Journal of
Mathematics
and Applications

JMA No 38, pp 33-48 (2015)

On some differential sandwich theorems using an extended generalized Sălăgean operator and extended Ruscheweyh operator

Loriana Andrei

Abstract: In this work we define a new operator using the extended generalized Sălăgean operator and extended Ruscheweyh operator. Denote by $DR_\lambda^{m,n}$ the Hadamard product of the extended generalized Sălăgean operator D_λ^m and extended Ruscheweyh operator R^n, given by $DR_\lambda^{m,n} : \mathcal{A}_\zeta^* \to \mathcal{A}_\zeta^*$, $DR_\lambda^{m,n} f(z,\zeta) = (D_\lambda^m * R^n) f(z,\zeta)$ and $\mathcal{A}_{n\zeta}^* = \{f \in \mathcal{H}(U \times \overline{U}),\ f(z,\zeta) = z + a_{n+1}(\zeta) z^{n+1} + \ldots,\ z \in U, \zeta \in \overline{U}\}$ is the class of normalized analytic functions with $\mathcal{A}_{1\zeta}^* = \mathcal{A}_\zeta^*$. The purpose of this paper is to introduce sufficient conditions for strong differential subordination and strong differential superordination involving the operator $DR_\lambda^{m,n}$ and also to obtain sandwich-type results.

AMS Subject Classification: *30C45*
Keywords and Phrases: *analytic functions, differential operator, differential subordination, differential superordination.*

1 Introduction

Denote by U the unit disc of the complex plane $U = \{z \in \mathbb{C} :\ |z| < 1\}$, $\overline{U} = \{z \in \mathbb{C} :\ |z| \leq 1\}$ the closed unit disc of the complex plane and $\mathcal{H}(U \times \overline{U})$ the class of analytic functions in $U \times \overline{U}$.

Let

$$\mathcal{A}_{n\zeta}^* = \{f \in \mathcal{H}(U \times \overline{U}),\ f(z,\zeta) = z + a_{n+1}(\zeta) z^{n+1} + \ldots,\ z \in U,\ \zeta \in \overline{U}\},$$

with $\mathcal{A}_{1\zeta}^* = \mathcal{A}_\zeta^*$, where $a_k(\zeta)$ are holomorphic functions in $\overline{U}$ for $k \geq 2$, and $\mathcal{H}^*[a,n,\zeta] = \{f \in \mathcal{H}(U \times \overline{U}),\ f(z,\zeta) = a + a_n(\zeta) z^n + a_{n+1}(\zeta) z^{n+1} + \ldots,\ z \in U, \zeta \in \overline{U}\}$, for $a \in \mathbb{C}$ and $n \in \mathbb{N}$, $a_k(\zeta)$ are holomorphic functions in $\overline{U}$ for $k \geq n$.

Generalizing the notion of differential subordinations, J.A. Antonino and S. Romaguera have introduced in [17] the notion of strong differential subordinations, which was developed by G.I. Oros and Gh. Oros in [18].

Definition 1.1 *[18] Let $f(z,\zeta)$, $H(z,\zeta)$ analytic in $U\times\overline{U}$. The function $f(z,\zeta)$ is said to be strongly subordinate to $H(z,\zeta)$ if there exists a function w analytic in U, with $w(0)=0$ and $|w(z)|<1$ such that $f(z,\zeta)=H(w(z),\zeta)$ for all $\zeta\in\overline{U}$. In such a case we write $f(z,\zeta)\prec\prec H(z,\zeta)$, $z\in U$, $\zeta\in\overline{U}$.*

Remark 1.1 *[18] (i) Since $f(z,\zeta)$ is analytic in $U\times\overline{U}$, for all $\zeta\in\overline{U}$, and univalent in U, for all $\zeta\in\overline{U}$, Definition 1.1 is equivalent to $f(0,\zeta)=H(0,\zeta)$, for all $\zeta\in\overline{U}$, and $f\left(U\times\overline{U}\right)\subset H\left(U\times\overline{U}\right)$.*

(ii) If $H(z,\zeta)\equiv H(z)$ and $f(z,\zeta)\equiv f(z)$, the strong subordination becomes the usual notion of subordination.

As a dual notion of strong differential subordination G.I. Oros has introduced and developed the notion of strong differential superordinations in [19].

Definition 1.2 *[19] Let $f(z,\zeta)$, $H(z,\zeta)$ analytic in $U\times\overline{U}$. The function $f(z,\zeta)$ is said to be strongly superordinate to $H(z,\zeta)$ if there exists a function w analytic in U, with $w(0)=0$ and $|w(z)|<1$, such that $H(z,\zeta)=f(w(z),\zeta)$, for all $\zeta\in\overline{U}$. In such a case we write $H(z,\zeta)\prec\prec f(z,\zeta)$, $z\in U$, $\zeta\in\overline{U}$.*

Remark 1.2 *[19] (i) Since $f(z,\zeta)$ is analytic in $U\times\overline{U}$, for all $\zeta\in\overline{U}$, and univalent in U, for all $\zeta\in\overline{U}$, Definition 1.2 is equivalent to $H(0,\zeta)=f(0,\zeta)$, for all $\zeta\in\overline{U}$, and $H\left(U\times\overline{U}\right)\subset f\left(U\times\overline{U}\right)$.*

(ii) If $H(z,\zeta)\equiv H(z)$ and $f(z,\zeta)\equiv f(z)$, the strong superordination becomes the usual notion of superordination.

Definition 1.3 *[1] We denote by Q^* the set of functions that are analytic and injective on $\overline{U}\times\overline{U}\backslash E(f,\zeta)$, where $E(f,\zeta)=\{y\in\partial U:\lim\limits_{z\to y}f(z,\zeta)=\infty\}$, and are such that $f'_z(y,\zeta)\neq 0$ for $y\in\partial U\times\overline{U}\backslash E(f,\zeta)$. The subclass of Q^* for which $f(0,\zeta)=a$ is denoted by $Q^*(a)$.*

For two functions $f(z,\zeta)=z+\sum_{j=2}^{\infty}a_j(\zeta)z^j$ and $g(z,\zeta)=z+\sum_{j=2}^{\infty}b_j(\zeta)z^j$ analytic in $U\times\overline{U}$, the Hadamard product (or convolution) of $f(z,\zeta)$ and $g(z,\zeta)$, written as $(f*g)(z,\zeta)$ is defined by

$$f(z,\zeta)*g(z,\zeta)=(f*g)(z,\zeta)=z+\sum_{j=2}^{\infty}a_j(\zeta)b_j(\zeta)z^j.$$

Definition 1.4 *([2]) For $f\in\mathcal{A}^*_\zeta$, $\lambda\geq 0$ and $m\in\mathbb{N}$, the extended generalized Sălăgean operator D^m_λ is defined by $D^m_\lambda:\mathcal{A}^*_\zeta\to\mathcal{A}^*_\zeta$,*

$$\begin{aligned}
D^0_\lambda f(z,\zeta) &= f(z,\zeta)\\
D^1_\lambda f(z,\zeta) &= (1-\lambda)f(z,\zeta)+\lambda z f'_z(z,\zeta)=D_\lambda f(z,\zeta)\\
&\ldots\\
D^{m+1}_\lambda f(z,\zeta) &= (1-\lambda)D^m_\lambda f(z,\zeta)+\lambda z\left(D^m_\lambda f(z,\zeta)\right)'_z=D_\lambda\left(D^m_\lambda f(z,\zeta)\right),
\end{aligned}$$

for $z \in U$, $\zeta \in \overline{U}$.

Remark 1.3 *If* $f \in \mathcal{A}_{\zeta}^{*}$ *and* $f(z,\zeta) = z + \sum_{j=2}^{\infty} a_j(\zeta) z^j$, *then* $D_{\lambda}^{m} f(z,\zeta) = z + \sum_{j=2}^{\infty} [1 + (j-1)\lambda]^m a_j(\zeta) z^j$, *for* $z \in U$, $\zeta \in \overline{U}$.

Definition 1.5 *([3]) For* $f \in \mathcal{A}_{\zeta}^{*}$, $m \in \mathbb{N}$, *the extended Ruscheweyh derivative* R^m *is defined by* $R^m : \mathcal{A}_{\zeta}^{*} \to \mathcal{A}_{\zeta}^{*}$,

$$\begin{aligned} R^0 f(z,\zeta) &= f(z,\zeta) \\ R^1 f(z,\zeta) &= z f_z'(z,\zeta) \\ &\ \ \ldots \\ (m+1) R^{m+1} f(z,\zeta) &= z \left(R^m f(z,\zeta)\right)_z' + m R^m f(z,\zeta), \end{aligned}$$

$z \in U$, $\zeta \in \overline{U}$.

Remark 1.4 *If* $f \in \mathcal{A}_{\zeta}^{*}$, $f(z,\zeta) = z + \sum_{j=2}^{\infty} a_j(\zeta) z^j$, *then* $R^m f(z,\zeta) = z + \sum_{j=2}^{\infty} \frac{(m+j-1)!}{m!(j-1)!} a_j(\zeta) z^j$, $z \in U$, $\zeta \in \overline{U}$.

In order to prove our strong subordination and strong superordination results, we make use of the following known results.

Lemma 1.1 *Let the function* q *be univalent in* $U \times \overline{U}$ *and* θ *and* ϕ *be analytic in a domain* D *containing* $q\left(U \times \overline{U}\right)$ *with* $\phi(w) \neq 0$ *when* $w \in q\left(U \times \overline{U}\right)$. *Set* $Q(z,\zeta) = z q_z'(z,\zeta) \phi(q(z,\zeta))$ *and* $h(z,\zeta) = \theta(q(z,\zeta)) + Q(z,\zeta)$. *Suppose that*

1. Q *is starlike univalent in* $U \times \overline{U}$ *and*
2. $Re\left(\frac{z h_z'(z,\zeta)}{Q(z,\zeta)}\right) > 0$ *for* $z \in U$, $\zeta \in \overline{U}$.

If p *is analytic with* $p(0,\zeta) = q(0,\zeta)$, $p\left(U \times \overline{U}\right) \subseteq D$ *and*

$$\theta(p(z,\zeta)) + z p_z'(z,\zeta) \phi(p(z,\zeta)) \prec\prec \theta(q(z,\zeta)) + z q_z'(z,\zeta) \phi(q(z,\zeta)),$$

then $p(z,\zeta) \prec\prec q(z,\zeta)$ *and* q *is the best dominant.*

Lemma 1.2 *Let the function* q *be convex univalent in* $U \times \overline{U}$ *and* ν *and* ϕ *be analytic in a domain* D *containing* $q\left(U \times \overline{U}\right)$. *Suppose that*

1. $Re\left(\frac{\nu_z'(q(z,\zeta))}{\phi(q(z,\zeta))}\right) > 0$ *for* $z \in U$, $\zeta \in \overline{U}$ *and*
2. $\psi(z,\zeta) = z q_z'(z,\zeta) \phi(q(z,\zeta))$ *is starlike univalent in* $U \times \overline{U}$.

If $p(z,\zeta) \in \mathcal{H}^*[q(0,\zeta), 1, \zeta] \cap Q^*$, *with* $p\left(U \times \overline{U}\right) \subseteq D$ *and* $\nu(p(z,\zeta)) + z p_z'(z) \phi(p(z,\zeta))$ *is univalent in* $U \times \overline{U}$ *and*

$$\nu(q(z,\zeta)) + z q_z'(z,\zeta) \phi(q(z,\zeta)) \prec\prec \nu(p(z,\zeta)) + z p_z'(z,\zeta) \phi(p(z,\zeta)),$$

then $q(z,\zeta) \prec\prec p(z,\zeta)$ *and* q *is the best subordinant.*

2 Main results

Extending the results from [11] to the class $\mathcal{A}_\zeta^*$ we obtain:

Definition 2.1 *([12]) Let $\lambda \geq 0$ and $n, m \in \mathbb{N}$. Denote by $DR_\lambda^{m,n} : \mathcal{A}_\zeta^* \to \mathcal{A}_\zeta^*$ the operator given by the Hadamard product of the extended generalized Sălăgean operator D_λ^m and the extended Ruscheweyh operator R^n,*

$$DR_\lambda^{m,n} f(z,\zeta) = (D_\lambda^m * R^n) f(z,\zeta),$$

for any $z \in U$, $\zeta \in \overline{U}$, and each nonnegative integers m, n.

Remark 2.1 *If $f \in \mathcal{A}_\zeta^*$ and $f(z,\zeta) = z + \sum_{j=2}^\infty a_j(\zeta) z^j$, then*
$DR_\lambda^{m,n} f(z,\zeta) = z + \sum_{j=2}^\infty [1+(j-1)\lambda]^m \frac{(n+j-1)!}{n!(j-1)!} a_j^2(\zeta) z^j$, for $z \in U$, $\zeta \in \overline{U}$.

Remark 2.2 *For $m = n$ we obtain the operator DR_λ^m studied in [13], [14], [15], [16], [4], [5], [6].*
For $\lambda = 1$, $m = n$, we obtain the Hadamard product SR^n [7] of the Sălăgean operator S^n and Ruscheweyh derivative R^n, which was studied in [8], [9], [10].

Using simple computation one obtains the next result.

Proposition 2.1 *For $m, n \in \mathbb{N}$ and $\lambda \geq 0$ we have For $m, n \in \mathbb{N}$ and $\lambda \geq 0$ we have*

$$DR_\lambda^{m+1,n} f(z,\zeta) = (1-\lambda) DR_\lambda^{m,n} f(z,\zeta) + \lambda z \left(DR_\lambda^{m,n} f(z,\zeta)\right)'_z \tag{2.1}$$

and

$$z\left(DR_\lambda^{m,n} f(z,\zeta)\right)'_z = (n+1) DR_\lambda^{m,n+1} f(z,\zeta) - n DR_\lambda^{m,n} f(z,\zeta). \tag{2.2}$$

Proof. We have

$$\begin{aligned}
DR_\lambda^{m+1,n} f(z,\zeta) &= z + \sum_{j=2}^\infty [1+(j-1)\lambda]^{m+1} \frac{(n+j-1)!}{n!(j-1)!} a_j^2(\zeta) z^j \\
&= z + \sum_{j=2}^\infty [(1-\lambda)+\lambda j][1+(j-1)\lambda]^m \frac{(n+j-1)!}{n!(j-1)!} a_j^2(\zeta) z^j \\
&= z + (1-\lambda) \sum_{j=2}^\infty [1+(j-1)\lambda]^m \frac{(n+j-1)!}{n!(j-1)!} a_j^2(\zeta) z^j \\
&\quad + \lambda \sum_{j=2}^\infty [1+(j-1)\lambda]^m \frac{(n+j-1)!}{n!(j-1)!} j a_j^2(\zeta) z^j \\
&= (1-\lambda) DR_\lambda^{m,n} f(z,\zeta) + \lambda z \left(DR_\lambda^{m,n} f(z,\zeta)\right)'_z,
\end{aligned}$$

and

$$
\begin{aligned}
&(n+1)\, DR_{\lambda}^{m,n+1} f(z,\zeta) - n DR_{\lambda}^{m,n} f(z,\zeta) \\
&= (n+1)\, z + (n+1) \sum_{j=2}^{\infty} [1+(j-1)\lambda]^m \frac{(n+j)!}{(n+1)!\,(j-1)!} a_j^2(\zeta)\, z^j \\
&\quad - nz - n \sum_{j=2}^{\infty} [1+(j-1)\lambda]^m \frac{(n+j-1)!}{n!\,(j-1)!} a_j^2(\zeta)\, z^j \\
&= z + (n+1) \sum_{j=2}^{\infty} [1+(j-1)\lambda]^m \frac{n+j}{n+1} \frac{(n+j-1)!}{n!\,(j-1)!} a_j^2(\zeta)\, z^j \\
&\quad - n \sum_{j=2}^{\infty} [1+(j-1)\lambda]^m \frac{(n+j-1)!}{n!\,(j-1)!} a_j^2(\zeta)\, z^j \\
&= z + \sum_{j=2}^{\infty} [1+(j-1)\lambda]^m \frac{(n+j-1)!}{n!\,(j-1)!} j a_j^2(z)\, z^j \\
&= z \left(DR_{\lambda}^{m,n} f(z,\zeta)\right)_z' .
\end{aligned}
$$

■

We begin with the following

Theorem 2.2 *Let* $\frac{DR_{\lambda}^{m+1,n} f(z,\zeta)}{DR_{\lambda}^{m,n} f(z,\zeta)} \in \mathcal{H}\left(U \times \overline{U}\right)$, $z \in U$, $\zeta \in \overline{U}$, $f \in \mathcal{A}_{\zeta}^{*}$, $m, n \in \mathbb{N}$, $\lambda \geq 0$ *and let the function* $q(z,\zeta)$ *be convex and univalent in* $U \times \overline{U}$ *such that* $q(0,\zeta) = 1$. *Assume that*

$$
\operatorname{Re}\left(1 + \frac{\alpha}{\mu} + \frac{2\beta}{\mu} q(z,\zeta) + \frac{z q''_{z^2}(z,\zeta)}{q'_z(z,\zeta)}\right) > 0, \quad z \in U,\ \zeta \in \overline{U}, \tag{2.3}
$$

for $\alpha, \beta, \mu, \in \mathbb{C}, \mu \neq 0,\ z \in U,\ \zeta \in \overline{U}$, *and*

$$
\begin{aligned}
\psi_{\lambda}^{m,n}(\alpha,\beta,\mu;z,\zeta) &:= \left(\frac{1-\lambda(n+1)}{\lambda}\mu + \alpha\right) \frac{DR_{\lambda}^{m+1,n} f(z,\zeta)}{DR_{\lambda}^{m,n} f(z,\zeta)} \\
&+ \mu(n+1)\left[1-\lambda(n+2)\right] \frac{DR_{\lambda}^{m,n+1} f(z,\zeta)}{DR_{\lambda}^{m,n} f(z,\zeta)} \\
&+ \lambda\mu(n+1)(n+2) \frac{DR_{\lambda}^{m,n+2} f(z,\zeta)}{DR_{\lambda}^{m,n} f(z,\zeta)} + \left(\beta - \frac{\mu}{\lambda}\right) \left(\frac{DR_{\lambda}^{m+1,n} f(z,\zeta)}{DR_{\lambda}^{m,n} f(z,\zeta)}\right)^2 .
\end{aligned} \tag{2.4}
$$

If q *satisfies the following strong differential subordination*

$$
\psi_{\lambda}^{m,n}(\alpha,\beta,\mu;z,\zeta) \prec\prec \alpha q(z,\zeta) + \beta\left(q(z,\zeta)\right)^2 + \mu z q'_z(z,\zeta), \tag{2.5}
$$

for, $\alpha, \beta, \mu \in \mathbb{C},\ \mu \neq 0$ *then*

$$
\frac{DR_{\lambda}^{m+1,n} f(z,\zeta)}{DR_{\lambda}^{m,n} f(z,\zeta)} \prec\prec q(z,\zeta), \quad z \in U,\ \zeta \in \overline{U}, \tag{2.6}
$$

and q is the best dominant.

Proof. Let the function p be defined by $p(z,\zeta) := \frac{DR_\lambda^{m+1,n} f(z,\zeta)}{DR_\lambda^{m,n} f(z,\zeta)}$, $z \in U$, $z \neq 0, \zeta \in \overline{U}$, $f \in \mathcal{A}_\zeta^*$. The function p is analytic in U and $p(0,\zeta) = 1$.

Differentiating with respect to z this function, we get
$zp'_z(z,\zeta) = \frac{z\left(DR_\lambda^{m+1,n} f(z,\zeta)\right)'_z}{DR_\lambda^{m,n} f(z,\zeta)} - \frac{DR_\lambda^{m+1,n} f(z,\zeta)}{DR_\lambda^{m,n} f(z,\zeta)} \frac{z\left(DR_\lambda^{m,n} f(z,\zeta)\right)'_z}{DR_\lambda^{m,n} f(z,\zeta)}$

By using the identity (2.1) and (2.2), we obtain

$$\begin{aligned} zp'_z(z,\zeta) &= \frac{1-\lambda(n+1)}{\lambda} \frac{DR_\lambda^{m+1,n} f(z,\zeta)}{DR_\lambda^{m,n} f(z,\zeta)} \\ &+(n+1)\left[1-\lambda(n+2)\right] \frac{DR_\lambda^{m,n+1} f(z,\zeta)}{DR_\lambda^{m,n} f(z,\zeta)} \\ &+\lambda(n+1)(n+2) \frac{DR_\lambda^{m,n+2} f(z,\zeta)}{DR_\lambda^{m,n} f(z,\zeta)} - \frac{1}{\lambda}\left(\frac{DR_\lambda^{m+1,n} f(z,\zeta)}{DR_\lambda^{m,n} f(z,\zeta)}\right)^2 \\ &+\lambda(n+1)(n+2) \frac{DR_\lambda^{m,n+2} f(z,\zeta)}{DR_\lambda^{m,n} f(z,\zeta)} - \frac{1}{\lambda}\left(\frac{DR_\lambda^{m+1,n} f(z,\zeta)}{DR_\lambda^{m,n} f(z,\zeta)}\right)^2 \quad (2.7) \end{aligned}$$

By setting $\theta(w) := \alpha w + \beta w^2$ and $\phi(w) := \mu$, $\alpha, \beta, \mu \in \mathbb{C}$, $\mu \neq 0$ it can be easily verified that θ is analytic in $\mathbb{C}$, ϕ is analytic in $\mathbb{C}\backslash\{0\}$ and that $\phi(w) \neq 0$, $w \in \mathbb{C}\backslash\{0\}$.

Also, by letting $Q(z,\zeta) = zq'_z(z,\zeta)\phi(q(z,\zeta)) = \mu zq'_z(z,\zeta)$,we find that $Q(z,\zeta)$ is starlike univalent in $U \times \overline{U}$.

Let $h(z,\zeta) = \theta(q(z,\zeta)) + Q(z,\zeta) = \alpha q(z,\zeta) + \beta(q(z,\zeta))^2 + \mu zq'_z(z,\zeta)$, $z \in U$, $\zeta \in \overline{U}$.

If we derive the function Q, with respect to z, perform calculations, we have $\operatorname{Re}\left(\frac{zh'_z(z,\zeta)}{Q(z,\zeta)}\right) = \operatorname{Re}\left(1 + \frac{\alpha}{\mu} + \frac{2\beta}{\mu} q(z,\zeta) + \frac{zq''_{z^2}(z,\zeta)}{q'_z(z,\zeta)}\right) > 0$.

By using (2.7), we obtain $\alpha p(z,\zeta) + \beta(p(z,\zeta))^2 + \mu zp'_z(z,\zeta) =$
$\left(\frac{1-\lambda(n+1)}{\lambda}\mu + \alpha\right) \frac{DR_\lambda^{m+1,n} f(z,\zeta)}{DR_\lambda^{m,n} f(z,\zeta)} + .\mu(n+1)\left[1-\lambda(n+2)\right] \frac{DR_\lambda^{m,n+1} f(z,\zeta)}{DR_\lambda^{m,n} f(z,\zeta)} + \lambda\mu(n+1)(n+2) \frac{DR_\lambda^{m,n+2} f(z,\zeta)}{DR_\lambda^{m,n} f(z,\zeta)} + \left(\beta - \frac{\mu}{\lambda}\right)\left(\frac{DR_\lambda^{m+1,n} f(z,\zeta)}{DR_\lambda^{m,n} f(z,\zeta)}\right)^2$.

By using (2.5), we have $\alpha p(z,\zeta) + \beta(p(z,\zeta))^2 + \mu zp'_z(z,\zeta) \prec\prec \alpha q(z,\zeta) + \beta(q(z,\zeta))^2 + \mu zq'_z(z,\zeta)$.

Therefore, the conditions of Lemma 1.1 are met, so we have $p(z,\zeta) \prec\prec q(z,\zeta)$, $z \in U$, $\zeta \in \overline{U}$, i.e. $\frac{DR_\lambda^{m+1,n} f(z,\zeta)}{DR_\lambda^{m,n} f(z,\zeta)} \prec\prec q(z,\zeta)$, $z \in U$, $\zeta \in \overline{U}$, and q is the best dominant.
■

Corollary 2.3 *Let* $q(z,\zeta) = \frac{\zeta+Az}{\zeta+Bz}$, $-1 \leq B < A \leq 1$, $m,n \in \mathbb{N}$, $\lambda \geq 0$, $z \in U$, $\zeta \in \overline{U}$. *Assume that (2.3) holds. If* $f \in \mathcal{A}_\zeta^*$ *and*

$$\psi_\lambda^{m,n}(\alpha,\beta,\mu;z,\zeta) \prec\prec \alpha\frac{\zeta+Az}{\zeta+Bz} + \beta\left(\frac{\zeta+Az}{\zeta+Bz}\right)^2 + \mu\frac{\zeta(A-B)z}{(\zeta+Bz)^2},$$

for $\alpha,\beta,\mu\in\mathbb{C}$, $\mu\neq 0$, $-1\leq B<A\leq 1$, *where* $\psi_{\lambda}^{m,n}$ *is defined in (2.4), then*

$$\frac{DR_{\lambda}^{m+1,n}f(z,\zeta)}{DR_{\lambda}^{m,n}f(z,\zeta)}\prec\prec\frac{\zeta+Az}{\zeta+Bz}$$

and $\frac{\zeta+Az}{\zeta+Bz}$ *is the best dominant.*

Proof. For $q(z,\zeta)=\frac{\zeta+Az}{\zeta+Bz}$, $-1\leq B<A\leq 1$, in Theorem 2.2 we get the corollary. ■

Corollary 2.4 *Let* $q(z,\zeta)=\left(\frac{\zeta+z}{\zeta-z}\right)^{\gamma}$, $m,n\in\mathbb{N}$, $\lambda\geq 0, z\in U$. *Assume that (2.3) holds. If* $f\in\mathcal{A}_{\zeta}^{*}$ *and*

$$\psi_{\lambda}^{m,n}(\alpha,\beta,\mu;z,\zeta)\prec\alpha\left(\frac{\zeta+z}{\zeta-z}\right)^{\gamma}+\beta\left(\frac{\zeta+z}{\zeta-z}\right)^{2\gamma}+\mu\frac{2\zeta\gamma z}{(\zeta-z)^{2}}\left(\frac{\zeta+z}{\zeta-z}\right)^{\gamma-1}$$

for $\alpha,\mu,\beta\in\mathbb{C}$, $0<\gamma\leq 1$, $\mu\neq 0$, *where* $\psi_{\lambda}^{m,n}$ *is defined in (2.4), then*

$$\frac{DR_{\lambda}^{m+1,n}f(z,\zeta)}{DR_{\lambda}^{m,n}f(z,\zeta)}\prec\prec\left(\frac{\zeta+z}{\zeta-z}\right)^{\gamma},$$

and $\left(\frac{\zeta+z}{\zeta-z}\right)^{\gamma}$ *is the best dominant.*

Proof. Corollary follows by using Theorem 2.2 for $q(z,\zeta)=\left(\frac{\zeta+z}{\zeta-z}\right)^{\gamma}$, $0<\gamma\leq 1$. ■

Theorem 2.5 *Let* q *be convex and univalent in* $U\times\overline{U}$, *such that* $q(0,\zeta)=1$, $m,n\in\mathbb{N}$, $\lambda\geq 0$. *Assume that*

$$\operatorname{Re}\left(\frac{q_{z}'(z,\zeta)}{\mu}(\alpha+2\beta q(z,\zeta))\right)>0,\ \text{for}\ \alpha,\mu,\beta\in\mathbb{C},\ \mu\neq 0, \tag{2.8}$$

$z\in U$, $\zeta\in\overline{U}$.
If $f\in\mathcal{A}_{\zeta}^{*}$, $\frac{DR_{\lambda}^{m+1,n}f(z,\zeta)}{DR_{\lambda}^{m,n}f(z,\zeta)}\in\mathcal{H}^{*}[q(0,\zeta),1,\zeta]\cap Q^{*}$ *and* $\psi_{\lambda}^{m,n}(\alpha,\beta,\mu;z,\zeta)$ *is univalent in* $U\times\overline{U}$, *where* $\psi_{\lambda}^{m,n}(\alpha,\beta,\mu;z,\zeta)$ *is as defined in (2.4), then*

$$\alpha q(z,\zeta)+\beta(q(z,\zeta))^{2}+\mu z q_{z}'(z,\zeta)\prec\prec\psi_{\lambda}^{m,n}(\alpha,\beta,\mu;z,\zeta), \tag{2.9}$$

$z\in U$, $\zeta\in\overline{U}$, *implies*

$$q(z,\zeta)\prec\prec\frac{DR_{\lambda}^{m+1,n}f(z,\zeta)}{DR_{\lambda}^{m,n}f(z,\zeta)},\quad z\in U,\ \zeta\in\overline{U}, \tag{2.10}$$

and q *is the best subordinant.*

Proof. Let the function p be defined by $p(z,\zeta) := \frac{DR_{\lambda}^{m+1,n}f(z,\zeta)}{DR_{\lambda}^{m,n}f(z,\zeta)}$, $z \in U$, $z \neq 0$, $\zeta \in \overline{U}$, $f \in \mathcal{A}_{\zeta}^{*}$.

By setting $\nu(w) := \alpha w + \beta w^2$ and $\phi(w) := \mu$ it can be easily verified that ν is analytic in $\mathbb{C}$, ϕ is analytic in $\mathbb{C}\backslash\{0\}$ and that $\phi(w) \neq 0$, $w \in \mathbb{C}\backslash\{0\}$.
Since $\frac{\nu'_z(q(z,\zeta))}{\phi(q(z,\zeta))} = \frac{q'_z(z,\zeta)}{\mu}(\alpha + 2\beta q(z,\zeta))$, it follows that

$$\operatorname{Re}\left(\frac{\nu'_z(q(z,\zeta))}{\phi(q(z,\zeta))}\right) = \operatorname{Re}\left(\frac{q'_z(z,\zeta)}{\mu}(\alpha + 2\beta q(z,\zeta))\right) > 0,$$

for $\mu, \xi, \beta \in \mathbb{C}$, $\mu \neq 0$.

By using (2.9) we obtain

$$\alpha q(z,\zeta) + \beta(q(z,\zeta))^2 + \mu z q'_z(z,\zeta) \prec\prec \alpha q(z,\zeta) + \beta(q(z,\zeta))^2 + \mu z q'_z(z,\zeta).$$

Using Lemma 1.2, we have

$$q(z,\zeta) \prec\prec p(z,\zeta) = \frac{DR_{\lambda}^{m+1,n}f(z,\zeta)}{DR_{\lambda}^{m,n}f(z,\zeta)}, \quad z \in U,\ \zeta \in \overline{U},$$

and q is the best subordinant. ■

Corollary 2.6 *Let* $q(z,\zeta) = \frac{\zeta + Az}{\zeta + Bz}$, $-1 \leq B < A \leq 1$, $m, n \in \mathbb{N}$, $\lambda \geq 0$. *Assume that (2.8) holds.*

If $f \in \mathcal{A}_{\zeta}^{*}, \frac{DR_{\lambda}^{m+1,n}f(z,\zeta)}{DR_{\lambda}^{m,n}f(z,\zeta)} \in \mathcal{H}^{*}[q(0,\zeta),1,\zeta] \cap Q^{*}$ *and*

$$\alpha\frac{\zeta + Az}{\zeta + Bz} + \beta\left(\frac{\zeta + Az}{\zeta + Bz}\right)^2 + \mu\frac{\zeta(A-B)z}{(\zeta + Bz)^2} \prec\prec \psi_{\lambda}^{m,n}(\alpha,\beta,\mu;z,\zeta),$$

for $\alpha, \mu, \beta \in \mathbb{C}$, $\mu \neq 0$, $-1 \leq B < A \leq 1$, *where* $\psi_{\lambda}^{m,n}$ *is defined in (2.4), then*

$$\frac{\zeta + Az}{\zeta + Bz} \prec\prec \frac{DR_{\lambda}^{m+1,n}f(z,\zeta)}{DR_{\lambda}^{m,n}f(z,\zeta)}$$

and $\frac{\zeta + Az}{\zeta + Bz}$ *is the best subordinant.*

Proof. For $q(z,\zeta) = \frac{\zeta + Az}{\zeta + Bz}$, $-1 \leq B < A \leq 1$ in Theorem 2.5 we get the corollary. ■

Corollary 2.7 *Let* $q(z,\zeta) = \left(\frac{\zeta + z}{\zeta - z}\right)^{\gamma}$, $m, n \in \mathbb{N}$, $\lambda \geq 0$. *Assume that (2.8) holds.*

If $f \in \mathcal{A}_{\zeta}^{*}, \frac{DR_{\lambda}^{m+1,n}f(z,\zeta)}{DR_{\lambda}^{m,n}f(z,\zeta)} \in \mathcal{H}^{*}[q(0,\zeta),1,\zeta] \cap Q^{*}$ *and*

$$\alpha\left(\frac{\zeta + z}{\zeta - z}\right)^{\gamma} + \beta\left(\frac{\zeta + z}{\zeta - z}\right)^{2\gamma} + \mu\frac{2\zeta\gamma z}{(\zeta - z)^2}\left(\frac{\zeta + z}{\zeta - z}\right)^{\gamma - 1} \prec\prec \psi_{\lambda}^{m,n}(\alpha,\beta,\mu;z,\zeta),$$

for $\alpha, \mu, \beta \in \mathbb{C}$, $0 < \gamma \leq 1$, $\mu \neq 0$, *where* $\psi_{\lambda}^{m,n}$ *is defined in (2.4), then*

$$\left(\frac{\zeta + z}{\zeta - z}\right)^{\gamma} \prec\prec \frac{DR_{\lambda}^{m+1,n} f(z,\zeta)}{DR_{\lambda}^{m,n} f(z,\zeta)}$$

and $\left(\frac{\zeta+z}{\zeta-z}\right)^{\gamma}$ *is the best subordinant.*

Proof. Corollary follows by using Theorem 2.5 for $q(z,\zeta) = \left(\frac{\zeta+z}{\zeta-z}\right)^{\gamma}$, $0 < \gamma \leq 1$. ■

Combining Theorem 2.2 and Theorem 2.5, we state the following sandwich theorem.

Theorem 2.8 *Let* q_1 *and* q_2 *be analytic and univalent in* $U \times \overline{U}$ *such that* $q_1(z,\zeta) \neq 0$ *and* $q_2(z,\zeta) \neq 0$, *for all* $z \in U$, $\zeta \in \overline{U}$, *with* $z(q_1)'_z(z,\zeta)$ *and* $z(q_2)'_z(z,\zeta)$ *being starlike univalent. Suppose that* q_1 *satisfies (2.3) and* q_2 *satisfies (2.8). If* $f \in \mathcal{A}^*_{\zeta}$, $\frac{DR_{\lambda}^{m+1,n} f(z,\zeta)}{DR_{\lambda}^{m,n} f(z,\zeta)} \in \mathcal{H}^*[q(0,\zeta),1,\zeta] \cap Q^*$ *and* $\psi_{\lambda}^{m,n}(\alpha,\beta,\mu;z,\zeta)$ *is as defined in (2.4) univalent in* $U \times \overline{U}$, *then*

$$\alpha q_1(z,\zeta) + \beta (q_1(z,\zeta))^2 + \mu z (q_1)'_z(z,\zeta) \prec\prec \psi_{\lambda}^{m,n}(\alpha,\beta,\mu;z,\zeta)$$
$$\prec\prec \alpha q_2(z,\zeta) + \beta (q_2(z,\zeta))^2 + \mu z (q_2)'_z(z,\zeta),$$

for $\alpha, \mu, \beta \in \mathbb{C}$, $\mu \neq 0$, *implies*

$$q_1(z,\zeta) \prec\prec \frac{DR_{\lambda}^{m+1,n} f(z,\zeta)}{DR_{\lambda}^{m,n} f(z,\zeta)} \prec\prec q_2(z,\zeta), \quad \delta \in \mathbb{C},\ \delta \neq 0,$$

and q_1 *and* q_2 *are respectively the best subordinant and the best dominant.*

For $q_1(z,\zeta) = \frac{\zeta+A_1 z}{\zeta+B_1 z}$, $q_2(z,\zeta) = \frac{\zeta+A_2 z}{\zeta+B_2 z}$, where $-1 \leq B_2 < B_1 < A_1 < A_2 \leq 1$, we have the following corollary.

Corollary 2.9 *Let* $m,n \in \mathbb{N}$, $\lambda \geq 0$. *Assume that (2.3) and (2.8) hold for* $q_1(z,\zeta) = \frac{\zeta+A_1 z}{\zeta+B_1 z}$ *and* $q_2(z,\zeta) = \frac{\zeta+A_2 z}{\zeta+B_2 z}$, *respectively. If* $f \in \mathcal{A}^*_{\zeta}$, $\frac{DR_{\lambda}^{m+1,n} f(z,\zeta)}{DR_{\lambda}^{m,n} f(z,\zeta)} \in \mathcal{H}^*[q(0,\zeta),1,\zeta] \cap Q^*$ *and*

$$\alpha \frac{\zeta + A_1 z}{\zeta + B_1 z} + \beta \left(\frac{\zeta + A_1 z}{\zeta + B_1 z}\right)^2 + \mu \frac{(A_1 - B_1)\zeta z}{(\zeta + B_1 z)^2} \prec\prec \psi_{\lambda}^{m,n}(\alpha,\beta,\mu;z,\zeta)$$

$$\prec\prec \alpha \frac{\zeta + A_2 z}{\zeta + B_2 z} + \beta \left(\frac{\zeta + A_2 z}{\zeta + B_2 z}\right)^2 + \mu \frac{(A_2 - B_2)\zeta z}{(\zeta + B_2 z)^2},$$

for $\alpha, \mu, \beta \in \mathbb{C}$, $\mu \neq 0$, $-1 \leq B_2 \leq B_1 < A_1 \leq A_2 \leq 1$, *where* $\psi_{\lambda}^{m,n}$ *is defined in (2.4), then*

$$\frac{\zeta + A_1 z}{\zeta + B_1 z} \prec\prec \frac{DR_{\lambda}^{m+1,n} f(z,\zeta)}{DR_{\lambda}^{m,n} f(z,\zeta)} \prec\prec \frac{\zeta + A_2 z}{\zeta + B_2 z},$$

hence $\frac{\zeta+A_1 z}{\zeta+B_1 z}$ *and* $\frac{\zeta+A_2 z}{\zeta+B_2 z}$ *are the best subordinant and the best dominant, respectively.*

Theorem 2.10 *Let* $\left(\frac{DR_\lambda^{m+1,n}f(z,\zeta)}{DR_\lambda^{m,n}f(z,\zeta)}\right)^\delta \in \mathcal{H}\left(U\times\overline{U}\right), f\in\mathcal{A}_\zeta^*,\ z\in U,\ \zeta\in\overline{U},\ \delta\in\mathbb{C},$ $\delta\neq 0,\ m,n\in\mathbb{N},\ \lambda\geq 0$ *and let the function* $q(z,\zeta)$ *be convex and univalent in* $U\times\overline{U}$ *such that* $q(0,\zeta)=1,\ \zeta\in\overline{U}$. *Assume that*

$$\operatorname{Re}\left(\frac{\alpha+\beta}{\beta}+\frac{zq''_{z^2}(z,\zeta)}{q'_z(z,\zeta)}\right)>0, \tag{2.11}$$

for $\alpha,\beta\in\mathbb{C},\ \beta\neq 0,\ z\in U,\ \zeta\in\overline{U}$, *and*

$$\begin{aligned}\psi_\lambda^{m,n}(\alpha,\beta;z,\zeta) &:= \left(\frac{DR_\lambda^{m+1,n}f(z,\zeta)}{DR_\lambda^{m,n}f(z,\zeta)}\right)^\delta \\ &\cdot\left[\alpha+\delta\beta\frac{1-\lambda(n+1)}{\lambda}+\delta\beta(n+1)\left[1-\lambda(n+2)\right]\frac{DR_\lambda^{m,n+1}f(z,\zeta)}{DR_\lambda^{m+1,n}f(z,\zeta)}\right.\\ &\left.+\delta\beta\lambda(n+1)(n+2)\frac{DR_\lambda^{m,n+2}f(z,\zeta)}{DR_\lambda^{m+1,n}f(z,\zeta)}-\frac{\delta\beta}{\lambda}\frac{DR_\lambda^{m+1,n}f(z,\zeta)}{DR_\lambda^{m,n}f(z,\zeta)}\right]\end{aligned} \tag{2.12}$$

If q *satisfies the following strong differential subordination*

$$\psi_\lambda^{m,n}(\alpha,\beta;z,\zeta)\prec\prec\alpha q(z,\zeta)+\beta zq'_z(z,\zeta), \tag{2.13}$$

for $\alpha,\beta\in\mathbb{C},\ \beta\neq 0,\ z\in U,\ \zeta\in\overline{U}$, *then*

$$\left(\frac{DR_\lambda^{m+1,n}f(z,\zeta)}{DR_\lambda^{m,n}f(z,\zeta)}\right)^\delta\prec\prec q(z,\zeta),\quad z\in U,\ \zeta\in\overline{U},\ \delta\in\mathbb{C},\ \delta\neq 0, \tag{2.14}$$

and q *is the best dominant.*

Proof. Let the function p be defined by $p(z,\zeta):=\left(\frac{DR_\lambda^{m+1,n}f(z,\zeta)}{DR_\lambda^{m,n}f(z,\zeta)}\right)^\delta$, $z\in U$, $z\neq 0,\ \zeta\in\overline{U},\ f\in\mathcal{A}_\zeta^*$. The function p is analytic in $U\times\overline{U}$ and $p(0,\zeta)=1$.
We have

$$\begin{aligned}zp'_z(z,\zeta) &= \delta z\left(\frac{DR_\lambda^{m+1,n}f(z,\zeta)}{DR_\lambda^{m,n}f(z,\zeta)}\right)^\delta\frac{DR_\lambda^{m,n}f(z,\zeta)}{DR_\lambda^{m+1,n}f(z,\zeta)}\left(\frac{DR_\lambda^{m+1,n}f(z,\zeta)}{DR_\lambda^{m,n}f(z,\zeta)}\right)'_z \\ &= \delta\left(\frac{DR_\lambda^{m+1,n}f(z,\zeta)}{DR_\lambda^{m,n}f(z,\zeta)}\right)^\delta\frac{DR_\lambda^{m,n}f(z,\zeta)}{DR_\lambda^{m+1,n}f(z,\zeta)} \\ &\cdot\left(\frac{z\left(DR_\lambda^{m+1,n}f(z,\zeta)\right)'_z}{DR_\lambda^{m,n}f(z,\zeta)}-\frac{DR_\lambda^{m+1,n}f(z,\zeta)}{DR_\lambda^{m,n}f(z,\zeta)}\frac{z\left(DR_\lambda^{m,n}f(z,\zeta)\right)'_z}{DR_\lambda^{m,n}f(z,\zeta)}\right).\end{aligned}$$

By using the identity (2.1) and (2.2), we obtain

$$\begin{aligned} zp'_z(z,\zeta) &= \delta\left(\frac{DR_\lambda^{m+1,n}f(z,\zeta)}{DR_\lambda^{m,n}f(z,\zeta)}\right)^{\delta}\frac{DR_\lambda^{m,n}f(z,\zeta)}{DR_\lambda^{m+1,n}f(z,\zeta)} \\ &\cdot\left[\left(\frac{1-\lambda(n+1)}{\lambda}\right)\frac{DR_\lambda^{m+1,n}f(z,\zeta)}{DR_\lambda^{m,n}f(z,\zeta)}+n+1\right) \\ &\cdot[1-\lambda(n+2)]\frac{DR_\lambda^{m,n+1}f(z,\zeta)}{DR_\lambda^{m,n}f(z,\zeta)}+\lambda(n+1)(n+2)\frac{DR_\lambda^{m,n+2}f(z,\zeta)}{DR_\lambda^{m,n}f(z,\zeta)} \\ &-\frac{1}{\lambda}\left(\frac{DR_\lambda^{m+1,n}f(z,\zeta)}{DR_\lambda^{m,n}f(z,\zeta)}\right)^2\Bigg] \end{aligned} \tag{2.15}$$

so, we obtain

$$zp'_z(z,\zeta)=\delta\left(\frac{DR_\lambda^{m+1,n}f(z,\zeta)}{DR_\lambda^{m,n}f(z,\zeta)}\right)^{\delta}\left[\frac{1-\lambda(n+1)}{\lambda}+\right.$$

$$(n+1)[1-\lambda(n+2)]\frac{DR_\lambda^{m,n+1}f(z,\zeta)}{DR_\lambda^{m+1,n}f(z,\zeta)}+$$

$$\left.\lambda(n+1)(n+2)\frac{DR_\lambda^{m,n+2}f(z,\zeta)}{DR_\lambda^{m+1,n}f(z,\zeta)}-\frac{1}{\lambda}\frac{DR_\lambda^{m+1,n}f(z,\zeta)}{DR_\lambda^{m,n}f(z,\zeta)}\right] \tag{2.16}$$

By setting $\theta(w) := \alpha w$ and $\phi(w) := \beta$, it can be easily verified that θ is analytic in $\mathbb{C}$, ϕ is analytic in $\mathbb{C}\backslash\{0\}$ and that $\phi(w) \neq 0$, $w \in \mathbb{C}\backslash\{0\}$.

Also, by letting $Q(z,\zeta) = zq'_z(z,\zeta)\phi(q(z,\zeta)) = \beta zq'_z(z,\zeta)$, we find that $Q(z,\zeta)$ is starlike univalent in $U \times \overline{U}$.

Let $h(z,\zeta) = \theta(q(z,\zeta)) + Q(z,\zeta) = \alpha q(z,\zeta) + \beta zq'_z(z,\zeta)$.

We have $\operatorname{Re}\left(\frac{zh'_z(z,\zeta)}{Q(z,\zeta)}\right) = \operatorname{Re}\left(\frac{\alpha+\beta}{\beta} + \frac{zq''_{z^2}(z,\zeta)}{q'_z(z,\zeta)}\right) > 0$.

By using (2.16), we obtain

$$\alpha p(z,\zeta)+\beta zp'_z(z,\zeta)=\left(\frac{DR_\lambda^{m+1,n}f(z,\zeta)}{DR_\lambda^{m,n}f(z,\zeta)}\right)^{\delta}$$

$$\cdot\left[\alpha+\delta\beta\frac{1-\lambda(n+1)}{\lambda}+\delta\beta(n+1)[1-\lambda(n+2)]\frac{DR_\lambda^{m,n+1}f(z,\zeta)}{DR_\lambda^{m+1,n}f(z,\zeta)}\right.$$

$$\left.+\delta\beta\lambda(n+1)(n+2)\frac{DR_\lambda^{m,n+2}f(z,\zeta)}{DR_\lambda^{m+1,n}f(z,\zeta)}-\frac{\delta\beta}{\lambda}\frac{DR_\lambda^{m+1,n}f(z,\zeta)}{DR_\lambda^{m,n}f(z,\zeta)}\right].$$

By using (2.13), we have $\alpha p(z,\zeta) + \beta zp'_z(z,\zeta) \prec\prec \alpha q(z,\zeta) + \beta zq'_z(z,\zeta)$.

From Lemma 1.1, we have $p(z,\zeta) \prec\prec q(z,\zeta)$, $z \in U$, $\zeta \in \overline{U}$, i.e. $\left(\frac{DR_\lambda^{m+1,n}f(z,\zeta)}{DR_\lambda^{m,n}f(z,\zeta)}\right)^{\delta} \prec\prec$ $q(z,\zeta)$, $z \in U$, $\zeta \in \overline{U}$, $\delta \in \mathbb{C}$, $\delta \neq 0$ and q is the best dominant. ■

Corollary 2.11 *Let* $q(z,\zeta)=\frac{\zeta+Az}{\zeta+Bz}$, $z\in U$, $\zeta\in\overline{U}$, $-1\le B<A\le 1$, $m,n\in\mathbb{N}$, $\lambda\ge 0$. *Assume that (2.11) holds. If* $f\in\mathcal{A}_\zeta^*$ *and*

$$\psi_\lambda^{m,n}(\alpha,\beta;z,\zeta)\prec\prec\alpha\frac{\zeta+Az}{\zeta+Bz}+\beta\frac{(A-B)\zeta z}{(\zeta+Bz)^2},$$

for $\alpha,\beta\in\mathbb{C}$, $\beta\neq 0$, $-1\le B<A\le 1$, *where* $\psi_\lambda^{m,n}$ *is defined in (2.12), then*

$$\left(\frac{DR_\lambda^{m+1,n}f(z,\zeta)}{DR_\lambda^{m,n}f(z,\zeta)}\right)^\delta\prec\prec\frac{\zeta+Az}{\zeta+Bz},\quad \delta\in\mathbb{C},\ \delta\neq 0,$$

and $\frac{\zeta+Az}{\zeta+Bz}$ *is the best dominant.*

Proof. For $q(z,\zeta)=\frac{\zeta+Az}{\zeta+Bz}$, $-1\le B<A\le 1$, in Theorem 2.10 we get the corollary. ∎

Corollary 2.12 *Let* $q(z,\zeta)=\left(\frac{\zeta+z}{\zeta-z}\right)^\gamma$, $m,n\in\mathbb{N}$, $\lambda\ge 0$. *Assume that (2.11) holds. If* $f\in\mathcal{A}_\zeta^*$ *and*

$$\psi_\lambda^{m,n}(\alpha,\beta,\mu;z,\zeta)\prec\prec\alpha\left(\frac{\zeta+z}{\zeta-z}\right)^\gamma+\beta\frac{2\gamma\zeta z}{(\zeta-z)^2}\left(\frac{\zeta+z}{\zeta-z}\right)^{\gamma-1},$$

for $\alpha,\beta\in\mathbb{C}$, $0<\gamma\le 1$, $\beta\neq 0$, *where* $\psi_\lambda^{m,n}$ *is defined in (2.12), then*

$$\left(\frac{DR_\lambda^{m+1,n}f(z,\zeta)}{DR_\lambda^{m,n}f(z,\zeta)}\right)^\delta\prec\prec\left(\frac{\zeta+z}{\zeta-z}\right)^\gamma,\quad \delta\in\mathbb{C},\ \delta\neq 0,$$

and $\left(\frac{\zeta+z}{\zeta-z}\right)^\gamma$ *is the best dominant.*

Proof. Corollary follows by using Theorem 2.10 for $q(z,\zeta)=\left(\frac{\zeta+z}{\zeta-z}\right)^\gamma$, $0<\gamma\le 1$. ∎

Theorem 2.13 *Let* q *be convex and univalent in* $U\times\overline{U}$ *such that* $q(0,\zeta)=1$. *Assume that*

$$\mathrm{Re}\left(\frac{\alpha}{\beta}q_z'(z,\zeta)\right)>0,\ \text{for } \alpha,\beta\in\mathbb{C},\ \beta\neq 0. \tag{2.17}$$

If $f\in\mathcal{A}_\zeta^*$, $\left(\frac{DR_\lambda^{m+1,n}f(z,\zeta)}{DR_\lambda^{m,n}f(z,\zeta)}\right)^\delta\in\mathcal{H}^*[q(0,\zeta),1,\zeta]\cap Q^*$ *and* $\psi_\lambda^{m,n}(\alpha,\beta;z,\zeta)$ *is univalent in* $U\times\overline{U}$, *where* $\psi_\lambda^{m,n}(\alpha,\beta;z,\zeta)$ *is as defined in (2.12), then*

$$\alpha q(z,\zeta)+\beta z q_z'(z,\zeta)\prec\prec\psi_\lambda^{m,n}(\alpha,\beta;z,\zeta) \tag{2.18}$$

implies

$$q(z,\zeta)\prec\prec\left(\frac{DR_\lambda^{m+1,n}f(z,\zeta)}{DR_\lambda^{m,n}f(z,\zeta)}\right)^\delta,\quad \delta\in\mathbb{C},\ \delta\neq 0,\ z\in U,\ \zeta\in\overline{U}, \tag{2.19}$$

and q *is the best subordinant.*

Proof. Let the function p be defined by $p(z,\zeta) := \left(\frac{DR_\lambda^{m+1,n} f(z,\zeta)}{DR_\lambda^{m,n} f(z,\zeta)}\right)^\delta$, $z \in U$, $z \neq 0$, $\zeta \in \overline{U}$, $\delta \in \mathbb{C}, \delta \neq 0$, $f \in \mathcal{A}_\zeta^*$. The function p is analytic in $U \times \overline{U}$ and $p(0,\zeta) = 1$.

By setting $\nu(w) := \alpha w$ and $\phi(w) := \beta$ it can be easily verified that ν is analytic in $\mathbb{C}$, ϕ is analytic in $\mathbb{C}\backslash\{0\}$ and that $\phi(w) \neq 0$, $w \in \mathbb{C}\backslash\{0\}$.

Since $\frac{\nu_z'(q(z,\zeta))}{\phi(q(z,\zeta))} = \frac{\alpha}{\beta} q_z'(z,\zeta)$, it follows that
$\operatorname{Re}\left(\frac{\nu_z'(q(z,\zeta))}{\phi(q(z,\zeta))}\right) = \operatorname{Re}\left(\frac{\alpha}{\beta} q_z'(z,\zeta)\right) > 0$, for $\alpha, \beta \in \mathbb{C}$, $\beta \neq 0$.

Now, by using (2.18) we obtain

$$\alpha q(z,\zeta) + \beta z q_z'(z,\zeta) \prec\prec \alpha q(z,\zeta) + \beta z q_z'(z,\zeta), \quad z \in U,\ \zeta \in \overline{U}.$$

From Lemma 1.2, we have

$$q(z,\zeta) \prec\prec p(z,\zeta) = \left(\frac{DR_\lambda^{m+1,n} f(z,\zeta)}{DR_\lambda^{m,n} f(z,\zeta)}\right)^\delta,$$

$z \in U$, $\zeta \in \overline{U}$, $\delta \in \mathbb{C}$, $\delta \neq 0$, and q is the best subordinant. ■

Corollary 2.14 *Let* $q(z,\zeta) = \frac{\zeta + Az}{\zeta + Bz}$, $-1 \leq B < A \leq 1$, $z \in U$, $\zeta \in \overline{U}$, $m, n \in \mathbb{N}$, $\lambda \geq 0$. *Assume that (2.17) holds. If* $f \in \mathcal{A}_\zeta^*$, $\left(\frac{DR_\lambda^{m+1,n} f(z,\zeta)}{DR_\lambda^{m,n} f(z,\zeta)}\right)^\delta \in \mathcal{H}^*[q(0,\zeta), 1, \zeta] \cap Q^*$, $\delta \in \mathbb{C}$, $\delta \neq 0$ *and*

$$\alpha \frac{\zeta + Az}{\zeta + Bz} + \beta \frac{(A-B)\zeta z}{(\zeta + Bz)^2} \prec\prec \psi_\lambda^{m,n}(\alpha, \beta; z, \zeta),$$

for $\alpha, \beta \in \mathbb{C}$, $\beta \neq 0$, $-1 \leq B < A \leq 1$, *where* $\psi_\lambda^{m,n}$ *is defined in (2.12), then*

$$\frac{\zeta + Az}{\zeta + Bz} \prec\prec \left(\frac{DR_\lambda^{m+1,n} f(z,\zeta)}{DR_\lambda^{m,n} f(z,\zeta)}\right)^\delta, \delta \in \mathbb{C},\ \delta \neq 0,$$

and $\frac{\zeta + Az}{\zeta + Bz}$ *is the best subordinant.*

Proof. For $q(z,\zeta) = \frac{\zeta + Az}{\zeta + Bz}$, $-1 \leq B < A \leq 1$, in Theorem 2.13 we get the corollary. ■

Corollary 2.15 *Let* $q(z,\zeta) = \left(\frac{\zeta + z}{\zeta - z}\right)^\gamma$, $m, n \in \mathbb{N}$, $\lambda \geq 0$. *Assume that (2.17) holds. If* $f \in \mathcal{A}_\zeta^*$, $\left(\frac{DR_\lambda^{m+1,n} f(z,\zeta)}{DR_\lambda^{m,n} f(z,\zeta)}\right)^\delta \in \mathcal{H}^*[q(0,\zeta), 1, \zeta] \cap Q^*$ *and*

$$\alpha \left(\frac{\zeta + z}{\zeta - z}\right)^\gamma + \beta \frac{2\gamma\zeta z}{(\zeta - z)^2} \left(\frac{\zeta + z}{\zeta - z}\right)^{\gamma - 1} \prec\prec \psi_\lambda^{m,n}(\alpha, \beta, \mu; z, \zeta),$$

for $\alpha,\beta\in\mathbb{C}$, $0<\gamma\leq 1$, $\beta\neq 0$, *where* $\psi_\lambda^{m,n}$ *is defined in (2.12), then*

$$\left(\frac{\zeta+z}{\zeta-z}\right)^\gamma \prec\prec \left(\frac{DR_\lambda^{m+1,n}f(z,\zeta)}{DR_\lambda^{m,n}f(z,\zeta)}\right)^\delta, \quad \delta\in\mathbb{C},\ \delta\neq 0,$$

and $\left(\frac{\zeta+z}{\zeta-z}\right)^\gamma$ *is the best subordinant.*

Proof. Corollary follows by using Theorem 2.13 for $q(z,\zeta)=\left(\frac{\zeta+z}{\zeta-z}\right)^\gamma$, $0<\gamma\leq 1$. ∎

Combining Theorem 2.10 and Theorem 2.13, we state the following sandwich theorem.

Theorem 2.16 *Let* q_1 *and* q_2 *be convex and univalent in* $U\times\overline{U}$ *such that* $q_1(z,\zeta)\neq 0$ *and* $q_2(z,\zeta)\neq 0$, *for all* $z\in U$, $\zeta\in\overline{U}$. *Suppose that* q_1 *satisfies (2.11) and* q_2 *satisfies (2.17). If* $f\in\mathcal{A}^*_\zeta$, $\left(\frac{DR_\lambda^{m+1,n}f(z,\zeta)}{DR_\lambda^{m,n}f(z,\zeta)}\right)^\delta \in \mathcal{H}^*[q(0,\zeta),1,\zeta]\cap Q^*$, $\delta\in\mathbb{C}$, $\delta\neq 0$ *and* $\psi_\lambda^{m,n}(\alpha,\beta;z,\zeta)$ *is as defined in (2.12) univalent in* $U\times\overline{U}$, *then*

$$\alpha q_1(z,\zeta)+\beta z(q_1)'_z(z,\zeta) \prec\prec \psi_\lambda^{m,n}(\alpha,\beta;z,\zeta)$$
$$\prec\prec \alpha q_2(z,\zeta)+\beta z(q_2)'_z(z,\zeta),$$

for $\alpha,\beta\in\mathbb{C}$, $\beta\neq 0$, *implies*

$$q_1(z,\zeta) \prec\prec \left(\frac{DR_\lambda^{m+1,n}f(z,\zeta)}{DR_\lambda^{m,n}f(z,\zeta)}\right)^\delta \prec\prec q_2(z,\zeta),$$

$z\in U$, $\zeta\in\overline{U}$, $\delta\in\mathbb{C}$, $\delta\neq 0$, *and* q_1 *and* q_2 *are respectively the best subordinant and the best dominant.*

For $q_1(z,\zeta)=\frac{\zeta+A_1z}{\zeta+B_1z}$, $q_2(z,\zeta)=\frac{\zeta+A_2z}{\zeta+B_2z}$, where $-1\leq B_2<B_1<A_1<A_2\leq 1$, we have the following corollary.

Corollary 2.17 *Let* $m,n\in\mathbb{N}$, $\lambda\geq 0$. *Assume that (2.11) and (2.17) hold for* $q_1(z,\zeta)=\frac{\zeta+A_1z}{\zeta+B_1z}$ *and* $q_2(z,\zeta)=\frac{\zeta+A_2z}{\zeta+B_2z}$, *respectively. If* $f\in\mathcal{A}^*_\zeta$, $\left(\frac{DR_\lambda^{m+1,n}f(z,\zeta)}{DR_\lambda^{m,n}f(z,\zeta)}\right)^\delta\in \mathcal{H}^*[q(0,\zeta),1,\zeta]\cap Q^*$ *and*

$$\alpha\frac{\zeta+A_1z}{\zeta+B_1z}+\beta\frac{(A_1-B_1)\zeta z}{(\zeta+B_1z)^2} \prec\prec \psi_\lambda^{m,n}(\alpha,\beta,\mu;z,\zeta)$$
$$\prec\prec \alpha\frac{\zeta+A_2z}{\zeta+B_2z}+\beta\frac{(A_2-B_2)\zeta z}{(\zeta+B_2z)^2}, \quad z\in U,\ \zeta\in\overline{U},$$

for $\alpha,\beta\in\mathbb{C}$, $\beta\neq 0$, $-1\leq B_2\leq B_1<A_1\leq A_2\leq 1$, *where* $\psi_\lambda^{m,n}$ *is defined in (2.4), then*

$$\frac{\zeta+A_1z}{\zeta+B_1z} \prec\prec \left(\frac{DR_\lambda^{m+1,n}f(z,\zeta)}{DR_\lambda^{m,n}f(z,\zeta)}\right)^\delta \prec\prec \frac{\zeta+A_2z}{\zeta+B_2z},$$

$z\in U$, $\zeta\in\overline{U}$, $\delta\in\mathbb{C}$, $\delta\neq 0$, *hence* $\frac{\zeta+A_1z}{\zeta+B_1z}$ *and* $\frac{\zeta+A_2z}{\zeta+B_2z}$ *are the best subordinant and the best dominant, respectively.*

References

[1] A. Alb Lupaş, *On special strong differential superordinations using Sălăgean and Ruscheweyh operators*, submitted Jokull Journal.

[2] A. Alb Lupaş, *On special strong differential subordinations using a generalized Sălăgean operator and Ruscheweyh derivative*, Journal of Concrete and Applicable Mathematics, Vol. 10, No.'s 1-2, 2012, 17-23.

[3] A. Alb Lupaş, G.I. Oros, Gh. Oros, *On special strong differential subordinations using Sălăgean and Ruscheweyh operators*, Journal of Computational Analysis and Applications, Vol. 14, No. 2, 2012, 266-270.

[4] A. Alb Lupaş, *A note on strong differential subordinations using a generalized Sălăgean operator and Ruscheweyh operator*, Acta Universitatis Apulensis No. 34/2013, 105-114.

[5] A. Alb Lupaş, *Certain strong differential superordinations using a generalized Sălăgean operator and Ruscheweyh operator*, Journal of Applied Functional Analysis, Vol. 7, No.'s 1-2, 2012, 62-68.

[6] A. Alb Lupaş, *A note on strong differential superordinations using a generalized Sălăgean operator and Ruscheweyh operator*, Stud. Univ. Babes-Bolyai Math. 57(2012), No. 2, 153–165.

[7] A. Alb Lupaş, *Certain strong differential subordinations using Sălăgean and Ruscheweyh operators*, Advances in Applied Mathematical Analysis, Volume 6, Number 1 (2011), 27–34.

[8] A. Alb Lupaş, *A note on strong differential subordinations using Sălăgean and Ruscheweyh operators*, Libertas Mathematica, tomus XXXI (2011), 15-21.

[9] A. Alb Lupaş, *Certain strong differential superordinations using Sălăgean and Ruscheweyh operators*, Acta Universitatis Apulensis No. 30/2012, 325-336.

[10] A. Alb Lupaş, *A note on strong differential superordinations using Sălăgean and Ruscheweyh operators*, Journal of Applied Functional Analysis, Vol. 7, No.'s 1-2, 2012, 54-61.

[11] L. Andrei, *On some differential sandwich theorems using a generalized Sălăgean operator and Ruscheweyh operator*, Journal of Computational Analysis and Applications (to appear).

[12] L. Andrei, *Differential Sandwich Theorems using an extending generalized Sălăgean operator and extended Ruscheweyh operator*, submitted GFTA 2014.

[13] L. Andrei, *Strong differential subordination results using a generalized Sălăgean operator and Ruscheweyh operator*, Journal of Computational Analysis and Applications (to appear).

[14] L. Andrei, *Some strong differential subordination results using a generalized Sălăgean operator and Ruscheweyh operator*, submitted International Journal of Modern Mathematical Sciences.

[15] L. Andrei, *Strong differential superordination results using a generalized Sălăgean operator and Ruscheweyh operator*, submitted Journal of Computational Mathematics.

[16] L. Andrei, *Some strong differential superordination results using a generalized Sălăgean operator and Ruscheweyh operator*, International Journal of Research and Reviews in Applied Sciences (to appear).

[17] J.A. Antonino, S. Romaguera, *Strong differential subordination to Briot-Bouquet differential equations*, Journal of Differential Equations, 114 (1994), 101-105.

[18] G.I. Oros, Gh. Oros, *Strong differential subordination*, Turkish Journal of Mathematics, 33 (2009), 249-257.

[19] G.I. Oros, *Strong differential superordination*, Acta Universitatis Apulensis, 19 (2009), 101-106.

DOI: 10.7862/rf.2015.3

Loriana Andrei
email: lori_andrei@yahoo.com
Department of Mathematics and Computer Science
University of Oradea
1 Universitatii street, 410087 Oradea, Romania
Received 16.03.2014