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# Two Classes of Infinitely Many Solutions for Fractional Schrödinger-Maxwell System With Concave-Convex Power Nonlinearities

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Abstract: Employing critical theory and concentration estimates, we establish the existence of two classes of infinitely many weak solutions fractional Schrödinger-Poisson system involving critical Sobolev growth. The first classe of solutions with negative energy is found by using of notion genus while the second one contains infinitely many weak solutions with positive energy via Fountain theorem.

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## 1. Introduction

In this paper we focus our attention on the following critical fractional system

$$\begin{cases} (-\Delta)^s u + u + \phi u = \lambda a(x)|u|^{r-2}u + b(x)|u|^{2_s^*-2}u & \text{in } \mathbb{R}^3, \\ (-\Delta)^t \phi = u^2 & \text{in } \mathbb{R}^3, \end{cases} \tag{1}$$

where $s \in (\frac{3}{4}, 1)$, $t \in (0,1)$ with $4s + 2t > 3$, $1 < r < 2 < 2_s^* := \frac{6}{3-2s}$, $\lambda$ is a positive parameter, $a(x), b(x) \in \mathcal{C}(\mathbb{R}^3)$.
The system (1) is made up of a fractional Schrödinger equation coupled to a fractional poisson equation. It is well known that the system (1) has a strong physical

significance, because it appears in many quantum mechanics modules (see for example [5, 14]) and in semiconductor theory [3], and so on. In recent years, there has been an increasing attention to this type of system on the existence and the multiplicity of positive solutions, see the following references [2, 6, 8, 10, 11, 12, 15, 16, 21]. To our knowledge, there are few recent articles dealing with the result of the existence of two classes of solutions of infinite types and different signs of energies. By using the truncation tip at the level of the functional to make it bounded from below and satisfied the condition of $(P.S)_c$ for any $c < 0$. Following the Ljusternick-Schnirelmann theory, we obtain a negative class with infinitely solutions. Via the Fountain Theorem, we obtain the second class of infinitely positive solutions.

$(A_1)$ Let $1 < r < 2 < 2_s^*$, $\sigma = \frac{2_s^*}{2_s^*-r}$ and $2_s^* = \frac{6}{3-2s}$, $a(x) \in \mathcal{C}(\mathbb{R}^3) \cap L^\sigma(\mathbb{R}^3)$, $b(x) \in \mathcal{C}(\mathbb{R}^3) \cap L^\infty(\mathbb{R}^3)$,

$(A_2)$ $a(x) > 0$ in some open bounded subset $\Omega$ of $\mathbb{R}^3$ with strictly positive Lebesgue measure,

$(G_1)$ Let $G$ be a subgroup of $O_3$, $\#G = \infty$, $a(x), b(x)$ are $G-$invariant,

$(G_2)$ $a(x) \in \mathcal{C}(\mathbb{R}^3, \mathbb{R}^+) \cap L^\sigma_G(\mathbb{R}^3)$, $b(x) \in \mathcal{C}(\mathbb{R}^3, \mathbb{R}^+)$, $b(x) = b(|x|)$ for any $x \in \mathbb{R}^3$ and $b(0) = b(\infty) = 0$.

Our first main result is the following:

**Theorem 1.1.**
*If $(A_1)$ and $(A_2)$ are satisfied. Then there exists $\lambda_0 > 0$ such that, for each $\lambda \in (0, \lambda_0)$, the problem (1) has infinitely many solutions with negative energy.*

Our next goal is the following:

**Theorem 1.2.**
*If $(G_1)$ and $(G_2)$ are satisfied. Then for all $\lambda > 0$ the problem (1) has infinitely many solutions with positive energy.*

The paper is organized as follows. In Section 2, we present some preliminaries results and we give the interval parameter $\lambda$ for which the energy functional is compact. In Section 3, when $\lambda$ is small enough, we prove the first Theorem 1.1 by application of genus. In Section 4, we give the proof of the second Theorem 1.2 without condition under the parameter $\lambda > 0$, we establish this result via Fountain theorem.

## 2. Functional framework and preliminary

For any $s \in (0,1)$, we define the homogeneous fractional Sobolev space $\mathcal{D}^{s,2}(\mathbb{R}^3)$ as follows

$$\mathcal{D}^{s,2}(\mathbb{R}^3) = \left\{ u \in L^{2_s^*}(\mathbb{R}^3) : \ |\xi|^s \hat{u}(\xi) \in L^2(\mathbb{R}^3) \right\},$$

which is the completion of $C_0^\infty(\mathbb{R}^3)$ under the norm

$$\|u\|_{\mathcal{D}^{s,2}(\mathbb{R}^3)} = \left(\int_{\mathbb{R}^3} |\,(-\Delta)^{\frac{s}{2}}\, u|^2 dx\right)^{1/2} = \left(\int_{\mathbb{R}^3} |\xi|^{2s}|\hat{u}(\xi)|^2 d\xi\right)^{1/2}.$$

The fractional Sobolev space $H^s(\mathbb{R}^3)$ can be described by means of the Fourier transform, i.e.

$$H^s(\mathbb{R}^3) = \left\{u \in L^2(\mathbb{R}^3) : \int_{\mathbb{R}^3} |\xi|^{2s}|\hat{u}(\xi)|^2 + |\hat{u}(\xi)|^2 d\xi < +\infty\right\},$$

which is a Hilbert space under the norm. In this case, the inner product and the norm are defined as

$$\langle u, v\rangle = \int_{\mathbb{R}^3} |\xi|^{2s}\hat{u}(\xi)\overline{\hat{v}(\xi)} + \hat{u}(\xi)\overline{\hat{v}(\xi)} d\xi,$$

$$\|u\|_{H^s} = \left(\int_{\mathbb{R}^3} |\xi|^{2s}|\hat{u}(\xi)|^2 + |\hat{u}(\xi)|^2 d\xi\right)^{1/2}.$$

From Plancherel's theorem we have $\|u\|_{L^2(\mathbb{R}^3)} = \|\hat{u}\|_{L^2(\mathbb{R}^3)}$ and $\||\xi|^s|\hat{u}|\|_{L^2(\mathbb{R}^3)} = \|\,(-\Delta)^{\frac{s}{2}}\, u\|_{L^2(\mathbb{R}^3)}$. Hence

$$\|u\|_{H^s} = \left(\int_{\mathbb{R}^3} |\,(-\Delta)^{\frac{s}{2}}\, u(x)|^2 + |u(x)|^2 dx\right)^{1/2} \quad \forall u \in H^s(\mathbb{R}^3).$$

In our context, the Sobolev constant is given by

$$\mathbb{S} := \frac{\int_{\mathbb{R}^3} |(-\Delta)^{\frac{s}{2}} u(x)|^2 + |u(x)|^2 dx}{\left(\int_{\mathbb{R}^3} |u|^{2_s^*} dx\right)^{\frac{2}{2_s^*}}}. \tag{2}$$

From the embedding results, we know that $H^s(\mathbb{R}^3)$ is continuously and compactly embedded in $L^p(\mathbb{R}^3)$ when $1 \leq p < 2_s^*$, where $2_s^* = \frac{6}{3-2s}$ and the embedding is continuous but not compact if $p = 2_s^*$. For more general facts about the fractional Laplacian we refer the reader to the paper [7].
From [20], the author has proved that if $4s + 2t \geq 3$, for each $u \in H^s(\mathbb{R}^3)$, the Lax-Milgram theorem implies that there exists a unique $\phi_u^t \in \mathcal{D}^{1,2}(\mathbb{R}^3)$ such that

$$\int_{\mathbb{R}^3} (-\Delta)^{\frac{t}{2}} \phi_u^t (-\Delta)^{\frac{t}{2}} v dx = \int_{\mathbb{R}^3} u^2 v dx$$

$\forall v \in \mathcal{D}^{1,2}(\mathbb{R}^3)$, that is $\phi_u^t$ is a weak solution of

$$(-\Delta)^t \phi_u^t = u^2, \;\; x \in \mathbb{R}^3$$

and the representation formula holds

$$\phi_u^t(x) = c_t \int_{\mathbb{R}^3} \frac{u^2(y)}{|x-y|^{3-2t}} dy, \;\; x \in \mathbb{R}^3, \;\; c_t = \pi^{-\frac{3}{2}} 2^{-2t} \frac{\Gamma(\frac{3-2t}{2})}{\Gamma(t)},$$

which is called $t-$Riesz potential.

The properties of the function $\phi_u^t$ are given in the following lemma (see [[20], Lemma 2.3]).

**Lemma 2.1.** *If $4s + 2t \geq 3$, then for any $u \in H^s(\mathbb{R}^3)$, we have*

**(i)** $\phi_u^t \geq 0$;

**(ii)** $\phi_u^t : H^s(\mathbb{R}^3) \to \mathcal{D}^{s,2}(\mathbb{R}^3)$, *is continuous and maps bounded sets into bounded sets;*

**(iii)** $\int_{\mathbb{R}^3} \phi_u^t u^2 dx \leq S_s^2 \|u\|^2_{\frac{12}{3+2t}} \leq C\|u\|^4_{H^s(\mathbb{R}^3)}$;

**(iv)** *If $u_n \rightharpoonup u$ in $H^s(\mathbb{R}^3)$, then $\phi_{u_n}^t \to \phi_u^t$ in $\mathcal{D}^{s,2}(\mathbb{R}^3)$, and* $\int_{\mathbb{R}^3} \phi_{u_n}^t u_n^2 dx \to \int_{\mathbb{R}^3} \phi_u^t u^2 dx$.

Substituting $\phi_u^t$ in (1), it reduces as follows

$$(-\Delta)^s u + u + \phi_u^t u = \lambda a(x)|u|^{r-2}u + b(x)|u|^{2_s^*-2}u \text{ in } \mathbb{R}^3,$$

To find solutions of (1), we will use a variational approach. Hence, we will associate a suitable functional to our problem. More precisely, the Euler-Lagrange functional related to problem (1) is given by $I_\lambda : H^s(\mathbb{R}^3) \to \mathbb{R}$ defined as follows

$$I_\lambda(u) = \frac{1}{2}\|u\|^2_{H^s} + \frac{1}{4}\int_{\mathbb{R}^3} \phi_u^t u^2 dx - \frac{\lambda}{r}\int_{\mathbb{R}^3} a(x)|u|^r dx - \frac{1}{2_s^*}\int_{\mathbb{R}^3} b(x)|u|^{2_s^*} dx.$$

Obviously, $I_\lambda \in C^1(H^s(\mathbb{R}^3), \mathbb{R})$ and its critical points are weak solutions to (1). We call $u \in H^s(\mathbb{R}^3)$ is a weak solution of (1) if

$$\begin{aligned} \langle I'_\lambda(u), v\rangle &= \int_{\mathbb{R}^3} (-\Delta)^{\frac{s}{2}} u \, (-\Delta)^{\frac{s}{2}} v dx + \int_{\mathbb{R}^3} uv dx + \int_{\mathbb{R}^3} \phi_u^t uv dx \\ &\quad - \lambda \int_{\mathbb{R}^3} a(x)|u|^{r-2} uv dx - \int_{\mathbb{R}^3} b(x)|u|^{2_s^*-2} uv dx = 0, \end{aligned}$$

for any $v \in H^s(\mathbb{R}^3)$.

Defined $N : H^s(\mathbb{R}^3) \to \mathbb{R}$ by $N(u) = \int_{\mathbb{R}^3} \phi_u^t u^2 dx$. The following lemma shows that the functional and possesses property which is similar to the well-known Brezis-Lieb lemma [4].

**Lemma 2.2.** *Assume that $4s + 2t > 3$. Let $u_n \rightharpoonup u$ in $H^s(\mathbb{R}^3)$ and $u_n \to u$ a.e. in $\mathbb{R}^3$. Then*

**(i)** $N(u_n - u) = N(u_n) - N(u) + o_n(1)$;

**(ii)** $N'(u_n - u) = N'(u_n) - N'(u) + o_n(1)$; *in $H^{-s}(\mathbb{R}^3)$.*

**Proof.** We can consult for example ([[20], Lemma 2.4]). □

Along the way one can easily the following lemma

**Lemma 2.3.** *Under the same conditions as the Lemma 2.2. Let $v_n = u_n - u \rightharpoonup 0$. Then*

$$\begin{cases} I_\lambda(v_n) \to c - I_\lambda(u), \\ I'_\lambda(v_n) \to 0. \end{cases} \tag{3}$$

We recall that

**Definition 1.** Let X be a Banach space

**(i)** For $c \in \mathbb{R}$, a sequence $\{u_n\} \subset H^s(\mathbb{R}^3)$ is a $(PS)_c$ for $I_\lambda$ if $I_\lambda(u_n) = c + o(1)$ and $I_\lambda^{'}(u_n) = o(1)$ strongly in $H^{-s}(\mathbb{R}^3)$ as $n \to +\infty$;

**(ii)** $I_\lambda$ satisfies the $(PS)_c$ condition in $X$ if any $(PS)_c$ sequence for $I_\lambda$ contains a convergent subsequence.

Let us show firstly the $(PS)_c$ sequence is bounded.

**Lemma 2.4.** *Let $c \in \mathbb{R}$. If $\{u_n\}$ is $(PS)_c$- sequence for $I_\lambda$, then $\{u_n\}$ is bounded in $H^s(\mathbb{R}^3)$.*

**Proof.** We have

$$I_\lambda(u_n) = c + o(1) \text{ and } I_\lambda^{'}(u_n) = o(1) \text{ in } H^{-s}(\mathbb{R}^3), \tag{4}$$

By contradiction, we assume that $\|u_n\|_{H^s} \to +\infty$.
Let $\widehat{u}_n = \frac{u_n}{\|u_n\|_{H^s}}$. Clearly, $\|\widehat{u}_n\|_{H^s} = 1$ is bounded in $H^s(\mathbb{R}^3)$. Up to a subsequence, we may assume that

$$\widehat{u}_n \rightharpoonup \widehat{u} \ \ in \ H^s(\mathbb{R}^3).$$

This implies

$$\widehat{u}_n \to \widehat{u} \quad in \ L^r(\mathbb{R}^3), \ 1 \le r < 2_s^*.$$

By (4), we have

$$\begin{aligned} c + o(1) = &\frac{1}{2}\|u_n\|_{H^s}^2 \|\widehat{u}_n\|^2 + \frac{1}{4}\|u_n\|_{H^s}^2 \int_{\mathbb{R}^3} \phi_{\widehat{u}_n}^t \widehat{u}_n^2 dx - \frac{1}{2_s^*}\|u_n\|_{H^s}^{2_s^*} \int_{\mathbb{R}^3} b(x)|\widehat{u}_n|^{2_s^*} dx \\ &- \frac{\lambda}{r}\|u_n\|_{H^s}^r \int_{\mathbb{R}^3} a(x)|\widehat{u}_n|^r dx, \ as \ n \to +\infty, \end{aligned}$$

and

$$\begin{aligned} o(1) = &\|u_n\|_{H^s}^2 \|\widehat{u}_n\|^2 + \|u_n\|_{H^s}^2 \int_{\mathbb{R}^3} \phi_{\widehat{u}_n}^t \widehat{u}_n^2 dx - \|u_n\|_{H^s}^{2_s^*} \int_{\Omega} b(x)|\widehat{u}_n|^{2_s^*} dx \\ &- \lambda\|u_n\|_{H^s}^r \int_{\mathbb{R}^3} a(x)|\widehat{u}_n|^r dx, \ as \ n \to +\infty. \end{aligned}$$

By using the above two equalities, we have

$$\begin{aligned} o(1) = &(\frac{1}{r} - \frac{1}{2})\|\widehat{u}_n\|_{H^s}^2 + (\frac{1}{r} - \frac{1}{4}) \int_{\mathbb{R}^3} \phi_{\widehat{u}_n}^t \widehat{u}_n^2 dx \\ &+ \left(\frac{1}{2_s^*} - \frac{1}{r}\right) \|u_n\|_{H^s}^{2_s^*-2} \int_{\mathbb{R}^3} b(x)|\widehat{u}_n|^{2_s^*} dx, \end{aligned}$$

as $n \to +\infty$, that is

$$\begin{aligned}&(\frac{1}{r}-\frac{1}{2})\|\widehat{u}_n\|^2_{H^s}+(\frac{1}{r}-\frac{1}{4})\int_{\mathbb{R}^3}\phi^t_{\widehat{u}_n}\widehat{u}_n^2dx\\ &=\left(\frac{1}{r}-\frac{1}{2_s^*}\right)\|u_n\|^{2_s^*-2}_{H^s}\int_{\mathbb{R}^3}b(x)|\widehat{u}_n|^{2_s^*}dx+o(1),\end{aligned}$$

as $n \to +\infty$.
This implies,

$$(\frac{1}{r}-\frac{1}{2})\|\widehat{u}_n\|^2_{H^s}+(\frac{1}{r}-\frac{1}{4})\int_{\mathbb{R}^3}\phi^t_{\widehat{u}_n}\widehat{u}_n^2dx \to +\infty,$$

as $n \to +\infty$. By Lemma 2.1 (iii), there $C > 0$ and $\|\widehat{u}_n\|_{H^s} = 1$ we have

$$\begin{aligned}+\infty \leftarrow &(\frac{1}{r}-\frac{1}{2})\|\widehat{u}_n\|^2_{H^s}+(\frac{1}{r}-\frac{1}{4})\int_{\mathbb{R}^3}\phi^t_{\widehat{u}_n}\widehat{u}_n^2dx\\ &\leq (\frac{1}{r}-\frac{1}{2})\|\widehat{u}_n\|^2_{H^s}+C\|\widehat{u}_n\|^4=(\frac{1}{r}-\frac{1}{2})+C,\ asn\to+\infty,\end{aligned}$$

which is a contradiction. Thus $\{u_n\}$ is bounded in $H^s(\mathbb{R}^3)$. □

**Lemma 2.5.** *There exists $\lambda_0 > 0$ such that for every $0 < \lambda < \lambda_0$ the functional $I_\lambda$ satisfies $(PS)_c$ for all $c < 0$.*

**Proof.** Consider a $(PS)_c$ sequence $\{u_n\}$ for $I_\lambda$ with $c < 0$. From Lemma 2.4 $\{u_n\}$ is bounded in $H^s(\mathbb{R}^3)$. Going if necessary to a subsequence, we can assume that

$$\begin{cases} u_n \rightharpoonup u, \text{ in } H^s(\mathbb{R}^3),\\ u_n \to u, \text{ in } L^r(\mathbb{R}^3),\ 1\leq r<2_s^*.\end{cases} \tag{5}$$

By Lemma 2.3 we have

$$\langle I_\lambda'(u),\varphi\rangle = 0 \text{ for any } \varphi\in H^s(\mathbb{R}^3). \tag{6}$$

With (4) and $\sigma = \frac{2_s^*}{2_s^*-r}$ and the Hölder Inequality we get

$$\begin{aligned}I_\lambda(u_n) &- \frac{1}{2_s^*}\langle I_\lambda'(u_n),u_n\rangle = c+o(1)\|u_n\|_{H^s}\to c<0\\ &\geq \left(\frac{1}{2}-\frac{1}{2_s^*}\right)\|u_n\|^2_{H^s}+\left(\frac{1}{2}-\frac{1}{2_s^*}\right)\int_{\mathbb{R}^3}\phi^t_{u_n}u_n^2dx+\lambda\left(\frac{1}{2_s^*}-\frac{1}{r}\right)\int_{\mathbb{R}^3}a(x)|u_n|^rdx\\ &\geq \left(\frac{1}{2}-\frac{1}{2^s_*}\right)\mathbb{S}|u_n|^2_{2_s^*}-\lambda\left(\frac{1}{r}-\frac{1}{2_s^*}\right)|a|^\sigma_\sigma|u_n|^r_{2_s^*}.\end{aligned}$$

Then, there exists some constant $C > 0$ such that

$$|u_n|_{2_s^*}\leq C\lambda^{\frac{1}{2-r}}, \tag{7}$$

and Brezis-Lieb Lemma [4] implies

$$|u|_{2_s^*} \le C\lambda^{\frac{1}{2-r}}. \tag{8}$$

By (6), note that

$$\|u\|_{H^s}^2 + \int_{\mathbb{R}^3} \phi_u^t u^2 dx = \lambda \int_{\mathbb{R}^3} a(x)|u|^r dx + \int_{\mathbb{R}^3} b(x)|u|^{2_s^*} dx, \tag{9}$$

also, using Lemma 2.3, Lemma 2.1 (iv) and (4) we have

$$\|v_n\|_{H^s}^2 = \int_{\mathbb{R}^3} b(x)|v_n|^{2_s^*} dx + o(1). \tag{10}$$

Now, we suppose that

$$\lim_{n\to+\infty} \|v_n\|_{H^s}^2 = \lim_{n\to+\infty} \int_{\mathbb{R}^3} b(x)|v_n|^{2_s^*} dx = l \neq 0.$$

By Sobolev inequality, we have

$$\begin{aligned} \|v_n\|_{H^s}^2 &\ge \mathbb{S}\left(\int_{\mathbb{R}^3} |v_n|^{2_s^*} dx\right)^{\frac{2}{2_s^*}} \\ &\ge \mathbb{S} b_\infty^{-\frac{(3-2s)}{3}} \left(\int_{\mathbb{R}^3} b(x)|v_n|^{2_s^*} dx\right)^{\frac{2}{2_s^*}}, \end{aligned}$$

which implies that

$$l \ge \mathbb{S}^{\frac{3}{2s}} b_\infty^{\frac{2s-3}{2s}}. \tag{11}$$

Let $1 \le r < 2 < 2_s^*$. By Lemmas 2.3, 2.1 (iv), (7),(9),(10), (11) and the Hölder inequality, we have

$$\begin{aligned} o(1) + c &= \frac{1}{2}\|u\|_{H^s}^2 + \frac{1}{4}\int_{\mathbb{R}^3} \phi_u^t u^2 dx - \frac{\lambda}{r}\int_{\mathbb{R}^3} a(x)|u|^r dx - \frac{1}{2_s^*}\int_{\mathbb{R}^3} b(x)|u|^{2_s^*} dx \\ &+ \frac{1}{2}\|v_n\|_{H^s}^2 - \frac{1}{2_s^*}\int_{\mathbb{R}^3} b(x)|v_n|^{2_s^*} dx + o(1) \\ &= \frac{1}{4}\|u\|_{H^s}^2 + \left(\frac{1}{2} - \frac{1}{2_s^*}\right)\|v_n\|_{H^s}^2 + \frac{1}{4}\left(\|u\|_{H^s}^2 + \int_{\mathbb{R}^3} \phi_u^t u^2 dx\right) \\ &- \frac{\lambda}{r}\int_{\mathbb{R}^3} a(x)|u|^r dx - \frac{1}{2_s^*}\int_{\mathbb{R}^3} b(x)|u|^{2_s^*} dx \\ &= \frac{1}{4}\|u\|_{H^s}^2 + \left(\frac{1}{2} - \frac{1}{2_s^*}\right)\|v_n\|_{H^s}^2 + \lambda\left(\frac{1}{4} - \frac{1}{r}\right)\int_{\mathbb{R}^3} a(x)|u|^r dx \\ &+ \left(\frac{1}{4} - \frac{1}{2_s^*}\right)\int_{\mathbb{R}^3} b(x)|u|^{2_s^*} dx \\ &\ge \frac{2s}{3}\mathbb{S}^{\frac{3}{2s}} b_\infty^{\frac{2s-3}{2s}} + \lambda\left(\frac{r-4}{4r}\right)|a|_\sigma^\sigma |u|_{2_s^*}^r. \\ &\ge \frac{2s}{3}\mathbb{S}^{\frac{3}{2s}} b_\infty^{\frac{2s-3}{2s}} + C\lambda^{\frac{2}{2-r}}\left(\frac{r-4}{4r}\right)|a|_\sigma^\sigma. \end{aligned}$$

Then there exists $K > 0$ such that

$$0 > c \geq \frac{2s}{3}\mathbb{S}^{\frac{3}{2s}} b_\infty^{\frac{2s-3}{2s}} - K\lambda^{\frac{2}{2-r}},$$

which is a contradiction for $\lambda$ small enough. Then, $l = 0$, that is, $u_n \to u$ strongly in $H^s(\mathbb{R}^3)$. □

# 3. Proof of the first Theorem 1.1

First by the Sobolev inequality we obtain

$$I_\lambda(u) \geq h(\|u\|_{H^s}), \tag{12}$$

where

$$h(x) = \frac{1}{2}x^2 - \frac{b_\infty}{2_s^*\mathbb{S}^{\frac{2_s^*}{2}}} x^{2_s^*} - \frac{\lambda}{r} C_r x^r.$$

An easy computation shows that, there exists $\lambda_* > 0$ such that for all $0 < \lambda < \lambda_*$, the real valued function $x \mapsto h(x)$ has exactly two positive zeros denoted by $R_0$, $R_1$ and the point $R$ is where $h$ attains its nonnegative maximum, verifies $R_0 < R < R_1$. We now introduce the following truncation of the functional $I_\lambda$. Take the nonincreasing function $\tau : \mathbb{R}^+ \to [0,1]$ and $C^\infty(\mathbb{R}^+)$ such that

$$\begin{cases} \tau(x) = 1 & \text{if } x < R_0, \\ \tau(x) = 0 & \text{if } x > R_1 \end{cases} \tag{13}$$

Let $\varphi(u) = \tau(\|u\|_{H^s})$. We consider the truncated functional

$$\tilde{I}_\lambda(u) = \frac{1}{2}\|u\|_{H^s}^2 + \frac{1}{4}\int_{\mathbb{R}^3} \phi_u^t u^2 dx - \frac{\lambda}{r}\int_{\mathbb{R}^3} a(x)|u|^r dx - \frac{1}{2_s^*}\int_{\mathbb{R}^3} b(x)|u|^{2_s^*}\varphi(u) dx. \tag{14}$$

Similar to 12, we have

$$\tilde{I}_\lambda(u) \geq \overline{h}(\|u\|_{H^s}) \tag{15}$$

where

$$\overline{h}(x) = \frac{1}{2}x^2 - \frac{b_\infty}{2_s^*\mathbb{S}^{\frac{2_s^*}{2}}} x^{2_s^*}\tau(x) - \frac{\lambda}{r} C_r x^r.$$

Clearly,

$$\overline{h}(x) \geq h(x) \tag{16}$$

for $x \geq 0$ and $\overline{h}(x) = h(x)$ if $0 \leq x \leq R_0$, $\overline{h}(x) \geq 0$, if $R_0 < x \leq R_1$ and if $x > R_1$, $\overline{h}(x) = x^r(\frac{1}{2}x^{2-r} - \frac{\lambda}{r}C_r)$ is strictly increasing and so $\overline{h}(x) > 0$, if $x > R_1$. Consequently

$$\overline{h}(x) \geq 0 \ \text{ for } \ x \geq R_0. \tag{17}$$

We have the following result.

**Lemma 3.1.** *This lemma can be expressed as three assertions:*

1. $\tilde{I}_\lambda \in \mathcal{C}^1(H^s(\mathbb{R}^3), \mathbb{R})$, *is even.*

2. *If* $\tilde{I}_\lambda(u_0) \leq 0$ *then* $\|u_0\|_{H^s} < R_0$. *Moreover,* $\tilde{I}_\lambda(u) = I_\lambda(u)$ *for all* $u$ *in a small enough neighborhood of* $u_0$.

3. *There exists* $\lambda_0 > 0$, *such that if* $0 < \lambda < \lambda_0$, *then* $\tilde{I}_\lambda$ *verifies a local Palais-Smale condition for* $c < 0$.

**Proof.** Since $\varphi \in \mathcal{C}^\infty(H^s(\mathbb{R}^3), \mathbb{R})$ and $\varphi(u) = 1$ for $u$ near 0, $\tilde{I}_\lambda \in \mathcal{C}^1(H^s(\mathbb{R}^3), \mathbb{R})$ and assertion **1** holds.
Note that $\tilde{I}_\lambda(u_0) \geq I_\lambda(u_0)$. By taking $\tilde{I}_\lambda(u_0) \leq 0$, we can deduce from 15 that

$$\overline{h}(\|u_0\|_{H^s}) \leq 0.$$

Then By (16) and (17) we have

$$\|u_0\|_{H^s} < R_0. \tag{18}$$

For the proof of (3), let $\{u_n\} \subset H^s(\mathbb{R}^3)$ is a $(PS)_c$ sequence $\tilde{I}_\lambda$, with $c < 0$. Then we may assume that $\tilde{I}_\lambda(u_n) < 0$, $\tilde{I}'_\lambda(u_n) \to 0$. By (2) and for $0 < \lambda < \lambda_0$, $\|u_n\|_{H^s} < R_0$, so $\tilde{I}_\lambda(u_n) = I_\lambda(u_n)$ and $\tilde{I}'_\lambda(u_n) = I'_\lambda(u_n)$. By Lemma 2.5, $I_\lambda$ satisfies $(PS)_c$ condition for $c < 0$, so there is a subsequence $\{u_n\}$ such that $u_n \to u$ in $H^s(\mathbb{R}^3)$. Thus $\tilde{I}_\lambda$ satisfies $(PS)_c$ condition for $c < 0$. □

We first recall some concepts and results in minimax theory.
Let $X$ be a Banach space, and $\sum$ denote all closed subsets of $X - \{0\}$ which are symmetric with respect to the origin. For $A \in \sum$, we define the genus $\gamma(A)$ by

$$\gamma(A) = \min\left\{k \in \mathbb{N} : \exists \phi \in C(A; \mathbb{R}^k - \{0\}), \phi(-x) = \phi(x)\right\},$$

if the minimum exists, and if such a minimum does not exist then we define $\gamma(A) = \infty$. The main properties of genus are contained in the following lemma (see[9] for the details).

**Lemma 3.2.** *Let* $A, B \in \sum$. *Then*

1. *If* $A \subset B$, *then* $\gamma(A) \leq \gamma(B)$.

2. *If there exists an odd homeomorphism between* $A$ *and* $B$, *then* $\gamma(A) = \gamma(B)$.

3. *If* $S^{N-1}$ *is the sphere in* $\mathbb{R}^N$, *then* $\gamma(S^{N-1}) = N$.

4. $\gamma(A \cup B) \leq \gamma(A) + \gamma(B)$.

5. *If* $\gamma(A) < \infty$, *then* $\gamma(\overline{A - B}) \geq \gamma(A) - \gamma(B)$.

6. *If* $A$ *is compact, then* $\gamma(A) < \infty$, *and there exists* $\delta > 0$ *such that* $\gamma(A) = \gamma(N_\delta(A))$ *where* $N_\delta(A) = \{x \in X : d(x, A) \leq \delta\}$.

7. *If* $X_0$ *is a subspace of* $X$ *with codimension* $k$, *and* $\gamma(A) > k$, *then* $A \cap X_0 \neq \emptyset$.

It is possible to prove the existence of level sets of $\tilde{I}_\lambda$ with arbitrarily large genus, more precisely:

**Lemma 3.3.** $\forall n \in \mathbb{N}\ \exists \epsilon(n) > 0$ *such that*

$$\gamma(\{u \in H^s(\mathbb{R}^3) : \tilde{I}_\lambda(u) \leq -\epsilon(n)\}) \geq n.$$

**Proof.** Let $\Omega$ is an open bounded subset with strictly positive Lebesgue measure such that $a(x) > 0$ in $\Omega$. Let $X_0^s(\Omega)$ be the function space defined as

$$X_0^s(\Omega) := \left\{u \in H^s(\mathbb{R}^3) : u = 0 \text{ a.e. in } \mathbb{R}^3 \backslash \Omega\right\}.$$

So, $X_0^s(\Omega) \subset H^s(\mathbb{R}^3)$. Observe that by [[7], Proposition 3.6] we have the following identity

$$\|u\|_{X_0^s(\Omega)} = \left(\int_\Omega |(-\Delta)^{s/2} u(x)|^2 + |u(x)|^2 dx\right)^{1/2} = \|u_n\|_{H^s}.$$

For $n \in \mathbb{N}$, we consider $E_n$ be a $n-$dimensional subspace of $X_0^s(\Omega)$. Let $u_n \in E_n$ with norm $\|u_n\|_{H^s} = 1$. By $(A_2)$ there exists a $c_n > 0$ such that

$$\int_\Omega a(x)|u_n|^r dx \geq c_n > 0.$$

For $0 < \rho < R_0$ and using Lemma 2.1 (iii), we get

$$\tilde{I}_\lambda(\rho u_n) \leq \frac{1}{2}\rho^2 + \frac{1}{4}C\rho^4 - \rho^{2_s^*} \int_{\mathbb{R}^3} b(x)|u_n|^{2_s^*} dx - \frac{\lambda}{r}\rho^r \int_\Omega a(x)|u_n|^r dx. \tag{19}$$

Since $E_n$ is a finite-dimensional space, all the norms in $E_n$ are equivalent. Thus we can define

$$\alpha_n := \inf\{\int_\Omega a(x)|u_n|^r dx : u_n \in E_n, \|u_n\|_{H^s} = 1\} \geq c_n > 0,$$

$$\beta_n := \inf\{\int_\Omega b(x)|u_n|^{2_s^*} dx : u_n \in E_n, \|u_n\|_{H^s} = 1\} > 0.$$

By using the definitions of $\alpha_n$, $\beta_n$ and inequality 19, we obtain

$$\tilde{I}_\lambda(\rho u_n) \leq \frac{1}{2}\rho^2 + \frac{1}{4}C\rho^4 - \rho^{2_s^*}\beta_n - \frac{\lambda}{r}\rho^r \alpha_n.$$

Then, there exists $\epsilon(n) > 0$ and $0 < \rho < R_0$ such that

$$\tilde{I}_\lambda(\rho u) \leq -\epsilon(n)$$

for $u \in E_n$ and $\|u_n\|_{H^s} = 1$. Let $S_\eta = \{u \in H^s(\mathbb{R}^3)/\|u\|_{H^s} = \eta\}$, so

$$S_\eta \cap E_n \subset \{u \in H^s(\mathbb{R}^3)/\tilde{I}_\lambda(u) \leq -\epsilon(n)\},$$

therefore, by Lemma 3.2 we see that

$$\gamma(\{u \in H^s(\mathbb{R}^3)/\tilde{I}_\lambda(u) \leq -\epsilon\}) \geq \gamma(S_\eta \cap E_n) \geq n.$$

□

We are now in a position to prove the first result.

**Proof of the Theorem 1.1.**
For $n \in \mathbb{N}$, we define

$$\Gamma_n = \{A \subset H^s(\mathbb{R}^3) - \{0\}/A \text{ is close, } A = -A, \gamma(A) \geq n\}.$$

Let us set

$$c_n = \min_{A \in \Gamma_n} \max_{u \in A} \tilde{I}_\lambda(u),$$

and

$$K_c = \{u \in H^s(\mathbb{R}^3 : \tilde{I}'_\lambda(u) = 0, \tilde{I}_\lambda(u) = c\},$$

and suppose $0 < \lambda < \lambda_*$ where $\lambda_*$ is the constant given by Lemma 3.1.
We claim if $n, r \in \mathbb{N}$ are such that $c = c_n = c_{n+1} = \cdots c_{n+r}$, then $\gamma(K_c) \geq r + 1$. For simplicity, we call

$$\tilde{I}_\lambda^{-\epsilon} = \{u \in H^s(\mathbb{R}^3)/\tilde{I}_\lambda(u) \leq -\epsilon\}.$$

By lemma 3.3 there exists $\epsilon(n) > 0$ such that $\gamma(\tilde{I}_\lambda^{-\epsilon}) \geq n$, for all $n \in \mathbb{N}$. Because $\tilde{I}_\lambda(u)$ is continuous and even,$\tilde{I}_\lambda^{-\epsilon} \in \Gamma_n$, then $c_n \leq -\epsilon(n) < 0$ for all $n$ in $\mathbb{N}$. But $\tilde{I}_\lambda$ is bounded from below, hence $c_n > -\infty$ for all $n$ in $\mathbb{N}$.
Let us assume that $c = c_n = c_{n+1} = ... = c_{n+r}$. Note that $c < 0$ therefore, $\tilde{I}_\lambda$ verifies the Plais-Smale condition in $c$, and it is easy to see that $K_c$ is a compact set.
If $\gamma(K_c) \leq r$, there exists a closed and symmetric set U verifying $K_c \subset U$, such that $\gamma(U) \leq r$. By the deformation lemma (see [19]), we have an odd homeomorphism $\eta : H^s(\mathbb{R}^3) \to H^s(\mathbb{R}^3)$, such that $\eta(\tilde{I}_\lambda^{c+\delta} - U) \subset \tilde{I}_\lambda^{c-\delta}$, for some $\delta > 0$. By definition,

$$c = c_n = \inf_{A \in \Gamma_{n+r}} \sup_{u \in A} \tilde{I}_\lambda(u).$$

There exists then $A \in \Gamma_{n+r}$, such that $\sup_{u \in A} \tilde{I}_\lambda(u) < c + \delta$. i.e $A \subset \tilde{I}_\lambda^{c+\delta}$,

$$\eta(A - U) \subset \eta(\tilde{I}_\lambda^{c+\delta} - U) \subset \tilde{I}_\lambda^{c-\delta}.$$

By Lemma 3.2 (5) again $\gamma(\overline{A - U}) \geq \gamma(A) - \gamma(U) \geq n$, and $\gamma(\eta(\overline{A - U})) \geq \gamma(\overline{A - U})) \geq n$.

Then, $\eta(\overline{A-U}) \in \Gamma_n$. Impossible, in fact $\eta(\overline{A-U}) \in \Gamma_n$ implies $\sup_{u\in\eta(\overline{A-U})} \tilde{I}_\lambda(u) \geq c_n = c$.
So we have proved that $\gamma(K_c) \geq r+1$. We are now ready to show that $I_\lambda$ has infinitely many critical point solutions. Note that $c_n$ is non-decreasing and strictly negative. We distinguish two cases.
**Case 1** Suppose that there are $1 < n_1 < \cdots n_i < \cdots$, satisfying

$$c_{n_1} < \cdots < c_{n_i} < \cdots.$$

In this case, we have infinitely many distinct critical points.
**Case 2** We assume in this case, that for some positive integer $n_0$, there is a $r \geq 1$ such that $c = c_{n_0} = c_{n_0+1} = \cdots = c_{n_0+r}$, then $\gamma(K_{c_{n_0}}) \geq r+1$ which shows that $K_{c_{n_0}}$ contains infinitely many distinct elements. Since $\tilde{I}_\lambda(u) = I_\lambda(u)$ if $\tilde{I}_\lambda(u) < 0$, we see that there are infinitely many critical points of $I_\lambda(u)$. The theorem 1.1 is proved. □

# 4. Proof of the second Theorem 1.2

In this section, we show the existence of infinitely many solutions via the Fountain Theorem [22].
We consider

$$H^s_G(\mathbb{R}^3) := \left\{u \in H^s(\mathbb{R}^3) : u(\tau x) = u(x), \tau \in G\right\},$$

where $G$ is a subgroup of the group of orthogonal linear transformations $O_3$. Let us consider the functional $I_{\lambda,G}:\ H^s_G(\mathbb{R}^3) \to \mathbb{R}$ as $I_{\lambda,G} = I_\lambda|_{H^s_G(\mathbb{R}^3)}$. By the principle of symmetric criticality of Krawcewicz-Marzantowicz [13], we know that $u$ is a critical point of $I_\lambda$ if and only if $u$ is a critical point of $I_{\lambda,G} = I_\lambda|_{H^s_G(\mathbb{R}^3)}$.

**Lemma 4.1.** *For any $\lambda > 0$, $s \in (\frac{3}{4}, 1)$ and $t \in (0,1)$ such that $4s + 2t > 3$, the functional $I_{\lambda,G}$ satisfies $(PS)_c$ for all $c \in \mathbb{R}$.*

**Proof.** Let $\{u_n\}$ in $H^s_G(\mathbb{R}^3)$ such that $I_{\lambda,G}(u_n) \to c$ and $I'_{\lambda,G}(u_n) \to 0$ strongly in $H^{-s}_G(\mathbb{R}^3)$. Following the same arguments as in the proof of Lemma 2.4 we have $\{u_n\}$ is bounded. Therefore, up to a subsequence, we may assume that

$$\begin{cases} u_n \rightharpoonup u, \text{ in } H^s(\mathbb{R}^3); \\ u_n \to u, \text{ in } L^r(\mathbb{R}^3),\ 1 \leq r < 2^*_s; \\ u_n(x) \to u(x), \text{ a.e. in } \mathbb{R}^3. \end{cases} \tag{20}$$

From the concentration-compactness alternative for bounded sequences in the fractional space $H^s_G(\mathbb{R}^3)$, see [[18], Theorem 2.2 ]: There exists a subsequence, still denoted by $\{u_n\}$, at most countable set $\Lambda$, a set of points $\{x_j\}_{j\in\Lambda} \subset \mathbb{R}^3$ and real numbers $\mu_j, \nu_j \in [0,\infty)$ such that

$$|(-\Delta)^{s/2}u_n|^2 \rightharpoonup d\mu \geq |(-\Delta)^{s/2}u|^2 + \sum_{j\in\Lambda} \mu_j\delta_{x_j},\ \mu_j = \mu(x_j), \tag{21}$$

$$|u_n|^{2_s^*} \rightharpoonup d\nu = |u|^{2_s^*} + \sum_{j\in\Lambda} \nu_j \delta_{x_j},\ \nu_j = \nu(x_j), \tag{22}$$

$$\mu_j \geq \mathbb{S}\nu_j^{\frac{2}{2_s^*}}. \tag{23}$$

We claim that the concentration of $\nu$ cannot occur at any $x \neq 0$. Now we suppose that there exists $x_j \neq 0$, where $j_0 \in \Lambda$ such that $\nu_{j_0} = \nu_{x_{j_0}} > 0$. The measure $\nu$ is $G-$invariant. For all $\tau \in G$, $\nu(x_{j_0}) = \nu(\tau x_{j_0}) > 0$. We know that $\#G = \infty$, thus

$$\nu\left(\{\tau x_{j_0} : \tau \in G\}\right) = \infty.$$

Note that the measure $\nu$ is finite, which is a contradiction. Then, for any $x_j \neq 0$ where $j \in \Lambda$, we get $\nu_j = \nu(x_j) = 0$. Now we suppose that $0 \notin \{x_j : j \in \Lambda\}$. In fact, assume $\varepsilon > 0$ small enough such that for any $0 \notin B_\varepsilon(0)$. Let $\varphi_\varepsilon \in \mathcal{C}_0^\infty(\mathbb{R}^3)$ be a cut-off function centered at 0 satisfying

$$0 \leq \varphi_\varepsilon \leq 1,\ \ \varphi_\varepsilon(x) = \begin{cases} 1 \text{ if } |x| \leq \frac{\varepsilon}{2}, \\ 0 \text{ if } |x| \geq \varepsilon. \end{cases}$$

Since $(\varphi_\varepsilon u_n)$ is bounded, $\langle I'_{\lambda,G}(u_n), \varphi_\varepsilon u_n\rangle \to 0$, that is

$$\begin{aligned} &\langle(-\Delta)^{\frac{s}{2}}(u_n), \varphi_\varepsilon (-\Delta)^{\frac{s}{2}}(u_n)\rangle + \langle(-\Delta)^{\frac{s}{2}}(u_n), u_n (-\Delta)^{\frac{s}{2}}(\varphi_\varepsilon)\rangle + \int_{\mathbb{R}^3} u_n^2 \varphi_\varepsilon dx \\ &+ \int_{\mathbb{R}^3} \phi_{u_n}^t u_n^2 \varphi_\varepsilon dx = \lambda \int_{\mathbb{R}^3} a(x)|u_n|^r \varphi_\varepsilon dx + \int_{\mathbb{R}^3} b(x)|u_n|^{2_s^*} \varphi_\varepsilon dx + o(1) \end{aligned}$$

$$\lim_{n\to+\infty} \langle(-\Delta)^{\frac{s}{2}}(u_n), \varphi_\varepsilon (-\Delta)^{\frac{s}{2}}(u_n)\rangle = \int_{\mathbb{R}^3} \varphi_\varepsilon d\mu \tag{24}$$

$$\lim_{n\to+\infty} \int_{\mathbb{R}^3} b(x)|u_n|^{2_s^*} \varphi_\varepsilon dx = \int_{\mathbb{R}^3} b(x)\varphi_\varepsilon d\nu = \int_{\mathbb{R}^3} b(x)|u|^{2_s^*} \varphi_\varepsilon dx + b(x_j)\nu_j \tag{25}$$

$$\begin{aligned} &\lim_{\varepsilon\to 0}\lim_{n\to 0} \left|\langle(-\Delta)^{\frac{s}{2}}(u_n), u_n (-\Delta)^{\frac{s}{2}}(\varphi_\varepsilon)\rangle\right| \\ &\leq \lim_{\varepsilon\to 0}\lim_{n\to 0} \left(\left(\int_{\mathbb{R}^3} |(-\Delta)^{\frac{s}{2}} u_n|^2 dx\right)^{1/2} \times \left(\int_{\mathbb{R}^3} |u_n|^2 |(-\Delta)^{\frac{s}{2}} \varphi_\varepsilon|^2 dx\right)^{1/2}\right) \\ &\leq C \lim_{\varepsilon\to 0} \left(\int_{\mathbb{R}^3} |u|^2 |(-\Delta)^{\frac{s}{2}} \varphi_\varepsilon|^2 dx\right)^{1/2} \\ &\leq C \lim_{\varepsilon\to 0} \left(\int_{B_\varepsilon(0)} |u|^{2_s^*}|dx\right)^{1/2_s^*} \left(\int_{B_\varepsilon(0)} (-\Delta)^{\frac{s}{2}} \varphi_\varepsilon|^{\frac{3}{s}} dx\right)^{\frac{s}{3}} \\ &\leq C \lim_{\varepsilon\to 0} \left(\int_{B_\varepsilon(0)} |u|^{2_s^*}|dx\right)^{1/2_s^*} = 0, \end{aligned} \tag{26}$$

and

$$\begin{aligned}
&\lim_{\varepsilon\to 0}\int_{\mathbb{R}^3} |\,(-\Delta)^{\frac{s}{2}}\, u|^2\varphi_\varepsilon dx=0\\
&\lim_{\varepsilon\to 0}\int_{\mathbb{R}^3} b(x)|u|^{2_s^*}\varphi_\varepsilon dx=0,\\
&\lim_{\varepsilon\to 0}\int_{\mathbb{R}^3} a(x)|u|^{r}\varphi_\varepsilon dx=0,\ \ \lim_{\varepsilon\to 0}\int_{\mathbb{R}^3} |u|^{2}\varphi_\varepsilon dx=0,\\
&\lim_{\varepsilon\to 0}\int_{\mathbb{R}^3} \phi_u^t u^2\varphi_\varepsilon dx=0.
\end{aligned}\tag{27}$$

Thus,

$$\mu(\{0\})=b(0)\nu(\{0\}).$$

Note that $b(0)=0$, then $\mu(\{0\})=0$. In the next step, we claim that the concentration of $\nu$ cannot occur at infinity.

$$\nu_\infty=\lim_{R\to+\infty}\limsup_{n\to+\infty}\int_{|x|>R}|u_n|^{2_s^*}dx,$$

$$\mu_\infty=\limsup_{n\to+\infty}\int_{x|>R}|\,(-\Delta)^{\frac{s}{2}}\, u_n|^2dx.$$

Hence, by using the concept of the concentration-compactness in ([17],[18]) at infinity, $\nu_\infty$ and $\mu_\infty$ exist and satisfy :

$$\limsup_{n\to+\infty}\int_{\mathbb{R}^3}|u_n|^{2_s^*}dx=\int_{\mathbb{R}^3}d\nu+\nu_\infty$$

$$\limsup_{n\to+\infty}\int_{\mathbb{R}^3}|\,(-\Delta)^{\frac{s}{2}}\, u_n|^2dx=\int_{\mathbb{R}^3}d\mu+\mu_\infty.$$

$$\mathbb{S}\nu_\infty^{2/2_s^*}\le\mu_\infty.\tag{28}$$

For any $R>0$, take a radially symmetric function $\chi_R\in\mathcal{C}^\infty(\mathbb{R}^3)$ such that $0\le\chi_R\le 1$, $\chi_R=1$ in $\mathbb{R}^3\backslash B_{2R}$, $\chi_R=0$ in $B_R$. It is easy to obtain that $\chi_R u_n$ is bounded on $H_G^s(\mathbb{R}^3)$. Then

$$\lim_{n\to+\infty}\langle I'_{\lambda,G}(u_n),\chi_R u_n\rangle=0.$$

We have

$$\begin{aligned}
&\langle(-\Delta)^{\frac{s}{2}}\,(u_n),\chi_R\,(-\Delta)^{\frac{s}{2}}\,(u_n)\rangle+\langle(-\Delta)^{\frac{s}{2}}\,(u_n),u_n\,(-\Delta)^{\frac{s}{2}}\,(\chi_R)\rangle+\int_{\mathbb{R}^3}u_n^2\chi_R dx\\
&+\int_{\mathbb{R}^3}\phi_{u_n}^t u_n^2\chi_R dx=\lambda\int_{\mathbb{R}^3}a(x)|u_n|^r\chi_R dx+\int_{\mathbb{R}^3}b(x)|u_n|^{2_s^*}\chi_R dx+o(1)
\end{aligned}$$

Similar to the proof of (26), we have

$$\lim_{R\to+\infty}\limsup_{n\to+\infty}\langle(-\Delta)^{\frac{s}{2}}(u_n),u_n\,(-\Delta)^{\frac{s}{2}}(\chi_R)\rangle \le C\lim_{R\to+\infty}\left(\int_{R<|x|<2R}|u|^{2_s^*}dx\right)^{1/2_s^*}=0.$$

Also,

$$\lim_{R\to+\infty}\limsup_{n\to+\infty}\int_{\mathbb{R}^3}a(x)|u_n|^r\chi_R dx=\lim_{R\to+\infty}\int_{\mathbb{R}^3}a(x)|u|^r\chi_R dx=0,$$

$$\lim_{R\to+\infty}\limsup_{n\to+\infty}\int_{\mathbb{R}^3}u_n^2\chi_R dx=\lim_{R\to+\infty}\int_{\mathbb{R}^3}u^2\chi_R dx=0,$$

$$\lim_{R\to+\infty}\limsup_{n\to+\infty}\int_{\mathbb{R}^3}\phi_{u_n}^t u_n^2\chi_R dx=\lim_{R\to+\infty}\int_{|x|>R}\phi_u^t u^2\chi_R dx=0.$$

Since $b(\infty)=0$,

$$\lim_{R\to+\infty}\limsup_{n\to+\infty}\int_{|x|>R}b(x)|u_n|^{2_s^*}dx=0.$$

Then,

$$\mu_\infty=\lim_{R\to+\infty}\limsup_{n\to+\infty}\int_{x|>R}|\,(-\Delta)^{\frac{s}{2}}\,u_n|^2dx\le\lim_{R\to+\infty}\limsup_{n\to+\infty}\int_{|x|>R}b(x)|u_n|^{2_s^*}dx=0.$$

Thus $\mu_\infty=0$. Then, from (28) we obtain $\nu_\infty=0$. Hence, up to a subsequence, we derive

$$\lim_{n\to+\infty}\int_{\mathbb{R}^3}|u_n|^{2_s^*}dx=\int_{\mathbb{R}^3}|u|^{2_s^*}dx.$$

By Brézis-Leib [4] $u_n\to u$ in $L_G^{2_s^*}(\mathbb{R}^3)$. Note that $b\in L_G^\infty(\mathbb{R}^3)$ we obtain

$$\lim_{n\to+\infty}\int_{\mathbb{R}^3}b(x)|u_n-u|^{2_s^*}dx=0.$$

Then $u_n\to u$ strongly in $H_G^s(\mathbb{R}^3)$. □

Since $H_G^s(\mathbb{R}^3)$ is separable (see [1]), there exist $\{e_n\}_{n\in\mathbb{N}}\subset H_G^s(\mathbb{R}^3)$ and $\{f_n\}_{n\in\mathbb{N}}\subset H_G^{-s}(\mathbb{R}^3)$ with

$$H_G^s(\mathbb{R}^3)=\overline{\operatorname{span}\{e_n\}_{n=1}^\infty},\;\; H_G^{-s}(\mathbb{R}^3)=\overline{\operatorname{span}\{f_n\}_{n=1}^\infty}$$

$$\langle f_i,e_j\rangle=\begin{cases}1 \text{ if } i=j,\\ 0 \text{ if } i\ne j,\end{cases}$$

where $\langle,\rangle$ is the duality pairing between $H_G^{-s}(\mathbb{R}^3)$ and $H_G^s(\mathbb{R}^3)$.

$$\text{Let } X_j=\operatorname{span}\{e_j\},\;\; Y_n=\bigoplus_{j=0}^{n}X_j,\;\; Z_n=\overline{\bigoplus_{j=n}^{\infty}X_j}.$$

Let

**Lemma 4.2.** *([22] Fountain theorem)*
*Consider an even functional* $I_{\lambda,G} \in \mathcal{C}(H^s_G(\mathbb{R}^3),\mathbb{R})$. *If, for every* $k \in \mathbb{N}$, *there exist* $\rho_k > r_k > 0$ *such that*

1. $\alpha_k := \max\left\{I_{\lambda,G}(u) : u \in Y_k, \|u\|_{H^s_G} = \rho_k\right\} \leq 0.$
2. $\beta_k := \inf\left\{I_{\lambda,G}(u) : u \in Z_k, \|u\|_{H^s_G} = \rho_k\right\} \to \infty$ *as* $k \to +\infty$.
3. $I_{\lambda,G}$ *satisfying (PS) condition for every* $c > 0$.

*Then* $I_{\lambda,G}$ *has an unbounded sequence of critical values.*

**Proof of Theorem 1.2.**
The functional $I_{\lambda,G}$ is even, $I_{\lambda,G} \in \mathcal{C}(H^s_G(\mathbb{R}^3),\mathbb{R})$. By Lemma 4.1 $I_{\lambda,G}$ satisfying $(PS)$ condition for any $c \in \mathbb{R}$. We only need to verify $I_{\lambda,G}$ satisfying (1) and (2) of Lemma 4.2. Since $X_j$ is a finite-dimensional subspace of $H^s_G(\mathbb{R}^3)$ for each $j \in \mathbb{N}$ and $b(x) > 0$ a.e. in $\mathbb{R}^3$, this implies that there exists a constant $\varepsilon_j > 0$ such that for all $v \in X_j$ with $\|v\|_{H^s_G} = 1$ we have

$$\int_{\mathbb{R}^3} b(x)|v|^{2^*_s}dx \geq \varepsilon_j.$$

On the other hand,
for any $u \in X_j\backslash\{0\}$, with $\|u\|_{H^s_G} = 1$ and by using the Lemma Sobolev inequality we get

$$\begin{aligned} I_{\lambda,G}(tu) &\leq \frac{t^2}{2}\|u\|^2_{H^s_G} + C\frac{t^4}{2}\|u\|^4_{H^s_G} - \frac{\lambda t^r}{r}\int_{\mathbb{R}^3} a(x)|u|^r dx - \frac{t^{2^*_s}}{2^*_s}\int_{\mathbb{R}^3} b(x)|u|^{2^*_s}dx \\ &\leq \frac{t^2}{2} + C\frac{t^4}{2} - \frac{t^{2^*_s}}{2^*_s}\varepsilon_j. \end{aligned}$$

Since $4 < 2^*_s$, there exists $t_j > 1$ such that $e_j = t_j u$ satisfies $I_{\lambda,G}(e_j) \leq 0$. This proves (1) of Lemma 4.2.
Define

$$\beta_j = \sup_{u\in Z_j, \|u\|_{H^s_G}=1}\left(\int_{\mathbb{R}^3} b(x)|u|^{2^*_s}dx\right)^{1/2^*_s}.$$

By the definition of $Z_j$, we get $u_j \rightharpoonup 0$ in $H^s_G(\mathbb{R}^3)$. Since $b(x)$ is continuous, $b(0) = 0$, $b(\infty) = 0$ and by the same argument using in Lemma 4.1 we see that a concentration of the measure $\nu$ can only occur at 0 and $\infty$. We deduce that

$$\int_{\mathbb{R}^3} b(x)|u_j|^{2^*_s}dx \to 0,$$

as $j \to \infty$, so

$$\beta_j \to 0.$$

For all $u \in Z_j$, we have

$$I_{\lambda,G}(u) \geq \frac{1}{2}\|u\|^2_{H^s_G} - \frac{\lambda C}{r}\|u\|^r_{H^s_G} - \frac{\beta_j^{2^*_s}}{2^*_s}\|u\|^{2^*_s}_{H^s_G}.$$

Let $u \in Z_j$, such that $\|u\|_{H^s_G} = A_j = \left(\frac{1}{\beta_j^{2^*_s}}\right)^{\frac{1}{2^*_s-2}}$ Since $\beta_j \to 0$ we have $A_j \to +\infty$ as $j \to +\infty$. Since $1 < r < 2$ we have

$$I_{\lambda,G}(u) \geq \left(\frac{1}{2} - \frac{1}{2^*_s}\right)A_j^2 - \frac{\lambda C}{r}A_j^r \to +\infty, \text{ as } j \to +\infty.$$

So, $I_{\lambda,G}$ satisfies (2). All the assumptions of Lemma 4.2 are satisfied. Therefore, this concludes the proof of Theorem 1.2. □

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