Majorization problems for classes of analytic functions

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ABSTRACT: The main object of the present paper is to investigate problems of majorization for certain classes of analytic functions of complex order defined by an operator related to the modified Bessel functions of first kind. These results are obtained by investigating appropriate class of admissible functions. Various known or new special cases of our results are-

1 Introduction

Let $\mathcal{A}$ be the class of functions of the form

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n, \tag{1.1}$$

which are analytic in the open unit disk $\mathbb{U} = \{z \in \mathbb{C} : |z| < 1\}$.

For given $g(z) = z + \sum_{n=2}^{\infty} b_n z^n \in \mathcal{A}$ the Hadamard product of f and g is denoted by

$$(f * g)(z) = z + \sum_{n=2}^{\infty} a_n b_n z^n, \quad z \in \mathbb{U}. \tag{1.2}$$

Note that $f * g \in \mathcal{A}$ which are analytic in the open disc $\mathbb{U}$.

We say that $f \in \mathcal{A}$ is subordinate to $g \in \mathcal{A}$ denoted by $f \prec g$ if there exists a Schwarz function ω which is analytic in $\mathbb{U}$ with $\omega(0) = 0$ and $|\omega(z)| < 1$ for all $z \in \mathbb{U}$, such that $f(z) = g(\omega(z))$ for $z \in \mathbb{U}$.

Note that, if the function g is univalent in $\mathbb{U}$, due to Miller and Mocanu [9], we have

$$f(z) \prec g(z) \Longleftrightarrow [f(0) = g(0) \text{ and } f(\mathbb{U}) \subset g(\mathbb{U})].$$

If f and g are analytic functions in $\mathbb{U}$, following MacGregor [8], we say that f is majorized by g in $\mathbb{U}$ that is $f(z) \ll g(z)$ if there exists a function ϕ, analytic in $\mathbb{U}$, such that

$$|\phi(z)| < 1 \text{ and } f(z) = \phi(z)g(z), \ z \in \mathbb{U}.$$

It is of interest to note that the notation of majorization is closely related to the concept of quasi-subordination between analytic functions.

Let $\mathcal{C}^*(\gamma)$ denote the class of starlike functions of complex order γ ($\gamma \in \mathbb{C} \setminus \{0\}$), satisfying the following condition

$$\frac{f(z)}{z} \neq 0 \text{ and } \Re\left(1 + \frac{1}{\gamma}\left[\frac{zf'(z)}{f(z)} - 1\right]\right) > 0, \ z \in \mathbb{U}.$$

In particular, the class

$$\mathcal{S}^*(\alpha, \lambda) := \mathcal{C}^*((1-\alpha)cos\lambda \ e^{-i\lambda}), \ |\lambda| < \frac{\pi}{2}; \ 0 \le \alpha \le 1$$

denotes the class of λ-spiral function of order α investigated by Libera [6]. Moreover, the classes

$$\widehat{\mathcal{S}}^*(\lambda) := \mathcal{S}^*(0, \lambda), \ \mathcal{S}^*(\alpha) := \mathcal{S}^*(\alpha, 0)$$

are the class of spiral functions introduced by Špaček [12] (see also [13]) and the class of starlike functions of order α, respectively. For $\alpha = 0$, we obtain the familiar class $\mathcal{S}^* := \mathcal{S}^*(0)$ of starlike functions.

We recall here a generalized Bessel function of first kind of order p denoted by $\omega_{p,b,c} =: \omega$ defined in [1] and given by

$$\omega(z) = \omega_{p,b,c}(z) = \sum_{n=0}^{\infty} \frac{(-1)^n \ c^n}{n! \ \Gamma(p + n + (b+1)/2)} \left(\frac{z}{2}\right)^{2n+p}, \quad z \in \mathbb{C} \qquad (1.3)$$

which is the particular solution of the second order linear homogeneous differential equation

$$z^2\omega''(z) + bz\omega'^2 - [p^2 + (1-b)]\omega(z) = 0, \qquad (1.4)$$

where $b, p, c \in \mathbb{C}$, which is natural generalization of Bessel's equation.

The differential equation (1.4) permits the study of Bessel function, modified Bessel function, spherical Bessel function and modified spherical Bessel functions all together. Solutions of (1.4) are referred to as the generalized Bessel function of order p. The particular solution given by (1.3) is called the generalized Bessel function of the first kind of order p. Although the series defined in (1.3) is convergent everywhere, the function $\omega_{p,b,c}$ is generally not univalent in $\mathbb{U}$.

It is of interest to note that when $b = c = 1$, we reobtain the Bessel function of the first kind $\omega_{p,1,1} = j_p$, and for $b = 1, c = -1$ the function $\omega_{p,1,-1}$ becomes the modified Bessel function I_p. Further note that $b = 2$ and $c = 1$ the function $w_{p,2,1}(z)$

reduces to $\sqrt{\frac{2}{\pi}}\ J_p(z)$ becomes the spherical Bessel function of the first kind of order p. Now, we consider the function $u_{p,b,c}(z)$ defined by the transformation

$$u_{p,b,c}(z) = 2^p \Gamma\left(p + \frac{b+1}{2}\right)\ z^{1-\frac{p}{2}}\ \omega_{p,b,c}\ (\sqrt{z}).$$

By using well known Pochhammer symbol (or the shifted factorial) defined, in terms of the familiar Gamma function, by

$$(a)_n := \frac{\Gamma(a+n)}{\Gamma(a)} = \begin{cases} 1 & (n=0), \\ \\ a(a+1)(a+2)\cdots(a+n-1) & (n=1,2,\ldots). \end{cases}$$

we can express $u_{p,b,c}(z)$ as

$$u_{p,b,c}(z) = z + \sum_{n=1}^{\infty} \frac{(-c/4)^n}{(m)_n}\frac{z^{n+1}}{n!}, \tag{1.5}$$

where $m = p + \frac{b+1}{2} \notin \mathbb{Z}_0^-$. This function is analytic on $\mathbb{C}$ and satisfies the second-order linear differential equation

$$4z^2 u''(z) + 2(2p+b+1)zu'(z) + czu(z) = 0.$$

Now, we consider the linear operator

$$\mathfrak{B}_m^c f : \mathcal{A} \to \mathcal{A}$$

defined by

$$\mathfrak{B}_m^c f(z) := u_{p,b,c}(z) * f(z) = z + \sum_{n=1}^{\infty} \frac{(-c/4)^n}{(m)_n\ (n)!} a_{n+1}\ z^{n+1}, \quad z \in \mathbb{U}, \tag{1.6}$$

where $m = p + \frac{b+1}{2} \notin \mathbb{Z}_0^-$. It is easy to verify from the definition (1.6) that

$$z(\mathfrak{B}_{m+1}^c f(z))' = m\mathfrak{B}_m^c f(z) - (m-1)\mathfrak{B}_{m+1}^c f(z). \tag{1.7}$$

We recall the special cases of $\mathcal{B}_m^c-$ operator due to Baricz et al [3].

- Setting $b = c = 1$ in (1.6) or (1.7), we obtain the operator $\mathcal{J}_p : \mathcal{A} \to \mathcal{A}$ related with Bessel function, given by

$$\mathcal{J}_p f(z) = zu_{p,1,1}(z) * f(z) = z + \sum_{n=1}^{\infty} \frac{(-1/4)^n}{(p+1)_n\ (n)!} a_{n+1}\ z^{n+1}, \quad z \in \mathbb{U} \tag{1.8}$$

and its recursive relation

$$z(\mathcal{J}_{p+1} f(z))' = (p+1)\mathcal{J}_p f(z) - p\mathcal{J}_{p+1} f(z), \quad z \in \mathbb{U}.$$

- Setting $b = 1$ and $c = -1$ in (1.6) or (1.7), we obtain the operator $\mathcal{I}_p : \mathcal{A} \to \mathcal{A}$ related with modified Bessel function, given by

$$\mathcal{I}_p f(z) = z u_{p,1,-1}(z) * f(z) = z + \sum_{n=1}^{\infty} \frac{(1/4)^n}{(p+1)_n\,(n)!} a_{n+1}\, z^{n+1}, \quad z \in \mathbb{U} \quad (1.9)$$

  and its recursive relation

$$z(\mathcal{I}_{p+1} f(z))' = (p+1)\mathcal{I}_p f(z) - p\mathcal{I}_{p+1} f(z), \quad z \in \mathbb{U}.$$

- Setting $b = 2$ and $c = 1$ in (1.6) or (1.7), we obtain the operator $\mathcal{K}_p : \mathcal{A} \to \mathcal{A}$ related with spherical Bessel function, given by

$$\mathcal{K}_p f(z) = z u_{p,2,1}(z) * f(z) = z + \sum_{n=1}^{\infty} \frac{(-1/4)^n}{(p+\frac{3}{2})_n\,(n)!} a_{n+1}\, z^{n+1}, \quad z \in \mathbb{U} \quad (1.10)$$

  and its recursive relation

$$z(\mathcal{K}_{p+1} f(z))' = (p+\frac{3}{2})\mathcal{K}_p f(z) - (p+\frac{1}{2})\mathcal{K}_{p+1} f(z), \quad z \in \mathbb{U}.$$

It is of interest to note that the function $\mathcal{B}_m^c$ given by (1.6) is an elementary transformation of the generalized hypergeometric function, i.e it is easy to see that $\mathcal{B}_m^c f(z) = z\, {}_0F_1\left(m; \frac{-c}{4}z\right) * f(z)$ and also $u_{p,b,c}(\frac{-4}{c}z) * f(z) = z\, {}_0F_1(m; z)$.

The generalized Bessel function is a recent topic of study in Geometric Function Theory (e.g. see the work of [1, 2, 3]). Using the $\mathcal{B}_m^c$ − linear operator due to Baricz et al [3] given by (1.6), we now define the following new subclass of $\mathcal{A}$.

Definition 1 *A function $f(z) \in \mathcal{A}$ is said to be in the class $\mathcal{S}_m^c(A, B; \gamma)$, if*

$$1 + \frac{1}{\gamma}\left[\frac{z(\mathcal{B}_{m+1}^c f(z))'}{\mathcal{B}_{m+1}^c f(z)} - 1\right] \prec \frac{1 + Az}{1 + Bz}, \qquad (1.11)$$

where $-1 \le B < A \le 1$; $\gamma, c, m \in \mathbb{C}$, $\gamma \ne 0$, $m \ne 0, -1, -2, \ldots$.

In particular, the class

$$\mathcal{S}_m^c(\gamma) := \mathcal{S}_m^c(1, -1; \gamma),$$

denote the class of functions $f \in \mathcal{A}$ satisfying the following condition:

$$\Re\left(1 + \frac{1}{\gamma}\left[\frac{z(\mathcal{B}_{m+1}^c f(z))'}{\mathcal{B}_{m+1}^c f(z)} - 1\right]\right) > 0, \quad z \in \mathbb{U}. \qquad (1.12)$$

Moreover, let us denote

$$\mathcal{S}_m^c(\alpha, \lambda) := \mathcal{S}_m^c((1-\alpha)cos\lambda\, e^{-i\lambda}), \quad \mathcal{S}_m^c(\alpha) := \mathcal{S}_m^c(\alpha, 0), \; |\lambda| < \frac{\pi}{2}; \; 0 \le \alpha \le 1.$$

Majorization problems for the class $\mathcal{S}^*$ had been studied by MacGregor [8]. Recently Altintas et al. [4] investigated a majorization problem for the class $\mathcal{C}^*(\gamma)$ and Goyal and Goswami [5] generalized these results for the class of analytic functions involving fractional operator. In this paper we investigated a majorization problem for the class $\mathcal{S}_m^c(A, B; \gamma)$ associated with Bessel functions and point out some special cases of our result.

2 The main results

First we show that the class $\mathcal{S}_m^c(A, B; \gamma)$ is not empty.

Theorem 1 *A function $f \in \mathcal{A}$ of the form (1.1) belongs to the class $\mathcal{S}_m^c(A, B; \gamma)$ if it satisfies the condition*

$$\sum_{n=2}^{\infty} d_n \left|a_n\right| \leq (B - A) \left|\gamma\right|, \tag{2.1}$$

where

$$d_n = \frac{(\left|c\right|/4)^{n-1} \left\{(B+1)(n-1) + (B-A)\left|\gamma\right|\right\}}{\left|(m)_{n-1}\right|\ (n-1)!}, \quad n = 2, 3 \ldots .$$

Proof. A function f of the form (1.1) belongs to the class $\mathcal{S}_m^c(A, B; \gamma)$ if and only if there exists a function $\omega,\ \left|\omega(z)\right| \leq \left|z\right|\ \ (z \in \mathbb{U})$, such that for $z \in \mathbb{U}$ we have

$$1 + \frac{1}{\gamma}\left[\frac{z(\mathfrak{B}_{m+1}^c f(z))'}{\mathfrak{B}_{m+1}^c f(z)} - 1\right] = \frac{1 + A\omega(z)}{1 + B\omega(z)}$$

or equivalently

$$z(\mathfrak{B}_{m+1}^c f(z))' - \mathfrak{B}_{m+1}^c f(z) = \omega(z)\left\{Bz(\mathfrak{B}_{m+1}^c f(z))' + [(B-A)\gamma - B]\mathfrak{B}_{m+1}^c f(z)\right\}.$$

Thus, it is sufficient to prove that for $z \in \mathbb{U}$ we have

$$\left|z(\mathfrak{B}_{m+1}^c f(z))' - \mathfrak{B}_{m+1}^c f(z)\right| - \left|Bz(\mathfrak{B}_{m+1}^c f(z))' + [(B-A)\gamma - B]\mathfrak{B}_{m+1}^c f(z)\right| \leq 0.$$

Indeed, letting $\left|z\right| = r\ \ (0 \leq r < 1)$ and $\alpha_n = \frac{(-c/4)^{n-1}}{(m)_{n-1}\ (n-1)!}$ we have

$$\left|z(\mathfrak{B}_{m+1}^c f(z))' - \mathfrak{B}_{m+1}^c f(z)\right| - \left|Bz(\mathfrak{B}_{m+1}^c f(z))' + [(B-A)\gamma - B]\mathfrak{B}_{m+1}^c f(z)\right|$$

$$= \left|\sum_{n=2}^{\infty}(n-1)\alpha_n a_n z^n\right| - \left|(B-A)\gamma z - \sum_{n=2}^{\infty}(Bn + (B-A)\gamma - B)\alpha_n a_n z^n\right|$$

$$\leq \sum_{n=2}^{\infty}(n-1)\left|\alpha_n\right|\left|a_n\right|r^{n-1} - (B-A)\left|\gamma\right| + \sum_{n=2}^{\infty}(Bn + (B-A)\left|\gamma\right| - B)\left|\alpha_n\right|\left|a_n\right|r^{n-1}$$

$$\leq \sum_{n=2}^{\infty}d_n\left|a_n\right|r^{n-1} - (B-A)\left|\gamma\right| \leq 0,$$

whence $f \in \mathcal{S}_m^c(A, B; \gamma)$. ∎

Remark 1 *By Theorem 1 we see that a function f of the form (1.1) belongs to the class $\mathcal{S}_m^c(A, B; \gamma)$ if it has "sufficiently small" coefficients. In particular, the functions*

$$f(z) = z + az^n, \quad z \in \mathbb{U},$$

where

$$\left|a\right| \leq \frac{(\left|c\right|/4)^n \left\{(B+1)(n-1) + (B-A)\left|\gamma\right|\right\}}{\left|(m)_n\right|\ (n)!\ (B-A)\left|\gamma\right|}$$

belong to the class $\mathcal{S}_m^c(A, B; \gamma)$. The convex combinations of these functions belong to the class $\mathcal{S}_m^c(A, B; \gamma)$ too.

Theorem 2 *Let $f \in \mathcal{A}$ and suppose that $g \in \mathcal{S}_m^c(A, B; \gamma)$ with $|m| \geq |\gamma(A-B)+mB|$. If $\mathfrak{B}_{m+1}^c f(z)$ is majorized by $\mathfrak{B}_{m+1}^c g(z)$, then*

$$|\mathfrak{B}_m^c f(z)| \leq |\mathfrak{B}_m^c g(z)|, \quad |z| \leq r_1, \tag{2.2}$$

where r_1 is the smallest positive root of the equation

$$|\gamma(A - B) + mB|r^3 - (|m| + 2|B|)\, r^2 - (|\gamma(A - B) + mB| + 2)\, r + |m| = 0. \tag{2.3}$$

Proof.   Since $g \in \mathcal{S}_m^c(A, B; \gamma)$, we find from (1.11) that

$$1 + \frac{1}{\gamma}\left(\frac{z(\mathfrak{B}_{m+1}^c g(z))'}{\mathfrak{B}_{m+1}^c g(z)} - 1\right) = \frac{1 + Aw(z)}{1 + Bw(z)}, \tag{2.4}$$

where w is analytic in $\mathbb{U}$, with $w(0)$ and $|w(z)| \leq |z|$ for all $z \in \mathbb{U}$.

From (2.4), we get

$$\frac{z(\mathfrak{B}_{m+1}^c g(z))'}{\mathfrak{B}_{m+1}^c g(z)} = \frac{1 + [\gamma(A - B) + B]w(z)}{1 + Bw(z)}. \tag{2.5}$$

Now, by applying the relation (1.7) in (2.5), we get

$$\frac{m\mathfrak{B}_m^c g(z)}{\mathfrak{B}_{m+1}^c g(z)} = \frac{m + [\gamma(A - B) + mB]w(z)}{1 + Bw(z)} \tag{2.6}$$

which yields that,

$$\left|\mathfrak{B}_{m+1}^c g(z)\right| \leq \frac{|m|\,[1 + |B|\,|z|]}{|m| - |\gamma(A - B) + mB|\,|z|}\left|\mathfrak{B}_m^c g(z)\right|. \tag{2.7}$$

Since $\mathfrak{B}_{m+1}^c f(z)$ is majorized by $\mathfrak{B}_{m+1}^c g(z)$, then there exist a function ϕ analytic in $\mathbb{U}$, with $\phi(0)$ and $|\phi(z)| \leq |z|$ for all $z \in \mathbb{U}$, such that

$$\mathfrak{B}_{m+1}^c f(z) = \phi(z)\mathfrak{B}_{m+1}^c g(z).$$

By differentiating with respect to z we get

$$z(\mathfrak{B}_{m+1}^c f(z))' = z\phi'(z)\mathfrak{B}_{m+1}^c g(z) + z\phi(z)(\mathfrak{B}_{m+1}^c g(z))'. \tag{2.8}$$

Noting that the Schwarz function ϕ satisfies (cf. [10])

$$\left|\phi'(z)\right| \leq \frac{1 - |\phi(z)|^2}{1 - |z|^2} \tag{2.9}$$

and using (1.7), (2.7) and (2.9) in (2.8), we have

$$|\mathfrak{B}_m^c f(z)| \leq \left(|\phi(z)| + \frac{1 - |\phi(z)|^2}{1 - |z|^2}\frac{(1 + |B||z|)\,|z|}{|m| - |\gamma(A - B) + mB||z|}\right)|\mathfrak{B}_m^c g(z)|,$$

Setting $|z| = r$ and $|\phi(z)| = \rho$, $0 \leq \rho \leq 1$ the above inequality leads us to the inequality

$$|\mathfrak{B}_m^c f(z)| \leq F(\rho, r) \ |\mathfrak{B}_m^c g(z)|, \qquad (2.10)$$

where

$$F(\rho, r) = \frac{\Phi(\rho)}{(1 - r^2)(|m| - |\gamma(A - B) + mB|r)},$$

with

$$\Phi(\rho) = -\rho^2(1 + |B|\,r)r + \rho(1 - r^2)(|m| - |mB + \gamma(A - B)|r) + r(1 + |B|\,r).$$

It is clear that if

$$\frac{(1 - r^2)(|m| - |mB + \gamma(A - B)|r)}{2(1 + |B|\,r)r} \geq 1,$$

then the function Φ takes its maximum value in the interval $\langle 0, 1 \rangle$ at $\rho = 1$. Since the above inequality holds for $0 \leq r \leq r_1 = r_1(\gamma, A, B)$, where r_1 is the smallest positive root of the equation (2.3), then there is $0 < F(\rho, r) \leq F(\rho, 1) = 1$ for $r \in \langle 0, r_1 \rangle$ and $\rho \in \langle 0, 1 \rangle$. This gives (2.2) and completes the proof. $\blacksquare$

Putting $A = 1$, $B = -1$ in Theorem 2, we have the following corollary:

Corollary 1 *Let $f \in \mathcal{A}$ and suppose that $g \in \mathcal{S}_m^c(\gamma)$ with $|m| \geq |2\gamma - m|$. If $\mathfrak{B}_{m+1}^c f(z)$ is majorized by $\mathfrak{B}_{m+1}^c g(z)$, then*

$$|\mathfrak{B}_m^c f(z)| \leq |\mathfrak{B}_m^c g(z)|, \ \ |z| \leq r_2, \qquad (2.11)$$

where r_2 is the smallest positive root of the equation

$$|2\gamma - m|r^3 - (|m| + 2)\,r^2 - (|2\gamma - m| + 2)\,r + |m| = 0, \qquad (2.12)$$

given by

$$r_2 = \frac{\kappa - \sqrt{\kappa^2 - 4|m|\ |2\gamma - m|}}{2|2\gamma - m|}, \ \ \kappa = (|m| + 2) + |2\gamma - m|.$$

Putting $\gamma = (1 - \alpha)cos\lambda e^{-i\lambda}$, $|\lambda| < \frac{\pi}{2}$; $0 \leq \alpha \leq 1$, in corollary 1, we have the following corollary.

Corollary 2 *Let $f \in \mathcal{A}$ and suppose that $g \in \mathcal{S}_m^c(\alpha, \lambda)$ with $|m| \geq |2(1-\alpha)cos\lambda e^{-i\lambda} - m|$. If $\mathfrak{B}_{m+1}^c f(z)$ is majorized by $\mathfrak{B}_{m+1}^c g(z)$, then*

$$\left|\mathfrak{B}_{m+1}^c f(z)\right| \leq |\mathfrak{B}_{m+1}^c g(z)|, \ \ |z| \leq r_3, \qquad (2.13)$$

where r_3 is the smallest positive root of the equation

$$|2(1 - \alpha)cos\lambda e^{-i\lambda} - m|r^3 - (|m| + 2)\,r^2 - \left(|2(1 - \alpha)cos\lambda e^{-i\lambda} - m| + 2\right)r + |m| = 0, \qquad (2.14)$$

given by

$$r_3 = \frac{\delta - \sqrt{\delta^2 - 4|m|\ |2(1 - \alpha)cos\lambda e^{-i\lambda} - m|}}{2|2(1 - \alpha)cos\lambda e^{-i\lambda} - m|} \qquad (2.15)$$

and

$$\delta = (|m| + 2) + |2(1 - \alpha)cos\lambda e^{-i\lambda} - m|.$$

Further, by taking $\lambda = 0$ we obtain the next corollary.

Corollary 3 *Let $f \in \mathcal{A}$ and suppose that $g \in \mathcal{S}_m^c(\alpha)$ with $\operatorname{Re} m \geq 1 - \alpha$. If $\mathfrak{B}_{m+1}^c f(z)$ is majorized by $\mathfrak{B}_{m+1}^c g(z)$, then*

$$|\mathfrak{B}_m^c f(z)| \leq |\mathfrak{B}_m^c g(z)|, \quad |z| \leq r_4, \tag{2.16}$$

where

$$r_4 = \frac{\delta - \sqrt{\delta^2 - 4|m| \ |2(1 - \alpha) - m|}}{2|2(1 - \alpha) - m|}$$

and

$$\delta = (|m| + 2) + |2(1 - \alpha) - m|.$$

For $\alpha = 0$ and $m = 1$ Corollary 3 reduces to the following result.

Corollary 4 *[8] Let $f \in \mathcal{A}$ and suppose that $g \in \mathcal{S}_1^c(0)$. If $\mathfrak{B}_2^c f(z)$ is majorized by $\mathfrak{B}_2^c g(z)$, then*

$$|\mathfrak{B}_1^c f(z)| \leq |\mathfrak{B}_1^c g(z)|, \quad |z| \leq r_5, \tag{2.17}$$

where $r_5 := 2 - \sqrt{3}$.

Concluding Remarks: Further specializing the parameters b, c one can define the various other interesting subclasses of $\mathcal{S}_m^c(A, B; \gamma)$, involving the types of Bessel functions as stated in equations (1.8) to (1.10), and one can easily derive the result as in Theorem 2 and the corresponding corollaries as mentioned above. The details involved may be left as an exercise for the interested reader.

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