

Structure of solutions of nonautonomous optimal control problems in metric spaces

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ABSTRACT: We establish turnpike results for a nonautonomous discrete-time optimal control system describing a model of economic dynamics.

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1 Introduction

The study of the existence, the structure and properties of (approximate) solutions of optimal control problems defined on infinite intervals and on sufficiently large intervals has recently been a rapidly growing area of research [4-8, 10, 11, 14, 15, 16, 18-20, 22, 23, 27, 30]. These problems arise in engineering [1, 32], in models of economic growth [2, 9, 12, 17, 21, 24, 25, 27-29, 31], in infinite discrete models of solid-state physics related to dislocations in one-dimensional crystals [3, 26] and in the theory of thermodynamical equilibrium for materials [13, 16].

In this paper we study the structure of approximate solutions of nonautonomous discrete-time optimal control systems arising in economic dynamics which are determined by sequences of lower semicontinuous objective functions.

For each nonempty set Y denote by $\mathcal{B}(Y)$ the set of all bounded functions $f : Y \rightarrow \mathbb{R}^1$ and for each $f \in \mathcal{B}(Y)$ set

$$\|f\| = \sup\{|f(y)| : y \in Y\}.$$

For each nonempty compact metric space Y denote by $C(Y)$ the set of all continuous functions $f : Y \rightarrow \mathbb{R}^1$.

Let (X, ρ) be a compact metric space with the metric ρ . The set $X \times X$ is equipped with the metric ρ_1 defined by

$$\rho_1((x_1, x_2), (y_1, y_2)) = \rho(x_1, y_1) + \rho(x_2, y_2), \quad (x_1, x_2), (y_1, y_2) \in X \times X.$$

For each integer $t \geq 0$ let Ω_t be a nonempty closed subset of the metric space $X \times X$.

Let $T \geq 0$ be an integer. A sequence $\{x_t\}_{t=0}^\infty \subset X$ is called a program if $(x_t, x_{t+1}) \in \Omega_t$ for all integers $t \geq 0$.

Let T_1, T_2 be integers such that $0 \leq T_1 < T_2$. A sequence $\{x_t\}_{t=T_1}^{T_2} \subset X$ is called a program if $(x_t, x_{t+1}) \in \Omega_t$ for all integers t satisfying $T_1 \leq t < T_2$.

We assume that there exists a program $\{x_t\}_{t=0}^\infty$. Denote by \mathcal{M} the set of all sequences of functions $\{f_t\}_{t=0}^\infty$ such that for each integer $t \geq 0$

$$f_t \in \mathcal{B}(\Omega_t) \tag{1.1}$$

and that

$$\sup\{\|f_t\| : t = 0, 1, \dots\} < \infty. \tag{1.2}$$

For each pair of sequences $\{f_t\}_{t=0}^\infty, \{g_t\}_{t=0}^\infty \in \mathcal{M}$ set

$$d(\{f_t\}_{t=0}^\infty, \{g_t\}_{t=0}^\infty) = \sup\{\|f_t - g_t\| : t = 0, 1, \dots\}. \tag{1.3}$$

It is easy to see that $d : \mathcal{M} \times \mathcal{M} \rightarrow [0, \infty)$ is a metric on \mathcal{M} and that the metric space (\mathcal{M}, d) is complete.

Let $\{f_t\}_{t=0}^\infty \in \mathcal{M}$. We consider the following optimization problems

$$\sum_{t=T_1}^{T_2-1} f_t(x_t, x_{t+1}) \rightarrow \min \text{ s. t. } \{x_t\}_{t=T_1}^{T_2} \text{ is a program,}$$

$$\sum_{t=T_1}^{T_2-1} f_t(x_t, x_{t+1}) \rightarrow \min \text{ s. t. } \{x_t\}_{t=T_1}^{T_2} \text{ is a program and } x_{T_1} = y,$$

$$\sum_{t=T_1}^{T_2-1} f_t(x_t, x_{t+1}) \rightarrow \min \text{ s. t. } \{x_t\}_{t=T_1}^{T_2} \text{ is a program and } x_{T_1} = y, x_{T_2} = z,$$

where $y, z \in X$ and integers T_1, T_2 satisfy $0 \leq T_1 < T_2$.

The interest in these discrete-time optimal problems stems from the study of various optimization problems which can be reduced to this framework, e. g., continuous-time control systems which are represented by ordinary differential equations whose cost integrand contains a discounting factor [12], the study of the discrete Frenkel-Kontorova model related to dislocations in one-dimensional crystals [3, 26] and the analysis of a long slender bar of a polymeric material under tension in [13, 16]. Similar optimization problems are also considered in mathematical economics [9, 17, 24, 28, 29, 31].

For each $y, z \in X$ and each pair of integers T_1, T_2 satisfying $0 \leq T_1 < T_2$ set

$$U(\{f_t\}_{t=0}^\infty, T_1, T_2) = \inf\left\{\sum_{t=T_1}^{T_2-1} f_t(x_t, x_{t+1}) : \{x_t\}_{t=T_1}^{T_2} \text{ is a program}\right\}, \quad (1.4)$$

$$U(\{f_t\}_{t=0}^\infty, T_1, T_2, y) = \inf\left\{\sum_{t=T_1}^{T_2-1} f_t(x_t, x_{t+1}) : \{x_t\}_{t=T_1}^{T_2} \text{ is a program and } x_{T_1} = y\right\}, \quad (1.5)$$

$$U(\{f_t\}_{t=0}^\infty, T_1, T_2, y, z) = \inf\left\{\sum_{t=T_1}^{T_2-1} f_t(x_t, x_{t+1}) : \{x_t\}_{t=T_1}^{T_2} \text{ is a program and } x_{T_1} = y, x_{T_2} = z\right\}. \quad (1.6)$$

Here we assume that the infimum over empty set is ∞ .

Denote by \mathcal{M}_{reg} the set of all sequences of functions $\{f_i\}_{i=0}^\infty \in \mathcal{M}$ for which there exist a program $\{x_t^f\}_{t=0}^\infty$ and constants $c_f > 0$, $\gamma_f > 0$ such that the following conditions hold:

- (C1) the function f_t is lower semicontinuous for all integers $t \geq 0$;
- (C2) for each pair of integers $T_1 \geq 0$, $T_2 > T_1$,

$$\sum_{t=T_1}^{T_2-1} f_t(x_t^f, x_{t+1}^f) \leq U(\{f_t\}_{t=0}^\infty, T_1, T_2) + c_f;$$

- (C3) for each $\epsilon > 0$ there exists $\delta > 0$ such that for each integer $t \geq 0$ and each $(x, y) \in \Omega_t$ satisfying $\rho(x, x_t^f) \leq \delta$, $\rho(y, x_{t+1}^f) \leq \delta$ we have

$$|f_t(x_t^f, x_{t+1}^f) - f_t(x, y)| \leq \epsilon;$$

- (C4) for each integer $t \geq 0$, each $(x_t, x_{t+1}) \in \Omega_t$ satisfying $\rho(x_t, x_t^f) \leq \gamma_f$ and each $(x'_{t+1}, x'_{t+2}) \in \Omega_{t+1}$ satisfying $\rho(x'_{t+2}, x'_{t+2}^f) \leq \gamma_f$ there is $x \in X$ such that

$$(x_t, x) \in \Omega_t, (x, x'_{t+2}) \in \Omega_{t+1};$$

moreover, for each $\epsilon > 0$ there exists $\delta \in (0, \gamma_f)$ such that for each integer $t \geq 0$, each $(x_t, x_{t+1}) \in \Omega_t$ and each $(x'_{t+1}, x'_{t+2}) \in \Omega_{t+1}$ satisfying $\rho(x_t, x_t^f) \leq \delta$ and $\rho(x'_{t+2}, x'_{t+2}^f) \leq \delta$ there is $x \in X$ such that

$$(x_t, x) \in \Omega_t, (x, x'_{t+2}) \in \Omega_{t+1}, \rho(x, x_{t+1}^f) \leq \epsilon.$$

Denote by $\bar{\mathcal{M}}_{reg}$ the closure of \mathcal{M}_{reg} in (\mathcal{M}, d) . Denote by $\mathcal{M}_{c,reg}$ the set of all sequences $\{f_i\}_{i=0}^\infty \in \mathcal{M}_{reg}$ such that $f_i \in C(\Omega_i)$ for all integers $i \geq 0$ and by $\bar{\mathcal{M}}_{c,reg}$ the closure of $\mathcal{M}_{c,reg}$ in (\mathcal{M}, d) .

We study the optimization problems stated above with the sequence of objective functions $\{f_i\}_{i=0}^\infty \in \mathcal{M}_{reg}$. Our study is based on the relation between these finite

horizon problems and the corresponding infinite horizon optimization problem determined by $\{f_i\}_{i=0}^\infty$. Note that the condition (C2) means that the program $\{x_i^f\}_{i=0}^\infty$ is an approximate solution of this infinite horizon problem.

We are interested in turnpike properties of approximate solutions of our optimization problems, which are independent of the length of the interval $T_2 - T_1$, for all sufficiently large intervals. To have these properties means that the approximate solutions of the problems are determined mainly by the objective functions, and are essentially independent of the choice of interval and endpoint conditions, except in regions close to the endpoints. Turnpike properties are well known in mathematical economics. The term was first coined by Samuelson in 1948 (see [26]) where he showed that an efficient expanding economy would spend most of the time in the vicinity of a balanced equilibrium path (also called a von Neumann path).

The paper is organized as follows. In Section 2 we present turnpike results and show the existence of optimal solutions over infinite horizon established in [31]. Our main results (Theorems 3.1 and 3.2) are stated in Section 3. Section 4 contains an example. Our auxiliary results are proved in Section 5. Section 6 contains the proof of Theorem 3.1 while Theorem 3.2 is proved in Section 7.

2 Preliminaries

Let $\{f_i\}_{i=0}^\infty \in \mathcal{M}_{reg}$, a program $\{x_i^f\}_{i=0}^\infty$, $c_f > 0$ and $\gamma_f > 0$ be such that (C1)-(C4) hold.

In [31] we proved the following useful result.

Proposition 2.1 *Let $S \geq 0$ be an integer and $\{x_i\}_{i=S}^\infty$ be a program. Then either the sequence $\{\sum_{i=S}^{T-1} f_i(x_i, x_{i+1}) - \sum_{i=S}^{T-1} f_i(x_i^f, x_{i+1}^f)\}_{T=S+1}^\infty$ is bounded or*

$$\lim_{T \rightarrow \infty} \left[\sum_{i=S}^{T-1} f_i(x_i, x_{i+1}) - \sum_{i=S}^{T-1} f_i(x_i^f, x_{i+1}^f) \right] = \infty.$$

A program $\{x_t\}_{t=S}^\infty$, where $S \geq 0$ is an integer, is called $(\{f_i\}_{i=0}^\infty)$ -good if the sequence

$$\left\{ \sum_{i=S}^{T-1} f_i(x_i, x_{i+1}) - \sum_{i=S}^{T-1} f_i(x_i^f, x_{i+1}^f) \right\}_{T=S+1}^\infty$$

is bounded [9, 27-29, 31].

We say that the sequence $\{f_i\}_{i=0}^\infty$ possesses an asymptotic turnpike property (or briefly (ATP)) [31] with $\{x_i^f\}_{i=0}^\infty$ being the turnpike if for each integer $S \geq 0$ and each $(\{f_i\}_{i=0}^\infty)$ -good program $\{x_i\}_{i=S}^\infty$,

$$\lim_{i \rightarrow \infty} \rho(x_i, x_i^f) = 0.$$

We say that the sequence $\{f_i\}_{i=0}^\infty$ possesses a turnpike property (or briefly (TP)) [31] if for each $\epsilon > 0$ and each $M > 0$ there exist $\delta > 0$ and a natural number L such

that for each pair of integers $T_1 \geq 0$, $T_2 \geq T_1 + 2L$ and each program $\{x_t\}_{t=T_1}^{T_2}$ which satisfies

$$\sum_{i=T_1}^{T_2-1} f_i(x_i, x_{i+1}) \leq \min\{U(\{f_i\}_{i=0}^\infty, T_1, T_2, x_{T_1}, x_{T_2}) + \delta, U(\{f_i\}_{i=0}^\infty, T_1, T_2) + M\},$$

the inequality $\rho(x_i, x_i^f) \leq \epsilon$ holds for all integers $i = T_1 + L, \dots, T_2 - L$.

The sequence $\{x_i^f\}_{i=0}^\infty$ is called the turnpike of $\{f_i\}_{i=0}^\infty$.

In [31] we proved the following results (see Theorems 2.1-2.4).

Theorem 2.1 *The sequence $\{f_i\}_{i=0}^\infty$ possesses the turnpike property if and only if $\{f_i\}_{i=0}^\infty$ possesses (ATP) and the following property:*

(P) *For each $\epsilon > 0$ and each $M > 0$ there exist $\delta > 0$ and a natural number L such that for each integer $T \geq 0$ and each program $\{x_t\}_{t=T}^{T+L}$ which satisfies*

$$\begin{aligned} & \sum_{i=T}^{T+L-1} f_i(x_i, x_{i+1}) \\ & \leq \min\{U(\{f_i\}_{i=0}^\infty, T, T+L, x_T, x_{T+L}) + \delta, U(\{f_i\}_{i=0}^\infty, T, T+L) + M\} \end{aligned}$$

there is an integer $j \in \{T, \dots, T+L\}$ for which $\rho(x_j, x_j^f) \leq \epsilon$.

The property (P) means that if a natural number L is large enough and a program $\{x_t\}_{t=T}^{T+L}$ is an approximate solution of the corresponding finite horizon problem, then there is $j \in \{T, \dots, T+L\}$ such that x_j is close to x_j^f .

We denote by $\text{Card}(A)$ the cardinality of the set A .

Theorem 2.2 *Assume that the sequence $\{f_i\}_{i=0}^\infty$ possesses (ATP) and the property (P), $\epsilon > 0$ and $M > 0$. Then there exists a natural number L such that for each pair of integers $T_1 \geq 0$, $T_2 > T_1 + L$ and each program $\{x_t\}_{t=T_1}^{T_2}$ which satisfies*

$$\sum_{t=T_1}^{T_2-1} f_t(x_t, x_{t+1}) \leq U(\{f_i\}_{i=0}^\infty, T_1, T_2) + M$$

the following inequality holds:

$$\text{Card}(\{t \in \{T_1, \dots, T_2\} : \rho(x_t, x_t^f) > \epsilon\}) \leq L.$$

Let $S \geq 0$ be an integer. A program $\{x_t\}_{t=S}^\infty$ is called $(\{f_i\}_{i=0}^\infty)$ -minimal [3, 26, 31] if for each integer $T > S$,

$$\sum_{t=S}^{T-1} f_t(x_t, x_{t+1}) = U(\{f_i\}_{i=0}^\infty, S, T, x_S, x_T).$$

A program $\{x_t\}_{t=S}^\infty$ is called $(\{f_i\}_{i=0}^\infty)$ -overtaking optimal [12, 27, 31] if for each program $\{x'_t\}_{t=S}^\infty$ satisfying $x_S = x'_S$,

$$\limsup_{T \rightarrow \infty} \left(\sum_{t=S}^{T-1} f_t(x_t, x_{t+1}) - \sum_{t=S}^{T-1} f_t(x'_t, x'_{t+1}) \right) \leq 0.$$

Theorem 2.3 Assume that the sequence $\{f_i\}_{i=0}^\infty$ possesses (ATP), $z \in X$, $S \geq 0$ is an integer and that there exists an $(\{f_i\}_{i=0}^\infty)$ -good program $\{x_t\}_{t=S}^\infty$ satisfying $x_S = z$. Then there exists an $(\{f_i\}_{i=0}^\infty)$ -overtaking optimal program $\{x_t^*\}_{t=S}^\infty$ satisfying $x_S^* = z$.

Theorem 2.4 Assume that the sequence $\{f_i\}_{i=0}^\infty$ possesses (ATP), $z \in X$, $S \geq 0$ is an integer and that there exists an $(\{f_i\}_{i=0}^\infty)$ -good program $\{\bar{x}_t\}_{t=S}^\infty$ satisfying $\bar{x}_S = z$. Let a program $\{x_t\}_{t=S}^\infty$ satisfy $x_S = z$. Then the following properties are equivalent.

- (i) $\{x_t\}_{t=S}^\infty$ is an $(\{f_i\}_{i=0}^\infty)$ -overtaking optimal program;
- (ii) the program $\{x_t\}_{t=S}^\infty$ is $(\{f_i\}_{i=0}^\infty)$ -minimal and $(\{f_i\}_{i=0}^\infty)$ -good;
- (iii) the program $\{x_t\}_{t=S}^\infty$ is $(\{f_i\}_{i=0}^\infty)$ -minimal and satisfies $\lim_{t \rightarrow \infty} \rho(x_t, x_t^f) = 0$.

3 Main results

Let $\{f_i\}_{i=0}^\infty \in \mathcal{M}_{reg}$, a program $\{x_i^f\}_{i=0}^\infty$, $c_f > 0$ and $\gamma_f > 0$ be such that (C1)-(C4) hold.

We say that the sequence $\{f_i\}_{i=0}^\infty$ possesses a strong asymptotic turnpike property (or briefly (SATP)) with $\{x_i^f\}_{i=0}^\infty$ being the turnpike if for each integer $S \geq 0$ and each $(\{f_i\}_{i=0}^\infty)$ -good program $\{x_i\}_{i=S}^\infty$,

$$\sum_{i=0}^{\infty} \rho(x_i, x_i^f) < \infty.$$

Clearly, (SATP) implies (ATP).

We say that the sequence $\{f_i\}_{i=0}^\infty$ possesses a strong turnpike property (or briefly (STP)) if for each $\epsilon > 0$ and each $M > 0$ there exist $\delta > 0$ and a natural number L such that for each pair of integers $T_1 \geq 0$, $T_2 \geq T_1 + 2L$ and each program $\{x_t\}_{t=T_1}^{T_2}$ which satisfies

$$\sum_{i=T_1}^{T_2-1} f_i(x_i, x_{i+1}) \leq \min\{U(\{f_i\}_{i=0}^\infty, T_1, T_2, x_{T_1}, x_{T_2}) + \delta, U(\{f_i\}_{i=0}^\infty, T_1, T_2) + M\},$$

the inequality $\sum_{i=T_1+L}^{T_2-L} \rho(x_i, x_i^f) \leq \epsilon$ holds.

The sequence $\{x_i^f\}_{i=0}^\infty$ is called the turnpike of $\{f_i\}_{i=0}^\infty$.

Clearly, (STP) implies (TP).

In this paper we prove the following two results which are extensions of Theorems 2.1 and 2.2 respectively.

Theorem 3.1 *The sequence $\{f_i\}_{i=0}^\infty$ possesses the strong turnpike property if and only if $\{f_i\}_{i=0}^\infty$ possesses (SATP) and the property (P).*

Theorem 3.2 *Assume that the sequence $\{f_i\}_{i=0}^\infty$ possesses (SATP) and the property (P), and $M > 0$. Then there exist a natural number L and $M_0 > 0$ such that for each pair of integers $T_1 \geq 0$, $T_2 > T_1 + L$ and each program $\{x_t\}_{t=T_1}^{T_2}$ which satisfies*

$$\sum_{t=T_1}^{T_2-1} f_t(x_t, x_{t+1}) \leq U(\{f_i\}_{i=0}^\infty, T_1, T_2) + M$$

the following inequality holds:

$$\sum_{i=T_1}^{T_2} \rho(x_i, x_i^f) \leq M_0.$$

4 An example

Let $\{f_i\}_{i=0}^\infty \in \mathcal{M}_{reg}$, a program $\{x_i^f\}_{i=0}^\infty$, $c_f > 0$ and $\gamma_f > 0$ be such that (C1)-(C4) hold.

Now we show that $\{f_i\}_{i=0}^\infty$ is approximated by elements of \mathcal{M}_{reg} possessing (STP). For each $r \in (0, 1)$ and all integers $i \geq 0$ set

$$f_i^{(r)}(x, y) = f_i(x, y) + r\rho(x, x_i^f), \quad (x, y) \in \Omega_i. \quad (4.1)$$

Clearly, $\{f_i^{(r)}\}_{i=0}^\infty \in \mathcal{M}_{reg}$ for all $r \in (0, 1)$ and $\lim_{r \rightarrow 0^+} d(\{f_i^{(r)}\}_{i=0}^\infty, \{f_i\}_{i=0}^\infty) = 0$.

Proposition 4.1 *Let $r \in (0, 1)$. Then $\{f_i^{(r)}\}_{i=0}^\infty$ possesses (STP) with $\{x_i^f\}_{i=0}^\infty$ being the turnpike.*

Proof. By Proposition 2.6 of [31], $\{f_i^{(r)}\}_{i=0}^\infty$ possesses (TP) with $\{x_i^f\}_{i=0}^\infty$ being the turnpike. It follows from Theorem 2.1 that $\{f_i^{(r)}\}_{i=0}^\infty$ has the property (P). In view of Theorem 3.1 it is sufficient to show that $\{f_i^{(r)}\}_{i=0}^\infty$ possesses (SATP).

Assume that $S \geq 0$ is an integer and that a program $\{x_i\}_{i=S}^\infty$ is $(\{f_i^{(r)}\}_{i=0}^\infty)$ -good. Then there is $c_1 > 0$ such that

$$\left| \sum_{t=S}^{T-1} f_t^{(r)}(x_t, x_{t+1}) - \sum_{t=S}^{T-1} f_t^{(r)}(x_t^f, x_{t+1}^f) \right| \leq c_1 \text{ for all integers } T > S. \quad (4.2)$$

By Proposition 2.1, (4.1) and (4.2), $\sum_{t=S}^\infty \rho(x_t, x_t^f) < \infty$. Thus (SATP) holds. Proposition 4.1 is proved.

5 Auxiliary results

We use the notation, definitions and assumptions introduced in Sections 1-3.

The following two results were obtained in [31].

Lemma 5.1 *Let an integer $S \geq 0$ and a program $\{x_i\}_{i=S}^\infty$ be $(\{f_i\}_{i=0}^\infty)$ -good. Then there is a number $c > 0$ such that for each pair of integers $T_1 \geq S$ and $T_2 > T_1$,*

$$\sum_{i=T_1}^{T_2-1} f_i(x_i, x_{i+1}) \leq U(\{f_i\}_{i=0}^\infty, T_1, T_2) + c$$

and the following property holds:

for each $\epsilon > 0$ there exists a natural number L such that for each integer $T_1 \geq L$ and each integer $T_2 > T_1$,

$$\sum_{i=T_1}^{T_2-1} f_i(x_i, x_{i+1}) \leq U(\{f_i\}_{i=0}^\infty, T_1, T_2, x_{T_1}, x_{T_2}) + \epsilon.$$

Lemma 5.2 *Let $\epsilon > 0$. Then there exists $\delta > 0$ such that for each pair of integers $T_1 > 0$, $T_2 > T_1 + 2$ and each program $\{x_i\}_{i=T_1}^{T_2}$ satisfying*

$$\rho(x_{T_1+1}, x_{T_1+1}^f) \leq \delta, \quad \rho(x_{T_2-1}, x_{T_2-1}^f) \leq \delta,$$

$$\sum_{i=T_1+1}^{T_2-2} f_i(x_i, x_{i+1}) \leq U(\{f_i\}_{i=0}^\infty, T_1 + 1, T_2 - 1, x_{T_1+1}, x_{T_2-1}) + \delta$$

there exists a program $\{\tilde{x}_i\}_{i=T_1-1}^{T_2+1}$ such that

$$\tilde{x}_{T_1-1} = x_{T_1-1}^f, \quad \tilde{x}_{T_2+1} = x_{T_2+1}^f, \quad \tilde{x}_i = x_i, \quad i = T_1 + 1, \dots, T_2 - 1$$

and that the following inequality holds:

$$\sum_{i=T_1-1}^{T_2} f_i(\tilde{x}_i, \tilde{x}_{i+1}) \leq \sum_{i=T_1-1}^{T_2} f_i(x_i^f, x_{i+1}^f) + \epsilon.$$

Lemma 5.3 *Assume that $\{f_i\}_{i=0}^\infty$ possesses (SATP) and let $\epsilon > 0$. Then there exist $\delta > 0$ and a natural number L such that for each pair of integers $T_2 > T_1 \geq L$ and each program $\{x_i\}_{i=T_1}^{T_2}$ satisfying*

$$x_{T_1} = x_{T_1}^f, \quad x_{T_2} = x_{T_2}^f, \quad \sum_{i=T_1}^{T_2-1} f_i(x_i, x_{i+1}) \leq \sum_{i=T_1}^{T_2-1} f_i(x_i^f, x_{i+1}^f) + \delta$$

the inequality $\sum_{t=T_1}^{T_2} \rho(x_t, x_t^f) \leq \epsilon$ holds.

Proof. Assume that the lemma is not true. Then there exist sequences of natural numbers $\{T_k\}_{k=1}^\infty, \{S_k\}_{k=1}^\infty$ such that for each natural number k ,

$$T_k < S_k < T_{k+1}$$

and there exists a program $\{x_i^{(k)}\}_{i=T_k}^{S_k}$ such that

$$x_{T_k}^{(k)} = x_{T_k}^f, \quad x_{S_k}^{(k)} = x_{S_k}^f, \quad (5.1)$$

$$\sum_{i=T_k}^{S_k-1} f_i(x_i^{(k)}, x_{i+1}^{(k)}) \leq \sum_{i=T_k}^{S_k-1} f_i(x_i^f, x_{i+1}^f) + 2^{-k}, \quad (5.2)$$

$$\sum_{i=T_k}^{S_k} \rho(x_i^{(k)}, x_i^f) > \epsilon. \quad (5.3)$$

Define a sequence $\{x_i\}_{i=0}^\infty \subset X$ as follows: for each integer $k \geq 1$,

$$x_i = x_i^{(k)}, \quad i = T_k, \dots, S_k, \quad (5.4)$$

$$x_i = x_i^f \text{ for all integers } i \geq 0 \text{ such that } i \notin \cup_{k=1}^\infty \{T_k, \dots, S_k\}. \quad (5.5)$$

By (5.1), (5.4) and (5.5) $\{x_i\}_{i=0}^\infty$ is a well-defined program. By (5.2), (5.4) and (5.5) for each integer $p \geq 1$,

$$\sum_{i=0}^{S_p} f_i(x_i, x_{i+1}) \leq \sum_{i=0}^{S_p} f_i(x_i^f, x_{i+1}^f) + \sum_{i=1}^p 2^{-i}.$$

Combined with Proposition 2.1 this implies that the program $\{x_i\}_{i=0}^\infty$ is $(\{f_i\}_{i=0}^\infty)$ -good. In view of (SATP),

$$\sum_{i=0}^\infty \rho(x_i, x_i^f) < \infty.$$

On the other hand, it follows from (5.3), (5.4) and (5.5) that

$$\sum_{i=0}^\infty \rho(x_i, x_i^f) \geq \sum_{k=0}^\infty \left(\sum_{i=T_k}^{S_k} \rho(x_i^{(k)}, x_i^f) \right) \geq \sum_{k=0}^\infty \epsilon = \infty.$$

The contradiction we have reached completes the proof of Lemma 5.3.

Lemma 5.4 *Assume that $\{f_i\}_{i=0}^\infty$ possesses (SATP) and let $\epsilon > 0$. Then there exist $\delta > 0$ and a natural number L such that for each pair of integers T_1, T_2 satisfying $T_1 > L, T_2 > T_1 + 2$ and each program $\{x_i\}_{i=T_1}^{T_2}$ satisfying*

$$\rho(x_{T_1+1}, x_{T_1+1}^f) \leq \delta, \quad \rho(x_{T_2-1}, x_{T_2-1}^f) \leq \delta, \quad (5.6)$$

$$\sum_{i=T_1+1}^{T_2-2} f_i(x_i, x_{i+1}) \leq U(\{f_i\}_{i=0}^{\infty}, T_1 + 1, T_2 - 1, x_{T_1+1}, x_{T_2-1}) + \delta \quad (5.7)$$

the following inequality holds:

$$\sum_{t=T_1+1}^{T_2-1} \rho(x_t, x_t^f) \leq \epsilon.$$

Proof. By Lemma 5.3, there exist $\delta_1 > 0$ and a natural number L such that for each pair of integers $S_2 > S_1 \geq L$ and each program $\{x_i\}_{i=S_1}^{S_2}$ satisfying

$$x_{S_i} = x_{S_i}^f, \quad i = 1, 2, \quad \sum_{i=S_1}^{S_2-1} f_i(x_i, x_{i+1}) \leq \sum_{i=S_1}^{S_2-1} f_i(x_i^f, x_{i+1}^f) + \delta_1 \quad (5.8)$$

we have

$$\sum_{i=S_1}^{S_2} \rho(x_i, x_i^f) \leq \epsilon. \quad (5.9)$$

By Lemma 5.2 there exist $\delta > 0$ such that for each pair of integers $T_1 > 0$, $T_2 > T_1 + 2$ and each program $\{x_i\}_{i=T_1}^{T_2}$ satisfying

$$\rho(x_{T_1+1}, x_{T_1+1}^f) \leq \delta, \quad \rho(x_{T_2-1}, x_{T_2-1}^f) \leq \delta,$$

$$\sum_{i=T_1+1}^{T_2-2} f_i(x_i, x_{i+1}) \leq U(\{f_i\}_{i=0}^{\infty}, T_1 + 1, T_2 - 1, x_{T_1+1}, x_{T_2-1}) + \delta$$

there exists a program $\{\tilde{x}_i\}_{i=T_1-1}^{T_2+1}$ such that

$$\tilde{x}_{T_1-1} = x_{T_1-1}^f, \quad \tilde{x}_{T_2+1} = x_{T_2+1}^f, \quad \tilde{x}_i = x_i, \quad i = T_1 + 1, \dots, T_2 - 1, \quad (5.10)$$

$$\sum_{i=T_1-1}^{T_2} f_i(\tilde{x}_i, \tilde{x}_{i+1}) \leq \sum_{i=T_1-1}^{T_2} f_i(x_i^f, x_{i+1}^f) + \delta_1. \quad (5.11)$$

Assume that an integer $T_1 > L$, an integer $T_2 > T_1 + 2$ and a program $\{x_i\}_{i=T_1}^{T_2}$ satisfies (5.6) and (5.7). By (5.6), (5.7) and the choice of δ , there exists a program $\{\tilde{x}_i\}_{i=T_1-1}^{T_2+1}$ which satisfies (5.10), (5.11). By (5.10), (5.11), the choice of δ_1 (see (5.8), (5.9)),

$$\sum_{i=T_1-1}^{T_2+1} \rho(\tilde{x}_i, x_i^f) \leq \epsilon.$$

Together with (5.10) this implies that

$$\sum_{i=T_1+1}^{T_2-1} \rho(x_i, x_i^f) \leq \epsilon.$$

Lemma 5.4 is proved.

6 Proof of Theorem 3.1

Assume that (STP) holds. Then (TP) holds and in view of Theorem 2.1, (ATP) and the the property (P) hold. Therefore

$$\lim_{t \rightarrow \infty} \rho(x_t, x_t^f) = 0 \quad (6.1)$$

for each integer $S \geq 0$ and each $(\{f_i\}_{i=0}^\infty)$ -good program $\{x_i\}_{i=S}^\infty$.

Let us show that (SATP) holds. Assume that $S \geq 0$ is an integer and a program $\{x_i\}_{i=S}^\infty$ is $(\{f_i\}_{i=0}^\infty)$ -good. Then (6.1) is true. By Lemma 5.1, there is $c > 0$ such that for all integers $T_1 \geq S$, $T_2 > T_1$,

$$\sum_{i=T_1}^{T_2-1} f_i(x_i, x_{i+1}) \leq U(\{f_i\}_{i=0}^\infty, T_1, T_2) + c. \quad (6.2)$$

By (STP), there exist $\delta > 0$ and a natural number L_0 such that for each pair of integers $T_1 \geq 0$, $T_2 \geq T_1 + 2L_0$ and each program $\{z_t\}_{t=T_1}^{T_2}$ which satisfies

$$\sum_{i=T_1}^{T_2-1} f_i(z_i, z_{i+1}) \leq \min\{U(\{f_i\}_{i=0}^\infty, T_1, T_2, z_{T_1}, z_{T_2}) + \delta, U(\{f_i\}_{i=0}^\infty, T_1, T_2) + c\} \quad (6.3)$$

we have

$$\sum_{i=T_1+L_0}^{T_2-L_0} \rho(z_i, x_i^f) \leq 1. \quad (6.4)$$

By Lemma 5.1, there exists a natural number $L_1 > S$ such that for each integer $T_1 \geq L_1$ and each integer $T_2 > T_1$,

$$\sum_{i=T_1}^{T_2-1} f_i(x_i, x_{i+1}) \leq U(\{f_i\}_{i=0}^\infty, T_1, T_2, x_{T_1}, x_{T_2}) + \delta. \quad (6.5)$$

Assume that integers

$$T_1 \geq L_1, \quad T_2 \geq T_1 + 2L_0. \quad (6.6)$$

Then (6.2) and (6.5) hold. In view of (6.2), (6.5), (6.6) and the choice of δ, L_0 ,

$$\sum_{i=T_1+L_0}^{T_2-L_0} \rho(x_i, x_i^f) \leq 1. \quad (6.7)$$

Since (6.7) holds for any pair of integers T_1, T_2 satisfying (6.6) we conclude that

$$\sum_{i=L_1+L_0}^{\infty} \rho(x_i, x_i^f) \leq 1.$$

This implies that $\sum_{i=0}^{\infty} \rho(x_i, x_i^f) < \infty$ and that (SATP) holds. Thus we have shown that (STP) implies (SATP) and the property (P).

Assume that (SATP) and the property (P) hold.

Let $\epsilon > 0$ and $M > 0$. By Lemma 5.4 there exist $\delta_0 > 0$ and a natural number L_0 such that for each pair of integers S_1, S_2 satisfying $S_1 > L_0$, $S_2 > S_1 + 2$ and each program $\{x_i\}_{i=S_1}^{S_2}$ satisfying

$$\rho(x_{S_1+1}, x_{S_1+1}^f) \leq \delta_0, \quad \rho(x_{S_2-1}, x_{S_2-1}^f) \leq \delta_0,$$

$$\sum_{i=S_1+1}^{S_2-2} f_i(x_i, x_{i+1}) \leq U(\{f_i\}_{i=0}^\infty, S_1 + 1, S_2 - 1, x_{S_1+1}, x_{S_2-1}) + \delta_0 \quad (6.8)$$

we have

$$\sum_{i=S_1+1}^{S_2-2} \rho(x_i, x_i^f) \leq \epsilon. \quad (6.9)$$

By the property (P) there exist

$$\delta \in (0, \delta_0) \quad (6.10)$$

and a natural number L_1 such that for each integer $T \geq 0$ and each program $\{x_t\}_{t=T}^{T+L_1}$ which satisfies

$$\begin{aligned} & \sum_{i=T}^{T+L_1-1} f_i(x_i, x_{i+1}) \\ & \leq \min\{U(\{f_i\}_{i=0}^\infty, T, T+L_1, x_T, x_{T+L_1}) + \delta, U(\{f_i\}_{i=0}^\infty, T, T+L_1) + 3c_f + M\}, \end{aligned} \quad (6.11)$$

there is an integer j such that

$$j \in \{T, \dots, T + L_1\}, \quad \rho(x_j, x_j^f) \leq \delta_0. \quad (6.12)$$

Choose a natural number

$$L \geq 4L_0 + 4L_1. \quad (6.13)$$

Assume that a pair of integers $T_1 \geq 0$, $T_2 \geq T_1 + 2L$ and that a program $\{x_t\}_{t=T_1}^{T_2}$ satisfies

$$\sum_{i=T_1}^{T_2-1} f_i(x_i, x_{i+1}) \leq \min\{U(\{f_i\}_{i=0}^\infty, T_1, T_2, x_{T_1}, x_{T_2}) + \delta, U(\{f_i\}_{i=0}^\infty, T_1, T_2) + M\}. \quad (6.14)$$

In order to complete the proof of the theorem it is sufficient to show that

$$\sum_{i=T_1+L}^{T_2-L} \rho(x_i, x_i^f) \leq \epsilon. \quad (6.15)$$

Let integers S_1, S_2 satisfy $T_1 < S_1 < S_2 < T_2$. By (6.14) and (C2),

$$\begin{aligned}
\sum_{i=S_1}^{S_2-1} f_i(x_i, x_{i+1}) &= \sum_{i=T_1}^{T_2-1} f_i(x_i, x_{i+1}) - \sum_{i=T_1}^{S_1-1} f_i(x_i, x_{i+1}) - \sum_{i=S_2}^{T_2-1} f_i(x_i, x_{i+1}) \\
&\leq U(\{f_i\}_{i=0}^{\infty}, T_1, T_2) + M - U(\{f_i\}_{i=0}^{\infty}, T_1, S_1) - U(\{f_i\}_{i=0}^{\infty}, S_2, T_2) \\
&\leq \sum_{i=T_1}^{T_2-1} f_i(x_i^f, x_{i+1}^f) + M - \sum_{i=T_1}^{S_1-1} f_i(x_i^f, x_{i+1}^f) + c_f - \sum_{i=S_2}^{T_2-1} f_i(x_i^f, x_{i+1}^f) + c_f \\
&= \sum_{i=S_1}^{S_2-1} f_i(x_i^f, x_{i+1}^f) + 2c_f + M \leq U(\{f_i\}_{i=0}^{\infty}, S_1, S_2) + 3c_f + M.
\end{aligned}$$

Thus

$$\sum_{i=S_1}^{S_2-1} f_i(x_i, x_{i+1}) \leq U(\{f_i\}_{i=0}^{\infty}, S_1, S_2) + 3c_f + M \quad (6.16)$$

for all pairs of integers S_1, S_2 satisfying $T_1 < S_1 < S_2 < T_2$.

By (6.13), (6.14), (6.16), the choice of δ (see (6.10)-(6.12)) there exist integers

$$\tau_1 \in \{L_1 + T_1 + 2L_0, \dots, T_1 + 2L_0 + 2L_1\}, \tau_2 \in \{T_2 - 2L_1, \dots, T_2 - L_1\} \quad (6.17)$$

such that

$$\rho(x_{\tau_i}, x_{\tau_i}^f) \leq \delta_0, \quad i = 1, 2. \quad (6.18)$$

By (6.13) and (6.17),

$$\tau_2 - \tau_1 \geq 2L_0 + L. \quad (6.19)$$

By (6.14) and (6.17),

$$\sum_{i=\tau_1}^{\tau_2-1} f_i(x_i, x_{i+1}) \leq U(\{f_i\}_{i=0}^{\infty}, \tau_1, \tau_2, x_{\tau_1}, x_{\tau_2}) + \delta. \quad (6.20)$$

By (6.19), (6.20), (6.17), (6.18), (6.10) and the choice of L_0 and δ_0 (see (6.7)-(6.9)),

$$\sum_{i=\tau_1}^{\tau_2} \rho(x_i, x_i^f) \leq \epsilon.$$

Together with (6.13) and (6.17) this implies (6.15). Theorem 3.1 is proved.

7 Proof of Theorem 3.2

Set

$$D_0 = \sup\{\rho(z_1, z_2) : z_1, z_2 \in X\}. \quad (7.1)$$

We suppose that the sum over empty set is zero. By (SATP), the property (P) and Theorem 3.1, $\{f_i\}_{i=0}^{\infty}$ possesses (STP). By (STP) there exist $\delta \in (0, 1)$ and a natural number L_0 such that the following property holds:

(a) for each pair of integers $\tau_1 \geq 0$, $\tau_2 \geq \tau_1 + 2L_0$ and each program $\{x_t\}_{t=\tau_1}^{\tau_2}$ which satisfies

$$\sum_{t=\tau_1}^{\tau_2-1} f_t(x_t, x_{t+1}) \leq \min\{U(\{f_i\}_{i=0}^{\infty}, \tau_1, \tau_2, x_{\tau_1}, x_{\tau_2}) + \delta, U(\{f_i\}_{i=0}^{\infty}, \tau_1, \tau_2) + 2M + 4c_f\}$$

the inequality

$$\sum_{i=\tau_1+L_0}^{\tau_2-L_0} \rho(x_i, x_i^f) \leq 1$$

holds.

Choose a natural number

$$L > (4L_0 + 3)(\delta^{-1}M + 1) \quad (7.2)$$

and

$$M_0 > 1 + (\delta^{-1}M + 2)(1 + 2D_0(2L_0 + 1)). \quad (7.3)$$

Assume that integers $T_1 \geq 0$, $T_2 > T_1 + L$ and that a program $\{x_t\}_{t=T_1}^{T_2}$ satisfies

$$\sum_{t=T_1}^{T_2-1} f_t(x_t, x_{t+1}) \leq U(\{f_i\}_{i=0}^{\infty}, T_1, T_2) + M. \quad (7.4)$$

Set

$$t_0 = T_1. \quad (7.5)$$

By induction we define a finite strictly increasing sequence of integers $\{t_i\}_{i=0}^q \subset [T_1, T_2]$ where q is a natural number such that:

$$t_q = T_2; \quad (7.6)$$

(b) for each integer i satisfying $0 \leq i < q - 1$,

$$\sum_{t=t_i}^{t_{i+1}-1} f_t(x_t, x_{t+1}) > U(\{f_i\}_{i=0}^{\infty}, t_i, t_{i+1}, x_{t_i}, x_{t_{i+1}}) + \delta; \quad (7.7)$$

(c) if an integer i satisfies $0 \leq i \leq q - 1$ and (7.7), then

$$t_{i+1} > t_i + 1 \text{ and } \sum_{t=t_i}^{t_{i+1}-2} f_t(x_t, x_{t+1}) \leq U(\{f_j\}_{j=0}^{\infty}, t_i, t_{i+1} - 1, x_{t_i}, x_{t_{i+1}-1}) + \delta. \quad (7.8)$$

Assume that an integer $p \geq 0$ and we have already defined a strictly increasing sequence of integers $\{t_i\}_{i=0}^p \subset [T_1, T_2]$ such that $t_p < T_2$ and that for each integer i satisfying $0 \leq i < p$, (7.7) and (7.8) hold. (Note that for $p = 0$ our assumption holds.) We define t_{p+1} .

There are two cases:

$$\sum_{t=t_p}^{T_2-1} f_t(x_t, x_{t+1}) \leq U(\{f_i\}_{i=0}^{\infty}, t_p, T_2, x_{t_p}, x_{T_2}) + \delta; \quad (7.9)$$

$$\sum_{t=t_p}^{T_2-1} f_t(x_t, x_{t+1}) > U(\{f_i\}_{i=0}^{\infty}, t_p, T_2, x_{t_p}, x_{T_2}) + \delta. \quad (7.10)$$

Assume that (7.9) holds. Then we set $q = p + 1$, $t_q = T_2$, the construction of the sequence is completed and the properties (b), (c) hold.

Assume that (7.10) holds. Set

$$t_{p+1} = \min\{S \in \{t_p + 1, \dots, T_2\} :$$

$$\sum_{t=t_p}^{S-1} f_t(x_t, x_{t+1}) > U(\{f_i\}_{i=0}^{\infty}, t_p, S, x_{t_p}, x_S) + \delta\}. \quad (7.11)$$

Clearly, t_{p+1} is well-defined. If $t_{p+1} = T_2$, then we set $q = p + 1$, the construction is completed and it is not difficult to see that (b) and (c) hold.

Assume that $t_{p+1} < T_2$. Then it is easy to see that the assumption made for p is also true for $p + 1$.

Clearly our construction is completed after a final number of steps and let $t_q = T_2$ be its last element, where q is a natural number. It follows from the construction that the properties (b) and (c) hold.

By (7.4) and the property (b)

$$\begin{aligned} M &\geq \sum_{t=T_1}^{T_2-1} f_t(x_t, x_{t+1}) - U(\{f_i\}_{i=0}^{\infty}, T_1, T_2) \\ &\geq \sum_{t=T_1}^{T_2-1} f_t(x_t, x_{t+1}) - U(\{f_i\}_{i=0}^{\infty}, T_1, T_2, x_{T_1}, x_{T_2}) \\ &\geq \sum_{i=0}^{q-1} \left\{ \sum_{t=t_i}^{t_{i+1}-1} f_t(x_t, x_{t+1}) - U(\{f_j\}_{j=0}^{\infty}, t_i, t_{i+1}, x_{t_i}, x_{t_{i+1}}) \right\} : \\ &\quad \text{is an integer, } 0 \leq i < q - 1 \geq \delta(q - 1), \\ &\quad q \leq \delta^{-1}M + 1. \end{aligned} \quad (7.12)$$

Set

$$A = \{i \in \{0, \dots, q - 1\} : t_{i+1} - t_i > 2L_0\}. \quad (7.13)$$

Let

$$j \in A. \quad (7.14)$$

By (b), (c) and (7.13) and (7.14),

$$\sum_{t=t_j}^{t_{j+1}-2} f_t(x_t, x_{t+1}) \leq U(\{f_i\}_{i=0}^{\infty}, t_j, t_{j+1} - 1, x_{t_j}, x_{t_{j+1}-1}) + \delta. \quad (7.15)$$

By (7.4), (7.13), (7.14) and (C2),

$$\begin{aligned} \sum_{t=t_j}^{t_{j+1}-2} f_t(x_t, x_{t+1}) &= \sum_{t=T_1}^{T_2-1} f_t(x_t, x_{t+1}) \\ &\quad - \sum \{f_t(x_t, x_{t+1}) : t \text{ is an integer, } T_1 \leq t < t_j\} - \sum_{t=t_{j+1}-1}^{T_2-1} f_t(x_t, x_{t+1}) \\ &\leq \sum_{t=T_1}^{T_2-1} f_t(x_t^f, x_{t+1}^f) + M + c_f \\ &\quad - \sum \{f_t(x_t^f, x_{t+1}^f) : t \text{ is an integer, } T_1 \leq t < t_j\} + c_f - \sum_{t=t_{j+1}-1}^{T_2-1} f_t(x_t^f, x_{t+1}^f) \\ &= \sum_{t=t_j}^{t_{j+1}-2} f_t(x_t^f, x_{t+1}^f) + M + 2c_f \leq U(\{f_i\}_{i=0}^{\infty}, t_j, t_{j+1} - 1) + M + 3c_f. \end{aligned} \quad (7.16)$$

By (7.13), (7.14), (7.15), (7.16) and property (a),

$$\sum_{t=t_j+L_0}^{t_{j+1}-1-L_0} \rho(x_t, x_t^f) \leq 1 \quad (7.17)$$

for all $j \in A$. By (7.5), (7.6), (7.13), (7.1), (7.17), (7.12) and (7.3),

$$\begin{aligned} \sum_{t=0}^T \rho(x_t, x_t^f) &\leq D_0 + \sum_{j=0}^{q-1} \left(\sum_{t=t_j}^{t_{j+1}-1} \rho(x_t, x_t^f) \right) \\ &= D_0 + \sum_{j \in A} \left(\sum_{t=t_j}^{t_{j+1}-1} \rho(x_t, x_t^f) \right) \\ &\quad + \sum \left\{ \sum_{t=t_j}^{t_{j+1}-1} \rho(x_t, x_t^f) : j \in \{0, \dots, q-1\} \setminus A \right\} \\ &\leq D_0 + \sum_{j \in A} (L_0 D_0 + 1 + D_0(L_0 + 1)) + q D_0(2L_0 + 1) \\ &\leq D_0 + q(1 + 2D_0(2L_0 + 1)) \leq (\delta^{-1}M + 2)(1 + 2D_0(2L_0 + 1)) < M_0. \end{aligned}$$

Theorem 3.2 is proved. *

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