a|)\int_1^R dr \\
&+ (M-|a|)\int_1^R \frac{M|b|r+M^2-|a|^2}{M(M-|a|)r^2+M|b|r+|a|(M-|a|)}dr \\
&= M\frac{R^n-1}{n} - (M-|a|)(R-1) \\
&+ \frac{|b|}{2}\int_1^R \frac{2M(M-|a|)r+M|b|}{M(M-|a|)r^2+M|b|r+|a|(M-|a|)}dr \\
&+ \frac{1}{2}\times\int_1^R \frac{2(M^2-|a|^2)(M-|a|)-M|b|^2}{M(M-|a|)r^2+M|b|r+|a|(M-|a|)}dr \\
&= M\frac{R^n-1}{n} - (M-|a|)(R-1) \\
&+ \frac{|b|}{2}\ln\frac{M(M-|a|)R^2+M|b|R+|a|(M-|a|)}{(M^2-|a|^2)+M|b|} \\
&+ \frac{2(M-|a|)(M^2-|a|^2)-M|b|^2}{2M(M-|a|)}\int_1^R \frac{1}{\left\{r+\frac{|b|}{2(M-|a|)}\right\}^2+\frac{|a|}{M}-\frac{|b|^2}{4(M-|a|)^2}}dr \\
&= M\frac{R^n-1}{n} - (M-|a|)(R-1) + \\
&\quad \frac{|b|}{2}\ln\frac{(M-|a|)MR^2+M|b|R+|a|(M-|a|)}{(M^2-|a|^2)+M|b|} + \\
&\quad \frac{2(M-|a|)(M^2-|a|^2)-M|b|^2}{2M(M-|a|)}D \text{ (by (1.6), (1.7) and (1.8))},
\end{aligned}
$$

which implies

$$
\begin{aligned}
M(p,R) \leq\ & M(p,1) + M\frac{R^n-1}{n} - (M-|a|)(R-1) + \\
& \frac{|b|}{2}\ln\frac{(M-|a|)MR^2+M|b|R+|a|(M-|a|)}{(M^2-|a|^2)+M|b|} + \\
& \frac{2(M-|a|)(M^2-|a|^2)-M|b|^2}{2M(M-|a|)}D
\end{aligned}
$$

and inequalities (1.9) and (1.10) follow respectively by using relations (1.1) and (1.2).

Further we consider the possibility $M = |a|$. The proof of inequalities (1.11) and (1.12) is similar to the proof of inequalities (1.9) and (1.10), with one change:

$$\text{inequality (3.3) instead of inequality (3.2)}$$

and so we omit the details. This completes the proof of Theorem 1.4.

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**Vinay Kumar Jain**
email: `vinayjain.kgp@gmail.com`
ORCID: 0000-0003-2382-2499
Mathematics Department
I.I.T.
Kharagpur - 721302
INDIA

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# The Existence of Monotonic Solutions of a Class of Quadratic Integral Equations of Volterra Type

*Osman Karakurt and Ömer Faruk Temizer**

ABSTRACT: Using the technique associated with measure of noncompactness we prove the existence of monotonic solutions of a class of quadratic integral equation of Volterra type in the Banach space of real functions defined and continuous on a bounded and closed interval.

*AMS Subject Classification:* 45D05, 45M20.
*Keywords and Phrases:* Nonlinear Volterra integral equations; Measure of noncompactness; Fixed point theorem.

## 1. Introduction

The theory of integral operators and integral equations is an important part of nonlinear analysis. This theory is frequently applicable in other branches of mathematics and mathematical physics, engineering, economics, biology as well in describing problems connected with real world [1, 2, 7, 9, 10, 11].

The aim of this paper is to investigate the existence of nondecreasing solutions of a class of a quadratic integral equations of Volterra type. We will look for solutions of those equations in the Banach space of real functions being defined and continuous on a bounded and closed interval. The main tool used in our investigation is the technique of measure of noncompactness which is frequently used in several branches of nonlinear analysis [4, 7, 5, 9].

We will apply the measure of noncompactness defined in [6] to proving the solvability of the considered equations in the class of monotonic functions.

The results of this paper generalize the results obtained earlier in the paper [3].

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## 2. Notation and auxiliary facts

Now, we are going to recall the basic results which are needed further on.

Assume that $E$ is a real Banach space with the norm $\|.\|$ and the zero element 0. Denote by $B(x, r)$ the closed ball centered at $x$ and with radius $r$ and by $B_r$ the ball $B(0, r)$. If $X$ is a nonempty subset of $E$ we denote by $\overline{X}$, Conv$X$ the closure and the convex closure of $X$, respectively.

With the symbols $\lambda X$ and $X + Y$ we denote the algebraic operations on the sets. Finally, let us denote by $\mathfrak{M}_E$ the family of all nonempty and bounded subsets of $E$ and by $\mathfrak{N}_E$ its subfamily consisting of all relatively compact sets.

**Definition 2.1** (See [4]). A function $\mu : \mathfrak{M}_E \to [0, \infty)$ is said to be a measure of noncompactness in the space $E$ if it is satisfies the following conditions

(1) The family $\ker\mu = \{\mathrm{X} \in \mathfrak{M}_\mathrm{E} : \mu(\mathrm{X}) = 0\} \neq \emptyset$ and $\ker\mu \subset \mathfrak{N}_\mathrm{E}$,

(2) $X \subset Y \Rightarrow \mu(X) \leq \mu(Y)$,

(3) $\mu(\overline{X}) = \mu(\mathrm{Conv}X) = \mu(X)$,

(4) $\mu(\lambda X + (1 - \lambda)Y) \leq \lambda\mu(X) + (1 - \lambda)\mu(Y), \quad \text{for}\lambda \in [0, 1]$,

(5) If $\{X_n\}_n$ is a sequence of closed sets from $\mathfrak{M}_E$ such that $X_{n+1} \subset X_n$ for $n = 1, 2, \ldots$ and if $\lim_{n\to\infty} \mu(X_n) = 0$, then the set $X_\infty = \bigcap_{n=1}^{\infty} X_n$ is nonempty.

The family $\ker\mu$ described above is called the kernel of the measure of noncompactness $\mu$. Further facts concerning measures of noncompactness and their properties may be found in [4].

Now, let us suppose that Q is a nonempty subset of the Banach space $E$ and the operator $F : \mathrm{Q} \to E$ is continuous and transforms bounded sets onto bounded ones. We say that $F$ satisfies the Darbo condition (with a constant $k \geq 0$) with respect to a measure of noncompactness $\mu$ if for any bounded subset $X$ of Q the following inequality holds:

$$\mu(FX) \leq k\mu(X).$$

If $F$ satisfies the Darbo condition with $k < 1$ then it is said to be a contraction with respect to $\mu$, [8]. For our further purposes we will only need the following fixed point theorem.

**Theorem 2.1.** *Let Q be a nonempty, bounded, closed and convex subset of the Banach space E and $\mu$ be a measure of noncompactness in E. Let $F : Q \to Q$ be a continuous transformation such that $\mu(FX) \leq k\mu(X)$ for any nonempty subset X of Q, where $k \in [0, 1)$ is a constant. Then, F has a fixed point in the set Q,* [3].

**Remark 1.** Under assumptions of the above theorem it can be shown that, the set $FixF$ of fixed points of $F$ belonging to Q is a member of $\ker\mu$. This observation allows us to characterize solutions of considered equations, [3].

In what follows, we will work in the classical Banach space $C[0, M]$ consisting of all real functions defined and continuous on the interval [0,M]. For convenience, we write $I = [0, M]$ and $C(I) = C[0, M]$. The space $C(I)$ is furnished by the standard norm $\|x\| = \max\{|x(t)| : t \in I\}$.

Now, we recall the definition of a measure of noncompactness in $C(I)$ which will be used in the sequel. That measure was introduced and studied in the paper [6].

To do this let us fix a nonempty and bounded subset $X$ of $C(I)$. For $\varepsilon > 0$ and $x \in X$ denote by $w(x, \varepsilon)$ the modulus of continuity of $x$ defined by

$$w(x, \varepsilon) = \sup\{|x(t) - x(s)| : t, s \in I, |t - s| \leq \varepsilon\}.$$

Further, let us put

$$w(X, \varepsilon) = \sup\{w(x, \varepsilon) : x \in X\},$$

$$w_0(X) = \lim_{\varepsilon \to 0} w(X, \varepsilon).$$

Next, let us define the following quantities

$$i(x) = \sup\{|x(s) - x(t)| - [x(s) - x(t)] : t, s \in I,\ t \leq s\},$$
$$i(X) = \sup\{i(x) : x \in X\}.$$

Observe that, $i(X) = 0$ if and only if all functions belonging to $X$ are nondecreasing on $I$. Finally, let us put

$$\mu(X) = w_0(X) + i(X).$$

It can be shown that, the function $\mu$ is a measure of noncompactness in the space $C(I)$ (see [6]). Moreover, the kernel ker$\mu$ consist of all sets $X$ belonging to $\mathfrak{M}_{C(I)}$ such that all functions from $X$ are equicontinuous and nondecreasing on the interval $I$.

# 3. Main result

In this section, we apply the above defined measure of noncompactness $\mu$ to the study of monotonic solutions of our integral equation.

We consider the following nonlinear integral equation of Volterra type

$$x(t) = a(\alpha(t)) + (Tx)(\beta(t)) \int_0^{\gamma(t)} v(t, \tau, x(\eta(\tau)))d\tau, \quad t \in I = [0, M]. \tag{3.1}$$

The functions $a(\alpha(t))$, $v(t, \tau, x(\eta(\tau)))$ and $(Tx)(\beta(t))$ appearing in this equation are given while $x = x(t)$ is an unknown function. This equation will be examined under the following assumptions:

$(i)$ $\alpha, \beta, \gamma, \eta$: $I \to I$ are continuous functions and $\alpha, \beta, \gamma$ are nondecreasing on $I$.

$(ii)$ The function $a \in C(I)$ is nondecreasing and nonnegative on the interval $I$.

$(iii)$ $v : I \times I \times \mathbb{R} \to \mathbb{R}$ is a continuous function such that $v : I \times I \times \mathbb{R}_+ \to \mathbb{R}_+$ and for arbitrarily fixed $\tau \in I$ and $x \in \mathbb{R}_+$ the function $t \to v(t, \tau, x)$ is nondecreasing on $I$.

$(iv)$ There exists a nondecreasing function $f : \mathbb{R}_+ \to \mathbb{R}_+$ such that the inequality $|v(t, \tau, x)| \leq f(|x|)$ holds for $t, \tau \in I$ and $x \in \mathbb{R}$.

$(v)$ The operator $T : C(I) \to C(I)$ is continuous and $T$ is a positive operator, i.e. $Tx \geq 0$ if $x \geq 0$.

$(vi)$ There exist nonnegative constants $c, d$ and $p > 0$ such that $|(Tx)(t)| \leq c + d\|x\|^p$ for each $x \in C(I)$ and all $t \in I$.

$(vii)$ The inequality $a(\|\alpha\|) + (c + dr^p)Mf(r) \leq r$ has a positive solution $r_0$.

$(viii)$ The operator $T$ in $B^+_{r_0} = \{x \in B_{r_0} : x(t) \geq 0,\ t \in I\}$ satisfies the inequality $\mu(TX) \leq \theta\mu(X)$ for the measure of noncompactness $\mu$ with a constant $\theta$ such that $Mf(r_0)\theta < 1$, where $\theta \in [0, 1)$.

Then, we have the following theorem:

**Theorem 3.1.** *Under the assumptions* $(i)$-$(viii)$ *the equation* (3.1) *has at least one solution* $x = x(t)$ *which belongs to the space* $C(I)$ *and is nondecreasing on the interval* $I$.

**Proof.** Let us consider the operator $V$ defined on the space $C(I)$ in the following way:

$$(Vx)(t) = a(\alpha(t)) + (Tx)(\beta(t)) \int_0^{\gamma(t)} v(t, \tau, x(\eta(\tau)))d\tau.$$

The proof will now proceed in two steps: firstly $V$ is continuous and secondly $V$ is contraction transformation on $B^+_{r_0} \subset C(I)$.

**Step 1.** In view of the assumptions $(i)$, $(ii)$, $(iii)$ and $(v)$ it follows that, the function $Vx$ is continuous on $I$ for any function $x \in C(I)$, i.e., $V$ transforms the space $C(I)$ into itself. Moreover, keeping in mind the assumptions $(iv)$ and $(vi)$ we get

$$\begin{aligned} |(Vx)(t)| &\leq |a(\alpha(t))| + |(Tx)(\beta(t))| \left| \int_0^{\gamma(t)} v(t, \tau, x(\eta(\tau)))d\tau \right| \\ &\leq a(\|\alpha\|) + (c + d\|x\|^p) \int_0^{\gamma(t)} f(|x(\eta(\tau))|)d\tau \\ &\leq a(\|\alpha\|) + (c + d\|x\|^p) \int_0^{\gamma(t)} f(\|x\|)d\tau \\ &\leq a(\|\alpha\|) + (c + d\|x\|^p)Mf(\|x\|). \end{aligned}$$

Hence, we obtain the inequality

$$\|Vx\| \leq a(\|\alpha\|) + (c + d\|x\|^p)Mf(\|x\|).$$

For $r_0 \geq \|x\|$ such that provide assumption $(vii)$, we get $\|Vx\| \leq r_0$. This shows that $V$ transforms the ball $B_{r_0}$ into itself i.e., $V : B_{r_0} \to B_{r_0}$.

Let us consider the operator $V$ on the subset $B^+_{r_0}$ of the ball $B_{r_0}$ defined by

$$B^+_{r_0} = \{x \in B_{r_0} : x(t) \geq 0, \quad t \in I\}.$$

Since the function $x$ defined as $x(t) = r_0$ for all $t \in I$ is a member of the set $B^+_{r_0}$, the set $B^+_{r_0}$ is nonempty. Since $B_{r_0}$ is bounded, $B^+_{r_0}$ is bounded. The inequalities

$$\lambda x(t) + (1 - \lambda)y(t) \geq 0$$

and

$$\|\lambda x + (1 - \lambda)y\| \leq \lambda\|x\| + (1 - \lambda)\|y\| \leq \lambda r_0 + (1 - \lambda)r_0 = r_0$$

hold for all $x, y \in B^+_{r_0}$, $t \in I$ and $\lambda$ such that $0 \leq \lambda \leq 1$. So, $B^+_{r_0}$ is convex.

Let us take a convergent sequence $(x_n) \subset B^+_{r_0} \subset B_{r_0}$ so that $\lim_{n\to\infty} x_n = x$. Since

$$\|x_n - x\| = \max_{t\in I} |x_n(t) - x(t)| \to 0 \quad (n \to \infty),$$

we get $\lim_{n\to\infty} x_n(t) = x(t)$. Hence, we have $x(t) \geq 0$ for all $t \in I$. Thus, $x \in B^+_{r_0}$ and $B^+_{r_0}$ is closed.

In view of these facts and assumptions $(i)$, $(ii)$, $(iii)$ and $(v)$ it follows that $V$ transforms the set $B^+_{r_0}$ into itself.

Now, we show that $V$ is continuous on the set $B^+_{r_0}$. To do this let us fix $\varepsilon > 0$ and take arbitrarily $x, y \in B^+_{r_0}$ such that $\|x - y\| \leq \varepsilon$. Then, for $t \in I$ we get the following inequalities:

$$\begin{aligned}
& |(Vx)(t) - (Vy)(t)| \\
= & \left|(Tx)(\beta(t)) \int_0^{\gamma(t)} v(t, \tau, x(\eta(\tau)))d\tau - (Ty)(\beta(t)) \int_0^{\gamma(t)} v(t, \tau, y(\eta(\tau)))d\tau\right| \\
\leq & \left|(Tx)(\beta(t)) \int_0^{\gamma(t)} v(t, \tau, x(\eta(\tau)))d\tau - (Ty)(\beta(t)) \int_0^{\gamma(t)} v(t, \tau, x(\eta(\tau)))d\tau\right| \\
+ & \left|(Ty)(\beta(t)) \int_0^{\gamma(t)} v(t, \tau, x(\eta(\tau)))d\tau - (Ty)(\beta(t)) \int_0^{\gamma(t)} v(t, \tau, y(\eta(\tau)))d\tau\right|
\end{aligned}$$

$$
\begin{aligned}
\leq\ & |(Tx)(\beta(t)) - (Ty)(\beta(t))| \int_0^{\gamma(t)} |v(t,\tau,x(\eta(\tau)))| d\tau \\
+\ & |(Ty)(\beta(t))| \int_0^{\gamma(t)} |v(t,\tau,x(\eta(\tau))) - v(t,\tau,y(\eta(\tau)))|\, d\tau \\
\leq\ & |(Tx)(\beta(t)) - (Ty)(\beta(t))| \int_0^{\gamma(t)} f(|x(\eta(\tau))|) d\tau \\
+\ & |(Ty)(\beta(t))| \int_0^{\gamma(t)} |v(t,\tau,x(\eta(\tau))) - v(t,\tau,y(\eta(\tau)))|\, d\tau \\
\leq\ & |(Tx)(\beta(t)) - (Ty)(\beta(t))| \int_0^{\gamma(t)} f(\|x\|) d\tau \\
+\ & |(Ty)(\beta(t))| \int_0^{\gamma(t)} |v(t,\tau,x(\eta(\tau))) - v(t,\tau,y(\eta(\tau)))|\, d\tau \\
\leq\ & |(Tx)(\beta(t)) - (Ty)(\beta(t))| \int_0^{\gamma(t)} f(r_0) d\tau \\
+\ & |(Ty)(\beta(t))| \int_0^{\gamma(t)} |v(t,\tau,x(\eta(\tau))) - v(t,\tau,y(\eta(\tau)))|\, d\tau \\
\leq\ & |(Tx - Ty)(\beta(t))| \int_0^{\gamma(t)} f(r_0) d\tau \\
+\ & (c + d\|y\|^p) \int_0^{\gamma(t)} |v(t,\tau,x(\eta(\tau))) - v(t,\tau,y(\eta(\tau)))|\, d\tau \\
\leq\ & \|Tx - Ty\| \int_0^{\gamma(t)} f(r_0) d\tau + (c + dr_0^p) \int_0^{\gamma(t)} \beta_{r_0}(\varepsilon) d\tau \\
\leq\ & \|Tx - Ty\| M f(r_0) + (c + dr_0^p) \beta_{r_0}(\varepsilon) M,
\end{aligned}
$$

where $\beta_{r_0}(\varepsilon)$ is defined as

$$\beta_{r_0}(\varepsilon) = \sup\{|v(t,\tau,x) - v(t,\tau,y)| :\ t,\tau \in I,\ x,y \in [0,r_0],\ |x-y| \leq \varepsilon\}.$$

From the above estimate we obtain the following inequality:

$$\|Vx - Vy\| \leq \|Tx - Ty\| M f(r_0) + (c + dr_0^p) M \beta_{r_0}(\varepsilon).$$

From the uniform cotinuity of the function $v$ on the set $I \times I \times [0, r_0]$ we have that $\beta_{r_0}(\varepsilon) \to 0$ as $\varepsilon \to 0$ and from the continuity of $T$, we have that $\|Tx - Ty\| \to 0$ as $\varepsilon \to 0$. The last inequality implies continuity of the operator $V$ on the set $B_{r_0}^+$.

**Step 2.** In what follows let us take a nonempty set $X \subset Br_0^+$. Further, fix arbitrarily a number $\varepsilon > 0$ and choose $x \in X$ and $t, s \in [0, M]$ such that $|t - s| \leq \varepsilon$. Without loss of generality we may assume that $t \leq s$. Then, in view of our assumptions we obtain

$$
\begin{aligned}
& |(Vx)(s)-(Vx)(t)| \\
\leq\ & |a(\alpha(s))-a(\alpha(t))| \\
+\ & \left|(Tx)(\beta(s))\int_0^{\gamma(s)} v(s,\tau,x(\eta(\tau)))d\tau-(Tx)(\beta(t))\int_0^{\gamma(t)} v(t,\tau,x(\eta(\tau)))d\tau\right| \\
\leq\ & w(a,w(\alpha,\varepsilon))+\left|[(Tx)(\beta(s))-(Tx)(\beta(t))]\int_0^{\gamma(s)} v(s,\tau,x(\eta(\tau)))d\tau\right| \\
+\ & \left|(Tx)(\beta(t))\int_0^{\gamma(s)} v(s,\tau,x(\eta(\tau)))d\tau-(Tx)(\beta(t))\int_0^{\gamma(s)} v(t,\tau,x(\eta(\tau)))d\tau\right| \\
+\ & \left|(Tx)(\beta(t))\int_0^{\gamma(s)} v(t,\tau,x(\eta(\tau)))d\tau-(Tx)(\beta(t))\int_0^{\gamma(t)} v(t,\tau,x(\eta(\tau)))d\tau\right| \\
\leq\ & w(a,w(\alpha,\varepsilon))+|(Tx)(\beta(s))-(Tx)(\beta(t))|\int_0^{\gamma(s)} |v(s,\tau,x(\eta(\tau)))|d\tau \\
+\ & |(Tx)(\beta(t))|\int_0^{\gamma(s)} |v(s,\tau,x(\eta(\tau)))-v(t,\tau,x(\eta(\tau)))|d\tau \\
+\ & |(Tx)(\beta(t)|\left|\int_{\gamma(t)}^{\gamma(s)} v(t,\tau,x(\eta(\tau)))d\tau\right| \\
\leq\ & w(a,w(\alpha,\varepsilon))+w(Tx,w(\beta,\varepsilon))\int_0^{\gamma(s)} f(r_0)d\tau+(c+dr_0^p)\int_0^{\gamma(s)} \xi_{r_0}(\varepsilon)d\tau \\
+\ & (c+dr_0^p)f(r_0)|\gamma(s)-\gamma(t)| \\
\leq\ & w(a,w(\alpha,\varepsilon))+w(Tx,w(\beta,\varepsilon))Mf(r_0)+(c+dr_0^p)M\xi_{r_0}(\varepsilon) \\
+\ & (c+dr_0^p)f(r_0)|\gamma(s)-\gamma(t)|,
\end{aligned}
$$

where $\xi_{r_0}(\varepsilon)$ is defined as

$$\xi_{r_0}(\varepsilon)=\sup\{|v(s,\tau,x))-v(t,\tau,x))|:\ t,s,\tau\in I,\ \ |s-t|\leq\varepsilon,\ \ x\in[0,r_0]\}.$$

Notice, that in view of the uniform continuity of the function $v$ on the set $I\times I\times[0,r_0]$ and from the uniform continuity of the function $\gamma$ on the interval $I$, we have $\xi_{r_0}(\varepsilon)\to 0$ as $\varepsilon\to 0$ and $(\gamma(s)-\gamma(t))\to 0$. Thus, we have the inequality

$$
\begin{aligned}
& |(Vx)(s)-(Vx)(t)| \\
\leq\ & w(a,w(\alpha,\varepsilon))+w(Tx,w(\beta,\varepsilon))Mf(r_0)+(c+dr_0^p)M\xi_{r_0}(\varepsilon) \\
+\ & (c+dr_0^p)f(r_0)|\gamma(s)-\gamma(t)|.
\end{aligned}
$$

If we take the supremum at this inequality over the $t$'s and $s$'s, we have the inequality

$$
\begin{aligned}
w(Vx,\varepsilon)\ \leq\ & w(a,w(\alpha,\varepsilon))+w(Tx,w(\beta,\varepsilon))Mf(r_0)+(c+dr_0^p)M\xi_{r_0}(\varepsilon) \\
+\ & (c+dr_0^p)f(r_0)w(\gamma,\varepsilon).
\end{aligned}
$$

If we take the supremum at this inequality over $x$'s, we have the following estimation

$$\begin{aligned} w(VX,\varepsilon) &\leq w(a,w(\alpha,\varepsilon))+w(TX,w(\beta,\varepsilon))Mf(r_0)+(c+dr_0^p)M\xi_{r_0}(\varepsilon) \\ &+ (c+dr_0^p)f(r_0)w(\gamma,\varepsilon). \end{aligned}$$

For $\varepsilon \to 0$, we have

$$w_0(VX) \leq Mf(r_0)w_0(TX). \tag{3.2}$$

On the other hand, let us fix arbitrarily $x \in X$ and $t,s \in I$ such that $t \leq s$. Then, we have the following estimate:

$$\begin{aligned} & |(Vx)(s)-(Vx)(t)|-[(Vx)(s)-(Vx)(t)] \\ =& \left|a(\alpha(s))+(Tx)(\beta(s))\int_0^{\gamma(s)} v(s,\tau,x(\eta(\tau)))d\tau\right. \\ -& \left. a(\alpha(t))-(Tx)(\beta(t))\int_0^{\gamma(t)} v(t,\tau,x(\eta(\tau)))d\tau\right| \\ -& \left[a(\alpha(s))+(Tx)(\beta(s))\int_0^{\gamma(s)} v(s,\tau,x(\eta(\tau)))d\tau\right. \\ -& \left. a(\alpha(t))-(Tx)(\beta(t))\int_0^{\gamma(t)} v(t,\tau,x(\eta(\tau)))d\tau\right] \\ \leq& [|a(\alpha(s))-a(\alpha(t))|-(a(\alpha(s))-a(\alpha(t)))] \\ +& \left|(Tx)(\beta(s))\int_0^{\gamma(s)} v(s,\tau,x(\eta(\tau)))d\tau-(Tx)(\beta(t))\int_0^{\gamma(t)} v(t,\tau,x(\eta(\tau)))d\tau\right| \\ -& \left[(Tx)(\beta(s))\int_0^{\gamma(s)} v(s,\tau,x(\eta(\tau)))d\tau-(Tx)(\beta(t))\int_0^{\gamma(t)} v(t,\tau,x(\eta(\tau)))d\tau\right] \\ \leq& \left|(Tx)(\beta(s))\int_0^{\gamma(s)} v(s,\tau,x(\eta(\tau)))d\tau-(Tx)(\beta(t))\int_0^{\gamma(s)} v(s,\tau,x(\eta(\tau)))d\tau\right| \\ +& \left|(Tx)(\beta(t))\int_0^{\gamma(s)} v(s,\tau,x(\eta(\tau)))d\tau-(Tx)(\beta(t))\int_0^{\gamma(t)} v(t,\tau,x(\eta(\tau)))d\tau\right| \\ -& \left[(Tx)(\beta(s))\int_0^{\gamma(s)} v(s,\tau,x(\eta(\tau)))d\tau-(Tx)(\beta(t))\int_0^{\gamma(s)} v(s,\tau,x(\eta(\tau)))d\tau\right] \\ -& \left[(Tx)(\beta(t))\int_0^{\gamma(s)} v(s,\tau,x(\eta(\tau)))d\tau-(Tx)(\beta(t))\int_0^{\gamma(t)} v(t,\tau,x(\eta(\tau)))d\tau\right] \\ \leq& [|(Tx)(\beta(s))-(Tx)(\beta(t))|-[(Tx)(\beta(s))-(Tx)(\beta(t))]]\int_0^{\gamma(s)} v(s,\tau,x(\eta(\tau)))d\tau \\ \leq& [|(Tx)(\beta(s))-(Tx)(\beta(t))|-[(Tx)(\beta(s))-(Tx)(\beta(t))]]Mf(r_0). \end{aligned}$$

If we take supremum on both sides of this inequality over the $t,s \in I=[0,M]$, we have the inequality

$$\begin{aligned} i(Vx) &\leq Mf(r_0)\sup[|(Tx)(\beta(s))-(Tx)(\beta(t))|-[(Tx)(\beta(s))-(Tx)(\beta(t))]] \\ &\leq Mf(r_0)i(Tx), \end{aligned}$$

since the function $\beta$ is nondecreasing. If we take supremum over the $x$'s, we get the inequality

$$i(VX) \leq Mf(r_0)i(TX). \tag{3.3}$$

Finally, from the inequalities (3.2) and (3.3), we obtain

$$\mu(VX) \leq Mf(r_0)\mu(TX) \leq Mf(r_0)\theta\mu(X).$$

From the assumption $(viii)$ which is

$$Mf(r_0)\theta < 1$$

and by applying Theorem 2.1, $V$ has a fixed point in the set $B^+_{r_0}$.

Let us remember that from Remark 1, the set Fix$V$ of fixed points of $V$ belonging to $C(I)$ is a member of ker$\mu$. i.e. $\mu(\text{Fix}V)=0$ and this implies $i(\text{Fix}V)=0$. Therefore the solutions are nondecreasing on $I$. Thus the proof is completed. □

**Corollary 3.1.** *We assume that the function $a$ is positive, the function $f$ is continuous and the assumptions $(i)$-$(vi)$ and $(viii)$ are provided in the Theorem 3.1. Let us take the inequality*

$$a(\|\alpha\|)+(c+d)Mf(1) < 1$$

*instead of $(vii)$. So, the function $h$ defined as*

$$h:[0,1]\to\mathbb{R}, h(r)=a(\|\alpha\|)+(c+dr^p)Mf(r)-r$$

*is continuous and*

$$h(0)=a(\|\alpha\|)+cMf(0)>0$$

*and*

$$h(1)=a(\|\alpha\|)+(c+d)Mf(1)-1<0.$$

*Thus, there exists at least one a number $r_0\in(0,1)$ such that $h(r_0)=0$. Consequently, all of the assumptions of the Theorem 3.1 hold and the equation (3.1) has at least one solution $x=x(t)\in B^+_{r_0}$.*

**Example 3.1.** Let us consider the equation

$$x(t)=\frac{t^2}{5}+\frac{1+x^2(t)}{2}\int_0^{t^2}\frac{\sin t+e^{x(\tau^2)}}{8+\tau}d\tau, \quad t\in I=[0,1], \tag{3.4}$$

where $\alpha(t) = t^2$, $\beta(t) = t$, $\gamma(t) = t^2$, $\eta(\tau) = \tau^2$, $a(s) = \frac{s}{5}$, $a(\alpha(t)) = \frac{t^2}{5}$ and the function $a$ is nondecreasing and positive and $\|\alpha\| = 1$, $a(\|\alpha\|) = \frac{1}{5}$.

$$v(t, \tau, x) = \frac{\sin t + e^x}{8 + \tau}$$

and

$$(Tx)(t) = \frac{1 + x^2(t)}{2}.$$

We have the following estimate

$$|v(t, \tau, x)| = \left| \frac{\sin t + e^x}{8 + \tau} \right| \leq \frac{1 + e^x}{8} \leq \frac{1 + e^{|x|}}{8} = f(|x|)$$

for all $t, \tau \in I$ and $x \in \mathbb{R}$. From the above equation we see that $f(x) = \frac{1+e^x}{8}$. Let us see that the operator $T$ is continuous. Let $x_0$ be arbitrarily element chosen from $C(I)$. For $\|x - x_0\| < \delta$, we have the following estimate:

$$\begin{aligned} \|Tx - Tx_0\| &= \max_{t \in I} \left| \frac{1 + x^2(t)}{2} - \frac{1 + x_0^2(t)}{2} \right| \\ &= \frac{1}{2} \max_{t \in I} \left| x^2(t) - x_0^2(t) \right| \\ &= \frac{1}{2} \max_{t \in I} \left[ |x(t) - x_0(t)||x(t) + x_0(t)| \right] \end{aligned}$$

and

$$|x(t)| = |x(t) - x_0(t) + x_0(t)| \leq |x(t) - x_0(t)| + |x_0(t)| \leq \|x - x_0\| + \|x_0\|$$

such that,

$$|x(t)| \leq \delta + \|x_0\|. \tag{3.5}$$

From the inequality (3.5), we obtain

$$|x(t) + x_0(t)| \leq |x(t)| + \|x_0\| \leq \delta + 2\|x_0\|.$$

Thus, we obtain

$$\begin{aligned} \|Tx - Tx_0\| &= \frac{1}{2} \max_{t \in I} \left[ |x(t) - x_0(t)||x(t) + x_0(t)| \right] \\ &\leq \frac{1}{2} (\delta + 2\|x_0\|) \max_{t \in I} |x(t) - x_0(t)| \\ &= \frac{1}{2} (\delta + 2\|x_0\|) \|x - x_0\|. \end{aligned}$$

Taking

$$\frac{1}{2} (\delta + 2\|x_0\|) \delta = \varepsilon$$

we get

$$\begin{aligned}
\delta^2 + 2\|x_0\|\delta - 2\varepsilon = 0 \quad &\Rightarrow \quad (\delta + \|x_0\|)^2 - \|x_0\|^2 - 2\varepsilon = 0 \\
&\Rightarrow \quad (\delta + \|x_0\|)^2 = \|x_0\|^2 + 2\varepsilon \\
&\Rightarrow \quad \delta + \|x_0\| = \sqrt{\|x_0\|^2 + 2\varepsilon}.
\end{aligned}$$

If $\delta$ is chosen as

$$\delta = \sqrt{\|x_0\|^2 + 2\varepsilon} - \|x_0\| > 0,$$

it is seen that the operator $T$ is continuous at the point $x_0$. Since $x_0$ is an arbitrarily element chosen from $C(I)$, $T$ is continuous on $C(I)$. On the other hand, for each $x \in C(I)$ and each $t \in I$ the inequality

$$|(Tx)(t)| \leq c + d\|x\|^p, \quad (p > 0)$$

is provided. Namely,

$$\left|\frac{1 + x^2(t)}{2}\right| \leq \frac{1}{2} + \frac{1}{2}|x^2(t)| = \frac{1}{2} + \frac{1}{2}|x(t)|^2 \leq \frac{1}{2} + \frac{1}{2}\|x\|^2, \quad c = \frac{1}{2}, \quad d = \frac{1}{2}, \quad p = 2.$$

There exists $r_0$ positive solution that provides the inequality

$$a(\|\alpha\|) + (c + dr^p)Mf(r) \leq r,$$

where $\|\alpha\| = 1$, $a(\|\alpha\|) = \frac{1}{5}$, $M = 1$. Any number $r_0$ which provides the inequality

$$0,375018 \leq r_0 \leq 1,65394$$

is a solution of the following inequality:

$$\frac{1}{5} + \frac{1}{8}(1 + e^r)\left(\frac{1}{2} + \frac{1}{2}r^2\right) \leq r.$$

For example $r_0 = 1$ is a solution of this inequality.

Let $X \neq \emptyset$, $X \subset B_{r_0}^+$, $x \in B_{r_0}^+$ and $t_1, t_2 \in I$. We have the following estimate:

$$\begin{aligned}
|(Tx(t_2)) - (Tx(t_1))| \quad &= \quad \left|\frac{1 + x^2(t_2)}{2} - \frac{1 + x^2(t_1)}{2}\right| \\
&\leq \quad \frac{1}{2}|x(t_2) + x(t_1)||x(t_2) - x(t_1)| \\
&\leq \quad \frac{1}{2}(|x(t_2)| + |x(t_1)|)|x(t_2) - x(t_1)| \\
&\leq \quad \frac{1}{2}(\|x\| + \|x\|)|x(t_2) - x(t_1)| \\
&\leq \quad \frac{1}{2}(2r_0)|x(t_2) - x(t_1)|
\end{aligned}$$

$$\sup_{t_1,t_2\in I} |(Tx)(t_2)) - (Tx)(t_1))| \leq \sup_{t_1,t_2\in I} |x(t_2) - x(t_1)|$$

i.e.

$$w(Tx, \varepsilon) \leq w(x, \varepsilon).$$

Thus, we have the following inequalities:

$$\sup_{x\in X} w(Tx, \varepsilon) \leq \sup_{x\in X} w(x, \varepsilon),$$

$$w(TX, \varepsilon) \leq w(X, \varepsilon),$$

$$\lim_{\varepsilon\to 0} w(TX, \varepsilon) \leq \lim_{\varepsilon\to 0} w(X, \varepsilon),$$

$$w_0(TX) \leq w_0(X). \tag{3.6}$$

Let $X \neq \emptyset$, $X \subset B^+_{r_0}$, $x \in B^+_{r_0}$, $t_1 \leq t_2$ and $t_1, t_2 \in I$. In this case we have the following estimate:

$$\begin{aligned}
& |(Tx)(t_2) - (Tx)(t_1)| - [(Tx)(t_2) - (Tx)(t_1)] \\
= & \left|\frac{1 + x^2(t_2) - 1 - x^2(t_1)}{2}\right| - \left[\frac{1 + x^2(t_2) - 1 - x^2(t_1)}{2}\right] \\
\leq & \frac{1}{2}|x(t_2) - x(t_1)||x(t_2) + x(t_1)| - \frac{1}{2}[(x(t_2) - x(t_1))(x(t_2) + x(t_1))] \\
= & \frac{1}{2}\left(|x(t_2)| + |x(t_1)|\right)\left[|x(t_2) - x(t_1)| - (x(t_2) - x(t_1))\right] \\
\leq & \frac{1}{2}(\|x\| + \|x\|)[|x(t_2) - x(t_1)| - (x(t_2) - x(t_1))] \\
\leq & \frac{1}{2}2r_0[|x(t_2) - x(t_1)| - (x(t_2) - x(t_1))] \\
= & |x(t_2) - x(t_1)| - [x(t_2) - x(t_1)].
\end{aligned}$$

Hence,

$$\begin{aligned}
& \sup_{t_1,t_2\in I} [|(Tx)(t_2) - (Tx)(t_1)| - [(Tx)(t_2) - (Tx)(t_1)]] \\
\leq & \sup_{t_1,t_2\in I} [|x(t_2) - x(t_1)| - [x(t_2) - x(t_1)]],
\end{aligned}$$

$$i(Tx) \leq i(x)$$

in view of the inequalities,

$$\sup_{x\in X} i(Tx) \leq \sup_{x\in X} i(x)$$

and so, we obtain

$$i(TX) \leq i(X). \tag{3.7}$$

From the inequalities (3.6) and (3.7), we get

$$\mu(TX) \leq \mu(X),$$

where $\theta$ can be taken as $\theta = 1$. In this case the inequality $Mf(r_0)\theta < 1$ holds. Because, for $\theta = 1$, $r_0 = 1$, $M = 1$ and $f(1) = \frac{1+e}{8}$, the inequality

$$Mf(1)\theta = \frac{1+e}{8} < 1$$

holds. Since all of our assumptions are satisfied, this equation has a nondecreasing solution on $B^+_{r_0}$.

**Remark 2.** In the Example 3.1, since

$$|(Tx)(t)| \leq \frac{1}{2} + \frac{1}{2}\|x\|^2$$

for all $x \in C(I)$ and $t \in I$, the condition $(v)$

$$|(Tx)(t)| \leq c + d\|x\|$$

in [3] does not hold. Hence, the result given in [3] is not applicable to the integral equation (3.4) in the Example 3.1.

**Example 3.2.** Let us consider the equation

$$x(t) = \frac{\sin(t-1+\frac{\pi}{2})}{5} + \frac{1+x^3(t)}{7}\int_0^{t^3} \frac{\tan t + e^{x(\tau^2)}}{2+\tau} d\tau, \ t \in I = [0,1], \tag{3.8}$$

where $\alpha(t) = t$, $\beta(t) = t$, $\gamma(t) = t^3$, $\eta(\tau) = \tau^2$, the function $a(t) = \frac{\sin(t-1+\frac{\pi}{2})}{5}$ is nondecreasing and positive and $a(\|\alpha\|) = \frac{1}{5}$. We have the following estimate:

$$|v(t,\tau,x)| = \left|\frac{\tan t + e^x}{2+\tau}\right| \leq \frac{\sqrt{3}+e^x}{2} \leq \frac{\sqrt{3}+e^{|x|}}{2} = f(|x|)$$

for all $t, \tau \in I$ and $x \in \mathbb{R}$. From the above equation, we see that $f(x) = \frac{\sqrt{3}+e^x}{2}$ and $(Tx)(t) = \frac{1+x^3(t)}{7}$. It is obvious that $T : C(I) \to C(I)$. Let us see that the operator $T$ is continuous. Let $x_0$ be an arbitrarily element chosen from $C(I)$. When $\|x - x_0\| < \delta$ we have the following estimate:

$$\begin{aligned}
\|Tx - Tx_0\| &= \max_{t\in I}\left|\frac{1+x^3(t)}{7} - \frac{1+x_0^3(t)}{7}\right| \\
&= \frac{1}{7}\max_{t\in I}|x^3(t) - x_0^3(t)| \\
&= \frac{1}{7}\max_{t\in I}[|x(t)-x_0(t)||x^2(t) + x(t)x_0(t) + x_0^2(t)|]
\end{aligned}$$

$$\begin{aligned}
|x^2(t)+x(t)x_0(t)+x_0^2(t)| &= |(x(t)-x_0(t))^2+3x(t)x_0(t)| \\
&\le |(x(t)-x_0(t))|^2+3|x(t)x_0(t)| \\
&< \delta^2+3|x(t)-x_0(t)+x_0(t)||x_0(t)| \\
&\le \delta^2+3\delta\|x_0\|+3\|x_0\|^2.
\end{aligned}$$

From the above inequalities, we obtain

$$\begin{aligned}
\|Tx-Tx_0\| &= \frac{1}{7}\max_{t\in I}[|x(t)-x_0(t)||x^2(t)+x(t)x_0(t)+x_0^2(t)|] \\
&< \frac{1}{7}(\delta^3+3\delta^2\|x_0\|+3\delta\|x_0\|^2) \\
&= \frac{1}{7}((\delta+\|x_0\|)^3-\|x_0\|^3) \\
&= \frac{1}{7}(\delta+\|x_0\|)^3-\frac{1}{7}\|x_0\|^3=\varepsilon \\
&\Rightarrow \frac{1}{7}(\delta+\|x_0\|)^3=\varepsilon+\frac{1}{7}\|x_0\|^3 \\
&\Rightarrow \delta+\|x_0\|=(7\varepsilon+\|x_0\|^3)^{\frac{1}{3}} \\
&\Rightarrow \delta=(7\varepsilon+\|x_0\|^3)^{\frac{1}{3}}-\|x_0\|>0.
\end{aligned}$$

If $\delta$ is chosen as $\delta=(7\varepsilon+\|x_0\|^3)^{\frac{1}{3}}-\|x_0\|>0$, it is seen that the operator $T$ is continuous at the point $x_0$. Since $x_0$ is an arbitrarily element chosen from $C(I)$, $T$ is continuous on $C(I)$. Since

$$|(Tx)(t)|=\left|\frac{1+x^3(t)}{7}\right|\le\frac{1}{7}+\frac{1}{7}|x^3(t)|=\frac{1}{7}+\frac{1}{7}|x(t)|^3\le\frac{1}{7}+\frac{1}{7}\|x\|^3,$$

$$c=\frac{1}{7},\quad d=\frac{1}{7},\quad p=3,$$

the inequality

$$|(Tx)(t)|\le c+d\|x\|^p,\quad (p>0)$$

holds. There exists positive solution $r_0$ that provides the inequality

$$a(\|\alpha\|)+(c+dr^p)Mf(r)\le r,$$

where $\|\alpha\|=1$, $a(\|\alpha\|)=\frac{1}{5}$, $M=1$.

Any number $r_0$ providing the inequality

$$0,386812\le r_0\le 1,32116$$

is a solution of the following inequality:

$$\left(\frac{1}{2}(\sqrt{3}+e^r)\right)\left(\frac{1}{7}+\frac{1}{7}r^3\right)+\frac{1}{5}\le r.$$

For example $r_0 = 1$ is a solution of this inequality.

For any $t_1, t_2 \in [0,1]$ such that $|t_2 - t_1| \le \varepsilon$, $\emptyset \ne X \subset B_{r_0}^+ = B_1^+$ and $x \in X$, we obtain

$$\begin{aligned} |(Tx)(t_2) - (Tx)(t_1)| &= \left| \frac{1 + x^3(t_2)}{7} - \frac{1 + x^3(t_1)}{7} \right| \\ &= \frac{1}{7}|x^3(t_2) - x^3(t_1)| \\ &\le \frac{1}{7}|x(t_2) - x(t_1)||x^2(t_2) + x(t_2)x(t_1) + x^2(t_1)| \\ &\le \frac{1}{7}|x(t_2) - x(t_1)|(|x^2(t_2)| + |x(t_2)x(t_1)| + |x^2(t_1)|) \\ &\le \frac{1}{7}|x(t_2) - x(t_1)|3\|x\|^2 \\ &\le \frac{3}{7}|x(t_2) - x(t_1)|. \end{aligned}$$

If we take the supremum on both sides of inequality over $t_1, t_2 \in I$ such that $|t_2 - t_1| \le \varepsilon$ and $x \in X$, we get

$$w(Tx, \varepsilon) \le \frac{3}{7} w(x, \varepsilon).$$

If we take the supremum at this inequality over $x \in X$, we get

$$w(TX, \varepsilon) \le \frac{3}{7} w(X, \varepsilon),$$

where, for $\varepsilon \to 0$, we obtain

$$w_0(TX) \le \frac{3}{7} w_0(X). \tag{3.9}$$

For any $t_1, t_2 \in [0,1]$ such that $t_1 \le t_2$, $\emptyset \ne X \subset B_{r_0}^+ = B_1^+$ and $x \in X$, we get

$$\begin{aligned} & |(Tx)(t_2) - (Tx)(t_1)| - [(Tx)(t_2) - (Tx)(t_1)] \\ = & \left| \frac{1 + x^3(t_2)}{7} - \frac{1 + x^3(t_1)}{7} \right| - \left[ \frac{1 + x^3(t_2)}{7} - \frac{1 + x^3(t_1)}{7} \right] \\ = & \frac{1}{7} \left| x^3(t_2) - x^3(t_1) \right| - \frac{1}{7}[x^3(t_2) - x^3(t_1)] \\ = & \frac{1}{7}|(x(t_2) - x(t_1))(x^2(t_2) + x(t_2)x(t_1) + x^2(t_1))| \\ - & \frac{1}{7}(x(t_2) - x(t_1))(x^2(t_2) + x(t_2)x(t_1) + x^2(t_1)) \\ = & \frac{1}{7}(x^2(t_2) + x(t_2)x(t_1) + x^2(t_1))[|x(t_2) - x(t_1)| - (x(t_2) - x(t_1))] \\ \le & \frac{3}{7}\|x\|^2[|x(t_2) - x(t_1)| - (x(t_2) - x(t_1))] \\ \le & \frac{3}{7}[|x(t_2) - x(t_1)| - (x(t_2) - x(t_1))]. \end{aligned}$$

If we take the supremum at this inequality over $t_1, t_2 \in [0,1]$ such that $t_1 \leq t_2$, we get

$$i(Tx) \leq \frac{3}{7} i(x).$$

If we take the supremum at this inequality over $x \in X$, we get

$$i(TX) \leq \frac{3}{7} i(X). \tag{3.10}$$

From the inequalities (3.9) and (3.10) we get

$$\mu(TX) \leq \frac{3}{7} \mu(X).$$

Then, $\theta$ can be taken as $\theta = \frac{3}{7}$. For $r = 1$, $\theta = \frac{3}{7}$ and $M = 1$, we get

$$Mf(1)\frac{3}{7} < 1.$$

Thus, all of our assumptions provide and hence this equation has a nondecreasing solution on $B^+_{r_0}$.

**Remark 3.** In the Example 3.2, since

$$|(Tx)(t)| \leq \frac{1}{7} + \frac{1}{7}\|x\|^3$$

for all $x \in C(I)$ and $t \in I$, the condition $(v)$

$$|(Tx)(t)| \leq c + d\|x\|$$

in [3] does not hold. Hence, the result given in [3] is not applicable to the integral equation (3.8) presented in the Example 3.2.

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**Osman Karakurt**
email: `osman-44@yandex.com`
ORCID: 0000-0002-4669-8470
Yeşilyurt G. N. Mesleki ve Teknik Anadolu Lisesi
Malatya
TURKEY

**Ömer Faruk Temizer***
email: `omer.temizer@inonu.edu.tr`
ORCID: 0000-0002-3843-5945
Eğitim Fakültesi, A-Blok
İnönü Üniversitesi
44280-Malatya
TURKEY
*Corresponding author

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# Number of Zeros of a Polynomial in a Specific Region with Restricted Coefficients

*Abdullah Mir, Abrar Ahmad and Adil Hussain Malik*

ABSTRACT: This paper focuses on the problem concerning the location and the number of zeros of polynomials in a specific region when their coefficients are restricted with special conditions. We obtain extensions of some classical results concerning the number of zeros of polynomials in a prescribed region by imposing the restrictions on the moduli of the coefficients, the real parts(only) of the coefficients, and the real and imaginary parts of the coefficients.

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*Keywords and Phrases:* Polynomial; Zeros; Eneström-Kakeya theorem.

## 1. Introduction

Locating zeros of polynomials with special conditions for the coefficients, in particular, the number of zeros of complex polynomials in a disk when their coefficients are restricted with special conditions has applications in many areas of applied mathematics, including linear control systems, electrical networks, root approximation and signal processing, and for this reason there is always a need for better and better estimates for the region containing some or all the zeros of a polynomial. A review on the location of zeros of polynomials can be found in ([1], [5], [8], [11]). If $P(z) = \sum_{j=0}^{n} a_j z^j$ is a polynomial of degree $n$ such that $a_n \geq a_{n-1} \geq ... \geq a_1 \geq a_0 > 0$, then $P(z)$ has all its zeros in $|z| \leq 1$. This famous result is known as Eneström-Kakeya theorem, for reference see (section 8.3 of [11]). In the literature, for example see ([1] - [12]), there exist various extensions and generalizations of Eneström-Kakeya theorem. Taking

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account of the restrictions on the coefficients of a polynomial allows for establishing improved bounds and here, in this paper, we impose some restrictions on the coefficients of polynomials in order to count the number of zeros in a specific region. The following result concerning the number of zeros of a polynomial in a closed disk can be found in Titchmarsh's classic "The Theory of Functions", see ([13], page 171, 2nd edition).

**Theorem A.** *Let $F(z)$ be analytic in $|z| \leq R$. Let $|F(z)| \leq M$ in $|z| \leq R$ and suppose $F(0) \neq 0$. Then for $0 < \delta < 1$, the number of zeros of $F(z)$ in the disk $|z| \leq R\delta$ does not exceed*

$$\frac{1}{\log \frac{1}{\delta}} \log \frac{M}{|F(0)|}.$$

Regarding the number of zeros in $|z| \leq \frac{1}{2}$ and by putting a restriction on the coefficients of a polynomial similar to that of the Eneström-Kakeya theorem, Mohammad [9] used a special case of Theorem A to prove the following result.

**Theorem B.** *If $P(z) = \sum_{j=0}^{n} a_j z^j$ is a polynomial of degree $n$ such that $0 < a_0 \leq a_1 \leq ... \leq a_n$, then the number of zeros of $P(z)$ in $|z| \leq \frac{1}{2}$ does not exceed*

$$1 + \frac{1}{\log 2} \log \left( \frac{a_n}{a_0} \right).$$

The above result of Mohammad [9] was generalized in different ways for example see ([1], [2], [4], [5], [11]). Using hypotheses related to those of Theorem B, very recently Qasim et al. [6] imposed a monotonic condition on the moduli and then on the real and imaginary parts of the coefficients of the Lucanary type of polynomials $P(z) = a_0 + \sum\limits_{j=\mu}^{n} a_j z^j$ and proved the following results.

**Theorem C.** *Let $P(z) = a_0 + \sum\limits_{j=\mu}^{n} a_j z^j, 1 \leq \mu \leq n-1, a_0 \neq 0$ be a polynomial of degree $n$. If for some real $\alpha$ and $\beta$*

$$|\arg\, a_j - \beta| \leq \alpha \leq \frac{\pi}{2}, \qquad \mu \leq j \leq n,$$

*and for some $t > 0$ and some $k$ with $\mu \leq k \leq n$,*

$$t^{\mu}|a_{\mu}| \leq ... \leq t^{k-1}|a_{k-1}| \leq t^{k}|a_k| \geq t^{k+1}|a_{k+1}| \geq ... \geq t^{n-1}|a_{n-1}| \geq t^{n}|a_n|,$$

*then the number of zeros of $P(z)$ in $|z| \leq \frac{1}{2}$ does not exceed*

$$\frac{1}{\log 2} \log \left( \frac{M}{|a_0|} \right),$$

*where*

$$\begin{aligned} M =& 2|a_0|t + |a_{\mu}|t^{\mu+1}(1 - \sin\alpha - \cos\alpha) + 2|a_k|t^{k+1}\cos\alpha + \\ & |a_n|t^{n+1}(1 - \sin\alpha - \cos\alpha) + 2\sum_{j=\mu}^{n} |a_j|t^{j+1}\sin\alpha. \end{aligned}$$

**Theorem D.** *Let* $P(z) = a_0 + \sum\limits_{j=\mu}^{n} a_j z^j, 1 \leq \mu \leq n-1, a_0 \neq 0$ *be a polynomial of degree* $n$ *with Re* $a_j = \alpha_j$ *and Im* $a_j = \beta_j$ *for* $\mu \leq j \leq n$*. Suppose that for some* $t > 0$ *and some* $k$ *with* $\mu \leq k \leq n$*, we have*

$$t^{\mu}\alpha_{\mu} \leq ... \leq t^{k-1}\alpha_{k-1} \leq t^k\alpha_k \geq t^{k+1}\alpha_{k+1} \geq ... \geq t^{n-1}\alpha_{n-1} \geq t^n\alpha_n,$$

*then the number of zeros of* $P(z)$ *in* $|z| \leq \frac{t}{2}$ *does not exceed*

$$\frac{1}{\log 2} \log\left(\frac{M}{|a_0|}\right),$$

*where*

$$\begin{aligned} M =& 2(|\alpha_0| + |\beta_0|)t + (|a_\mu| - \alpha_\mu)t^{\mu+1} + 2|\alpha_k|t^{k+1} + \\ & (|\alpha_n| - \alpha_n)t^{n+1} + 2\sum_{j=\mu}^{n} |\beta_j|t^{j+1}. \end{aligned}$$

**Theorem E.** *Let* $P(z) = a_0 + \sum\limits_{j=\mu}^{n} a_j z^j, 1 \leq \mu \leq n-1,\ a_0 \neq 0$ *be a polynomial of degree* $n$ *with Re* $a_j = \alpha_j$ *and Im* $a_j = \beta_j$ *for* $\mu \leq j \leq n$*. Suppose that for some* $t > 0$ *and some* $k$ *with* $\mu \leq k \leq n$*, we have*

$$t^{\mu}\alpha_{\mu} \leq ... \leq t^{k-1}\alpha_{k-1} \leq t^k\alpha_k \geq t^{k+1}\alpha_{k+1} \geq ... \geq t^{n-1}\alpha_{n-1} \geq t^n\alpha_n$$

*and for some* $\mu \leq l \leq n$ *we have*

$$t^{\mu}\beta_{\mu} \leq ... \leq t^{l-1}\beta_{l-1} \leq t^k\beta_l \geq t^{l+1}\beta_{l+1} \geq ... \geq t^{n-1}\beta_{n-1} \geq t^n\beta_n,$$

*then the number of zeros of* $P(z)$ *in* $|z| \leq \frac{t}{2}$ *does not exceed*

$$\frac{1}{\log 2} \log\left(\frac{M}{|a_0|}\right),$$

*where*

$$\begin{aligned} M =& 2(|\alpha_0| + |\beta_0|)t + (|a_\mu| - \alpha_\mu + |\beta_\mu| - \beta_\mu)t^{\mu+1} \\ & + 2(\alpha_k t^{k+1} + \beta_l t^{l+1}) + (|\alpha_n| - \alpha_n + |\beta_\mu| - \beta_\mu)t^{n+1}. \end{aligned}$$

In this paper, we further weaken the hypotheses of the above results and prove the following.

## 2. Main results

**Theorem 1.** *Let* $P(z) = a_0 + \sum_{j=\mu}^{n} a_j z^j, 1 \leq \mu < n,\ a_0 \neq 0$, *where for some* $t > 0$ *and some* $\mu \leq k \leq n$,

$$t^{\mu}|a_{\mu}| \leq ... \leq t^{k-1}|a_{k-1}| \leq t^{k}|a_k| \geq t^{k+1}|a_{k+1}| \geq ... \geq t^{n-1}|a_{n-1}| \geq t^{n}|a_n|$$

*and* $|\arg\ a_j - \beta| \leq \alpha \leq \frac{\pi}{2}$ *for* $\mu \leq j \leq n$, *for some real* $\alpha$ *and* $\beta$. *Then for* $0 < \delta < 1$, *the number of zeros of* $P(z)$ *in the disk* $|z| \leq \delta t$ *does not exceed*

$$\frac{1}{\log\frac{1}{\delta}} \log\frac{M}{|a_0|},$$

*where*

$$\begin{aligned} M =& 2|a_0|t + (|a_{\mu}|t^{\mu+1} + |a_n|t^{n+1})(1 - \cos\alpha - \sin\alpha) \\ &+ 2|a_k|t^{k+1}\cos\alpha + 2\sum_{j=\mu}^{n} |a_j|t^{j+1}\sin\alpha. \end{aligned}$$

Notice that when $t = 1$ in Theorem 1, we get the following.

**Corollary 1.** *Let* $P(z) = a_0 + \sum_{j=\mu}^{n} a_j z^j, 1 \leq \mu < n, a_0 \neq 0$, *where for some* $\mu \leq k \leq n$,

$$|a_{\mu}| \leq ... \leq |a_{k-1}| \leq |a_k| \geq |a_{k+1}| \geq ... \geq |a_{n-1}| \geq |a_n|$$

*and* $|\arg\ a_j - \beta| \leq \alpha \leq \frac{\pi}{2}$ *for* $\mu \leq j \leq n$, *for some real* $\alpha$ *and* $\beta$. *Then for* $0 < \delta < 1$, *the number of zeros of* $P(z)$ *in the disk* $|z| \leq \delta$ *does not exceed*

$$\frac{1}{\log\frac{1}{\delta}} \log\frac{M}{|a_0|},$$

*where*

$$\begin{aligned} M =& 2|a_0| + (|a_{\mu}| + |a_n|)(1 - \cos\alpha - \sin\alpha) \\ &+ 2|a_k|\cos\alpha + 2\sum_{j=\mu}^{n} |a_j|\sin\alpha. \end{aligned}$$

Clearly for $\delta = \frac{1}{2}$, Theorem 1 reduces to Theorem C and Corollary 1 reduces to Corollary 1.1 of Qasim et al. [6]. With $t = 1$ and $k = n$ in Theorem 1, the hypothesis becomes $|a_{\mu}| \leq ... \leq |a_{n-1}| \leq |a_n|$, and the value of M becomes $2|a_0| + (|a_{\mu}| + |a_n|)(1 - \cos\alpha - \sin\alpha) + 2|a_n|\cos\alpha + 2\sum_{j=\mu}^{n} |a_j|\sin\alpha$, and hence Theorem 1 implies Corollary 1.2 of Qasim et al. [6]. In the same way for $t = 1$, $k = \mu$ and for $\delta = \frac{1}{2}$, Theorem 1 implies Corollary 1.3 of Qasim et al. [6].

**Theorem 2.** *Let* $P(z) = a_0 + \sum_{j=\mu}^{n} a_j z^j, 1 \leq \mu < n,\ a_0 \neq 0,$ *where* $Re\ a_j = \alpha_j$ *and* $Im\ a_j = \beta_j$ *for* $\mu \leq j \leq n$. *Suppose that for some* $t > 0$ *and some* $k$ *with* $\mu \leq k \leq n$, *we have*

$$t^\mu \alpha_\mu \leq ... \leq t^{k-1}\alpha_{k-1} \leq t^k \alpha_k \geq t^{k+1}\alpha_{k+1} \geq ... \geq t^{n-1}\alpha_{n-1} \geq t^n \alpha_n.$$

*Then for* $0 < \delta < 1$, *the number of zeros of* $P(z)$ *in the disk* $|z| \leq \delta t$ *does not exceed*

$$\frac{1}{\log\frac{1}{\delta}} \log\frac{M}{|a_0|},$$

*where*

$$\begin{aligned} M =& 2(|\alpha_0| + |\beta_0|)t + (|\alpha_\mu| - \alpha_\mu)t^{\mu+1} \\ &+ 2\alpha_k t^{k+1} + (|\alpha_n| - \alpha_n)t^{n+1} + 2\sum_{j=\mu}^{n} |\beta_j| t^{j+1}. \end{aligned}$$

**Remark 1.** For $\delta = \frac{1}{2}$, Theorem 2 reduces to Theorem D.

Notice that with $t = 1$ in Theorem 2, we get the following.

**Corollary 2.** *Let* $P(z) = a_0 + \sum_{j=\mu}^{n} a_j z^j, 1 \leq \mu < n,\ a_0 \neq 0,$ *where* $Re\ a_j = \alpha_j$ *and* $Im\ a_j = \beta_j$ *for* $\mu \leq j \leq n$. *Suppose we have for some* $\mu \leq k \leq n$,

$$\alpha_\mu \leq ... \leq \alpha_{k-1} \leq \alpha_k \geq \alpha_{k+1} \geq ... \geq \alpha_{n-1} \geq \alpha_n.$$

*Then for* $0 < \delta < 1$, *the number of zeros of* $P(z)$ *in the disk* $|z| < \delta$ *does not exceed*

$$\frac{1}{\log\frac{1}{\delta}} \log\frac{M}{|a_0|},$$

*where* $M = 2(|\alpha_0| + |\beta_0|) + (|\alpha_\mu| - \alpha_\mu) + 2\alpha_k + (|\alpha_n| - \alpha_n) + 2\sum_{j=\mu}^{n} |\beta_j|$.
Clearly for $\delta = \frac{1}{2}$, the Corollary 2 reduces to Corollary 2.1 of Qasim et al. [6].
With $t = 1, k = n$ in Theorem 2, the hypothesis becomes $\alpha_\mu \leq ... \leq \alpha_{n-1} \leq \alpha_n, 1 \leq \mu < n$ and the value of $M$ becomes

$$2(|\alpha_0| + |\beta_0|) + (|\alpha_\mu| - \alpha_\mu) + (|\alpha_n| + \alpha_n) + 2\sum_{j=\mu}^{n} |\beta_j|,$$

therefore, Corollary 2.2 of Qasim et al. [6] follows from Theorem 2.
By manipulating the parameter $k$, $\mu$ and $t$, we easily get Corollary 2.3 and Corollary 2.4 of Qasim et al. [6] from Theorem 2.
Finally, we put the monotonicity-type condition on the real and imaginary parts of

the coefficient of $P(z) = a_0 + \sum_{j=\mu}^{n} a_j z^j$ and get the following result.

**Theorem 3.** *Let* $P(z) = a_0 + \sum_{j=\mu}^{n} a_j z^j, 1 \leq \mu < n,\ a_0 \neq 0$ *where* $Re\ a_j = \alpha_j$ *and* $Im\ a_j = \beta_j$ *for* $\mu \leq j \leq n$. *Suppose that for some* $t > 0$, *for some* $\mu \leq k \leq n$, *we have*

$$t^{\mu}\alpha_{\mu} \leq ... \leq t^{k-1}\alpha_{k-1} \leq t^{k}\alpha_k \geq t^{k+1}\alpha_{k+1} \geq ... \geq t^{n-1}\alpha_{n-1} \geq t^{n}\alpha_n$$

*and for some* $\mu \leq l \leq n$, *we have*

$$t^{\mu}\beta_{\mu} \leq ... \leq t^{l-1}\beta_{l-1} \leq t^{k}\beta_l \geq t^{l+1}\beta_{l+1} \geq ... \geq t^{n-1}\beta_{n-1} \geq t^{n}\beta_n.$$

*Then for* $0 < \delta < 1$, *the number of zeros of* $P(z)$ *in the disk* $|z| \leq \delta t$ *does not exceed*

$$\frac{1}{\log\frac{1}{\delta}}\log\frac{M}{|a_0|},$$

*where*

$$\begin{aligned} M &= 2(|\alpha_0| + |\beta_0|)t + (|a_{\mu}| - \alpha_{\mu} + |\beta_{\mu}| - \beta_{\mu})t^{\mu+1} \\ &\quad +2(\alpha_k t^{k+1} + \beta_l t^{l+1}) + (|\alpha_n| - \alpha_n + |\beta_{\mu}| - \beta_{\mu})t^{n+1}. \end{aligned}$$

Taking $\delta = \frac{1}{2}$ in Theorem 3, we get Theorem E. Theorem 3 gives several corollaries with hypotheses concerning monotonicity of real and imaginary parts. For example, with $t = 1$, we have the following result.

**Corollary 3.** *Let* $P(z) = a_0 + \sum_{j=\mu}^{n} a_j z^j, 1 \leq \mu < n,\ a_0 \neq 0$, *where* $Re\ a_j = \alpha_j$ *and* $Im\ a_j = \beta_j$ *for* $\mu \leq j \leq n$. *Suppose that for some* $\mu \leq k \leq n$, *we have*

$$\alpha_{\mu} \leq ... \leq \alpha_{k-1} \leq \alpha_k \geq \alpha_{k+1} \geq ... \geq \alpha_{n-1} \geq \alpha_n$$

*and for some* $\mu \leq l \leq n$, *we have*

$$\beta_{\mu} \leq ... \leq \beta_{l-1} \leq \beta_l \geq \beta_{l+1} \geq ... \geq \beta_{n-1} \geq \beta_n.$$

*Then for* $0 < \delta < 1$, *the number of zeros of* $P(z)$ *in the disk* $|z| \leq \delta$ *does not exceed*

$$\frac{1}{\log\frac{1}{\delta}}\log\frac{M}{|a_0|},$$

*where*

$$M = 2(|\alpha_0| + |\beta_0|) + (|a_{\mu}| - \alpha_{\mu} + |\beta_{\mu}| - \beta_{\mu}) + 2(\alpha_k + \beta_l) + (|\alpha_n| - \alpha_n + |\beta_{\mu}| - \beta_{\mu}).$$

With $t = 1$ and $k = l = n$ in Theorem 3, we get the following.

**Corollary 4.** *Let* $P(z) = a_0 + \sum_{j=\mu}^{n} a_j z^j, 1 \leq \mu < n,\ a_0 \neq 0,$ *where Re* $a_j = \alpha_j$ *and Im* $a_j = \beta_j$ *for* $\mu \leq j \leq n$. *Suppose that for some* $\mu \leq k \leq n$, *we have*

$$\alpha_\mu \leq ... \leq \alpha_{n-1} \leq \alpha_n$$

*and*

$$\beta_\mu \leq ... \leq \beta_{n-1} \leq \beta_n.$$

*Then for* $0 < \delta < 1$, *the number of zeros of* $P(z)$ *in the disk* $|z| \leq \delta$ *does not exceed*

$$\frac{1}{\log\frac{1}{\delta}}\log\frac{M}{|a_0|},$$

*where*

$$M = 2(|\alpha_0| + |\beta_0|) + (|a_\mu| - \alpha_\mu + |\beta_\mu| - \beta_\mu) + (|\alpha_n| + \alpha_n + |\beta_\mu| + \beta_\mu).$$

For $t = 1,\ k = l = \mu$ in Theorem 3, we get the following

**Corollary 5.** *Let* $P(z) = a_0 + \sum_{j=\mu}^{n} a_j z^j, 1 \leq \mu < n,\ a_0 \neq 0,$ *where Re* $a_j = \alpha_j$ *and Im* $a_j = \beta_j$ *for* $\mu \leq j \leq n$. *Suppose that*

$$\alpha_\mu \geq ... \geq \alpha_{n-1} \geq \alpha_n$$

*and*

$$\beta_\mu \geq ... \geq \beta_{n-1} \geq \beta_n.$$

*Then for* $0 < \delta < 1$, *the number of zeros of* $P(z)$ *in the disk* $|z| \leq \delta$ *does not exceed*

$$\frac{1}{\log\frac{1}{\delta}}\log\frac{M}{|a_0|},$$

*where*

$$M = 2(|\alpha_0| + |\beta_0|) + (|a_\mu| + \alpha_\mu + |\beta_\mu| + \beta_\mu) + (|\alpha_n| - \alpha_n + |\beta_\mu| - \beta_\mu).$$

## 3. Proofs of theorems

We need the following lemma for the proofs of theorems.

**Lemma 1.** *For any two complex numbers* $b_0$ *and* $b_1$ *such that* $|b_0| \geq |b_1|$. *Suppose* $|\arg\ b_j - \beta| \leq \alpha \leq \frac{\pi}{2}$, *for* $j = 0, 1$ *for some real* $\alpha$ *and* $\beta$, *then*

$$|b_0 - b_1| \leq (|b_0| - |b_1|)\cos\alpha + (|b_0| + |b_1|)\sin\alpha.$$

The above lemma is due to Govil and Rahman [5].

**Proof of Theorem 1.** Consider the polynomial

$$\begin{aligned}
F(z) &= (t-z)P(z)\\
&= (t-z)\left(a_0 + \sum_{j=\mu}^{n} a_j z^j\right)\\
&= a_0 t + \sum_{j=\mu}^{n} t a_j z^j - a_0 z - \sum_{j=\mu}^{n} a_j z^{j+1}\\
&= a_0(t-z) + \sum_{j=\mu}^{n} t a_j z^j - \sum_{j=\mu+1}^{n+1} a_{j-1} z^j\\
&= a_0(t-z) + t a_\mu z^\mu + \sum_{j=\mu+1}^{n} (t a_j - a_{j-1}) z^j - a_n z^{n+1}.
\end{aligned}$$

For $|z| = t$, we have

$$\begin{aligned}
|F(z)| &\le 2t|a_0| + |a_\mu| t^{\mu+1} + \sum_{j=\mu+1}^{n} |t a_j - a_{j-1}| t^j + |a_n| t^{n+1}\\
&= 2t|a_0| + |a_\mu| t^{\mu+1} + \sum_{j=\mu+1}^{k} |t a_j - a_{j-1}| t^j + \sum_{j=k+1}^{n} |a_{j-1} - t a_j| t^j + |a_n| t^{n+1}.
\end{aligned}$$

Using Lemma 1 with $b_0 = a_j t$ and $b_1 = a_{j-1}$ when $\mu + 1 \le j \le k$ and with $b_0 = a_{j-1}$ and $b_1 = a_j t$ when $k+1 \le j \le n$,

$$\begin{aligned}
|F(z)| &\le 2t|a_0| + |a_\mu| t^{\mu+1} + \sum_{j=\mu+1}^{k} \left\{(|a_j| t - |a_{j-1}|)\cos\alpha + (|a_j| t + |a_{j-1}|)\sin\alpha\right\} t^j\\
&\quad + \sum_{j=k+1}^{n} \left\{(|a_{j-1}| - |a_j| t)\cos\alpha + (|a_j| t + |a_{j-1}|)\sin\alpha\right\} t^j + |a_n| t^{n+1}\\
&= 2|a_0| t + |a_\mu| t^{\mu+1} + \sum_{j=\mu+1}^{k} |a_j| t^{j+1}\cos\alpha - \sum_{j=\mu+1}^{k} |a_{j-1}| t^j \cos\alpha\\
&\quad + \sum_{j=\mu+1}^{k} |a_j| t^{j+1}\sin\alpha + \sum_{j=\mu+1}^{k} |a_{j-1}| t^j \sin\alpha + \sum_{j=k+1}^{n} |a_{j-1}| t^j \cos\alpha\\
&\quad - \sum_{j=k+1}^{n} |a_j| t^{j+1}\cos\alpha + \sum_{j=k+1}^{n} |a_{j-1}| t^j \sin\alpha + \sum_{j=k+1}^{n} |a_j| t^{j+1}\sin\alpha + |a_n| t^{n+1}
\end{aligned}$$

$$
\begin{aligned}
&= 2|a_0|t + |a_\mu|t^{\mu+1} - |a_\mu|t^{\mu+1}\cos\alpha + |a_k|t^{k+1}\cos\alpha + |a_\mu|t^{\mu+1}\sin\alpha \\
&+ |a_k|t^{k+1}\sin\alpha + 2\sum_{j=\mu+1}^{k-1}|a_j|t^{j+1}\sin\alpha + |a_k|t^{k+1}\cos\alpha - |a_n|t^{n+1}\cos\alpha + |a_k|t^{k+1}\sin\alpha \\
&+ |a_n|t^{n+1}\sin\alpha + 2\sum_{j=k+1}^{n-1}|a_j|t^{j+1}\sin\alpha + |a_n|t^{n+1}. \\
&= 2|a_0|t + |a_\mu|t^{\mu+1} + |a_\mu|t^{\mu+1}(\sin\alpha - \cos\alpha) + 2\sum_{j=\mu+1}^{n-1}|a_j|t^{j+1}\sin\alpha \\
&+ 2|a_k|t^{k+1}\cos\alpha + (\sin\alpha - \cos\alpha + 1)|a_n|t^{n+1} \\
&= 2|a_0|t + |a_\mu|t^{\mu+1}(1 - \sin\alpha - \cos\alpha) + 2|a_k|t^{k+1}\cos\alpha + |a_n|t^{n+1}(1 - \sin\alpha - \cos\alpha) \\
&+ 2\sum_{j=\mu}^{n}|a_j|t^{j+1}\sin\alpha. \\
&= M(say).
\end{aligned}
$$

Now $F(z)$ is analytic in $|z| \leq t$ and $|F(z)| \leq M$ for $|z| = t$. So by Theorem A and the Maximum Modulus Theorem, the number of zeros of $F$ (and hence of $P$) in $|z| \leq \delta t$ is less than or equal to

$$\frac{1}{\log\frac{1}{\delta}}\ \log\left(\frac{M}{|a_0|}\right).$$

Hence the Theorem 1 follows.

**Proof of Theorem 2.** As in the proof of Theorem 1,

$$
\begin{aligned}
F(z) &= (t-z)P(z) \\
&= a_0(t-z) + ta_\mu z^\mu + \sum_{j=\mu+1}^{n}(ta_j - a_{j-1})z^j - a_n z^{n+1},
\end{aligned}
$$

and so

$$
\begin{aligned}
F(z) &= (\alpha_0 + i\beta_0)(t-z) + (\alpha_\mu + i\beta_\mu)tz^\mu + \sum_{j=\mu+1}^{n}(\alpha_j t - \alpha_{j-1})z^j \\
&+ i\sum_{j=\mu+1}^{n}(\beta_j t - \beta_{j-1})z^j - (\alpha_n + i\beta_n)z^{n+1}.
\end{aligned}
$$

For $|z|=t$, we have

$$
\begin{aligned}
|F(z)| &\leq 2(|\alpha_0|+|\beta_0|)t+(|\alpha_\mu|+|\beta_\mu|)t^{\mu+1}+\sum_{j=\mu+1}^{n}|\alpha_j t-\alpha_{j-1}|t^j \\
&+\sum_{j=\mu+1}^{n}(|\beta_j|t+|\beta_{j-1}|)t^j+(|\alpha_n|+|\beta_n|)t^{n+1} \\
&=2(|\alpha_0|+|\beta_0|)t+(|\alpha_\mu|+|\beta_\mu|)t^{\mu+1}+\sum_{j=\mu+1}^{n}(\alpha_j t-\alpha_{j-1})t^j \\
&+\sum_{j=k+1}^{n}(\alpha_{j-1}-\alpha_j t)t^j+|\beta_\mu|t^{\mu+1}+2\sum_{j=\mu+1}^{n-1}|\beta_j|t^{j+1}+|\beta_n|t^{n+1} \\
&+(|\alpha_n|+|\beta_n|)t^{n+1} \\
&=2(|\alpha_0|+|\beta_0|)t+(|\alpha_\mu|-\alpha_\mu)t^{\mu+1}+2\alpha_k t^{k+1}+(|\alpha_n|-\alpha_n)t^{n+1} \\
&+2\sum_{j=\mu}^{n}|\beta_j|t^{j+1} \\
&=M
\end{aligned}
$$

The result follows as in the proof of Theorem 1.

**Proof of Theorem 3.** As in the proof of Theorem 2,

$$
\begin{aligned}
F(z) &= (t-z)P(z) \\
&= a_0(t-z)+ta_\mu z^\mu+\sum_{j=\mu+1}^{n}(a_j t-a_{j-1})z^j-a_n z^{n+1} \\
&= (\alpha_0+i\beta_0)(t-z)+(\alpha_\mu+i\beta_\mu)tz^\mu+\sum_{j=\mu+1}^{n}(\alpha_j t-\alpha_{j-1})z^j \\
&+i\sum_{j=\mu+1}^{n}(\beta_j t-\beta_{j-1})z^j-(\alpha_n+i\beta_n)z^{n+1}.
\end{aligned}
$$

For $|z|=t$, we have

$$
\begin{aligned}
|F(z)| &\leq 2(|\alpha_0|+|\beta_0|)t+(|\alpha_\mu|+|\beta_\mu|)t^{\mu+1}+\sum_{j=\mu+1}^{n}|\alpha_j t-\alpha_{j-1}|t^j \\
&+\sum_{j=\mu+1}^{n}|\beta_j t-\beta_{j-1}|t^j+(|\alpha_n|+|\beta_n|)t^{n+1}
\end{aligned}
$$

$$= 2(|\alpha_0| + |\beta_0|)t + (|\alpha_\mu| + |\beta_\mu|)t^{\mu+1} + \sum_{j=\mu+1}^{k} (\alpha_j t - \alpha_{j-1})t^j + \sum_{j=k+1}^{n} (\alpha_{j-1} - \alpha_j t)t^j$$
$$+ \sum_{j=\mu+1}^{l} (\beta_j t - \beta_{j-1})t^j + \sum_{j=l+1}^{n} (\beta_{j-1} - \beta_j t)t^j + (|\alpha_n| + |\beta_n|)t^{n+1}$$
$$= 2(|\alpha_0| + |\beta_0|)t + (|\alpha_\mu| - \alpha_\mu + |\beta_\mu| - \beta_\mu)t^{\mu+1} + 2(\alpha_k t^{k+1} + \beta_l t^{l+1})$$
$$+ (|\alpha_n| - \alpha_n + |\beta_n| - \beta_n)t^{n+1}$$
$$= M.$$

The result now follows as in the proof of Theorem 1.

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**Abdullah Mir**
email: `mabdullah_mir@yahoo.co.in`
Department of Mathematics
University of Kashmir
Srinagar, 190006
INDIA

**Abrar Ahmad**
email: `abrarahmad1100@gmail.com`
Department of Mathematics
University of Kashmir
Srinagar, 190006
INDIA

**Adil Hussain Malik**
email: `malikadil6909@gmail.com`
Department of Mathematics
University of Kashmir
Srinagar, 190006
INDIA

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# A Minimax Approach to Mapping Partial Interval Uncertainties into Point Estimates

*Vadim Romanuke*

Abstract: A problem of simultaneously reducing a group of interval uncertainties is considered. The intervals are positively normalized. There is a constraint, by which the sum of any point estimates taken from those intervals is equal to 1. Hence, the last interval is suspended. For mapping the interval uncertainties into point estimates, a minimax decision-making method is suggested. The last interval's point estimate is then tacitly found. Minimax is applied to a maximal disbalance between a real unknown amount and a guessed amount. These amounts are interpreted as aftermaths of the point estimation. According to this model, the decision-maker is granted a pure strategy, whose components are the most appropriate point estimates. Such strategy is always single. Its components are always less than the right endpoints. The best mapping case is when we obtain a totally regular strategy whose components are greater than the left endpoints. The irregular strategy's components admitting many left endpoints are computed by special formulae. The worst strategy exists, whose single component is greater than the corresponding left endpoint. Apart from the point estimation, irregularities in the decision-maker's optimal strategy may serve as an evidence of the intervals' incorrectness. The irregularity of higher ranks is a criterion for correcting the intervals.

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# 1. Introduction

Interval estimates are unavoidable in modeling processes whose parameters are generated with a variety of stochastic or weakly controllable factors. It is a poor practice to deal with such estimates because conclusions and inferences from them come "intervally" uncertain (e.g., [4, 13, 49, 19]). For reducing interval uncertainties, both expert judgments and statistically observed data are used [1, 21, 23]. Ideally, this is Bayesian decision making. Practically, a faithful elicitation of the probability and utility functions is almost always prohibitive [23, 18, 46]. Estimations based on expert judgments allow narrowing the interval in a limited number of steps [41, 50]. When no expert or other statistical data are available, decisions on reducing intervals are made with special policies/strategies [4, 47, 20, 2].

On the other hand, point estimates are much less reliable. As the interval becomes shorter, reliability of the interval estimate requires more statistical data [9, 36]. For performing a decision analysis involving fuzzy logic, we need to substantiate a membership function and a method of defuzzification [22, 43]. Their substantiation also relies on statistical observations. In general, a proper/valid mapping of the interval estimate into a point estimate takes huge statistics [17, 47, 46]. Is the mapping possible without statistical data? Can we "guess" the most appropriate point value that could substitute the interval? These questions are answered positively only by using a conception of the least risk under the worst conditions that could happen. In other words, this is about minimax principle.

In decision making, minimax principle allows forgetting the absence of any statistics. In contrast to decisions using expected value or expected utility, being non-probabilistic is a key feature of minimax decision making. Minimax robustness is used in statistical decision theory, where a deterministic parameter is estimated under uncertainty [3, 35]. Minimax is also used in designing linear estimators, where solutions of convex optimization problems give the optimal, minimax regret-minimizing linear estimator [32, 6, 17]. The main ground for trusting the use of minimax principle is that it minimizes losses [42, 26]. Minimaxing risks/damages is a common routine in studying processes with highly volatile and non-controllable parameters [4, 19, 14, 3]. Then game models are involved. Their solutions may propose both pure and mixed strategies [44, 33]. Pure strategies are quite acceptable for a decision-maker, whereas mixed ones contain probabilities (or probabilistic measures). Practical realization of probabilities is inseparably associated with relative statistical frequencies. The frequencies tend to the corresponding probabilities by some statistical data [46, 39, 9, 11]. But if we have those data, we can map the interval estimate into a point estimate without minimaxing. Then no game modeling is needed anymore.

Availability of statistical data excludes reasonability of the minimax mixed strategies. Therefore, only pure strategies are reasonable to make non-statistical decisions on point estimates. This task becomes severer for multiple interval uncertainties representing a group of connected/intertwined parameters [13, 19, 34, 12, 8]. Indeed, if intervals are closed, then we have to map a multidimensional hyperparallelepiped into a point of the same dimensionality.

## 2. Background, related works, and motivation

Generally, uncertainty of a group of $N$ parameters studied in $\mathbb{R}^N$ by $N \in \mathbb{N} \backslash \{1\}$ consists in that we can choose its value $\mathbf{X}$ from a subset $\mathbb{X} \subset \mathbb{R}^N$:

$$\mathbf{X} = [x_k]_{1 \times N} \in \bigtimes_{k=1}^{N} \mathbb{X}_k = \mathbb{X} \subset \mathbb{R}^N \text{ by } |\mathbb{X}_n| > 1 \text{ or } \mu_{\mathbb{R}} (\mathbb{X}_n) \neq 0 \;\forall\, n = \overline{1, N}. \quad (1)$$

If subset $\mathbb{X}$ is finite, then $\mu_{\mathbb{R}^N} (\mathbb{X}) = 0$ and only condition $|\mathbb{X}_n| > 1 \;\forall\, n = \overline{1, N}$ in (1) remains relevant. Both conditions in (1) are relevant for infinite subset $\mathbb{X}$. The uncertainty becomes partial if $\mathbb{X}$ is a closed $N$-dimensional hyperparallelepiped within the nonnegative orthant of $\mathbb{R}^N$, and $\sum\limits_{n=1}^{N} x_n$ is a constant value [44, 38]. It is very convenient to consider the normalized uncertainty [13, 21, 23, 9]:

$$x_k \in [a_k;\; b_k] = \mathbb{X}_k \subset (0;\; 1)$$
$$\text{by } a_k < b_k \;\forall\, k = \overline{1, N-1} \text{ and } x_N = 1 - \sum_{n=1}^{N-1} x_n > 0. \quad (2)$$

It is important to note that (2) is followed with an inequality

$$\sum_{k=1}^{N-1} a_k < \sum_{k=1}^{N-1} b_k < 1. \quad (3)$$

If we knew a probabilistic measure $\mathscr{F}_n (x_n)$ over $\mathbb{X}_n$ (being a generatrix to the Lebesgue–Stieltjes integral), where

$$\int_{a_n}^{b_n} \mathrm{d}\mathscr{F}_n (x_n) = 1,$$

the $n$-th interval uncertainty could be reduced as follows:

$$\mathrm{M}_{\mathscr{F}_n} (\Theta) = \int_{a_n}^{b_n} x_n \mathrm{d}\mathscr{F}_n (x_n). \quad (4)$$

Value (4) is a point estimate [17], which is the mathematical expectation of a random variable $\Theta$ with its values $x_n \in [a_n;\; b_n]$ by the probabilistic measure $\mathscr{F}_n (x_n)$. In particular, knowing a probability density function $f (x_n)$, which is $f_n (x_n) \,\mathrm{d}x_n = \mathrm{d}\mathscr{F}_n (x_n)$, the point estimate becomes

$$\mathrm{M}_{f_n} (\Theta) = \int_{a_n}^{b_n} x_n f_n (x_n) \,\mathrm{d}x_n.$$

However, as it was mentioned above, eliciting a probability function is not an easy process. All the more that there are simultaneously $N$ such functions (see, e.g., [29, 49]), although the $N$-th function (for cases with $N > 1$) is needless for the partial uncertainty, where value

$$1 - \sum_{n=1}^{N-1} \mathrm{M}_{\mathscr{F}_n}(\Theta) \quad \text{or} \quad 1 - \sum_{n=1}^{N-1} \mathrm{M}_{f_n}(\Theta)$$

can be accepted as a point estimate of the $N$-th interval. Besides, those $N-1$ probability functions must be constant through a given amount of time [15, 17, 5, 31]. Otherwise, the point estimates intended for this amount become fluent. Thus the estimation loses its sense.

Without statistical data, the guessed values are used instead of the intervals in (2). While "guessing", the minimax principle is applied in order to prevent the most inappropriate guess. The prevention is understood as smoothing over the most negative effect [40]. The effect is computed as a ratio between an aftermath of what is real (becoming known for us only after some period of time) and an aftermath of our guess (the point estimation aftermaths). Such aftermath is a function $\rho$ defined on every interval [17, 7, 48, 44]. If we say that $y_k$ is a point estimate for the $k$-th interval but $x_k$ is a real value (valid for the given amount of time), then the effect is $\dfrac{\rho(x_k)}{\rho(y_k)}$. Owing to (2), only $N-1$ point estimates

$$\begin{aligned} \mathbf{X}_{N-1} &= [x_k]_{1\times(N-1)} \\ &\in \left\{\mathbf{X}_{N-1} \in \mathbb{R}^{N-1} |\ x_n \in [a_n;\, b_n] = \mathbb{X}_n \subset (0;1)\ \forall\, n = \overline{1,\, N-1}\right\} \end{aligned} \tag{5}$$

are guessed as

$$\begin{aligned} \mathbf{Y}_{N-1} &= [y_k]_{1\times(N-1)} \\ &\in \left\{\mathbf{Y}_{N-1} \in \mathbb{R}^{N-1} |\ y_m \in [a_m;\, b_m] = \mathbb{Y}_m \subset (0;1)\ \forall\, m = \overline{1,\, N-1}\right\}. \end{aligned} \tag{6}$$

Then the most negative effect is a function of $2N-2$ variables:

$$\begin{aligned} K(\mathbf{X}_{N-1},\, \mathbf{Y}_{N-1}) &= K\left(\{x_i\}_{i=1}^{N-1},\ \{y_j\}_{j=1}^{N-1}\right) \\ &= \max\left\{\left\{\frac{\rho(x_k)}{\rho(y_k)}\right\}_{k=1}^{N-1},\ \frac{\rho\left(1 - \sum\limits_{n=1}^{N-1} x_n\right)}{\rho\left(1 - \sum\limits_{m=1}^{N-1} y_m\right)}\right\}. \end{aligned} \tag{7}$$

In fact, (6) is a strategy of the decision-maker in the problem of choosing the best point estimates (Figure 1). The best version of this strategy is found as the second player's optimal strategy in a game with kernel (7) on a hyperparallelepiped

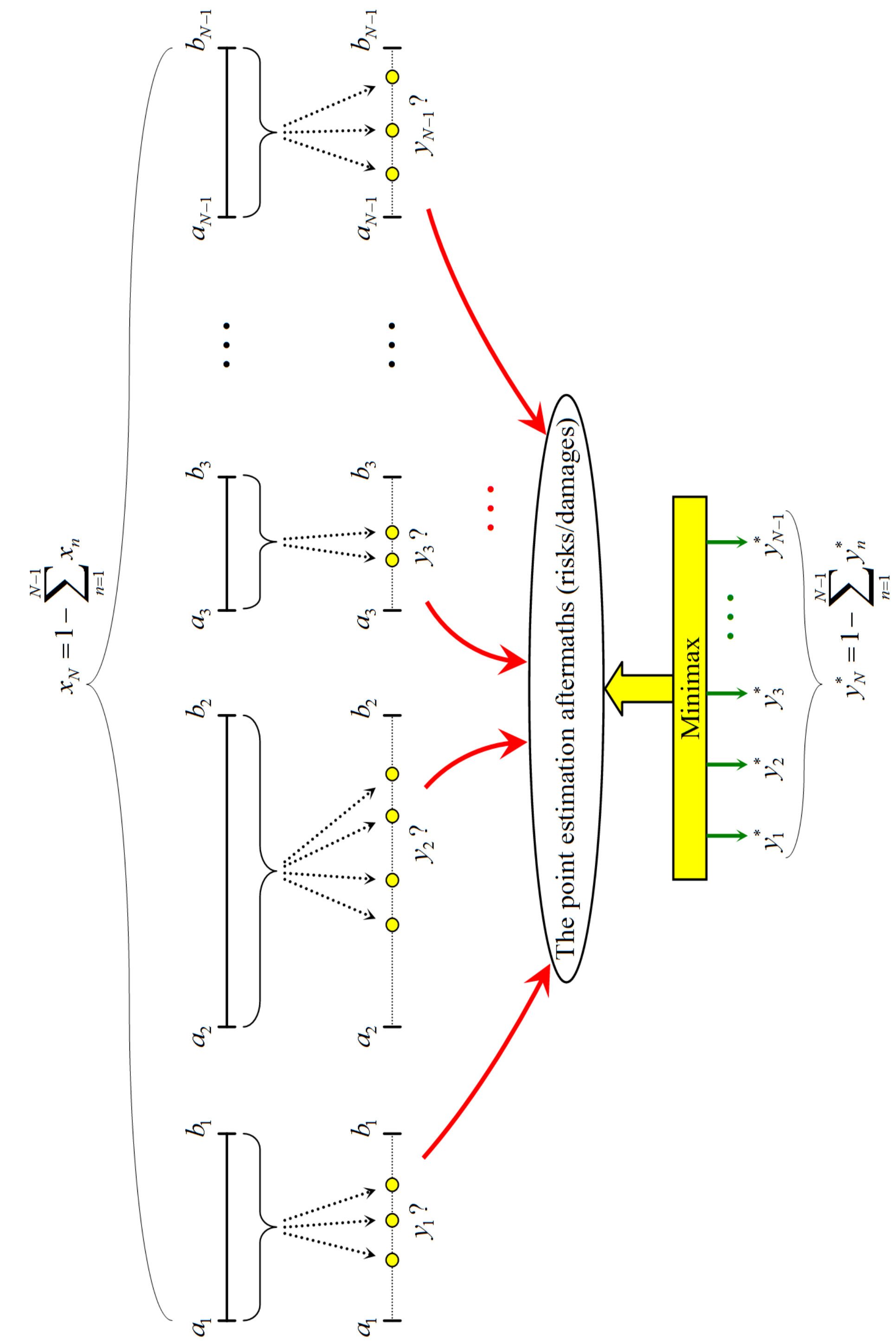


Figure 1: Conception of choosing the best point estimates with minimax

$$\left\{\mathop{\times}_{n=1}^{N-1}\mathbb{X}_n\right\}\times\left\{\mathop{\times}_{m=1}^{N-1}\mathbb{Y}_m\right\}\subset\left\{\mathop{\times}_{n=1}^{N-1}(0;\,1)\right\}\times\left\{\mathop{\times}_{m=1}^{N-1}(0;\,1)\right\}\subset\mathbb{R}^{2N-2}, \tag{8}$$

where (5) and (6) are pure strategies of the first and second players, respectively [44, 45]. The first player's pure strategy set is $\mathop{\times}_{n=1}^{N-1}\mathbb{X}_n$, and the second player's pure strategy set is $\mathop{\times}_{m=1}^{N-1}\mathbb{Y}_m$.

The simplest case of the effect computation is just $\rho(y_k)=y_k$ (see [45, 38]). Then the game kernel (7) is

$$\begin{aligned} K(\mathbf{X}_{N-1},\,\mathbf{Y}_{N-1}) &= K\left(\{x_i\}_{i=1}^{N-1},\,\{y_j\}_{j=1}^{N-1}\right) \\ &= \max\left\{\left\{\frac{x_k}{y_k}\right\}_{k=1}^{N-1},\,\frac{1-\sum\limits_{n=1}^{N-1}x_n}{1-\sum\limits_{m=1}^{N-1}y_m}\right\}. \end{aligned} \tag{9}$$

In a trivial case, when $N=2$, kernel (9) is

$$K(x_1,\,y_1)=\max\left\{\frac{x_1}{y_1},\,\frac{1-x_1}{1-y_1}\right\}$$

and the best minimax decision is [45]

$$y_1^*=\frac{b_1}{1+b_1-a_1}.$$

For the general case, in the game with kernel (9) on hyperparallelepiped (8), the decision-maker (as the second player) has a pure optimal strategy $\mathbf{Y}_{N-1}^*=[y_k^*]_{1\times(N-1)}$ (see [38]).

The case with kernel (7) is still unsolved. Shall the decision-maker possess a pure optimal strategy in this case just for certain types of the function $\rho$? Will there be any peculiarities (or special cases) in computing components of strategy $\mathbf{Y}_{N-1}^*$? Can there be a continuum of such strategies? If it can, then what is a routine to select a single unique strategy? Answers to these open questions may become a fair contribution to the field of non-statistical interval uncertainty reduction.

## 3. Goals and tasks to be fulfilled

The goal is to find a decision-maker's optimal strategy $\mathbf{Y}_{N-1}^*$, if any, in the game with kernel

$$K(\mathbf{X}_{N-1},\,\mathbf{Y}_{N-1}) = K\left(\{x_i\}_{i=1}^{N-1},\,\{y_j\}_{j=1}^{N-1}\right)$$

$$= \max \left\{ \left\{ \frac{x_k^q}{y_k^q} \right\}_{k=1}^{N-1}, \frac{\left(1 - \sum\limits_{n=1}^{N-1} x_n\right)^q}{\left(1 - \sum\limits_{m=1}^{N-1} y_m\right)^q} \right\} \quad (10)$$

on hyperparallelepiped (8) for any $q > 0$. Along with point estimates $\{y_m^*\}_{m=1}^{N-1}$ of the first $N-1$ intervals, this strategy will allow to find a point estimate of the $N$-th interval as

$$y_N^* = 1 - \sum_{m=1}^{N-1} y_m^*. \quad (11)$$

For achieving the goal, the following tasks are to be fulfilled:

1. To ascertain whether the second player has an optimal pure strategy $\mathbf{Y}_{N-1}^*$ in the game.
2. If strategy $\mathbf{Y}_{N-1}^*$ exists, to state conditions for finding its components.
3. To give examples of finding components of strategy $\mathbf{Y}_{N-1}^*$.
4. To discuss its applicability and significance.

## 4. Decision-maker's optimal strategy

Before we get started, an abbreviation of the game notation should be given.

**Definition 1.** The game with kernel (10) on hyperparallelepiped (8) for any $q > 0$ is called partial uncertainty reduction game (PURG).

The positive $q$ is not emphasized in PURG. The reason is going to be made plain below.

**Theorem 1.** *The decision-maker has a pure optimal strategy*

$$\mathbf{Y}_{N-1}^* = [y_k^*]_{1 \times (N-1)}$$

*in PURG. Components of this strategy are*

$$y_k^* = \frac{b_k}{1 + \sum\limits_{n=1}^{N-1} (b_n - a_n)} \qquad \forall\, k = \overline{1, N-1} \quad (12)$$

*if only*

$$\frac{b_k}{1 + \sum\limits_{n=1}^{N-1} (b_n - a_n)} \geqslant a_k \qquad \forall\, k = \overline{1, N-1}. \quad (13)$$

**Proof.** Let

$$L_k\left(x_k,\ y_k\right)=\frac{x_k^q}{y_k^q}\quad\text{for}\quad k=\overline{1,\ N-1}\quad\text{and}$$

$$L_N\left(\{x_i\}_{i=1}^{N-1},\ \{y_j\}_{j=1}^{N-1}\right)=\frac{\left(1-\sum\limits_{n=1}^{N-1}x_n\right)^q}{\left(1-\sum\limits_{m=1}^{N-1}y_m\right)^q}. \tag{14}$$

So, kernel (10) is a maximum of $N$ functions (14). Partial derivatives of the second order of those functions with respect to components of the second player's pure strategy are:

$$\begin{aligned}\frac{\partial^2}{\partial y_k^2}L_k\left(x_k,\ y_k\right) &= \frac{\partial^2}{\partial y_k^2}\left(\frac{x_k^q}{y_k^q}\right)=\frac{\partial}{\partial y_k}\left(-\frac{qx_k^q}{y_k^{q+1}}\right)\\ &= \frac{q\left(q+1\right)x_k^q}{y_k^{q+2}}\ \text{for}\ k=\overline{1,\ N-1}\end{aligned} \tag{15}$$

and

$$\begin{aligned}\frac{\partial^2}{\partial y_k^2}L_N\left(\{x_i\}_{i=1}^{N-1},\ \{y_j\}_{j=1}^{N-1}\right) &= \frac{\partial^2}{\partial y_k^2}\left(\frac{\left(1-\sum\limits_{n=1}^{N-1}x_n\right)^q}{\left(1-\sum\limits_{m=1}^{N-1}y_m\right)^q}\right)\\ &= \frac{\partial}{\partial y_k}\left(\frac{q\left(1-\sum\limits_{n=1}^{N-1}x_n\right)^q}{\left(1-\sum\limits_{m=1}^{N-1}y_m\right)^{q+1}}\right)\\ &= \frac{q\left(q+1\right)\left(1-\sum\limits_{n=1}^{N-1}x_n\right)^q}{\left(1-\sum\limits_{m=1}^{N-1}y_m\right)^{q+2}}\ \text{for}\ k=\overline{1,\ N-1}.\end{aligned} \tag{16}$$

Each of those $2N-2$ derivatives (15) and (16) is positive. This implies that functions

$$\left\{\{L_k\left(x_k,\ y_k\right)\}_{k=1}^{N-1},\ L_N\left(\{x_i\}_{i=1}^{N-1},\ \{y_j\}_{j=1}^{N-1}\right)\right\}$$

are strictly convex [44]. Then, owing to [37], function (10) is strictly convex itself. Therefore, PURG is convex. Owing to the game strict convexity, the second player herein has a pure strategy, which is

$$\begin{aligned}\mathbf{Y}^{*}_{N-1} &\in \arg\left\{\min_{\mathbf{Y}_{N-1}\in \bigtimes_{m=1}^{N-1} \mathbb{Y}_m}\left\{\max_{\mathbf{X}_{N-1}\in \bigtimes_{n=1}^{N-1} \mathbb{X}_n} K\left(\mathbf{X}_{N-1},\ \mathbf{Y}_{N-1}\right)\right\}\right\}\\ &= \arg\left\{\min_{\mathbf{Y}_{N-1}\in \bigtimes_{m=1}^{N-1} [a_m;\, b_m]}\left\{\max_{\mathbf{X}_{N-1}\in \bigtimes_{n=1}^{N-1} [a_n;\, b_n]} K\left(\mathbf{X}_{N-1},\ \mathbf{Y}_{N-1}\right)\right\}\right\}.\end{aligned}$$

Further, we have:

$$\begin{aligned}&\max_{\mathbf{X}_{N-1}\in \bigtimes_{n=1}^{N-1} [a_n;\, b_n]} K\left(\mathbf{X}_{N-1},\ \mathbf{Y}_{N-1}\right)\\ &= \max_{\mathbf{X}_{N-1}\in \bigtimes_{n=1}^{N-1} [a_n;\, b_n]}\left\{\max\left\{\left\{\frac{x_k^q}{y_k^q}\right\}_{k=1}^{N-1},\ \frac{\left(1-\sum\limits_{i=1}^{N-1} x_i\right)^q}{\left(1-\sum\limits_{j=1}^{N-1} y_j\right)^q}\right\}\right\}\\ &= \max\left\{\left\{\max_{x_k\in[a_k;\, b_k]}\left\{\frac{x_k^q}{y_k^q}\right\}\right\}_{k=1}^{N-1},\ \max_{\mathbf{X}_{N-1}\in \bigtimes_{n=1}^{N-1} [a_n;\, b_n]}\left\{\frac{\left(1-\sum\limits_{i=1}^{N-1} x_i\right)^q}{\left(1-\sum\limits_{j=1}^{N-1} y_j\right)^q}\right\}\right\}\\ &= \max\left\{\left\{\left(\frac{b_k}{y_k}\right)^q\right\}_{k=1}^{N-1},\ \left(\frac{1-\sum\limits_{i=1}^{N-1} a_i}{1-\sum\limits_{j=1}^{N-1} y_j}\right)^q\right\}. \qquad (17)\end{aligned}$$

Maximum (17) is a function of $N-1$ variables $\{y_k\}_{k=1}^{N-1}$. This function consists of $N$ upper parts of hyperbolic hypersurfaces raised to the power $q$. Its minimum on $\bigtimes_{m=1}^{N-1} \mathbb{Y}_m$ is reached when all those $N$ parts are equal [45, 38]:

$$\min_{\mathbf{Y}_{N-1}\in \bigtimes_{m=1}^{N-1} [a_m;\, b_m]}\left\{\max_{\mathbf{X}_{N-1}\in \bigtimes_{n=1}^{N-1} [a_n;\, b_n]} K\left(\mathbf{X}_{N-1},\ \mathbf{Y}_{N-1}\right)\right\}$$

$$= \min_{\mathbf{Y}_{N-1} \in \bigtimes_{m=1}^{N-1} [a_m;\, b_m]} \left\{ \max \left\{ \left\{ \left( \frac{b_k}{y_k} \right)^q \right\}_{k=1}^{N-1},\ \left( \frac{1 - \sum\limits_{i=1}^{N-1} a_i}{1 - \sum\limits_{j=1}^{N-1} y_j} \right)^q \right\} \right\}$$

$$= \max \left\{ \left\{ \left( \frac{b_k}{y_k^*} \right)^q \right\}_{k=1}^{N-1},\ \left( \frac{1 - \sum\limits_{i=1}^{N-1} a_i}{1 - \sum\limits_{j=1}^{N-1} y_j^*} \right)^q \right\} = v_{\mathrm{opt}},$$

where the optimal game value

$$v_{\mathrm{opt}} = \left( \frac{b_k}{y_k^*} \right)^q = \left( \frac{1 - \sum\limits_{i=1}^{N-1} a_i}{1 - \sum\limits_{j=1}^{N-1} y_j^*} \right)^q \qquad \forall\, k = \overline{1,\, N-1}. \tag{18}$$

It is apparent that we can state

$$\frac{b_k}{y_k^*} = \frac{1 - \sum\limits_{i=1}^{N-1} a_i}{1 - \sum\limits_{j=1}^{N-1} y_j^*} \qquad \forall\, k = \overline{1,\, N-1} \tag{19}$$

instead of (18). It follows from (18) that

$$y_m^* = \frac{b_m}{b_k} y_k^* \qquad \forall\, m = \overline{1,\, N-1} \quad \text{and} \quad \forall\, k = \overline{1,\, N-1} \tag{20}$$

along with that

$$y_k^* \left( 1 - \sum_{i=1}^{N-1} a_i \right) = b_k \left( 1 - \sum_{j=1}^{N-1} y_j^* \right) \qquad \forall\, k = \overline{1,\, N-1}. \tag{21}$$

Ratio (20) allows to see that

$$b_k y_m^* = b_k \frac{b_m}{b_k} y_k^* = b_m y_k^*. \tag{22}$$

Equality (22) implies that we can exchange indices. This allows to re-write the right term in (21) as follows:

$$b_k \left( 1 - \sum_{j=1}^{N-1} y_j^* \right) = b_k - \sum_{j=1}^{N-1} b_k y_j^* = b_k - \sum_{j=1}^{N-1} b_j y_k^* = b_k - y_k^* \sum_{j=1}^{N-1} b_j. \tag{23}$$

Now, we plug the last term of (23) into the right-hand side of (21):

$$y_k^* \left(1 - \sum_{i=1}^{N-1} a_i\right) = b_k - y_k^* \sum_{j=1}^{N-1} b_j,$$

whence

$$y_k^* = \frac{b_k}{1 + \sum\limits_{j=1}^{N-1} b_j - \sum\limits_{i=1}^{N-1} a_i} = \frac{b_k}{1 + \sum\limits_{n=1}^{N-1} (b_n - a_n)}.$$

However, value (12) can really be the $k$-th component of the decision-maker's optimal strategy if

$$\frac{b_k}{1 + \sum\limits_{n=1}^{N-1} (b_n - a_n)} \in [a_k;\ b_k]\,.$$

Inequality

$$\frac{b_k}{1 + \sum\limits_{n=1}^{N-1} (b_n - a_n)} \leqslant b_k \qquad \forall\, k = \overline{1,\, N-1} \tag{24}$$

is always true because

$$\frac{1}{1 + \sum\limits_{n=1}^{N-1} (b_n - a_n)} \leqslant 1$$

and

$$1 \leqslant 1 + \sum_{n=1}^{N-1} (b_n - a_n), \tag{25}$$

where

$$\sum_{n=1}^{N-1} (b_n - a_n) > 0 \tag{26}$$

that follows (3). Finally, just requirement (13) remains. □

Thus the decision-maker's optimal strategy does not depend upon $q$. The optimal game value (18), showing the poorest ratio between the point estimation aftermaths, however, is directly influenced with $q$.

**Example 1.** Consider an example construing the sense of requirement (13). Suppose that $N = 3$, $a_1 = 0.3$, $b_1 = 0.6$, $a_2 = 0.25$, $b_2 = 0.3$. Then

$$\frac{b_1}{1 + \sum\limits_{n=1}^{2} (b_n - a_n)} = \frac{0.6}{1 + 0.6 - 0.3 + 0.3 - 0.25} = \frac{4}{9} > \frac{3}{10} = a_1$$

but

$$\frac{b_2}{1+\sum\limits_{n=1}^{2}(b_n-a_n)} = \frac{0.3}{1+0.6-0.3+0.3-0.25} = \frac{2}{9} < \frac{1}{4} = a_2.$$

Thus requirement (13) is violated here for the second component. This counterexample prompts to consider the following theorem of necessity.

**Theorem 2.** *To have the $k$-th component as* (12) *in PURG, it is necessary that the inequality*

$$\frac{b_k-a_k}{\sum\limits_{n=1}^{N-1}(b_n-a_n)} \geqslant a_k \tag{27}$$

*be true.*

**Proof.** Requirement (13) for the $k$-th component is expanded as follows:

$$b_k \geqslant a_k + a_k \sum_{n=1}^{N-1}(b_n-a_n),$$

$$b_k - a_k \geqslant a_k \left( \sum_{m=1}^{N-1} b_m - \sum_{m=1}^{N-1} a_m \right),$$

whence, dividing with owing to (3), we get inequality (27). □

It will be shown below that condition (27) is not sufficient for having the $k$-th component as (12). But it is sufficient to say that if condition (27) is violated, then the $k$-th component is not computed as (12). This is, so to speak, a rejection of conditions in Theorem 1. Henceforward, the following definitions become important.

**Definition 2.** The second player's optimal strategy $\mathbf{Y}^*_{N-1}$ in PURG is called regular if its components are computed as (12). A component of a regular strategy is called regular.

**Definition 3.** The regular optimal strategy $\mathbf{Y}^*_{N-1}$ in PURG is called totally regular if

$$y_k^* \in (a_k;\, b_k) \qquad \forall\, k = \overline{1,\, N-1}. \tag{28}$$

A component of a totally regular strategy is called totally regular.

**Definition 4.** The second player's optimal strategy $\mathbf{Y}^*_{N-1}$ in PURG is called irregular if at least a one inequality in requirement (13) is violated. A component of an irregular strategy is called irregular.

These definitions facilitate in treating various types of optimal strategy $\mathbf{Y}^*_{N-1}$. It easy to see that if inequality (27) holds for every $k = \overline{1,\, N-1}$, then $\mathbf{Y}^*_{N-1}$ is regular. Obviously, a regular strategy $\mathbf{Y}^*_{N-1}$ consists of $N-1$ (all) regular components. They are computed straightforwardly by (12). If strategy $\mathbf{Y}^*_{N-1}$ has an irregular component, this strategy is not a regular one. Before finding irregular strategies, the span of the regular component is asserted below.

**Theorem 3.** *Whichever regular strategy $\mathbf{Y}^*_{N-1}$ in PURG is, its $k$-th component*

$$y^*_k \in [a_k;\, b_k) \qquad \forall\, k = \overline{1,\, N-1}. \tag{29}$$

**Proof.** As an inequality in requirement (13) can be violated, then an occurrence $y^*_k = a_k$ is possible. Owing to (26), inequality (25) holds strictly. Hence, inequality (24) holds strictly as well. □

Do we know a PURG, in which strategy $\mathbf{Y}^*_{N-1}$ is totally regular? Actually, such a PURG exists, although the intervals seem "regular" themselves [38].

**Theorem 4.** *In PURG whose hyperparallelepiped* (8) *is a hypercube*

$$\left\{ \bigtimes_{n=1}^{N-1} [a;\, b] \right\} \times \left\{ \bigtimes_{m=1}^{N-1} [a;\, b] \right\} \subset \left\{ \bigtimes_{n=1}^{N-1} (0;\, 1) \right\} \times \left\{ \bigtimes_{m=1}^{N-1} (0;\, 1) \right\} \subset \mathbb{R}^{2N-2} \tag{30}$$

*by*

$$\{[a_k;\, b_k] = [a;\, b]\}_{k=1}^{N-1}, \tag{31}$$

*the decision-maker has a totally regular strategy $\mathbf{Y}^*_{N-1}$ whose components are identical:*

$$y^*_k = \frac{b}{1 + (N-1)\,(b-a)} \qquad \forall\, k = \overline{1,\, N-1}. \tag{32}$$

**Proof.** Formula (32) is directly obtained by plugging (31) into (12). We know from (3) that

$$\sum_{k=1}^{N-1} a_k = (N-1)\,a < 1.$$

So,

$$\frac{1}{N-1} > a$$

and

$$\frac{b-a}{(N-1)\,(b-a)} > a. \tag{33}$$

Inequality (33) is referred to Theorem 2 by considering inequality (27) with the strict sign that gives (28). □

Thus identical or "regular" intervals (31) generate totally regular and identical components (32). In this way, interval uncertainties are "regulated" if we are allowed to slightly adjust endpoints of different intervals. Of course, this is not always possible in practical situations. If endpoints of different intervals are not adjustable, then condition (13) is not guaranteed. Now the question is whether requirement (13) can be violated entirely, i.e., inequality (27) fails $\forall\, k = \overline{1,\, N-1}$.

**Theorem 5.** *Inequality* (27) *in PURG holds strictly at least for a one* $k \in \left\{\overline{1, N-1}\right\}$.

**Proof.** Assume that inequality (27) nonstrictly fails $\forall\, k = \overline{1, N-1}$. This implies that

$$\frac{b_k - a_k}{\sum\limits_{n=1}^{N-1} (b_n - a_n)} \leqslant a_k \qquad \forall\, k = \overline{1, N-1}.$$

Then, summing up both sides, we get

$$\frac{\sum\limits_{k=1}^{N-1} (b_k - a_k)}{\sum\limits_{n=1}^{N-1} (b_n - a_n)} = 1 \leqslant \sum_{k=1}^{N-1} a_k$$

that is impossible. This contradiction proves the theorem assertion. □

So, strategy $\mathbf{Y}^*_{N-1}$ is irregular if inequality (27) fails at least for a one $k \in \left\{\overline{1, N-1}\right\}$. How is $y^*_k$ then computed? What is strategy $\mathbf{Y}^*_{N-1}$ then, after all? The following theorem answers this question partially.

**Theorem 6.** *In PURG, let*

$$\frac{b_u}{1 + \sum\limits_{n=1}^{N-1} (b_n - a_n)} < a_u \quad \text{for} \quad u \in \mathscr{U} \subset \left\{\overline{1, N-1}\right\} \quad \text{by} \quad \mathscr{U} \neq \emptyset. \tag{34}$$

*Then* $|\mathscr{U}|$ *components of an irregular strategy* $\mathbf{Y}^*_{N-1}$ *are found as*

$$y^*_u = a_u \qquad \forall\, u \in \mathscr{U} \subset \left\{\overline{1, N-1}\right\}. \tag{35}$$

*The rest* $N - 1 - |\mathscr{U}|$ *components are computed as*

$$y^*_k = \frac{b_k \left(1 - \sum\limits_{w \in \mathscr{U}} a_w\right)}{1 + \sum\limits_{n=1}^{N-1} (b_n - a_n) - \sum\limits_{w \in \mathscr{U}} b_w} \qquad \forall\, k \in \left\{\overline{1, N-1}\right\} \backslash \mathscr{U} \tag{36}$$

*if only*

$$\frac{b_k \left(1 - \sum\limits_{w \in \mathscr{U}} a_w\right)}{1 + \sum\limits_{n=1}^{N-1} (b_n - a_n) - \sum\limits_{w \in \mathscr{U}} b_w} \geqslant a_k \qquad \forall\, k \in \left\{\overline{1, N-1}\right\} \backslash \mathscr{U}. \tag{37}$$

**Proof.** Let

$$y_u^{**} = \frac{b_u}{1 + \sum\limits_{n=1}^{N-1} (b_n - a_n)} \quad \text{for} \quad u \in \mathscr{U} \subset \left\{\overline{1,\, N-1}\right\}. \tag{38}$$

Due to (34), points (38) cannot be components of $\mathbf{Y}_{N-1}^*$ because $y_u^{**} \notin [a_u;\ b_u]$. So,

$$y_u^* > y_u^{**} \qquad \forall\, u \in \mathscr{U}. \tag{39}$$

Let (18) be re-written as

$$\left(\frac{b_u}{y_u^{**}}\right)^q = \left(\frac{b_k}{y_k^*}\right)^q = \left(\frac{1 - \sum\limits_{i=1}^{N-1} a_i}{1 - \sum\limits_{z \in \left\{\overline{1,\, N-1}\right\} \setminus \mathscr{U}} y_z^* - \sum\limits_{w \in \mathscr{U}} y_w^{**}}\right)^q$$
$$\forall\, u \in \mathscr{U} \quad \text{and} \quad \forall\, k \in \left\{\overline{1,\, N-1}\right\} \setminus \mathscr{U}. \tag{40}$$

Taking into account (39), from (40) we get

$$\left(\frac{b_u}{y_u^*}\right)^q < \left(\frac{b_k}{y_k^*}\right)^q < \left(\frac{1 - \sum\limits_{i=1}^{N-1} a_i}{1 - \sum\limits_{z \in \left\{\overline{1,\, N-1}\right\} \setminus \mathscr{U}} y_z^* - \sum\limits_{w \in \mathscr{U}} y_w^*}\right)^q$$
$$\forall\, u \in \mathscr{U} \quad \text{and} \quad \forall\, k \in \left\{\overline{1,\, N-1}\right\} \setminus \mathscr{U} \tag{41}$$

instead of (18). Inequalities (41) are equivalent to inequalities

$$\frac{b_u}{y_u^*} < \frac{b_k}{y_k^*} < \frac{1 - \sum\limits_{i=1}^{N-1} a_i}{1 - \sum\limits_{z \in \left\{\overline{1,\, N-1}\right\} \setminus \mathscr{U}} y_z^* - \sum\limits_{w \in \mathscr{U}} y_w^*}$$
$$\forall\, u \in \mathscr{U} \quad \text{and} \quad \forall\, k \in \left\{\overline{1,\, N-1}\right\} \setminus \mathscr{U}. \tag{42}$$

Similarly to getting equality (19), the second player here endeavors to minimize the right term in (42) to get equality

$$v_{\text{opt}} = \frac{b_k}{y_k^*} = \frac{1 - \sum\limits_{i=1}^{N-1} a_i}{1 - \sum\limits_{z \in \left\{\overline{1,\, N-1}\right\} \setminus \mathscr{U}} y_z^* - \sum\limits_{w \in \mathscr{U}} y_w^*} \qquad \forall\, k \in \left\{\overline{1,\, N-1}\right\} \setminus \mathscr{U} \tag{43}$$

that is still possible by setting

$$y_k^* < \frac{b_k}{1 + \sum\limits_{n=1}^{N-1} (b_n - a_n)} \quad \text{for} \quad \frac{b_k}{1 + \sum\limits_{n=1}^{N-1} (b_n - a_n)} > a_k. \tag{44}$$

Setting simultaneously $y_w^* > a_w$ only increases the right term in (43). To decrease this term, it would be necessary to decrease $y_k^*$ for some $k \in \left\{\overline{1,\, N-1}\right\} \setminus \mathscr{U}$ by (44) more. But then the term $\dfrac{b_k}{y_k^*}$ would be increased as well. Such behavior of the second player contradicts with its optimality principle, where $v_{\text{opt}}$ is tried for minimization. Therefore, setting (35) is optimal, and the rest $N - 1 - |\mathscr{U}|$ components are roots of equations (43). Similarly to (20),

$$y_m^* = \frac{b_m}{b_k} y_k^* \;\; \forall\, m \in \left\{\overline{1,\, N-1}\right\} \setminus \mathscr{U} \quad \text{and} \quad \forall\, k \in \left\{\overline{1,\, N-1}\right\} \setminus \mathscr{U}, \tag{45}$$

whence exchanging indices (22) is true. On the other hand, we get $N - 1 - |\mathscr{U}|$ equations

$$\begin{aligned} &y_k^* \left(1 - \sum_{i=1}^{N-1} a_i\right) \\ &\quad = b_k \left(1 - \sum_{z \in \left\{\overline{1,\, N-1}\right\} \setminus \mathscr{U}} y_z^* - \sum_{w \in \mathscr{U}} y_w^*\right) \qquad \forall\, k \in \left\{\overline{1,\, N-1}\right\} \setminus \mathscr{U} \end{aligned} \tag{46}$$

from (43). Equations (46) are re-written with (35) and (45) as follows:

$$\begin{aligned} &b_k \left(1 - \sum_{z \in \left\{\overline{1,\, N-1}\right\} \setminus \mathscr{U}} y_z^* - \sum_{w \in \mathscr{U}} y_w^*\right) \\ &\quad = b_k \left(1 - \sum_{w \in \mathscr{U}} a_w\right) - b_k \sum_{z \in \left\{\overline{1,\, N-1}\right\} \setminus \mathscr{U}} y_z^* \\ &\quad = b_k \left(1 - \sum_{w \in \mathscr{U}} a_w\right) - y_k^* \sum_{z \in \left\{\overline{1,\, N-1}\right\} \setminus \mathscr{U}} b_z \\ &\quad = b_k \left(1 - \sum_{w \in \mathscr{U}} a_w\right) - y_k^* \left(\sum_{i=1}^{N-1} b_i - \sum_{w \in \mathscr{U}} b_w\right) \quad \forall\, k \in \left\{\overline{1,\, N-1}\right\} \setminus \mathscr{U}. \end{aligned} \tag{47}$$

Now, we plug the last term of (47) into the right-hand side of (46):

$$y_k^* \left(1 - \sum_{i=1}^{N-1} a_i\right)$$

$$= \quad b_k \left( 1 - \sum_{w \in \mathscr{U}} a_w \right) - y_k^* \left( \sum_{i=1}^{N-1} b_i - \sum_{w \in \mathscr{U}} b_w \right) \quad \forall\, k \in \left\{ \overline{1,\, N-1} \right\} \setminus \mathscr{U}\,,$$

whence (36) follows on. However, value (36) can really be the $k$-th component of the decision-maker's optimal strategy if

$$\frac{b_k \left( 1 - \sum\limits_{w \in \mathscr{U}} a_w \right)}{1 + \sum\limits_{n=1}^{N-1} (b_n - a_n) - \sum\limits_{w \in \mathscr{U}} b_w} \in [a_k;\, b_k] \qquad \forall\, k \in \left\{ \overline{1,\, N-1} \right\} \setminus \mathscr{U}\,.$$

From inequality

$$\frac{b_k \left( 1 - \sum\limits_{w \in \mathscr{U}} a_w \right)}{1 + \sum\limits_{n=1}^{N-1} (b_n - a_n) - \sum\limits_{w \in \mathscr{U}} b_w} \leqslant b_k \qquad \forall\, k \in \left\{ \overline{1,\, N-1} \right\} \setminus \mathscr{U} \tag{48}$$

successively we have:

$$\frac{1 - \sum\limits_{w \in \mathscr{U}} a_w}{1 + \sum\limits_{n=1}^{N-1} (b_n - a_n) - \sum\limits_{w \in \mathscr{U}} b_w} \leqslant 1,$$

$$1 - \sum_{w \in \mathscr{U}} a_w \leqslant 1 + \sum_{n=1}^{N-1} (b_n - a_n) - \sum_{w \in \mathscr{U}} b_w,$$

$$0 \leqslant \sum_{n=1}^{N-1} (b_n - a_n) - \sum_{w \in \mathscr{U}} (b_w - a_w) = \sum_{z \in \left\{ \overline{1,\, N-1} \right\} \setminus \mathscr{U}} (b_z - a_z). \tag{49}$$

Inequality (49) is always true, so inequality (48) holds as well. Finally, just requirement (37) remains. $\square$

According to Theorem 5, in PURG $\exists\, k \in \left\{ \overline{1,\, N-1} \right\}$ such that the $k$-th inequality in requirement (13) holds. Therefore, the maximal number of inequalities (34) is $N-2$, i.e., $|\mathscr{U}| \leqslant N-2$. So, Example 1 gave a case with the maximal number by $N=3$, where $\mathscr{U} = \{2\}$. For that case, $y_2^* = a_2 = 0.25$ and

$$y_1^* = \frac{b_1\,(1 - a_2)}{1 + \sum\limits_{n=1}^{2} (b_n - a_n) - b_2} = \frac{0.6 \cdot 0.75}{1 + 0.6 - 0.3 + 0.3 - 0.25 - 0.3} = \frac{3}{7} > \frac{3}{10} = a_1.$$

**Example 2.** Consider an example construing the sense of requirement (37). Suppose that $N = 4$, $a_1 = 0.1$, $b_1 = 0.5$, $a_2 = 0.15$, $b_2 = 0.2$, $a_3 = 0.16$, $b_3 = 0.25$. Then

$$\sum_{n=1}^{3}(b_n - a_n) = 0.5 - 0.1 + 0.2 - 0.15 + 0.25 - 0.16 = 0.54,$$

$$\frac{b_1}{1 + \sum\limits_{n=1}^{3}(b_n - a_n)} = \frac{0.5}{1 + 0.54} = \frac{25}{77} > \frac{1}{10} = a_1$$

and

$$\frac{b_3}{1 + \sum\limits_{n=1}^{3}(b_n - a_n)} = \frac{0.25}{1 + 0.54} = \frac{25}{154} > \frac{4}{25} = a_3,$$

but

$$\frac{b_2}{1 + \sum\limits_{n=1}^{3}(b_n - a_n)} = \frac{0.2}{1 + 0.54} = \frac{10}{77} < \frac{3}{20} = a_2.$$

So, $\mathscr{U} = \{2\}$ and, according to Theorem 6, $y_2^* = a_2 = 0.15$,

$$\frac{b_1(1 - a_2)}{1 + \sum\limits_{n=1}^{3}(b_n - a_n) - b_2} = \frac{0.5 \cdot (1 - 0.15)}{1 + 0.54 - 0.2} = \frac{85}{268} > \frac{1}{10} = a_1,$$

but

$$\frac{b_3(1 - a_2)}{1 + \sum\limits_{n=1}^{3}(b_n - a_n) - b_2} = \frac{0.25 \cdot (1 - 0.15)}{1 + 0.54 - 0.2} = \frac{85}{536} < \frac{4}{25} = a_3.$$

Thus requirement (37) is violated here for the third component. This counterexample shows that Theorem 6 does not conclude the question of finding an irregular strategy $\mathbf{Y}_{N-1}^*$ in PURG. This question also needs knowing the span of the irregular strategy component.

**Theorem 7.** *Whichever irregular strategy* $\mathbf{Y}_{N-1}^*$ *in PURG is, its* $k$*-th component is* (29)*. Besides,*

$$\frac{b_k\left(1 - \sum\limits_{w \in \mathscr{U}} a_w\right)}{1 + \sum\limits_{n=1}^{N-1}(b_n - a_n) - \sum\limits_{w \in \mathscr{U}} b_w} < \frac{b_k}{1 + \sum\limits_{n=1}^{N-1}(b_n - a_n)} < b_k$$

$$\forall\, k \in \left\{\overline{1,\, N-1}\right\} \backslash \mathscr{U}. \tag{50}$$

**Proof.** Let components of an irregular strategy $\mathbf{Y}^*_{N-1}$ be (35) and (36). The right inequality in (50) has been already proved in Theorem 3. Consider the difference between the left and right terms in the left inequality in (50) divided by $b_k$:

$$\frac{\left(1-\sum\limits_{w\in\mathscr{U}} a_w\right)\left(1+\sum\limits_{n=1}^{N-1}(b_n-a_n)\right)-\left(1+\sum\limits_{n=1}^{N-1}(b_n-a_n)-\sum\limits_{w\in\mathscr{U}} b_w\right)}{\left(1+\sum\limits_{n=1}^{N-1}(b_n-a_n)-\sum\limits_{w\in\mathscr{U}} b_w\right)\left(1+\sum\limits_{n=1}^{N-1}(b_n-a_n)\right)}$$

$$\begin{aligned} &= \left(1+\sum_{n=1}^{N-1}(b_n-a_n)-\sum_{w\in U} a_w\left(1+\sum_{n=1}^{N-1}(b_n-a_n)\right)-1-\sum_{n=1}^{N-1}(b_n-a_n)\right.\\ &+\left.\sum_{w\in U} b_w\right)\left(1+\sum_{n=1}^{N-1}(b_n-a_n)-\sum_{w\in U} b_w\right)^{-1}\left(1+\sum_{n=1}^{N-1}(b_n-a_n)\right)^{-1}\\ &= \frac{-\sum\limits_{w\in\mathscr{U}} a_w\left(1+\sum\limits_{n=1}^{N-1}(b_n-a_n)\right)+\sum\limits_{w\in\mathscr{U}} b_w}{\left(1+\sum\limits_{n=1}^{N-1}(b_n-a_n)-\sum\limits_{w\in\mathscr{U}} b_w\right)\left(1+\sum\limits_{n=1}^{N-1}(b_n-a_n)\right)}. \end{aligned} \tag{51}$$

Summing up both sides of (34), we get

$$\frac{\sum\limits_{u\in\mathscr{U}} b_u}{1+\sum\limits_{n=1}^{N-1}(b_n-a_n)} < \sum_{u\in\mathscr{U}} a_u,$$

$$\sum_{u\in\mathscr{U}} b_u-\sum_{u\in\mathscr{U}} a_u\left(1+\sum_{n=1}^{N-1}(b_n-a_n)\right)<0,$$

whence the numerator in the last term of (51) is negative. This confirms the double inequality (50). □

As requirement (37) can be violated at least for a one $k$, irregularity of strategy $\mathbf{Y}^*_{N-1}$ by Theorem 6 and property (50) of span (29) of its components need supplementation. An irregular strategy $\mathbf{Y}^*_{N-1}$, wherein

$$\frac{b_k\left(1-\sum\limits_{w\in\mathscr{U}} a_w\right)}{1+\sum\limits_{n=1}^{N-1}(b_n-a_n)-\sum\limits_{w\in\mathscr{U}} b_w} < a_k \quad \text{by} \quad k\in\left\{\overline{1,\,N-1}\right\}\setminus\mathscr{U},$$

acquires deeper irregularity. Therefore, components of the irregular strategy should be distinguished by ranks of their irregularity. Particularly, this is about downward-biasing of their values in accordance to (50).

**Definition 5.** The $u$-th component (35) of the second player's optimal strategy $\mathbf{Y}_{N-1}^{*}$ in PURG is called a simple irregular component of the first rank (SIC-1) if condition (34) holds. A set of those components (SICs-1) is called a left strategy subset of the first rank (LSS-1).

**Definition 6.** The $k$-th component (36) of the second player's optimal strategy $\mathbf{Y}_{N-1}^{*}$ in PURG is called a downward-biased irregular component of the first rank (DBIC-1). A set of those components (DBICs-1) is called a biased strategy subset of the first rank (BSS-1).

Clearly, an irregular strategy $\mathbf{Y}_{N-1}^{*}$ contains at least an SIC-1. Owing to Theorem 5 we know that number of SICs-1 does not exceed $N-2$. Going deeper with irregularity, can requirement (37) be violated entirely?

**Theorem 8.** *Whichever nonempty LSS-1 in PURG is,* $\exists\, l \in \left\{\overline{1,\, N-1}\right\} \backslash \mathscr{U}$ *such that*

$$\frac{b_l \left(1 - \sum\limits_{w \in \mathscr{U}} a_w\right)}{1 + \sum\limits_{n=1}^{N-1} (b_n - a_n) - \sum\limits_{w \in \mathscr{U}} b_w} > a_l. \tag{52}$$

**Proof.** Assume that inequality (37) holds with the reverse sign implying that

$$\frac{b_k \left(1 - \sum\limits_{w \in \mathscr{U}} a_w\right)}{1 + \sum\limits_{n=1}^{N-1} (b_n - a_n) - \sum\limits_{w \in \mathscr{U}} b_w} \leqslant a_k \qquad \forall\, k \in \left\{\overline{1,\, N-1}\right\} \backslash \mathscr{U}. \tag{53}$$

Then, summing up both sides of (53) over $k \in \left\{\overline{1,\, N-1}\right\} \backslash \mathscr{U}$, we get

$$\frac{\sum\limits_{k \in \left\{\overline{1,\, N-1}\right\} \backslash \mathscr{U}} b_k \left(1 - \sum\limits_{w \in \mathscr{U}} a_w\right)}{1 + \sum\limits_{n=1}^{N-1} (b_n - a_n) - \sum\limits_{w \in \mathscr{U}} b_w} \leqslant \sum\limits_{k \in \left\{\overline{1,\, N-1}\right\} \backslash \mathscr{U}} a_k,$$

$$\sum\limits_{k \in \left\{\overline{1,\, N-1}\right\} \backslash \mathscr{U}} b_k \left(1 - \sum\limits_{w \in \mathscr{U}} a_w\right) \leqslant \sum\limits_{k \in \left\{\overline{1,\, N-1}\right\} \backslash \mathscr{U}} a_k \left(1 + \sum\limits_{n=1}^{N-1} (b_n - a_n) - \sum\limits_{w \in \mathscr{U}} b_w\right),$$

$$\sum_{k \in \left\{\overline{1,\, N-1}\right\} \backslash \mathscr{U}} b_k \left(1 - \sum_{w \in \mathscr{U}} a_w\right)$$

$$\leqslant \sum_{k \in \left\{\overline{1,\, N-1}\right\} \backslash \mathscr{U}} a_k \left(1 - \sum_{n=1}^{N-1} a_n + \sum_{j \in \left\{\overline{1,\, N-1}\right\} \backslash \mathscr{U}} b_j\right),$$

$$\sum_{k \in \left\{\overline{1,\, N-1}\right\} \backslash \mathscr{U}} b_k \left(1 - \sum_{w \in \mathscr{U}} a_w\right) - \sum_{k \in \left\{\overline{1,\, N-1}\right\} \backslash \mathscr{U}} a_k \sum_{j \in \left\{\overline{1,\, N-1}\right\} \backslash \mathscr{U}} b_j$$

$$\leqslant \sum_{k \in \left\{\overline{1,\, N-1}\right\} \backslash \mathscr{U}} a_k \left(1 - \sum_{n=1}^{N-1} a_n\right),$$

$$\sum_{k \in \left\{\overline{1,\, N-1}\right\} \backslash \mathscr{U}} b_k \left(1 - \sum_{w \in \mathscr{U}} a_w - \sum_{k \in \left\{\overline{1,\, N-1}\right\} \backslash \mathscr{U}} a_k\right)$$

$$\leqslant \sum_{k \in \left\{\overline{1,\, N-1}\right\} \backslash \mathscr{U}} a_k \left(1 - \sum_{n=1}^{N-1} a_n\right),$$

$$\sum_{k \in \left\{\overline{1,\, N-1}\right\} \backslash \mathscr{U}} b_k \left(1 - \sum_{n=1}^{N-1} a_n\right) \leqslant \sum_{k \in \left\{\overline{1,\, N-1}\right\} \backslash \mathscr{U}} a_k \left(1 - \sum_{n=1}^{N-1} a_n\right),$$

$$\sum_{k \in \left\{\overline{1,\, N-1}\right\} \backslash \mathscr{U}} b_k \leqslant \sum_{k \in \left\{\overline{1,\, N-1}\right\} \backslash \mathscr{U}} a_k, \tag{54}$$

that is impossible. The refuted assumption implies that $\exists\, l \in \left\{\overline{1,\, N-1}\right\} \backslash \mathscr{U}$ such that (52) holds. $\square$

Although requirement (37) cannot be violated entirely, some of its inequalities may turn false (see Example 2). Then the corresponding irregular optimal strategy $\mathbf{Y}^*_{N-1}$ does not contain BSS-1.

**Definition 7.** An irregular optimal strategy $\mathbf{Y}^*_{N-1}$ of the second player in PURG is called an irregular strategy of the first rank (IS-1) if it consists only of LSS-1 and BSS-1.

We found an IS-1 in Example 1. Example 2 showed a case, where DBICs-1 cannot be computed by (36), although $y_2^* = 0.15$ turned to be a single SIC-1. That is why the irregular strategy $\mathbf{Y}^*_{N-1}$ in Example 2 cannot be an IS-1. Apparently, irregular strategies can have different ranks of their irregularity. Deeper irregularity implies a higher rank of irregular components. As irregular strategies $\mathbf{Y}^*_{N-1}$ in PURG may be of higher ranks, Theorem 1 and Theorem 6 must be generalized.

**Theorem 9.** *Let* $\mathscr{U}_h \subset \left\{\overline{1,\, N-1}\right\}$ *be a subset of those indices within the set* $\left\{\overline{1,\, N-1}\right\}$ *in PURG, for which inequality*

$$\frac{b_{u_h}\left(1-\sum\limits_{w\in\mathscr{A}_{h-1}} a_w\right)}{1+\sum\limits_{n=1}^{N-1}(b_n-a_n)-\sum\limits_{w\in\mathscr{A}_{h-1}} b_w} < a_{u_h}$$

$$\text{for}\quad \mathscr{A}_{h-1}=\bigcup_{q=1}^{h-1}\mathscr{U}_q \quad \text{and} \quad u_h\in\mathscr{U}_h \tag{55}$$

*holds successively starting from* $h=1$ *up to some* $h\in\mathbb{N}$ *and*

$$\bigcap_{q=1}^{h}\mathscr{U}_q=\emptyset \quad \text{for} \quad \mathscr{U}_q\neq\emptyset \quad \forall\, q=\overline{1,\, h}. \tag{56}$$

*Then the maximal value of the index* $h$ *is constrained:*

$$h\leqslant N-2. \tag{57}$$

*Besides,*

$$\mathscr{A}_h=\bigcup_{q=1}^{h}\mathscr{U}_q$$

*and* $|\mathscr{A}_h|$ *components of an irregular strategy* $\mathbf{Y}^*_{N-1}$ *are found as*

$$y^*_{u_r}=a_{u_r} \quad \forall\, u_r\in\mathscr{U}_r\subset\left\{\overline{1,\; N-1}\right\} \quad \text{by} \quad r=\overline{1,\, h}. \tag{58}$$

*The rest* $N-1-|\mathscr{A}_h|$ *components are computed as*

$$y^*_k=\frac{b_k\left(1-\sum\limits_{w\in\mathscr{A}_h} a_w\right)}{1+\sum\limits_{n=1}^{N-1}(b_n-a_n)-\sum\limits_{w\in\mathscr{A}_h} b_w} \qquad \forall\, k\in\left\{\overline{1,\, N-1}\right\}\backslash\mathscr{A}_h \tag{59}$$

*if only*

$$\frac{b_k\left(1-\sum\limits_{w\in\mathscr{A}_h} a_w\right)}{1+\sum\limits_{n=1}^{N-1}(b_n-a_n)-\sum\limits_{w\in\mathscr{A}_h} b_w}\geqslant a_k \qquad \forall\, k\in\left\{\overline{1,\, N-1}\right\}\backslash\mathscr{A}_h. \tag{60}$$

**Proof.** If $h = 1$ then $\mathscr{A}_{h-1} = \emptyset$ and it falls directly within conditions of Theorem 6. By induction for $h \geqslant 2$, the reasoning for (58)–(60) under (55) and (56) is the same as that in Theorem 6, where $\mathscr{U}$ along the proof is substituted with $\mathscr{A}_h$, formula (38) is substituted with

$$y_u^{**} = \frac{b_u \left(1 - \sum\limits_{w \in \mathscr{A}_{h-1}} a_w\right)}{1 + \sum\limits_{n=1}^{N-1} (b_n - a_n) - \sum\limits_{w \in \mathscr{A}_{h-1}} b_w} \quad \text{for} \quad u \in \mathscr{U} \subset \left\{\overline{1,\, N-1}\right\}$$

and formula (44) is substituted with

$$y_k^* < \frac{b_k \left(1 - \sum\limits_{w \in \mathscr{A}_{h-1}} a_w\right)}{1 + \sum\limits_{n=1}^{N-1} (b_n - a_n) - \sum\limits_{w \in \mathscr{A}_{h-1}} b_w} \quad \text{for} \quad \frac{b_k \left(1 - \sum\limits_{w \in \mathscr{A}_{h-1}} a_w\right)}{1 + \sum\limits_{n=1}^{N-1} (b_n - a_n) - \sum\limits_{w \in \mathscr{A}_{h-1}} b_w} > a_k.$$

Thus, it remains only to prove (57). The maximal number of non-overlapping nonempty subsets of the set $\left\{\overline{1,\, N-1}\right\}$ is $N-1$ (meaning not every possible combination, but "pieces" of the whole set once broken into them). In this case, they all are singletons. If $h = N - 1$ then

$$\mathscr{A}_{N-1} = \bigcup_{q=1}^{N-1} \mathscr{U}_q = \left\{\overline{1,\, N-1}\right\}$$

and the $k$-th component (59) would be

$$\begin{aligned} y_k^* &= \frac{b_k \left(1 - \sum\limits_{w \in \mathscr{A}_{N-1}} a_w\right)}{1 + \sum\limits_{n=1}^{N-1} (b_n - a_n) - \sum\limits_{w \in \mathscr{A}_{N-1}} b_w} \\ &= \frac{b_k \left(1 - \sum\limits_{w=1}^{N-1} a_w\right)}{1 + \sum\limits_{n=1}^{N-1} (b_n - a_n) - \sum\limits_{w=1}^{N-1} b_w} = \frac{b_k \left(1 - \sum\limits_{w=1}^{N-1} a_w\right)}{1 - \sum\limits_{n=1}^{N-1} a_n} = b_k \end{aligned}$$

by $k \in \left\{\overline{1,\, N-1}\right\} \setminus \mathscr{A}_{N-1} = \emptyset$. Hence, the case $h = N-1$ is impossible. If $h = N-2$ then the subset

$$\mathscr{A}_{N-2} = \bigcup_{q=1}^{N-2} \mathscr{U}_q \tag{61}$$

consists of $N-2$ indices, and thus $k \in \left\{\overline{1,\, N-1}\right\} \backslash \mathscr{A}_{N-2}$ exists as a single index. So, from inequality (60) we have that

$$\frac{b_k \left(1 - \sum\limits_{w \in \mathscr{A}_{N-2}} a_w\right)}{1 + \sum\limits_{n=1}^{N-1} (b_n - a_n) - \sum\limits_{w \in \mathscr{A}_{N-2}} b_w} \geqslant a_k, \tag{62}$$

$$\frac{b_k \left(1 - \sum\limits_{w \in \mathscr{A}_{N-2}} a_w\right)}{1 + b_k - \sum\limits_{n=1}^{N-1} a_n} \geqslant a_k,$$

$$b_k \left(1 - \sum_{w \in \mathscr{A}_{N-2}} a_w\right) \geqslant a_k \left(1 + b_k - \sum_{n=1}^{N-1} a_n\right),$$

$$b_k \left(1 - \sum_{w \in \mathscr{A}_{N-2}} a_w\right) - a_k b_k \geqslant a_k \left(1 - \sum_{n=1}^{N-1} a_n\right),$$

$$b_k \left(1 - \sum_{w \in \mathscr{A}_{N-2}} a_w - a_k\right) \geqslant a_k \left(1 - \sum_{n=1}^{N-1} a_n\right),$$

$$b_k \left(1 - \sum_{n=1}^{N-1} a_n\right) \geqslant a_k \left(1 - \sum_{n=1}^{N-1} a_n\right). \tag{63}$$

Inequality (63) is true, so that confirms the case $h = N-2$ is possible. Possibility of cases $h < N-2$ is inductively verifiable. □

The following definitions generalize ranking the irregularity of the decision-maker's optimal strategy.

**Definition 8.** The $u_h$-th component $y^*_{u_h} = a_{u_h}$ of the second player's optimal strategy $\mathbf{Y}^*_{N-1}$ in PURG is called a simple irregular component of the $h$-th rank (SIC-$h$) if condition (55) holds, $h = \overline{1,\, N-2}$. A set of those components (SICs-$h$) is called a left strategy subset of the $h$-th rank (LSS-$h$).

**Definition 9.** The $k$-th component (59) of the second player's optimal strategy $\mathbf{Y}^*_{N-1}$ in PURG is called a downward-biased irregular component of the $h$-th rank (DBIC-$h$), $h = \overline{1,\, N-2}$. A set of those components (DBICs-$h$) is called a biased strategy subset of the $h$-th rank (BSS-$h$).

**Definition 10.** An irregular optimal strategy $\mathbf{Y}^*_{N-1}$ of the second player in PURG is called an irregular strategy of the $h$-th rank (IS-$h$) if it includes an LSS-$h$ and does not contain irregular components of ranks higher than $h$.

According to Theorem 9, IS-$h$ always contains SICs-$r$ $\forall\, r = \overline{1,\, h}$ and BSS-$h$. Every LSS-$r$ contains at least a one component. IS-$h$ does not contain BSS-$r$ $\forall\, r = \overline{1,\, h-1}$. The irregular optimal strategy of the highest rank, which is, literally, IS-$(N-2)$, has a captivating property.

**Theorem 10.** *IS-$(N-2)$ in PURG has a one and only one component, which is greater than the corresponding left endpoint.*

**Proof.** Using (61) from the proof of Theorem 9, we have (58) for $N-2$ indices. Here, inequality (62) holds for a single index $k \in \left\{\overline{1,\, N-1}\right\} \backslash \mathscr{A}_{N-2}$. But inequality (63) always holds strictly. Therefore, inequality (62) holds strictly as well. □

Reverting to Example 2, its decision-maker's optimal strategy $\mathbf{Y}^*_{N-1}$ is computable now: $\mathscr{U}_1 = \{2\}$ and a single SIC-1 is $y_2^* = a_2 = 0.15$, $\mathscr{U}_2 = \{3\}$ and a single SIC-2 is $y_3^* = a_3 = 0.16$. Finally, a single DBIC-2 is

$$y_1^* = \frac{b_1\,(1 - a_2 - a_3)}{1 + \sum\limits_{n=1}^{3} (b_n - a_n) - b_2 - b_3} = \frac{0.5 \cdot (1 - 0.15 - 0.16)}{1 + 0.54 - 0.2 - 0.25} = \frac{69}{218} > \frac{1}{10} = a_1.$$

This IS-2 is an instance that supports the assertion of Theorem 10. Inequality (52) within the assertion of Theorem 8 holds as well. Similar to Theorem 9, which generalizes Theorem 6, an inductive generalization to Theorem 8 is important as well.

**Theorem 11.** *Whichever nonempty IS-$h$ in PURG is, $\exists\, l \in \left\{\overline{1,\, N-1}\right\} \backslash \mathscr{A}_h$ such that*

$$\frac{b_l \left(1 - \sum\limits_{w \in \mathscr{A}_h} a_w\right)}{1 + \sum\limits_{n=1}^{N-1} (b_n - a_n) - \sum\limits_{w \in \mathscr{A}_h} b_w} > a_l \quad \text{for} \quad h \in \left\{\overline{1,\, N-2}\right\}. \tag{64}$$

**Proof.** Assume that inequality (60) holds with the reverse sign implying that

$$\frac{b_k \left(1 - \sum\limits_{w \in \mathscr{A}_h} a_w\right)}{1 + \sum\limits_{n=1}^{N-1} (b_n - a_n) - \sum\limits_{w \in \mathscr{A}_h} b_w} \leqslant a_k \qquad \forall\, k \in \left\{\overline{1,\, N-1}\right\} \backslash \mathscr{A}_h.$$

Then, $\mathscr{U}$ is substituted with $\mathscr{A}_h$, and we get the same reasoning as that in Theorem 8, finally coming to contradiction (54). The refuted assumption implies that $\exists\, l \in \left\{\overline{1,\, N-1}\right\} \backslash \mathscr{A}_h$ such that (64) holds. □

According to Theorem 11, a case of $y_k^* = a_k\ \forall\, k \in \left\{\overline{1,\, N-1}\right\}$ in PURG is impossible. Nevertheless, it is revealed that a potential DBIC-$h$ is always less than a potential DBIC-$(h-1)$.

**Theorem 12.** *DBICs-h and DBICs-$(h-1)$ obey the inequality*

$$\frac{b_k\left(1-\sum\limits_{w\in\mathscr{A}_h} a_w\right)}{1+\sum\limits_{n=1}^{N-1}(b_n-a_n)-\sum\limits_{w\in\mathscr{A}_h} b_w} < \frac{b_k\left(1-\sum\limits_{z\in\mathscr{A}_{h-1}} a_z\right)}{1+\sum\limits_{n=1}^{N-1}(b_n-a_n)-\sum\limits_{z\in\mathscr{A}_{h-1}} b_z}$$
$$\forall\, k \in \left\{\overline{1,\, N-1}\right\}\backslash\mathscr{A}_h \quad \textit{and} \quad \forall\, h = \overline{2,\, N-2}. \tag{65}$$

**Proof.** The difference between the left and right terms in inequality (65) divided by $b_k$ is:

$$\left(\left(1-\sum_{w\in\mathscr{A}_h} a_w\right)\left(1+\sum_{n=1}^{N-1}(b_n-a_n)-\sum_{z\in\mathscr{A}_{h-1}} b_z\right)\right.$$
$$\left.-\left(1-\sum_{z\in\mathscr{A}_{h-1}} a_z\right)\left(1+\sum_{n=1}^{N-1}(b_n-a_n)-\sum_{w\in\mathscr{A}_h} b_w\right)\right)\left(1+\sum_{n=1}^{N-1}(b_n-a_n)\right.$$
$$\left.-\sum_{w\in\mathscr{A}_h} b_w\right)^{-1}\left(1+\sum_{n=1}^{N-1}(b_n-a_n)-\sum_{z\in\mathscr{A}_{h-1}} b_z\right)^{-1}. \tag{66}$$

Using that $\mathscr{A}_h = \mathscr{A}_{h-1}\cup\mathscr{U}_h$ and

$$\sum_{w\in\mathscr{A}_h} a_w = \sum_{z\in\mathscr{A}_{h-1}} a_z + \sum_{l\in\mathscr{U}_h} a_l,$$
$$\sum_{w\in\mathscr{A}_h} b_w = \sum_{z\in\mathscr{A}_{h-1}} b_z + \sum_{l\in\mathscr{U}_h} b_l,$$

the expanded numerator in (66) is:

$$1+\sum_{n=1}^{N-1}(b_n-a_n)-\sum_{z\in\mathscr{A}_{h-1}} b_z-\sum_{w\in\mathscr{A}_h} a_w$$
$$-\sum_{w\in\mathscr{A}_h} a_w\sum_{n=1}^{N-1}(b_n-a_n)+\sum_{w\in\mathscr{A}_h} a_w\sum_{z\in\mathscr{A}_{h-1}} b_z$$
$$-\quad 1-\sum_{n=1}^{N-1}(b_n-a_n)+\sum_{w\in\mathscr{A}_h} b_w+\sum_{z\in\mathscr{A}_{h-1}} a_z$$

$$
\begin{aligned}
&+ \sum_{z\in\mathscr{A}_{h-1}} a_z \sum_{n=1}^{N-1}(b_n - a_n) - \sum_{z\in\mathscr{A}_{h-1}} a_z \sum_{w\in\mathscr{A}_h} b_w \\
&= -\sum_{l\in\mathscr{U}_h} a_l - \sum_{l\in\mathscr{U}_h} a_l \sum_{n=1}^{N-1}(b_n - a_n) \\
&+ \sum_{w\in\mathscr{A}_h} a_w \sum_{z\in\mathscr{A}_{h-1}} b_z + \sum_{l\in\mathscr{U}_h} b_l - \sum_{z\in\mathscr{A}_{h-1}} a_z \sum_{w\in\mathscr{A}_h} b_w \\
&= -\sum_{l\in\mathscr{U}_h} a_l \left(1 + \sum_{n=1}^{N-1}(b_n - a_n)\right) + \left(\sum_{z\in\mathscr{A}_{h-1}} a_z + \sum_{l\in\mathscr{U}_h} a_l\right) \sum_{z\in\mathscr{A}_{h-1}} b_z \\
&+ \sum_{l\in\mathscr{U}_h} b_l - \sum_{z\in\mathscr{A}_{h-1}} a_z \left(\sum_{z\in\mathscr{A}_{h-1}} b_z + \sum_{l\in\mathscr{U}_h} b_l\right) \\
&= -\sum_{l\in\mathscr{U}_h} a_l \left(1 + \sum_{n=1}^{N-1}(b_n - a_n)\right) \\
&+ \sum_{l\in\mathscr{U}_h} a_l \sum_{z\in\mathscr{A}_{h-1}} b_z + \sum_{l\in\mathscr{U}_h} b_l - \sum_{z\in\mathscr{A}_{h-1}} a_z \sum_{l\in\mathscr{U}_h} b_l \\
&= -\sum_{l\in\mathscr{U}_h} a_l \left(1 + \sum_{n=1}^{N-1}(b_n - a_n) - \sum_{z\in\mathscr{A}_{h-1}} b_z\right) \\
&+ \sum_{l\in\mathscr{U}_h} b_l \left(1 - \sum_{z\in\mathscr{A}_{h-1}} a_z\right). \qquad (67)
\end{aligned}
$$

Summing up both sides of (55), we get

$$
\frac{\displaystyle\sum_{u_h\in\mathscr{U}_h} b_{u_h} \left(1 - \sum_{w\in\mathscr{A}_{h-1}} a_w\right)}{\displaystyle 1 + \sum_{n=1}^{N-1}(b_n - a_n) - \sum_{w\in\mathscr{A}_{h-1}} b_w} < \sum_{u_h\in\mathscr{U}_h} a_{u_h},
$$

$$
\sum_{u_h\in\mathscr{U}_h} b_{u_h} \left(1 - \sum_{w\in\mathscr{A}_{h-1}} a_w\right) - \sum_{u_h\in\mathscr{U}_h} a_{u_h} \left(1 + \sum_{n=1}^{N-1}(b_n - a_n) - \sum_{w\in\mathscr{A}_{h-1}} b_w\right) < 0,
$$

whence the last term of (67) is negative. Thus, difference (66) is negative, so inequality (65) holds. □

Henceforward, strategy $\mathbf{Y}_{N-1}^{*}$ in PURG is either regular or irregular. Whichever strategy $\mathbf{Y}_{N-1}^{*}$ is, its $k$-th component is (29) following Theorem 7 and Theorem 12.

# 5. Application

Branches, where PURG and the strategy $\mathbf{Y}_{N-1}^{*}$ may be applied, relate to multiparametric systems whose statistics either are very poor or include a high volatility. These systems must consist of $N$ objects, over which some action is accomplished. Delivering capacities, loading/charging up to uncertain levels, time windowing/scheduling are the most attractable examples of the action [44, 45, 19, 40, 10]. Volumes or periods thereby are either non-predictable or their adjustment takes significant time/resources/energy.

An instance of the PURG's application is an incipient consumer service, where amounts of demands at the start are given as intervals, although the grand total is fixed. The $N$-th consumer is a fictional one, whose "demand" is an unclaimed amount. This amount is usually delivered backward or utilized. If all consumers are equal (without priorities), the PURG-based model with (30)–(32) is acceptable.

If the amounts are areas to be appropriately divided, then $q = 2$. Cases with $q = 2$ relate mostly to areas/squares when intervals are given in feet, yards, miles, etc. If the amounts are measured in cubature (say, for gallonage, barrels, or cubic meters), then $q = 3$. Cases with $q = 3$ relate to volumes when intervals are given similarly. The power $q$, although not influencing on the strategy $\mathbf{Y}_{N-1}^{*}$, stands for the factual results and the point estimation aftermaths.

PURG can be used in preparing data for interval analysis, without mapping intervals into points. The data preparation purports refinement of intervals. If an irregularity is revealed then the intervals may be considered inappropriate for operating over them. Thus, the presence of irregular components in $\mathbf{Y}_{N-1}^{*}$ is a criterion for preventing improper interval operations. Only regular strategy $\mathbf{Y}_{N-1}^{*}$ or $\mathbf{Y}_{N-1}^{*}$ with just DBICs-1 and a few SICs-1 might admit operations over the corresponding $N - 1$ intervals.

**Example 3.** Let $N = 5$, $a_1 = 0.175$, $b_1 = 0.225$, $a_2 = 0.2$, $b_2 = 0.225$, $a_3 = 0.2$, $b_3 = 0.25$, $a_4 = 0.15$, $b_4 = 0.275$. Here, first of all,

$$\sum_{n=1}^{4} (b_n - a_n) = 0.225 - 0.175 + 0.225 - 0.2 + 0.25 - 0.2 + 0.275 - 0.15 = 0.25,$$

$$\frac{b_1}{1 + \sum_{n=1}^{4} (b_n - a_n)} = \frac{0.225}{1 + 0.25} = \frac{9}{50} > \frac{7}{40} = a_1,$$

$$\frac{b_2}{1 + \sum_{n=1}^{4} (b_n - a_n)} = \frac{0.225}{1 + 0.25} = \frac{9}{50} < \frac{1}{5} = a_2,$$

$$\frac{b_3}{1+\sum\limits_{n=1}^{4}(b_n-a_n)} = \frac{0.25}{1+0.25} = \frac{1}{5} = a_3,$$

$$\frac{b_4}{1+\sum\limits_{n=1}^{4}(b_n-a_n)} = \frac{0.275}{1+0.25} = \frac{11}{50} > \frac{3}{20} = a_4,$$

whence $\mathscr{U}_1 = \{2\}$ and $y_2^* = a_2 = 0.2$. Then

$$\frac{b_1(1-a_2)}{1+\sum\limits_{n=1}^{4}(b_n-a_n)-b_2} = \frac{0.225\cdot(1-0.2)}{1+0.25-0.225} = \frac{36}{205} > \frac{7}{40} = a_1,$$

$$\frac{b_3(1-a_2)}{1+\sum\limits_{n=1}^{4}(b_n-a_n)-b_2} = \frac{0.25\cdot(1-0.2)}{1+0.25-0.225} = \frac{8}{41} < \frac{1}{5} = a_3,$$

$$\frac{b_4(1-a_2)}{1+\sum\limits_{n=1}^{4}(b_n-a_n)-b_2} = \frac{0.275\cdot(1-0.2)}{1+0.25-0.225} = \frac{44}{205} > \frac{3}{20} = a_4,$$

whence $\mathscr{U}_2 = \{3\}$ and $y_3^* = a_3 = 0.2$. Further,

$$\frac{b_1(1-a_2-a_3)}{1+\sum\limits_{n=1}^{4}(b_n-a_n)-b_2-b_3} = \frac{0.225\cdot(1-0.2-0.2)}{1+0.25-0.225-0.25} = \frac{27}{155} < \frac{7}{40} = a_1,$$

$$\frac{b_4(1-a_2-a_3)}{1+\sum\limits_{n=1}^{4}(b_n-a_n)-b_2-b_3} = \frac{0.275\cdot(1-0.2-0.2)}{1+0.25-0.225-0.25} = \frac{33}{155} > \frac{3}{20} = a_4,$$

whence $\mathscr{U}_3 = \{1\}$ and $y_1^* = a_1 = 0.175$. Finally, according to Theorem 10,

$$\begin{aligned} y_4^* &= \frac{b_4(1-a_2-a_3-a_1)}{1+\sum\limits_{n=1}^{4}(b_n-a_n)-b_2-b_3-b_1} \\ &= \frac{0.275\cdot(1-0.2-0.2-0.175)}{1+0.25-0.225-0.25-0.225} = \frac{17}{80} > \frac{3}{20} = a_4. \end{aligned}$$

The found IS-3 is obviously poor. But after corrections of the first three intervals to [0.165; 0.225], [0.17; 0.235], [0.19; 0.26], [0.15; 0.275] a totally regular strategy is obtained. The intervals herein are made wider for 20 %, 160 %, and 40 %, respectively (the second interval was initially too short).

It is quite clear that the cost $v_{\text{opt}}$ of mapping interval uncertainties into point estimates by the minimax approach is always lesser than the cost by any other point estimation:

$$\begin{aligned} v_{\text{opt}} &= \max\left\{\left\{\left(\frac{b_k}{y_k^*}\right)^q\right\}_{k=1}^{N-1},\ \left(\frac{1-\sum\limits_{i=1}^{N-1} a_i}{1-\sum\limits_{j=1}^{N-1} y_j^*}\right)^q\right\} \\ &\leqslant \max\left\{\left\{\left(\frac{b_k}{y_k^{\langle 0\rangle}}\right)^q\right\}_{k=1}^{N-1},\ \left(\frac{1-\sum\limits_{i=1}^{N-1} a_i}{1-\sum\limits_{j=1}^{N-1} y_j^{\langle 0\rangle}}\right)^q\right\} = v^{\langle 0\rangle}, \end{aligned} \tag{68}$$

where $\left\{y_m^{\langle 0\rangle}\right\}_{m=1}^{N-1}$ are point estimates the first $N-1$ intervals, and

$$y_N^{\langle 0\rangle} = 1 - \sum_{m=1}^{N-1} y_m^{\langle 0\rangle}.$$

Point estimates $\left\{y_m^{\langle 0\rangle}\right\}_{m=1}^{N-1}$ can be obtained, for example, by simply finding an arithmetic/geometric mean of left and right endpoints for every interval. The numerical analysis, based on randomly generated positive intervals $\{[a_k;\ b_k]\}_{k=1}^{N-1}$ by (3), shows that non-minimax cost $v^{\langle 0\rangle}$ for both arithmetic and geometric mean approaches is far greater that that of minimax (Figure 2). The cost of choosing the point estimates as values of random variables is obviously the worst.

The average costs of the point estimation inevitably grow as $N$ increases. The minimax cost growth is significantly slower (Figure 3). Moreover, it looks like it has an asymptote, which is less than 1.3 for a reasonably wide sequence of $N$. The growths of the cost for both arithmetic and geometric mean approaches along with the random point choice do not seem to have such asymptotes. Nonetheless, the worst minimax cost $v_{\text{opt}} > 1.4$ (before averaging in Figure 3).

The advantage of the minimax approach reflected in Figure 3 strengthens as the number of intervals to be mapped into point estimates increases. Despite the minimax cost expectedly grows, prevention of the worst event is the main goal. This is especially important for branches where consequences of mistaken decisions will cost far much more than $v_{\text{opt}}$ by (68), e.g., power management systems planning [4, 32], allocation of resources for sustainability [24, 19, 25], nuclear data processing (1) and design of reactor in-core monitoring systems [16], reliable flight trajectories [28], cybersecurity [30, 25].

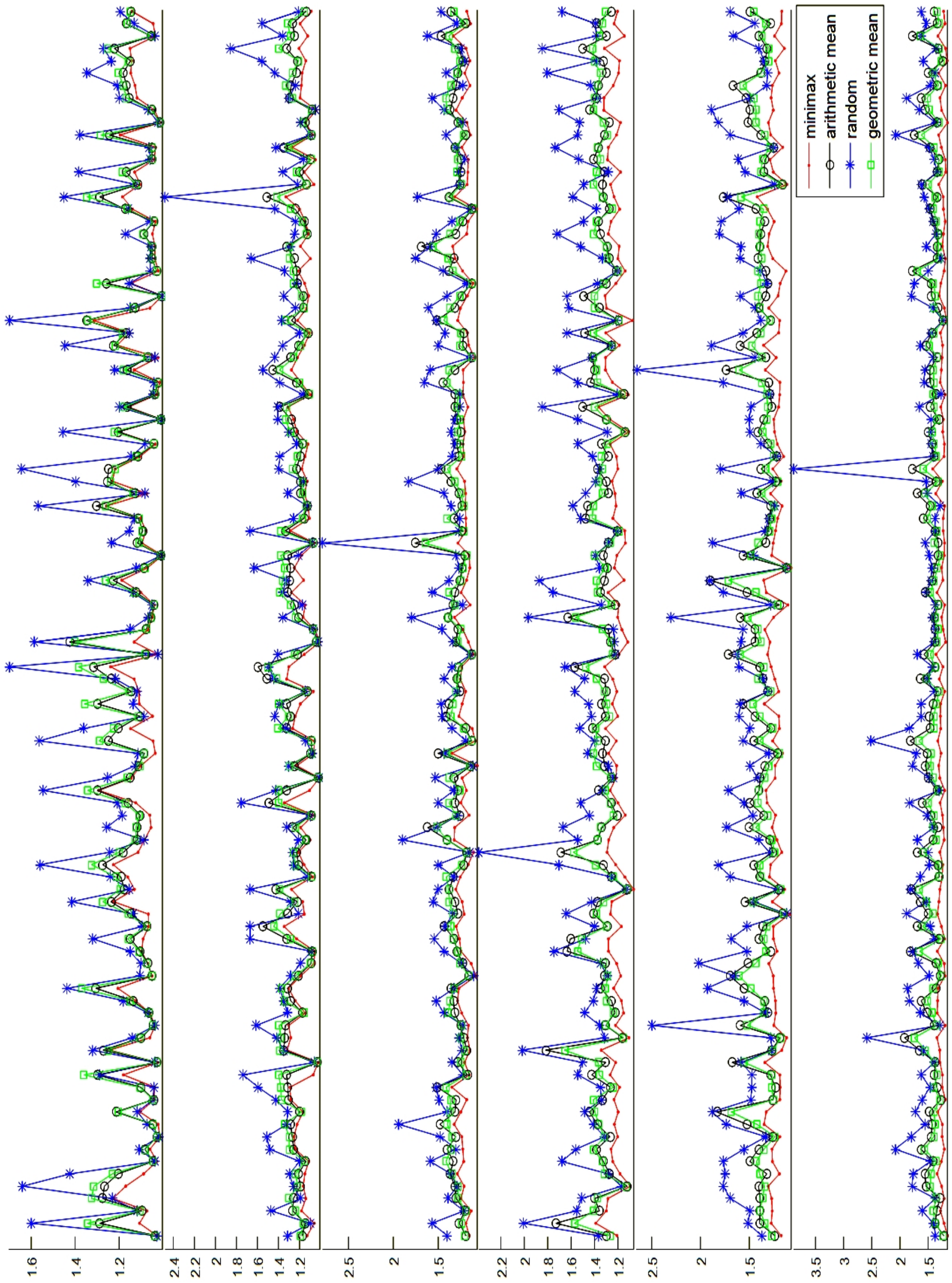


Figure 2: Costs of the point estimation (for $q = 1$) for six series of 100 sets of intervals by increasing $N$ from 2 through 7 (moving from the top to bottom), where the maximal ratio of the right and left endpoints is not greater than 2

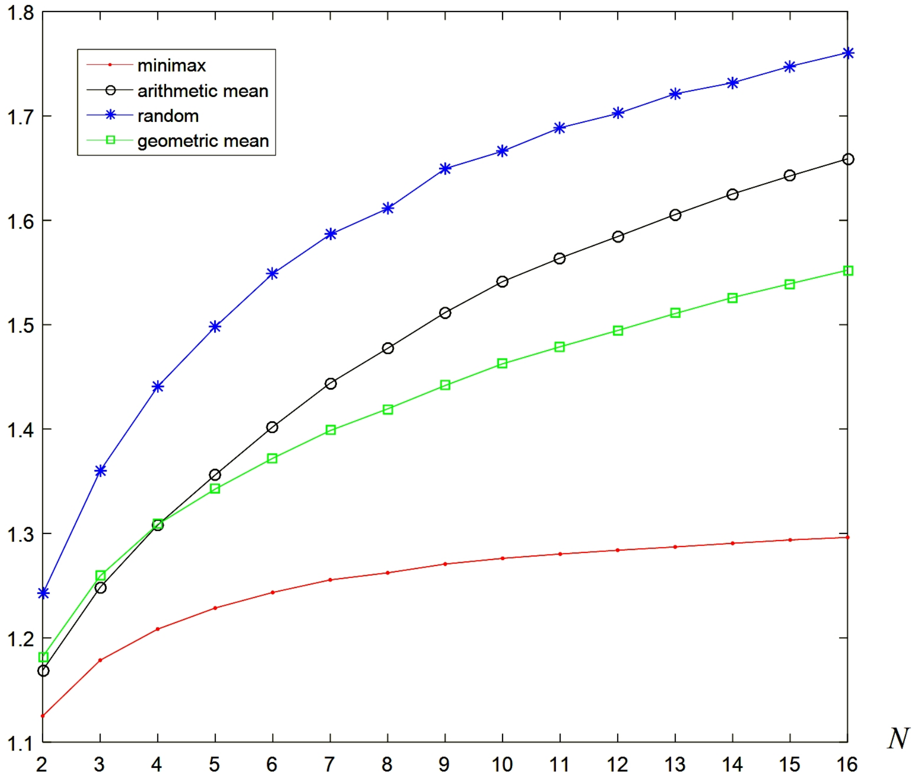


Figure 3: The average costs of the point estimation (for $q = 1$) for 15 series of 5000 sets of intervals, where the maximal ratio of the right and left endpoints is not greater than 2

## 6. Discussion

An apparent demerit of the PURG-based model is that the decision-maker's optimal strategy $\mathbf{Y}^*_{N-1}$ becomes inefficient as statistical data are accumulated. When statistical data are still insufficient for Bayesian decisions, the minimax strategy can be softened with expectations [27, 51]. A suitable moment to stop using the minimax approach along with the PURG-based model is when the respective probability density functions can be elicited from the accumulated statistics. However, strategy $\mathbf{Y}^*_{N-1}$ appears efficient when external conditions change frequently/quickly (i.e., are highly volatile) regardless of whether statistical data are sufficient or not.

Figure 4 shows interrelation of the proven theorems. Simpler cases are enveloped in more general cases. Theorem 4 describes an ideal version of PURG. It is naively believed that approximating to such PURG will give a totally regular strategy for the decision-maker.

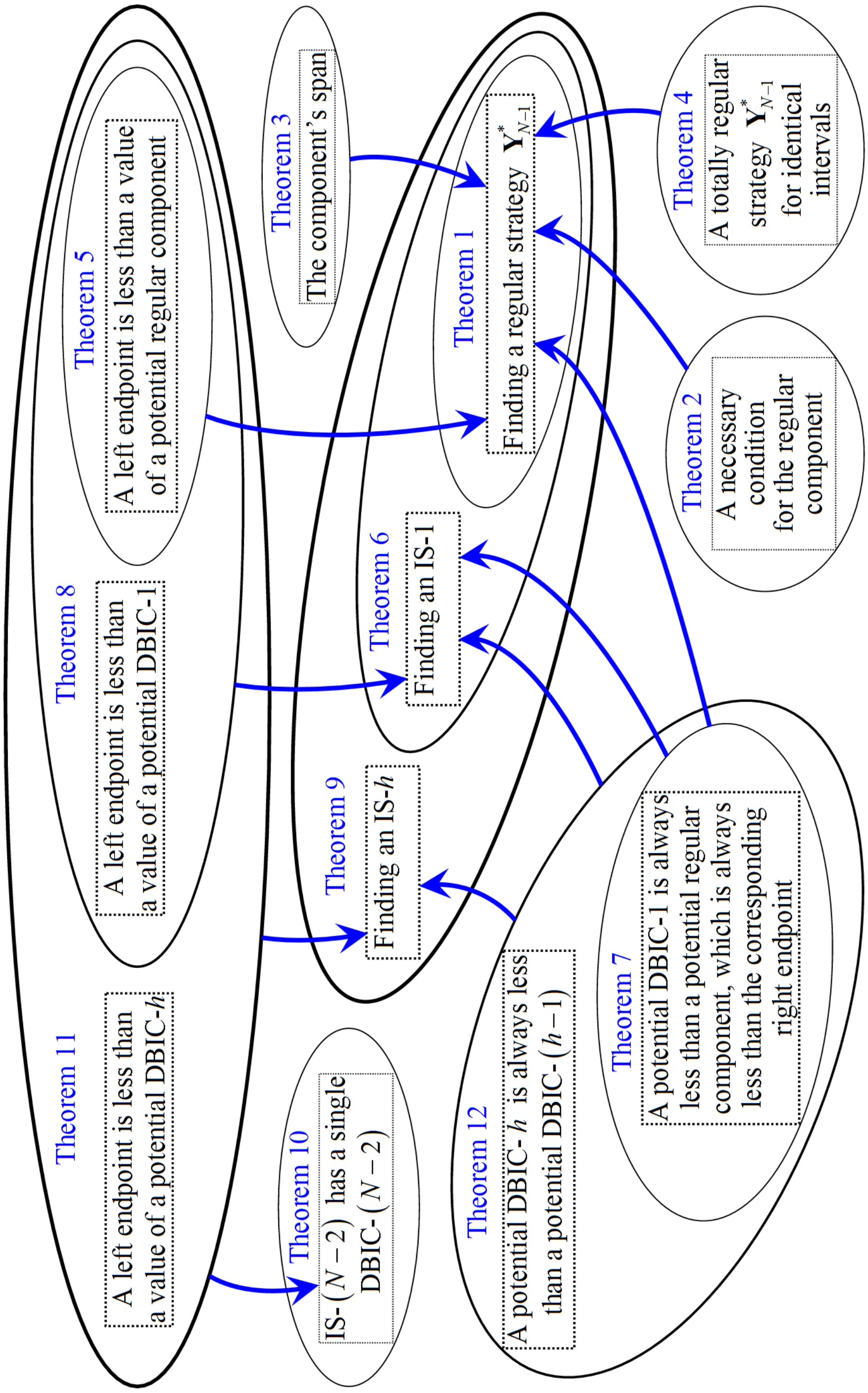


Figure 4: Assertions and purposes of the proven theorems

Normalization to 1 allows considering a fluent grand total, but the endpoints of $N-1$ intervals must be fixed. This means that the portion of a particular interval should be always fixed. This is the fundamental step while dealing with real interval data.

Simple realizability in practice is a merit of the PURG-based model. Another one is a non-sophisticated routine of building PURG and finding the minimax strategy. This routine is of two stages. Firstly, grand total and boundaries of objects' portions are normalized. Secondly, components $\{y_k^*\}_{k=1}^{N-1}$ are computed, whichever formulae are used. Moreover, the solution is independent of $q$. Such independency imparts some universality to PURG.

Strategy $\mathbf{Y}_{N-1}^*$ is always single, there are no PURGs with a continuum of such strategies. This saves from solving a subsequent decision making problem that lies in selecting a single unique strategy. Additionally, the minimax irregular strategy is ranked, that facilitate in comparing different intervals given before the point estimation. Intervals which are mapped into simple irregular components of higher ranks are less reliable.

Notwithstanding the intervals and IS-3 in Example 3, a series of numerical tests confirms that cases of deeply irregular components (high ranks of irregularity) are unlikely. Unlikelihood strengthens when endpoints of the same type (left or right) are close by value, especially if they are identical. A confluent case is PURG on hypercube (30) granting a totally regular strategy with components (32).

## 7. Conclusion

The PURG-based model, including formulae for $\{y_k^*\}_{k=1}^{N-1}$ and (11), is a contribution to the field of non-statistical interval uncertainty reduction. This is a model of maximal disbalance between a real unknown amount and a guessed amount. These amounts are interpreted as aftermaths of the point estimation. This model grants a pure strategy $\mathbf{Y}_{N-1}^*$ whose components are the most appropriate point estimates. The appropriateness is founded on the minimax principle. This is a subtle optimization model addressing minimum information processes. The approach will work efficiently only by when no information is available but those interval estimates. Such a situation is common in processing raw data of difficult measurements and measurements with jeopardy (for instance, concentrations of river/air/ground pollutants, radiation control, etc.), where statistics are too small to estimate, e.g., a probability distribution and find point estimates of its main characteristics (expectance and variance). The minimax approach is applicable for cases just like those mentioned above, when the risk of a biased decision should be as minimal as possible until additional information becomes available.

Main benefits of PURG and $\mathbf{Y}_{N-1}^*$ are simplicity and acceptability of any time period for practical realization. This is because we do not have expected values, but only ready-to-go values. In other words, decisions based on minimax strategy $\mathbf{Y}_{N-1}^*$ are implemented on-the-fly, whichever changes of initial intervals are, and the result of such an implementation comes instantly. If something new emerges frequently (resulting in frequently "floating" intervals), the minimax reaction by strategy $\mathbf{Y}^*_{N-1}$ is immediate owing to its components are re-computed easily.

Apart from the point estimation, the PURG-based model contributes to the decision making theory by deciding on appropriateness of the initial interval estimations grouped as (2) by (3). Irregularities in the decision-maker's optimal strategy $\mathbf{Y}^*_{N-1}$ may serve as an evidence of incorrectly setting the corresponding intervals' endpoints. Furthermore, higher ranks of irregularity might reject the corresponding intervals and send them for correction before starting interval calculus. More specifically, a totally regular strategy $\mathbf{Y}^*_{N-1}$ satisfying condition (28) is the best case for mapping partial interval uncertainties into point estimates. A regular strategy $\mathbf{Y}^*_{N-1}$ found by (12) but failing with (28) produces a suspicion of that not all the intervals are evaluated correctly. An irregular strategy of the first rank (IS-1) found by (34)–(37) is a feature of poorly evaluated intervals. Then, correction of the intervals corresponding to LSS-1 (a set of SICs-1) is an option. Deeper irregularity implying a higher rank of irregular components directs that option to compulsorily correcting the intervals which correspond to left strategy subsets of all the ranks reached. Therefore, LSS-2 and left strategy subsets of higher ranks play a role of indicators to which intervals should be necessarily corrected.

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**Vadim Romanuke**
email: `v.romanuke@amw.gdynia.pl`
ORCID: 0000-0003-3543-3087
Faculty of Navigation and Naval Weapons
Polish Naval Academy
Gdynia
POLAND

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# Finite Blaschke Products and Decomposition

*Sümeyra Uçar and Nihal Yılmaz Özgür*

Abstract: Let $B(z)$ be a finite Blaschke product of degree $n$. We consider the problem when a finite Blaschke product can be written as a composition of two nontrivial Blaschke products of lower degree related to the condition $B \circ M = B$ where $M$ is a Möbius transformation from the unit disk onto itself.

*AMS Subject Classification:* 30J10, 30D05.
*Keywords and Phrases:* Finite Blaschke products; Composition of Blaschke products.

## 1. Introduction

It is known that a Möbius transformation from the unit disc $\mathbb{D}$ onto itself is of the following form:

$$M(z) = c\frac{z-\alpha}{1-\overline{\alpha}z}, \tag{1.1}$$

where $\alpha \in \mathbb{D}$ and $c$ is a complex constant of modulus one (see [5] and [8]).

The rational function

$$B(z) = c\prod_{k=1}^{n}\frac{z-a_k}{1-\overline{a_k}z}$$

is called a finite Blaschke product of degree $n$ for the unit disc where $|c| = 1$ and $|a_k| < 1$, $1 \leq k \leq n$.

Blaschke products of the following form are called canonical Blaschke products:

$$B(z) = z\prod_{k=1}^{n-1}\frac{z-a_k}{1-\overline{a_k}z},\ |a_k| < 1 \text{ for } 1 \leq k \leq n-1. \tag{1.2}$$

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It is well-known that every canonical Blaschke product $B$ of degree $n$, is associated with a unique Poncelet curve (for more details see [2], [4] and [9]).

Decomposition of finite Blaschke products is an interesting matter studied by many researchers by the use of a point $\lambda$ on the unit circle $\partial\mathbb{D}$ and the points $z_1, z_2, ..., z_n$ on the unit circle $\partial\mathbb{D}$ such that $B(z_1) = ... = B(z_n) = \lambda$. For example, using circles passing through the origin, it was given the determination of these points for the Blaschke products written as composition of two nontrivial Blaschke products of lower degree (see [11] and [12]). On the other hand, decomposition of finite Blaschke products is related to the condition $B \circ M = B$ where $M$ is a Möbius transformation of the form (1.1) and different from the identity (see [1] and [8]). Some of the recent studies about decomposibility of finite Blaschke products can be found in [3].

In this paper we consider the relationship between the following two questions for a given canonical finite Blaschke product:

$Q1$) Is there a Möbius transformation $M$ such that $B \circ M = B$ and $M$ is different from the identity?

$Q2$) Can $B$ be written as a composition $B = B_2 \circ B_1$ of two finite Blaschke products of lower degree?

Also, the above problems have been considered in details due to group theory in [7]. In the present study, we focus on a special class of finite Blaschke products (canonical finite Blaschke products).

In Section 2, we recall some known theorems about these questions. In Section 3 we give some theorems and examples related to the above two questions.

## 2. Preliminaries

In this section we give some information about decomposition of finite Blaschke products written as $B \circ M = B$ where $M$ is a Möbius transformation different from the identity. In [8], it was proved that the set of continuous functions $M$ from the unit disc into the unit disc such that $B \circ M = B$ is a cyclic group if $B$ is a finite Blaschke product. In [1], the condition $B \circ M = B$ was used in the following theorem.

**Theorem 2.1.** (*See [1], Theorem* 3.1 *on page* 335) *Let $B$ be a finite Blaschke product of degree $n$. Suppose $M \neq I$ is holomorphic from $\mathbb{D}$ into $\mathbb{D}$ such that $B \circ M = B$. Then*:

(*i*) *$M$ is a Möbius transformation,*

(*ii*) *There is a positive integer $k \geq 2$ such that the iterates, $M, ...M^{k-1}$ are all distinct but $M^k = I$.*

(*iii*) *$k$ divides $n$.*

(*iv*) *There is a $\gamma \in \mathbb{D}$ such that $M(\gamma) = \gamma$.*

(*v*) *B can be written as a composition* $B = B_2 \circ B_1$ *of finite Blaschke products with the degree* $B_1 = k$ *and the degree of* $B_2 = n/k$. $B_1$ *may be taken to be*

$$B_1(z) = \left(\frac{z-\gamma}{1-\overline{\gamma}z}\right)^k .$$

But the condition $B \circ M = B$ is not necessary for a decomposition of finite Blaschke products (see [1] for more details).

It follows from Theorem 2.1 that if a finite Blaschke product $B$ can be written as $B \circ M = B$, then $B$ can be decomposed into a composition of two finite Blaschke products of lower order. However the following theorem gives necessary and sufficient conditions for the question $Q1$.

**Theorem 2.2.** (*See [8], Proposition* 4.1 *on page* 202) *Let* $B$ *be finite Blaschke product of degree* $n \geq 1$, $B(z) = \prod\limits_{k=1}^{n} \frac{z-a_k}{1-\overline{a_k}z}$ *with* $a_k \in \mathbb{D}$ *for* $1 \leq k \leq n$. *Let* $M$ *be a Möbius transformation from* $\mathbb{D}$ *onto* $\mathbb{D}$. *The following assertions are equivalent*:

(*i*) $(B \circ M)(z) = B(z), z \in \mathbb{C}, |z| \leq 1.$

(*ii*) $M(\{a_1, a_2, ..., a_n\}) = \{a_1, a_2, ..., a_n\}$ *and there exists* $z_0 \in \overline{\mathbb{D}} \backslash \{a_1, a_2, ..., a_n\}$ *such that* $(B \circ M)(z_0) = B(z_0)$.

Using the following proposition given in [8], we know how to construct a finite Blaschke product of degree $n \geq 1$ satisfying the condition $B \circ M = B$ where M is a Möbius transformation from $\mathbb{D}$ into $\mathbb{D}$ different from identity.

**Proposition 2.1.** (*See [8], Proposition* 4.2 *on page* 203) *Let* $n$ *be a positive integer and let* $M$ *be a Möbius transformation from* $\mathbb{D}$ *into* $\mathbb{D}$ *such that* $M^n(0) = 0$ *and* $\{0, M(0), ..., M^{n-1}(0)\}$ *is a set of* $n$ *distinct points in* $\mathbb{D}$. *Consider the finite Blaschke product* $B(z) = z \prod\limits_{k=1}^{n-1} \frac{z-M^k(0)}{1-\overline{M^k(0)}z}$. *Then the group* $G$ *of the invariants of* $B$ *is generated by* $M$.

From [10], we know the following theorem and we will use this theorem in the next chapter.

**Theorem 2.3.** *Let*

$$A(z) = \prod_{k=1}^{n} \frac{z-a_k}{1-\overline{a_k}z} \text{ and } B(z) = \prod_{k=1}^{n} \frac{z-b_k}{1-\overline{b_k}z}$$

*with* $a_k$ *and* $b_k \in \mathbb{D} = \{|z| < 1\}$ *for* $k = 1, 2, ..., n$. *Suppose that* $A(\lambda_k) = B(\lambda_k)$ *for* $n$ *distinct points* $\lambda_1, ..., \lambda_n$ *in* $\mathbb{D}$. *Then* $A \equiv B$.

# 3. Blaschke products of degree n

Let $B$ be a canonical Blaschke product of degree $n$ and following [8], let $Z(B)$ denotes the set of the elements $z \in \mathbb{D}$ such that $B(z) = 0$. In this section we consider the relationship between the questions (Q1) and (Q2).

Now we give the following theorem for the Blaschke products of degree $n$.

**Theorem 3.1.** *Let $M(z) = c\frac{z-\alpha}{1-\overline{\alpha}z}$ be a Möbius transformation different from the identity from the unit disc into itself and $B(z) = z\prod\limits_{k=1}^{n-1}\frac{z-a_k}{1-\overline{a_k}z}$ be a canonical Blaschke product of degree $n$. Then $B \circ M = B$ if and only if $M(z) = c\frac{z-a_j}{1-\overline{a_j}z}$ with $|a_j| = |a_l|$ for some $a_j, a_l$ $(0 \leq j, l \leq n-1, j \neq l)$, $c = -\frac{a_j}{a_l}$ and the equation*

$$M^{n-1}(0) - a_i = 0, \ (1 \leq i \leq n-1) \tag{3.1}$$

*is satisfied by the non-zero zeros of $B$.*

**Proof. Necessity:** Let $M(z) = c\frac{z-\alpha}{1-\overline{\alpha}z}$ be a Möbius transformation different from the identity from the unit disc into itself, $B(z) = z\prod\limits_{k=1}^{n-1}\frac{z-a_k}{1-\overline{a_k}z}$ be a canonical Blaschke product of degree $n$ and $B \circ M = B$. From Proposition 2.1, we know $Z(B) = \{0, M(0), ..., M^{n-1}(0)\}$ and $M^n(0) = 0$. Without loss of generality, let us take $a_1 = M(0)$. Then we find

$$a_1 = -c\alpha \tag{3.2}$$

Let $a_j = M^j(0)$ $(2 \leq j \leq n-1)$ and then we find the following equations:

$$a_2 = M^2(0) = -\frac{c\alpha(1+c)}{1+c|\alpha|^2}, \ a_3 = M^3(0)$$

$$\vdots$$

$$a_{n-1} = M^{n-1}(0). \tag{3.3}$$

By Theorem 2.1, then it should be $M^n(0) = 0$. Using the equation (3.3) we have

$$M^n(0) = M\left(M^{n-1}(0)\right) = M(a_{n-1}) = 0$$

and so we get

$$a_{n-1} = \alpha.$$

By the equation (3.2) we find $c = -\frac{a_1}{a_{n-1}}$ and hence $|a_1| = |a_{n-1}|$. If we take $a_1 = a_j$ and $a_{n-1} = a_l$ then the proof follows.

**Sufficiency:** For the points 0 and $a_k$ $(1 \leq k \leq n-1)$ in $\mathbb{D}$, we have

$$(B \circ M)(0) = B(0) \text{ and } (B \circ M)(a_k) = B(a_k),$$

by the equation (3.1). Then, by Theorem 2.3 we obtain

$$B \circ M \equiv B.$$

$\square$

Notice that if all $a_k = 0$, then we know that $M(z) = e^{\frac{2\pi i}{n}}$ (see [8] on page 202). From [8], we know the following corollary for the Blaschke products of degree 3.

**Corollary 3.1.** (*See [8], page* 205) *Let $G$ be the cyclic group which is composed of the transformations $M$ such that $B \circ M = B$. Then we have the following assertions*:

(*i*) *If $B(z) = z^3$, $G$ is generated by $M(z) = e^{2i\pi/3}z$, $z \in \overline{\mathbb{D}}$.*

(*ii*) *If $Z(B)$ contains a non-zero point in $\mathbb{D}$, $B(z) = z\frac{z-a_1}{1-\overline{a_1}z}\frac{z+\overline{c}a_1}{1+c\overline{a_1}z}$ where $\alpha \in \mathbb{D}\backslash\{0\}$ and where $c + \overline{c} = -1 - |a_1|^2$. In this case the group $G$ is generated by $M(z) = c\frac{z+\overline{c}a_1}{1+c\overline{a_1}z}$.*

However, as an application of Theorem 3.1, we give the following corollary in our form for degree 3.

**Corollary 3.2.** *Let $M(z) = c\frac{z-a_1}{1-\overline{a_1}z}$ be a Möbius transformation different from the identity from the unit disc onto itself and $B(z) = z\frac{z-a}{1-\overline{a}z}\frac{z-b}{1-\overline{b}z}$ be a Blaschke product of degree 3. Then $B \circ M = B$ if and only if $M(z) = c\frac{z-b}{1-\overline{b}z}$ with $|a| = |b|$ and some $c$ where $c$ is a root of the equation $c^2 + c(1 + |a|^2) + 1 = 0$ with $|c| = 1$.*

As an other application of Theorem 3.1, a similar corollary can be given for the Blaschke products of prime degrees. We give the following corollaries and examples for degree 5 and 7.

**Corollary 3.3.** *Let $M(z) = c\frac{z-\alpha}{1-\overline{\alpha}z}$ be a Möbius transformation different from the identity from the unit disc onto itself and $B(z) = z\prod\limits_{k=1}^{4}\frac{z-a_k}{1-\overline{a_k}z}$ be a Blaschke product of degree 5. Then $B \circ M = B$ if and only if $M(z) = c\frac{z-a_l}{1-\overline{a_l}z}$ with $|a_j| = |a_l|$ for some $(0 < j, l \leq 4)$, $c = -\frac{a_j}{a_l}$ and the equation*

$$4c^2a_l|a_l|^2 + 3ca_l|a_l|^2 + 3c^3a_l|a_l|^2 + c^4a_l + c^3a_l + c^2a_l + ca_l + a_l + c^2a_l|a_l|^4 = 0, \quad (3.4)$$

*is satisfied by the non-zero zeros of $B$.*

**Example 3.1.** Let $B(z) = z\prod\limits_{k=1}^{4}\frac{z-a_k}{1-\overline{a_k}z}$ be a Blaschke product of degree 5. From Proposition 2.1, we know that $Z(B) = \{0, M(0), ..., M^4(0)\}$, so we can take

$$\begin{aligned} a_1 &= M(0), \\ a_2 &= M^2(0), \\ a_3 &= M^3(0), \\ a_4 &= M^4(0). \end{aligned} \quad (3.5)$$

Let $a_l = \frac{1}{2}$. Using the equation (3.4), we obtain $c = -0.856763 - i0.515711$ and

$$M(z) = \frac{(0.856763 + i0.515711)(1 - 2z)}{2 - z}.$$

Using the equation (3.5), we find

$$\begin{aligned} a_1 &= 0.428381 + i0.257855, \\ a_2 &= 0.278236 - i0.188486, \\ a_3 &= 0.141178 + 0.304977i, \\ a_4 &= 0.5. \end{aligned}$$

Then we find $(B \circ M)(z) = B(z)$ for the points $z \in \mathbb{D}$.

**Corollary 3.4.** *Let $M(z) = c\frac{z-\alpha}{1-\overline{\alpha}z}$ be a Möbius transformation different from the identity from the unit disc into itself and $B(z) = z\prod_{k=1}^{6}\frac{z-a_k}{1-\overline{a_k}z}$ be a Blaschke product of degree 7. Then $B \circ M = B$ if and only if $M(z) = c\frac{z-a_l}{1-\overline{a_l}z}$ with $|a_j| = |a_l|$ for some $(0 < j, l \leq 6)$, $c = -\frac{a_j}{a_l}$ and the equation*

$$\begin{aligned} & a_l + 5ca_l|a_l|^2 + 8c^2a_l|a_l|^2 + 9c^3a_l|a_l|^2 + 8c^4a_l|a_l|^2 \\ & +5c^5a_l|a_l|^2 + 6c^2a_l|a_l|^4 + 9c^3a_l|a_l|^4 + 6c^4a_l|a_l|^4 \\ & +c^3a_l|a_l|^6 + ca_l + c^2a_l + c^3a_l + c^4a_l + c^5a_l + c^6a_l = 0 \end{aligned} \tag{3.6}$$

*is satisfied by the non-zero zeros of $B$.*

**Example 3.2.** Let $B(z) = z\prod_{k=1}^{6}\frac{z-a_k}{1-\overline{a_k}z}$ be a Blaschke product of degree 7. From Proposition 2.1, we know that $Z(B) = \{0, M(0), ..., M^6(0)\}$, so we can take

$$\begin{aligned} a_1 &= M(0), \\ a_2 &= M^2(0), \\ a_3 &= M^3(0), \\ a_4 &= M^4(0), \\ a_5 &= M^5(0), \\ a_6 &= M^6(0). \end{aligned} \tag{3.7}$$

Let $a_l = \frac{1}{2}$. Using the equation (3.6), we obtain $c = 0.217617 - i0.976034$ and

$$M(z) = \frac{-(0.217617 - i0.976034)(2z - 1)}{2 - z}.$$

Using the equation (3.7), we find

$$\begin{aligned} a_1 &= -0.108809 + i0.488017, \\ a_2 &= 163605 + i0.702141, \\ a_3 &= 0.40682 + i0.679542, \\ a_4 &= 0.574725 + i0.54495, \\ a_5 &= 0.64971 + i0.312482, \\ a_6 &= 0.5. \end{aligned}$$

Then we obtain $(B \circ M)(z) = B(z)$ for the points $z \in \mathbb{D}$.

Now we consider the canonical Blaschke products of degree 4. At first, from [8], we can give the following corollary for a Blaschke product $B$ of degree 4 .

**Corollary 3.5.** (*See [8], page* 204) *Let $G$ be a cyclic group which is composed of the transformations $M$ such that $B \circ M = B$. Then we have the following assertions*:

(*i*) *If $B(z) = z^4$, $G$ is generated by $M(z) = iz$, $z \in \overline{\mathbb{D}}$.*

(*ii*) *If $Z(B)$ contains a non-zero point in $\mathbb{D}$, there are two cases*:

(*a*)

$$B(z) = z\frac{z - a_1}{1 - \overline{a_1}z}\frac{z - a_2}{1 - \overline{a_2}z}\frac{z - \frac{a_1 - a_2}{1 - \overline{a_1}a_2}}{1 - z\left(\frac{\overline{a_1} - \overline{a_2}}{1 - a_1\overline{a_2}}\right)},$$

*where $a_1$ or $a_2$ is non-equal to* 0. *In this case, $M(z) = -\frac{z - a_1}{1 - \overline{a_1}z}$ and thus $M$ does not generate $G$ since the degree of $M$ is equal to* 2.

(*b*)

$$B(z) = z\frac{z - a_1}{1 - \overline{a_1}z}\frac{z - M(a_1)}{1 - \overline{M(a_1)}z}\frac{z - M^2(a_1)}{1 - \overline{M^2(a_1)}z},$$

*with $a_1 \in \mathbb{D}$. In this case $M(z) = c\frac{z + \overline{c}a_1}{1 + c\overline{a_1}z}$ generates $G$ with $|c| = 1$ and $c + \overline{c} = -2|a_1|^2$.*

In this case there is a nice relation between decomposition of the finite Blaschke products $B$ of order 4 and the Poncelet curves associated with them. From [6] and [13], we know the following theorem.

**Theorem 3.2.** *For any Blaschke product $B$ of order* 4, *$B$ is a composition of two Blaschke products of degree* 2, *that is $B(z) = (f \circ g)(z)$ where $f(z) = z\frac{z - \alpha}{1 - \overline{\alpha}z}$, $g(z) = z\frac{z - \beta}{1 - \overline{\beta}z}$, $\alpha = -a_1a_2$ and $\beta = \frac{a_1 + a_2 - a_1a_2(\overline{a_1} + \overline{a_2})}{1 - |a_1a_2|^2}$, if and only if the Poncelet curve $E$ of this Blaschke product is an ellipse with the equation*

$$E : |z - a_1| + |z - a_2| = |1 - \overline{a_1}a_2|\sqrt{\frac{|a_1|^2 + |a_2|^2 - 2}{|a_1|^2|a_2|^2 - 1}}.$$

It is also known that the decomposition of some Blaschke products $B$ of degree 4 is linked with the case that Poncelet curve of this Blaschke product is an ellipse with a nice geometric property.

**Theorem 3.3.** (*See [13], Theorem* 5.2 *on page* 103) *Let* $a_1, a_2$ *and* $a_3$ *be three distinct nonzero complex numbers with* $|a_i| < 1$ *for* $1 \leq i \leq 3$ *and* $B(z) = z\prod_{i=1}^{3} \frac{z-a_i}{1-\overline{a_i}z}$ *be a Blaschke product of degree* 4 *with the condition that one of its zeros, say* $a_1$,*satisfies the following equation:*

$$a_1 + \overline{a_1}a_2a_3 = a_2 + a_3.$$

*Then the Poncelet curve associated with* $B$ *is the ellipse* $E$ *with the equation*

$$E : |z - a_2| + |z - a_3| = |1 - \overline{a_2}a_3| \sqrt{\frac{|a_2|^2 + |a_3|^2 - 2}{|a_2|^2 |a_3|^2 - 1}}.$$

Let $B(z)$ be given as in the statement of Theorem 3.3. For any $\lambda \in \partial\mathbb{D}$, let $z_1, z_2, z_3$ and $z_4$ be the four distinct points satisfying $B(z_1) = B(z_2) = B(z_3) = B(z_4) = \lambda$. Then the Poncelet curve associated with $B$ is an ellipse $E$ with foci $a_2$ and $a_3$ and the lines joining $z_1, z_3$ and $z_2, z_4$ pass through the point $a_1$.

**Example 3.3.** Let $a_1 = \frac{2}{3}, a_2 = \frac{1}{2} - i\frac{1}{2}, a_3 = \frac{1}{2} + i\frac{1}{2}$ and $B(z) = z\prod_{i=1}^{3} \frac{z-a_i}{1-\overline{a_i}z}$. The Poncelet curve associated with $B$ is an ellipse with foci $a_2$ and $a_3$ (see Figure 1).

However decomposition of a Blaschke product $B$ is not always linked with the Poncelet curve of the Blaschke product, as we will see in the following theorem.

**Theorem 3.4.** (*See [11], Theorem* 4.2 *in page* 69) *Let* $a_1, a_2, ..., a_{2n-1}$ *be* $2n - 1$ *distinct nonzero complex numbers with* $|a_k| < 1$ *for* $1 \leq k \leq 2n - 1$ *and* $B(z) = z\prod_{k=1}^{2n-1} \frac{z-a_k}{1-\overline{a_k}z}$ *be a Blaschke product of degree* $2n$ *with the condition that one of its zeros, say* $a_1$, *satisfies the following equations*:

$$\begin{aligned} a_1 + \overline{a_1}a_2a_3 &= a_2 + a_3. \\ a_1 + \overline{a_1}a_4a_5 &= a_4 + a_5. \\ &\cdots \\ a_1 + \overline{a_1}a_{2n-2}a_{2n-1} &= a_{2n-2} + a_{2n-1}. \end{aligned}$$

(*i*) *If* $L$ *is any line through the point* $a_1$, *then for the points* $z_1$ *and* $z_2$ *at which* $L$ *intersects* $\partial\mathbb{D}$, *we have* $B(z_1) = B(z_2)$.

(*ii*) *The unit circle* $\partial\mathbb{D}$ *and any circle through the points* 0 *and* $\frac{1}{\overline{a_1}}$ *have exactly two distinct intersection points* $z_1$ *and* $z_2$. *Then we have* $B(z_1) = B(z_2)$ *for these intersection points.*

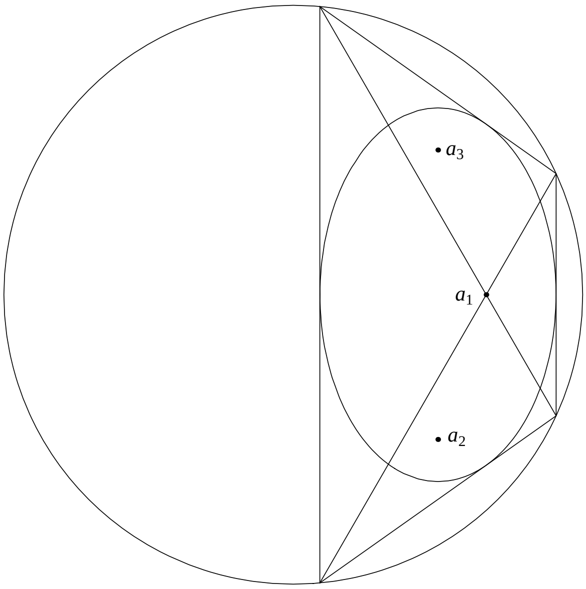


Figure 1:

From the proof of Theorem 3.4, we know that Blaschke product $B$ can be written as $B(z) = B_2 \circ B_1(z)$ where

$$B_1(z) = z\frac{z - a_1}{1 - \overline{a}_1 z} \text{ and } B_2(z) = z\frac{(z + a_2 a_3)(z + a_4 a_5)...(z + a_{2n-2}a_{2n-1})}{(1 + \overline{a_2 a_3}z)(1 + \overline{a_4 a_5}z)...(1 + \overline{a_{2n-2}a_{2n-1}}z)}.$$

Now we investigate under what conditions $B \circ M = B$ such that $M$ is different from the identity for the Blaschke product given in Theorem 3.4. For $n = 2$, we can give the following theorem.

**Theorem 3.5.** *Let $a_1, a_2$ and $a_3$ be three distinct nonzero complex numbers with $|a_k| < 1$ for $1 \leq k \leq 3$ and $B(z) = z\prod_{k=1}^{3} \frac{z-a_k}{1-\overline{a}_k z}$ be a Blaschke product of degree 4 with the condition that one of its zeros, say $a_1$, satisfies the following equation*:

$$a_1 + \overline{a_1}a_2 a_3 = a_2 + a_3.$$

*(i) If $M(z) = -\frac{z-a_1}{1-\overline{a}_1 z}$ then we have $B = B_2 \circ B_1$ and $B = B \circ M$ only when $a_3 = \frac{a_1 - a_2}{1 - \overline{a}_1 a_2}$.*

*(ii) Let $M(z) = c\frac{z+\overline{c}a_1}{1+c\overline{a}_1 z}$ with the conditions $c + \overline{c} = -2|a_1|^2$, $a_2 = M(a_1)$ and $a_3 = M^2(a_1)$. If $a_1$ and $c$ with $|c| = 1$ satisfy the following equation*

$$a_1 c + 2a_1 c^2 |a_1|^2 + a_1 c^3 |a_1|^2 + a_1 - a_1 |a_1|^2 = 0, \tag{3.8}$$

*then we have $B = B_2 \circ B_1$ and $B = B \circ M$.*

**Proof.** We use the equation $a_1 + \overline{a_1} a_2 a_3 = a_2 + a_3$, Theorem 3.4 and Corollary 3.5.

$(i)$ Let $M(z) = -\frac{z-a_1}{1-\overline{a}_1 z}$. Then by Corollary 3.5, the condition $B = B \circ M$ implies $a_3 = \frac{a_1 - a_2}{1-\overline{a}_1 a_2}$.

$(ii)$ Let $M(z) = c\frac{z+\overline{c}a_1}{1+c\overline{a}_1 z}$. From Corollary 3.5, it should be $a_2 = M(a_1)$, $a_3 = M^2(a_1)$ and $c + \overline{c} = -2\left|a_1\right|^2$. Then we obtain

$$a_2 = \frac{a_1(1+c)}{1+c\left|a_1\right|^2} \text{ and } a_3 = \frac{a_1(1-\left|a_1\right|^2)}{1+2c\left|a_1\right|^2 + c^2\left|a_1\right|^2}.$$

If we substitute these values of $a_2$ and $a_3$ in the equation $a_1 + \overline{a_1} a_2 a_3 = a_2 + a_3$, we have the following equation

$$a_1 c + 2a_1 c^2 \left|a_1\right|^2 + a_1 c^3 \left|a_1\right|^2 + a_1 - a_1\left|a_1\right|^2 = 0.$$

Also in both cases we know that $B$ has a decomposition as $B = B_2 \circ B_1$ by Theorem 3.3. Thus the proof is completed. $\square$

Now, we give two examples for the both cases of the above theorem.

**Example 3.4.** Let $B$ be a Blaschke product and $M$ be a Möbius transformation of the following forms:

$$B(z) = z\frac{z-a_1}{1-\overline{a}_1 z}\frac{z-a_2}{1-\overline{a}_2 z}\frac{z-\left(\frac{a_1-a_2}{1-\overline{a}_1 a_2}\right)}{1-z\left(\frac{\overline{a}_1-\overline{a}_2}{1-a_1\overline{a}_2}\right)} \text{ and } M(z) = \frac{-z+a_1}{1-\overline{a}_1 z}.$$

For $a_1 = \frac{1}{2}$ and $a_2 = \frac{1}{2} - \frac{i}{2}$ we obtain

$$B(z) = \frac{z(z-\frac{1}{2})(z-\frac{1}{2}+\frac{i}{2})(z-\frac{1}{5}-\frac{3i}{5})}{(-\frac{1}{2}z+1)(1-z(\frac{1}{2}+\frac{i}{2}))(1-z(\frac{1}{5}-\frac{3i}{5}))} \text{ and } M(z) = \frac{-z+\frac{1}{2}}{1-\frac{1}{2}z}.$$

Then we find $(B \circ M)(z) = B(z)$ and $B(z) = (B_2 \circ B_1)(z)$ for the points $z \in \mathbb{D}$.

**Example 3.5.** Let $B$ be a Blaschke product and $M$ be a Möbius transformation of the following forms:

$$B(z) = z\frac{z-a_1}{1-\overline{a_1}z}\frac{z-M(a_1)}{1-\overline{M(a_1)}z}\frac{z-M^2(a_1)}{1-\overline{M^2(a_1)}z} \text{ and } M(z) = c\frac{z+\overline{c}a_1}{1+c\overline{a_1}z}.$$

For $a_1 = \frac{2}{3}$, solving the equation $a_1 c + 2a_1 c^2\left|a_1\right|^2 + a_1 c^3 \left|a_1\right|^2 + a_1 - a_1\left|a_1\right|^2 = 0$ we find $c = -1$. Then we have $B$ and $M$ of the following forms:

$$B(z) = \frac{z^2(-\frac{2}{3}+z)^2}{(1-\frac{2}{3}z)^2} \text{ and } M(z) = -\frac{-\frac{2}{3}+z}{1-\frac{2}{3}z}$$

Then we find $(B \circ M)(z) = B(z)$ and $B(z) = (B_2 \circ B_1)(z)$ for the points $z \in \mathbb{D}$.

From the above discussions, we can say that decomposition of a finite Blaschke product $B$ is linked with its zeros. But for a finite Blaschke product $B$ of degree 4, this case is also linked with the Poncelet curve of $B$.

Using Theorem 3.1 and Theorem 3.4, for the Blaschke product of degree $2n$, we give the following result.

**Corollary 3.6.** *Let $a_1, a_2, ..., a_{2n-1}$ be $2n-1$ distinct nonzero complex numbers with $|a_k| < 1$ for $1 \leq k \leq 2n-1$ and $B(z) = z \prod_{k=1}^{2n-1} \frac{z-a_k}{1-\overline{a_k}z}$ be a Blaschke products of degree $2n$ with the condition that one of its zeros, say $a_1$, satisfies the following equations:*

$$\begin{aligned} a_1 + \overline{a_1}a_2a_3 &= a_2 + a_3 \\ a_1 + \overline{a_1}a_4a_5 &= a_4 + a_5 \\ &\cdots \\ a_1 + \overline{a_1}a_{2n-2}a_{2n-1} &= a_{2n-2} + a_{2n-1}. \end{aligned}$$

*Let $M(z) = c\frac{z-a_{2n-1}}{1-z\overline{a_{2n-1}}}$ with the conditions $|c| = 1$, $M^{2n-1}(0) - a_{2n-1} = 0$, $a_1 = M(0), a_2 = M^2(0), ..., a_{2n-1} = M^{2n-1}(0)$. If $a_{2n-1}$ and $c$ satisfy following equations:*

$$\begin{aligned} M(0) + \overline{M(0)}M^2(0)M^3(0) &= M^2(0) + M^3(0) \\ M(0) + \overline{M(0)}M^4(0)M^5(0) &= M^4(0) + M^5(0) \\ &\cdots \\ M(0) + \overline{M(0)}M^{2n-2}(0)M^{2n-1}(0) &= M^{2n-2}(0) + M^{2n-1}(0) \end{aligned}$$

*Then we have $B = B_2 \circ B_1$ and $B \circ M = B$.*

**Proof.** The proof is obvious from Theorem 3.1 and Theorem 3.4. □

**Example 3.6.** Let $B$ be a Blaschke product and $M$ be a Möbius transformation of the following forms:

$$B(z) = z\frac{z-M(0)}{1-\overline{M(0)}z}\frac{z-M^2(0)}{1-\overline{M^2(0)}z}\frac{z-M^3(0)}{1-\overline{M^3(0)}z}\frac{z-M^4(0)}{1-\overline{M^4(0)}z}\ \frac{z-M^5(0)}{1-\overline{M^5(0)}z}$$

and

$$M(z) = c\frac{z-a_5}{1-\overline{a_5}z}.$$

From Corollary 3.6, it should be

$$M(0) + \overline{M(0)}M^2(0)M^3(0) = M^2(0) + M^3(0) \tag{3.9}$$

and

$$M(0) + \overline{M(0)}M^4(0)M^5(0) = M^4(0) + M^5(0). \tag{3.10}$$

We know that $a_1 = M(0) = -ca_5, a_2 = M^2(0) = -ca_5\frac{(1+c)}{1+c|a_5|^2}, a_3 = M^3(0) = -ca_5\frac{\left(1+c+c^2+c|a_5|^2\right)}{1+2c|a_5|^2+c^2|a_5|^2}$ and $a_4 = M^4(0) = -ca_5\frac{(1+c)\left(1+c^2+2c|a_5|^2\right)}{1+c|a_5|^2\left(3+2c+c^2+c|a_5|^2\right)}$. Writing these values in the equations (3.9) and (3.10), we have

$$\begin{aligned} & ca_5 + 2c^2a_5 + c^3a_5 + c^3a_5\left|a_5\right|^2 + c^4a_5\left|a_5\right|^2 - 2c^3a_5\left|a_5\right|^4 \\ & -c^4a_5\left|a_5\right|^4 - ca_5\left|a_5\right|^2 - c^2a_5\left|a_5\right|^2 - c^2a_5\left|a_5\right|^4 = 0 \end{aligned} \tag{3.11}$$

and

$$\begin{aligned} & a_5 - c^2a_5 - c^3a_5 - c^4a_5 + 3ca_5\left|a_5\right|^2 + c^2a_5\left|a_5\right|^4 + 2c^2a_5\left|a_5\right|^2 + c^4a_5\left|a_5\right|^2 \\ & +c^3a_5\left|a_5\right|^4 - ca_5\left|a_5\right|^2 - 2c^2a_5\left|a_5\right|^4 - a_5\left|a_5\right|^2 - 2ca_5\left|a_5\right|^4 = 0. \end{aligned} \tag{3.12}$$

For $a_5 = \frac{1}{2}$, solving the equations (3.11) and (3.12) we find $c = -1$. Then we have $B$ and $M$ of the following forms:

$$B(z) = z\frac{(2z-1)(z-0.5)^2\left(z-1.4803\times10^{-16}\right)\left(z-7.40149\times10^{-17}\right)}{(2-z)(1-0.5z)^2\left(1-1.4803\times10^{-16}z\right)\left(1-7.40149\times10^{-17}z\right)}$$

and

$$M(z) = \frac{2z-1}{z-2}.$$

Then for the points $z \in \mathbb{D}$, we find

$$(B \circ M)(z) = B(z) \text{ and } B(z) = (B_2 \circ B_1)(z)$$

where

$$B_1(z) = z\frac{z-0.5}{1-0.5z} \text{ and } B_2(z) = z\frac{z+3.70074\times10^{-17}}{1+3.70074\times10^{-17}z}\frac{z+7.40149\times10^{-17}}{1+7.40149\times10^{-17}z}.$$

**Corollary 3.7.** *Let $a_1, a_2, ..., a_{3n-1}$ be $3n-1$ distinct nonzero complex numbers with $|a_k| < 1$ for $1 \le k \le 3n-1$ and $B(z) = z\prod_{k=1}^{3n-1}\frac{z-a_k}{1-\overline{a_k}z}$ be a Blaschke products of degree $3n$ with the condition that one of its zeros, say $a_1$, satisfies the following equations:*

$$\begin{aligned} a_1 + a_2 + a_3a_4a_5\overline{a_1a_2} &= a_3 + a_4 + a_5, \\ a_1a_2 + a_3a_4a_5\left(\overline{a_1} + \overline{a_2}\right) &= a_3a_4 + a_3a_5 + a_4a_5, \\ &\cdots \\ a_1 + a_2 + a_{3n-3}a_{3n-2}a_{3n-1}\overline{a_1a_2} &= a_{3n-3} + a_{3n-2} + a_{3n-1}, \\ a_1a_2 + a_{3n-3}a_{3n-2}a_{3n-1}\left(\overline{a_1} + \overline{a_2}\right) &= a_{3n-3}a_{3n-2} + a_{3n-3}a_{3n-1} + a_{3n-2}a_{3n-1}. \end{aligned}$$

*Let $M(z) = c\frac{z-a_{3n-1}}{1-z\overline{a_{3n-1}}}$ with the conditions $|c| = 1$, $M^{3n-1}(0) - a_{3n-1} = 0$, $a_1 = M(0), a_2 = M^2(0), ..., a_{3n-1} = M^{3n-1}(0)$. If $a_{3n-1}$ and $c$ satisfy following* *equations:*

$$
\begin{aligned}
& M(0)+M^2(0)+M^3(0)\,M^4(0)\,M^5(0)\,\overline{M(0)\,M^2(0)} \\
& = M^3(0)+M^4(0)+M^5(0), \\
& M(0)\,M^2(0)+M^3(0)\,M^4(0)\,M^5(0)\left(\overline{M(0)}+\overline{M^2(0)}\right) \\
& = M^3(0)\,M^4(0)+M^3(0)\,M^5(0)+M^4(0)\,M^5(0), \\
& \ldots \\
& M(0)+M^2(0)+M^{3n-3}(0)\,M^{3n-2}(0)\,M^{3n-1}(0)\,\overline{M(0)\,M^2(0)} \\
& = M^{3n-3}(0)+M^{3n-2}(0)+M^{3n-1}(0), \\
& M(0)\,M^2(0)+M^{3n-3}(0)\,M^{3n-2}(0)\,M^{3n-1}(0)\left(\overline{M(0)}+\overline{M^2(0)}\right) \\
& = M^{3n-3}(0)\,M^{3n-2}(0)+M^{3n-3}(0)\,M^{3n-1}(0)+M^{3n-2}(0)\,M^{3n-1}(0).
\end{aligned}
$$

*Then we have $B = B_2 \circ B_1$ and $B \circ M = B$.*

**Proof.** Let $a_1, a_2, ..., a_{3n-1}$ be $3n-1$ distinct nonzero complex numbers with $|a_k| < 1$ for $1 \le k \le 3n-1$ and $B(z) = z \prod\limits_{k=1}^{3n-1} \frac{z-a_k}{1-\overline{a_k}z}$ be a Blaschke product of degree $3n$ with the condition that two of its zeros, say $a_1$ and $a_2$, satisfies the following equations:

$$
\begin{aligned}
a_1+a_2+a_3a_4a_5\overline{a_1a_2} &= a_3+a_4+a_5, \\
a_1a_2+a_3a_4a_5\left(\overline{a_1}+\overline{a_2}\right) &= a_3a_4+a_3a_5+a_4a_5, \\
a_1+a_2+a_6a_7a_8\overline{a_1a_2} &= a_6+a_7+a_8, \\
a_1a_2+a_6a_7a_8\left(\overline{a_1}+\overline{a_2}\right) &= a_6a_7+a_6a_8+a_7a_8, \\
& \ldots \\
a_1+a_2+a_{3n-3}a_{3n-2}a_{3n-1}\overline{a_1a_2} &= a_{3n-3}+a_{3n-2}+a_{3n-1}, \\
a_1a_2+a_{3n-3}a_{3n-2}a_{3n-1}\left(\overline{a_1}+\overline{a_2}\right) &= a_{3n-3}a_{3n-2}+a_{3n-3}a_{3n-1}+a_{3n-2}a_{3n-1}.
\end{aligned}
$$

By Theorem 4.4 on page 71 in [11], we know that B$(z)$ can be written as a composition of two Blaschke products of degree 3 and $n$ as $B(z) = (B_2 \circ B_1)(z)$ where

$$B_1(z) = \frac{z(z-a_1)(z-a_2)}{(1-\overline{a_1}z)(1-\overline{a_2}z)}$$

and

$$B_2(z) = \frac{z(z-a_3a_4a_5)(z-a_6a_7a_8)\ldots(z-a_{3n-3}a_{3n-2}a_{3n-1})}{(1-\overline{a_3a_4a_5}z)(1-\overline{a_6a_7a_8}z)\ldots(1-\overline{a_{3n-3}a_{3n-2}a_{3n-1}}z)}$$

Then, the rest of the proof is clear from Theorem 3.1. □

**Example 3.7.** Let $B$ be a Blaschke product and $M$ be a Möbius transformation of the following forms:

$$B(z) = z\frac{z-M(0)}{1-\overline{M(0)}z}\,\frac{z-M^2(0)}{1-\overline{M^2(0)}z}\,\frac{z-M^3(0)}{1-\overline{M^3(0)}z}\,\frac{z-M^4(0)}{1-\overline{M^4(0)}z}\,\frac{z-M^5(0)}{1-\overline{M^5(0)}z}$$

and

$$M(z) = c\frac{z - a_5}{1 - \overline{a_5}z}.$$

From Corollary 3.7, it should be

$$\begin{aligned} M(0) + M^2(0) + M^3(0)\, M^4(0)\, M^5(0)\, \overline{M(0)\, M^2(0)} \\ = M^3(0) + M^4(0) + M^5(0) \end{aligned} \tag{3.13}$$

and

$$\begin{aligned} M(0)\, M^2(0) + M^3(0)\, M^4(0)\, M^5(0) \left(\overline{M(0)} + \overline{M^2(0)}\right) \\ = M^3(0)\, M^4(0) + M^3(0)\, M^5(0) + M^4(0)\, M^5(0)\,. \end{aligned} \tag{3.14}$$

We know that $a_1 = M(0) = -ca_5, a_2 = M^2(0) = -ca_5 \frac{(1+c)}{1+c|a_5|^2}, a_3 = M^3(0) = -ca_5 \frac{\left(1+c+c^2+c|a_5|^2\right)}{1+2c|a_5|^2+c^2|a_5|^2}$ and $a_4 = M^4(0) = -ca_5 \frac{(1+c)\left(1+c^2+2c|a_5|^2\right)}{1+c|a_5|^2\left(3+2c+c^2+c|a_5|^2\right)}$. Writing these values in the equations (3.13) and (3.14), we have

$$\begin{aligned}
&-ca_5 \left(1 + c|a_5|^2\right) \left(1 + 2c|a_5|^2 + c^2|a_5|^2\right) \left(1 + c|a_5|^2 \left(3 + 2c + c^2 + c|a_5|^2\right)\right) \left(1 + \overline{c}|a_5|^2\right) \\
&-ca_5 (1 + c) \left(1 + 2c|a_5|^2 + c^2|a_5|^2\right) \left(1 + c|a_5|^2 \left(3 + 2c + c^2 + c|a_5|^2\right)\right) \left(1 + \overline{c}|a_5|^2\right) \\
&+a_5 |a_5|^4 \left(1 + c + c^2 + c|a_5|^2\right) (1 + c) \left(1 + c^2 + 2c|a_5|^2\right) (1 + \overline{c}) \left(1 + c|a_5|^2\right) \\
&+ca_5 \left(1 + c + c^2 + c|a_5|^2\right) \left(1 + c|a_5|^2\right) \left(1 + c|a_5|^2 \left(3 + 2c + c^2 + c|a_5|^2\right)\right) \left(1 + \overline{c}|a_5|^2\right) \\
&+ca_5 (1 + c) \left(1 + c^2 + 2c|a_5|^2\right) (1 + c|a_5|)^2 \left(1 + 2c|a_5|^2 + c^2|a_5|^2\right) (1 + \overline{c}|a_5|)^2 \\
&-a_5 \left(1 + c|a_5|^2\right) \left(1 + 2c|a_5|^2 + c^2|a_5|^2\right) \left(1 + c|a_5|^2 \left(3 + 2c + c^2 + c|a_5|^2\right)\right) \left(1 + \overline{c}|a_5|^2\right) = 0.
\end{aligned} \tag{3.15}$$

and

$$\begin{aligned}
&c^2 a_5^2 (1 + c) \left(1 + 2c|a_5|^2 + c^2|a_5|^2\right) \left(1 + c|a_5|^2 \left(3 + 2c + c^2 + c|a_5|^2\right)\right) \left(1 + \overline{c}|a_5|^2\right) \left(1 + \overline{c}|a_5|^2\right) \\
&+c^2 a_5^3 \left(1 + c + c^2 + c|a_5|^2\right) (1 + c) \left(1 + c^2 + 2c|a_5|^2\right) \left(-\overline{ca_5} \left(1 + \overline{c}|a_5|^2\right) - \overline{ca_5} (1 + \overline{c})\right) \\
&\cdot \left(1 + c|a_5|^2\right) - c^2 a_5^2 \left(1 + c + c^2 + c|a_5|^2\right) (1 + c) \left(1 + c^2 + 2c|a_5|^2\right) \left(1 + \overline{c}|a_5|^2\right) \left(1 + c|a_5|^2\right) \\
&+ca_5^2 \left(1 + c + c^2 + c|a_5|^2\right) \left(1 + c|a_5|^2\right) \left(1 + c|a_5|^2 \left(3 + 2c + c^2 + c|a_5|^2\right)\right) \left(1 + \overline{c}|a_5|^2\right) \\
&+ca_5^2 (1 + c) \left(1 + c^2 + 2c|a_5|^2\right) \left(1 + c|a_5|^2\right) \left(1 + 2c|a_5|^2 + c^2|a_5|^2\right) \left(1 + \overline{c}|a_5|^2\right) = 0.
\end{aligned} \tag{3.16}$$

For $a_5 = \frac{1}{2}$, solving the equations (3.15) and (3.16) we find $c = -0.625 + i0.780625$ Then we have $B$ and $M$ of the following forms:

$$B(z) = z \frac{(2z - 1) \left(z - 0.5 + 6.39697 \times 10^{-11} i\right) (z - 0.3125 + 0.390312i)^2 \left(z - 2.51094 \times 10^{-11} - 1.05107 \times 10^{-10} i\right)}{(2 - z) \left(1 - z \left(0.5 + 6.39697 \times 10^{-11} i\right)\right) (1 - z (0.3125 + 0.390312i))^2 \left(1 - z \left(2.51094 \times 10^{-11} - 1.05107 \times 10^{-10} i\right)\right)} \tag{3.17}$$

and

$$M(z) = (0.625 - 0.780625i) \frac{1 - 2z}{2 - z}.$$

Then for the points $z \in \mathbb{D}$, we find

$$(B \circ M)(z) = B(z) \text{ and } B(z) = (B_2 \circ B_1)(z)$$

where

$$B_1(z) = z\frac{\left(z - 0.5 + 6.39697 \times 10^{-11}i\right)(z - 0.3125 + 0.390312i)}{(1 - z\,(0.5 + 6.39697 \times 10^{-11}i))\,(1 - z\,(0.3125 + 0.390312i))}$$

and

$$B_2(z) = z\frac{z - 2.44357 \times 10^{-11} - 1.15227 \times 10^{-11}i}{1 - z\,(2.44357 \times 10^{-11} - 1.15227 \times 10^{-11}i)}.$$

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**Sümeyra Uçar**
email: `sumeyraucar@balikesir.edu.tr`
ORCID: 0000-0002-6628-526X
Department of Mathematics
Balıkesir University
10145 Balıkesir
TURKEY

**Nihal Yılmaz Özgür**
email: `nihal@balikesir.edu.tr`
ORCID: 0000-0002-8152-1830
Department of Mathematics
Balıkesir University
10145 Balıkesir
TURKEY

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