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On Some Inclusion in the Set Theory

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Abstract: This note contains the proof of an inclusion in the set theory. In that proof we use only basic laws appearing in the set theory. More precisely, using some basic laws of the set theory we provide the proof of an inclusion which is applied in the proof of certain theorem of the classical measure theory. The presented paper has an elementary character. Only the basic tools of the set theory are involved.

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1. Introduction

The purpose of this concise note is to present the proof of some inclusion of the set theory connected with a basic theorem concerning the property of the relation appearing in the measure theory which is called the equivalence with respect to a measure (cf. [5]; see also [6]).

In order to present the relation, assume that X is a nonempty set and S is a σ-field of some subsets of X, i. e., S is a family of some subsets of X which is σ-additive (i. e., if $A_i \in S$ for $i = 1, 2, ...$ then $\bigcup_{i=1}^{\infty} A_i \in S$) and such that $A \setminus B \in S$ for arbitrary sets $A, B \in S$. Further, let m be a measure defined on S, i. e., $m : S \to \overline{\mathbb{R}}_+ = [0, +\infty]$ is σ-additive (that means $m\left(\bigcup_{i=1}^{\infty} A_i\right) = \sum_{i=1}^{\infty} m(A_i)$ for any sequence of sets belonging to S which are pairwise disjoint) and such that $m(\emptyset) = 0$.

We say that sets $A, B \in S$ *are equivalent with respect to the measure* m (we write $A \approx B$) if

$$m(A \setminus B) = m(B \setminus A) = 0.$$

It is easily seen that the relation of the equivalence with respect to the measure m is reflexive and symmetric. Thus, to prove that the relation $\approx$ is an equivalence relation it is only sufficient to prove that it is transitive, i. e., that the following implication holds

$$A \approx B \quad \text{and} \quad B \approx C \implies A \approx C \tag{1.1}$$

for arbitrary sets $A, B, C \in S$. The implication (1.1) will be proved if we show the following inclusion for arbitrary sets A, B, C :

$$A \setminus C \subset (A \setminus B) \cup (B \setminus C). \tag{1.2}$$

Observe that the proof of inclusion (1.2) can be performed in the standard way with the help of the transition of our problem to mathematical logic and then it is not difficult. However, it is interesting to conduct that proof by the use of basic laws of the theory of sets.
Since we have not found such a proof in popular mathematical literature (cf. [1–4,7,8]), we are going to present it in what follows.

2. Main result

The previously announced result is formulated in the form of the following theorem.

Theorem. *Let A, B, C be arbitrary sets. Then inclusion (1.2) holds.*

Proof. Denote by P the set appearing on the right-hand side of inclusion (1.2) i. e., $P = (A \setminus B) \cup (B \setminus C)$. Then, applying the well-known equality

$$X \setminus Y = X \cap Y'$$

we get

$$P = (A \setminus B) \cup (B \setminus C) = A \cap B' \cup B \cap C'.$$

Hence, in view of the distributivity of the union over the intersection, we obtain

$$\begin{aligned} P &= [(A \cap B') \cup B] \cap [(A \cap B') \cup C'] \\ &= [(A \cup B) \cap (B' \cup B)] \cap [(A \cup C') \cap (B' \cup C')] \\ &= (A \cup B) \cap (A \cup C') \cap (B' \cup C'). \end{aligned}$$

Now, applying the law of the associativity, we get

$$P = (B' \cup C') \cap [(A \cup C') \cap (A \cup B)]. \tag{2.1}$$

Further, using two times the law of the distributivity of the union over the intersection, we obtain consecutively the following equalities for the set L, where L denotes the left-hand side of inclusion (1.2):

$$\begin{aligned} L = A \setminus C = A \cap C' &= (A \cap C') \cup \emptyset = (A \cap C') \cup (B \cap B') \\ &= [(A \cap C') \cup B] \cap [(A \cap C') \cup B'] \\ &= [(A \cup B) \cap (C' \cup B)] \cap [(A \cup B') \cap (C' \cup B')]. \end{aligned}$$

Next, in virtue of the law of the associativity for the intersection, we get

$$L = [(A \cup B) \cap (A \cup B')] \cap [(C' \cup B) \cap (C' \cup B')].$$

Hence, taking into account the fact that the union is distributive over the intersection, we derive the equality

$$\begin{aligned} L &= [A \cup (B \cap B')] \cap [(C' \cup B) \cap (C' \cup B')] \\ &= A \cap [(C' \cup B) \cap (C' \cup B')]. \end{aligned}$$

Further, in view of the associativity of the intersection and the distributivity of the intersection over the union, we obtain

$$\begin{aligned} L &= (C' \cup B') \cap [A \cap (C' \cup B)] \\ &= (B' \cup C') \cap [(A \cap C') \cup (A \cap B)]. \end{aligned} \tag{2.2}$$

Now, comparing expressions (2.1) and (2.2) we see that in order to prove inclusion (1.2) it is sufficient to show that

$$(A \cap C') \cup (A \cap B) \subset (A \cup C') \cap (A \cup B). \tag{2.3}$$

To this end, similarly as before, let us denote

$$L = (A \cap C') \cup (A \cap B), \quad P = (A \cup C') \cap (A \cup B).$$

Then, keeping in mind the distributivity of the intersection over the union, we have

$$\begin{aligned} P = (A \cup C') \cap (A \cup B) &= [(A \cup B) \cap A] \cup [(A \cup B) \cap C'] \\ &= [(A \cap A) \cup (A \cap B)] \cup [(A \cap C') \cup (B \cap C')] \\ &= [A \cup (A \cap B)] \cup [(A \cap C') \cup (B \cap C')] \\ &= [(A \cap C') \cup (A \cap B)] \cup [A \cup (B \cap C')]. \end{aligned} \tag{2.4}$$

Finally, taking into account the form of the set L and (2.4) we conclude that

$$L = (A \cap C') \cup (A \cap B) \subset [(A \cap C') \cup (A \cap B)] \cup [A \cup (B \cap C')] = P.$$

Obviously, the above obtained inclusion completes the proof since we showed the desired inclusion (2.3).

□

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