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On circularly symmetric functions

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Abstract: Let $D \subset \mathbb{C}$ and $0 \in D$. A set D is circularly symmetric if for each $\varrho \in \mathbb{R}^+$ a set $D \cap \{\zeta \in \mathbb{C} : |\zeta| = \varrho\}$ is one of three forms: an empty set, a whole circle, a curve symmetric with respect to the real axis containing ϱ. A function $f \in \mathcal{A}$ is circularly symmetric if $f(\Delta)$ is a circularly symmetric set. The class of all such functions we denote by X. The above definitions were given by Jenkins in [2].

In this paper besides X we also consider some of its subclasses: $X(\lambda)$ and $Y \cap S^*$ consisting of functions in X with the second coefficient fixed and univalent starlike functions respectively. According to the suggestion, in Abstract we add one more paragraph at the end of the section:

For $X(\lambda)$ we find the radii of starlikeness, starlikeness of order α, univalence and local univalence. We also obtain some distortion results. For $Y \cap S^*$ we discuss some coefficient problems, among others the Fekete-Szegö ineqalities.

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1 The class of circularly symmetric functions and some its subclasses.

Let $\tilde{\mathcal{A}}$ denote the class of all functions analytic in $\Delta \equiv \{\zeta \in \mathbb{C} : |\zeta| < 1\}$ and let $\mathcal{A}$ denote the class of all functions analytic in Δ normalized by $f(0) = f'(0) - 1 = 0$. Similar notation is applied to the class of typically real functions, i.e. functions satisfying the following condition: $\operatorname{Im} z \operatorname{Im} f(z) \geq 0$ for $z \in \Delta$. The set of all analytic and typically real functions is denoted by $\tilde{T}$; the subset of $\tilde{T}$ consisting of normalized functions is denoted by T. Hence $T = \tilde{T} \cap \mathcal{A}$. It follows from the definition of a typically real function that $z \in \Delta^+ \Leftrightarrow f(z) \in \mathbb{C}^+$ and $z \in \Delta^- \Leftrightarrow f(z) \in \mathbb{C}^-$. The symbols Δ^+, Δ^-, $\mathbb{C}^+$, $\mathbb{C}^-$ mean the following open sets: the upper and the lower half of the unit disk Δ and the upper and the lower halfplane.

In this paper we focus on so called circularly symmetric functions, which were defined by Jenkins in [2]. Let us start with the following definitions.

Let $D \subset \mathbb{C}$ and $0 \in D$.

Definition 1. A set D is circularly symmetric if for each $\varrho \in \mathbb{R}^+$ a set $D \cap \{\zeta \in \mathbb{C} : |\zeta| = \varrho\}$ is one of three forms: an empty set, a whole circle, a curve symmetric with respect to the real axis containing ϱ.

Definition 2. A function $f \in \mathcal{A}$ is circularly symmetric if $f(\Delta)$ is a circularly symmetric set. The class of all such functions we denote by X.

In fact, Jenkins claimed more than it was stated in the above definition. He considered only these circularly symmetric functions which are univalent. This assumption is rather restrictive. Furthermore, there are no objections to reject it. The number of interesting problems appear while discussing non-univalent circularly symmetric functions. For these reasons we decided to define a circularly symmetric function as in Definition 2. In order to distinguish the classes of non-univalent and univalent circularly symmetric functions we will denote the latter by Y.

Besides X we will also consider some of its subclasses: $X(\lambda)$ and $Y \cap S^*$ consisting of functions in X with the fixed second coefficient of the Taylor series expansion and univalent starlike functions respectively. As it was shown in [2], for all $r \in (0,1)$ and for a circularly symmetric function f the expression $|f(re^{i\varphi})|$ is a nonincreasing function for $\varphi \in (0,\pi)$ and a nondecreasing function for $\varphi \in (\pi, 2\pi)$. From this fact and the equality

$$-\frac{\partial}{\partial\varphi}\left(\log|f(re^{i\varphi})|\right) = \operatorname{Im}\left(re^{i\varphi}\frac{f'(re^{i\varphi})}{f(re^{i\varphi})}\right)$$

it follows that on the circle $|z| = r$ there is

$$\operatorname{Im}\frac{zf'(z)}{f(z)} \geq 0 \quad \text{if and only if} \quad \operatorname{Im} z \geq 0\ .$$

Hence

Theorem 1. *[2]*

$$f \in X \Leftrightarrow \frac{zf'(z)}{f(z)} \in \tilde{T}\ .$$

The condition $\frac{zf'(z)}{f(z)} \in \tilde{T}$ is not sufficient for univalence of f. We have only

Theorem 2. *If $f \in Y$ then $\frac{zf'(z)}{f(z)} \in \tilde{T}$.*

According to Theorem 1, all coefficients of the Taylor expansion of $f \in X$ are real.

Some other results concerning Y one can find in [1] and [4].

Similar, but more general, functions were discussed by Libera in [3]. He considered so called disk-like functions. The functions f of this class have the property: there exists a number ϱ depending on f that for each fixed r, $r \in (\varrho, 1]$, there exist numbers φ_1, φ_2 depending on r that $|f(re^{i\varphi})|$ is decreasing if φ increases in some interval $I_1 = (\varphi_1, \varphi_2)$ and increasing in $I_2 = (\varphi_2, \varphi_1 + 2\pi)$. The class of these functions Libera denoted by $\mathcal{D}$. In particular, if f has real coefficients and $|f(re^{i\varphi})|$ is increasing on the lower half of the circle $|z| = r$ and is decreasing on the upper half of this circle, then f is a circularly symmetric function. Although $\mathcal{D}$ is more general than X, some of the results of the paper [3] are still valid for the class X.

Let us assume that f is of the form $f(z) = z + \lambda z^2 + \ldots$. From Theorem 1 it follows that a function

$$\frac{1}{\lambda}\left(\frac{zf'(z)}{f(z)} - 1\right)$$

is in T; let us denote it by $h(z)$. Hence

$$f(z) = z \exp\left(\lambda \int_0^z \frac{h(\zeta)}{\zeta} d\zeta\right) . \tag{1}$$

Applying the very well known relation between T and $CVR(i)$ consisting of functions with real coefficients g which are convex in the direction of the imaginary axis and normalized by $g(0) = g'(0) - 1 = 0$, we obtain

Corollary 1.

$$f \in X \Leftrightarrow f(z) = z \exp\{\lambda g(z)\}\ ,\ g \in CVR(i)\ ,\ \lambda > 0\ . \tag{2}$$

The conclusion similar to the above corollary one can find in the paper of Libera (corollary on page 253).

Basing on the equivalence (2) we can define the subclass of X containing these circularly symmetric functions for which the second coefficient is fixed and equal to $\lambda \geq 0$. We denote this class by $X(\lambda)$. For $\lambda = 0$ the set $X(0)$ has only one element - the identity function. We shall present the properties of $X(\lambda)$ in next section.

2 Properties of $X(\lambda)$.

Theorem 3. *The radius of starlikeness for $X(\lambda)$ is equal to $r_{S^*}(X(\lambda)) = r_\lambda$, where $r_\lambda = \frac{1}{4}\left(\sqrt{\lambda+4} - \sqrt{\lambda}\right)^2$. The extremal function is $f_\lambda(z) = z\exp\left(\lambda\frac{z}{1+z}\right)$.*

Proof

It follows from (1) that $\frac{zf'(z)}{f(z)} = 1 + \lambda z g'(z) = 1 + \lambda h(z)$, where $g \in CVR(i)$, $h \in T$. The well-known estimate of the real part of a typically real function leads to

$$\mathrm{Re}\,\frac{zf'(z)}{f(z)} \geq 1 - \lambda\frac{r}{(1-r)^2}\ .$$

Therefore, $\mathrm{Re}\,\frac{zf'(z)}{f(z)} \geq 0$ if and only if $r \leq \frac{2}{2+\lambda+\sqrt{\lambda^2+4\lambda}}$, or equivalently, if $r \leq r_\lambda$. Equality in the above estimate holds for $h(z) = \frac{z}{(1+z)^2}$ and $z = -r$. It means that the extremal function is f_λ. ■

The result of Theorem 3 can be generalized in order to finding the radius of starlikeness of order α, $\alpha \in [0, 1)$. It sufficies to replace the inequality $1 - \lambda\frac{r}{(1-r)^2} \geq 0$ by $1 - \lambda\frac{r}{(1-r)^2} \geq \alpha$. Hence

Theorem 4. *The radius of starlikeness of order α, $\alpha \in [0,1)$ for $X(\lambda)$ is equal to* $r_{S^*(\alpha)}(X(\lambda)) = \frac{1}{4}\left(\sqrt{\frac{\lambda}{1-\alpha}+4}-\sqrt{\frac{\lambda}{1-\alpha}}\right)^2$. *The extremal function is* $f_\lambda(z) = z\exp\left(\lambda\frac{z}{1+z}\right)$.

Observe that for all $f \in X(\lambda)$ the condition $\frac{zf'(z)}{f(z)} \neq 0$ holds if only $z \in \Delta_{r_\lambda}$. Moreover, for f_λ and $z = -r_\lambda$ there is

$$\left.\frac{zf'(z)}{f(z)}\right|_{z=-r_\lambda} = \left.\left(1+\lambda\frac{z}{(1+z)^2}\right)\right|_{z=-r_\lambda} = 1-\lambda\frac{r_\lambda}{(1-r_\lambda)^2} = 0 \ .$$

This results in

Theorem 5. *The radius of local univalence for $X(\lambda)$ is equal to $r_{LU}(X(\lambda)) = r_\lambda$.*

Because of $r_{S^*} \le r_S \le r_{LU}$, which is true for any class of analytic functions, we obtain

Corollary 2. *The radius of univalence for $X(\lambda)$ is equal to $r_S(X(\lambda)) = r_\lambda$.*

It is known that the second coefficients of the Taylor expansion of functions in the following subclasses of $\mathcal{A}$ consisting of: convex functions, univalent functions and locally univalent functions have the upper bounds: 1, 2, 4 respectively. For this reason it is worth observing that

$$r_S(X(1)) = \frac{1}{2}(3-\sqrt{5}) \ , \ r_S(X(2)) = 2-\sqrt{3} \ , \ r_S(X(4)) = (\sqrt{2}-1)^2 \ .$$

Theorem 6. *If $f \in X(\lambda)$ and $r = |z| \in (0,1)$ then*

$$r\exp\left(\frac{-\lambda r}{1-r}\right) \le |f(z)| \le r\exp\left(\frac{\lambda r}{1-r}\right) \ , \tag{3}$$

Equalities in the above estimates hold for $f(z) = z\exp\left(\frac{\lambda z}{1+z}\right)$, $z = -r$ and $f(z) = z\exp\left(\frac{\lambda z}{1-z}\right)$, $z = r$ respectively.

Proof
For $g \in CVR(i)$ the exact estimate holds (see for example [5])

$$|\operatorname{Re} g(z)| \le \frac{r}{1-r} \ , \tag{4}$$

with equality for $g(z) = \frac{z}{1+z}$, $z = -r$ and $g(z) = \frac{z}{1-z}$, $z = r$ respectively. From Corollary 1 it follows that

$$|f(z)| = |z|\exp\left(\lambda \operatorname{Re} g(z)\right) \ .$$

Combining it with (4) completes the proof. ■

Theorem 7. *If $f \in X(\lambda)$ and $r = |z| \in (0,1)$ then*

$$|f'(z)| \le \left(1 + \lambda \frac{r}{(1-r)^2}\right) \exp\left(\frac{\lambda r}{1-r}\right) \ , \tag{5}$$

Equality in the above estimate holds for $f(z) = z \exp\left(\frac{\lambda z}{1-z}\right)$ and $z = r$.

Proof
From Corollary 1 and (1) we have

$$|f'(z)| = \left|\frac{f(z)}{z}\right| |1 + \lambda h(z)| \ ,$$

where $h \in T$. Applying Theorem 6 and the estimate of the modulus of a function in T in the above equality leads to the assertion. ■

According to Theorems 6 and 7, both $|f(z)|$ and $|f'(z)|$ can be arbitrarily large while considering functions in the whole class X, not only functions with the second coefficient fixed.

3 Properties of $Y \cap S^*$.

In the paper of Szapiel [4] one can find the relation between the class of circurally symmetric functions which are starlike with the class of typically real functions T:

Theorem 8.

$$f \in Y \cap S^* \Leftrightarrow \frac{zf'(z)}{f(z)} \in \tilde{T} \cap P_R \ .$$

Szapiel also proved the representation formula for functions in the class $\mathcal{R}^2 = \{q \in \mathcal{A} : q = p^2, p \in \tilde{T} \cap P_R\}$. Namely, $q \in \mathcal{R}^2$ if and only if

$$q(z) = \int_{-1}^{1} \frac{(1+z)^2}{1 - 2zt + z^2} d\mu(t) \ . \tag{6}$$

From this formula one can establish the relationship between $\mathcal{R}^2$ and T:

$$q \in \mathcal{R}^2 \Leftrightarrow g \in T \ , \tag{7}$$

where

$$q(z) = (1+z)^2 \frac{g(z)}{z} \ .$$

From the above we get

Corollary 3.

$$f \in Y \cap S^* \Leftrightarrow \frac{zf'(z)}{f(z)} = (1+z)\sqrt{\frac{g(z)}{z}} \ , \ g \in T. \tag{8}$$

Examples. Putting functions of the class T into (8) we obtain associated functions from $Y \cap S^*$:
1. If $g(z) = z$, then $\frac{zf'(z)}{f(z)} = 1 + z$ and hence $f(z) = ze^z$.
2. If $g(z) = \frac{z}{(1+z)^2}$, then $\frac{zf'(z)}{f(z)} = 1$ and hence $f(z) = z$.
3. If $g(z) = \frac{z}{(1-z)^2}$, then $\frac{zf'(z)}{f(z)} = \frac{1+z}{1-z}$ and hence $f(z) = \frac{z}{(1-z)^2}$.

Many of the properties of $Y \cap S^*$ follow directly from obvious inclusion $Y \cap S^* \subset S^*$ and the fact that the Koebe function $f(z) = \frac{z}{(1-z)^2}$, which is starlike, belongs also to $Y \cap S^*$. This observation gives us the following sharp results:

1. If $f \in Y \cap S^*$ and $f(z) = \sum_{n=1}^{\infty} a_n z^n$, then $|a_n| \leq n$.
2. If $f \in Y \cap S^*$ and $r = |z| \in (0,1)$, then $\frac{r}{(1+r)^2} \leq |f(z)| \leq \frac{r}{(1-r)^2}$.
3. If $f \in Y \cap S^*$ and $r = |z| \in (0,1)$, then $\frac{1-r}{(1+r)^3} \leq |f'(z)| \leq \frac{1+r}{(1-r)^3}$.
4. Every function in $Y \cap S^*$ is convex in the disk $|z| < 2 - \sqrt{3}$.
5. Every function in $Y \cap S^*$ is strongly starlike of order α in the disk $|z| < \tan(\alpha\frac{\pi}{4})$.

Now we shall find the lower bounds of the second and the third coefficients in $Y \cap S^*$. Let $f(z) = z + \sum_{n=2}^{\infty} a_n z^n \in Y \cap S^*$ and $g(z) = z + \sum_{n=2}^{\infty} b_n z^n \in T$. From (8) we conclude

$$2a_2 = b_2 + 2 \;, \tag{9}$$
$$4a_3 - {a_2}^2 = b_3 + 2b_2 + 1 \;. \tag{10}$$

Let us denote by $A_{2,3}(A)$ a set $\{(a_2(f), a_3(f)) : f \in A\}$. This set for T is known: $A_{2,3}(T) = \{(x,y) : -2 \leq x \leq 2, x^2 - 1 \leq y \leq 3\}$. This results in the following bound for a function in $Y \cap S^*$:

$$0 \leq a_2 \leq 2 \;. \tag{11}$$

Taking into account (9) and (10) in $A_{2,3}(T)$ we obtain

Theorem 9.

$$A_{2,3}(Y \cap S^*) = \left\{(x,y) : 0 \leq x \leq 2, \frac{1}{4}\left(5x^2 - 4x\right) \leq y \leq \frac{1}{4}\left(x^2 + 4x\right)\right\} \;.$$

Consequently

Corollary 4. *Let $f \in Y \cap S^*$ have the Taylor series expansion $f(z) = z + \sum_{n=2}^{\infty} a_n z^n$. Then $-\frac{1}{5} \leq a_3 \leq 3$.*

From (10) and $A_{2,3}(T)$ it follows that

$$4a_3 - {a_2}^2 = b_3 + 2b_2 + 1 \geq {b_2}^2 + 2b_2 \geq -1$$

and

$$4a_3 - {a_2}^2 = b_3 + 2b_2 + 1 \leq {b_2}^2 + 4 \leq 8 \;.$$

Hence

Theorem 10. *Let $f \in Y \cap S^*$ have the Taylor series expansion $f(z) = z + \sum_{n=2}^{\infty} a_n z^n$. Then $-\frac{1}{4} \leq a_3 - \frac{1}{4}{a_2}^2 \leq 2$.*

The points of intersection of two parabolas from Theorem 9 coincide with two pairs of coefficients (a_2, a_3) of the functions $f_1(z) = z$ and $f_2(z) = \frac{z}{(1-z)^2}$. From Theorem 10 it follows that the class $Y \cap S^*$ is not a convex set because the set $A_{2,3}(Y \cap S^*)$ is not convex.

Basing on Theorem 9 one can derive so called the Fekete-Szegö ineqalities for $Y \cap S^*$.

Theorem 11. *Let $f \in Y \cap S^*$ be of the form $f(z) = z + \sum_{n=2}^{\infty} a_n z^n$. Then*

$$\begin{cases} \frac{1}{4\mu-5} & \mu \leq 1 \\ 3-4\mu & \mu \geq 1 \end{cases} \leq a_3 - \mu {a_2}^2 \leq \begin{cases} 3-4\mu & \mu \leq \frac{1}{2} \\ \frac{1}{4\mu-1} & \mu \geq \frac{1}{2} \end{cases} .$$

Proof
Assume that $f(z) = z + \sum_{n=2}^{\infty} a_n z^n \in Y \cap S^*$. Let us denote by Q a function $Q(a_2, a_3) = a_3 - \mu {a_2}^2$. With a fixed $\mu \in \mathbb{R}$ the function Q achieves its extremal value on the boundary of the set $A_{2,3}(Y \cap S^*)$.

Let us consider two functions

$$Q_1(x) = Q(x, \frac{1}{4}x^2 + x) = x^2 \left(\frac{1}{4} - \mu \right) + x$$

and

$$Q_2(x) = Q(x, \frac{5}{4}x^2 - x) = x^2 \left(\frac{5}{4} - \mu \right) - x \ .$$

For $x \in [0, 2]$ the inequality

$$Q_1(x) \geq Q_2(x)$$

holds; hence

$$\begin{aligned} (12) \quad & \max\{Q : f \in Y \cap S^*\} \\ & = \max\{Q(x,y) : (x,y) \in A_{2,3}(Y \cap S^*)\} \\ & = \max\{Q_1(x) : x \in [0,2]\} \end{aligned}$$

and

$$\begin{aligned} (13) \quad & \min\{Q : f \in Y \cap S^*\} \\ & = \min\{Q(x,y) : (x,y) \in A_{2,3}(Y \cap S^*)\} \\ & = \min\{Q_2(x) : x \in [0,2]\} \ . \end{aligned}$$

The function Q_1 for $\mu \leq \frac{1}{2}$ is strictly increasing in $[0, 2]$; thus $\max\{Q_1(x) : x \in [0, 2]\} = Q_1(2)$. For $\mu > \frac{1}{2}$ the function Q_1 increases in $(0, x_1)$ and decreases in $(x_1, 2)$, where $x_1 = \frac{2}{4\mu-1}$. This results in $\max\{Q_1(x) : x \in [0, 2]\} = Q_1(x_1)$.

Similarly, the function Q_2 for $\mu < 1$ decreases in $(0, x_2)$ and increases in $(x_2, 2)$, where $x_2 = \frac{2}{5-4\mu}$. Hence $\min\{Q_2(x) : x \in [0,2]\} = Q_2(x_2)$. For $\mu \geq 1$ the function Q_2 is strictly decreasing in $[0,2]$, so $\min\{Q_2(x) : x \in [0,2]\} = Q_2(2)$. ■

Taking $\mu = 0$ or $\mu = \frac{1}{4}$ we obtain previously obtained results from Corollary 4 and from Theorem 10.

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