Journal of
Mathematics
and Applications
JMA No 38, pp 133-150 (2015)

The random of lacunary statistical on χ^2 over p-metric spaces defined by Musielak

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ABSTRACT: Mursaleen introduced the concepts of statistical convergence in random 2-normed spaces. Recently Mohiuddine and Aiyup defined the notion of lacunary statistical convergence and lacunary statistical Cauchy in random 2-normed spaces. In this paper, we define and study the notion of lacunary statistical convergence and lacunary of statistical Cauchy sequences in random on χ^2 over $p-$ metric spaces defined by Musielak and prove some theorems which generalizes Mohiuddine and Aiyup results.

AMS Subject Classification: *analytic sequence, double sequences, χ^2 space, Musielak - modulus function, Random $p-$ metric space, Lacunary sequence, Statistical convergence*
Keywords and Phrases: *40A05,40C05,40D05*

1 Introduction

The concept of statistical convergence play a vital role not only in pure mathematics but also in other branches of science involving mathematics, especially in information theory, computer science, biological science, dynamical systems, geographic information systems, population modeling, and motion planning in robotics.

The notion of statistical convergence was introduced by Fast and Schoenberg independently. Over the years and under different names statistical convergence has been discussed in the theory of fourier analysis, ergodic theory and number the- ory. Later on it was further investigated by Fridy , Šalát , Çakalli , Maio and Kocinac , Miller , Maddox , Leindler , Mursaleen and Alotaibi , Mursaleen and Edely , and many others. In the recent years, generalizations of statistical convergence have appeared in the study of strong integral summability and the structure of ideals of bounded continuous func- tions on Stone-Čech compactification of the natural numbers. Moreover statistical convergence is closely related to the concept of convergence in probability.

Throughout w, χ and Λ denote the classes of all, gai and analytic scalar valued

single sequences, respectively.
We write w^2 for the set of all complex sequences (x_{mn}), where $m, n \in \mathbb{N}$, the set of positive integers. Then, w^2 is a linear space under the coordinate wise addition and scalar multiplication.

Some initial works on double sequence spaces is found in Bromwich [2]. Later on, they were investigated by Hardy [3], Moricz [6], Moricz and Rhoades [7], Basarir and Solankan [1], Tripathy [8], Turkmenoglu [9], and many others.

We procure the following sets of double sequences:

$$\mathcal{M}_u(t) := \left\{(x_{mn}) \in w^2 : sup_{m,n\in N} |x_{mn}|^{t_{mn}} < \infty\right\},$$

$$\mathcal{C}_p(t) := \left\{(x_{mn}) \in w^2 : p - \lim_{m,n\to\infty} |x_{mn} - l|^{t_{mn}} = 1 \, for \, some \, l \in \mathbb{C}\right\},$$

$$\mathcal{C}_{0p}(t) := \left\{(x_{mn}) \in w^2 : p - \lim_{m,n\to\infty} |x_{mn}|^{t_{mn}} = 1\right\},$$

$$\mathcal{L}_u(t) := \left\{(x_{mn}) \in w^2 : \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} |x_{mn}|^{t_{mn}} < \infty\right\},$$

$$\mathcal{C}_{bp}(t) := \mathcal{C}_p(t) \bigcap \mathcal{M}_u(t) \text{ and } \mathcal{C}_{0bp}(t) = \mathcal{C}_{0p}(t) \bigcap \mathcal{M}_u(t);$$

where $t = (t_{mn})$ is the sequence of strictly positive reals t_{mn} for all $m, n \in \mathbb{N}$ and $p - lim_{m,n\to\infty}$ denotes the limit in the Pringsheim's sense. In the case $t_{mn} = 1$ for all $m, n \in \mathbb{N}; \mathcal{M}_u(t), \mathcal{C}_p(t), \mathcal{C}_{0p}(t), \mathcal{L}_u(t), \mathcal{C}_{bp}(t)$ and $\mathcal{C}_{0bp}(t)$ reduce to the sets $\mathcal{M}_u, \mathcal{C}_p, \mathcal{C}_{0p}, \mathcal{L}_u, \mathcal{C}_{bp}$ and $\mathcal{C}_{0bp}$, respectively. Now, we may summarize the knowledge given in some document related to the double sequence spaces. Gökhan and Colak [11,12] have proved that $\mathcal{M}_u(t)$ and $\mathcal{C}_p(t), \mathcal{C}_{bp}(t)$ are complete paranormed spaces of double sequences and gave the $\alpha-, \beta-, \gamma-$ duals of the spaces $\mathcal{M}_u(t)$ and $\mathcal{C}_{bp}(t)$. Quite recently, in her PhD thesis, Zelter [13] has essentially studied both the theory of topological double sequence spaces and the theory of summability of double sequences. Mursaleen and Edely [14] have independently introduced the statistical convergence and Cauchy for double sequences and given the relation between statistical convergent and strongly Cesàro summable double sequences. Altay and Basar [15] have defined the spaces $\mathcal{BS}, \mathcal{BS}(t), \mathcal{CS}_p, \mathcal{CS}_{bp}, \mathcal{CS}_r$ and $\mathcal{BV}$ of double sequences consisting of all double series whose sequence of partial sums are in the spaces $\mathcal{M}_u, \mathcal{M}_u(t), \mathcal{C}_p, \mathcal{C}_{bp}, \mathcal{C}_r$ and $\mathcal{L}_u$, respectively, and also examined some properties of those sequence spaces and determined the $\alpha-$ duals of the spaces $\mathcal{BS}, \mathcal{BV}, \mathcal{CS}_{bp}$ and the $\beta(\vartheta)-$ duals of the spaces $\mathcal{CS}_{bp}$ and $\mathcal{CS}_r$ of double series. Basar and Sever [16] have introduced the Banach space $\mathcal{L}_q$ of double sequences corresponding to the well-known space ℓ_q of single sequences and examined some properties of the space $\mathcal{L}_q$. Quite recently Subramanian and Misra [17] have studied the space $\chi^2_M(p, q, u)$ of double sequences and gave some inclusion relations.

The class of sequences which are strongly Cesàro summable with respect to a modulus was introduced by Maddox [5] as an extension of the definition of strongly Cesàro summable sequences. Cannor [18] further extended this definition to a definition of strong $A-$ summability with respect to a modulus where $A = (a_{n,k})$ is a nonnegative regular matrix and established some connections between strong $A-$ summability, strong $A-$ summability with respect to a modulus, and $A-$ statistical convergence. In [19] the notion of convergence of double sequences was presented by A. Pringsheim. Also, in [20]-[21], and [22] the four dimensional matrix transformation $(Ax)_{k,\ell} = \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} a_{k\ell}^{mn} x_{mn}$ was studied extensively by Robison and Hamilton.

We need the following inequality in the sequel of the paper. For $a, b, \geq 0$ and $0 < p < 1$, we have

$$(a+b)^p \leq a^p + b^p \tag{1.1}$$

The double series $\sum_{m,n=1}^{\infty} x_{mn}$ is called convergent if and only if the double sequence (s_{mn}) is convergent, where $s_{mn} = \sum_{i,j=1}^{m,n} x_{ij} (m, n \in \mathbb{N})$.

A sequence $x = (x_{mn})$is said to be double analytic if $sup_{mn} |x_{mn}|^{1/m+n} < \infty$. The vector space of all double analytic sequences will be denoted by Λ^2. A sequence $x = (x_{mn})$ is called double gai sequence if $((m+n)! |x_{mn}|)^{1/m+n} \to 0$ as $m, n \to \infty$. The double gai sequences will be denoted by χ^2. Let $\phi = \{all finite sequences\}$.

Consider a double sequence $x = (x_{ij})$. The $(m,n)^{th}$ section $x^{[m,n]}$ of the sequence is defined by $x^{[m,n]} = \sum_{i,j=0}^{m,n} x_{ij} \Im_{ij}$ for all $m, n \in \mathbb{N}$; where $\Im_{ij}$ denotes the double sequence whose only non zero term is a $\frac{1}{(i+j)!}$ in the $(i,j)^{th}$ place for each $i, j \in \mathbb{N}$.

Let M and Φ are mutually complementary modulus functions. Then, we have:
(i) For all $u, y \geq 0$,

$$uy \leq M(u) + \Phi(y), (Young's\, inequality)[See[10]] \tag{1.2}$$

(ii) For all $u \geq 0$,

$$u\eta(u) = M(u) + \Phi(\eta(u)). \tag{1.3}$$

(iii) For all $u \geq 0$, and $0 < \lambda < 1$,

$$M(\lambda u) \leq \lambda M(u) \tag{1.4}$$

Lindenstrauss and Tzafriri [4] used the idea of Orlicz function to construct Orlicz sequence space

$$\ell_M = \left\{x \in w : \sum_{k=1}^{\infty} M\left(\frac{|x_k|}{\rho}\right) < \infty,\ for\, some\, \rho > 0\right\},$$

The space ℓ_M with the norm

$$\|x\| = inf\left\{\rho > 0 : \sum_{k=1}^{\infty} M\left(\frac{|x_k|}{\rho}\right) \leq 1\right\},$$

becomes a Banach space which is called an Orlicz sequence space. For $M(t) = t^p\,(1 \leq p < \infty)$, the spaces ℓ_M coincide with the classical sequence space ℓ_p.

A sequence $f = (f_{mn})$ of modulus function is called a Musielak-modulus function. A sequence $g = (g_{mn})$ defined by

$$g_{mn}(v) = sup\{|v|\,u - (f_{mn})(u) : u \geq 0\}, m, n = 1, 2, \cdots$$

is called the complementary function of a Musielak-modulus function f. For a given Musielak modulus function f, the Musielak-modulus sequence space t_f is defined as follows

$$t_f = \left\{ x \in w^2 : I_f(|x_{mn}|)^{1/m+n} \to 0\, as\, m, n \to \infty \right\},$$

where I_f is a convex modular defined by

$$I_f(x) = \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} f_{mn}(|x_{mn}|)^{1/m+n}, x = (x_{mn}) \in t_f.$$

We consider t_f equipped with the Luxemburg metric

$$d(x, y) = sup_{mn} \left\{ inf \left(\sum_{m=1}^{\infty} \sum_{n=1}^{\infty} f_{mn} \left(\frac{|x_{mn}|^{1/m+n}}{mn} \right) \right) \leq 1 \right\}$$

If X is a sequence space, we give the following definitions:

(i)$X^{'}$= the continuous dual of X;

(ii)$X^{\alpha} = \left\{ a = (a_{mn}) : \sum_{m,n=1}^{\infty} |a_{mn} x_{mn}| < \infty,\, for\, each\, x \in X \right\};$

(iii)$X^{\beta} = \left\{ a = (a_{mn}) : \sum_{m,n=1}^{\infty} a_{mn} x_{mn}\, is\, convegent,\, foreach\, x \in X \right\};$

(iv)$X^{\gamma} = \left\{ a = (a_{mn}) : sup_{mn} \geq 1 \left| \sum_{m,n=1}^{M,N} a_{mn} x_{mn} \right| < \infty, foreachx \in X \right\};$

(v)$let\, X\, beanFK - space \supset \phi;\, then\, X^{f} = \left\{ f(\Im_{mn}) : f \in X^{'} \right\};$

(vi)$X^{\delta} = \left\{ a = (a_{mn}) : sup_{mn} |a_{mn} x_{mn}|^{1/m+n} < \infty,\, foreach\, x \in X \right\};$

$X^{\alpha}.X^{\beta}, X^{\gamma}$ are called $\alpha - (orK\ddot{o}the - Toeplitz)$dual of $X, \beta - (or\, generalized - K\ddot{o}the - Toeplitz)\, dual\, of X, \gamma -\, dual\, of\, X,\, \delta -\, dual\, of X\, respectively. X^{\alpha}$ is defined by Gupta and Kamptan [10]. It is clear that $X^{\alpha} \subset X^{\beta}$ and $X^{\alpha} \subset X^{\gamma}$, but $X^{\beta} \subset X^{\gamma}$ does not hold, since the sequence of partial sums of a double convergent series need not to be bounded.

The notion of difference sequence spaces (for single sequences) was introduced by Kizmaz as follows

$$Z(\Delta) = \{x = (x_k) \in w : (\Delta x_k) \in Z\}$$

for $Z = c, c_0$ and ℓ_∞, where $\Delta x_k = x_k - x_{k+1}$ for all $k \in \mathbb{N}$.
Here c, c_0 and ℓ_∞ denote the classes of convergent,null and bounded sclar valued single sequences respectively. The difference sequence space bv_p of the classical space ℓ_p is introduced and studied in the case $1 \leq p \leq \infty$ by Başar and Altay and in the case $0 < p < 1$ by Altay and Başar in [15]. The spaces $c(\Delta), c_0(\Delta), \ell_\infty(\Delta)$ and bv_p are Banach spaces normed by

$$\|x\| = |x_1| + sup_{k \geq 1} |\Delta x_k| \text{ and } \|x\|_{bv_p} = \left(\sum_{k=1}^{\infty} |x_k|^p \right)^{1/p}, (1 \leq p < \infty).$$

Later on the notion was further investigated by many others. We now introduce the following difference double sequence spaces defined by

$$Z(\Delta) = \left\{ x = (x_{mn}) \in w^2 : (\Delta x_{mn}) \in Z \right\}$$

where $Z = \Lambda^2, \chi^2$ and $\Delta x_{mn} = (x_{mn} - x_{mn+1}) - (x_{m+1n} - x_{m+1n+1}) = x_{mn} - x_{mn+1} - x_{m+1n} + x_{m+1n+1}$ for all $m, n \in \mathbb{N}$. The generalized difference double notion has the following representation: $\Delta^m x_{mn} = \Delta^{m-1} x_{mn} - \Delta^{m-1} x_{mn+1} - \Delta^{m-1} x_{m+1n} + \Delta^{m-1} x_{m+1n+1}$, and also this generalized difference double notion has the following binomial representation:

$$\Delta^m x_{mn} = \sum_{i=0}^{m} \sum_{j=0}^{m} (-1)^{i+j} \binom{m}{i} \binom{m}{j} x_{m+i,n+j}.$$

2 Definition and Preliminaries

Let $n \in \mathbb{N}$ and X be a real vector space of dimension w, where $n \leq w$. A real valued function $d_p(x_1, \ldots, x_n) = \|(d_1(x_1), \ldots, d_n(x_n))\|_p$ on X satisfying the following four conditions:
(i) $\|(d_1(x_1), \ldots, d_n(x_n))\|_p = 0$ if and and only if $d_1(x_1), \ldots, d_n(x_n)$ are linearly dependent,
(ii) $\|(d_1(x_1), \ldots, d_n(x_n))\|_p$ is invariant under permutation,
(iii) $\|(\alpha d_1(x_1), \ldots, d_n(x_n))\|_p = |\alpha| \, \|(d_1(x_1), \ldots, d_n(x_n))\|_p, \alpha \in \mathbb{R}$
(iv) $d_p\left((x_1, y_1), (x_2, y_2) \cdots (x_n, y_n)\right) = \left(d_X(x_1, x_2, \cdots x_n)^p + d_Y(y_1, y_2, \cdots y_n)^p\right)^{1/p}$ for $1 \leq p < \infty$; (or)
(v) $d\left((x_1, y_1), (x_2, y_2), \cdots (x_n, y_n)\right) := \sup\left\{d_X(x_1, x_2, \cdots x_n), d_Y(y_1, y_2, \cdots y_n)\right\}$,
for $x_1, x_2, \cdots x_n \in X, y_1, y_2, \cdots y_n \in Y$ is called the p product metric of the Cartesian product of n metric spaces is the p norm of the n-vector of the norms of the n subspaces.

A trivial example of p product metric of n metric space is the p norm space is $X = \mathbb{R}$ equipped with the following Euclidean metric in the product space is the p norm:

$$\begin{aligned} \|(d_1(x_1), \ldots, d_n(x_n))\|_E &= sup\left(|det(d_{mn}(x_{mn}))|\right) \\ &= sup \left(\begin{vmatrix} d_{11}(x_{11}) & d_{12}(x_{12}) & \ldots & d_{1n}(x_{1n}) \\ d_{21}(x_{21}) & d_{22}(x_{22}) & \ldots & d_{2n}(x_{1n}) \\ . & & & \\ . & & & \\ . & & & \\ d_{n1}(x_{n1}) & d_{n2}(x_{n2}) & \ldots & d_{nn}(x_{nn}) \end{vmatrix} \right) \end{aligned}$$

where $x_i = (x_{i1}, \cdots x_{in}) \in \mathbb{R}^n$ for each $i = 1, 2, \cdots n$.
If every Cauchy sequence in X converges to some $L \in X$, then X is said to be complete with respect to the $p-$ metric. Any complete $p-$ metric space is said to be $p-$ Banach metric space.

Let X be a linear metric space. A function $w : X \to \mathbb{R}$ is called paranorm, if
(1) $w(x) \geq 0$, for all $x \in X$;
(2) $w(-x) = w(x)$, for all $x \in X$;
(3) $w(x+y) \leq w(x) + w(y)$, for all $x, y \in X$;
(4) If (σ_{mn}) is a sequence of scalars with $\sigma_{mn} \to \sigma$ as $m, n \to \infty$ and (x_{mn}) is a sequence of vectors with $w(x_{mn} - x) \to 0$ as $m, n \to \infty$, then $w(\sigma_{mn} x_{mn} - \sigma x) \to 0$ as $m, n \to \infty$.
A paranorm w for which $w(x) = 0$ implies $x = 0$ is called total paranorm and the pair (X, w) is called a total paranormed space. It is well known that the metric of any linear metric space is given by some total paranorm (see [23], Theorem 10.4.2, p.183).

By the convergence of a double sequence we mean the convergence on the Pringsheim sense that is, a double sequence $x = (x_{mn})$ has Prinsheim limit L (denoted by $P - limx = L$) provided that given $\epsilon > 0$ there exists $n \in \mathbb{N}$ such that $|x_{mn} - L| < \epsilon$. We shall write more briefly as $P-$ convergent.

The double sequence $\theta_{rs} = \{(m_r, n_s)\}$ is called double lacunary sequence if there exist two increasing of integers such that

$$m_0 = 0, \varphi_r = m_r - m_{r-1} \to \infty \text{ as } r \to \infty \text{ and}$$

$$n_0 = 0, \varphi_s = n_s - n_{s-1} \to \infty \text{ as } s \to \infty.$$

Notations: $m_{rs} = m_r n_s, h_{rs} = \varphi_r \bar{\varphi}_s, \theta_{rs}$ is determined by

$$I_{rs} = \{(m,n) : m_{r-1} < m \leq m_r \, and \, n_{s-1} < n \leq n_s\},$$

$$q_r = \tfrac{m_r}{m_{r-1}}, \bar{q_s} = \tfrac{n_s}{n_{s-1}} \text{ and } q_{rs} = q_r \bar{q_s}.$$

The notion of $\lambda-$ double gai and double analytic sequences as follows: Let $\lambda = (\lambda_{mn})_{m,n=0}^{\infty}$ be a strictly increasing sequences of positive real numbers tending to infinity, that is

$$0 < \lambda_{00} < \lambda_{11} < \cdots \quad \text{and } \lambda_{mn} \to \infty \, as \, m, n \to \infty$$

and said that a sequence $x = (x_{mn}) \in w^2$ is $\lambda-$ convergent to 0, called a the $\lambda-$ limit of x, if $\mu_{mn}(x) \to 0 \, as \, m, n \to \infty$, where

$$\begin{aligned} \mu_{mn}(x) &= \frac{1}{\varphi_{rs}} \sum_{m \in I_{rs}} \sum_{n \in I_{rs}} (\Delta^{m-1} \lambda_{m,n} - \Delta^{m-1} \lambda_{m,n+1} - \\ & \quad \Delta^{m-1} \lambda_{m+1,n} + \Delta^{m-1} \lambda_{m+1,n+1}) |x_{mn}|^{1/m+n}. \end{aligned}$$

The sequence $x = (x_{mn}) \in w^2$ is $\lambda-$ double analytic if $sup_{uv} |\mu_{mn}(x)| < \infty$. If $\lim_{mn} x_{mn} = 0$ in the ordinary sense of convergence, then

$$\lim_{mn} \left(\frac{1}{\varphi_{rs}} \sum_{m \in I_{rs}} \sum_{n \in I_{rs}} \left(\Delta^{m-1} \lambda_{m,n} - \Delta^{m-1} \lambda_{m,n+1} - \Delta^{m-1} \lambda_{m+1,n} + \Delta^{m-1} \lambda_{m+1,n+1} \right) \right.$$
$$\left. ((m+n)! \, |x_{mn} - 0|)^{1/m+n} \right) = 0.$$

This implies that

$$\lim_{mn} |\mu_{mn}(x) - 0| = \lim_{mn} \left| \left(\frac{1}{\varphi_{rs}} \sum_{m \in I_{rs}} \sum_{n \in I_{rs}} \left(\Delta^{m-1}\lambda_{m,n} - \Delta^{m-1}\lambda_{m,n+1} \right.\right.\right.$$
$$\left.\left.\left. -\Delta^{m-1}\lambda_{m+1,n} + \Delta^{m-1}\lambda_{m+1,n+1} \right) \left((m+n)! \, \|x_{mn} - 0\| \right)^{1/m+n} \right) \right| = 0.$$

which yields that $\lim_{uv} \mu_{mn}(x) = 0$ and hence $x = (x_{mn}) \in w^2$ is $\lambda-$ convergent to 0.

Let I^2- be an admissible ideal of $2^{\mathbb{N}\times\mathbb{N}}, \theta_{rs}$ be a double lacunary sequence, $f = (f_{mn})$ be a Musielak-modulus function and $\left(X, \|(d(x_1), d(x_2), \cdots, d(x_{n-1}))\|_p\right)$ be a $p-$metric space, $q = (q_{mn})$ be double analytic sequence of strictly positive real numbers. By $w^2(p-X)$ we denote the space of all sequences defined over $\left(X, \|(d(x_1), d(x_2), \cdots, d(x_{n-1}))\|_p\right)$. The following inequality will be used throughout the paper. If $0 \le q_{mn} \le supq_{mn} = H, K = max\left(1, 2^{H-1}\right)$ then

$$|a_{mn} + b_{mn}|^{q_{mn}} \le K \left\{ |a_{mn}|^{q_{mn}} + |b_{mn}|^{q_{mn}} \right\} \tag{2.1}$$

for all m, n and $a_{mn}, b_{mn} \in \mathbb{C}$. Also $|a|^{q_{mn}} \le max\left(1, |a|^H\right)$ for all $a \in \mathbb{C}$.
In the present paper we define the following sequence spaces:

$$\begin{aligned} &\left[\chi^{2q}_{f\mu}, \|(d(x_1), d(x_2), \cdots, d(x_{n-1}))\|^{\varphi}_p\right]^{I^2}_{\theta_{rs}} \\ &= \left\{ r, s \in I_{rs} : \left[f_{mn}\left(\|\mu_{mn}(x), (d(x_1), d(x_2), \cdots, d(x_{n-1}))\|_p \right) \right]^{q_{mn}} \ge \epsilon \right\} \\ &\in I^2 \end{aligned}$$

$$\begin{aligned} &\left[\Lambda^{2q}_{f\mu}, \|(d(x_1), d(x_2), \cdots, d(x_{n-1}))\|^{\varphi}_p\right]^{I^2}_{\theta_{rs}} \\ &= \left\{ r, s \in I_{rs} : \left[f_{mn}\left(\|\mu_{mn}(x), (d(x_1), d(x_2), \cdots, d(x_{n-1}))\|_p \right) \right]^{q_{mn}} \ge K \right\} \\ &\in I^2, \end{aligned}$$

If we take $f_{mn}(x) = x$, we get

$$\begin{aligned} &\left[\chi^{2q}_{f\mu}, \|(d(x_1), d(x_2), \cdots, d(x_{n-1}))\|^{\varphi}_p\right]^{I^2}_{\theta_{rs}} \\ &= \left\{ r, s \in I_{rs} : \left[\left(\|\mu_{mn}(x), (d(x_1), d(x_2), \cdots, d(x_{n-1}))\|_p \right) \right]^{q_{mn}} \ge \epsilon \right\} \\ &\in I^2, \end{aligned}$$

$$\begin{aligned} &\left[\Lambda^{2q}_{f\mu}, \|(d(x_1), d(x_2), \cdots, d(x_{n-1}))\|^{\varphi}_p\right]^{I^2}_{\theta_{rs}} \\ &= \left\{ r, s \in I_{rs} : \left[\left(\|\mu_{mn}(x), (d(x_1), d(x_2), \cdots, d(x_{n-1}))\|_p \right) \right]^{q_{mn}} \ge K \right\} \\ &\in I^2, \end{aligned}$$

If we take $q = (q_{mn}) = 1$, we get

$$\begin{aligned} &\left[\chi^2_{f\mu}, \|(d(x_1), d(x_2), \cdots, d(x_{n-1}))\|_p^{\varphi}\right]^I_{\theta_{rs}} \\ &= \left\{r, s \in I_{rs} : \left[f_{mn}\left(\|\mu_{mn}(x), (d(x_1), d(x_2), \cdots, d(x_{n-1}))\|_p\right)\right] \geq \epsilon\right\} \\ &\in I^2 \end{aligned}$$

$$\begin{aligned} &, \left[\Lambda^2_{f\mu}, \|(d(x_1), d(x_2), \cdots, d(x_{n-1}))\|_p^{\varphi}\right]^{I^2}_{\theta_{rs}} | \\ &= \left\{r, s \in I_{rs} : \left[f_{mn}\left(\|\mu_{mn}(x), (d(x_1), d(x_2), \cdots, d(x_{n-1}))\|_p\right)\right] \geq K\right\} \\ &\in I^2, \end{aligned}$$

In the present paper we plan to study some topological properties and inclusion relation between the above defined sequence spaces.

$$\left[\chi^{2q}_{f\mu}, \|(d(x_1), d(x_2), \cdots, d(x_{n-1}))\|_p^{\varphi}\right]^{I^2}_{\theta_{rs}}$$

and

$$\left[\Lambda^{2q}_{f\mu}, \|(d(x_1), d(x_2), \cdots, d(x_{n-1}))\|_p^{\varphi}\right]^{I^2}_{\theta_{rs}}$$

which we shall discuss in this paper.

3 Main Results

3.1 Theorem

Let $f = (f_{mn})$ be a Musielak-modulus function, $q = (q_{mn})$ be a double analytic sequence of strictly positive real numbers, the sequence spaces

$$\left[\chi^{2q}_{f\mu}, \|(d(x_1), d(x_2), \cdots, d(x_{n-1}))\|_p^{\varphi}\right]^{I^2}_{\theta_{rs}}$$

and $\left[\Lambda^{2q}_{f\mu}, \|(d(x_1), d(x_2), \cdots, d(x_{n-1}))\|_p^{\varphi}\right]^{I^2}_{\theta_{rs}}$ are linear spaces.

Proof: It is routine verification. Therefore the proof is omitted.

3.2 Theorem

Let $f = (f_{mn})$ be a Musielak-modulus function, $q = (q_{mn})$ be a double analytic sequence of strictly positive real numbers, the sequence space

$$\left[\chi^{2q}_{f\mu}, \|(d(x_1), d(x_2), \cdots, d(x_{n-1}))\|_p^{\varphi}\right]^{I^2}_{\theta_{rs}}$$

is a paranormed space with respect to the paranorm defined by
$g(x) = inf$
$\left\{\left[f_{mn}\left(\|\mu_{mn}(x), (d(x_1), d(x_2), \cdots, d(x_{n-1}))\|_p\right)\right]^{q_{mn}} \leq 1\right\}$.

Proof: Clearly $g(x) \geq 0$ for

$$x = (x_{mn}) \in \left[\chi^{2q}_{f\mu}, \|(d(x_1), d(x_2), \cdots, d(x_{n-1}))\|^{\varphi}_p\right]^{I^2}_{\theta_{rs}}$$

Since $f_{mn}(0) = 0$, we get $g(0) = 0$.

Conversely, suppose that $g(x) = 0$, then

$$inf\left\{\left[f_{mn}\left(\|\mu_{mn}(x), (d(x_1), d(x_2), \cdots, d(x_{n-1}))\|_p\right)\right]^{q_{mn}} \leq 1\right\} = 0.$$

Suppose that $\mu_{mn}(x) \neq 0$ for each $m, n \in \mathbb{N}$. Then

$$\|\mu_{mn}(x), (d(x_1), d(x_2), \cdots, d(x_{n-1}))\|^{\varphi}_p \to \infty.$$

It follows that

$$\left(\left[f_{mn}\left(\|\mu_{mn}(x), (d(x_1), d(x_2), \cdots, d(x_{n-1}))\|_p\right)\right]^{q_{mn}}\right)^{1/H} \to \infty$$

which is a contradiction. Therefore $\mu_{mn}(x) = 0$. Let

$$\left(\left[f_{mn}\left(\|\mu_{mn}(x), (d(x_1), d(x_2), \cdots, d(x_{n-1}))\|_p\right)\right]^{q_{mn}}\right)^{1/H} \leq 1$$

and

$$\left(\left[f_{mn}\left(\|\mu_{mn}(y), (d(x_1), d(x_2), \cdots, d(x_{n-1}))\|_p\right)\right]^{q_{mn}}\right)^{1/H} \leq 1$$

Then by using Minkowski's inequality, we have

$$\begin{aligned}
&\left(\left[f_{mn}\left(\|\mu_{mn}(x+y), (d(x_1), d(x_2), \cdots, d(x_{n-1}))\|_p\right)\right]^{q_{mn}}\right)^{1/H} \\
&\leq \left(\left[f_{mn}\left(\|\mu_{mn}(x), (d(x_1), d(x_2), \cdots, d(x_{n-1}))\|_p\right)\right]^{q_{mn}}\right)^{1/H} \\
&\quad + \left(\left[f_{mn}\left(\|\mu_{mn}(y), (d(x_1), d(x_2), \cdots, d(x_{n-1}))\|_p\right)\right]^{q_{mn}}\right)^{1/H}.
\end{aligned}$$

So we have

$$\begin{aligned}
g(x+y) &= inf\left\{\left[f_{mn}\left(\|\mu_{mn}(x+y), (d(x_1), d(x_2), \cdots, d(x_{n-1}))\|_p\right)\right]^{q_{mn}} \leq 1\right\} \\
&\leq \inf\left\{\left[f_{mn}\left(\|\mu_{mn}(x), (d(x_1), d(x_2), \cdots, d(x_{n-1}))\|_p\right)\right]^{q_{mn}} \leq 1\right\} \\
&\quad + \inf\left\{\left[f_{mn}\left(\|\mu_{mn}(y), (d(x_1), d(x_2), \cdots, d(x_{n-1}))\|_p\right)\right]^{q_{mn}} \leq 1\right\}
\end{aligned}$$

Therefore,

$$g\left(x+y\right)\leq g\left(x\right)+g\left(y\right).$$

Finally, to prove that the scalar multiplication is continuous. Let λ be any complex number. By definition,

$$g\left(\lambda x\right)=inf\left\{\left[f_{mn}\left(\left\|\mu_{mn}\left(\lambda x\right),\left(d\left(x_1\right),d\left(x_2\right),\cdots,d\left(x_{n-1}\right)\right)\right\|_p\right)\right]^{q_{mn}}\leq 1\right\}.$$

Then

$$g\left(\lambda x\right)=\inf\left\{\left(\left(\left|\lambda\right|t\right)^{q_{mn}/H}:\left[f_{mn}\left(\left\|\mu_{mn}\left(\lambda x\right),\left(d\left(x_1\right),d\left(x_2\right),\cdots,d\left(x_{n-1}\right)\right)\right\|_p\right)\right]^{q_{mn}}\leq 1\right\}$$

where $t=\frac{1}{|\lambda|}$. Since $|\lambda|^{q_{mn}}\leq max\left(1,|\lambda|^{supq_{mn}}\right)$, we have

$$g\left(\lambda x\right)\leq max\left(1,|\lambda|^{supq_{mn}}\right)$$

$$\inf\left\{t^{q_{mn}/H}:\left[f_{mn}\left(\left\|\mu_{mn}\left(\lambda x\right),\left(d\left(x_1\right),d\left(x_2\right),\cdots,d\left(x_{n-1}\right)\right)\right\|_p\right)\right]^{q_{mn}}\leq 1\right\}$$

This completes the proof.

3.3 Theorem

(i) If the Musielak modulus function (f_{mn}) satisfies Δ_2- condition, then

$$\begin{aligned}&\left[\chi^{2q}_{f\mu},\left\|\mu_{mn}\left(x\right),\left(d\left(x_1\right),d\left(x_2\right),\cdots,d\left(x_{n-1}\right)\right)\right\|^{\varphi}_p\right]^{I^{2\alpha}}_{\theta_{rs}}\\ =\;&\left[\chi^{2q\mu}_{g},\left\|\mu_{uv}\left(x\right),\left(d\left(x_1\right),d\left(x_2\right),\cdots,d\left(x_{n-1}\right)\right)\right\|^{\varphi}_p\right]^{I^{2}}_{\theta_{rs}}.\end{aligned}$$

(ii) If the Musielak modulus function (g_{mn}) satisfies Δ_2- condition, then

$$\begin{aligned}&\left[\chi^{2q\mu}_{g},\left\|\mu_{mn}\left(x\right),\left(d\left(x_1\right),d\left(x_2\right),\cdots,d\left(x_{n-1}\right)\right)\right\|^{\varphi}_p\right]^{I^{2\alpha}}_{\theta_{rs}}\\ =\;&\left[\chi^{2q}_{f\mu},\left\|\mu_{mn}\left(x\right),\left(d\left(x_1\right),d\left(x_2\right),\cdots,d\left(x_{n-1}\right)\right)\right\|^{\varphi}_p\right]^{I^{2}}_{\theta_{rs}}\end{aligned}$$

Proof: Let the Musielak modulus function (f_{mn}) satisfies Δ_2- condition, we get

$$\begin{aligned}&\left[\chi^{2q\mu}_{g},\left\|\mu_{mn}\left(x\right),\left(d\left(x_1\right),d\left(x_2\right),\cdots,d\left(x_{n-1}\right)\right)\right\|^{\varphi}_p\right]^{I^{2}}_{\theta_{rs}}\subset\\ &\left[\chi^{2q}_{f\mu},\left\|\mu_{mn}\left(x\right),\left(d\left(x_1\right),d\left(x_2\right),\cdots,d\left(x_{n-1}\right)\right)\right\|^{\varphi}_p\right]^{I^{2\alpha}}_{\theta_{rs}}\end{aligned}\tag{3.1}$$

To prove the inclusion

$$\begin{aligned}&\left[\chi^{2q}_{f\mu},\left\|\mu_{mn}\left(x\right),\left(d\left(x_1\right),d\left(x_2\right),\cdots,d\left(x_{n-1}\right)\right)\right\|^{\varphi}_p\right]^{I^{2\alpha}}_{\theta_{rs}}\\ \subset\;&\left[\chi^{2q\mu}_{g},\left\|\mu_{mn}\left(x\right),\left(d\left(x_1\right),d\left(x_2\right),\cdots,d\left(x_{n-1}\right)\right)\right\|^{\varphi}_p\right]^{I^{2}}_{\theta_{rs}},\end{aligned}$$

let

$$a \in \left[\chi^{2q}_{f\mu}, \|\mu_{mn}(x), (d(x_1), d(x_2), \cdots, d(x_{n-1}))\|^{\varphi}_{p}\right]^{I^{2\alpha}}_{\theta rs}.$$

Then for all $\{x_{mn}\}$ with

$$(x_{mn}) \in \left[\chi^{2q}_{f\mu}, \|\mu_{mn}(x), (d(x_1), d(x_2), \cdots, d(x_{n-1}))\|^{\varphi}_{p}\right]^{I^{2}}_{\theta_{rs}}$$

we have

$$\sum_{m=1}^{\infty}\sum_{n=1}^{\infty} |x_{mn}a_{mn}| < \infty. \tag{3.2}$$

Since the Musielak modulus function (f_{mn}) satisfies Δ_2- condition, then

$$(y_{mn}) \in \left[\chi^{2q}_{f\mu}, \|\mu_{mn}(x), (d(x_1), d(x_2), \cdots, d(x_{n-1}))\|^{\varphi}_{p}\right]^{I^{2}}_{\theta_{rs}},$$

we get

$$\sum_{m=1}^{\infty}\sum_{n=1}^{\infty} \left|\frac{\varphi_{rs}y_{mn}a_{mn}}{\Delta^m\lambda_{mn}(m+n)!}\right| < \infty.$$

by (3.2). Thus

$$\begin{aligned}(\varphi_{rs}a_{mn}) &\in \left[\chi^{2q}_{f\mu}, \|\mu_{mn}(x), (d(x_1), d(x_2), \cdots, d(x_{n-1}))\|^{\varphi}_{p}\right]^{I^{2}}_{\theta_{rs}} \\ &= \left[\chi^{2q\mu}_{g}, \|\mu_{mn}(x), (d(x_1), d(x_2), \cdots, d(x_{n-1}))\|^{\varphi}_{p}\right]^{I^{2}}_{\theta_{rs}}\end{aligned}$$

and hence

$$(a_{mn}) \in \left[\chi^{2q\mu}_{g}, \|\mu_{mn}(x), (d(x_1), d(x_2), \cdots, d(x_{n-1}))\|^{\varphi}_{p}\right]^{I^{2}}_{\theta_{rs}}.$$

This gives that

$$\begin{aligned}&\left[\chi^{2q}_{f\mu}, \|\mu_{mn}(x), (d(x_1), d(x_2), \cdots, d(x_{n-1}))\|^{\varphi}_{p}\right]^{I^{2\alpha}}_{\theta_{rs}} \\ &\subset \left[\chi^{2q\mu}_{g}, \|\mu_{mn}(x), (d(x_1), d(x_2), \cdots, d(x_{n-1}))\|^{\varphi}_{p}\right]^{I^{2}}_{\theta_{rs}}\end{aligned}$$

we are granted with (3.1) and (3.3)

$$\begin{aligned}&\left[\chi^{2q}_{f\mu}, \|\mu_{mn}(x), (d(x_1), d(x_2), \cdots, d(x_{n-1}))\|^{\varphi}_{p}\right]^{I^{2\alpha}}_{\theta_{rs}} \\ &= \left[\chi^{2q\mu}_{g}, \|\mu_{mn}(x), (d(x_1), d(x_2), \cdots, d(x_{n-1}))\|^{\varphi}_{p}\right]^{I^{2}}_{\theta_{rs}}\end{aligned}$$

(ii) Similarly, one can prove that

$$\left[\chi_g^{2q\mu}, \|\mu_{mn}(x), (d(x_1), d(x_2), \cdots, d(x_{n-1}))\|_p^{\varphi}\right]_{\theta_{rs}}^{I^{2\alpha}} \subset \left[\chi_{f\mu}^{2q}, \|\mu_{mn}(x), (d(x_1), d(x_2), \cdots, d(x_{n-1}))\|_p^{\varphi}\right]_{\theta_{rs}}^{I^2}$$

if the Musielak modulus function (g_{mn}) satisfies Δ_2- condition.

3.4 Proposition

If $0 < q_{mn} < p_{mn} < \infty$ for each m and m, then

$$\left[\Lambda_{f\mu}^{2q}, \|\mu_{mn}(x), (d(x_1), d(x_2), \cdots, d(x_{n-1}))\|_p^{\varphi}\right]_{\theta_{rs}}^{I^2} \subseteq \left[\Lambda_{f\mu}^{2p}, \|\mu_{mn}(x), (d(x_1), d(x_2), \cdots, d(x_{n-1}))\|_p^{\varphi}\right]_{\theta_{rs}}^{I^2}$$

Proof: The proof is standard, so we omit it.

3.5 Proposition

(i) If $0 < inf q_{mn} \leq q_{mn} < 1$ then

$$\left[\Lambda_{f\mu}^{2q}, \|\mu_{mn}(x), (d(x_1), d(x_2), \cdots, d(x_{n-1}))\|_p^{\varphi}\right]_{\theta_{rs}}^{I^2} \subset \left[\Lambda_{f\mu}^{2}, \|\mu_{mn}(x), (d(x_1), d(x_2), \cdots, d(x_{n-1}))\|_p^{\varphi}\right]_{\theta_{rs}}^{I^2}.$$

(ii) If $1 \leq q_{mn} \leq sup q_{mn} < \infty$, then

$$\left[\Lambda_{f\mu}^{2}, \|\mu_{mn}(x), (d(x_1), d(x_2), \cdots, d(x_{n-1}))\|_p^{\varphi}\right]_{\theta_{rs}}^{I^2} \subset \left[\Lambda_{f\mu}^{2q}, \|\mu_{mn}(x), (d(x_1), d(x_2), \cdots, d(x_{n-1}))\|_p^{\varphi}\right]_{\theta_{rs}}^{I^2}$$

Proof: The proof is standard, so we omit it.

3.6 Proposition

Let $f^{'} = \left(f^{'}_{mn}\right)$ and $f^{''} = \left(f^{''}_{mn}\right)$ are sequences of Musielak functions, we have

$$\left[\Lambda^{2q}_{f^{'}\mu}, \|\mu_{mn}(x), (d(x_1), d(x_2), \cdots, d(x_{n-1}))\|^{\varphi}_{p}\right]^{I^2}_{\theta_{rs}} \bigcap \left[\Lambda^{2q}_{f^{''}\mu}, \|\mu_{mn}(x), (d(x_1), d(x_2), \cdots, d(x_{n-1}))\|^{\varphi}_{p}\right]^{I^2}_{\theta_{rs}} \subseteq \left[\Lambda^{2q}_{f^{'}+f^{''}\mu}, \|\mu_{mn}(x), (d(x_1), d(x_2), \cdots, d(x_{n-1}))\|^{\varphi}_{p}\right]^{I^2}_{\theta_{rs}}$$

Proof: The proof is easy so we omit it.

3.7 Proposition

For any sequence of Musielak functions $f = (f_{mn})$ and $q = (q_{mn})$ be double analytic sequence of strictly positive real numbers. Then

$$\left[\chi^{2q}_{f\mu}, \|\mu_{mn}(x), (d(x_1), d(x_2), \cdots, d(x_{n-1}))\|^{\varphi}_{p}\right]^{I^2}_{\theta_{rs}} \subset \left[\Lambda^{2q}_{f\mu}, \|\mu_{mn}(x), (d(x_1), d(x_2), \cdots, d(x_{n-1}))\|^{\varphi}_{p}\right]^{I^2}_{\theta_{rs}}.$$

Proof: The proof is easy so we omit it.

3.8 Proposition

The sequence space $\left[\Lambda^{2q}_{f\mu}, \|\mu_{mn}(x), (d(x_1), d(x_2), \cdots, d(x_{n-1}))\|^{\varphi}_{p}\right]^{I^2}_{\theta_{rs}}$ is solid

Proof: Let $x = (x_{mn}) \in \left[\Lambda^{2q}_{f\mu}, \|\mu_{mn}(x), (d(x_1), d(x_2), \cdots, d(x_{n-1}))\|^{\varphi}_{p}\right]^{I^2}_{\theta_{rs}}$, (i.e)

$$sup_{mn}\left[\Lambda^{2q}_{f\mu}, \|\mu_{mn}(x), (d(x_1), d(x_2), \cdots, d(x_{n-1}))\|^{\varphi}_{p}\right]^{I^2}_{\theta_{rs}} < \infty.$$

Let (α_{mn}) be double sequence of scalars such that $|\alpha_{mn}| \leq 1$ for all $m, n \in \mathbb{N} \times \mathbb{N}$. Then we get

$$\sup_{mn}\left[\Lambda^{2q}_{f\mu}, \|\mu_{mn}(\alpha x), (d(x_1), d(x_2), \cdots, d(x_{n-1}))\|^{\varphi}_{p}\right]^{I^2}_{\theta_{rs}} \leq \sup_{mn}\left[\Lambda^{2q}_{f\mu}, \|\mu_{mn}(x), (d(x_1), d(x_2), \cdots, d(x_{n-1}))\|^{\varphi}_{p}\right]^{I^2}_{\theta_{rs}}.$$

This completes the proof.

3.9 Proposition

The sequence space $\left[\Lambda^{2q}_{f\mu}, \|\mu_{mn}(x), (d(x_1), d(x_2), \cdots, d(x_{n-1}))\|^{\varphi}_{p}\right]^{I^2}_{\theta_{rs}}$ is monotone

Proof: The proof follows from Proposition 3.8.

3.10 Proposition

If $f = (f_{mn})$ be any Musielak function. Then

$$\left[\Lambda^{2q}_{f\mu}, \|\mu_{mn}(x), (d(x_1), d(x_2), \cdots, d(x_{n-1}))\|^{\varphi^*}_{p}\right]^{I^2}_{\theta_{rs}} \subset \left[\Lambda^{2q}_{f\mu}, \|\mu_{mn}(x), (d(x_1), d(x_2), \cdots, d(x_{n-1}))\|^{\varphi^{**}}_{p}\right]^{I^2}_{\theta_{rs}}$$

if and only if $\sup_{r,s\geq 1} \frac{\varphi^*_{rs}}{\varphi^{**}_{rs}} < \infty$.

Proof: Let

$$x \in \left[\Lambda^{2q}_{f\mu}, \|\mu_{mn}(x), (d(x_1), d(x_2), \cdots, d(x_{n-1}))\|^{\varphi^*}_{p}\right]^{I^2}_{\theta_{rs}}$$

and

$$N = sup_{r,s\geq 1} \frac{\varphi^*_{rs}}{\varphi^{**}_{rs}} < \infty.$$

Then we get

$$\left[\Lambda^{2q}_{f\mu}, \|\mu_{mn}(x), (d(x_1), d(x_2), \cdots, d(x_{n-1}))\|^{\varphi^{**}_{rs}}_{p}\right]^{I^2}_{\theta_{rs}} = \mathbb{N}\left[\Lambda^{2q}_{f\mu}, \|\mu_{mn}(x), (d(x_1), d(x_2), \cdots, d(x_{n-1}))\|^{\varphi^*_{rs}}_{p}\right]^{I^2}_{\theta_{rs}} = 0.$$

Thus $x \in \left[\Lambda^{2q}_{f\mu}, \|\mu_{mn}(x), (d(x_1), d(x_2), \cdots, d(x_{n-1}))\|^{\varphi^{**}}_{p}\right]^{I^2}_{\theta_{rs}}$.

Conversely, suppose that

$$\left[\Lambda^{2q}_{f\mu}, \|\mu_{mn}(x), (d(x_1), d(x_2), \cdots, d(x_{n-1}))\|^{\varphi^*}_{p}\right]^{I}_{N_\theta} \subset \left[\Lambda^{2qu}_{f\mu}, \|\mu_{mn}(x), (d(x_1), d(x_2), \cdots, d(x_{n-1}))\|^{\varphi^{**}}_{p}\right]^{I^2}_{\theta_{rs}}$$

and

$$x \in \left[\Lambda^{2q}_{f\mu}, \|\mu_{mn}(x), (d(x_1), d(x_2), \cdots, d(x_{n-1}))\|^{\varphi^*}_{p}\right]^{I^2}_{\theta_{rs}}.$$

Then

$$\left[\Lambda^{2q}_{f\mu}, \|\mu_{mn}(x), (d(x_1), d(x_2), \cdots, d(x_{n-1}))\|^{\varphi^*}_{p}\right]^{I^2}_{\theta_{rs}} < \epsilon,$$

for every $\epsilon > 0$. Suppose that $\sup_{r,s\geq 1} \frac{\varphi^*_{rs}}{\varphi^{**}_{rs}} = \infty$, then there exists a sequence of members (rs_{jk}) such that $\lim_{j,k\to\infty} \frac{\varphi^*_{jk}}{\varphi^{**}_{jk}} = \infty$. Hence, we have

$$\left[\Lambda^{2q}_{f\mu}, \|\mu_{mn}(x), (d(x_1), d(x_2), \cdots, d(x_{n-1}))\|_p^{\varphi^*_{rs}}\right]^{I^2}_{\theta_{rs}} = \infty.$$

Therefore

$$x \notin \left[\Lambda^{2q}_{f\mu}, \|\mu_{mn}(x), (d(x_1), d(x_2), \cdots, d(x_{n-1}))\|_p^{\varphi^{**}}\right]^{I^2}_{\theta_{rs}},$$

which is a contradiction. This completes the proof.

3.11 Proposition

If $f = (f_{mn})$ be any Musielak function. Then

$$\left[\Lambda^{2q}_{f\mu}, \|\mu_{mn}(x), (d(x_1), d(x_2), \cdots, d(x_{n-1}))\|_p^{\varphi^*}\right]^{I^2}_{\theta_{rs}}$$
$$= \left[\Lambda^{2q}_{f\mu}, \|\mu_{mn}(x), (d(x_1), d(x_2), \cdots, d(x_{n-1}))\|_p^{\varphi^{**}}\right]^{I^2}_{\theta_{rs}}$$

if and only if

$$\sup_{r,s\geq 1} \frac{\varphi^*_{rs}}{\varphi^{**}_{rs}} < \infty, \quad \sup_{r,s\geq 1} \frac{\varphi^{**}_{rs}}{\varphi^*_{rs}} > \infty.$$

Proof: It is easy to prove so we omit.

3.12 Proposition

The sequence space

$$\left[\chi^{2q}_{f\mu}, \|\mu_{mn}(x), (d(x_1), d(x_2), \cdots, d(x_{n-1}))\|_p^{\varphi}\right]^{I^2}_{\theta_{rs}}$$

is not solid

Proof: The result follows from the following example.

Example: Consider

$$x = (x_{mn}) = \begin{pmatrix} 1 & 1 & ... & 1 \\ 1 & 1 & ... & 1 \\ . & & & \\ . & & & \\ . & & & \\ 1 & 1 & ... & 1 \end{pmatrix} \in \left[\chi^{2q}_{f\mu}, \|\mu_{mn}(x), (d(x_1), d(x_2), \cdots, d(x_{n-1}))\|_p^{\varphi}\right]^{I^2}_{\theta_{rs}}.$$

Let

$$\alpha_{mn} = \begin{pmatrix} -1^{m+n} & -1^{m+n} & ... & -1^{m+n} \\ -1^{m+n} & -1^{m+n} & ... & -1^{m+n} \\ . & & & \\ . & & & \\ . & & & \\ -1^{m+n} & -1^{m+n} & ... & -1^{m+n} \end{pmatrix},$$

for all $m, n \in \mathbb{N}$. Then

$$\alpha_{mn} x_{mn} \notin \left[\chi^{2q}_{f\mu}, \|\mu_{mn}(x), (d(x_1), d(x_2), \cdots, d(x_{n-1}))\|^{\varphi}_{p}\right]^{I^2}_{\theta_{rs}}.$$

Hence

$$\left[\chi^{2q}_{f\mu}, \|\mu_{mn}(x), (d(x_1), d(x_2), \cdots, d(x_{n-1}))\|^{\varphi}_{p}\right]^{I^2}_{\theta_{rs}}$$

is not solid.

3.13 Proposition

The sequence space $\left[\chi^{2q}_{f\mu}, \|\mu_{mn}(x), (d(x_1), d(x_2), \cdots, d(x_{n-1}))\|^{\varphi}_{p}\right]^{I^2}_{\theta_{rs}}$ is not monotone

Proof: The proof follows from Proposition 3.12.

Competing Interests: Authors have declared that no competing interests exist.

Acknowledgement: The authors thank the referee for his careful reading of the manuscript and comments that improved the presentation of the paper.

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DOI: 10.7862/rf.2015.11

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Received 4.07.2014