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Remarkable identities

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Abstract: In the paper a number of identities involving even powers of the values of functions tangent, cotangent, secans and cosecans are proved. Namely, the following relations are shown:

$$\begin{aligned}\sum_{j=1}^{m-1} f^{2n}\left(\frac{\pi j}{2m}\right) &= w_f(m),\\ \sum_{j=0}^{m-1} f^{2n}\left(\frac{2j+1}{4m}\pi\right) &= v_f(m),\\ \sum_{j=1}^{m} f^{2n}\left(\frac{\pi j}{2m+1}\right) &= \tilde{w}_f(m),\end{aligned}$$

where m, n are positive integers, f is one of the functions: tangent, cotangent, secans or cosecans and $w_f(x)$, $v_f(x)$, $\tilde{w}_f(x)$ are some polynomials from $\mathbb{Q}[x]$.

One of the remarkable identities is the following:

$$\sum_{j=0}^{m-1} \sin^{-2}\frac{(2j+1)\pi}{2m} = m^2, \qquad \text{provided } m \geq 1.$$

Some of these identities are used to find, by elementary means, the sums of the series of the form $\sum_{j=1}^{\infty}\frac{1}{j^{2n}}$, where n is a fixed positive integer. One can also notice that Bernoulli numbers appear in the leading coefficients of the polynomials $w_f(x)$, $v_f(x)$ and $\tilde{w}_f(x)$.

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In [7] the following formulas have been proved

$$\sum_{j=1}^{m} \cot^2\frac{\pi j}{2m+1} = \frac{m(2m-1)}{3}, \qquad \sum_{j=1}^{m} \sin^{-2}\frac{\pi j}{2m+1} = \frac{2m(m+1)}{3}, \tag{1}$$

where $m \in \mathbb{N}_1$. By $\mathbb{N}_k$ for a positive integer k we mean $\mathbb{N} \setminus \{0, 1, 2 \dots, k-1\}$. The above identities were then used in an elementary proof of the formula $\sum_{k=1}^{\infty} \frac{1}{k^2} = \frac{\pi^2}{6}$.

In this paper we develop the ideas from [7] to prove more generalized identities than (1). Next we use some of them to find the sum of $\sum_{k=1}^{\infty} \frac{1}{k^{2n}}$, where $n \in \mathbb{N}_1$. The general identities given in this article yield, in particular, the following identity of uncommon beauty

$$\sum_{j=0}^{m-1} \sin^{-2} \frac{2j+1}{2m}\pi = m^2, \qquad m \in \mathbb{N}_1.$$

Some elementary methods of finding the sums of the series of the form $\sum_{j=1}^{\infty} \frac{1}{j^{2n}}$ may be found for example in [1], [3], [5], [6], [8].

We start by recalling some basic facts on symmetric polynomials in m variables. Put

$$\sigma_n = \sum_{j=1}^{m} x_j^n \qquad \text{for } n \in \mathbb{N}_1,$$

$$\tau_k = \sum_{1 \le j_1 < j_2 < \dots < j_k \le m} x_{j_1} x_{j_2} \cdots x_{j_k} \qquad \text{for } k \in \{1, 2, \dots, m\}.$$

Moreover, for the convenience set $\tau_k = 0$ for $k > m$.

The following lemma comes from [2].

Lemma 1 (Newton)**.** *Let $n \in \mathbb{N}_1$, then*

$$\sigma_n - \tau_1\sigma_{n-1} + \tau_2\sigma_{n-2} - \dots + (-1)^{n-1}\tau_{n-1}\sigma_1 + (-1)^n n\tau_n = 0. \tag{2}$$

In view of Lemma 1 we have

$$\sigma_n = \det \begin{pmatrix} (-1)^{n+1}n\tau_n & -\tau_1 & \tau_2 & \dots & (-1)^{n-2}\tau_{n-2} & (-1)^{n-1}\tau_{n-1} \\ (-1)^n(n-1)\tau_{n-1} & 1 & -\tau_1 & \dots & (-1)^{n-3}\tau_{n-3} & (-1)^{n-2}\tau_{n-2} \\ (-1)^{n-1}(n-2)\tau_{n-2} & 0 & 1 & \dots & (-1)^{n-4}\tau_{n-4} & (-1)^{n-3}\tau_{n-3} \\ \vdots & \vdots & \ddots & \ddots & \vdots & \vdots \\ -2\tau_2 & 0 & 0 & \dots & 1 & -\tau_1 \\ \tau_1 & 0 & 0 & \dots & 0 & 1 \end{pmatrix} \tag{3}$$

for every $n \in \mathbb{N}_1$. Indeed, putting in (2) instead of n respectively $n-1, n-2, \dots, 1$ we get, together with (2), the system of n equations in n variables: $\sigma_1, \dots, \sigma_n$. Such a system is a Cramer's system and by the Cramer's rule we get (3).

From now on by $D_{\tan}$ and $D_{\cot}$ we denote the domains of the trigonometric functions tangent and cotangent, respectively.

Lemma 2. *The following identities hold true:*

(A) $\frac{\sin 2mx}{\cos^{2m} x} \cot x = \sum_{j=0}^{m} \binom{2m}{2j+1} (-1)^j \tan^{2j} x, \qquad (m, x) \in \mathbb{N} \times (D_{\tan} \cap D_{\cot})$;

(B) $\frac{\cos 2mx}{\cos^{2m} x} = \sum_{j=0}^{m} \binom{2m}{2j} (-1)^j \tan^{2j} x, \qquad (m, x) \in \mathbb{N} \times D_{\tan}$;

(C) $\frac{\sin(2m+1)x}{\cos^{2m+1}x}\cot x = \sum_{j=0}^{m}\binom{2m+1}{2j+1}(-1)^j\tan^{2j}x, \qquad (m,x)\in\mathbb{N}\times(D_{\tan}\cap D_{\cot})$;

(D) $\frac{\sin(2m+1)x}{\sin^{2m+1}x} = \sum_{j=0}^{m}\binom{2m+1}{2j+1}(-1)^j\cot^{2m-2j}x, \qquad (m,x)\in\mathbb{N}\times D_{\cot}$;

(E) $\frac{\cos(2m+1)x}{\cos^{2m+1}x} = \sum_{j=0}^{m}\binom{2m+1}{2j}(-1)^j\tan^{2j}x, \qquad (m,x)\in\mathbb{N}\times D_{\tan}$;

(F) $\frac{\cos(2m+1)x}{\sin^{2m+1}x}\tan x = \sum_{j=0}^{m}\binom{2m+1}{2j}(-1)^j\cot^{2m-2j}x, \qquad (m,x)\in\mathbb{N}\times(D_{\tan}\cap D_{\cot})$.

Proof. It is a known fact that

$$\sum_{j=0}^{k}\binom{k}{j}\cos^{k-j}x(\mathrm{i}\sin x)^j = (\cos x+\mathrm{i}\sin x)^k = \cos kx+\mathrm{i}\sin kx$$

for $k\in\mathbb{N}$ and $x\in\mathbb{R}$. Putting $k=2m$ in the above equation and comparing real and imaginary parts of the both sides we obtain (A) and (B). Similarly, with $k=2m+1$ we get (C), (D), (E) and (F). □

Now we prove the following result.

Theorem 1. *For every $m\in\mathbb{N}_2$ and any $n\in\mathbb{N}_1$,*

$$\sigma_{n,m}(A) = \sum_{j=1}^{m-1}\tan^{2n}\frac{\pi j}{2m} = \sum_{j=1}^{m-1}\cot^{2n}\frac{\pi j}{2m},$$

where $\sigma_{n,m}(A)$ denotes the determinant given by (3) *in which $\tau_j = \frac{\binom{2m}{2j+1}}{2m}$ for $j\in\{1,2,\ldots,n\}$.*

Proof. Replace in the identity (A) of Lemma 2, $\tan^2 x$ by t and set

$$w_A(t) = \sum_{j=0}^{m}\binom{2m}{2j+1}(-1)^j t^j, \tag{4}$$

then $w_A(t)$ is a polynomial of order $m-1$ in the real variable t.

On the other hand, substituting $\frac{\pi l}{2m}$, where $l\in\{1,2,\ldots,m-1\}$, for x in (A) we get

$$0 = \sum_{j=0}^{m}\binom{2m}{2j+1}(-1)^j\tan^{2j}\frac{\pi l}{2m}, \qquad l\in\{1,2,\ldots,m-1\}.$$

Hence and by (4) we obtain

$$w_A(t) = (-1)^{m-1}\binom{2m}{2m-1}\prod_{j=1}^{m-1}\left(t-\tan^2\frac{\pi j}{2m}\right) = (-1)^{m-1}2m\prod_{j=1}^{m-1}\left(t-\tan^2\frac{\pi j}{2m}\right).$$

This and the Vieta's formulas give

$$\sum_{1\le k_1<k_2<\cdots<k_j\le m-1} \tan^2\frac{\pi k_1}{2m}\tan^2\frac{\pi k_2}{2m}\cdots\tan^2\frac{\pi k_j}{2m} = \frac{\binom{2m}{2j+1}}{2m}$$

and in view of (3) we have

$$\sigma_{n,m}(A) = \sum_{j=1}^{m-1}\tan^{2n}\frac{\pi j}{2m}.$$

As $\tan\frac{\pi j}{2m} = \cot\frac{\pi(m-j)}{2m}$ for $j\in\{1,2,\dots,m-1\}$ we get

$$\sum_{j=1}^{m-1}\tan^{2n}\frac{\pi j}{2m} = \sum_{j=1}^{m-1}\cot^{2n}\frac{\pi(m-j)}{2m} = \sum_{j=1}^{m-1}\cot^{2n}\frac{\pi j}{2m},$$

which completes the proof. □

Theorem 2. *For every $m,n\in\mathbb{N}_1$ the following identity holds true:*

$$\sigma_{n,m}(B) = \sum_{j=0}^{m-1}\tan^{2n}\frac{2j+1}{4m}\pi = \sum_{j=0}^{m-1}\cot^{2n}\frac{2j+1}{4m}\pi,$$

where $\sigma_{n,m}(B)$ denotes the determinant given by (3) *in which $\tau_j = \binom{2m}{2j}$ for $j\in\{1,2,\dots,n\}$.*

Proof. Similarly as in the proof of Theorem 1, replace in the right hand side of the identity (B) of Lemma 2, $\tan^2 x$ by t and set

$$w_B(t) = \sum_{j=0}^{m}\binom{2m}{2j}(-1)^j t^j.$$

Next, substitute $\frac{2l+1}{4m}\pi$, where $l\in\{0,1,\dots,m-1\}$, for x in (B). This yields

$$0 = \sum_{j=0}^{m}\binom{2m}{2j}(-1)^j\tan^{2j}\frac{2l+1}{4m}\pi, \qquad l\in\{0,1,\dots,m-1\}.$$

Hence and by the definition of $w_B(t)$ we get

$$w_B(t) = (-1)^m\prod_{j=0}^{m-1}\left(t-\tan^2\frac{2j+1}{4m}\pi\right),$$

which in view of the Vieta's formulas gives

$$\sum_{1\le k_1<k_2<\cdots<k_j\le m-1}\tan^2\frac{2k_1+1}{4m}\tan^2\frac{2k_2+1}{4m}\cdots\tan^2\frac{2k_j+1}{4m} = \binom{2m}{2j}.$$

By this and (3),

$$\sigma_{n,m}(B) = \sum_{j=0}^{m-1} \tan^{2n} \frac{2j+1}{4m}\pi.$$

Using the same argument as in the proof of Theorem 1 we get

$$\sum_{j=0}^{m-1} \tan^{2n} \frac{2j+1}{2m}\pi = \sum_{j=0}^{m-1} \cot^{2n} \frac{2j+1}{4m}\pi$$

and the proof is completed. □

Using identities (C) and (D) of Lemma 2 and the same method as in proofs of Theorems 1 and 2 one may obtain

Theorem 3. *For every $m, n \in \mathbb{N}_1$ the following identity holds true:*

$$\sigma_{n,m}(C) = \sum_{j=1}^{m} \tan^{2n} \frac{\pi j}{2m+1},$$

where $\sigma_{n,m}(C)$ denotes the determinant given by (3) *in which $\tau_j = \binom{2m+1}{2j}$ for $j \in \{1, 2, \dots, n\}$.*

Theorem 4. *For every $m, n \in \mathbb{N}_1$ the following identity holds true:*

$$\sigma_{n,m}(D) = \sum_{j=1}^{m} \cot^{2n} \frac{\pi j}{2m+1},$$

where $\sigma_{n,m}(D)$ denotes the determinant given by (3) *in which $\tau_j = \frac{1}{2m+1}\binom{2m+1}{2j+1}$ for $j \in \{1, 2, \dots, n\}$.*

Finally, applying the same reasoning as in the proof of Theorem 1 from (E) and (F) of Lemma 2 we have

Theorem 5. *For every $m, n \in \mathbb{N}_1$ the following identity holds true:*

$$\sigma_{n,m}(E) = \sum_{j=0}^{m-1} \tan^{2n} \frac{2j+1}{2(2m+1)}\pi,$$

where $\sigma_{n,m}(E)$ denotes the determinant given by (3) *in which $\tau_j = \frac{1}{2m+1}\binom{2m+1}{2j+1}$ for $j \in \{1, 2, \dots, n\}$.*

Theorem 6. *For every $m, n \in \mathbb{N}_1$ the following identity holds true:*

$$\sigma_{n,m}(F) = \sum_{j=0}^{m-1} \cot^{2n} \frac{2j+1}{2(2m+1)}\pi,$$

where $\sigma_{n,m}(F)$ denotes the determinant given by (3) *in which $\tau_j = \binom{2m+1}{2j}$ for $j \in \{1, 2, \dots, n\}$.*

The following formulas

$$\cot^{2n} x = \left(\frac{1-\sin^2 x}{\sin^2 x}\right)^n, \qquad \tan^{2n} x = \left(\frac{1-\cos^2 x}{\cos^2 x}\right)^n$$

yield

Lemma 3. *The following identities hold true:*

(G) $\sum_{j=0}^{n-1} \binom{n}{j}(-1)^j \sin^{2j-2n} x = (-1)^{n-1} + \cot^{2n} x, \qquad (n,x) \in \mathbb{N}_1 \times D_{\cot};$

(H) $\sum_{j=0}^{n-1} \binom{n}{j}(-1)^j \cos^{2j-2n} x = (-1)^{n-1} + \tan^{2n} x, \qquad (m,x) \in \mathbb{N}_1 \times D_{\tan}.$

Lemma 4. *Assume that $n \in \mathbb{N}_1$ and $x \in D_{\cot}$, then*

$$\frac{1}{\sin^{2n} x} = \det \begin{pmatrix} (-1)^{n-1} + \cot^{2n} x & -\binom{n}{1} & \binom{n}{2} & -\binom{n}{3} & \dots & (-1)^{n-1}\binom{n}{n-1} \\ (-1)^{n-2} + \cot^{2n-2} x & 1 & -\binom{n-1}{1} & \binom{n-1}{2} & \dots & (-1)^{n-2}\binom{n-1}{n-2} \\ (-1)^{n-3} + \cot^{2n-4} x & 0 & 1 & -\binom{n-2}{1} & \dots & (-1)^{n-3}\binom{n-2}{n-3} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 1 + \cot^2 x & 0 & 0 & 0 & \dots & 1 \end{pmatrix}.$$

Proof. Replacing n in (G) (Lemma 3) by $n-1, n-2, \dots, 1$, respectively we get, together with (G), the system of n equations in n variables:

$$\frac{1}{\sin^{2n} x}, \frac{1}{\sin^{2n-2} x}, \dots, \frac{1}{\sin^2 x}.$$

Such a system is a Cramer's system and the assertion follows by the Cramer's rule. □

Using (H) in the same manner as in Lemma 4 we obtain

Lemma 5. *Let $n \in \mathbb{N}_1$ and $x \in D_{\tan}$, then*

$$\frac{1}{\cos^{2n} x} = \det \begin{pmatrix} (-1)^{n-1} + \tan^{2n} x & -\binom{n}{1} & \binom{n}{2} & -\binom{n}{3} & \dots & (-1)^{n-1}\binom{n}{n-1} \\ (-1)^{n-2} + \tan^{2n-2} x & 1 & -\binom{n-1}{1} & \binom{n-1}{2} & \dots & (-1)^{n-2}\binom{n-1}{n-2} \\ (-1)^{n-3} + \tan^{2n-4} x & 0 & 1 & -\binom{n-2}{1} & \dots & (-1)^{n-3}\binom{n-2}{n-3} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 1 + \tan^2 x & 0 & 0 & 0 & \dots & 1 \end{pmatrix}.$$

To shorten notation from now on we set

$$\mu(a_n, a_{n-1}, \dots, a_1) = \det \begin{pmatrix} a_n & -\binom{n}{1} & \binom{n}{2} & -\binom{n}{3} & \dots & (-1)^{n-1}\binom{n}{n-1} \\ a_{n-1} & 1 & -\binom{n-1}{1} & \binom{n-1}{2} & \dots & (-1)^{n-2}\binom{n-1}{n-2} \\ a_{n-2} & 0 & 1 & -\binom{n-2}{1} & \dots & (-1)^{n-3}\binom{n-2}{n-3} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ a_1 & 0 & 0 & 0 & \dots & 1 \end{pmatrix},$$

thus the identities of Lemmas 4 and 5 can be written as

$$\frac{1}{\sin^{2n} x} = \mu \left((-1)^{n-1} + \cot^{2n} x, (-1)^{n-2} + \cot^{2n-2} x, \ldots, 1 + \cot^2 x\right) \tag{5}$$

and

$$\frac{1}{\cos^{2n} x} = \mu \left((-1)^{n-1} + \tan^{2n} x, (-1)^{n-2} + \tan^{2n-2} x, \ldots, 1 + \tan^2 x\right), \tag{6}$$

respectively.

Theorem 7. *For every $m \in \mathbb{N}_2$ and each $n \in \mathbb{N}_1$ the following identity holds true:*

$$\begin{aligned} \sum_{j=1}^{m-1} \sin^{-2n} \frac{\pi j}{2m} = \sum_{j=1}^{m-1} \cos^{-2n} \frac{\pi j}{2m} \qquad (7) \\ = \mu \left((-1)^{n-1}(m-1) + \sigma_{n,m}(A), (-1)^{n-2}(m-1) + \sigma_{n-1,m}(A)\right., \\ \left. \ldots, (m-1) + \sigma_{1,m}(A)\right), \end{aligned}$$

where the numbers $\sigma_{k,m}(A)$ for $k \in \{1, 2, \ldots, n\}$ are defined in Theorem 1.

Proof. In view of (5) we can write

$$\sin^{-2n} \frac{\pi j}{2m} = \mu \left((-1)^{n-1} + \cot^{2n} \frac{\pi j}{2m}, (-1)^{n-2} + \cot^{2n-2} \frac{\pi j}{2m}, \ldots, 1 + \cot^2 \frac{\pi j}{2m}\right)$$

for $j \in \{1, 2, \ldots, m-1\}$. This by the definition of μ, properties od determinants and Theorem 1 gives

$$\begin{aligned} &\sum_{j=1}^{m-1} \sin^{-2n} \frac{\pi j}{2m} \\ &= \mu \left(\sum_{j=1}^{m-1} \left((-1)^{n-1} + \cot^{2n} \frac{\pi j}{2m}\right), \sum_{j=1}^{m-1} \left((-1)^{n-2} + \cot^{2n-2} \frac{\pi j}{2m}\right), \right. \\ &\qquad \left. \ldots, \sum_{j=1}^{m-1} \left(1 + \cot^2 \frac{\pi j}{2m}\right)\right) \\ &= \mu \left((-1)^{n-1}(m-1) + \sigma_{n,m}(A), (-1)^{n-2}(m-1) + \sigma_{n-1,m}(A)\right., \\ &\qquad \left. \ldots, (m-1) + \sigma_{1,m}(A)\right). \end{aligned}$$

The same reasoning applies to the second identity. □

Analysis similar to that in the proof of Theorem 7 and the use of Theorems 2 – 6 give

Theorem 8. *For every $n, m \in \mathbb{N}_1$ the following identities holds true:*

$$\begin{aligned}&\sum_{j=0}^{m-1} \sin^{-2n} \frac{2j+1}{4m}\pi = \sum_{j=0}^{m-1} \cos^{-2n} \frac{2j+1}{4m}\pi \\ &= \mu\left((-1)^{n-1}m + \sigma_{n,m}(B), (-1)^{n-2}m + \sigma_{n-1,m}(B), \ldots, m + \sigma_{1,m}(B)\right),\end{aligned} \tag{8}$$

$$\begin{aligned}&\sum_{j=1}^{m} \sin^{-2n} \frac{\pi j}{2m+1} \\ &= \mu\left((-1)^{n-1}m + \sigma_{n,m}(D), (-1)^{n-2}m + \sigma_{n-1,m}(D), \ldots, m + \sigma_{1,m}(D)\right),\end{aligned} \tag{9}$$

$$\begin{aligned}&\sum_{j=1}^{m} \cos^{-2n} \frac{\pi j}{2m+1} \\ &= \mu\left((-1)^{n-1}m + \sigma_{n,m}(C), (-1)^{n-2}m + \sigma_{n-1,m}(C), \ldots, m + \sigma_{1,m}(C)\right),\end{aligned} \tag{10}$$

$$\begin{aligned}&\sum_{j=0}^{m-1} \sin^{-2n} \frac{(2j+1)\pi}{2(2m+1)} \\ &= \mu\left((-1)^{n-1}m + \sigma_{n,m}(F), (-1)^{n-2}m + \sigma_{n-1,m}(F), \ldots, m + \sigma_{1,m}(F)\right),\end{aligned} \tag{11}$$

$$\begin{aligned}&\sum_{j=0}^{m-1} \cos^{-2n} \frac{(2j+1)\pi}{2(2m+1)} \\ &= \mu\left((-1)^{n-1}m + \sigma_{n,m}(E), (-1)^{n-2}m + \sigma_{n-1,m}(E), \ldots, m + \sigma_{1,m}(E)\right),\end{aligned} \tag{12}$$

where $\sigma_{k,m}(B)$, $\sigma_{k,m}(C)$, $\sigma_{k,m}(D)$, $\sigma_{k,m}(E)$, $\sigma_{k,m}(F)$ for $k \in \{1, 2, \ldots, n\}$ are defined in Theorems 2 – 6.

Now we show that the general identities from Theorems 1 – 8 yield some particular equalities, including the one considered by the authors as remarkable.

Theorem 9. *If $m \in \mathbb{N}$, then*

$$\sum_{j=1}^{m-1} \sin^{-2} \frac{\pi j}{m} = \frac{m^2-1}{3}, \qquad \textit{provided } m \geq 2, \tag{13}$$

$$\sum_{j=1}^{m-1} \cot^{2} \frac{\pi j}{m} = \frac{(m-1)(m-2)}{3}, \qquad \textit{provided } m \geq 2, \tag{14}$$

$$\sum_{j=0}^{m-1} \sin^{-2} \frac{(2j+1)\pi}{2m} = m^2, \qquad \textit{provided } m \geq 1, \tag{15}$$

Proof. According to Theorem 1 we have

$$\sum_{j=1}^{m-1} \tan^2 \frac{\pi j}{2m} = \sum_{j=1}^{m-1} \cot^2 \frac{\pi j}{2m} = \frac{1}{2m}\binom{2m}{3}. \tag{16}$$

On the other hand, in view of

$$\tan^2 x + \cot^2 x = \frac{4}{\sin^2 2x} - 2$$

we get

$$\sum_{j=1}^{m-1} \tan^{2n} \frac{\pi j}{2m} + \sum_{j=1}^{m-1} \cot^{2n} \frac{\pi j}{2m} = 4\sum_{j=1}^{m-1} \sin^{-2} \frac{\pi j}{m} - 2(m-1).$$

Combining this with (16) gives

$$\sum_{j=1}^{m-1} \sin^{-2} \frac{\pi j}{m} = \frac{m^2-1}{3}$$

for $m \geq 2$. This proves (13).

To prove (14) notice that the identity

$$\cot^2 x - \frac{1}{\sin^2 x} = -1$$

yields

$$\sum_{j=1}^{m-1} \cot^2 \frac{\pi j}{m} - \sum_{j=1}^{m-1} \sin^{-2} \frac{\pi j}{m} = -(m-1), \qquad m \geq 2.$$

Thus by (13) we obtain (14).

Finally we show the remarkable (15). Theorem 8 leads to

$$\sum_{j=0}^{m-1} \sin^{-2} \frac{(2j+1)\pi}{4m} = \sum_{j=0}^{m-1} \cos^{-2} \frac{(2j+1)\pi}{4m} = m + \binom{2m}{2} = 2m^2 \tag{17}$$

for $m \geq 1$. Since

$$\frac{1}{\sin^2 x} + \frac{1}{\cos^2 x} = \frac{4}{\sin^2 2x}$$

we have

$$\sum_{j=0}^{m-1} \sin^{-2} \frac{(2j+1)\pi}{4m} + \sum_{j=0}^{m-1} \cos^{-2} \frac{(2j+1)\pi}{4m} = 4\sum_{j=0}^{m-1} \sin^{-2} \frac{(2j+1)\pi}{2m}, \qquad m \geq 1,$$

which by (17) implies (15), and the theorem follows. □

Next we use the the identities proved here to find the sums of the series of the form $\sum_{k=1}^{\infty} \frac{1}{k^{2n}}$, where $n \in \mathbb{N}_1$. We begin with the following lemma.

Lemma 6. *Let $n \in \mathbb{N}_1$, then expression $\sigma_{n,m}(A)$, defined in Theorem 1, is a value of some polynomial from $\mathbb{Q}[x]$, where $x = m$. The order of such a polynomial does not exceed $2n$.*

Proof. The proof is by induction on n. For $n = 1$ we have

$$\sigma_{1,m}(A) = \frac{1}{2m}\binom{2m}{3} = \frac{2}{3}m^2 - m - \frac{1}{3},$$

and the assertion follows. Fix $n \geq 2$ Assuming Lemma 6 to hold for any $k \in \mathbb{N}_1$, $k \leq n-1$ we prove it for n. By (2),

$$\sigma_{n,m}(A) = \sum_{j=1}^{n-1}(-1)^{j-1}\tau_j\sigma_{n-j,m}(A) - (-1)^n n\tau_n,$$

where $\tau_j = \frac{1}{2m}\binom{2m}{2j+1}$ for $j \in \{1, 2, \ldots, n\}$. Hence by the inductive assumption $\sigma_{n,m}(A)$ is a value of some polynomial from $\mathbb{Q}[x]$ of order not greater than $2n$ with $x = m$, as claimed. □

Theorem 10. *For every $n \in \mathbb{N}_1$,*

$$\sum_{j=1}^{\infty}\frac{1}{j^{2n}} = \lim_{m\to\infty}\frac{\pi^{2n}\sigma_{n,m}(A)}{(2m)^{2n}},$$

where $\sigma_{n,m}(A)$ is defined in Theorem 1.

Proof. Observe that

$$0 < \cot x < \frac{1}{x} < \frac{1}{\sin x} \qquad x \in \left(0, \frac{\pi}{2}\right),$$

thus

$$\cot^{2n}\frac{\pi j}{2m} < \left(\frac{2m}{\pi j}\right)^{2n} < \frac{1}{\sin^{2n}\frac{\pi j}{2m}}$$

and in consequence

$$\sum_{j=1}^{m-1}\cot^{2n}\frac{\pi j}{2m} < \left(\frac{2m}{\pi}\right)^{2n}\sum_{j=1}^{m-1}\frac{1}{j^{2n}} < \sum_{j=1}^{m-1}\frac{1}{\sin^{2n}\frac{\pi j}{2m}}$$

for $n \in \mathbb{N}_1$, $m \in \mathbb{N}_2$ and $j \in \{1, 2, \ldots, n\}$. By the definitions of $\sigma_{n,m}(A)$ and the function μ we have

$$\frac{\pi^{2n}\sigma_{n,m}(A)}{(2m)^{2n}} < \sum_{j=1}^{m-1}\frac{1}{j^{2n}} < \frac{\pi^{2n}}{(2m)^{2n}}\mu((-1)^{n-1}(m-1) + \sigma_{n,m}(A),$$
$$(-1)^{n-2}(m-1) + \sigma_{n-1,m}(A), \tag{18}$$
$$\ldots, m-1+\sigma_{1,m}(A)).$$

The formula for μ and the properties of determinants give

$$\begin{aligned}
&\mu\left((-1)^{n-1}(m-1)+\sigma_{n,m}(A),(-1)^{n-2}(m-1)+\sigma_{n-1,m}(A),\ldots,m-1+\sigma_{1,m}(A)\right)\\
&\quad= (m-1)\mu((-1)^{n-1},(-1)^{n-2},\ldots,(-1)^{n-n})+\mu(\sigma_{n,m}(A),\sigma_{n-1,m}(A),\ldots,\sigma_{1,m}(A))\\
&\quad= (m-1)C_1+\sigma_{n,m}(A)+C_2\sigma_{n-1,m}(A)+\ldots+C_n\sigma_{1,m}(A),
\end{aligned}$$

where $C_1,\ldots,C_n$ are constants depending on n. Hence by Lemma 6 and inequality (18) we obtain

$$\lim_{m\to\infty}\frac{\pi^{2n}\sigma_{n,m}(A)}{(2m)^{2n}}\le\sum_{j=1}^{\infty}\frac{1}{j^{2n}}\le\lim_{m\to\infty}\frac{\pi^{2n}\sigma_{n,m}(A)}{(2m)^{2n}},$$

which establishes the formula. □

Remark 1. Note that in the proof Theorem 10 (the last step of the proof) we have actually proved more, namely that the order of the polynomial from Lemma 6 equals exactly $2n$. Indeed, if it was not true, we would have

$$\lim_{m\to\infty}\frac{\pi^{2n}\sigma_{n,m}(A)}{(2m)^{2n}}=0$$

and consequently

$$\sum_{j=1}^{\infty}\frac{1}{j^{2n}}\le 0,$$

which is impossible.

Remark 2. Treating $\sigma_{n,m}(A)$ as a polynomial in m of order $2n$ we have

$$\lim_{m\to\infty}\frac{\pi^{2n}\sigma_{n,m}(A)}{(2m)^{2n}}=a_{2n}\frac{\pi^{2n}}{4^n},$$

where a_{2n} denotes the leading coefficient of $\sigma_{n,m}(A)$. On the other hand,

$$B_{2n}\frac{2^{2n-1}\pi^{2n}}{(2n)!}(-1)^{n-1}=\sum_{j=1}^{\infty}\frac{1}{j^{2n}},\qquad n\in\mathbb{N}_1,$$

where B_{2n} stands for the $2n$-th Bernoulli number (see [4], p.320). Thus we get the following relation between Bernoulli numbers and the coefficients of $\sigma_{n,m}(A)$

$$a_{2n}=B_{2n}\frac{2^{4n-1}}{(2n)!}(-1)^{n-1},\qquad n\in\mathbb{N}_1.$$

Remark 3. Similarly as Theorem 10 one can show that

$$\sum_{j=1}^{\infty}\frac{1}{j^{2n}}=\lim_{m\to\infty}\frac{\pi^{2n}\sigma_{n,m}(B)}{(2m)^{2n}}=\lim_{m\to\infty}\frac{\pi^{2n}\sigma_{n,m}(D)}{(2m)^{2n}}=\lim_{m\to\infty}\frac{\pi^{2n}\sigma_{n,m}(F)}{(2m)^{2n}},\qquad n\in\mathbb{N}_1.$$

where $\sigma_{n,m}(B)$, $\sigma_{n,m}(D)$ and $\sigma_{n,m}(F)$ are defined in Theorems 2, 4 and 6, respectively.

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