

# On the Analytic $\alpha$ -Lipschitz Vector-Valued Operators

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**ABSTRACT:** Let  $(X, d)$  be a non-empty compact metric space in  $\mathbb{C}$ ,  $(B, \| \cdot \|)$  be a commutative unital Banach algebra over the scalar field  $\mathbb{F}(= \mathbb{R} \text{ or } \mathbb{C})$  and  $\alpha \in \mathbb{R}$  with  $0 < \alpha \leq 1$ . In this work, first we define the analytic  $\alpha$ -Lipschitz  $B$ -valued operators on  $X$  and denote the Banach algebra of all these operators by  $Lip_A^\alpha(X, B)$ . When  $B = \mathbb{F}$ , we write  $Lip_A^\alpha(X)$  instead of  $Lip_A^\alpha(X, B)$ . Then we study some interesting results about  $Lip_A^\alpha(X, B)$ , including the relationship between  $Lip_A^\alpha(X, B)$  with  $Lip_A^\alpha(X)$  and  $B$ , and also characterize the characters on  $Lip_A^\alpha(X, B)$ .

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## 1. Introduction

Throughout this paper, let  $(X, d)$  be a compact metric space in  $\mathbb{C}$ ,  $(B, \| \cdot \|)$  be a commutative unital Banach algebra over the scalar field  $\mathbb{F}(= \mathbb{R} \text{ or } \mathbb{C})$  with unit  $\mathbf{e}$ ,  $C(X, B)$  be the set of all  $B$ -valued continuous operators on  $X$ , and also  $\alpha \in \mathbb{R}$  with  $0 < \alpha \leq 1$ .

The *dual space* of  $B$  is the vector space  $B^*$  whose elements are the continuous linear functionals on  $B$ . The set of all multiplicative functionals on  $B$  is called *spectrum* of  $B$ ; we denote it by  $\sigma(B)$ . Suppose that throughout this article,  $\Lambda \in \sigma(B)$  is arbitrary and fixed. Since  $\sigma(B)$  is a subset of the closed unit ball of  $B^*$ ,  $\| \Lambda \|$  is bounded, where

$$\| \Lambda \| = \sup \{ | \Lambda x | : x \in B, \| x \| \leq 1 \}.$$

When  $B = \mathbb{F}$ , take  $\Lambda$  as the identity function  $\Lambda x = x$ .

Consider the set  $Y$  as follows

$$Y := \{(x, y) : x, y \in X, x \neq y\}.$$

For an operator  $f : X \rightarrow B$  and any  $(x, y) \in Y$  define

$$L_f^\alpha(x, y) := \frac{|(\Lambda o f)(x) - (\Lambda o f)(y)|}{d^\alpha(x, y)},$$

where  $d^\alpha(x, y) = (d(x, y))^\alpha$  and  $0 < \alpha \leq 1$ . Now define

$$p_\alpha(f) := \sup_{x \neq y} L_f^\alpha(x, y), \quad 0 < \alpha \leq 1,$$

which is called the *Lipschitz constant* of  $f$ . Also for  $0 < \alpha \leq 1$  define

$$Lip^\alpha(X, B) := \{f \in C(X, B) : p_\alpha(f) < +\infty\},$$

and for  $0 < \alpha < 1$  define

$$lip^\alpha(X, B) := \{f \in Lip^\alpha(X, B) : \lim_{d(x, y) \rightarrow 0} L_f^\alpha(x, y) = 0\}.$$

The elements of  $Lip^\alpha(X, B)$  and  $lip^\alpha(X, B)$  are called *big* and *little*  $\alpha$ -Lipschitz  $B$ -valued operators, respectively.

Now, for each  $\lambda \in \mathbb{F}$ ,  $x \in X$  and  $f, g \in C(X, B)$  define

$$(f + g)(x) := f(x) + g(x), \quad (\lambda f)(x) := \lambda f(x),$$

and the uniform norm  $\| \cdot \|_\infty$  on  $C(X, B)$  by

$$\| f \|_\infty := \sup_{x \in X} \| f(x) \| \quad ; \quad f \in C(X, B).$$

Also for any  $f \in Lip^\alpha(X, B)$  define

$$\| f \|_\alpha := p_\alpha(f) + \| f \|_\infty.$$

It is easy to see that  $(C(X, B), \| \cdot \|_\infty)$  becomes a Banach algebra over  $\mathbb{F}$ .

Cao, Zhang and Xu in [6] proved that  $(Lip^\alpha(X, B), \| \cdot \|_\alpha)$  is a Banach space over  $\mathbb{F}$  and  $(lip^\alpha(X, B), \| \cdot \|_\alpha)$  is a closed linear subspace of  $(Lip^\alpha(X, B), \| \cdot \|_\alpha)$  when  $B$  is a Banach space. We also studied some of the properties of these algebras in [14-17] when  $B$  is a commutative unital Banach algebra.

Note that for  $\alpha = 1$  and  $B = \mathbb{F}$ , the space  $Lip^1(X, \mathbb{F})$  consisting of all Lipschitz functions from  $X$  into  $\mathbb{F}$  ( $= \mathbb{R}$  or  $\mathbb{C}$ ) has a series of interesting and important properties, which has been studied by many mathematicians, including the first of them Sherbert [13]. In [7, 18] some properties of Lipschitz scalar-valued functions are mentioned.

Let  $D$  be an open subset of  $X$ . An operator  $f$  of  $D$  into  $B$  is said to be *analytic* on  $D$  if, for every continuous linear functional  $\phi \in B^*$ , the scalar-valued function  $\phi o f$

is analytic on  $D$  in the usual sense. Note that we do not require  $D$  to be connected. For a full discussion of analytic complex-valued and vector-valued functions, see [2, 7]. The algebra of all continuous  $B$ -valued operators on  $X$  whose analytic in interior  $X$  is denoted by  $A(X, B)$ . We write  $A(X)$  instead of  $A(X, \mathbb{F})$  ( $\mathbb{F} = \mathbb{R}$  or  $\mathbb{C}$ ). Some of the properties of these algebras have been studied by certain mathematicians, see [1, 3-5, 8-11].

Finally, in this article, we introduce the analytic  $\alpha$ -Lipschitz  $B$ -valued operator algebras  $Lip_A^\alpha(X, B)$  and we characterize their characters, also we study the relationship between of  $Lip_A^\alpha(X, B)$  and  $B$ . We prove the main results of the article in several theorems.

## 2. Lip-analytic Operators

In this section, we introduce the analytic  $\alpha$ -Lipschitz vector-valued operator algebras  $Lip_A^\alpha(X, B)$ , and we study some of their properties.

We write  $C(X)$  and  $Lip^\alpha(X)$  instead of  $C(X, \mathbb{F})$  and  $Lip^\alpha(X, \mathbb{F})$  respectively. By the Stone-Weierstrass theorem, we have

**Theorem 2.1.** [7].  $A(X)$  is uniformly dense in  $C(X)$ .

It is obvious that  $A(X, B)$  is a subalgebra of  $C(X, B)$ . We have

**Theorem 2.2.**  $A(X, B)$  is uniformly dense in  $C(X, B)$ .

**Proof.** Let  $\epsilon > 0$  and  $f \in C(X, B)$  be arbitrary. We show that there exists  $g \in A(X, B)$  such that  $\|f - g\|_\infty < \epsilon$ . Since  $f \in C(X, B)$ ,  $\Lambda f \in C(X)$ . Then by Theorem 2.1, there is  $h \in A(X)$  such that  $\|\Lambda f - h\|_\infty < \epsilon$ . So

$$\sup_{x \in X} |(\Lambda f)(x) - h(x)| < \epsilon.$$

Since  $\Lambda(\mathbf{e}) = 1$ ,  $h(x) = \Lambda(h(x)\mathbf{e})$  for all  $x \in X$ . Then

$$\sup_{x \in X} |\Lambda(f(x)) - \Lambda(h(x)\mathbf{e})| < \epsilon.$$

Hence

$$\sup_{x \in X} |\Lambda((f - h.\mathbf{e})(x))| < \epsilon.$$

Since  $\Lambda \in \sigma(B)$  is arbitrary,  $\sup_{x \in X} \|(f - h.\mathbf{e})(x)\| < \epsilon$ . Thus  $\|f - h.\mathbf{e}\|_\infty < \epsilon$ . Now, take  $g := h.\mathbf{e}$ . Since  $h \in A(X)$  and  $\mathbf{e} \in B$ ,  $g \in A(X, B)$ . Therefore we conclude that  $\|f - g\| < \epsilon$  where  $g \in A(X, B)$ .  $\square$

We have the similar Theorem 2.1 for the algebra of Lipschitz scalar-valued functions:

**Theorem 2.3.** [18].  $Lip^\alpha(X)$  is uniformly dense in  $C(X)$ .

**Theorem 2.4.**  $Lip^\alpha(X, B)$  is uniformly dense in  $C(X, B)$ .

**Proof.** Let  $\epsilon > 0$  and  $f \in C(X, B)$  be arbitrary. We show that there exists  $h \in Lip^\alpha(X, B)$  such that  $\|h - f\|_\infty < \epsilon$ . Since  $f \in C(X, B)$ ,  $\Lambda of \in C(X)$ . So by Theorem 2.3, there exists  $g \in Lip^\alpha(X)$  such that  $\|g - \Lambda of\|_\infty < \epsilon$ . Define

$$\eta : \mathbb{C} \rightarrow B$$

$$\eta(\lambda) := \lambda \mathbf{e}.$$

Since  $g$  is continuous,  $\eta og$  is continuous. Also

$$\begin{aligned} p_\alpha(\eta og) &= \sup_{x \neq y} L_{\eta og}^\alpha(x, y) \\ &= \sup_{x \neq y} \frac{\|(\eta og)(x) - (\eta og)(y)\|}{d^\alpha(x, y)} \\ &= \sup_{x \neq y} \frac{\|g(x)\mathbf{e} - g(y)\mathbf{e}\|}{d^\alpha(x, y)} \quad (\|\mathbf{e}\| = 1) \\ &\leq p_\alpha(g) < \infty. \end{aligned}$$

So  $\eta og \in Lip^\alpha(X, B)$ . Set  $h := \eta og$ . Now we show that  $\|h - f\|_\infty < \epsilon$ . Since  $\Lambda(\mathbf{e}) = 1$ , for all  $x \in X$  we have

$$\begin{aligned} |\Lambda(g(x)\mathbf{e} - f(x))| &= |g(x) - (\Lambda of)(x)| \\ &\leq \|g - \Lambda of\|_\infty \\ &< \epsilon. \end{aligned}$$

This implies that

$$|\Lambda(\eta og - f)(x)| < \epsilon, \quad x \in X.$$

Since  $\Lambda \in \sigma(B)$  is arbitrary,  $\|(\eta og - f)(x)\| < \epsilon$  for all  $x \in X$ . Consequently,  $\|\eta og - f\|_\infty < \epsilon$  or  $\|h - f\|_\infty < \epsilon$ . This completes the proof.  $\square$

**Corollary 2.5.** By using Theorems 2.2 and 2.4, each element of  $A(X, B)$  can be approximated by elements of  $Lip^\alpha(X, B)$  with sup-norm. Also each element of  $Lip^\alpha(X, B)$  can be approximated by elements of  $A(X, B)$  with sup-norm.

**Definition 2.6.** Let  $D$  be an open subset of  $X$ . An operator  $f$  of  $D$  into  $B$  is said to be Lip-analytic on  $D$  if  $f \in Lip^\alpha(X, B) \cap A(X, B)$ .

The algebra of all Lip-analytic  $B$ -valued operators on  $X$  whose analytic in interior  $X$  is denoted by  $Lip_A^\alpha(X, B)$ . When  $B = \mathbb{F}$ , we write  $Lip_A^\alpha(X)$  instead of  $Lip_A^\alpha(X, B)$ .

By Theorems 2.2 and 2.4, we can prove that:

**Theorem 2.7.**  $Lip_A^\alpha(X, B)$  is uniformly dense in  $C(X, B)$ .

Let  $E_1$  and  $E_2$  be linear spaces. From [12], a *tensor product* of  $E_1$  and  $E_2$  is a pair  $(T, \tau)$ , where  $T$  is a linear space and  $\tau : E_1 \times E_2 \rightarrow T$  is a bilinear map with the following (universal) property: For each linear space  $F$  and for each bilinear map  $V : E_1 \times E_2 \rightarrow F$ , there is a unique linear map  $U : T \rightarrow F$  such that  $V = U\tau$ .

We shall also use the standard notation for tensor products, we write  $E_1 \otimes E_2$  for  $T$  and  $x_1 \otimes x_2 = \tau(x_1, x_2)$  for  $x_1 \in E_1$  and  $x_2 \in E_2$ . If  $Z \in E_1 \otimes E_2$ , then there is  $m \in \mathbb{N}$ , and for each  $j = 1, 2$  there are  $x_j^{(1)}, \dots, x_j^{(m)} \in E_j$  such that  $Z = \sum_{k=1}^m x_1^{(k)} \otimes x_2^{(k)}$ .

Let  $E_1$  and  $E_2$  be Banach spaces with dual spaces  $E_1^*$  and  $E_2^*$ . Then we define for  $Z \in E_1 \otimes E_2$

$$\| Z \|_\epsilon = \sup \left\{ |\langle Z, \phi_1 \otimes \phi_2 \rangle| : \phi_j \in N_1[0, E_j^*], j = 1, 2 \right\},$$

where

$$Z = \sum_{k=1}^m x_1^{(k)} \otimes x_2^{(k)} ; (m \in \mathbb{N}, x_j^{(k)} \in E_j, j = 1, 2, 1 \leq k \leq m),$$

and

$$\begin{aligned} \langle Z, \phi_1 \otimes \phi_2 \rangle &= \left\langle \sum_{k=1}^m x_1^{(k)} \otimes x_2^{(k)}, \phi_1 \otimes \phi_2 \right\rangle \\ &= (\phi_1 \otimes \phi_2) \left( \sum_{k=1}^m x_1^{(k)} \otimes x_2^{(k)} \right) \\ &= \sum_{k=1}^m (\phi_1 \otimes \phi_2)(x_1^{(k)} \otimes x_2^{(k)}) \\ &= \sum_{k=1}^m \phi_1(x_1^{(k)}) \phi_2(x_2^{(k)}), \end{aligned}$$

and  $N_1[0, E_j^*]$  is closed ball in  $E_j^*$  with radius 1 centered at 0. We call  $\| \cdot \|_\epsilon$  the *injective norm* on  $E_1 \otimes E_2$ .

Let  $(E_1, \| \cdot \|_1)$  and  $(E_2, \| \cdot \|_2)$  be Banach spaces. Then their *injective tensor product*  $E_1 \check{\otimes} E_2$  is the completion of  $E_1 \otimes E_2$  with respect to  $\| \cdot \|_\epsilon$ . For every  $Z \in E_1 \check{\otimes} E_2$ , we have

$$\| Z \|_\epsilon = \sup \left\{ \| (id \otimes \phi)(Z) \|_1 : \phi \in N_1[0, E_2^*] \right\},$$

where

$$(id \otimes \phi)(a \otimes b) = a\phi(b) ; (a \in E_1, b \in E_2).$$

**Definition 2.8.** Let  $E_1$  and  $E_2$  be Banach spaces. A norm  $\| \cdot \|$  on  $E_1 \otimes E_2$  is called a *cross norm* if

$$\| x_1 \otimes x_2 \| = \| x_1 \| \| x_2 \| \quad (x_1 \in E_1, x_2 \in E_2).$$

**Proposition 2.9.** [12]. Let  $E_1$  and  $E_2$  be Banach spaces. Then  $\| \cdot \|_\epsilon$  is a cross norm on  $E_1 \otimes E_2$ .

### 3. The Main Results

In this section, we present the main results of the article.

**Theorem 3.1.**  $Lip_A^\alpha(X, B)$  is isometrically isomorphic to  $Lip_A^\alpha(X) \otimes B$ .

**Proof.** It is straightforward to prove that the mapping

$$Lip_A^\alpha(X) \times B \rightarrow Lip_A^\alpha(X, B), \quad (f, b) \mapsto fb \quad (3.1)$$

is bilinear. So from the defining property of the algebraic tensor product  $Lip_A^\alpha(X) \otimes B$ , it follows that (1) extends to a linear map

$$\begin{aligned} S : Lip_A^\alpha(X) \otimes B &\longrightarrow Lip_A^\alpha(X, B) \\ S(f \otimes b) &:= fb, \end{aligned}$$

where

$$(fb)(x) := f(x)b; \quad (x \in X).$$

Then

$$\begin{aligned} \|S(f \otimes b)\|_\alpha &= \|fb\|_\alpha = \|fb\|_\infty + p_\alpha(fb) \\ &= \|f\|_\infty \|b\| + p_\alpha(f) \|b\| \\ &= (\|f\|_\infty + p_\alpha(f)) \|b\| \\ &= \|f\|_\alpha \|b\| \\ &= \|f \otimes b\|_\epsilon. \end{aligned}$$

Therefore  $S$  is an isometry and thus injective with closed range. It remains to be shown that it has dense range as well.

Let  $f \in Lip_A^\alpha(X, B)$  and  $\epsilon > 0$ . Being the continuous image of a compact space,  $K := f(X) \subset B$  is compact. We may thus find  $b_1, \dots, b_n \in B$  such that  $K \subset \cup_{i=1}^n N(b_i, \epsilon)$ , where  $N(b_i, \epsilon)$  is a neighborhood with radius  $\epsilon$  centered at  $b_i$ . Let  $U_j := f^{-1}(N(b_j, \epsilon))$  for  $j = 1, \dots, n$ . Choose  $f_1, \dots, f_n \in Lip_A^\alpha(X, B)$  such that  $supp(f_j) \subset U_j$  for  $j = 1, \dots, n$ , and  $\Lambda o(\sum_{i=1}^n f_i) = 1$ . Then for every  $x \in X$  we have

$$\begin{aligned} \left\| \left( f - \sum_{i=1}^n S(\Lambda o f_i \otimes b_i) \right) (x) \right\| &= \left\| \left( f - \sum_{i=1}^n (\Lambda o f_i) b_i \right) (x) \right\| \\ &= \left\| f(x) - \sum_{i=1}^n (\Lambda o f_i)(x) b_i \right\| \\ &= \left\| f(x) \left( \Lambda o \left( \sum_{i=1}^n f_i \right) \right) (x) - \sum_{i=1}^n (\Lambda o f_i)(x) b_i \right\| \\ &= \left\| f(x) \sum_{i=1}^n (\Lambda o f_i)(x) - \sum_{i=1}^n (\Lambda o f_i)(x) b_i \right\| \end{aligned}$$

$$\begin{aligned}
 &= \left\| \sum_{i=1}^n (\Lambda o f_i)(x)(f(x) - b_i) \right\| \\
 &\leq \sum_{i=1}^n |(\Lambda o f_i)(x)| \|f(x) - b_i\|.
 \end{aligned}$$

It is easy to see that the right hand side of the above relation is less than  $\epsilon$ . So we conclude that  $\overline{R_S} = Lip_A^\alpha(X, B)$ . This completes the proof.  $\square$

With an argument similar to the proof of Theorem 3.1, we can prove that:

**Theorem 3.2.**  $A(X, B)$  is isometrically isomorphic to  $A(X) \otimes B$ .

Define the canonical embedding

$$j : Lip_A^\alpha(X) \rightarrow Lip_A^\alpha(X, B)$$

$$j(h) := h \otimes \mathbf{e},$$

such that

$$(h \otimes \mathbf{e})(x) := h(x)\mathbf{e}; \quad x \in X.$$

By Theorem 3.1, the map  $j$  is well defined. Let  $\chi$  be an arbitrary and fixed character on  $Lip_A^\alpha(X, B)$ . Then there is  $z \in X$  such that  $\chi o j$  is the evaluation at  $z$ , indeed  $\chi o j = \delta_z$  where  $\delta_z(f) = f(z)$ .

Define  $\varphi(\omega) := \omega - z, (\omega \in X)$ . It is clear that  $\varphi \in A(X)$ , and we have

$$\begin{aligned}
 p_\alpha(\varphi) &= \sup_{x \neq y} \frac{|\varphi(x) - \varphi(y)|}{|x - y|^\alpha} = \sup_{x \neq y} \frac{|(x - z) - (y - z)|}{|x - y|^\alpha} \\
 &= \sup_{x \neq y} |x - y|^{1-\alpha} < \infty.
 \end{aligned}$$

So  $\varphi \in Lip^\alpha(X)$ , and consequently  $\varphi \in Lip_A^\alpha(X)$ .

Now consider

$$I := \{f \in Lip_A^\alpha(X, B) : f(z) = 0\}.$$

It is obvious that  $I$  is nonempty and an ideal in  $Lip_A^\alpha(X, B)$ .

**Theorem 3.3.**  $I$  is contained in the kernel of  $\chi$ .

**Proof.** Let  $f \in I$  be arbitrary. Then  $f \in A(X, B)$ . So  $f$  has a Taylor series expansion  $f(\omega) = \sum_{n=1}^\infty \frac{f^{(n)}(z)}{n!}(\omega - z)^n$  around  $z$ . Define

$$g(\omega) := \begin{cases} \frac{f(\omega)}{\omega - z} & ; \quad \omega \neq z, \\ f'(z) & ; \quad \omega = z. \end{cases}$$

It is clear that  $\Lambda \log$  is analytic in the interior of  $X$ , so  $g \in A(X, B)$ . For  $\omega = z$ , it is obvious that  $g \in Lip_A^\alpha(X, B)$ , and for  $\omega \neq z$  we have

$$f(\omega) = (\omega - z)g(\omega) = \varphi(\omega)g(\omega).$$

It can be easily proved that  $g \in Lip_A^\alpha(X, B)$ . Then for every  $\omega \in X$  with  $\omega \neq z$ , we have

$$\begin{aligned} f(\omega) &= \varphi(\omega)g(\omega) = \varphi(\omega)\mathbf{e}g(\omega) \\ &= (\varphi \otimes \mathbf{e})(\omega)g(\omega) = ((\varphi \otimes \mathbf{e})g)(\omega) \\ &= (j(\varphi)g)(\omega). \end{aligned}$$

So  $f = j(\varphi)g$ . Therefore

$$\begin{aligned} \chi(f) &= \chi(j(\varphi)g) = \chi(j(\varphi))\chi(g) \\ &= (\chi \circ j)(\varphi)\chi(g) = \delta_z(\varphi)\chi(g) \\ &= \varphi(z)\chi(g) = 0 \times \chi(g) = 0. \end{aligned}$$

So  $f \in \ker \chi$ , and that means  $I \subset \ker \chi$ . This completes the proof.  $\square$

**Theorem 3.4.** *Every character  $\chi$  on  $Lip_A^\alpha(X, B)$  is of form  $\chi = \psi \circ \delta_z$  for some character  $\psi$  on  $B$  and some  $z \in X$ , where  $\delta_z(f) = f(z)$ .*

**Proof.** Let  $\chi$  be an arbitrary character on  $Lip_A^\alpha(X, B)$ . Then there is  $z \in X$  such that  $\chi \circ j$  is the evaluation at  $z$ , indeed  $\chi \circ j = \delta_z$  where  $\delta_z(f) = f(z)$ . Define

$$I := \{f \in Lip_A^\alpha(X, B) : f(z) = 0\}.$$

By Theorem 3.3,  $I$  is contained in the kernel of  $\chi$ . It is clear that  $\ker \delta_z = I$ . Therefore  $\ker \delta_z \subset \ker \chi$ . We obtain the desired factorization  $\chi = \psi \circ \delta_z$  for some character  $\psi$  on  $B$ .  $\square$

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