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Supra b-compact and supra b-Lindelöf spaces

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ABSTRACT: In this paper we introduce the notion of supra b-compact spaces and investigate its several properties and characterizat eions. Also we introduce and study the notion of supra b-Lindelöf spaces.

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1.Introduction and preliminaries

In 1983, A. S. Mashhour et al. [3] introduced the supra topological spaces. In 1996, D. Andrijevic [1] introduced and studied a class of generalized open sets in a topological space called b-open sets. This type of sets discussed by El-Atike [2] under the name of γ-open sets. In 2010, O. R. Sayed et al. [4] introduced and studied a class of sets and maps between topological spaces called supra b-open sets and supra b-continuous functions respectively. Now we introduce the concepts of supra b-compact and supra b-Lindelöf spaces and investigate several properties for these concepts.

Throughout this paper (X, τ), (Y, ρ) and (Z, σ) (or simply X, Y and Z) denote topological spaces on which no separation axioms are assumed unless explicitly stated. For a subset A of (X, τ), the closure and the interior of A in X are denoted by $Cl(A)$ and $Int(A)$, respectively. The complement of A is denoted by $X - A$. In the space (X, τ), a subset A is said to be b-open [1] if $A \subseteq Cl(Int(A)) \cup Int(Cl(A))$. The family of all b-open sets of (X, τ) is denoted by $BO(X)$. A subcollection $\mu \subseteq 2^X$ is called a supra topology [3] on X if $X \in \mu$ and μ is closed under arbitrary union. (X, μ) is called a supra topological space. The elements of μ are said to be supra open in (X, μ) and the complement of a supra open set is called a supra closed set. The supra closure of a set A, denoted by $Cl^\mu(A)$, is the intersection of all supra closed sets including A. The supra interior of a set A, denoted by $Int^\mu(A)$, is the union of all supra open sets included in A. The supra topology μ on X is associated with the topology τ if $\tau \subseteq \mu$.

Definition 1.1 *[4] Let (X, μ) be a supra topological space. A set A is called a supra b-open set if $A \subseteq Cl^{\mu}(Int^{\mu}(A)) \cup Int^{\mu}(Cl^{\mu}(A))$. The complement of a supra b-open set is called a supra b-closed set.*

Theorem 1.2 *[4]. (i) Arbitrary union of supra b-open sets is always supra b-open.*

(ii) Finite intersection of supra b-open sets may fail to be supra b-open.

Definition 1.3 *[4] The supra b-closure of a set A, denoted by $Cl^{\mu}_{b}(A)$, is the intersection of supra b-closed sets including A. The super b-interior of a set A, denoted by $Int^{\mu}_{b}(A)$, is the union of supra b-open sets included in A.*

2. Supra b-compact and supra b-Lindelöf spaces

Definition 2.1 *A collection $\{U_{\alpha} : \alpha \in \Delta\}$ of supra b-open sets in a supra topological space (X, μ) is called a supra b-open cover of a subset B of X if $B \subseteq \cup\{U_{\alpha} : \alpha \in \Delta\}$.*

Definition 2.2 *A supra topological space (X, μ) is called supra b-compact (resp. supra b-Lindelöf) if every supra b-open cover of X has a finite (resp. countable) subcover.*

The proof of the following theorem is straightforward and thus omitted.

Theorem 2.3 *If X is finite (resp. countable) then (X, μ) is supra b-compact (resp. supra b-Lindelöf) for any supra topology μ on X.*

Definition 2.4 *A subset B of a supra topological space (X, μ) is said to be supra b-compact (resp. supra b-Lindelöf) relative to X if, for every collection $\{U_{\alpha} : \alpha \in \Delta\}$ of supra b-open subsets of X such that $B \subseteq \cup\{U_{\alpha} : \alpha \in \Delta\}$, there exists a finite (resp. countable) subset Δ_0 of Δ such that $B \subseteq \cup\{U_{\alpha} : \alpha \in \Delta_0\}$.*

Notice that if (X, μ) is a supra topological space and $A \subseteq X$ then $\mu_A = \{U \cap A : U \in \mu\}$ is a supra topology on A.

(A, μ_A) is called a supra subspace of (X, μ).

Definition 2.5 *A subset B of a supra topological space (X, μ) is said to be supra b-compact (resp. supra b-Lindelöf) if B is supra b-compact (resp. supra b-Lindelöf) as a supra subspace of X.*

Theorem 2.6 *Every supra b-closed subset of a supra b-compact space X is supra b-compact relative to X.*

Prof: Let A be a supra b-closed subset of X and $\tilde{U}$ be a cover of A by supra b-open subsets of X. Then $\tilde{U}^{*} = \tilde{U} \cup \{X - A\}$ is a supra b-open cover of X. Since X is supra b-compact, $\tilde{U}^{*}$ has a finite subcover $\tilde{U}^{**}$ for X. Now $\tilde{U}^{**} - \{X - A\}$ is a finite subcover of $\tilde{U}$ for A, so A is supra b-compact relative to X. ■

Theorem 2.7 *Every supra b-closed subset of a supra b-Lindelöf space X is supra b-Lindelöf relative to X.*

Prof: Similar to the proof of the above theorem. ■

Theorem 2.8 *Every supra subspace of a supra topological space (X, μ) is supra b-compact relative to X if and only if every supra b-open subspace of X is supra b-compact relative to X.*

Prof: $\Rightarrow$) Is clear.

$\Leftarrow$) Let Y be a supra subspace of X and let $\tilde{U} = \{U_\alpha : \alpha \in \Delta\}$ be a cover of Y by supra b-open sets in X. Now let $V = \cup\tilde{U}$, then V is a supra b-open subset of X, so it is supra b-compact relative to X. But $\tilde{U}$ is a cover of V so $\tilde{U}$ has a finite subcover $\tilde{U}^*$ for V. Then $V \subseteq \cup\tilde{U}^*$ and therefore $Y \subseteq V \subseteq \cup\tilde{U}^*$. So $\tilde{U}^*$ is a finite subcover of $\tilde{U}$ for Y. Then Y is supra b-compact relative to X. ■

Theorem 2.9 *Every supra subspace of a supra topological space (X, μ) is supra b-Lindelöf relative to X if and only if every supra b-open subspace of X is supra b-Lindelöf relative to X.*

Prof: Similar to the proof of the above theorem. ■

For a family $\tilde{A}$ of subsets of X, if all finite intersection of the elements of $\tilde{A}$ are non-empty, we say that $\tilde{A}$ has the finite intersection property.

Theorem 2.10 *A supra topological space (X, μ) is supra b-compact if and only if every supra b-closed family of subsets of X having the finite intersection property, has a non-empty intersection.*

Prof: $\Rightarrow$) Let $\tilde{A} = \{A_\alpha : \alpha \in \Delta\}$ be a supra b-closed family of subsets of X which has the finite intersection property. Suppose that $\cap\{A_\alpha : \alpha \in \Delta\} = \phi$. Let $\tilde{U} = \{X - A_\alpha : \alpha \in \Delta\}$ then $\tilde{U}$ is a supra b-open cover of X. Then $\tilde{U}$ has a finite subcover $\tilde{U}' = \{X - A_{\alpha_1}, X - A_{\alpha_2}, ..., X - A_{\alpha_n}\}$. Now $\tilde{A}' = \{A_{\alpha_1}, A_{\alpha_2}, ..., A_{\alpha_n}\}$ is a finite subfamily of $\tilde{A}$ with $\cap\{A_{\alpha_i} : i = 1, 2, ..., n\} = \phi$ which is a contradiction.

$\Leftarrow$) Let $\tilde{U} = \{U_\alpha : \alpha \in \Delta\}$ be a supra b-open cover of X. Suppose that $\tilde{U}$ has no finite subcover. Now $\tilde{A} = \{X - U_\alpha : \alpha \in \Delta\}$ is a supra b-closed family of subsets of X which has the finite intersection property. So by assumption we have $\cap\{X - U_\alpha : \alpha \in \Delta\} \neq \phi$. Then $\cup\{U_\alpha : \alpha \in \Delta\} \neq X$ which is a contradiction. ■

The proof of the following theorem is straightforward and thus omitted.

Theorem 2.11 *The finite (resp. countable) union of supra b-compact (resp. supra b-Lindelöf) sets relative to a supra topological space X is supra b-compact (resp. supra b-Lindelöf) relative to X.*

Theorem 2.12 *Let A be a supra b-compact (resp. supra b-Lindelöf) set relative to a supra topological space X and B be a supra b-closed subset of X. Then $A \cap B$ is supra b-compact (resp. supra b-Lindelöf) relative to X.*

Prof: We will show the case when A is supra b-compact relative to X, the other case is similar. Suppose that $\tilde{U} = \{U_\alpha : \alpha \in \Delta\}$ is a cover of $A \cap B$ by supra b-open sets in X. Then $\tilde{O} = \{U_\alpha : \alpha \in \Delta\} \cup \{X - B\}$ is a cover of A by supra b-open sets in X, but A is supra b-compact relative to X, so there exist $\alpha_1, \alpha_2, ..., \alpha_n \in \Delta$ such that $A \subseteq (\cup\{U_{\alpha_i} : i = 1, 2, ..., n\}) \cup (X - B)$. Then $A \cap B \subseteq \cup\{(U_{\alpha_i} \cap B) : i = 1, 2, ..., n\} \subseteq \cup\{U_{\alpha_i} : i = 1, 2, ..., n\}$. Hence, $A \cap B$ is supra b-compact relative to X. ■

Definition 2.13 *[4] Let (X, τ) and (Y, ρ) be two topological spaces and μ be an associated supra topology with τ. A function $f : (X, \tau) \to (Y, \rho)$ is called a supra b-continuous function if the inverse image of each open set in Y is a supra b-open set in X.*

Theorem 2.14 *A supra b-continuous image of a supra b-compact space is compact.*

Prof: Let $f : X \to Y$ be a supra b-continuous function from a supra b-compact space X onto a topological space Y. Let $\tilde{O} = \{V_\alpha : \alpha \in \Delta\}$ be an open cover of Y. Then $\tilde{U} = \{f^{-1}(V_\alpha) : \alpha \in \Delta\}$ is a supra b-open cover of X. Since X is supra b-compact, $\tilde{U}$ has a finite subcover say $\{f^{-1}(V_{\alpha_1}), f^{-1}(V_{\alpha_2}), ..., f^{-1}(V_{\alpha_n})\}$. Now $\{V_{\alpha_1}, V_{\alpha_2}, ..., V_{\alpha_n}\}$ is a finite subcover of $\tilde{O}$ for Y. ■

Theorem 2.15 *A supra b-continuous image of a supra b-Lindelöf space is Lindelöf.*

Prof: Similar to the proof of the above theorem. ■

Definition 2.16 *Let (X, τ) and (Y, ρ) be two topological spaces and μ, η be associated supra topologies with τ and ρ respectively. A function $f : (X, \tau) \to (Y, \rho)$ is called a supra b-irresolute function if the inverse image of each supra b-open set in Y is a supra b-open set in X.*

Theorem 2.17 *If a function $f : X \to Y$ is supra b-irresolute and a subset B of X is supra b-compact relative to X, then $f(B)$ is supra b-compact relative to Y.*

Prof: Let $\tilde{O} = \{V_\alpha : \alpha \in \Delta\}$ be a cover of $f(B)$ by supra b-open subsets of Y. Then $\tilde{U} = \{f^{-1}(V_\alpha) : \alpha \in \Delta\}$ is a cover of B by supra b-open subsets of X. Since B is supra b-compact relative to X, $\tilde{U}$ has a finite subcover $\tilde{U}^* = \{f^{-1}(V_{\alpha_1}), f^{-1}(V_{\alpha_2}), ..., f^{-1}(V_{\alpha_n})\}$ for B. Now $\{V_{\alpha_1}, V_{\alpha_2}, ..., V_{\alpha_n}\}$ is a finite subcover of $\tilde{O}$ for $f(B)$. So $f(B)$ is supra b-compact relative to Y. ■

Theorem 2.18 *If a function $f : X \to Y$ is supra b-irresolute and a subset B of X is supra b-Lindelöf relative to X, then $f(B)$ is supra b-Lindelöf relative to Y.*

Prof: Similar to the proof of the above theorem. ■

Definition 2.19 *[4]. A function $f : (X, \tau) \to (Y, \rho)$ is called a supra b-open function if the image of each open set in X is a supra b-open set in (Y, η).*

The proof of the following theorem is straightforward and thus omitted.

Theorem 2.20 *Let $f : (X, \tau) \to (Y, \rho)$ be a supra b-open surjection and η be a supra topology associated with ρ. If (Y, η) is supra b-compact (resp. supra b-Lindelöf) then (X, τ) is compact (resp. Lindelöf).*

Definition 2.21 *A subset F of a supra topological space (X, μ) is called supra b-F_σ-set if $F = \cup\{F_i : i = 1, 2, ...\}$ where F_i is a supra b-closed subset of X for each $i = 1, 2, ...$.*

Theorem 2.22 *A supra b-F_σ-set F of a supra b-Lindelöf space X is supra b-Lindelöf relative to X.*

Prof: Let $F = \cup\{F_i : i = 1, 2, ...\}$ where F_i is a supra b-closed subset of X for each $i = 1, 2, ...$. Let $\tilde{U}$ be a cover of F by supra b-open sets in X, then $\tilde{U}$ is a cover of F_i for each $i = 1, 2, ...$ by supra b-open subsets of X. Since F_i is supra b-Lindelöf relative to X, $\tilde{U}$ has a countable subcover $\tilde{U}_i = \{U_{i_1}, U_{i_2}, ...\}$ for F_i for each $i = 1, 2, ...$. Now $\cup\{\tilde{U}_i : i = 1, 2, ...\}$ is a countable subcover of $\tilde{U}$ for F. So F is supra b-Lindelöf relative to X. ■

References

[1] D. Andrijevic, *On b-open sets*, Mat. Vesnik, 48 (1996), 59 - 64.

[2] A. A. El-Atik, *A study on some types of mappings on topological spaces*, MSc Thesis, Egypt, Tanta University, 1997.

[3] A. S. Mashhour, A. A. Allam, F. S. Mahmoud and F. H. Khedr, *On supra topological spaces*, Indian J. Pure Appl. Math., 14(4) (1983), 502 - 510.

[4] O. R. Sayed and T. Noiri, *On supra b-open sets and supra b-continuity on topological spaces*, Eur. J. Pure Appl. Math., 3 (2010), 295 - 302.

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