Preserving subordination and superordination results of generalized Srivastava-Attiya operator

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ABSTRACT: In this paper, we obtain some subordination and superordination-preserving results of the generalized Srivastava-Attyia operator. Sandwich-type result is also obtained.

1. Introduction

Let $H(U)$ be the class of functions analytic in $U = \{z \in \mathbb{C} : |z| < 1\}$ and $H[a, n]$ be the subclass of $H(U)$ consisting of functions of the form $f(z) = a + a_n z^n + a_{n+1} z^{n+1} + ...$, with $H_0 = H[0, 1]$ and $H = H[1, 1]$. Denote $A(p)$ by the class of all analytic functions of the form

$$f(z) = z^p + \sum_{n=1}^{\infty} a_{p+n} z^{p+n} \quad (p \in \mathbb{N} = \{1, 2, 3, ...\} ; z \in U) \tag{1.1}$$

and let $A(1) = A$. For $f, F \in H(U)$, the function $f(z)$ is said to be subordinate to $F(z)$, or $F(z)$ is superordinate to $f(z)$, if there exists a function $\omega(z)$ analytic in U with $\omega(0) = 0$ and $|\omega(z)| < 1 (z \in U)$, such that $f(z) = F(\omega(z))$. In such a case we write $f(z) \prec F(z)$. If F is univalent, then $f(z) \prec F(z)$ if and only if $f(0) = F(0)$ and $f(U) \subset F(U)$ (see [14] and [15]).

Let $\phi : \mathbb{C}^2 \times U \to \mathbb{C}$ and $h(z)$ be univalent in U. If $p(z)$ is analytic in U and satisfies the first order differential subordination:

$$\phi\left(p(z), zp'(z); z\right) \prec h(z), \tag{1.2}$$

then $p(z)$ is a solution of the differential subordination (1.2). The univalent function $q(z)$ is called a dominant of the solutions of the differential subordination (1.2) if $p(z) \prec q(z)$ for all $p(z)$ satisfying (1.2). A univalent dominant $\tilde{q}$ that satisfies $\tilde{q} \prec q$ for all dominants of (1.2) is called the best dominant. If $p(z)$ and $\phi\left(p(z), zp'(z); z\right)$ are univalent in U and if $p(z)$ satisfies the first order differential superordination:

$$h(z) \prec \phi\left(p(z), zp'(z); z\right),\tag{1.3}$$

then $p(z)$ is a solution of the differential superordination (1.3). An analytic function $q(z)$ is called a subordinant of the solutions of the differential superordination (1.3) if $q(z) \prec p(z)$ for all $p(z)$ satisfying (1.3). A univalent subordinant $\tilde{q}$ that satisfies $q \prec \tilde{q}$ for all subordinants of (1.3) is called the best subordinant (see [14] and [15]).

The general Hurwitz-Lerch Zeta function $\Phi(z, s, a)$ is defined by:

$$\Phi(z, s, a) = \sum_{n=0}^{\infty} \frac{z^n}{(n+a)^s} ,\tag{1.4}$$

$(a \in \mathbb{C}\backslash\mathbb{Z}_0^-; \ \mathbb{Z}_0^- = \{0, -1, -2, ...\}; s \in \mathbb{C}$ when $|z| < 1; R\{s\} > 1$ when $|z| = 1)$.

For interesting properties and characteristics of the Hurwitz-Lerch Zeta function $\Phi(z, s, a)$ (see [3], [8], [9], [11] and [19]).

Recently, Srivastava and Attiya [18] introduced the linear operator $L_{s,b} : A \to A$, defined in terms of the Hadamard product by

$$L_{s,b}(f)(z) = G_{s,b}(z) * f(z) \ (z \in U; b \in \mathbb{C}\backslash\mathbb{Z}_0^-; s \in \mathbb{C}),\tag{1.5}$$

where for convenience,

$$G_{s,b} = (1+b)^s[\Phi(z, s, b) - b^{-s}] \ (z \in U).\tag{1.6}$$

The Srivastava-Attiya operator $L_{s,b}$ contains among its special cases, the integral operators introduced and investigated by Alexander [1], Libera [7] and Jung et al. [6].

Analogous to $L_{s,b}$, Liu [10] defined the operator $J_{p,s,b} : A(p) \to A(p)$ by

$$J_{p,s,b}(f)(z) = G_{p,s,b}(z) * f(z) \ (z \in U; b \in \mathbb{C}\backslash\mathbb{Z}_0^-; s \in \mathbb{C}; p \in \mathbb{N}),\tag{1.7}$$

where

$$G_{p,s,b} = (1+b)^s[\Phi_p(z, s, b) - b^{-s}]$$

and

$$\Phi_p(z, s, b) = \frac{1}{b^s} + \sum_{n=0}^{\infty} \frac{z^{n+p}}{(n+1+b)^s}.\tag{1.8}$$

It is easy to observe from (1.7) and (1.8) that

$$J_{p,s,b}(f)(z) = z^p + \sum_{n=1}^{\infty} \left(\frac{1+b}{n+1+b}\right)^s a_{n+p} z^{n+p}.\tag{1.9}$$

We note that

(i) $J_{p,0,b}(f)(z) = f(z)$;

(ii) $J_{1,1,0}(f)(z) = Lf(z) = \int_0^z \frac{f(t)}{t} dt$, where the operator L was introduced by Alexander [1];

(iii) $J_{1,s,b}(f)(z) = L_{s,b}f(z)$ $\left(s \in \mathbb{C}, b \in \mathbb{C}\backslash\mathbb{Z}_0^-\right)$, where the operator $L_{s,b}$ was introduced by Srivastava and Attiya [18];

(iv) $J_{p,1,\nu+p-1}(f)(z) = F_{\nu,p}(f(z))$ $(\nu > -p, p \in \mathbb{N})$, where the operator $F_{\nu,p}$ was introduced by Choi et al. [4];

(v) $J_{p,\alpha,p}(f)(z) = I_p^\alpha f(z)$ $(\alpha \geq 0, p \in \mathbb{N})$, where the operator I_p^α was introduced by Shams et al. [17];

(vi) $J_{p,m,p-1}(f)(z) = J_p^m f(z)$ $(m \in \mathbb{N}_0 = \mathbb{N} \cup \{0\}, p \in \mathbb{N})$, where the operator J_p^m was introduced by El-Ashwah and Aouf [5];

(vii) $J_{p,m,p+l-1}(f)(z) = J_p^m(l)f(z)$ $(m \in \mathbb{N}_0, p \in \mathbb{N}, l \geq 0)$, where the operator $J_p^m(l)$ was introduced by El-Ashwah and Aouf [5].

It follows from (1.9) that:

$$z\left(J_{p,s+1,b}(f)(z)\right)' = (b+1)J_{p,s,b}(f)(z) - (b+1-p)J_{p,s+1,b}(f)(z). \qquad (1.10)$$

To prove our results, we need the following definitions and lemmas.

Definition 1 [14] . *Denote by F the set of all functions $q(z)$ that are analytic and injective on $\bar{U}\backslash E(q)$ where*

$$E(q) = \left\{\zeta \in \partial U : \lim_{z \to \zeta} q(z) = \infty\right\}$$

and are such that $q'(\zeta) \neq 0$ for $\zeta \in \partial U\backslash E(q)$. Further let the subclass of F for which $q(0) = a$ be denoted by $F(a)$, $F(0) \equiv F_0$ and $F(1) \equiv F_1$.

Definition 2 [15]. *A function $L(z,t)$ $(z \in U, t \geq 0)$ is said to be a subordination chain if $L(\cdot,t)$ is analytic and univalent in U for all $t \geq 0$, $L(z,\cdot)$ is continuously differentiable on $[0;1)$ for all $z \in U$ and $L(z,t_1) \prec L(z,t_2)$ for all $0 \leq t_1 \leq t_2$.*

Lemma 1 [16]. *The function $L(z,t) : U \times [0;1) \longrightarrow \mathbb{C}$ of the form*

$$L(z,t) = a_1(t)z + a_2(t)z^2 + ... \quad (a_1(t) \neq 0; t \geq 0)$$

and $\lim_{t \to \infty} |a_1(t)| = \infty$ is a subordination chain if and only if

$$\text{Re}\left\{\frac{z\partial L(z,t)/\partial z}{\partial L(z,t)/\partial t}\right\} > 0 \quad (z \in U, t \geq 0).$$

Lemma 2 [12]. *Suppose that the function $\mathcal{H} : \mathbb{C}^2 \to \mathbb{C}$ satisfies the condition*

$$\text{Re}\{\mathcal{H}(is;t)\} \leq 0$$

for all real s and for all $t \leq -n\left(1 + s^2\right)/2$, $n \in \mathbb{N}$. If the function $p(z) = 1 + p_n z^n + p_{n+1}z^{n+1} + ...$ is analytic in U and

$$\text{Re}\left\{\mathcal{H}\left(p(z); zp'(z)\right)\right\} > 0 \quad (z \in U),$$

then $\operatorname{Re}\{p(z)\} > 0$ *for* $z \in U$.

Lemma 3 [13]. *Let* $\kappa, \gamma \in \mathbb{C}$ *with* $\kappa \neq 0$ *and let* $h \in H(U)$ *with* $h(0) = c$. *If* $\operatorname{Re}\{\kappa h(z) + \gamma\} > 0 \, (z \in U)$, *then the solution of the following differential equation:*

$$q(z) + \frac{zq'(z)}{\kappa q(z) + \gamma} = h(z) \quad (z \in U; q(0) = c)$$

is analytic in U *and satisfies* $\operatorname{Re}\{\kappa q(z) + \gamma\} > 0$ *for* $z \in U$.

Lemma 4 [14]. *Let* $p \in F(a)$ *and let* $q(z) = a + a_n z^n + a_{n+1} z^{n+1} + ...$*be analytic in* U *with* $q(z) \neq a$ *and* $n \geq 1$. *If* q *is not subordinate to* p, *then there exists two points* $z_0 = r_0 e^{i\theta} \in U$ *and* $\zeta_0 \in \partial U \backslash E(q)$ *such that*

$$q(U_{r_0}) \subset p(U); \quad q(z_0) = p(\zeta_0) \quad \text{and} \quad z_0 p'(z_0) = m\zeta_0 p'(\zeta_0) \quad (m \geq n).$$

Lemma 5 [15]. *Let* $q \in H[a; 1]$ *and* $\varphi : \mathbb{C}^2 \to \mathbb{C}$. *Also set* $\varphi\left(q(z), zq'(z)\right) = h(z)$. *If* $L(z, t) = \varphi\left(q(z), tzq'(z)\right)$ *is a subordination chain and* $p \in H[a; 1] \cap F(a)$, *then*

$$h(z) \prec \varphi\left(p(z), zp'(z)\right),$$

implies that $q(z) \prec p(z)$. *Furthermore, if* $\varphi\left(q(z), zq'(z)\right) = h(z)$ *has a univalent solution* $q \in F(a)$, *then* q *is the best subordinant.*

In the present paper, we aim to prove some subordination-preserving and superordination-preserving properties associated with the integral operator $J_{p,s,b}$. Sandwich-type result involving this operator is also derived.

2. Main results

Unless otherwise mentioned, we assume throughout this section that $b \in \mathbb{C} \backslash \mathbb{Z}_0^-, s \in \mathbb{C}$, $\operatorname{Re}(b) > 0$, $p \in \mathbb{N}$ and $z \in \mathbb{U}$.

Theorem 1. *Let* $f, g \in A(p)$ *and*

$$\operatorname{Re}\left\{1 + \frac{z\phi''(z)}{\phi'(z)}\right\} > -\delta \qquad \left(\phi(z) = \frac{J_{p,s-1,b}(g)(z)}{z^p}; z \in U\right), \qquad (2.1)$$

where δ *is given by*

$$\delta = \frac{1 + |b+1|^2 - \left|1 - (b+1)^2\right|}{4\,[1 + \operatorname{Re}(b)]} \qquad (z \in U). \qquad (2.2)$$

Then the subordination condition

$$\frac{J_{p,s-1,b}(f)(z)}{z^p} \prec \frac{J_{p,s-1,b}(g)(z)}{z^p} \qquad (2.3)$$

implies that

$$\frac{J_{p,s,b}(f)(z)}{z^p} \prec \frac{J_{p,s,b}(g)(z)}{z^p} \tag{2.4}$$

and the function $\frac{J_{p,s,b}(g)(z)}{z^p}$ is the best dominant.

Proof. Let us define the functions $F(z)$ and $G(z)$ in U by

$$F(z) = \frac{J_{p,s,b}(f)(z)}{z^p} \quad \text{and} \quad G(z) = \frac{J_{p,s,b}(g)(z)}{z^p} \quad (z \in U) \tag{2.5}$$

and without loss of generality we assume that $G(z)$ is analytic, univalent on $\bar{U}$ and

$$G'(\zeta) \neq 0 \quad (|\zeta| = 1).$$

If not, then we replace $F(z)$ and $G(z)$ by $F(\rho z)$ and $G(\rho z)$, respectively, with $0 < \rho < 1$. These new functions have the desired properties on $\bar{U}$, so we can use them in the proof of our result and the results would follow by letting $\rho \to 1$.

We first show that, if

$$q(z) = 1 + \frac{zG''(z)}{G'(z)} \quad (z \in U), \tag{2.6}$$

then

$$\text{Re}\{q(z)\} > 0 \quad (z \in U).$$

From (1.10) and the definition of the functions G, ϕ, we obtain that

$$\phi(z) = G(z) + \frac{zG'(z)}{b+1}. \tag{2.7}$$

Differentiating both sides of (2.7) with respect to z yields

$$\phi'(z) = \left(1 + \frac{1}{b+1}\right)G'(z) + \frac{zG''(z)}{b+1}. \tag{2.8}$$

Combining (2.6) and (2.8), we easily get

$$1 + \frac{z\phi''(z)}{\phi'(z)} = q(z) + \frac{zq'(z)}{q(z)+b+1} = h(z) \quad (z \in U). \tag{2.9}$$

It follows from (2.1) and (2.9) that

$$\text{Re}\{h(z) + b + 1\} > 0 \quad (z \in U). \tag{2.10}$$

Moreover, by using Lemma 3, we conclude that the differential equation (2.9) has a solution $q(z) \in H(U)$ with $h(0) = q(0) = 1$. Let

$$\mathcal{H}(u,v) = u + \frac{v}{u+b+1} + \delta,$$

where δ is given by (2.2). From (2.9) and (2.10), we obtain $\mathrm{Re}\left\{\mathcal{H}\left(q(z); zq^{'}(z)\right)\right\} > 0 \ (z \in U)$.

To verify the condition

$$\mathrm{Re}\left\{\mathcal{H}\left(i\vartheta; t\right)\right\} \leq 0 \quad \left(\vartheta \in \mathbb{R}; t \leq -\frac{1+\vartheta^2}{2}\right), \tag{2.11}$$

we proceed as follows:

$$\begin{aligned}\mathrm{Re}\left\{\mathcal{H}\left(i\vartheta; t\right)\right\} & = \mathrm{Re}\left\{i\vartheta + \frac{t}{b+1+i\vartheta} + \delta\right\} = \frac{t\left(1+\mathrm{Re}\left(b\right)\right)}{\left|b+1+i\vartheta\right|^2} + \delta \\ & \leq -\frac{\Upsilon\left(b, \vartheta, \delta\right)}{2\left|b+1+i\vartheta\right|^2},\end{aligned}$$

where

$$\Upsilon\left(b, \vartheta, \delta\right) = \left[1 + \mathrm{Re}\left(b\right) - 2\delta\right]\vartheta^2 - 4\delta\,\mathrm{Im}\left(b\right)\vartheta - 2\delta\left|b+1\right|^2 + 1 + \mathrm{Re}\left(b\right). \tag{2.12}$$

For δ given by (2.2), the coefficient of ϑ^2 in the quadratic expression $\Upsilon\left(b, \vartheta, \delta\right)$ given by (2.12) is positive or equal to zero. To check this, put $b+1 = c$, so that

$$1 + \mathrm{Re}\left(b\right) = c_1 \quad \text{and} \quad \mathrm{Im}\left(b\right) = c_2.$$

We thus have to verify that

$$c_1 - 2\delta \geq 0,$$

or

$$c_1 \geq 2\delta = \frac{1 + |c|^2 - \left|1 - c^2\right|}{2c_1}.$$

This inequality will hold true if

$$2c_1^2 + \left|1 - c^2\right| \geq 1 + |c|^2 = 1 + c_1^2 + c_2^2,$$

that is, if

$$\left|1 - c^2\right| \geq 1 - \mathrm{Re}\left(c^2\right),$$

which is obviously true. Moreover, the quadratic expression $\Upsilon\left(b, \vartheta, \delta\right)$ by ϑ in (2.12) is a perfect square for the assumed value of δ given by (2.2). Hence we see that (2.11) holds. Thus, by using Lemma 2, we conclude that

$$\mathrm{Re}\left\{q\left(z\right)\right\} > 0 \quad \left(z \in U\right),$$

that is, that G defined by (2.5) is convex (univalent) in U. Next, we prove that the subordination condition (2.3) implies that

$$F\left(z\right) \prec G\left(z\right),$$

for the functions F and G defined by (2.5). Consider the function $L(z,t)$ given by

$$L(z,t) = G(z) + \frac{(1+t)\,zG^{'}(z)}{b+1} \quad (0 \le t < \infty; z \in U).$$
(2.13)

We note that

$$\left. \frac{\partial L(z,t)}{\partial z}\right|_{z=0} = G^{'}(0)\left(1 + \frac{1+t}{b+1}\right) \ne 0 \quad (0 \le t < \infty; z \in U; \mathrm{Re}\,\{b+1\} > 0).$$

This show that the function

$$L(z,t) = a_1(t)\,z + ... ,$$

satisfies the condition $a_1(t) \ne 0\ (0 \le t < \infty)$. Further, we have

$$\mathrm{Re}\left\{\frac{z\partial L(z,t)/\partial z}{\partial L(z,t)/\partial t}\right\} = \mathrm{Re}\,\{b+1+(1+t)\,q(z)\} > 0 \quad (0 \le t < \infty; z \in U).$$

Since $G(z)$ is convex and $\mathrm{Re}\,\{b+1\} > 0$. Therefore, by using Lemma 1, we deduce that $L(z,t)$ is a subordination chain. It follows from the definition of subordination chain that

$$\phi(z) = G(z) + \frac{zG^{'}(z)}{b+1} = L(z,0)$$

and

$$L(z,0) \prec L(z,t) \quad (0 \le t < \infty),$$

which implies that

$$L(\zeta,t) \notin L(U,0) = \phi(U) \quad (0 \le t < \infty; \zeta \in \partial U).$$
(2.14)

If F is not subordinate to G, by using Lemma 4, we know that there exist two points $z_0 \in U$ and $\zeta_0 \in \partial U$ such that

$$F(z_0) = G(\zeta_0) \quad \text{and} \quad z_0 F^{'}(z_0) = (1+t)\,\zeta_0 G^{'}(\zeta_0) \quad (0 \le t < \infty).$$
(2.15)

Hence, by using (2.5), (2.13),(2.15) and (2.3), we have

$$L(\zeta_0,t) = G(\zeta_0) + \frac{(1+t)\,\zeta_0 G^{'}(\zeta_0)}{b+1} = F(z_0) + \frac{z_0 F^{'}(z_0)}{b+1} = \frac{J_{p,s-1,b}(f)(z_0)}{z_0^p} \in \phi(U).$$

This contradicts (2.14). Thus, we deduce that $F \prec G$. Considering $F = G$, we see that the function G is the best dominant. This completes the proof of Theorem 1.

We now derive the following superordination result.

Theorem 2. *Let $f, g \in A(p)$ and*

$$\mathrm{Re}\left\{1 + \frac{z\phi^{''}(z)}{\phi^{'}(z)}\right\} > -\delta \quad \left(\phi(z) = \frac{J_{p,s-1,b}(g)(z)}{z^p}; z \in U\right),$$
(2.16)

where δ is given by (2.2) *. If the function* $\frac{J_{p,s-1,b}(f)(z)}{z^p}$ *is univalent in U and* $\frac{J_{p,s,b}(f)(z)}{z^p} \in F,$ *then the superordination condition*

$$\frac{J_{p,s-1,b}(g)(z)}{z^p} \prec \frac{J_{p,s-1,b}(f)(z)}{z^p} \qquad (2.17)$$

implies that

$$\frac{J_{p,s,b}(g)(z)}{z^p} \prec \frac{J_{p,s,b}(f)(z)}{z^p} \qquad (2.18)$$

and the function $\frac{J_{p,s,b}(g)(z)}{z^p}$ *is the best subordinant.*

Proof. Suppose that the functions F, G and q are defined by (2.5) and (2.6), respectively. By applying similar method as in the proof of Theorem 1, we get

$$\mathrm{Re}\left\{q\left(z\right)\right\} > 0 \quad \left(z \in U\right).$$

Next, to arrive at our desired result, we show that $G \prec F$. For this, we suppose that the function $L\left(z,t\right)$ be defined by (2.13). Since G is convex, by applying a similar method as in Theorem 1, we deduce that $L\left(z,t\right)$ is subordination chain. Therefore, by using Lemma 5, we conclude that $G \prec F$. Moreover, since the differential equation

$$\phi\left(z\right) = G\left(z\right) + \frac{zG'\left(z\right)}{b+1} = \varphi\left(G\left(z\right), zG'\left(z\right)\right)$$

has a univalent solution G, it is the best subordinant. This completes the proof of Theorem 2.

Combining the above-mentioned subordination and superordination results involving the operator $J_{p,s,b}$, the following "sandwich-type result" is derived.

Theorem 3. *Let* $f, g_j \in A\left(p\right)\left(j = 1, 2\right)$ *and*

$$\mathrm{Re}\left\{1 + \frac{z\phi_j''\left(z\right)}{\phi_j'\left(z\right)}\right\} > -\delta \quad \left(\phi_j\left(z\right) = \frac{J_{p,s-1,b}(g_j)(z)}{z^p}\left(j = 1, 2\right); z \in U\right),$$

where δ is given by (2.2) *. If the function* $\frac{J_{p,s-1,b}(f)(z)}{z^p}$ *is univalent in U and* $\frac{J_{p,s,b}(f)(z)}{z^p} \in F,$ *then the condition*

$$\frac{J_{p,s-1,b}(g_1)(z)}{z^p} \prec \frac{J_{p,s-1,b}(f)(z)}{z^p} \prec \frac{J_{p,s-1,b}(g_2)(z)}{z^p} \qquad (2.19)$$

implies that

$$\frac{J_{p,s,b}(g_1)(z)}{z^p} \prec \frac{J_{p,s,b}(f)(z)}{z^p} \prec \frac{J_{p,s,b}(g_2)(z)}{z^p} \qquad (2.20)$$

and the functions $\frac{J_{p,s,b}(g_1)(z)}{z^p}$ *and* $\frac{J_{p,s,b}(g_2)(z)}{z^p}$ *are, respectively, the best subordinant and the best dominant.*

Remark. *(i) Putting $b = p$ and $s = \alpha$ $(\alpha \geq 0, p \in \mathbb{N})$ in our results of this paper, we obtain the results obtained by Aouf and Seoudy [2];*

(ii) Specializing the parameters s and b in our results of this paper, we obtain the results for the corresponding operators $F_{\nu,p}$, J_p^m and $J_p^m\left(l\right)$ which are defined in the introduction.

References

[1] J. W. Alexander, *Functions which map the interior of the unit circle upon simple regions*, Ann. Math. Ser., 2, 17 (1915), 12-22.

[2] M. K. Aouf and T. M. Seoudy, *Some preserving subordination and superordination results of certain integral operator*, Int. J. Open Problems Comput. Math., (to appear).

[3] J. H. Choi and H. M. Srivastava, *Certain families of series associated with the Hurwitz-Lerch Zeta function*, Appl. Math. Comput., 170 (2005), 399-409.

[4] J. H. Choi, M. Saigo and H.M. Srivastava, *Some inclusion properties of a certain family of integral operators*, J. Math. Anal. Appl., 276 (2002), 432-445.

[5] R. M. El-Ashwah and M. K. Aouf, *Some properties of new integral operator*, Acta Univ. Apulensis, 24 (2010), 51-61.

[6] T. B. Jung , Y. C. Kim and H. M. Srivastava, *The Hardy space of analytic functions associated with certain one-parameter families of integral operators*, J. Math. Anal. Appl., 176 (1993), 138-147.

[7] R. J. Libera, *Some classes of regular univalent functions*, Proc. Amer. Math. Soc., 16 (1969), 755-758.

[8] S.-D. Lin and H. M. Srivastava, *Some families of the Hurwitz-Lerch Zeta functions and associated fractional derivative and other integral representations*, Appl. Math. Comput., 154 (2004), 725-733.

[9] S.-D. Lin, H. M. Srivastava and P.-Y. Wang, *Some expansion formulas for a class of generalized Hurwitz-Lerch Zeta functions*, Integral Transforms Spec. Funct., 17 (2006), 817-827.

[10] J.-L. Liu, *Subordinations for certain multivalent analytic functions associated with the generalized Srivastava-Attiya operator*, Integral Transforms Spec. Funct., 18 (2007), 207-216.

[11] Q.-M. Luo and H. M. Srivastava, *Some generalizations of the Apostol-Bernoulli and Apostol-Euler polynomials*, J. Math. Anal. Appl., 308 (2005), 290-302.

[12] S. S. Miller and P. T. Mocanu, *Differential subordinations and univalent functions*, Michigan Math. J., 28 (1981), no. 2, 157–172.

[13] S. S. Miller and P. T. Mocanu, *Univalent solutions of Briot-Bouquet differential equations*, J. Differential Equations, 56 (1985), no. 3, 297–309.

[14] S. S. Miller and P. T. Mocanu, *Differential Subordinations: Theory and Applications, Series on Monographs and Textbooks in Pure and Applied Mathematics*, Vol. 225, Marcel Dekker, New York and Basel, 2000.

[15] S. Miller and P. T. Mocanu, *Subordinants of differential superordinations*, Complex Variables. Theory Appl., 48 (2003), no. 10, 815–826.

[16] Ch. Pommerenke, *Univalent Functions*, Vandenhoeck and Ruprecht, Göttingen, 1975.

[17] S. Shams, S. R. Kulkarni, and J. M. Jahangiri, *Subordination properties of p–valent functions defined by integral operators*, Internat. J. Math. Math. Sci., Vol. 2006, Art. ID 94572, 1–3.

[18] H. M. Srivastava and A. A. Attiya, *An integral operator associated with the Hurwitz-Lerch Zeta function and differential subordination*, Integral Transforms Spec. Funct., 18 (2007), 207-216.

[19] H. M. Srivastava and J. Choi, *Series associated with the Zeta and Related Functions*, Kluwer Academic Publishers, Dordrecht, Boston, London, 2001.

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