ISSN 1733-6775

Journal of Mathematics and Applications

vol. 32 (2010)

JMA

Department of Mathematics
Rzeszów University of Technology
Rzeszów, Poland

Journal of Mathematics and Applications

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vol. 32 (2010)

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http://www.jma.prz.rzeszow.pl

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Text pepared to print in LATEX

p-ISSN 1733-6775

Publishing House of the Rzeszów University of Technology

Printed in March 2010
(22/10)

Journal of Mathematic and Applications
vol. 32 (2010)
Contents

1. **W. Aziz, H. Leiva, N. Merentes, J.L. Sánchez:** *Functions of two variables with bounded φ – variation in the sense of Riesz*
2. **Urszula Bednarz, Janusz Sokół***: On T-neighborhoods of analytic functions*
3. **K.K. Dixit, S. Porwal:** *Convolution of the Subclass of Salagean-type Harmonic Univalent Functions with Negative Coefficients*
4. **J. Dziok, J. Ohriska:** *Applications of a first order differential subordination for analytic functions*
5. **Jarosław Kapeluszny:** *Some remarks on a fixed point property for multivalued mapping*
6. **Sh. Khosravianarab, S.R. Kulkarni, R.K. Raina:** *On a certain class of analytic functions involving hypergeometric functions*
7. **Samit Kumar Majumder, Sujit Kumar Sardar:** *On cartesian product of fuzzy prime and fuzzy semiprime ideals of semigroups*
8. **S. Mokhtari, A. Kordi, A. Moussavi, A. Ahmadi:** *On LI-ideals of lattice implication algebras*
9. **A. Provotar, L. Katerynych:** *Homeopath: Diagnostic information system*
10. **Roxana Şendruţiu:** *On certain functions with positive real part*
11. **D. Tsedenbayar, D. Batnasan, L. Hadhuu:** *An interpolation of polynomials with application to some Volterra operator pencils*
12. **Katarzyna Wilczek:** *The harmonic center of a trilateral and the Apollonius point of a triangle*
13. **A. Yousefian Darani:** *Notes on the ideal-based zero-divisior graph*
14. **Paweł Zaprawa:** *On coefficients' estimates of concave univalent functions*

Journal of
Mathematics
and Applications

No 32, pp 5-23 (2010)

Functions of two variables with bounded $\varphi-$variation in the sense of Riesz

W. Aziz, H. Leiva, N. Merentes, J. L. Sánchez

Submitted by: *Józef Banaś*

ABSTRACT: In this paper we introduce the concept of bounded φ-variation function, in the sense of Riesz, defined in a rectangle $I_a^b[a,b] \times [a,b] \subset \mathbb{R}^2$. We prove that the linear space $BV_\varphi^R(I_a^b)$ generated by the class $V_\varphi^R(I_a^b)$ of all φ-bounded variation functions is a Banach algebra. Moreover, we give necessary and sufficient conditions for the Nemytskii operator acting in the space $BV_\varphi^R(I_a^b)$ to be globally Lipschitz.

AMS Subject Classification: *26B30, 26B35*
Key Words and Phrases: *Bounded variation in the sense of Riesz, variation space, Banach algebra*

1. Introduction

In 1881, C. Jordan in [9], introduced the notion of bounded variation function such as it is known today. With the years this concept was generalized in several ways, depending on its usefulness in the context of some theories. In 1910, F. Riesz in [16], defined the concept of $p-$ bounded variation function, with $1 < p < \infty$, to show that the dual space of $L_p[a,b]$ is $L_q[a,b]$ $(\frac{1}{p} + \frac{1}{q} = 1)$. Moreover, he proved that these functions are absolutely continuous with derivatives in the space $L_p[a,b]$ (Riesz lemma).

In 1937 L. C. Young [19] considered the set Φ of all nondecreasing and continues functions $\varphi : [0,+\infty) \longrightarrow [0,+\infty)$ with $\varphi(0) = 0$ and $\varphi(t) \longrightarrow +\infty$ if $t \longrightarrow +\infty$ and generalized the work of Wiener [18].

In 1953 Yu. Medved'ev (see [13]), generalized the concept of bounded variation in the Riesz sense to a class of φ-bounded variation functions.

Subsequently, V.V. Chistyakov reconsidered in [4] the works of Vitali (1904) and Hardy (1905) presenting the total bounded variation in a rectangle I_a^b of $\mathbb{R}^2$. Also, he proved that the class of total bounded variation functions $BV(I_a^b;\mathbb{R})$ is a Banach algebra endowed with the norm $\|f\| = |f(a)| + TV(f; I_a^b)$, for $f \in BV(I_a^b;\mathbb{R})$ and $\|f \cdot g\| \leq 4\|f\| \cdot \|g\|$ with $f(a) = f(a_1, a_2)$ and $TV(I_a^b;\mathbb{R}) = V_{[a_1,b_1]}(f) + V_{[a_2,b_2]}(f) + V_{I_a^b}(f)$. Furthermore, he characterized the composition operator (Nemytskii) on these spaces satisfying the global Lipschitz condition.

In this paper we introduce the concept of bounded φ-variation function in the sense of Riesz, defined on the rectangle $I_a^b = [a, b] \times [a, b] \subset \mathbb{R}^2$ and we prove that the linear space $BV_\varphi^R(I_a^b)$ generated by the class $V_\varphi^R(I_a^b)$ of all φ-bounded variation functions is a Banach algebra. Moreover, we give necessary and sufficient conditions for the Nemytskii operator acting in the space $BV_\varphi^R(I_a^b)$ to be globally Lipschitz.

2. φ–total bounded variation in the sense of Riesz

In this section we introduce the concept of φ–total bounded variation in the sense of Riesz, and we prove that the class of such functions is a linear space.
Following the definition of φ–bounded variation in the sense of Riesz given in [13] and the generalization the total bounded variation in the Hardy spaces given in [4], we introduce the notion of φ–bounded variation in the sense of Riesz for functions f defined on the rectangle $I_a^b \subset \mathbb{R}^2$.

Let us introduce the following notation: $\Delta s_j = s_j - s_{j-1}$, $\Delta t_i = t_i - t_{i-1}$ and

$$\Delta_{10} f(t_i, s_j) = f(t_i, s_j) - f(t_{i-1}, s_j),$$

$$\Delta_{01} f(t_i, s_j) = f(t_i, s_j) - f(t_i, s_{j-1}),$$

$$\Delta_{11} f(t_i, s_j) = f(t_{i-1}, s_{j-1}) + f(t_i, s_j) - f(t_{i-1}, s_j) - f(t_i, s_{j-1}).$$

Assume that φ is a fixed function in the class Φ (see Introduction).

Definition 2.1. The φ–total bounded variation in the sense of Riesz is defined as follows:

(a) Let $x_2 \in [a_2, b_2]$. Consider the function $f(\cdot, x_2) : [a_1, b_1] \times \{x_2\} \longrightarrow \mathbb{R}$. The φ–variation in the sense of Riesz of the function $f(\cdot, x_2)$ of one variable defined by $f(\cdot, x_2)(t) = f(t, x_2)$, $t \in [a_1, b_1]$, on the interval $[x_1, y_1]$, is the quantity

$$V^R_{\varphi,[x_1,\,y_1]}(f(\cdot, x_2)) := \sup_{\Pi_1} \sum_{i=1}^{m} \varphi\left[\frac{|\Delta_{10} f(t_i, x_2)|}{|\Delta t_i|}\right] |\Delta t_i|, \tag{1}$$

where the supremum is taken over all partitions $\Pi_1 = \{t_i\}_{i=0}^m$ ($m \in \mathbb{N}$) of the interval $[x_1, y_1]$.

(b) A similar applies to the variation $V_{\varphi,[x_2,y_2]}$ if $x_1 \in [a_1, b_1]$ is fixed and $[x_2, y_2]$ is a subinterval of $[a_2, b_2]$. That is, for the function $f(x_1, \cdot) : \{x_1\} \times [a_2, b_2] \longrightarrow \mathbb{R}$ we define φ–variation in the sense Riesz, as the quantity

$$V^R_{\varphi,[x_2,\,y_2]}(f(x_1, \cdot)) := \sup_{\Pi_2} \sum_{j=1}^{n} \varphi\left[\frac{|\Delta_{01} f(x_1, s_j)|}{|\Delta s_j|}\right] |\Delta s_j|, \tag{2}$$

where the supremum is taken over the set of all partitions $\Pi_2 = \{s_j\}_{j=0}^n$ $(n \in \mathbb{N})$ of the interval $[x_2, y_2]$.

(c) The φ–bidimensional variation in the sense of Riesz is defined by the formula

$$V_\varphi^R(f) := \sup_{\Pi_1, \Pi_2} \sum_{i=1}^m \sum_{j=1}^n \varphi \left[\frac{|\Delta_{11} f(t_i, s_j)|}{|\Delta t_i||\Delta s_j|} \right] \cdot |\Delta t_i||\Delta s_j|, \tag{3}$$

where the supremum is taken over the set of all partitions (Π_1, Π_2) of the rectangle $I_a^b \subset \mathbb{R}^2$.

(d) The φ–total bounded variation in the sense of Riesz of the function $f : I_a^b \longrightarrow \mathbb{R}$ is denoted by $TV_\varphi^R(f)$ and is defined as follows:

$$TV_\varphi^R(f) = TV_\varphi^R(f, I_a^b) := V_{\varphi,[a_1,b_1]}^R(f(\cdot, a_2)) + V_{\varphi,[a_2,b_2]}^R(f(a_1, \cdot)) + V_\varphi^R(f), \tag{4}$$

provided $TV_\varphi^R(f) < \infty$.

The class of all the functions $f : I_a^b \longrightarrow \mathbb{R}$ having φ–total bounded variation in the sense of Riesz is denoted by $V_\varphi^R(I_a^b)$. Other words, we have:

$$V_\varphi^R(I_a^b) = V_\varphi^R(I_a^b, \mathbb{R}) := \{f : I_a^b \longrightarrow \mathbb{R} : \quad TV_\varphi^R(f) < \infty\}. \tag{5}$$

Example 1. Let $f : I_a^b \longrightarrow \mathbb{R}$ be defined by the formula $f(x_1, x_2) = (ax_1 + bx_2)^2$, where $a, b \in \mathbb{R}$. Then, it is easily seen that $f \in V_\varphi^R(I_a^b)$.

Now, we give the definition allowing us to characterize φ-functions.

Definition 2.2. Let $\varphi \in \Phi$. If $\lim_{t\to\infty} \sup \dfrac{\varphi(t)}{t} = \infty$, then we say that φ satisfies the condition ∞_1.

Theorem 2.3. *Assume that $\varphi \in \Phi$ and $f : I_a^b \longrightarrow \mathbb{R}$. Then:*

(a) $TV_\varphi^R(f) \geq 0$ for all functions $f \in V_\varphi^R(I_a^b)$.

(b) The function $TV_\varphi^R(\cdot) : V_\varphi^R(I_a^b) \longrightarrow \mathbb{R}$ is even, that is $TV_\varphi^R(f) = TV_\varphi^R(-f)$.

(c) If $f \in V_\varphi^R(I_a^b)$, then f is bounded in I_a^b.

(d) $TV_\varphi^R(f) = 0$ if and only if $f = const$.

(e) φ is convex if and only if $TV_\varphi^R(\cdot)$ is convex.

(f) $V_\varphi^R(I_a^b) \subset BV(I_a^b)$.

(g) If $\overline{\lim_{t\to\infty}} \dfrac{\varphi(t)}{t}$ is finite then $V_\varphi^R(I_a^b) = BV(I_a^b)$.

Proof. In order to verify part (a), it is sufficient to use the definition of bounded variation. To prove part (b) we used the properties of the function φ and the fact that the absolute value function $|\cdot|$ is even.
(c) It can be done by contradiction.
(d) The first implication can be easily verified by contradiction, while the converse one is trivial.
(e) Suppose that φ is convex and $f, g : I_a^b \longrightarrow \mathbb{R}$, $\alpha, \beta \in [0, 1]$ are such that $\alpha+\beta = 1$. Given the partitions $\Pi_1 : a_1 = t_0 < \cdots < t_m = b_1$ and $\Pi_2 : a_2 = s_0 < \cdots < s_n = b_2$ of the intervals $[a_1, b_1]$ and $[a_2, b_2]$, respectively. Then we have:

$$\begin{aligned}
&\alpha TV_\varphi^R(f) + \beta TV_\varphi^R(g)\\
&= \alpha V^R_{\varphi,[a_1,b_1]}(f) + \alpha V^R_{\varphi,[a_2,b_2]}(f) + \alpha V_\varphi^R(f) + \beta V^R_{\varphi,[a_1,b_1]}(g)\\
&\quad + \beta V^R_{\varphi,[a_2,b_2]}(g) + \beta V_\varphi^R(g)\\
&= \sup_{\Pi_1} \sum_{i=1}^{m} \left[\alpha\varphi\left[\frac{|\Delta_{10} f(t_i, x_2)|}{|\Delta t_i|}\right] + \beta\varphi\left[\frac{|\Delta_{10} g(t_i, x_2)|}{|\Delta t_i|}\right]\right] |\Delta t_i|\\
&\quad + \sup_{\Pi_2} \sum_{j=1}^{n} \left[\alpha\varphi\left[\frac{|\Delta_{01} f(x_1, s_j)|}{|\Delta s_j|}\right] + \beta\varphi\left[\frac{|\Delta_{01} g(x_1, s_j)|}{|\Delta s_j|}\right]\right] |\Delta s_j|\\
&\quad + \sup_{\Pi_1,\Pi_2} \sum_{i=1}^{m}\sum_{j=1}^{n} \left[\alpha\varphi\left[\frac{|\Delta_{11} f(t_i, s_j)|}{|\Delta t_i||\Delta s_j|}\right] + \beta\varphi\left[\frac{|\Delta_{11} g(t_i, s_j)|}{|\Delta t_i||\Delta s_j|}\right]\right] |\Delta t_i||\Delta s_j|.
\end{aligned}$$

Hence, taking into account that φ is convex and nondecreasing, we get

$$\begin{aligned}
&\alpha TV_\varphi^R(f) + \beta TV_\varphi^R(g)\\
&\geq \sup_{\Pi_1} \sum_{i=1}^{m} \varphi\left[\frac{|\Delta_{10}(\alpha f+\beta g)(t_i, x_2)|}{|\Delta t_i|}\right] |\Delta t_i|\\
&\quad + \sup_{\Pi_2} \sum_{j=1}^{n} \varphi\left[\frac{|\Delta_{01}(\alpha f+\beta g)(x_1, s_j)|}{|\Delta s_j|}\right] |\Delta s_j|\\
&\quad + \sup_{\Pi_1,\Pi_2} \sum_{i=1}^{m}\sum_{j=1}^{n} \varphi\left[\frac{|\Delta_{11}(\alpha f+\beta g)(t_i, s_j)|}{|\Delta t_i||\Delta s_j|}\right] \cdot |\Delta t_i||\Delta s_j|\\
&= V^R_{\varphi,[a_1,b_1]}(\alpha f+\beta g) + V^R_{\varphi,[a_2,b_2]}(\alpha f+\beta g) + V_\varphi^R(\alpha f+\beta g)\\
&= TV_\varphi^R(\alpha f+\beta g).
\end{aligned}$$

Therefore,

$$TV_\varphi^R(\alpha f+\beta g) \leq \alpha TV_\varphi^R(f) + \beta TV_\varphi^R(g).$$

Now, suppose that $TV_\varphi^R(\cdot)$ is convex and let $x, y \in [0, +\infty)$. Further, let $f, g : I_a^b \longrightarrow \mathbb{R}$ be functions defined by the formulas:

$$f(t, s) = x \cdot (t - s) \text{ and } g(t, s) = y \cdot (t - s), \quad t \in [a_1, b_1], \ s \in [a_2, b_2].$$

Take $\alpha, \beta \in [0, 1]$ such that $\alpha + \beta = 1$. Then we obtain:

$$\begin{aligned} V^R_{\varphi,[a_1,\, b_1]}(\alpha f + \beta g) &= \sup_{\Pi_1} \sum_{i=1}^{m} \varphi \left[\frac{|\Delta_{10}(\alpha f + \beta g)(t_i, x_2)|}{|\Delta t_i|} \right] \cdot |\Delta t_i| \\ &= \sup_{\Pi_1} \sum_{i=1}^{m} \varphi \left[\frac{|(\alpha x + \beta y)|\Delta t_i||}{|\Delta t_i|} \right] \cdot |\Delta t_i| \\ &= \varphi(\alpha x + \beta y) \cdot |b_1 - a_1|. \end{aligned}$$

Hence we get

$$V^R_{\varphi,[a_1,\, b_1]}(\alpha f + \beta g) = \varphi(\alpha x + \beta y) \cdot |b_1 - a_1|.$$

In a similar way, we obtain

$$V^R_{\varphi,[a_2,\, b_2]}(\alpha f + \beta g) = \varphi(\alpha x + \beta y) \cdot |b_2 - a_2|.$$

Next, we have the following equality:

$$V^R_{\varphi,[a_1,\, b_1]}(\alpha f + \beta g) = \sup_{\Pi_1, \Pi_2} \sum_{i=1}^{m} \sum_{j=1}^{n} \varphi \left[\frac{|\Delta_{11}(\alpha f + \beta g)(t_i, s_j))|}{|\Delta t_i||\Delta s_j|} \right] |\Delta t_i||\Delta s_j| = 0.$$

In addition, we obtain

$$V^R_{\varphi,[a_1,\, b_1]}(f) = \sup_{\Pi_1} \sum_{i=1}^{m} \varphi \left[\frac{|\Delta_{10} f(t_i, x_2)|}{\Delta t_i} \right] \cdot |\Delta t_i| = \varphi(x)|b_1 - a_1|.$$

Further observe that, $V^R_{\varphi,[a_2,\, b_2]}(f) = \varphi(y)|b_2 - a_2|$ and $V^R_\varphi(f) = 0$. Similarly, $V^R_{\varphi,[a_1,\, b_1]}(g) = \varphi(x)|b_1 - a_1|$, $V^R_{\varphi,[a_2,\, b_2]}(g) = \varphi(y)|b_2 - a_2|$ and $V^R_\varphi(g) = 0$. Taking into account the convexity of TV^R_φ, we obtain:

$$\begin{aligned} \varphi(\alpha x + \beta y) &\Big[|b_1 - a_1| + |b_2 - a_2| \Big] TV^R_\varphi(\alpha f + \beta g) \\ \leq & \Big(\alpha\varphi(x) + \beta\varphi(y) \Big) \cdot \Big[|b_1 - a_1| + |b_2 - a_2| \Big]. \end{aligned}$$

Since $b_i - a_i \neq 0$; for $i = 1, 2$ we have

$$\varphi(\alpha x + \beta y) \leq \alpha\varphi(x) + \beta\varphi(y) \quad \alpha, \beta \in [0, 1], \ \alpha + \beta = 1.$$

Therefore, $\varphi(\cdot)$ is convex.

(f) Consider $f \in V_\varphi^R(I_a^b)$ and take the partitions $\Pi_1 : a_1 = t_0 < ... < t_m = b_1$, $\Pi_2 : a_2 = s_0 < ... < s_n = b_2$ of the intervals $[a_1, b_1]$ and $[a_2, b_2]$, respectively. Let us put:

$$\begin{aligned} \sigma_1 &:= \left\{ i : \left(\frac{|\Delta_{10} f(t_i, x_2)|}{|\Delta t_i|} \right) \leq 1 \right\}, \\ \sigma_2 &:= \left\{ j : \left(\frac{|\Delta_{01} f(x_1, s_j)|}{|\Delta s_j|} \right) \leq 1 \right\}, \\ \sigma_3 &:= \left\{ (i,j) : \left(\frac{|\Delta_{11} f(t_i, s_j)|}{|\Delta s_j||\Delta t_i|} \right) \leq 1 \right\}. \end{aligned}$$

Then, we get the following estimate

$$\begin{aligned} & \sum_{i=1}^m |\Delta_{10} f(t_i, x_2)| \sum_{i=1}^m \left(\frac{|\Delta_{10} f(t_i, x_2)|}{|\Delta t_i|} \right) |\Delta t_i| \\ & \leq |b_1 - a_1| + \frac{1}{\varphi(1)} \sum_{i=1}^m \varphi \left[\frac{|\Delta_{10} f(t_i, x_2)|}{|\Delta t_i|} \right] |\Delta t_i|. \end{aligned}$$

Hence we get

$$V_{[a_1, b_1]}(f) \leq |b_1 - a_1| + \frac{1}{\varphi(1)} V^R_{\varphi, [a_1, b_1]}(f) < \infty.$$

This allows us to deduce that $V^R_{\varphi,[a_1, b_1]}(f)$ is finite. Proceeding in a similar way we obtain that $V^R_{\varphi,[a_2, b_2]}(f)$ is also finite. So, we only have to verify that $V_\varphi^R(f, I_a^b)$ is finite to conclude that $V_\varphi^R(I_a^b) \subset BV(I_a^b)$. In fact, we have:

$$\begin{aligned} & \sum_{i=1}^m \sum_{j=1}^n |\Delta_{11} f(t_i, s_j)| \\ & \leq \sum_{i,j \in \sigma_3} |\Delta t_i||\Delta s_j| + \sum_{i,j \notin \sigma_3} \varphi \left(\frac{|\Delta_{11} f(t_i, s_j)|}{|\Delta t_i||\Delta s_j|} \right) |\Delta t_i||\Delta s_j| \\ V_{I_a^b}(f) & \leq A(I_a^b) + \frac{1}{\varphi(1)} V_\varphi^R(f), \end{aligned}$$

where $A(I_a^b)$ is the area of the rectangle I_a^b. Hence, we infer that

$$\begin{aligned} TV(f) &= V_{[a_1, b_1]}(f) + V_{[a_2, b_2]}(f) + V_{I_a^b}(f) \\ &\leq |b_1 - a_1| + \frac{1}{\varphi(1)} V^R_{\varphi,[a_1, b_1]}(f) + |b_2 - a_2| + \frac{1}{\varphi(1)} V^R_{\varphi,[a_2, b_2]}(f) \\ &\quad + A(I_a^b) + \frac{1}{\varphi(1)} V_\varphi^R(f, I_a^b) < \infty. \end{aligned}$$

Thus, $V_\varphi^R(I_a^b) \subset BV(I_a^b)$.

(g) Suppose that

$$0 < \lim_{t\to\infty} \sup \frac{\varphi(t)}{t} = r < \infty.$$

Then, for a fixed $\varepsilon > 0$ we can find t_0 such that

$$\sup_{t>T} \frac{\varphi(t)}{t} - r < \varepsilon \qquad \text{for} \quad T > t_0.$$

Consequently, we obtain:

$$\sup_{t>T} \frac{\varphi(t)}{t} < \varepsilon + r \qquad \text{for} \quad T > t_0$$

or equivalently

$$\varphi(t) < (\varepsilon + r)t \qquad \text{for} \quad t > t_0.$$

Other words, there are $t_0 > 0$ and $k > 0$ such that

$$\varphi(t) < kt \qquad \text{for} \quad t > t_0. \tag{6}$$

Now, take $f \in V_\varphi^R(I_a^b)$ and let Π_1, Π_2 be partitions of $[a_1, b_1]$ and $[a_2, a_2]$, respectively. Consider the following sets:

$$\begin{aligned}
C_{t_o} : &= \{i : \left(\frac{|\Delta_{10} f(t_i, x_2)|}{|\Delta t_i|}\right) \geq t_o\}, \\
C'_{t_o} : &= \{j : \left(\frac{|\Delta_{01} f(x_1, s_j)|}{|\Delta s_j|}\right) \geq t_o\}, \\
C''_{t_o} : &= \{(i,j) : \left(\frac{|\Delta_{11} f(t_i, s_j)|}{|\Delta s_j||\Delta t_i|}\right) \geq t_o\}.
\end{aligned}$$

Then, we get

$$\sum_{i=1}^{m} \varphi\left[\frac{|\Delta_{10} f(t_i, x_2)|}{|\Delta t_i|}\right] |\Delta t_i| \leq kV_{[a_1, b_1]}(f) + \varphi(t_o)(b_1 - a_1).$$

Therefore, $V_{\varphi,[a_1, b_1]}^R(f) \leq kV_{[a_1, b_1]}(f) + \varphi(t_o)(b_1 - a_1) < \infty$. Similarly, we obtain that $V_{\varphi,[a_2, b_2]}^R(f) \leq kV_{[a_2, b_2]}(f) + \varphi(t_o)(b_2 - a_2)$. Further, we prove that $V_\varphi^R(f) \leq kV_{I_a^b}(f) + \varphi(t_o)A(I_a^b)$. Indeed, we have

$$\sum_{i=1}^{m}\sum_{j=1}^{n} \varphi\left(\frac{|\Delta_{11} f(t_i, s_j)|}{|\Delta t_i||\Delta s_j|}\right) |\Delta t_i||\Delta s_j| \leq kV_{I_a^b}(f) + \varphi(t_o)A(I_a^b) < \infty.$$

Hence $V_\varphi^R(f) \leq kV_{I_a^b}(f) + \varphi(t_o)A(I_a^b)$. This implies

$$TV_\varphi^R(f) < kTV(f) + \varphi(t_o)\left[(b_1 - a_1) + (b_2 - a_2) + A(I_a^b)\right] < \infty$$

and consequently

$$BV(I_a^b) \subset V_\varphi^R(I_a^b).$$

On the other hand by (e) we conclude that

$$BV(I_a^b) = V_\varphi^R(I_a^b).$$

This completes the proof. ■

Keeping in mind Definition 2.2 from now on we will assume that φ is a convex function such that

$$\lim_{t\to\infty} \frac{\varphi(t)}{t} = \infty.$$

Remark 2.4. From Theorem 2.3 (b) and (d) follows that $V_\varphi^R(I_a^b)$ is a symmetric and convex subset of the linear space X consisting of all functions $f : I_a^b \longrightarrow \mathbb{R}$. Then the linear space $\left\langle V_\varphi^R(I_a^b) \right\rangle$ generated by $V_\varphi^R(I_a^b)$ may be written in the form

$$\left\langle V_\varphi^R(I_a^b) \right\rangle := \Big\{ f \in \mathbf{X} : \text{ there is } \lambda > 0 \text{ such that } \lambda f \in V_\varphi^R(I_a^b) \Big\}.$$

Denote by $BV_\varphi^R(I_a^b;\mathbb{R})$ the space of functions of φ-bounded variation in the sense of Riesz. Thus

$$\begin{aligned} BV_\varphi^R(I_a^b;\mathbb{R}) &:= \Big\{ f : I_a^b \longrightarrow \mathbb{R} : TV_\varphi^R(\lambda f) < +\infty \text{ for some } \lambda > 0 \Big\} \\ &= \left\langle V_\varphi^R(I_a^b) \right\rangle. \end{aligned}$$

Remark 2.5. Observe that the set $BV_\varphi^R(I_a^b)$ is an algebra with usual operations on functions.

Moreover the set

$$\mathcal{A} = \Big\{ f : I_a^b \longrightarrow \mathbb{R} : TV_\varphi^R(f) \leq 1 \Big\} \tag{7}$$

is absorbent and balanced, so the Minkowski functional associated to the set $\mathcal{A}$ is a semi-norm.

Remark 2.6. Since the set $\Big\{ \varepsilon > 0 : TV_\varphi^R(u/\varepsilon) \leq 1 \Big\}$ is nonempty, therefore the following definition has sense.

Definition 2.7. Let $\varphi \in \Phi$ be a convex function and let $\|\cdot\|_{\varphi,0} : BV_{\varphi,0}^R(I_a^b) \longrightarrow \mathbb{R}_+$ be defined by the formula

$$\|f\|_{\varphi,0} := \inf \Big\{ \varepsilon > 0 : TV_\varphi^R(f/\varepsilon) \leq 1 \Big\},$$

with $BV^R_{\varphi,0}(I^b_a) := \{f \in BV^R_\varphi(I^b_a) : \ f(a) = 0\}$.

Then, $BV^R_\varphi(I^b_a;\mathbb{R})$ has Banach space structure with respect to the norm

$$\|f\|^R_\varphi := |f(a)| + \inf\left\{\varepsilon > 0 : TV^R_\varphi(f/\varepsilon) \leq 1\right\} \quad \text{for} \quad f \in BV^R_\varphi(I^b_a;\mathbb{R}).$$

Theorem 2.8. *Let $\varphi \in \Phi$ be convex. Then $\left(BV^R_\varphi(I^b_a), \|\cdot\|^R_\varphi\right)$ is a Banach space.*

3. The Banach algebra $BV^R_\varphi(I^b_a)$

The techniques and methods used in this section are similar to those used by V.V. Chistyakov in [4].

The first main result of this section is contained in the following theorem:

Theorem 3.1. *The space $\left(BV^R_\varphi(I^b_a;\mathbb{R}), \|\cdot\|^R_\varphi\right)$ is a Banach algebra. In addition, $\|f\cdot g\|^R_\varphi \leq \|f\|^R_\varphi \cdot \|g\|^R_\varphi$ for $f, g \in BV^R_\varphi(I^b_a;\mathbb{R})$.*

Proof. We know that: $\|f\|^R_\varphi := |f(a)| + \|f - f(a)\|^R_{\varphi,0}$ and $\|f\|^R_{\varphi,0} := \inf\Big\{\varepsilon > 0 : TV^R_\varphi(f/\varepsilon) \leq 1\Big\}$. Hence we obtain

$$\begin{aligned}
&\|f\cdot g\|^R_\varphi = |(fg)(a)| + \|(fg) - (fg)(a)\|^R_{\varphi,0}\\
&= \ |f(a)\cdot g(a)| + \|f\cdot g + f\cdot g(a) - f\cdot g(a) - f(a)\cdot g(a)\|^R_{\varphi,0}\\
&= \ |f(a)\cdot g(a)| + \|f[g - g(a)] + [f - f(a)]g(a)\|^R_{\varphi,0}\\
&\leq \ |f(a)|\cdot|g(a)| + \|f[g - g(a)]\|^R_{\varphi,0} + \|[f - f(a)]g(a)\|^R_{\varphi,0}\\
&\leq \ |f(a)|\cdot|g(a)| + \|f\|^R_{\varphi,0}\cdot\|g - g(a)\|^R_{\varphi,0} + \|f - f(a)\|^R_{\varphi,0}\cdot|g(a)|\\
&= \ |f(a)|\cdot|g(a)| + \|f - f(a) + f(a)\|^R_{\varphi,0}\cdot\|g - g(a)\|^R_{\varphi,0} + \|f - f(a)\|^R_{\varphi,0}\cdot|g(a)|\\
&\leq \ |f(a)|\cdot|g(a)| + \Big[\|f - f(a)\|^R_{\varphi,0} + |f(a)|\Big]\|g - g(a)\|^R_{\varphi,0} + \|f - f(a)\|^R_{\varphi,0}\cdot|g(a)|\\
&= \ \Big[|f(a)| + \|f - f(a)\|^R_{\varphi,0}\Big]\cdot\Big[|g(a)| + \|g - g(a)\|^R_{\varphi,0}\Big]\\
&= \ \|f\|^R_\varphi\cdot\|g\|^R_\varphi.
\end{aligned}$$

Thus, the proof is complete. ■

4. The composition operator on $BV^R_\varphi(I^b_a;\mathbb{R})$

The objective of this section is to characterize the composition (Nemystkii) operator on the space $BV^R_\varphi(I^b_a;\mathbb{R})$ of functions of φ-total bounded variation in the sense of Riesz $BV^R_\varphi(I^b_a)$. The main result in this section (Theorem 5.1) will be proved without the notion of *left–left* regularization and *left–left continuity* of two variable functions.

Let us define the Luxemburg functional on the linear space $BV_\phi^R(I_a^b;\mathbb{R})$ by putting:

$$\mathcal{P}_\phi(f) := \inf\Big\{r>0 : TV_\phi^R(f/r)\le 1\Big\},\ f\in BV_\phi^R(I_a^b),\ \phi\in\Phi. \tag{8}$$

Since the mapping $\mathcal{P}_\phi$ is the Minkowski functional of a convex set $E_\phi = \Big\{f \in BV_\phi^R(I_a^b) : TV_\phi^R(f)\le 1\Big\}$, the zero mapping is contained in $Ker(E_\phi)$ and $\lambda E_\phi \subset E_\phi$ for all λ such that $|\lambda|<1$.

Next, we shall give the lemma which will be used in the proof of our main result.

Lemma 4.1. *Assume that $\phi\in\Phi$ is convex and $f\in BV_\phi^R(I_a^b;\mathbb{R})$. Then*

(a) If $\mathcal{P}_\phi(f)>0$ then $TV_\phi^R(f/\mathcal{P}_\phi(f))\le 1$.

(b) If $r>0$, then $TV_\phi^R(f/r)\le 1$ if and only if $\mathcal{P}_\phi(f)\le r$.

(c) If $r>0$ and $TV_\phi^R(f/\mathcal{P}_\phi(f))=1$ then $\mathcal{P}_\phi(f)=r$.

Proof. (a) The definition of $\mathcal{P}_\phi(f)$ implies that $TV_\phi^R(f/r)\le 1$ for all $r>\mathcal{P}_\phi(f)$. Let us choose a sequence $r_n>\mathcal{P}_\phi(f)$, $n\in\mathbb{N}$, which converges to $\mathcal{P}_\phi(f)$ when $n\to\infty$. Then $f/r_n\to f/\mathcal{P}_\phi(f)$ uniformly in I_a^b since I_a^b is closed. Hence we obtain

$$TV_\phi^R(f/\mathcal{P}_\phi(f))\le \lim_{n\to\infty}\inf TV_\phi^R(f/r_n)\le 1.$$

Consequently we deduce that $\mathcal{P}_\phi(f)\in\{r>0 : TV_\phi^R(f/r)\le 1\} := \Lambda$ and $\mathcal{P}_\phi(f)=\min\Lambda$.

(b) If $TV_\phi^R(f/r)\le 1$ then from the definition given by (8) we obtain that $\mathcal{P}_\phi(f)\le r$.

Conversely, if $\mathcal{P}_\phi(f)=r$, then $TV_\phi^R(f/r)\le 1$ by (b). Now, we shall show that

$$\text{If } \mathcal{P}_\phi(f)<r, \text{ then } TV_\phi^R(f/r)<1. \tag{9}$$

Indeed, if $\mathcal{P}_\phi(f)=0$, then f is a constant mapping and $TV_\phi^R(f/r)=0$ (see Theorem 2.3 (d)). Suppose that $\mathcal{P}_\phi(f)>0$. From the convexity of $TV_\phi^R(\cdot)$ (see Theorem 2.3 (e)) and from (a) we get:

$$\begin{aligned}
TV_\phi^R(f/r) &= TV_\phi^R\Big(\frac{\mathcal{P}_\phi(f)}{r}\cdot\frac{f}{\mathcal{P}_\phi(f)}+\Big(1-\frac{r}{\mathcal{P}_\phi(f)}\Big)c\Big)\\
&\le \frac{\mathcal{P}_\phi(f)}{r}\cdot TV_\phi^R\Big(\frac{f}{\mathcal{P}_\phi(f)}\Big)+\Big(1-\frac{r}{\mathcal{P}_\phi(f)}\Big)TV_\phi^R(c)\\
&= (\mathcal{P}_\phi(f)/r)TV_\phi^R\Big(\frac{f}{\mathcal{P}_\phi(f)}\Big)\\
&\le (\mathcal{P}_\phi(f)/r)<1.
\end{aligned}$$

(c) Assume that $TV_\phi^R(f/r)=1$. From part (b), if $\mathcal{P}_\phi(f)>r$ then $TV_\phi^R(f/r)>1$, which contradicts the assumption. If $\mathcal{P}_\phi(f)<r$, then from (8) we obtain $TV_\phi^R(f/r)<1$. Therefore, $\mathcal{P}_\phi(f)=r$. This ends the proof. ∎

5. Characterization of globally Lipschitzian composition operators

The following theorem is the main result of this work which extends the results of Matkowski in the case when the composition operator is defined on the space $BV_\varphi^R(I_a^b;\mathbb{R})$.

Theorem 5.1. *Let $\varphi \in \Phi$ be a convex function satisfying the condition ∞_1 and let $H : \mathbb{R}^{I_a^b} \longrightarrow \mathbb{R}^{I_a^b}$ be the composition operator generated by the function $h : I_a^b \times \mathbb{R} \to \mathbb{R}$ and defined by the formula*

$$(Hf)(t,s) = h\big(t,s,f(t,s)\big),$$

for $f \in \mathbb{R}^{I_a^b}$, $(t,s) \in I_a^b$. If H maps $BV_\varphi^R(I_a^b;\mathbb{R})$ into itself and is globally Lipschitzian, then the following condition is satisfied

$$\big|h(x,u_1) - h(x,u_2)\big| \le \delta\big|u_1 - u_2\big|, \tag{10}$$

for each $x \in I_a^b$ and all $u_1, u_2 \in \mathbb{R}$. Moreover, there exist two functions $h_0, h_1 \in BV_\varphi(I_a^b;\mathbb{R})$ such that

$$h(x,u) = h_0(x) + h_1(x)u, \tag{11}$$

for $x \in I_a^b$ and $u \in \mathbb{R}$. Conversely, if $h_0, h_1 \in BV_\varphi(I_a^b,\mathbb{R})$ and $h(x,u) = h_0(x) + h_1(x)u$, for $x \in I_a^b$ and for $u \in \mathbb{R}$, then H maps the space $BV_\varphi^R(I_a^b)$ into itself and is globally Lipschitzian.

Proof. Notice that in the proof we apply the technique similar to those from [3, 4]. At the beginning, for arbitrarily fixed $\alpha, \beta \in \mathbb{R}$, $\alpha < \beta$, let us put

$$\eta_{\alpha,\beta}(t) = \begin{cases} 0 & \text{for } \ t \le \alpha \\ \dfrac{t-\alpha}{\beta-\alpha} & \text{for } \ \alpha \le t \le \beta \\ 1 & \text{for } \ t \ge \beta. \end{cases} \tag{12}$$

Observe that $\eta_{\alpha,\beta} : \mathbb{R} \to \mathbb{R}$ and is Lipschitzian.

We divide the proof into three steps.

Step 1. We prove inequality (10). To this end we show first an auxiliary inequality which will be frequently used in our reasoning.
Since $H : BV_\varphi^R(I_a^b) \longrightarrow BV_\varphi^R(I_a^b)$ is Lipschitzian, there exists a constant $\mu > 0$ such that $\|Hf_1 - Hf_2\|_\varphi^R \le \mu\|f_1 - f_2\|_{BV_\varphi^R}$ for $f_1, f_2 \in BV_\varphi^R(I_a^b)$. The definition of the norm $\|\cdot\|_{BV_\varphi^R}$ implies that $\mathcal{P}_\varphi(Hf_1 - Hf_2) \le \mu\|f_1 - f_2\|_{BV_\varphi^R}$. From Lemma 4.1 (c) we infer that if $\|f_1 - f_2\|_{BV_\varphi^R} > 0$, then the last inequality is equivalent to the following one

$$TV_\varphi^R\left(\frac{Hf_1 - Hf_2}{\mu\|f_1 - f_2\|_{BV_\varphi^R}}\right) \le 1.$$

From the definitions of the operators TV_φ^R and H we deduce that for all $x = (x_1, x_2)$, $y = (y_1, y_2) \in \mathbb{R}^2$, $x_1 < y_1$, $x_2 < y_2$ we have that

$$\varphi\left(\frac{\|(Hf_1 - Hf_2)(x_1, x_2) - (Hf_1 - Hf_2)(y_1, y_2)\|}{\mu\|f_1 - f_2\|_{BV_\varphi^R}\|y - x\|}\right)\|y - x\| \leq 1.$$

Hence, taking the inverse function, we get

$$\begin{aligned}\Big|h(x, f_1(x)) - h(x, f_2(x)) - h(y, f_1(y)) + h(y, f_2(y))\Big| &\leq \\ \mu\|f_1 - f_2\|_{BV_\varphi^R}\|y - x\|\varphi^{-1}\Big(1\Big/\|y - x\|\Big).\end{aligned} \tag{13}$$

Now, we consider the following four cases: (i) $a_1 < x_1 \leq b_1$ and $a_2 < x_2 \leq b_2$, (ii) $a_1 < x_1 \leq b_1$ and $x_2 = a_2$, (iii) $x_1 = a_1$ and $a_2 < x_2 \leq b_2$, (iv) $x_1 = a_1$ and $x_2 = a_2$.

Thus, assume that u_1, u_2 are arbitrarily fixed real numbers and $\mathcal{H} = Hf_1 - Hf_2$.

Case (i). Define two functions f_1, f_2 on the space $BV_\varphi^R(I_a^b; \mathbb{R})$ by putting

$$\left\{\begin{aligned} f_1(y_1, y_2) &:= \Big[\eta_{a_1, x_1}(y_1) + \eta_{a_2, x_2}(y_2)\Big](u_1 - u_2)/2, \\ f_2(y_1, y_2) &:= \Big[\eta_{a_1, x_1}(y_1) - \eta_{a_2, x_2}(y_2)\Big](u_1 - u_2)/2, \end{aligned}\right. \quad a_j \leq y_j \leq b_j, \quad j = 1, 2 \tag{14}$$

for $a_j \leq y_j \leq b_j$ $(j = 1, 2)$.

Since $f_j(a) = 0$ $(j = 1, 2)$ we have that

$$V_{\varphi,[a_1,b_1]}^R\left(\frac{f_1 - f_2}{r}(\cdot, a_2)\right) = 0 = V_\varphi^R\left(\frac{f_1 - f_2}{r}, I_a^b\right)$$

and

$$V_{\varphi,[a_2,b_2]}^R\left(\frac{(f_1 - f_2)}{r}(a_1, \cdot)\right)\varphi\left(\frac{|u_1 - u_2|}{r|x_2 - a_2|}\right)|x_2 - a_2|.$$

If we choose $r > 0$ such that

$$\begin{aligned} TV_\varphi^R\left(\frac{f_1 - f_2}{r}\right) V_{\varphi,[a_1,b_1]}^R\left(\frac{f_1 - f_2}{r}(\cdot, a_2)\right) + V_{\varphi,[a_2,b_2]}^R\left(\frac{f_1 - f_2}{r}(a_1, \cdot)\right) \\ + V_\varphi^R\left(\frac{f_1 - f_2}{r}\right)\varphi\left(\frac{|u_1 - u_2|}{r|x_2 - a_2|}\right)|x_2 - a_2| = 1, \end{aligned}$$

then from Lemma 4.1 we obtain

$$\|f_1 - f_2\|_{BV_\varphi^R} = \mathcal{P}_\varphi(f_1 - f_2) = r = \frac{|u_1 - u_2|}{|x_2 - a_2|\varphi^{-1}\left(1/|x_2 - a_2|\right)}. \tag{15}$$

Next, putting (14) into (13), for all $a_1 < x_1 < y_1 \leq b_1$, $a_2 < x_2 < y_2 \leq b_2$ and $u_1, u_2 \in BV_\varphi^R(I_a^b)$ we get:

$$\begin{aligned}
&\Big|h(x_1,x_2,u_1) - h(x_1,x_2,u_2)\Big| \\
&\quad \leq \ \mu \|f_1 - f_2\|_{BV_\varphi^R} |x_2 - a_2| \varphi^{-1}\Big(1/|x_2 - a_2|\Big) \\
&\quad = \ \mu \frac{|u_1 - u_2|}{|x_2 - a_2|\varphi^{-1}\Big(1/|x_2 - a_2|\Big)} \cdot |x_2 - a_2|\varphi^{-1}\Big(1/|x_2 - a_2|\Big) \\
&\quad = \ \mu |u_1 - u_2|.
\end{aligned}$$

Hence we have that h is a Lipschitzian function.

Case (ii). We define the functions

$$f_j(y_1,y_2) = \eta_{a_1,x_1}(y_1)u_j \quad \text{for} \quad a_j \leq y_j \leq b_j \quad (j = 1,2). \tag{16}$$

Observe that $f_j(a) = 0$ $(j = 1,2)$ and

$$V_{\varphi,[a_1,b_1]}^R \left(\frac{f_1 - f_2}{r}(\cdot, a_2)\right) \varphi\left(\frac{|u_1 - u_2|}{r|x_1 - a_1|}\right) |x_1 - a_1|,$$

$$V_{\varphi,[a_2,b_2]}^R \left(\frac{f_1 - f_2}{r}(a_1, \cdot)\right) = 0 = V_\varphi^R\left(\frac{f_1 - f_2}{r}, I_a^b\right).$$

If we choose $r > 0$ such that

$$TV_\varphi^R\left(\frac{f_1 - f_2}{r}\right) \varphi\left(\frac{|u_1 - u_2|}{r|x_1 - a_1|}\right) |x_1 - a_1| = 1, \tag{17}$$

then from Lemma 4.1 we obtain

$$\|f_1 - f_2\|_{BV_\varphi^R} = \mathcal{P}_\varphi(f_1 - f_2) = r = \frac{|u_1 - u_2|}{|x_1 - a_1|\varphi^{-1}\left(1/|x_1 - a_1|\right)}. \tag{18}$$

Now, linking (17) and (13), for all $a_1 < x_1 \leq b_1$, $a_2 < x_2 \leq b_2$ and $u_1, u_2 \in BV_\varphi^R(I_a^b)$ we get:

$$\Big|h(x_1, a_2, f_1(x_1,a_2)) - h(x_1, a_2, f_2(x_1, a_2))\Big| \leq \mu |u_1 - u_2|. \tag{19}$$

Case (iii). This case can be done in a similar way as case (ii). We only have to define the functions $f_1, f_2 \in BV_\phi^R(I_a^b)$, by putting

$$f_j(y_1,y_2) = \eta_{a_2,x_2}(y_2)u_j \quad \text{for} \quad a_j \leq y_j \leq b_j \quad (j = 1,2).$$

Case (iv). Consider the functions $f_1, f_2 \in BV_\varphi^R(I_a^b)$ defined by the formula

$$f_j(y_1,y_2) := [1 - \eta_{a_1,b_1}(y_1)]u_j/2 \quad \text{for} \quad a_j \leq y_j \leq b_j \ (j = 1,2).$$

Then we have that $f_j(a) = u_j$ $(j = 1, 2)$, $f_j(b) = 0$ $(j = 1, 2)$ and we obtain $\mathcal{H}(b) = 0$. Moreover,

$$V^R_{\varphi,[a_1,b_1]}\left(\frac{f_1 - f_2}{r}(\cdot, a_2)\right) = \varphi\left(\frac{|u_1 - u_2|}{2|b_1 - a_1|}\right)|b_1 - a_1|,$$
$$V^R_{\varphi,[a_2,b_2]}\left(\frac{f_1 - f_2}{r}(a_1, \cdot)\right) = 0 = V^R_{\varphi}\left(\frac{f_1 - f_2}{r}, I^b_a\right).$$

If we take $r > 0$ such that

$$TV^R_{\varphi}\Big(\frac{f_1 - f_2}{r}; I^b_a\Big) = \varphi\left(\frac{|u_1 - u_2|}{2|b_1 - a_1|}\right)|b_1 - a_1| = 1,$$

then from Lemma 4.1 we get

$$\|f_1 - f_2\|^R_{\varphi} \;=\; \mathcal{P}(f_1 - f_2) = r\frac{|u_2 - u_1|}{|b_1 - a_1|\varphi^{-1}\left(\dfrac{1}{|b_1 - a_1|}\right)}.$$

Thus, from estimate (13) we deduce that

$$\big|h(a_1, a_2, u_1) - h(a_1, a_2, u_2)\big| \le \mu|u_2 - u_1|$$

and therefore h is Lipschitzian.

Step 2. We shall prove estimate (11). To this end, let us fix $x_1 \in (a_1, b_1]$ and $x_2 \in (a_2, b_2]$. Put $x = (x_1, x_2)$. Further, for each $m \in \mathbb{N}$ we consider the partitions:

$$a_1 < \alpha_1 < \beta_1 < \alpha_2 < \beta_2 < \cdots < \alpha_m < \beta_m < x_1,$$
$$a_2 < \overline{\alpha}_1 < \overline{\beta}_1 < \overline{\alpha}_2 < \overline{\beta}_2 < \cdots < \overline{\alpha}_m < \overline{\beta}_m < x_2.$$

Next, consider two auxiliary functions: $\eta_{\alpha,\beta} : [a_1, b_1] \to [0, 1]$ and $\overline{\eta}_{\alpha,\beta} : [a_2, b_2] \to [0, 1]$ defined in the following way:

$$\eta_m(t) := \begin{cases} 0 & \text{for } \; a_1 \le t \le \alpha_1 \\ \eta_{\alpha_i,\beta_i}(t) & \text{for } \; \alpha_i \le t \le \beta_i, \quad i = 1, 2, \cdots, m \\ 1 - \eta_{\beta_i,\alpha_{i+1}}(t) & \text{for } \; \beta_i \le t \le \alpha_{i+1}, \quad i = 1, 2, \cdots, m-1 \\ 1 & \text{for } \; \beta_m \le t \le b_1, \end{cases} \tag{20}$$

$$\overline{\eta}_m(s) := \begin{cases} 0 & \text{for } \; a_2 \le s \le \overline{\alpha}_1 \\ \eta_{\overline{\alpha}_i,\overline{\beta}_i}(s) & \text{for } \; \overline{\alpha}_i \le s \le \overline{\beta}_i, \quad i = 1, 2, \cdots, m \\ 1 - \eta_{\overline{\beta}_i,\overline{\alpha}_{i+1}}(s) & \text{for } \; \overline{\beta}_i \le s \le \overline{\alpha}_{i+1}, \quad i = 1, 2, \cdots, m-1 \\ 1 & \text{for } \; \overline{\beta}_m \le s \le b_2. \end{cases} \tag{21}$$

Now, observe that the following inequality holds:

$$V^R_{\varphi,[a_1,b_1]}(\mathcal{H}) \le \|Hf_1 - Hf_2\|^R_{\varphi} \le \mu\|f_1 - f_2\|^R_{\varphi}.$$

The above inequality can be expressed equivalently in the following way:

$$\sup_{\xi}\sum_{i=1}^{m}\varphi\left(\frac{\left|\mathcal{H}(t_i,a_2)-\mathcal{H}(t_{i-1},a_2)\right|}{|\Delta t_i|}\right)|\Delta t_i|\leq\mu\|f_1-f_2\|_{\varphi}^{R}.$$

In particular, we have

$$\sum_{i=1}^{m}\varphi\left(\frac{\left|\mathcal{H}(\beta_i,\overline{\beta_i})-\mathcal{H}(\alpha_i,\overline{\alpha_i})\right|}{|\beta_i-\alpha_i|}\right)|\beta_i-\alpha_i|\leq\mu\|f_1-f_2\|_{\varphi}^{R}. \tag{22}$$

Further on, for arbitrary numbers $u_1,u_2\in\mathbb{R}$, we define the functions $f_1,\ f_2$ by putting:

$$f_j(y_1,y_2)=\frac{1}{2}\Big[\eta_m(y_1)+\overline{\eta}_m(y_2)\Big]u_1+(2-j)u_2,\quad a_j\leq y_j\leq b_j\quad (j=1,2).$$

Observe, that

$$f_1(\alpha_i,\overline{\alpha_i})-f_2(\beta_i,\overline{\beta_i})=u_2-u_1$$

and consequently

$$\|f_1-f_2\|_{\varphi}^{R}=|u_1-u_2|.$$

Since $\mathcal{H}=Hf_1-Hf_2$, from (23) we get

$$\sum_{i=1}^{m}\varphi\left(\frac{\left|(Hf_1-Hf_2)(\beta_i,\overline{\beta_i})-(Hf_1-Hf_2)(\alpha_i,\overline{\alpha_i})\right|}{|\beta_i-\alpha_i|}\right)|\beta_i-\alpha_i|\leq\mu\|f_1-f_2\|_{\varphi}^{R}.$$

Consequently, we obtain

$$\sum_{i=1}^{m}\varphi\left(\frac{\left|h(\beta_i,\overline{\beta_i},f_1(\beta_i,\overline{\beta_i}))-h(\beta_i,\overline{\beta_i},f_2(\beta_i,\overline{\beta_i}))-h(\alpha_i,\overline{\alpha_i},f_1(\alpha_i,\overline{\alpha_i}))\right.}{|\beta_i-\alpha_i|}\right.$$
$$\left.\frac{\left.+h(\alpha_i,\overline{\alpha_i},f_2(\alpha_i,\overline{\alpha_i}))\right|}{}\right)|\beta_i-\alpha_i|\leq\mu\|f_1-f_2\|_{\varphi}^{R}.$$

Thus, from the definition of f_1 and f_2, we deduce that $f_1(\beta_i,\overline{\beta}_i)=u_1+u_2$, $f_2(\beta_i,\overline{\beta}_i)=u_1$, $f_1(\alpha_i,\overline{\alpha}_i)=u_2$ and $f_2(\alpha_i,\overline{\alpha}_i)=0$. This yields

$$\sum_{i=1}^{m}\varphi\left(\frac{\left|h(\beta_i,\overline{\beta_i},u_1+u_2)-h(\beta_i,\overline{\beta_i},u_1)-h(\alpha_i,\overline{\alpha_i},u_2)+h(\alpha_i,\overline{\alpha_i},0)\right|}{|\beta_i-\alpha_i|}\right)\cdot$$
$$|\beta_i-\alpha_i|\leq\mu|u_2-u_1|. \tag{23}$$

Since all constant functions of two variables defined on I_a^b belong to the space $BV_{\varphi}^{R}(I_a^b;\mathbb{R})$ and H maps this space into itself, we infer that the functions $h(\cdot,u)[x\mapsto$

$h(x,u)]$ belong to $BV_\varphi^R(I_a^b;\mathbb{R})$ for all $u \in \mathbb{R}$. Taking into account the absolute continuity of this function and passing to limit in (24) as $(\alpha_i, \overline{\alpha}_i) \to (\beta_i - 0, \overline{\beta}_i - 0)$, we obtain

$$\sum_{i=1}^{m} \varphi\left(\frac{\Big|h(x_1,x_2,u_1+u_2) - h(x_1,x_2,u_1) - h(x_1,x_2,u_2) + h(x_1,x_2,0)\Big|}{|\beta_i - \alpha_i|} \right).$$

$$|\beta_i - \alpha_i| \le \mu |u_2 - u_1|.$$

Hence

$$\varphi\left(\frac{\Big|h(x,u_1+u_2) - h(x,u_1) - h(x,u_2) + h(x,0)\Big|}{|\beta_i - \alpha_i|} \right) \le \frac{\mu|u_2-u_1|}{|\beta_i - \alpha_i|}.$$

From the above estimate we infer the following one

$$0 \le \frac{\Big|h(x,u_1+u_2) - h(x,u_1) - h(x,u_2) + h(x,0)\Big|}{|\beta_i - \alpha_i|}$$
$$\le \lim_{\alpha_i \to \beta_i - 0} (\beta_i - \alpha_i)\varphi^{-1}\left(\frac{|u_2-u_1|}{|\beta_i - \alpha_i|}\right).$$

Consequently, we get

$$\Big|h(x,u_1+u_2) - h(x,u_1) - h(x,u_2) + h(x,0)\Big| = 0$$

or, equivalently

$$h(x,u_1+u_2) - h(x,u_1) - h(x,u_2) + h(x,0) = 0.$$

Finally, we obtain the equality

$$h(x,u_1+u_2) + h(x,0) = h(x,u_1) + h(x,u_2), \tag{24}$$

being valid for all $x_1 \in (a_1,b_1],\ x_2 \in (a_2,b_2],\ $ and $u_1,\ u_2 \in \mathbb{R}$.

Now, let $x_1 \in (a_1,b_1]$ and $x_2 = b_2$. Consider the partitions $a_1 < \alpha_1 < \beta_1 < \alpha_2 < \beta_2 < \cdots < \alpha_m < \beta_m < x_1$ and $a_2 < \overline{\alpha}_1 < \overline{\beta}_1 < \overline{\alpha}_2 < \overline{\beta}_2 < \cdots < \overline{\alpha}_m < \overline{\beta}_m < b_2$. Similarly as before we obtain (24). Then passing to the limit when $(\alpha_1, \overline{\beta}_m) \to (x_1 - 0, x_2 + 0)$ in equation (24) we obtain again equality (25).

The cases $x_1 = a_1$ and $x_2 \in (a_2,b_2]$ or $x_1 = a_1$ and $x_2 = a_2$ can be treated in a similar way. Thus, we have

$$h(x,u_1+u_2) + h(x,0) = h(x,u_1) + h(x,u_2), \tag{25}$$

for all $x=(x_1,x_2)\in I_a^b$ and $f_1,\ f_2\in\mathbb{R}$.

To complete the proof of (11), for a fixed $x\in I_a^b$ we define the function $T_x:\mathbb{R}\to\mathbb{R}$, by putting

$$T_x(u)=h(x,u)-h(x,0).$$

Now, let us write equality (26) in the form

$$T_x(u_1+u_2)=T_x(u_1)+T_x(u_2),\quad \text{where}\quad u_1,u_2\in\mathbb{R}.$$

This proves that T_x is an additive operator. From inequality (10) and the definition of $h(\cdot,u)$, we get:

$$\Big|T_x(u_1)-T_x(u_2)\Big|\leq\mu|u_1-u_2|\quad \text{for}\quad u_1,\ u_2\in\mathbb{R}. \tag{26}$$

Thus T_x is a Lipschitzian mapping on $\mathbb{R}$.

In what follows let us define the mapping $h_0:\ I_a^b\longrightarrow\mathbb{R}$ by the formula $h_0(x)=h(x,0)$ for $x\in I_a^b$. Next, let $h_1:\ I_a^b\longrightarrow\mathbb{R}$ be defined as

$$h_1(x)u=T_x(u)\quad \text{for}\quad x\in I_a^b,\quad u\in\mathbb{R}.$$

Then we have

$$h(x,u)=T_x(u)+h(x,0)=h_1(x)u+h_0(x).$$

Since $h_0(\cdot)=h(\cdot,0)$ and $h_1(\cdot)=h(\cdot,1)-h(\cdot,0)$ then $h_0,\ h_1\in BV_\varphi^R(I_a^b;\mathbb{R})$. Therefore

$$h(x,u)=h_0(x)+h_1(x)u$$

for all $x\in I_a^b$ and $u\in\mathbb{R}$, with $h_0,h_1\in BV_\varphi^R(I_a^b)$.

Step 3. Conversely, suppose that the composition operator H is given by the formula

$$(Hf)(x)=h_0(x)+h_1(x)f(x)\quad \text{for}\quad x\in I_a^b,\quad f\in BV_\varphi^R(I_a^b).$$

Since $BV_\varphi^R(I_a^b)$ is an algebra we deduce that

$$TV_\varphi^R(Hf)\leq\|Hf\|_\varphi<\infty.$$

Thus, $Hf\in BV_\varphi^R(I_a^b)$. Other words, H maps $BV_\varphi^R(I_a^b)$ into itself. Hence we obtain

$$\|H(f_1)-H(f_2)\|_\varphi^R\quad\leq\quad\|h_1\|_\varphi^R\|f_1-f_2\|_\varphi^R. \tag{27}$$

This shows that H is Lipschitzian and completes the proof. ■

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Received 08.07.2009

J o u r n a l of
Mathematics
and Applications

No 32, pp 25-32 (2010)

On T-neighborhoods of analytic functions

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ABSTRACT: Let $\mathcal{A}$ denote the class of functions f analytic in the unit disk $\mathcal{U} = \{z \in \mathbf{C} : |z| < 1\}$ of the form $f(z) = z + \sum_{n=2}^{\infty} a_n z^n$. Given a sequence $T = \{T_n\}_{n=2}^{\infty}$ consisting of positive numbers, the T_δ-neighborhood ($\delta > 0$) of the function f is defined as

$$TN_\delta(f) = \left\{ g(z) = z + \sum_{n=2}^{\infty} b_n z^n \in \mathcal{A} : \sum_{n=2}^{\infty} T_n |a_n - b_n| \leq \delta \right\}.$$

In this paper we consider $TN_\delta(f)$ with the sequence of the form

$$T = \left\{ \frac{1}{n^2(n-1)} \right\}_{n=2}^{\infty}$$

to obtain some results about T_δ-neighborhoods of several classes of analytic functions. One of the considered problems is to find a number $\delta^*(A, B)$ such that

$$\delta^*(A, B) = \inf \left\{ \delta > 0 : TN_\delta(f) \supset B \quad for\ all \quad f \in A \right\},$$

where the sets $A, B \subset \mathcal{A}$ are given. Some results and open problems on the classes of starlike, convex, concave, close-to-convex and univalent functions are presented.

AMS Subject Classification: *Primary 30C45, Secondary 30C50, 40A05*
Key Words and Phrases: *Analytic functions; Starlike functions; Convex functions; Close-to-convex functions; Concave functions; Hadamard product; Neighborhood of function; T-neighborhood of function*

1. Introduction

Let $\mathcal{A}$ denote the class of functions f analytic in the unit disk $\mathcal{U} = \{z \in \mathbf{C} : |z| < 1\}$ of the form $f(z) = z + \sum\limits_{n=2}^{\infty} a_n z^n$. Given a sequence $T = \{T_n\}_{n=2}^{\infty}$ consisting of positive numbers, the T_δ-neighborhood ($\delta > 0$) of the function f is defined as

$$TN_\delta(f) = \left\{ g(z) = z + \sum_{n=2}^{\infty} b_n z^n \in \mathcal{A} : \sum_{n=2}^{\infty} T_n |a_n - b_n| \le \delta \right\}.$$

If $T = \{n\}_{n=2}^{\infty}$ then T_δ-neighborhood becomes the δ-neighborhood $N_\delta(f)$ introduced by St. Ruscheweyh [13]. He proved that if $f \in \mathcal{C}$ then $N_{1/4}(f) \subset \mathcal{S}^*$, where $\mathcal{C}$, $\mathcal{S}^*$ denote the well known classes of convex and starlike functions, respectively. In this way he generalized the earlier result that $N_1(z) \subset \mathcal{S}^*$. Some results of this type one can find in [16], [8], [9], [10]. The T_δ-neighborhood was introduced in [15], where the authors considered the problem of finding a sufficient condition $f \in \mathcal{A}$ that implies the existence of $TN_\delta(f)$ being contained in a given subclass. They proved a number of theorems showing the importance of convolutions in the study of T_δ-neighborhoods and considered for an arbitrary normal family $\mathcal{T} \subset \mathcal{A}$ of functions $t(z) = z + \sum\limits_{k=2}^{\infty} t_k z^k$ the sequence $T = \{\tau_k\}_{k=2}^{\infty}$ such that

$$\tau_n = \sup\{|t_n| : t \in \mathcal{T}\} > 0 \quad (n = 2, 3, ...).$$

For $f(z) = z + \sum\limits_{k=2}^{\infty} a_k z^k$ and $g(z) = z + \sum\limits_{k=2}^{\infty} b_k z^k$ the convolution or Hadamard product of f and g is $(f * g)(z) = z + \sum\limits_{k=2}^{\infty} a_k b_k z^k$.

An interesting problem of stability of convolution on certain classes by using the δ- neighborhoods was considered in [12], [11]. For work on this problem see also the papers [5], [6], [4]. Let $\mathcal{S}$ denote the subclass of $\mathcal{A}$ of functions univalent in $\mathcal{U}$. Let us consider the following sequence of nonnegative reals

$$T = \left\{ \frac{1}{n^2(n-1)} \right\}_{n=2}^{\infty} \tag{1.1}$$

In this paper we will use the above sequence to obtain the results about T_δ-neighborhoods. The motivation of choice the sequence (1.1) is the convergence of the series $\sum_{n=2}^{\infty} T_n |a_n - b_n|$ for $|a_n| \le n, |b_n| \le n'$.

2. Main results

Theorem 1. *If $f(z) = z + \sum\limits_{n=2}^{\infty} a_n z^n$, $g(z) = z + \sum\limits_{n=2}^{\infty} b_n z^n$ and $|a_n| \le n, |b_n| \le n$, $n = 2, 3, 4, ...$, then $g \in TN_2(f)$, where T is given in (1.1).*

Proof. We have

$$\sum_{n=2}^{\infty} T_n|a_n - b_n| \leq \sum_{n=2}^{\infty} \frac{2n}{n^2(n-1)} = \sum_{n=2}^{\infty} \frac{2}{n(n-1)} = 2,$$

so $g \in TN_2(f)$.

$\square$

It is well known that if $\mathcal{S}$, $\mathcal{S}^*$, $\mathcal{C}$ and $\mathcal{K}$ denote the well-known classes of univalent, starlike, convex and close-to-convex functions respectively then $\mathcal{C} \subset \mathcal{S}^* \subset \mathcal{K} \subset \mathcal{S}$ and if $f \in \mathcal{S}^*$ then $|a_n| \leq n$ while if $f \in \mathcal{C}$ then $|a_n| \leq 1$. As a direct application of Theorem 1 we obtain T_δ-neighborhood information for $\mathcal{S}^*$ and $\mathcal{K}$.

Corollary 1. *If f belongs to one of the classes $\mathcal{S}^*, \mathcal{K}, \mathcal{S}$, then $TN_2(f) \supset \mathcal{S}$.*

The result will change if we consider the class of convex functions $\mathcal{C}$.

Corollary 2. *If $f \in \mathcal{C}$ then $TN_x(f) \supset \mathcal{S}$, where $x = 3 - \frac{\pi^2}{6} = 1,355...$.*

Proof. If $f(z) = z + \sum_{n=2}^{\infty} a_n z^n \in \mathcal{C}$, then $|a_n| \leq 1, n \geq 2$. Thus if $g(z) = z + \sum_{n=2}^{\infty} b_n z^n \in \mathcal{S}$, then we have

$$\begin{aligned} \sum_{n=2}^{\infty} T_n|a_n - b_n| &\leq \sum_{n=2}^{\infty} \frac{n+1}{n^2(n-1)} = 2\sum_{n=2}^{\infty} \frac{1}{n(n-1)} - \sum_{n=2}^{\infty} \frac{1}{n^2} \\ &= 2 - \left(\frac{\pi^2}{6} - 1\right) = 3 - \frac{\pi^2}{6} = 1,355... . \end{aligned}$$

$\square$

An interesting problem is to find the smallest number δ^* such that $TN_{\delta^*}(f) \supset \mathcal{S}$ for each $f \in \mathcal{S}$. Let us denote for $A, B \subset \mathcal{S}$

$$\delta^*(A, B) = \inf\{\delta : TN_\delta(f) \supset B \ \forall f \in A\}.$$

Theorem 2. *The following inequalities are valid*

$$1,386\ldots = 2\ln 2 \leq \delta^*(\mathcal{S}, \mathcal{S}) \leq 2.$$

Proof. It is well-known that Koebe function and its rotations belong to the class $\mathcal{S}$. Thus the functions

$$f(z) = \frac{z}{(1-z)^2} = z + \sum_{n=2}^{\infty} nz^n,$$

$$g(z) = \frac{z}{(1+z)^2} = z + \sum_{n=2}^{\infty} (-1)^{n+1} n z^n$$

are in $\mathcal{S}$ and

$$\begin{aligned}
\sum_{n=2}^{\infty} T_n |a_n - b_n| &= \sum_{k=1}^{\infty} \frac{2 \cdot 2k}{(2k)^2(2k-1)} \\
&= 2 \sum_{k=1}^{\infty} \frac{1}{2k(2k-1)} = 2\left[1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \dots\right] \\
&= 2 \sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{k} = 2 \ln 2 = 1,386\dots\ .
\end{aligned}$$

Therefore $\delta^*(\mathcal{S}, \mathcal{S})$ cannot be smaller than $2\ln 2$ and by Corollary 1 the number $\delta^*(\mathcal{S}, \mathcal{S})$ is less or equal to 2.

□

The functions f and g in the proof of Theorem 2 are starlike and close-to-convex so as a direct application of Theorem 2 we obtain the following corollary.

Corollary 3. *The following inequalities are valid*

$$2 \ln 2 \le \delta^*(A, B) \le 2,$$

where A and B is one of the classes $\mathcal{S}$, $\mathcal{S}^*$ or $\mathcal{K}$.

Theorem 3. *The following inequalities are valid*

$$1,20876... = \frac{\pi^2}{12} + \ln 4 - 1 \le \delta^*(\mathcal{C}, \mathcal{S}) \le 3 - \frac{\pi^2}{6} = 1,355...\ .$$

Proof. Let

$$\begin{aligned}
f(z) &= \frac{z}{1+z} = z - z^2 + z^3 - z^4 + ... = z + \sum_{n=2}^{\infty} b_n z^n, \\
g(z) &= \frac{z}{(1-z)^2} = z + \sum_{n=2}^{\infty} n z^n = z + \sum_{n=2}^{\infty} a_n z^n.
\end{aligned}$$

Then $f \in \mathcal{C}$ and $g \in \mathcal{S}$ and

$$\begin{aligned}
\sum_{n=2}^{\infty} T_n|a_n - b_n| &= \sum_{n=2}^{\infty} \frac{n+(-1)^n}{n^2(n-1)} \\
&= \sum_{k=1}^{\infty} \frac{2k+1}{(2k)^2(2k-1)} + \sum_{k=1}^{\infty} \frac{(2k+1)-1}{(2k+1)^2((2k+1)-1)} \\
&= \sum_{k=1}^{\infty} \frac{(2k-1)+2}{(2k)^2(2k-1)} + \sum_{k=1}^{\infty} \frac{1}{(2k+1)^2} \\
&= \sum_{k=1}^{\infty} \frac{2}{(2k)^2(2k-1)} + \sum_{k=1}^{\infty} \frac{1}{(2k)^2} + \sum_{k=1}^{\infty} \frac{1}{(2k+1)^2} \\
&= \sum_{k=1}^{\infty} \frac{2}{(2k)^2(2k-1)} + \sum_{n=2}^{\infty} \frac{1}{n^2} \\
&= \sum_{k=1}^{\infty} \frac{2}{2k(2k-1)} - \sum_{k=1}^{\infty} \frac{2}{(2k)^2} + \sum_{n=2}^{\infty} \frac{1}{n^2} \\
&= \sum_{n=2}^{\infty} \frac{1}{n^2} - \frac{1}{2}\sum_{k=1}^{\infty} \frac{1}{k^2} + 2\left[\frac{1}{1\cdot 2} + \frac{1}{3\cdot 4} + \frac{1}{5\cdot 6} + \frac{1}{7\cdot 8} + \dots\right] \\
&= -1 + \frac{1}{2}\sum_{n=1}^{\infty} \frac{1}{n^2} + 2\left[1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \dots\right] \\
&= -1 + \frac{\pi^2}{12} + 2\ln 2.
\end{aligned}$$

The upper bound we obtain from Corollary 2.

□

In [3] the authors considered functions f that are meromorphic and univalent in the open unit disk $\mathcal{U} = \{z : |z| < 1\}$ holomorphic at zero and have the expansion $f(z) = z + a_2z^2 + a_3z^3 + \dots$. If, in addition, the complement of $f(\mathcal{U})$ with respect to $\overline{\mathbf{C}}$ is convex, then f is called a concave univalent function. The class of such functions is denoted by $\mathcal{C}o$. The main result of the paper [2] is that if $f \in \mathcal{C}o$, then $|a_n| \geq 1$ for all $n > 1$ and equality holds if and only if $f(z) = z/(1-\eta z), |\eta| = 1$. This result was conjectured earlier in [3]. In [2] the authors considered the class $\mathcal{C}o(p)$ of concave functions that have a pole at the point p. The same authors proved in [1] a stronger result that if $f \in \mathcal{C}o(1)$ that is f is analytic in $\mathcal{U}$ with $f(1) = \infty$, then

$$\left|a_n - \frac{n+1}{2}\right| \leq \frac{n-1}{2} \quad \text{for } \ \mathrm{n} \geq 2 \tag{1.2}$$

and equality holds only for the function

$$f_\theta(z) = \frac{2z - (1-e^{i\theta})z^2}{2(1-z)^2}. \tag{1.3}$$

It is easy to see that if $f \in \mathcal{C}o(1)$, then the complement of $f(\mathcal{U})$ can be represented as the union of a set of mutually disjoint half-lines (the endpoint of one half-line can lie on the another half-line) so $f(\mathcal{U})$ is a linearly accessible domain in the strict sense. For example see Fig.1 below. It is known, see [7], [14], that the set of all functions that are regular in $\mathcal{U}$ with the usual normalization and such that $f(\mathcal{U})$ is a linearly accessible domain in the strict sense is identical with the set of close-to convex functions $\mathcal{K}$. Therefore we obtain the next corollary.

Corollary 4. $\mathcal{C}o(1) \subset \mathcal{K}$.

Let us consider the "central" function with respect to coefficient in the class $\mathcal{C}o(1)$

$$f_c(z) = \frac{1}{2}\left[\frac{z}{1-z} + \frac{z}{(1-z)^2}\right] = z + \sum_{n=1}^{\infty} \frac{1+n}{2} z^n. \tag{1.4}$$

In order to prove that $f_c \in \mathcal{C}o(1)$ notice that for $z = \exp(i\varphi)$, $\varphi \in [0, 2\pi)$, we have

$$f_c(e^{i\varphi}) = \frac{2e^{i\varphi} - e^{2i\varphi}}{2(1-e^{i\varphi})^2} = \frac{\cos\varphi - 2 + i\sin\varphi}{4(1-\cos\varphi)} =: x + iy.$$

Therefore the complement of $f_c(\mathcal{U})$ with respect to $\overline{\mathbf{C}}$ is a convex region bounded by the parabola $\gamma : y^2 = -\frac{1}{2}x - \frac{3}{16}$, $x \in (-\infty, -\frac{3}{8})$. Moreover $f_c(1) = \infty$ and $f_c(\mathcal{U})$ is a linearly accessible domain in the strict sense so it is univalent and close-to-convex, Fig.1.

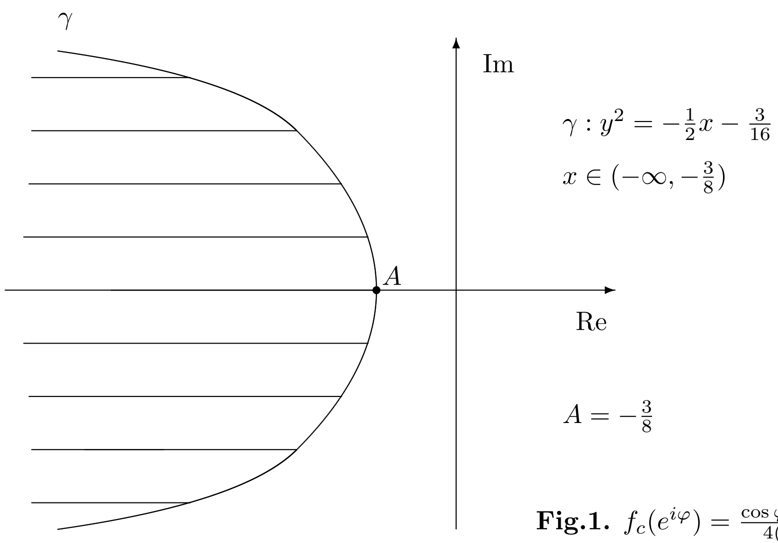

Fig.1. $f_c(e^{i\varphi}) = \frac{\cos\varphi - 2 + i\sin\varphi}{4(1-\cos\varphi)}$

Theorem 4. *We have the following relation*

$$\mathcal{C}o(1) \subset TN_\delta(f_c),$$

where $\delta = \frac{1}{2}(\frac{\pi^2}{6} - 1) = 0,32...$.

Proof. Let $f(z) = z + \sum_{n=2}^{\infty} a_n z^n$ be in $\mathcal{C}o(1)$. Thus from (1.2) we have

$$\frac{1}{n^2(n-1)} \left| a_n - \frac{n+1}{2} \right| \leq \frac{1}{2n^2}, \quad n \geq 2.$$

Therefore

$$\sum_{n=2}^{\infty} \frac{|a_n - (n+1)/2|}{n^2(n-1)} \leq \sum_{n=2}^{\infty} \frac{1}{2n^2} = \frac{1}{2}\left[\frac{\pi^2}{6} - 1\right],$$

so $f \in TN_\delta(f_c)$.

□

3. Open problem and conjectures

Conjecture 1. *Is it true that*

$$\delta^*(A, B) = \ln 4,$$

where A or B is one of the classes $\mathcal{S}^*$, $\mathcal{K}$, $\mathcal{S}$?

Conjecture 2. *Is it true that*

$$\delta^*(\mathcal{C}o(1), \mathcal{S}) = \delta^*(\mathcal{C}, \mathcal{S}) = \delta^*(\mathcal{C}, \mathcal{S}^*) = \delta^*(\mathcal{C}, \mathcal{K})$$

and it is equal $\ln 4 + \frac{\pi^2}{12} - 1$*?*

Open problem. *Are there the "central" functions satisfying an analogous relation as in Theorem 4 in the classes $\mathcal{C}$, $\mathcal{C}o(p)$, $\mathcal{S}^*$, $\mathcal{K}$ or $\mathcal{S}$?*

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Received 7.10.2009

Journal of
Mathematics
and Applications
No 32, pp 39−46 (2010)

Applications of a first order differential subordination for analytic functions

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ABSTRACT: In the paper, we define classes of analytic functions, in terms of subordination. By using Jack's Lemma and the Briot-Bouquet differential subordination we obtain some inclusion relations for defined classes

AMS Subject Classification: *30C45, 26A33*
Key Words and Phrases: *Analytic functions, differential subordination, Jack's Lemma, linear operator, convex functions*

1. Introduction

Let $\mathcal{A}$ denote the class of functions which are *analytic* in $\mathcal{U} := \mathcal{U}(1)$, where $\mathcal{U}(R) := \{z : |z| < R\}$, $0 < R \le 1$. By Ω we denote the class of the Schwarz functions, i.e. the class of functions $\omega \in \mathcal{A}$, such that

$$\omega(0) = 0, \ |\omega(z)| < 1 \quad (z \in \mathcal{U}).$$

For complex parameters β, γ and functions $h \in \mathcal{A}$, $\omega \in \Omega$, we consider the first-order differential equation of the form

$$q(z) + \frac{zq'(z)}{\beta q(z) + \gamma} = (h \circ \omega)(z), \quad q(0) = h(0) = 1. \tag{1}$$

If there exist a function $\omega \in \Omega$, such that the function $q \in \mathcal{A}$ is a solution of the Cauchy problem (1) then we write

$$q(z) + \frac{zq'(z)}{\beta q(z) + \gamma} \prec h(z). \tag{2}$$

The expression (2) is a first-order differential subordination and it is called the Briot-Bouquet differential subordination.

More general, we say that a function $f \in \mathcal{A}$ is *subordinate* to a function $F \in \mathcal{A}$, and write $f(z) \prec F(z)$, if and only if there exists a function $\omega \in \Omega$, such that

$$f(z) = (F \circ \omega)(z) \quad (z \in \mathcal{U}).$$

Moreover, we say that f is subordinate to F in $\mathcal{U}(R)$, if $f(Rz) \prec F(Rz)$. We shall write

$$f(z) \prec_R F(z)$$

in this case. In particular, if F is univalent in $\mathcal{U}$ we have the following equivalence (cf. [10]):

$$f(z) \prec F(z) \iff f(0) = F(0) \text{ and } f(\mathcal{U}) \subset F(\mathcal{U}).$$

Let $\mathcal{A}_0$ we denote class of functions $f \in A$ of the form:

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n, \tag{3}$$

By $f * g$ denote the *Hadamard product* (*or convolution*) of f, $g \in \mathcal{A}_0$, defined by

$$(f * g)(z) = \left(\sum_{n=1}^{\infty} a_n z^n\right) * \left(\sum_{n=1}^{\infty} b_n z^n\right) := \sum_{n=1}^{\infty} a_n b_n z^n.$$

Let λ, σ be complex numbers. We consider the linear operator $D_\sigma^\lambda : \mathcal{A}_0 \to \mathcal{A}_0$ defined by

$$D_\sigma^\lambda f(z) = \left(f * h_{\lambda,\sigma}\right)(z),$$

where

$$h_{\lambda,\sigma}(z) = \sum_{n=1}^{\infty} \left(\frac{n+\sigma}{1+\sigma}\right)^{\lambda} z^n \quad (z \in \mathcal{U}).$$

For a function $f \in \mathcal{A}_0$ we have

$$(1+\sigma) D_\sigma^{\lambda+1} f(z) = z\left[D_\sigma^\lambda f(z)\right]' + \sigma D_\sigma^\lambda f(z). \tag{4}$$

The linear operator D_σ^λ $(\lambda \in N)$ was introduced by Cho and Srivastava [1] (see also [13]). It is closely related to the multiplier transformations studied by Flett [5], and also to the differential-integral operator introduced by Sălăgean [11].

A function f belonging to the class $\mathcal{A}_0$ is said to be *starlike* in $\mathcal{U}(r)$ if and only if

$$\Re\left(\frac{zf'(z)}{f(z)}\right) > 0 \quad (z \in \mathcal{U}(r);\ 0 < r \leqq 1).$$

A function f belonging to the class $\mathcal{A}$ is said to be *convex* in $\mathcal{U}(r)$ if and only if

$$\Re\left(1 + \frac{zf''(z)}{f'(z)}\right) > 0 \quad (z \in \mathcal{U}(r);\ 0 < r \leqq 1).$$

Note that all functions starlike or convex in $\mathcal{U}(r)$ are univalent in $\mathcal{U}(r)$. Let h be a function convex in $\mathcal{U}$ with $h(0) = 1$, and let t be complex number. We denote by $\mathcal{V}(t, \lambda, \sigma; h)$ the class of functions $f \in \mathcal{A}_0$ satisfying the following condition:

$$z^{-1}\left[(1-t)\, D_\sigma^\lambda f(z) + tD_\sigma^{\lambda+1} f(z)\right] \prec h(z), \tag{5}$$

in terms of subordination.

Moreover, we define the class $\mathcal{W}(t, \lambda; h)$ of functions $f \in \mathcal{A}_0$ satisfying the following condition:

$$\frac{(1-t)\, D_\sigma^{\lambda+1} f(z) + tD_\sigma^{\lambda+2} f(z)}{(1-t)\, D_\sigma^\lambda f(z) + tD_\sigma^{\lambda+1} f(z)} \prec h(z). \tag{6}$$

In particular for real constants $A, B,\ -1 \leqq A < B \leq 1$, we denote

$$\begin{aligned}
\mathcal{V}(t, \lambda, \sigma; A, B) &= \mathcal{V}\left(t, \lambda, \sigma; \frac{1+Az}{1+Bz}\right), \\
\mathcal{W}(t, \lambda, \sigma; A, B) &= \mathcal{W}\left(t, \lambda, \sigma; \frac{1}{1+\sigma}\left(\frac{1+Az}{1+Bz} + \sigma\right)\right).
\end{aligned}$$

For suitable chosen parameters t, λ, σ, A, B classes defined above was investigated by many authors, see [1], [2], [3], [8], [9] and [12].

In the paper we present some inclusion relations for the defined classes.

2. Main results

We shall need the following lemmas.

Lemma 1.[7] *Let w be a nonconstant function analytic in $\mathcal{U}(r)$ with $w(0) = 0$. If*

$$|w(z_0)| = \max\{|w(z)|;\ |z| \leq |z_0|\} \quad (z_0 \in \mathcal{U}\mathrm{r})),$$

then there exists a real number k $(k \geq 1)$, such that

$$z_0 w'(z_0) = kw(z_0).$$

Lemma 2.[4] *Let h be a convex function in U, with*

$$\mathrm{Re}[\beta h(z) + \gamma] > 0 \quad (z \in \mathcal{U})$$

If a function q satisfies the Briot-Bouquet differential subordination (2) in $\mathcal{U}(R)$, i.e

$$q(z) + \frac{zq'(z)}{\beta q(z) + \gamma} \prec_R h(z),$$

then

$$q(z) \prec_R h(z).$$

Remark 1. If we put $\beta = 0,\ R = 1$ in Lemma 2 we obtain result due to Hallenbeck and Ruscheweyh **[6]**.

Making use of above lemmas, we get the following two theorems.

Theorem 1. *If* $\mathrm{Re}(\sigma) > -1$, *then*

$$\mathcal{V}(t, \lambda + m, \sigma; h) \subset \mathcal{V}(t, \lambda, \sigma; h) \quad (m \in \mathbf{N}).$$

Proof. It is clear that it is sufficient to prove the theorem for $m = 1$. Let a function f belong to the class $\mathcal{V}(t, \lambda + 1, \sigma; h)$ or equivalently

$$z^{-1}\left[(1-t)\, D_\sigma^{\lambda+1} f(z) + t D_\sigma^{\lambda+2} f(z)\right] \prec h(z). \tag{7}$$

It is sufficient to verify the condition (5). The function

$$q(z) = z^{-1}\left[(1-t)\, D_\sigma^{\lambda} f(z) + t D_\sigma^{\lambda+1} f(z)\right] \tag{8}$$

is analytic in $\mathcal{U}$ and $q(0) = 1$. Taking the derivative of (8) we get

$$z^{-1}\left[(1-t)\, D_\sigma^{\lambda+1} f(z) + t D_\sigma^{\lambda+2} f(z)\right] = q(z) + \frac{zq'(z)}{1+\sigma} \quad (z \in \mathcal{U})\,. \tag{9}$$

Thus by (7) we have

$$q(z) + \frac{zq'(z)}{1+\sigma} \prec h(z).$$

Lemma 2 now yields

$$q(z) \prec h(z).$$

Thus by (8) $f \in \mathcal{V}(t, \lambda, \sigma; h)$ and this proves Theorem 1.

Theorem 2. *If*

$$\mathrm{Re}[\,(1+\sigma)\, h(z)] > 0 \quad (z \in \mathcal{U})\,,$$

then

$$\mathcal{W}(t, \lambda{+}m, \sigma; h(z)) \subset \mathcal{W}(t, \lambda, \sigma; h(z)) \quad (m \in \mathbf{N}).$$

Proof. It is clear that it is sufficient to prove the theorem for $m = 1$. Let a function f belong to the class $\mathcal{W}(t, \lambda{+}1, \sigma; h(z))$ or equivalently

$$\frac{(1-t)\, D_\sigma^{\lambda+2} f(z) + t D_\sigma^{\lambda+3} f(z)}{(1-t)\, D_\sigma^{\lambda+1} f(z) + t D_\sigma^{\lambda+2} f(z)} \prec h(z) \tag{10}$$

It is sufficient to verify condition (6). If we put

$$R = \sup\left\{r :\ (1-t)\, D_\sigma^{\lambda} f(z) + t D_\sigma^{\lambda+1} f(z) \neq 0,\ z \in \mathcal{U}(r)\right\},$$

then the function

$$q(z)=\frac{(1-t)\,D_\sigma^{\lambda+1}f(z)+tD_\sigma^{\lambda+2}f(z)}{(1-t)\,D_\sigma^{\lambda}f(z)+tD_\sigma^{\lambda+1}f(z)} \tag{11}$$

is analytic in $\mathcal{U}(R)$ and $q(0)=1$. Taking the logarithmic derivative of (11) and applying (4) we get

$$\frac{(1-t)\,D_\sigma^{\lambda+2}f(z)+tD_\sigma^{\lambda+3}f(z)}{(1-t)\,D_\sigma^{\lambda+1}f(z)+tD_\sigma^{\lambda+2}f(z)}=q(z)+\frac{zq'(z)}{(1+\sigma)\,q(z)}\quad(z\in\mathcal{U}(R))\,. \tag{12}$$

Thus by (10) we have

$$q(z)+\frac{zq'(z)}{(1+\sigma)\,q(z)}\prec_R h(z).$$

Lemma 2 now yields

$$q(z)\prec_R h(z). \tag{13}$$

By (11) it suffices to verify that $R=1$. From (4), (11) and (13) we conclude that the function $H(z)=(1-t)\,D_\sigma^{\lambda}f(z)+tD_\sigma^{\lambda+1}f(z)$ is starlike in $\mathcal{U}(R)$ and consequently it is univalent in $\mathcal{U}(R)$. Thus we see that $H(z)$ cannot vanish on $|z|=R$ if $R<1$. Hence $R=1$ and this proves Theorem 1.

Putting $h(z)=\frac{1+Az}{1+Bz}$ and $h(z)=\frac{1}{1+\sigma}\left(\frac{1+Az}{1+Bz}+\sigma\right)$ in Theorems 1 and 2, respectively, we obtain the following two corollaries.

Corollary 1. *If* $\operatorname{Re}(\sigma)>-1$*, then*

$$\mathcal{V}(t,\lambda+m,\sigma;A,B)\subset\mathcal{V}(t,\lambda,\sigma;A,B)\quad(m\in\mathbf{N}).$$

Corollary 2. *If* $\sigma>0$*, then*

$$\mathcal{W}(t,\lambda+m,\sigma;A,B)\subset\mathcal{W}(t,\lambda,\sigma;A,B)\quad(m\in\mathbf{N}).$$

Using Lemma 1 we show the following sufficient conditions for functions to belong to the class $\mathcal{W}(t,\lambda;A,B)$.

Theorem 3. *Let* σ,A,B *be real numbers,* $\sigma>-1$, $0\le B\le 1$, $-B\le A<2AB-B$. *If a function* $f\in\mathcal{A}_0$ *satisfies the following inequality:*

$$\left|\frac{(1-t)\,D_\sigma^{\lambda+2}f(z)+tD_\sigma^{\lambda+3}f(z)}{(1-t)\,D_\sigma^{\lambda+1}f(z)+tD_\sigma^{\lambda+2}f(z)}-1\right|<\frac{(B-A)\,(2+\sigma-\sigma A-A)-2AB}{(1+\sigma)\,(1+B)\,(1-A)}\quad(z\in\mathcal{U})\,, \tag{14}$$

then f *belongs to the class* $\mathcal{W}(t,\lambda;A,B)$.

Proof. Let a function f belong to the class $\mathcal{A}_0$. Putting

$$q(z)=\frac{1+Aw(z)}{1+Bw(z)}\quad(z\in\mathcal{U}(R)) \tag{15}$$

in (12), we obtain

$$\frac{(1-t)\,D_{\sigma}^{\lambda+2}f(z)+tD_{\sigma}^{\lambda+3}f(z)}{(1-t)\,D_{\sigma}^{\lambda+1}f(z)+tD_{\sigma}^{\lambda+2}f(z)}=\frac{1+Aw(z)}{1+Bw(z)}+\frac{1}{1+\sigma}\left(\frac{Azw'(z)}{1+Aw(z)}-\frac{Bzw'(z)}{1+Bw(z)}\right).$$

Consequently, we have

$$F(z)=w(z)\left\{\frac{zw'(z)}{(1+\sigma)\,w(z)}\left(\frac{A}{1+Aw(z)}-\frac{B}{1+Bw(z)}\right)-\frac{B-A}{1+Bw(z)}\right\}, \tag{16}$$

where

$$F(z)=\frac{(1-t)\,D_{\sigma}^{\lambda+2}f(z)+tD_{\sigma}^{\lambda+3}f(z)}{(1-t)\,D_{\sigma}^{\lambda+1}f(z)+tD_{\sigma}^{\lambda+2}f(z)}-1.$$

By (6), (11) and (15) it is sufficient to verify that w is analytic in U and

$$|w(z)|<1 \quad (z\in\mathcal{U}).$$

Now, suppose that there exists a point $z_0\in\mathcal{U}(R)$, such that

$$|w(z_0)|=1,\ |w(z)|<1 \ \ (|z|<|z_0|).$$

Then, applying Lemma 1, we can write

$$z_0w'(z_0)=kw(z_0),\ w(z_0)=e^{i\theta} \quad (k\geq 1).$$

Combining these with (16) , we obtain

$$\begin{aligned}
|F(z_0)| &= \left|\frac{k}{1+\sigma}\left(\frac{-A}{1+Ae^{i\theta}}+\frac{B}{1+Be^{i\theta}}\right)+\frac{B-A}{1+Be^{i\theta}}\right|\\
&\geq \frac{k}{1+\sigma}\mathrm{Re}\left(\frac{-A}{1+Ae^{i\theta}}+\frac{B}{1+Be^{i\theta}}\right)+\frac{B-A}{1+B}\\
&\geq \frac{k}{1+\sigma}\left(\frac{-A}{1-A}+\frac{B}{1+B}\right)+\frac{B-A}{1+B}\\
&\geq \frac{(B-A)\,(2+\sigma-\sigma A-A)-2AB}{(1+\sigma)\,(1+B)\,(1-A)}.
\end{aligned}$$

Since this result contradicts (14) we conclude that w is the analytic function in $\mathcal{U}(R)$ and $|w(z)|<1$ $(z\in\mathcal{U}\,(R))$. Applying the same methods as in the proof of Theorem 2 we obtain $R=1$, which completes the proof of Theorem 3.

Putting $t=0,\ A=2\alpha-1$ and $B=1$ in Corollaries 1 and 2 and Theorem 3 we obtain following relationships for the operator D_{σ}^{λ}.

Corollary 3. *Let $Re\,(\sigma)>-1$, $m\in\mathbf{N}$. If a function $f\in A_0$ satisfies the following inequality:*

$$\mathrm{Re}\left(\frac{D_{\sigma}^{\lambda+m}f(z)}{z}\right)>\alpha,$$

then

$$\mathrm{Re}\left(\frac{D_\sigma^\lambda f(z)}{z}\right) > \alpha.$$

Corollary 4. *Let* $0 \le \alpha < 1$, $\sigma > 0$ *and* $m \in \mathbf{N}$. *If a function* $f \in \mathcal{A}_0$ *satisfies the following inequality:*

$$\mathrm{Re}\left\{\frac{D_\sigma^{\lambda+m+1} f(z)}{D_\sigma^{\lambda+m} f(z)}\right\} > \frac{\alpha+\sigma}{1+\sigma} \quad (z \in \mathcal{U}),$$

then

$$\mathrm{Re}\left\{\frac{D_\sigma^{\lambda+1} f(z)}{D_\sigma^{\lambda} f(z)}\right\} > \frac{\alpha+\sigma}{1+\sigma} \quad (z \in \mathcal{U}).$$

Corollary 5. *Let* $m \in \mathbf{N}$, $\sigma > 0$ *and* $0 \le \alpha < 2/3$. *If a function* $f \in \mathcal{A}_0$ *satisfies the following inequality:*

$$\left|\frac{D_\sigma^{\lambda+2} f(z)}{D_\sigma^{\lambda+1} f(z)} - 1\right| < 1 + (1+\sigma)(1-\alpha) - \frac{\alpha}{2(1-\alpha)} \quad (z \in \mathcal{U}),$$

then

$$\mathrm{Re}\left\{\frac{D_\sigma^{\lambda+1} f(z)}{D_\sigma^{\lambda} f(z)}\right\} > \frac{\alpha+\sigma}{1+\sigma} \quad (z \in \mathcal{U}).$$

Remark 2. Putting $\sigma = 0$ in Corollary 3, 4 and 5 we can obtain the results for the Sălăgean operator, which was introduced by Sălăgean[11]. Putting moreover $\lambda = 0$ or $\lambda = 1$ and $m = 1$ in Corollary 4 and 5 we obtain the sufficient conditions for starlikeness of order α and convexity of order α, respectively.

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Received 21.04.2009

J o u r n a l of
Mathematics
and Applications
No 32, pp 47-52 (2010)

Some remarks on a fixed point property for multivalued mapping

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Abstract: In [5] T. Domínguez Benavides and B. Gavira proved that Banach spaces with $\rho_X^{'}(0) < 1/2$ satisfy the fixed point property for nonexpansive compact convex valued multivalued mappings. We give some simplification of the proof of this theorem

AMS Subject Classification: *Primary 47H10, Secondary 47H04, 47H09*
Key Words and Phrases: *Multivalued mappings, Fixed point property, Nonexpansive mappings, Uniform smoothness*

Introduction

A Fixed Point Theory for multivalued mappings (that is, mappings such that the image of a point is a set) has many applications in Applied Sciences. Thus, the extension of the known fixed point results for singlevalued mappings to the setting of multivalued mappings looks like a very natural problem. Some theorems about the existence of fixed point for singlevalued nonexpansive mappings have already been extended to the multivalued case, In 1969 S.B. Nadler [12] extended the Banach Contraction Mapping Principle in complete metric space. However, many other questions remain open, for instance, the possibility of extending the well-known Kirk's Theorem, that is, whether reflexive Banach spaces with normal structure have the fixed point property for multivalued nonexpansive mappings. Until now, the answer is unknown. Since there exist different geometrical properties of Banach spaces which imply normal structure and reflexivity (for example, uniform convexity, uniform smoothness and nearly uniform convexity), it is natural to study if these properties imply the FPP for multivalued nonexpansive mappings. In 1974 T.C. Lim [10] proved the existence of a fixed point for a nonexpansive mapping defined from a closed bounded convex subset C of a uniformly convex Banach space X into the compact subsets of C. The original proof of Lim's Theorem combined Edelstein's method of asymptotic centers and transfinite induction. The study of certain properties of the asymptotic center of sequences led W.A. Kirk and S. Massa in 1990 to a generalization of Lim's Theorem, proving the existence of a fixed point for a nonexpansive mapping defined from a closed bounded convex subset C of a Banach space into compact convex subsets of C, under the hypothesis of compactness of the asymptotic center in C of every bounded sequence. The example given by T. Kuczumow and S. Prus [9] shows that this method gives no results in nearly uniformly convex spaces.

1. Preliminaries

In this section we introduce some notions and known results related to the existence of fixed points for multivalued nonexpansive mappings. We denote by $CB(X)$ the family of all nonempty closed bounded subsets of Banach space X and by $K(X)$ (resp. $KC(X)$) the family of all nonempty compact (resp. compact convex) subsets of X. On $CB(X)$ we have the Hausdorff metric H given by

$$H(A,B) = \max\left\{\sup_{a\in A} d(a,B), \sup_{b\in B} d(b,A)\right\}, \quad A,B \in CB(X)$$

where for $x \in X$ and $A \subset X$ $d(x,A) = \inf\{\|x-y\| : y \in A\}$.

Definition 1.1 *A multivalued mapping $T : C \to CB(X)$ is said to be k-contractive if*

$$H(Tx,Ty) \le k\,\|x-y\|, \quad x,y \in C, k \in [0,1).$$

and T is said to be nonexpansive if

$$H(Tx,Ty) \le \|x-y\|, \quad x,y \in C.$$

The concept of asymptotic center of sequences was firstly considered by M. Edelstein [6] and later it was extended to nets by T.C. Lim [11].

Definition 1.2 *Let C be a nonempty subset of a Banach space X, D a directed set and $\{x_\alpha\}_{\alpha\in D}$ a bounded net in X. The asymptotic radius and the asymptotic center of the net $\{x_\alpha\}_{\alpha\in D}$ in C are defined by*

$$\begin{aligned} r(C,\{x_\alpha\}) &= \inf\left\{\limsup_{\alpha} \|x_\alpha - x\| : x \in C\right\}, \\ A(C,\{x_\alpha\}) &= \left\{x \in C : \limsup_{\alpha} \|x_\alpha - x\| = r(C,\{x_\alpha\})\right\}. \end{aligned}$$

Recall that, if D is a bounded subset of X, the Chebyshev radius of D relative to a set C is defined by

$$r_c(D) = \inf\{\sup\{\|x-y\| : y \in D\} : x \in C\}.$$

Now we recall the $(DL)_\alpha$-condition (see [7]), which is the main tool in order to assure the existence of fixed point.

Definition 1.3 *A Banach space X is said to satisfy the $(DL)_\alpha$-condition with respect to a topology τ if there exists $\lambda \in [0,1)$ such that for every τ-compact convex subset C of X and for every bounded ultranet $\{x_\alpha\}$ in C*

$$r_c(A(C, \{x_\alpha\})) \leq \lambda r(C, \{x_\alpha\}).$$

Theorem 1.1 *Let C be a nonempty τ-compact closed bounded convex subset of a Banach space X and $T : C \to KC(C)$ be a nonexpansive mapping. If X satisfies the $(DL)_\alpha$-condition with respect to a topology τ, then T has a fixed point.*

Let us consider now the concept of uniform smoothness of Banach space.

Definition 1.4 *A Banach space X is said to be uniformly smooth if*

$$\rho_X^{'}(0) = 0,$$

where ρ_X is the modulus of smoothness of X, defined for $\lambda \geq 0$ by

$$\rho_X(\lambda) = \sup\left\{\frac{1}{2}(\|x + \lambda y\| + \|x - \lambda y\|) - 1 : \|x\| \leq 1 \, \|y\| \leq 1\right\}.$$

Its known (see [13]) that if $\rho_X^{'}(0) < 1/2$, then X is reflexive and has uniform normal structure.

2. Main result

In [5] T. Domínguez Benavides and B. Gavira proved that uniformly smooth Banach spaces satisfy the fixed point property for multivalued mappings. Indeed they proved this fact under weaker assumption $\rho_X^{'}(0) < 1/2$ (recall that X is uniformly smooth if and only if $\rho_X^{'}(0) = 0$). We give now some simplifications of the proof of this theorem.

Theorem 2.1 *Let C be a nonempty, convex, closed, bounded subset of a Banach space X such that $\rho^{'}(0) < 1/2$, and $T : C \to KC(C)$ be a nonexpansive mapping. Then T has a fixed point, that is, there exist $x \in C$ such that $x \in Tx$.*

Proof. Let $\{x_\alpha\}_{\alpha \in D}$ be a ultranet in $\mathcal{C}$. Denote $r = r(C, \{x_\alpha\})$, $A = A(C, \{x_\alpha\}$ and $r_C = r_C(A(C, \{x_\alpha\})$. Since C is a ω-compact set, the ultranet $\{x_\alpha\}_{\alpha \in D}$ is weakly convergent to a point $x \in C$. Furthermore, $\lim_{\alpha \in \mathcal{D}} \|x_\alpha - x\|$ exists for each $x \in C$. Let $z_1 \in A$ fixed and let $z \in A$ arbitrary. Then we have $\lim_\alpha \|x_\alpha - z_1\| = \lim_\alpha \|x_\alpha - z\| = r$. Let $\mathcal{U}$ be a free ultrafilter on the set $\mathcal{D}$ containing the filter generated by $\mathcal{B} = \{B_\alpha : \alpha \in \mathcal{D}\}$ with $B_\alpha = \{\beta \in \mathcal{D} : \beta \geq \alpha\}$. In the ultrapower $X_\mathcal{U}$ of X we consider

$$\widetilde{v} = \frac{1}{r}\{x_\alpha - z\}_\mathcal{U} \in S_{X_\mathcal{U}} \quad \widetilde{w} = \frac{1}{r}\{x_\alpha - z_1\}_\mathcal{U} \in S_{X_\mathcal{U}}.$$

Then for arbitrary $m \geq 1$ and $\beta \in (0,1)$ we have

$$\begin{aligned}
\|\widetilde{v} - \beta\widetilde{w}\|_{\mathcal{U}} &= \frac{1}{r}\lim_{\mathcal{U}} \|x_\alpha - z - \beta(x_\alpha - z_1)\| \\
&= \frac{1}{r}c\left\|x_\alpha - z - \frac{m-1}{m}\beta(x_\alpha - z_1) - \frac{1}{m}\beta(x_\alpha - z_1)\right\| \\
&\geq \frac{1}{r}\lim_{\alpha}\left\|x_\alpha - z - \frac{m-1}{m}\beta(x_\alpha - z_1)\right\| - \frac{1}{r}\frac{\beta}{m}\lim_{\alpha}\|x_\alpha - z_1\| \\
&\geq \frac{1}{r}\left\|(1 - \frac{m-1}{m}\beta)x + \frac{m-1}{m}\beta z_1 - z\right\| - \frac{\beta}{m}.
\end{aligned}$$

On the other hand,

$$\frac{1}{1+\beta}z + \frac{\beta}{1+\beta}z_1 \in A \text{ for every } \beta > 0,$$

and so we have

$$\begin{aligned}
\|\widetilde{v} + \beta\widetilde{w}\|_{\mathcal{U}} &= \frac{1}{r}\lim_{\mathcal{U}} \|x_\alpha - z + \beta(x_\alpha - z_1)\| \\
&= \frac{1}{r}\lim_{\alpha}(1+\beta)\left\|x_\alpha - (\frac{1}{1+\beta}z + \frac{\beta}{1+\beta}z_1)\right\| = 1+\beta.
\end{aligned}$$

Thus, we deduce

$$\begin{aligned}
&\|\widetilde{v} - \beta\widetilde{w}\|_{\mathcal{U}} + \|\widetilde{v} + \beta\widetilde{w}\|_{\mathcal{U}} \\
&\quad\geq \frac{1}{r}\left\|(1 - \frac{m-1}{m}\beta)x + \frac{m-1}{m}\beta z_1 - z\right\| - \frac{\beta}{m} + 1 + \beta
\end{aligned}$$

for every $z \in A$. Then we have

$$\begin{aligned}
2(\rho_{X_{\mathcal{U}}}(\beta) + 1) &\geq \frac{1}{r}\sup_{z\in A}\left\|(1 - \frac{m-1}{m}\beta)x + \frac{m-1}{m}\beta z_1 - z\right\| - \frac{\beta}{m} + 1 + \beta \\
&\geq \frac{r_C}{r} - \frac{\beta}{m} + 1 + \beta,\ m \geq 1.
\end{aligned}$$

The last inequality is true for every $m \geq 1$, so we obtain the following inequality

$$\rho_{X_{\mathcal{U}}}(\beta) \geq \frac{r_C}{2r} + \frac{\beta}{2} - \frac{1}{2}.$$

Since $X_{\mathcal{U}}$ is finitely representable in X, we have $\rho_{X_{\mathcal{U}}}(\beta) = \rho_X(\beta)$ for all β and thus

$$\begin{aligned}
\rho_X(\beta) &\geq \frac{r_C}{2r} + \frac{\beta}{2} - \frac{1}{2}, \\
r_C &\leq (2\rho_X(\beta) - \beta + 1)\, r.
\end{aligned}$$

Using the previous inequality, we can deduce that if $\rho^{'}(0) < 1/2$ (which is equivalent to $\rho_X(\beta) < \beta/2$ for some β, because $\rho_X(\beta)/\beta$ is increasing) then

$$2\rho_X(\beta) - \beta + 1 < 1 \text{ for some } \beta \in (0,1)\,.$$

The condition $\rho^{'}(0) < 1/2$ implies that X is reflexive. Thus, as a consequence of Theorem 1.4, we obtain a sufficient condition so that a Banach space X has the fixed point property for multivalued nonexpansive mappings. ■

Remark 2.1 *In original proof authors used the modulus of squareness which was given by C. Beníytez, K. Przeslawski and D. Yost in [3].*

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Received 20.01.2009

Journal of
Mathematics
and Applications
No 32, pp 53-61 (2010)

On a certain class of analytic functions involving hypergeometric functions

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Abstract: A certain class of functions analytic in the open unit disk and defined in terms of hypergeometric functions is introduced and investigated. We establish starlikeness, convexity and spirallikeness properties for this class of functions. Special cases and some useful consequences of our main results are aslo mentioned

AMS Subject Classification: *Primary 26A33, 30C45, 33C20, Secondary 30C75*
Key Words and Phrases: *Starlike functions; Convex functions; Spirallike functions; Hypergeometric Functions*

1. Introduction and Preliminaries

Let $\mathcal{A}$ be the class of *normalized* functions $F(z)$ of the form

$$F(z) = z + \sum_{k=2}^{\infty} a_k z^k, \tag{1.1}$$

which are analytic in the open unit disk $\mathcal{U} = \{z : z \in \mathbb{C}, |z| < 1\}$. The subclasses of the class $\mathcal{A}$ denoted by $\mathcal{S}^*(\alpha)$, $\mathcal{K}(\alpha)$ and $\mathcal{S}_p(\mu, \alpha)$ are, respectively, the subclasses of starlike functions of order α $(0 \leq \alpha < 1)$ in $\mathcal{U}$, the convex functions of order α $(0 \leq \alpha < 1)$ in $\mathcal{U}$, and the μ-spirallike functions of order α $(0 \leq \alpha < 1)$ in $\mathcal{U}$.

Corresponding to the functions given by

$$f(z) = 1 + \sum_{k=1}^{\infty} d_k z^k \text{ and } g(z) = 1 + \sum_{k=1}^{\infty} e_k z^k, \tag{1.2}$$

we introduce and investigate a class of functions $\mathcal{H}^{\lambda}_{a_1,b_1,c_1;a_2,b_2;c_2}(z)$ defined by

$$\mathcal{H}(z) = \mathcal{H}^{\lambda}_{a_1,b_1,c_1;a_2,b_2;c_2}(z) = \frac{({}_2F_1(a_1, b_1; c_1; z) * f(z)) - 1}{({}_2F_1(a_2, b_2; c_2; z) * g(z))^{\lambda}} \ (\lambda > 0;\ a_1 b_1 d_1 = c_1), \tag{1.3}$$

which is analytic in the open unit disc $\mathcal{U}$. We assume here and throughout this paper that the functions occurring in the numerator and denominator of the right-hand side of (1.3) are well defined in the open unit disk $\mathcal{U}$, and that

$$a_i > 0, b_i > 0, c_i > 0 \; (i = 1, 2).$$

The function ${}_2F_1(a, b; c; z)$ occurring in (1) is the Gaussian hypergeometric function defined by ([5])

$${}_2F_1(a, b; c; z) = \sum_{k=0}^{\infty} \frac{(a)_k (b)_k z^k}{(c)_k (1)_k} \quad (z \in \mathcal{U}),$$

where $a, b, c \in \mathbb{C}$ $(c \neq 0, -1, -2, ...)$, and $(a)_k$ is the Pochhammer symbol (or shifted factorial when $a \in \mathbb{N}$) defined in terms of the Gamma functions by

$$(a)_k = \frac{\Gamma(a+k)}{\Gamma(a)} = \begin{cases} 1, & k=0; \\ a(a+1)...(a+k-1), & k \in \mathbb{N}. \end{cases}$$

By applying elementary calculations, we observe (see below (2.3) also) that

$$\mathcal{H}(0) = \mathcal{H}'(0) - 1 = 0,$$

which asserts that the class $\mathcal{H}(z) \in \mathcal{A}$. On putting $a_1 = a_2 = d_1 = 1$, $b_1 = c_1$, $b_2 = c_2$ and $d_j = 0$ $(j \geq 2)$, we can write

$$F^{\lambda}(z) = \frac{z}{(1 + \sum\limits_{k=1}^{\infty} e_k z^k)^{\lambda}}, \tag{1.4}$$

which was investigated by Raina and Bansal [3] and contains as special cases the classes due to Fukui *et al.* [1], Mitrinovic [2] and Reade *et al.* [4]. In this paper we investigate the geometric properties of starlikeness, convexity and spirallikeness for the function class $\mathcal{H}(z)$.

Throughout this paper (for convenience sake), we let F_k and G_k denote the following:

$$F_k = \frac{(a_1)_k (b_1)_k}{(c_1)_k (1)_k}, \quad G_k = \frac{(a_2)_k (b_2)_k}{(c_2)_k (1)_k}. \tag{1.5}$$

2. Main Results

Theorem 1. *Let $\mathcal{H}(z)$ be defined by* (1.3). *Then $\mathcal{H}(z) \in S^*(\alpha)$ $(0 \leq \alpha < 1)$, provided that*

$$\sum_{k=2}^{\infty} F_k |d_k| \left(k - \alpha + \lambda \sum_{k=1}^{\infty} k G_k |e_k| \right) + \frac{1}{2} \sum_{k=1}^{\infty} A_k G_k |e_k|$$

$$+\sum_{k=1}^{\infty} G_k|e_k| \sum_{k=2}^{\infty}(k-\alpha)F_k|d_k| \leq 1-\alpha, \tag{2.1}$$

where F_k and G_k are given by (1.5) *and* $A_k = k\lambda + |k\lambda - 2(1-\alpha)|$.

Proof. In order to prove that $\mathcal{H}(z) \in \mathcal{S}^*(\alpha)$ $(0 \leq \alpha < 1)$, it is sufficient to show that

$$\left|\frac{1-\frac{z\mathcal{H}'(z)}{\mathcal{H}(z)}}{1-2\alpha+\frac{z\mathcal{H}'(z)}{\mathcal{H}(z)}}\right| < 1 \quad (z \in \mathcal{U}). \tag{2.2}$$

Differentiating (1.3) with respect to z, we get

$$\frac{z\mathcal{H}'(z)}{\mathcal{H}(z)} = \frac{\sum\limits_{k=1}^{\infty} kF_k d_k z^k}{\sum\limits_{k=1}^{\infty} F_k d_k z^k} - \lambda \frac{\sum\limits_{k=1}^{\infty} kG_k e_k z^k}{1+\sum\limits_{k=1}^{\infty} G_k e_k z^k}. \tag{2.3}$$

Using (2.2) and (2.3), we get

$$\left|\frac{1-\frac{z\mathcal{H}'(z)}{\mathcal{H}(z)}}{1-2\alpha+\frac{z\mathcal{H}'(z)}{\mathcal{H}(z)}}\right| = \left|\frac{C(z)}{D(z)}\right|,$$

where

$$C(z) = \sum_{k=2}^{\infty}(1-k)F_k d_k z^k \left(1+\sum_{k=1}^{\infty} G_k e_k z^k\right) + \lambda \sum_{k=1}^{\infty} kG_k e_k z^k \sum_{k=1}^{\infty} F_k d_k z^k$$

and

$$D(z) = \sum_{k=1}^{\infty}(1-2\alpha+k)\, F_k d_k z^k \left(1+\sum_{k=1}^{\infty} G_k e_k z^k\right) - \lambda \sum_{k=1}^{\infty} kG_k e_k z^k \sum_{k=1}^{\infty} F_k d_k z^k.$$

It follows that

$$\begin{aligned} |C(z)| \quad \leq \quad & \sum_{k=2}^{\infty}(k-1)F_k|d_k| + \lambda\sum_{k=1}^{\infty} kG_k|e_k| \\ & + \sum_{k=2}^{\infty}(k-1)F_k|d_k| \sum_{k=1}^{\infty} G_k|e_k| + \lambda \sum_{k=1}^{\infty} kG_k|e_k| \sum_{k=2}^{\infty} F_k|d_k| \end{aligned} \tag{2.4}$$

and

$$\begin{aligned} |D(z)| \quad \geq \quad & 2(1-\alpha) - \sum_{k=2}^{\infty}(1-2\alpha+k)F_k|d_k| - \sum_{k=1}^{\infty} |k\lambda - 2(1-\alpha)|\, G_k|e_k| \\ & - \sum_{k=1}^{\infty} G_k|e_k| \sum_{k=2}^{\infty}(1-2\alpha+k)\, F_k|d_k| - \lambda \sum_{k=1}^{\infty} kG_k|e_k| \sum_{k=2}^{\infty} F_k|d_k|. \end{aligned} \tag{2.5}$$

Making use of the inequalities (2.1), (2.4) and (2.5), the assertion (2.2) is established which proves Theorem 1.

The next result gives sufficient conditions such that the function $\mathcal{H}(z)$ defined by (1.3) belongs to $\mathcal{K}(\alpha)$ $(0 \leq \alpha < 1)$.

Theorem 2. *Let $\mathcal{H}(z)$ be defined by* (1.3). *Then $\mathcal{H}(z) \in \mathcal{K}(\alpha)$ $(0 \leq \alpha < 1)$, provided that there exist numbers $p, q > 0$, where $\frac{1}{p} + \frac{1}{q} \leq 1$, satisfying the following inequalities:*

$$\sum_{k=1}^{\infty} [pk(\lambda+1) + 1 - \alpha]\, G_k |e_k| \leq 1 - \alpha \tag{2.6}$$

and

$$\begin{aligned}
&\sum_{k=1}^{\infty} |k\lambda - 1|\, (kq + 1 - \alpha)\, G_k |e_k| \\
&\quad + \left(1 + \sum_{k=1}^{\infty} G_k |e_k|\right) \sum_{k=2}^{\infty} k(1 - \alpha + q(k-1)) F_k |d_k| \\
&\quad + q\lambda \sum_{k=1}^{\infty} k^2 G_k\, |e_k| \sum_{k=2}^{\infty} F_k\, |d_k| \\
&\quad + \sum_{k=1}^{\infty} kG_k\, |e_k| \sum_{k=2}^{\infty} (q\, |\lambda + k(1-\lambda)| + \lambda(1-\alpha)) F_k\, |d_k| \leq 1 - \alpha.
\end{aligned} \tag{2.7}$$

Proof. Let the inequalities (2.6) and (2.7) be satisfied for the function $\mathcal{H}(z)$. We prove that

$$\Re\left\{1 + \frac{z\mathcal{H}''(z)}{\mathcal{H}'(z)}\right\} > \alpha \quad (z \in \mathcal{U}). \tag{2.8}$$

After some calculations, we get

$$1 + \frac{z\mathcal{H}''(z)}{\mathcal{H}'(z)} = 1 - \left[(\lambda+1)\frac{\sum\limits_{k=1}^{\infty} kG_k e_k z^k}{1 + \sum\limits_{k=1}^{\infty} G_k e_k z^k} + \frac{M(z)}{N(z)}\right],$$

where

$$\begin{aligned}
M(z) \quad = \quad & \lambda \sum_{k=1}^{\infty} k^2 G_k e_k z^{k-1} \sum_{k=1}^{\infty} F_k d_k z^k - \sum_{k=1}^{\infty} k(k-1) F_k d_k z^{k-1} \left(1 + \sum_{k=1}^{\infty} G_k e_k z^k\right) \\
& - \sum_{k=1}^{\infty} kG_k e_k z^{k-1} \sum_{k=1}^{\infty} F_k d_k (\lambda + k(1-\lambda)) z^k.
\end{aligned}$$

and

$$\begin{aligned} N(z) &= z+\sum_{k=2}^{\infty} kF_k d_k z^k + \sum_{k=1}^{\infty} G_k e_k z^{k+1} + \sum_{k=1}^{\infty} G_k e_k z^k \sum_{k=2}^{\infty} kF_k d_k z^k \\ &\quad -\lambda\left[\sum_{k=1}^{\infty} kG_k e_k z^{k-1} + \sum_{k=1}^{\infty} kG_k e_k z^{k-1} \sum_{k=2}^{\infty} F_k d_k z^k\right]. \end{aligned}$$

It readily follows that

$$\begin{aligned} \Re\left\{1+\frac{z\mathcal{H}''(z)}{\mathcal{H}'(z)}\right\} &= 1-\Re\left[(\lambda+1)\frac{\sum\limits_{k=1}^{\infty} kG_k e_k z^k}{1+\sum\limits_{k=1}^{\infty} G_k e_k z^k} + \frac{M(z)}{N(z)}\right] \\ &\geq 1-\left|\frac{(\lambda+1)\sum\limits_{k=1}^{\infty} kG_k e_k z^k}{1+\sum\limits_{k=1}^{\infty} G_k e_k z^k}\right| - \left|\frac{M(z)}{N(z)}\right| \end{aligned} \tag{2.9}$$

and in view of (2.6), we infer that

$$\left|\frac{(\lambda+1)\sum\limits_{k=1}^{\infty} kG_k e_k z^k}{1+\sum\limits_{k=1}^{\infty} G_k e_k z^k}\right| \leq \frac{(\lambda+1)\sum\limits_{k=1}^{\infty} kG_k |e_k|}{1-\sum\limits_{k=1}^{\infty} G_k |e_k|} \leq \frac{1-\alpha}{p} \quad (0\leq\alpha<1). \tag{2.10}$$

Also

$$\begin{aligned} |M(z)| &\leq \sum_{k=1}^{\infty} |k(k\lambda-1)| G_k\,|e_k| + \lambda\sum_{k=1}^{\infty} k^2 G_k\,|e_k| \sum_{k=2}^{\infty} F_k\,|d_k| \\ &\quad + \sum_{k=2}^{\infty} k(k-1)F_k\,|d_k|\left(1+\sum_{k=1}^{\infty} G_k\,|e_k|\right) \\ &\quad + \sum_{k=1}^{\infty} kG_k\,|e_k| \sum_{k=2}^{\infty} |\lambda+k(1-\lambda)|\,F_k\,|d_k| \end{aligned}$$

and

$$\begin{aligned} |N(z)| &\geq \left(1-\sum_{k=1}^{\infty} |k\lambda-1|\,G_k\,|e_k|\right) - \sum_{k=2}^{\infty} kF_k\,|d_k|\left(1+\sum_{k=1}^{\infty} G_k\,|e_k|\right) \\ &\quad -\lambda\sum_{k=1}^{\infty} kG_k\,|e_k| \sum_{k=2}^{\infty} F_k\,|d_k|. \end{aligned}$$

Making use of (2.7), we get

$$\left|\frac{M(z)}{N(z)}\right| \leq \frac{1-\alpha}{q}. \tag{2.11}$$

Applying (2.9) to (2.11), we conclude that the inequality (2.8) holds true and the proof is complete.

Our next theorem gives a sufficient condition under which the function $\mathcal{H}(z)$ defined by (1.3) is a $\mu-$spirallike function of order α $(0 \leq \alpha < 1)$.

Theorem 3. *Let $\mathcal{H}(z)$ be defined by (1.3). Then $\mathcal{H}(z) \in \mathcal{S}_p(\mu, \alpha)$ ($0 \leq \alpha < 1$, $|\mu| < \frac{\pi}{2}$), provided that*

$$\begin{aligned} &\sum_{k=2}^{\infty} \left(\left|1 - ke^{i\mu}\right| + |A_k|\right) F_k|d_k| \\ &\quad + \sum_{k=1}^{\infty} \left(\left|1 + e^{i\mu}(k\lambda - 1)\right| + \left|1 - 2\alpha cos\mu - e^{i\mu}(k\lambda - 1)\right|\right) G_k|e_k| \\ &\quad + \sum_{k=1}^{\infty} G_k|e_k| \sum_{k=2}^{\infty} \left(\left|1 - ke^{i\mu}\right| + |A_k|\right) F_k|d_k| \\ &\quad + 2\lambda \sum_{k=1}^{\infty} kG_k|e_k| \sum_{k=2}^{\infty} F_k|d_k| \\ \leq\ & |A_1| - \left|1 - e^{i\mu}\right|, \end{aligned} \tag{2.12}$$

where $A_k = 1 - 2\alpha cos\mu + ke^{i\mu}$.

Proof. Suppose the inequality (2.12) holds true. We prove that $\mathcal{H}(z) \in \mathcal{S}_p(\mu, \alpha)$ $(0 \leq \alpha < 1, |\mu| < \frac{\pi}{2})$. It is sufficient to show that

$$\left| \frac{1 - e^{i\mu} \frac{z\mathcal{H}'(z)}{\mathcal{H}(z)}}{(1 - 2\alpha cos\mu) + e^{i\mu} \frac{z\mathcal{H}'(z)}{\mathcal{H}(z)}} \right| < 1 \ \ (0 \leq \alpha < 1, |\mu| < \frac{\pi}{2}, z \in \mathcal{U}). \tag{2.13}$$

From (2.13), we obtain

$$\left| \frac{1 - e^{i\mu} \frac{z\mathcal{H}'(z)}{\mathcal{H}(z)}}{(1 - 2\alpha cos\mu) + e^{i\mu} \frac{z\mathcal{H}'(z)}{\mathcal{H}(z)}} \right| = \left| \frac{L(z)}{Q(z)} \right|, \tag{2.14}$$

where

$$\begin{aligned} L(z) \ &= \ \sum_{k=1}^{\infty} \left(1 - ke^{i\mu}\right) F_k d_k z^k + \sum_{k=1}^{\infty} \left(1 + (k\lambda - 1)\, e^{i\mu}\right) G_k e_k z^{k+1} \\ &\quad + \lambda e^{i\mu} \sum_{k=1}^{\infty} kG_k e_k z^k \sum_{k=2}^{\infty} F_k d_k z^k + \sum_{k=2}^{\infty} \left(1 - ke^{i\mu}\right) F_k d_k z^k \sum_{k=1}^{\infty} G_k e_k z^k \end{aligned}$$

and

$$\begin{aligned} Q(z) &= A_1 z + \sum_{k=2}^{\infty} A_k F_k d_k z^k + \sum_{k=1}^{\infty} \left(1 - 2\alpha cos\mu + (1 - k\lambda)e^{i\mu}\right) G_k e_k z^{k+1} \\ &\quad -\lambda e^{i\mu} \sum_{k=1}^{\infty} k G_k e_k z^k \sum_{k=2}^{\infty} F_k d_k z^k + \sum_{k=1}^{\infty} G_k e_k z^k \sum_{k=2}^{\infty} A_k F_k d_k z^k, \end{aligned}$$

and A_k is given by$A_k = 1 - 2\alpha cos\mu + ke^{i\mu}$. But

$$\begin{aligned} |L(z)| &\leq \sum_{k=1}^{\infty} \left|1 - ke^{i\mu}\right| F_k |d_k| + \sum_{k=1}^{\infty} \left|1 + (k\lambda - 1)e^{i\mu}\right| G_k |e_k| \\ &\quad + \lambda \sum_{k=1}^{\infty} k G_k |e_k| \sum_{k=2}^{\infty} F_k |d_k| + \sum_{k=2}^{\infty} \left|1 - ke^{i\mu}\right| F_k |d_k| \sum_{k=1}^{\infty} G_k |e_k| \end{aligned} \tag{2.15}$$

and

$$\begin{aligned} |Q(z)| &\geq |A_1| - \sum_{k=2}^{\infty} |A_k| F_k \, |d_k| - \sum_{k=1}^{\infty} \left|1 - 2\alpha \cos\mu + (1 - k\lambda)e^{i\mu}\right| G_k \, |e_k| \\ &\quad -\lambda \sum_{k=1}^{\infty} k G_k \, |e_k| \sum_{k=2}^{\infty} F_k \, |d_k| - \sum_{k=2}^{\infty} |A_k| \, F_k \, |d_k| \sum_{k=1}^{\infty} G_k \, |e_k| \, . \end{aligned} \tag{2.16}$$

Under the condition (2.12), and by virtue of (2.14) to (2.16) the above inequality (2.13) holds true, which proves Theorem 3.

It may be observed that for $\mu = 0$, Theorem 3 eventually corresponds to Theorem 1.

3. Some Consequences of Main Results

Since the class of functions defined by (1.3) involves the familiar Gaussian hypergeometric function, therefore, Theorems 1-3 would find applications to several classes of functions involving special functions which arise from the Gaussian hypegeometric function by suitably specializing the parameters. For these special cases, one may refer, for instance, to Srivastava and Karlsson [5].

Remark 1. For

$$a_1 = a_2 = d_1 = 1, b_i = c_i (i = 1, 2), d_j = 0 (j \geq 2),$$

Theorems 1-3 after some elementary simplification reduce to the results [3, pp. 404-405, Theorems 1-3].

If we set

$$a_1 = a_2 = 1, b_i = c_i, (i = 1, 2), \alpha = 0$$

in Theorem 1, we get the following.

Corollary 1. *Let* $f(z) = 1 + \sum_{k=1}^{\infty} d_k z^k$ *and* $g(z) = 1 + \sum_{k=1}^{\infty} e_k z^k$. *Then the function* $\mathcal{W}(z)$ *defined by*

$$\mathcal{W}(z) = \frac{\sum_{k=1}^{\infty} d_k z^k}{\left(1 + \sum_{k=1}^{\infty} e_k z^k\right)^{\lambda}} \ (\lambda > 0), \tag{3.1}$$

is a starlike function provided that

$$\sum_{k=2}^{\infty} |d_k| \left(k + \lambda \sum_{k=1}^{\infty} k|e_k|\right) + \sum_{k=1}^{\infty} (k\lambda - 1)\,|e_k| + \sum_{k=1}^{\infty} |e_k| \sum_{k=2}^{\infty} k|d_k| \leq 1. \tag{3.2}$$

An interesting consequence of Theorem 1 occurs when $\lambda = 1$, and Theorem 1(in this special case on performing simple calculations) yield the following result.

Corollary 2. *The function* $\mathcal{H}^1_{a_1,b_1,c_1;a_2,b_2,c_2}(z)$ *defined by* (1.3) *is in* $\mathcal{S}^*(\alpha)$ $(0 \leq \alpha < 1)$, *provided that*

$$\begin{aligned}&\sum_{k=2}^{\infty} F_k\,|d_k| \left(k - \alpha + \sum_{k=1}^{\infty} kG_k\,|e_k|\right) \\ &\qquad + \sum_{k=2}^{\infty} (k + \alpha - 1)\,G_k\,|e_k| + \sum_{k=1}^{\infty} G_k\,|e_k| \sum_{k=2}^{\infty} (k - \alpha)\,F_k\,|d_k| \\ &\qquad < \begin{cases} (1-\alpha)\left(1 - \frac{a_2 b_2}{c_2}\,|e_1|\right), & 0 \leq \alpha < \frac{1}{2}, \\ 1 - \alpha\left(1 + \frac{a_2 b_2}{c_2}\,|e_1|\right), & \frac{1}{2} \leq \alpha < 1. \end{cases}\end{aligned} \tag{3.3}$$

Next, if we set

$$a_1 = a_2 = 1, b_i = c_i (i = 1, 2), e_i = e_{n+j} = 0\,(1 \leq i \leq n-1, j \in \mathbb{N}), e_n \neq 0$$

in (1.3), then Theorem 2 for the function $\mathcal{V}(z)$ defined by

$$\mathcal{V}(z) = \frac{\sum_{k=1}^{\infty} d_k z^k}{(1 + e_n z^n)^{\lambda}} \tag{3.4}$$

leads to the following result.

Corollary 3. *Let the function* $\mathcal{V}(z)$ *be defined by* (3.4). *Then* $\mathcal{V}(z) \in \mathcal{K}(\alpha)$ $(0 \leq \alpha < 1)$, *provided that there exist numbers* $p, q > 0$ *such that* $\frac{1}{p} + \frac{1}{q} \leq 1$ *and the following inequalities hold:*

$$(pn(\lambda + 1) + 1 - \alpha)\,|e_n| \leq 1 - \alpha \tag{3.5}$$

and

$$\sum_{k=2}^{\infty} \left[k(\delta_k - q)\,(1 + |e_n|) + n\,(q\,|\lambda + k(1-\lambda)| + \lambda\delta_n)\,|e_n|\right]$$
$$\leq \quad 1 - \alpha - |\lambda n - 1|\,\delta_n\,|e_n|\,, \tag{3.6}$$

where $\delta_n = qn + 1 - \alpha$.

Acknowledgement. Authors express their sincerest thanks to the referee for his invaluable comments.

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Received 12.02.2009, *Revisted 6.06.2009*

Journal of
Mathematics
and Applications
No 32, pp 63-66 (2010)

On cartesian product of fuzzy prime and fuzzy semiprime ideals of semigroups

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Submitted by: *Hari Mohan Srivastava*

ABSTRACT: The purpose of this paper is to study some properties of cartesian product of fuzzy prime and fuzzy semiprime ideals of semigroups

AMS Subject Classification: *08A72, 20M12, 3F55*
Key Words and Phrases: *Semigroup, Fuzzy prime ideal, Fuzzy semiprime ideal*

1. Introduction

In 1965 the concept of fuzzy subset was introduced by Zadeh[5]. Fuzzy subgroups and its important properties were defined and established by Rosenfeld[3]. The notion of fuzzy ideals of a semigroup was introduced by Kuroki[2]. Ersoy, Tepecik and Demir[1] analyzed cartesian product of fuzzy prime ideals of rings. The aim of this paper is to analyze some properties of cartesian product of fuzzy ideals, fuzzy prime ideals and fuzzy semiprime ideals of semigroups.

2. Preliminaries

In this section we review some basic definitions which will be required in the sequel. In what follows unless otherwise mentioned S stands for a semigroup.

Definition 2.1. [5] *A fuzzy subset of a non-empty set X is a function $\mu : X \to [0,1]$.*

Definition 2.2. [1] *Let μ be a fuzzy subset of a set S. Then for $t \in [0,1]$ the set μ_t=$\{x \in S : \mu(x) \geq t\}$ is called t-level subset or simply level subset of μ.*

Definition 2.3. [2] *A non-empty fuzzy subset μ of a semigroup S is called a fuzzy left ideal(fuzzy right ideal) of S if $\mu(xy) \geq \mu(y)$(resp. $\mu(xy) \geq \mu(x)$) $\forall x, y \in S$.*

Definition 2.4. [2] *A non-empty fuzzy subset μ of a semigroup S is called a fuzzy ideal of S if it is a fuzzy left ideal and a fuzzy right ideal of S.*

3. Fuzzy Prime and Fuzzy Semiprime Ideals

Definition 3.1. [4] *A fuzzy ideal μ of a semigroup S is called a fuzzy prime ideal of S if $\mu(xy) = \max\{\mu(x), \mu(y)\} \forall x, y \in S$.*

Definition 3.2. [4] *A fuzzy ideal μ of a semigroup S is called a fuzzy semiprime ideal of S if $\mu(x) \geq \mu(x^2) \forall x \in S$.*

Theorem 3.1. *Let S be a semigroup and μ be a non-empty fuzzy subset of S. Then the following are equivalent: (1) μ is a fuzzy prime ideal of S, (2) for any $t \in [0,1]$, the t-level subset μ_t of μ(if it is non-empty) is a prime ideal of S.*

Proof. Let μ be a fuzzy prime ideal of S. Let $t \in [0,1]$ be such that μ_t is non-empty. Let for $x, y \in S, xy \subseteq \mu_t$. Then $\mu(xy) \geq t$. Since μ is a fuzzy prime ideal of S, it follows that $\max\{\mu(x), \mu(y)\} \geq t$. So $\mu(x) \geq t$ or $\mu(y) \geq t$. Consequently, $x \in \mu_t$ or $y \in \mu_t$. Hence μ_t is a prime ideal of S.

Conversely, let every non-empty level subset μ_t of μ be a prime ideal of S. Let $x, y \in S$ and $\mu(xy) = t$. Then $\mu(xy) \geq t$ and $xy \in \mu_t$. So μ_t is non-empty and $xy \subseteq \mu_t$. Since μ_t is a prime ideal of $S, x \in \mu_t$ or $y \in \mu_t$. So $\mu(x) \geq t$ or $\mu(y) \geq t$. So $\max\{\mu(x), \mu(y)\} \geq t, i.e., \max\{\mu(x), \mu(y)\} \geq \mu(xy)$......(1). Again since μ is a fuzzy ideal of S, so we have $\mu(xy) \geq \mu(x)$ and $\mu(xy) \geq \mu(y)$. Then $\mu(xy) \geq \max\{\mu(x), \mu(y)\}$.....(2). Combining (1) and (2), we have $\mu(xy) = \max\{\mu(x), \mu(y)\}$. Hence μ is a fuzzy prime ideal of S. □

Theorem 3.2. *Let S be a semigroup and μ be a non-empty fuzzy subset of S. Then the following are equivalent: (1) μ is a fuzzy semiprime ideal of S, (2) for any $t \in [0,1]$, the t-level subset μ_t of μ(if it is non-empty) is a semiprime ideal of S.*

Proof. Let μ be a fuzzy semiprime ideal of S. Let $t \in [0,1]$ be such that μ_t is non-empty. Let for $x \in S, x^2 \in \mu_t$. Then $\mu(x^2) \geq t$. Since μ is a fuzzy semiprime ideal of S, then $\mu(x) \geq \mu(x^2)$. It follows that $\mu(x) \geq t$. Consequently, $x \in \mu_t$. Hence μ_t is a semiprime ideal of S.

Conversely, let every non-empty level subset μ_t of μ be a semiprime ideal of S. Let $x \in S$ and $\mu(x^2) = t$. Then $\mu(x^2) \geq t$ and $x^2 \in \mu_t$. So μ_t is non-empty. Since μ_t is a semiprime ideal of $S, x \in \mu_t$. So $\mu(x) \geq t \Rightarrow \mu(x) \geq \mu(x^2)$. Hence μ is a fuzzy semiprime ideal of S. □

4. Cartesian Product of Fuzzy Completely Prime and Fuzzy Completely Semiprime Ideals

Definition 4.1. [1] Let μ and σ be two fuzzy subsets of a set X. Then the cartesian product of μ and σ is defined by $(\mu \times \sigma)(x, y) = \min\{\mu(x), \sigma(y)\} \forall x, y \in X$.

Lemma 4.1. *Let μ and σ be two fuzzy subsets of a set X and $t \in [0,1]$. Then $(\mu \times \sigma)_t = \mu_t \times \sigma_t$.*

Proof. Let $(x, y) \in \mu_t \times \sigma_t \Leftrightarrow x \in \mu_t$ and $y \in \sigma_t \Leftrightarrow \mu(x) \geq t$ and $\sigma(y) \geq t \Leftrightarrow \min\{\mu(x), \sigma(y)\} \geq t \Leftrightarrow (\mu \times \sigma)(x, y) \geq t \Leftrightarrow (x, y) \in (\mu \times \sigma)_t$. Hence $(\mu \times \sigma)_t = \mu_t \times \sigma_t$. $\square$

Proposition 4.1. *Let μ and σ be two fuzzy left ideals(fuzzy right ideals, fuzzy ideals) of a semigroup S. Then $\mu \times \sigma$ is a fuzzy left ideal(resp. fuzzy right ideal, fuzzy ideal) of $S \times S$.*

Proof. Let μ and σ be two fuzzy left ideals of S and $(a, b), (c, d) \in S \times S$. Then $(\mu \times \sigma)\{(a, b)(c, d)\} = (\mu \times \sigma)(ac, bd) = \min\{\mu(ac), \sigma(bd)\} \geq \min\{\mu(c), \sigma(d)\}$ (since μ and σ are fuzzy left ideals of S) $= (\mu \times \sigma)(c, d)$. Hence $\mu \times \sigma$ is a fuzzy left ideal of $S \times S$. Similarly we can prove the other cases also. $\square$

Proposition 4.2. *Let μ and σ be two fuzzy prime ideals of a semigroup S. Then $\mu \times \sigma$ is a fuzzy prime ideal of $S \times S$.*

Proof. By Proposition 4.3, $\mu \times \sigma$ is a fuzzy ideal of $S \times S$. Let $(a, b), (c, d) \in S \times S$. Then $(\mu \times \sigma)\{(a, b)(c, d)\} = (\mu \times \sigma)(ac, bd) = \min\{\mu(ac), \sigma(bd)\} = \min[\max\{\mu(a), \mu(c)\}, \max\{\sigma(b), \sigma(d)\}]$(since μ and σ are fuzzy prime ideals of S) $= \max[\min\{\mu(a), \sigma(b)\}, \min\{\mu(c), \sigma(d)\}] = \max\{(\mu \times \sigma)(a, b), (\mu \times \sigma)(c, d)\}$. Hence $(\mu \times \sigma)$ is a fuzzy prime ideal of $S \times S$. $\square$

Proposition 4.3. *Let μ and σ be two fuzzy semiprime ideals of a semigroup S. Then $\mu \times \sigma$ is a fuzzy semiprime ideal of $S \times S$.*

Proof. By Proposition 4.3, $\mu \times \sigma$ is a fuzzy ideal of $S \times S$. Let $(a, b) \in S \times S$. Then $(\mu \times \sigma)(a, b) = \min\{\mu(a), \sigma(b)\} \geq \min\{\mu(a^2), \sigma(b^2)\}$(since μ and σ are fuzzy semiprime ideals of S) $= (\mu \times \sigma)(a^2, b^2) = (\mu \times \sigma)(a, b)^2$. Hence $(\mu \times \sigma)$ is a fuzzy semiprime ideal of $S \times S$. $\square$

Proposition 4.4. *Let μ and σ be two fuzzy prime ideals of a semigroup S. Then the level subset $(\mu \times \sigma)_t, t \in Im(\mu \times \sigma)$ is a prime ideal of $S \times S$.*

Proof. By Proposition 4.4, $\mu \times \sigma$ is a fuzzy prime ideal of $S \times S$. Let for $(x, y), (m, n) \in S \times S, (x, y)(m, n) \in (\mu \times \sigma)_t$. Then $(\mu \times \sigma)\{(x, y)(m, n)\} \geq t \Rightarrow (\mu \times \sigma)(xm, yn) \geq t \Rightarrow \min\{\mu(xm), \sigma(yn)\} \geq t \Rightarrow \mu(xm) \geq t$ and $\sigma(yn) \geq t \Rightarrow xm \in \mu_t$ and $yn \in \sigma_t \Rightarrow x \in \mu_t$ or $m \in \mu_t$ and $y \in \sigma_t$ or $n \in \sigma_t$(since μ_t and σ_t are prime ideals of S($cf.$ Theorem 3.3)). Hence $(x, y) \in \mu_t \times \sigma_t$ or $(m, n) \in \mu_t \times \sigma_t$. Since by Lemma 4.2, $(\mu \times \sigma)_t = \mu_t \times \sigma_t$, we deduce that $(x, y) \in (\mu \times \sigma)_t$ or $(m, n) \in (\mu \times \sigma)_t$. Consequently, $(\mu \times \sigma)_t$ is a prime ideal of $S \times S$. $\square$

Proposition 4.5. *If the level subset $(\mu \times \sigma)_t, t \in Im(\mu \times \sigma)$ of $\mu \times \sigma$ is a prime ideal of $S \times S$ then $(\mu \times \sigma)$ is a fuzzy prime ideal of $S \times S$.*

Proof. By Theorem 3.3, the proof follows immediately. $\square$

Proposition 4.6. *Let μ and σ be two fuzzy semiprime ideals of a semigroup S. Then the level subset $(\mu \times \sigma)_t, t \in Im(\mu \times \sigma)$ is a semiprime ideal of $S \times S$.*

Proof. By Proposition 4.5, $\mu \times \sigma$ is a fuzzy semiprime ideal of $S \times S$. Let for $(x, y) \in S \times S, (x, y)(x, y) \in (\mu \times \sigma)_t$. Then $(\mu \times \sigma)\{(x, y)(x, y)\} \geq t \Rightarrow (\mu \times \sigma)(x^2, y^2) \geq t \Rightarrow \min\{\mu(x^2), \sigma(y^2)\} \geq t \Rightarrow \mu(x^2) \geq t$ and $\sigma(y^2) \geq t \Rightarrow x^2 \in \mu_t$ and $y^2 \in \sigma_t \Rightarrow x \in \mu_t$ and $y \in \sigma_t$(since μ_t and σ_t are semiprime ideals of $S(cf.$ Theorem 3.4)). Thus $(x, y) \in \mu_t \times \sigma_t = (\mu \times \sigma)_t(cf.$ Lemma 4.2). Hence $(\mu \times \sigma)_t$ is a semiprime ideal of $S \times S$. □

Proposition 4.7. *If the level subset $(\mu \times \sigma)_t, t \in Im(\mu \times \sigma)$ of $\mu \times \sigma$ is a semiprime ideal of $S \times S$ then $(\mu \times \sigma)$ is a fuzzy semiprime ideal of $S \times S$.*

Proof. By Theorem 3.4, the proof follows immediately. □

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Received 11.08.2009

J o u r n a l of
Mathematics
and Applications

No 32, pp 67-74 (2010)

On LI-ideals of lattice implication algebras

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Submitted by: *Bronisław Wajnryb*

Abstract: We introduce the notions of a positive implicative LI-ideal and an associative LI-ideal in a lattice implication algebra and discuss some of their properties. Connections to related classes are investigated and equivalent conditions for both a positive implicative LI-ideal and an associative LI-ideal are provided

AMS Subject Classification: *03G10, 06B10, 54E15*
Key Words and Phrases: *Lattice implication algebra, Filter*

Introduction

Non-classical logic has become a considerable formal tool for computer science and artificial intelligence to deal with fuzzy information and uncertainty information. Many-valued logic, a great extension and development of classical logic [1], has always been a crucial direction in non-classical logic. In order to research the many-valued logical system whose propositional value is given in a lattice, in 1990 Xu [10,13] proposed the concept of lattice implication algebra. Since then this logical algebra has been extensively investigated by several researchers (see e.g. [2, 7, 8]). In [11] Xu and Qin introduced and studied the notions of filters and implicative filters in lattice implication algebras. In a lattice implication algebra, filters are important substructures, they play a significant role in studying the structure and the properties of lattice implication algebras. In[4], Jun et al. introduced the notions of positive implicative filters and associative filters in lattice implication algebras, and investigated some of their properties. In [5], Jun et al. defined and studied the notion of LI-ideals in lattice implication algebras. In this paper, as an extension of the above-mentioned works, we introduce the notions of a positive implicative LI-ideal and an associative LI-ideal in a lattice implication algebra and discuss some of their properties. Connections to related classes are investigated and equivalent conditions for both a positive implicative LI-ideal and an associative LI-ideal are provided.

Definition 1.1 *A lattice implication algebra is defined to be a bounded lattice* $(L; \vee, \wedge, 0, 1)$ *with order-reversing involution "$\prime$" and a binary operation "$\to$". In the sequel the binary operation "$\to$" will be denoted by juxtaposition. In a lattice implication algebra L, the following hold:*
(I1) $x(yz) = y(xz)$;
(I2) $xx = 1$;
(I3) $xy = y'x'$;
(I4) $xy = yx = 1 \Rightarrow x = y$;
(I5) $(xy)y = (yx)x$;
(L1) $(x \vee y)z = (xz) \wedge (yz)$;
(L2) $(x \wedge y)z = (xz) \vee (yz)$;
for all $x, y, z \in L$.

A lattice implication algebra L is called a lattice H-implication algebra if it satisfies $x \vee y \vee ((x \wedge y)z) = 1$ for all $x, y, z \in L$. We can define a partial ordering $\leq$ on a lattice implication algebra L by $x \leq y$ if and only if $xy = 1$.

Definition 1.2 *([12,2.1 and 2.2]) In a lattice implication algebra L, the following hold:*
(P1) $0x = 1, 1x = x$ *and* $x1 = 1$.
(P2) $xy \leq (yz)(xz)$.
(P3) $x \leq y$ *implies* $yz \leq xz$ *and* $zx \leq zy$.
(P4) $x' = x0$.
(P5) $x \vee y = (xy)y$.
(P6) $((yx)y')' = x \wedge y = ((xy)x')'$.
(P7) $x \leq (xy)y$.
In a lattice H-implication algebra L, the following hold:
(P8) $x(xy) = xy$.
(P9) $x(yz) = (xy)(xz)$.

Definition 1.3 *([11]) A subset F of L is called a filter of L if it satisfies for all* $x, y \in L$, $(F1)$ $1 \in F$,
$(F2)$ $x \in F$ *and* $xy \in F$ *imply* $y \in F$.

Definition 1.4 *([11]) A subset F of L is called an implicative filter of L if it satisfies* $(F1)$ *and*
$(F3)$ $x(yz) \in F$ *and* $xy \in F$ *imply* $xz \in F$ *for all* $x, y, z \in L$.

Definition 1.5 *([4]) A subset F of L is called a positive implicative filter of L if it satisfies* $(F1)$ *and*
$(F4)$ $x((yz)y) \in F$ *and* $x \in F$ *imply* $y \in F$ *for all* $x, y, z \in L$.

Definition 1.6 *([5]) Let L be a lattice implication algebra. A non-empty subset I of L is called an LI-ideal of L if it satisfies* (I_1) $0 \in I$ *and*
(I_2) $(xy)' \in I$ *and* $y \in I$ *imply* $x \in I$.

Definition 1.7 *([9]) A non-empty subset I of a lattice implication algebra L is said to be an implicative LI-ideal (briefly, ILI-ideal) of L if it satisfies (I_1) and*
(I_3) *$(((xy)'y)'z)' \in I$ and $z \in I$ imply $(xy)' \in I$ for all $x, y, z \in L$.*

1. Positive implicative LI-ideals

Definition 2.1 *A non-empty subset I of a lattice implication algebra L is said to be a positive implicative LI-ideal (briefly, $PILI$-ideal) of L if it satisfies (I_1) and*
(I_4) *$((y(zy)')'x)' \in I$ and $x \in I$ imply $y \in I$ for all $x, y, z \in L$.*

Lemma 2.1 *Every LI-ideal of L has the following property:*
$x \leq y$ and $y \in I$ imply $x \in I$.

Proof. $x \leq y$ so $xy = 1$ then $(xy)' = 0 \in I$ by $y \in I$ we have $x \in I$. ■

Lemma 2.2 *Let I be a non-empty subset of L. Then I is an LI-ideal of L if and only if for all $x, y \in I$ and $z \in L$, $(zy)' \leq x$ implies $z \in I$.*

Proof. Let $(zy)' \leq x$. Then $(zy)'x = 1$ so $((zy)'x)' = 0 \in I$. Since I is a LI-ideal and $x \in I$ then $(zy)' \in I$, also, $y \in I$ so $z \in I$.
Conversely, since $I \neq \emptyset$ there exist $x \in I$ which $(0x)' \leq x$ then $0 \in I$.
Now let $(xy)', y \in I$. Since $(xy)' \leq (xy)'$, Putting $X = (xy)'$, $Y = y$ and $Z = x$, thus $Z = x \in I$. ■

Theorem 2.1 *Let I be a non-empty subset of L. If I is an ILI-ideal of L, then I is an LI-ideal of L.*

Proof. In the definition of ILI-ideal, by replacing $Y = 0$ and $Z = y$, for all $X, Y, Z \in L$, we get the result.
■

Theorem 2.2 *Let I be a non-empty subset of L. If I is an $PILI$-ideal of L, then I is an LI-ideal of L.*

Proof. In the definition of $PILI$-ideal putting $X = y$, $Y = x$ and $Z = x$, for all $X, Y, Z \in L$, we have the result.
■

Theorem 2.3 *Let I be an LI-ideal of L. Then I is an $PILI$-ideal of L if and only if for all $x, y \in L$,*
(LI5) $(x(yx)')' \in I$ implies $x \in I$.

Proof. Assume that I is a positive implicative LI-ideal of L and let $X = 0$, $Y = x$ and $Z = y$. Then

$$((Y(ZY)')'X)' \in I \ \ \textit{and} \ \ X \in I \ \ \textit{imply} \ \ Y \in I,$$

i.e.,

$$((x(yx)')'0)' \in I \ \ and \ \ 0 \in I \ \ imply \ \ x \in I.$$

It means that, $(x(yx)')' \in I$ implies $x \in I$.
Conversely, since I is an LI-ideal of L, $(y(zy)')' \in I$. Hence, by $(LI5)$, $y \in I$. ■

Theorem 2.4 *Let I be an LI-ideal of a lattice implication algebra L. Then the following are equivalent:*
(i) *I is an ILI-ideal of L;*
(ii) *$((xy)'y)' \in I$ implies $(xy)' \in I$, for all $x, y \in L$;*
(iii) *$((xy)'z)' \in I$ implies $((xz)'(yz)')' \in I$, for all $x, y, z \in L$;*
(iv) *$(((xy)'z)'u)' \in I$ and $u \in I$imply $((xz)'(yz)')' \in I$, for all $x, y, z, u \in L$;*
(v) *$((xy)'z)' \in I$ and $(yz)' \in I$ imply $(xz)' \in I$.*

Proof. By [9, Theorem 3.8], we have $(i) \Leftrightarrow (ii) \Leftrightarrow (iii) \Leftrightarrow (iv)$. Therefore it is sufficient to show $(v) \Rightarrow (ii)$ and $(iv) \Rightarrow (v)$.
$(v) \Rightarrow (ii)$: In (v) putting $z = y$, we have $((xy)'y)' \in I$. and $(yy)' \in I$ which imply $(xy)' \in I$. Hence $((xy)'y)' \in I$ and $0 \in I$ imply $(xy)' \in I$. Thus $(v) \Rightarrow (ii)$.
$(iv) \Rightarrow (v)$: In (iv) putting $u = 0$, we have $(((xy)'z)'0)' \in I$ and $0 \in I$, which imply $((xz)'(yz)')' \in I$ for all $x, y, z \in L$. So $(((xy)'z)' \in I$ implies $((xz)'(yz)')' \in I$, for all $x, y, z \in L$.
Since, by (v), $(yz)' \in I$ and I is an LI-ideal of L, so $(xz)' \in I$. ■

The converse of Theorems 1 and 2 are not correct, by the following examples.

Example 2.1 *Let $L = \{0, a, b, c, d, 1\}$ be a set with a partial ordering. Define a unary operation "′" and a binary operation denoted by juxtaposition on L as follows:*

	0	a	b	c	d	1
0	1	1	1	1	1	1
a	c	1	b	c	b	1
b	d	a	1	b	a	1
c	a	a	1	1	a	1
d	b	1	1	b	1	1
1	0	a	b	c	d	1

x	x'
0	1
a	c
b	d
c	a
d	b
1	0

Define $\vee$- and $\wedge$- operations on L as follows:

$$x \vee y = (xy)y,$$
$$x \wedge y = ((x'y')y')',$$

for all $x, y \in L$. Then L is a lattice implication algebra. It is easy to check that $I = \{0, c\}$ is an LI-ideal of L. Notice that I is neither an ILI-ideal nor an $PILI$-ideal. For both of them take $x = a$ and $y = d$ in the definitions of an ILI-ideal and an $PILI$-ideal, we deduce that:
$((xy)'y)' = ((ad)'d)' = (b'd)' = 0 \in I$ *but* $(xy)' = (ad)' = d \notin I$, *and*
$(y(xy)')' = (d(ad)')' = (db')' = 0 \in I$ *but* $y = d \notin I$.

Theorem 2.5 *Let I be a non-empty subset of L. If I is a $PILI$-ideal of L, then it is an ILI-ideal of L.*

Proof. Let $((xy)'z)' \in I$ and $(yz)' \in I$. It is sufficient to show that $(xz)' \in I$. We have,

$$((xy)'z)' = (z'(xy))' = (x(z'y))' = (x(y'z))' = (y'(xz))' = ((xz)'y)'.$$

On the other hand, since for each $u, v, w \in L$ we have $(uv) \leq (vw)(uw)$, we deduce that
$(xz)'y \leq (yz)((xz)'z) \Rightarrow ((yz)((xz)'z))' \leq ((xz)'y)'$
$\Rightarrow (((xz)'z)'(yz)')' \leq ((xz)'y)'$.
Now by lemma 2, since $((xz)'y)', (yz)' \in I$, we have $((xz)'z)' \in I$. Also
$((xz)'z)' = (((xz)'0)'z)'$
$= (((xz)'(0z)')'z)'$
$= (((xz)'((xx)'z)')'z)'$
$= (((xz)'((xz)'x)')'z)'$.
Hence by $(I5)$, we have

$$((x(x(xz)')')'z)' = ((xz)'(x(xz)')')'.$$

By putting $u = (xz)'$ and $v = x$, from the last equation we get $(u(vu)')' \in I$. Since I is a positive implicative LI-ideal of L, $u \in I$. Thus $(xz)' \in I$. ∎

Theorem 2.6 *In a lattice implication algebra, any $PILI$-ideal is an ILI-ideal. Conversely, in a lattice H-implication algebra, any ILI-ideal is a $PILI$-ideal.*

Proof. Theorem 5 shows that if I is $PILI$-ideal of L then it is an ILI-ideal. Conversely assume that I is an ILI-ideal of L and $((y(zy)')'x)' \in I$ and $x \in I$. Then we have $(y(zy)')' \in I$ and

$$(y(zy)')' = ((zy)y')' = ((y'z')y')' = (y')'.$$

Since L is H-implication, we get the last equation. So $y = (y(zy)')' \in I$. ∎

Theorem 2.7 *Let S be a non-empty subset of a lattice implication algebra L. Assume that $S' = \{x' : x \in S\}$. Then S is a positive implicative filter of L if and only if S' is a $PILI$-ideal of L.*

Proof. Let S be a positive implicative filter of L. Then $1 \in S$, so $1' = 0 \in S'$.
Let $((y(zy)')'x)' \in S'$ and $x \in S'$. There exist some $u, v \in S$ such that $u = (y(zy)')'x$ and $v = x'$. So $x'((y'z')y') = x'(y(zy)') = u \in S$ and $v = x' \in S$. It follows that $y' \in S$, as S is a positive implicative filter. Hence $y \in S'$. Thus S' is a $PILI$-ideal of L.
Conversely, if S' is a $PILI$-ideal of L, then $0 \in S'$. So $1 = 0' \in (S')' = S$. Let $x((yz)y) \in S$ and $x \in S$. Then $(x((yz)y))' \in S'$ and $x' \in S'$. So $(((yz)y)'x')' \in S'$ and $x' \in S'$. Hence $((y'(z'y')')'x')' \in S'$ and $x' \in S'$. Thus $y' \in S'$, as S' is a $PILI$-ideal of L. That is $y \in S$, hence S is a positive implicative filter of L. ∎

Theorem 2.8 *Let I and J be two LI-ideals of a lattice implication algebra L with $I \subseteq J$. If I is a $PILI$-ideal of L, then so is J.*

Proof. Let $(x(yx)')' \in J$. Take $t = (x(yx)')'$, $X = (xt)'$ and $Y = x$. Then $(YX)' = (x(xt)')' = (x(x(x(yx)')')')' = ((x(yx)')')' = t'$. So
$(X(YX)')' = ((xt)'t')' = (t(xt))' = (x(tt))' = 0 \in I$ and so $X \in I$ by I is $PILI$-ideal of L. Since $I \subseteq J$, $(xt)' = X \in J$. $t \in J$ imply that $x \in J$. So J is $PILI$-ideal of L. ■

2. Associative LI-ideals

Definition 3.1 *Let x be a fixed element of L. A subset I of L is called an associative LI-ideal of L with respect to x if it satisfies (I_1) and*
(I_5) $((zy)'x)' \in I$ and $(yx)' \in I$ imply $z \in I$.
An associative LI-ideal with respect to all $x \neq 1$ is called an associative LI-ideal.

An associative LI-ideal with respect to 1 is whole algebra L. An associative LI-ideal with respect to 0 is coincident with an LI-ideal.

Example 3.1 *Let $L = \{0, a, b, c, d, 1\}$ be a set as a partial ordering. Define a unary operation "$'$" and a binary operation denoted by juxtaposition on L as follows:*

	0	a	b	c	d	1
0	1	1	1	1	1	1
a	d	1	a	c	c	1
b	c	1	1	c	c	1
c	b	a	b	1	a	1
d	a	1	a	1	1	1
1	0	a	b	c	d	1

x	x'
0	1
a	d
b	c
c	b
d	a
1	0

Define $\vee$- and $\wedge$- operations on L as follows:

$$x \vee y = (xy)y,$$
$$x \wedge y = ((x'y')y')',$$

for all $x, y \in L$. Then L is a lattice implication algebra. It is easy to check that $I = \{0, c, d\}$ is an LI-ideal of L.It is easy to check that I is an associative LI-ideal of L with respect to $0, c, d$.
Take $x = z = a$ and $y = c$,
$((ac)'a)' = (c'a)' = 0 \in I$ and $(ca)' = d \in I$ but $z = a \notin I$, so I is not an associative LI-ideal of L with respect to a.

Proposition 3.1 *Every associative LI-ideal with respect to x contains x itself.*

Proof. If $x = 0$ then $((zy)'0)' \in I$ and $(y0)' \in I$ imply $z \in I$.
So $(zy)' \in I$ and $y \in I$ imply $z \in I$, i.e., I is an LI-ideal of L that contains 0.
If $x = 1$ then $I = L$.
If $x \neq 0, 1$, take $y = 0$ and $z = x$ then $((x0)'x)' = (xx)' = 0 \in I$ and $(0x)' = 0 \in I$ imply $x \in I$. ■

Theorem 3.1 *Every associative LI-ideal of L is LI-ideal of L.*

Proof. If $(xy)' \in I$ and $y \in I$ then $((xy)'0)' \in I$ and $(y0)' \in I$. Since I is an associative LI-ideal of L then $x \in I$. ■

Theorem 3.2 *Let I is an LI-ideal of L. I is an associative LI-ideal if and only if $((zy)'x)' \in I$ implies $(z(yx)')' \in I$.*

Proof. (⇐) If $((zy)'x)' \in I$ and $(yx)' \in I$ then $(z(yx)')' \in I$ and $(yx)' \in I$. Since I is an LI-ideal of L then $z \in I$.
(⇒) Let $((zy)'x)' \in I$ then

$$\begin{aligned}(((z(yx)')'(zy)')'x)' &= (((z(yx)')'x)'(zy)')' \\ &= (((zx)'(yx)')'(zy)')' \\ &= 1' = 0 \in I\end{aligned}$$

The last equation is comes from $zy \le (yx)(zx)$ which implies $((zx)'(yx)')' \le (zy)'$. From assumption $((zy)'x)' \in I$ and I is an associative LI-ideal so $(z(yx)')' \in I$. ■

Theorem 3.3 *Let I is an LI-ideal of L. I is an associative LI-ideal if and only if $((yx)'x)' \in I$ implies $y \in I$.*

Proof. (⇒) $((yx)'x)' \in I$ then $(y(xx)')' \in I$ so $(y0)' = y \in I$.

$$\begin{aligned}(\Leftarrow)\quad ((((z(yx)')'x)'x)'((zy)'x)')' &= (((((z(yx)')'x)'x)'((zy)'x)')'0)' \\ &= (((((z(yx)')'x)'x)'((zy)'x)')'(((zx)'(yx)')'(zy)')')' \\ &= (((((z(yx)')'x)'x)'((zy)'x)')'(((z(yx)')'x)'(zy)')')' \\ &= (((((z(yx)')'x)'x)'(((z(yx)')'x)'(zy)')')'((zy)'x)')' \\ &= \le (((zy)'x)'((zy)'x)')' = 0.\end{aligned}$$

So $((((z(yx)')'x)'x)'((zy)'x)')' \in I$, $(((z(yx)')'x)'x)' \in I$

Assumption gives us $(z(yx)')' \in I$. By last theorem I is an associative LI-ideal. Equation in second line comes from $zy \le (yx)(zx)$ so $((zx)'(yx)')' \le (zy)'$ that $(((zx)'(yx)')'(zy)')' = 0$.
Inequality in second line comes from $xy \le (zx)(zy)$ so $((zy)'(zx)')' \le (xy)'$. ■

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Received 12.02.2009, Revisted 5.06.2009

Journal of
Mathematics
and Applications
No 32, pp 75-84 (2010)

Homeopath: Diagnostic information system

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ABSTRACT: Different approaches to the expert diagnostic systems of the Homeopat systems family are considered. The problems of representing the knowledge about models of data domains for such systems are investigated

AMS Subject Classification: *93B35, 93C42*
Key Words and Phrases: *homeopast system, diagnostc system*

1. Introduction

One of the most important problem of the knowledge-based systems is the knowledge representation. This is explained by the fact that a form of representation has a significant influence on the characteristics and the properties of the system. In order to manipulate different knowledge from the real world using a computer, it is needed to make a modeling (or formalization). In this paper the several models of associative knowledge representation (in the HOMEOPAT system) which are specified by the semantic networks are proposed. The properties of the corresponding systems differ in effectiveness of different problems solving as well as in approaches to their solving. Specifically, the advantages of the "precise" model of the semantic network [1-2] include the fact that insignificant modification can be dealt by the most powerful method of theorem proving - the resolution rule. The advantages of "imprecise" model [3,10] lead to the possibilities in solving the problems with fuzzy target settings. Neural networking models [4] allow the creation of dynamic diagnostic systems by using the learning, therefore it make possible to scale the sphere of covered problems.

2. Logical model of the associative knowledge representation

In this section the general questions of knowledge representation using semantic network, and the ways of its logical description are described. As known, the basic functional element of semantic network is the structure which consist two components - nodes and arcs which connect nodes. Each node represents some term, and arc is a relationship between a pair of terms. It is possible to claim that each of such pairs describes a fact. The nodes are marked with names of corresponding relationship. The arcs have direction, due to which there appears a "subject-object" relationship between the terms. Each node can be connected with any number of other nodes, which makes up a network of facts. From the logical point of view the structure of semantic network can be considered as a predicate with two arguments. Argument is node, and a predicate is a directed arc, which connects these nodes. The correct choice of relationship give the possibility to describe the complex collections of facts. Generally, the logical model of system knowledge is built in terms of formal system, which is represented by set of the following components:

1. Two finite alphabets K_1 and K_2;
2. Symbols x,y – variables, which get the values from K_1 and K_2;
3. Two single-placed predicate symbols P, R and one double-placed – Q;
4. Implication sign " $\rightarrow$ " and punctuation sign ",";
5. A finite sequence A_1, A_2, ... , A_k of the correctly build formulas to according with the definition below.

The corresponding elements of sets K_1 and K_2 are marked as c_1, c_2, ... , c_n and p_1, p_2, ... , p_m.

The symbols of the alphabet K_1 and variable x are called a term of the system above alphabet K_1. The symbols of the alphabet K_2 and variable y are called a theorem above K_2.

An atomic formula is an expression like $P(t_1)$, $R(t_2)$, $Q(t_1, t_2)$, where t_1, t_2 are terms above alphabets K_1 and K_2.

The correctly built formula (CBF) of the system is an atomic formula and expressions like:

$P(t_1) \rightarrow Q(t_2, t_1) \rightarrow R(t_2)$;

$R(t_2) \rightarrow Q(t_2, t_1) \rightarrow P(t_1)$.

CBF without variables is a claim. Consequential claim of the system is any axiom or formula without variables, which can be consequently received from the axioms by a finite use of the following rules, which are called the consequential rules:

1. Replacement of symbols x, y with symbols from K_1 K_2;
2. Consequence, resulting formula X_2 from X_1 and $X_1 \rightarrow X_2$ in condition, that X_1 is atomic formula.

Let's bring up the interpretation of the system of predicates:
$P(x)$ – "patient has symptom x";
$Q(y, x)$ – "medicine y has symptom X";
$R(y)$ – "prescribe y to the patient".
This way the formal knowledge system can be specified by a set of axioms:
$Q(x, y)$;
$P(x) \to Q(y, x) \to R(y)$;
$R(y) \to Q(y, x) \to P(x)$;
$P(x)$.
If alphabets K_1 and K_2 represent the sets of drugs and symptoms, the factual model of knowledge can be described by axioms of this system, putting the elements of corresponding alphabets instead of x and y variables.

Let's show that if a patient has symptom x, he should be prescribed the drug y. Let system has the following claims:

$$P(x); \tag{1}$$

$$P(x) \to Q(y, x) \to R(y); \tag{2}$$

$$Q(y, x); \tag{3}$$

$$R(y), \tag{4}$$

where (1), (2), (3) are axioms, (4) is the claim that we should prove.

In order to use the resolutions method, we need to represent the claim in the disjunctive form and take the negation of the conclusion. So we have the prove:

$$P(x); \tag{5}$$

$$\sim P(x) \vee \sim Q(y, x) \vee R(y) \quad \text{from(2)}; \tag{6}$$

$$Q(y, x); \tag{7}$$

$$\sim \; R(y) \quad \text{conclusion negation}; \tag{8}$$

$$\sim Q(y, x) \vee R(y) \quad \text{from (1) and (2)}; \tag{9}$$

$$R(y) \text{ from (3) and (5)}; \tag{10}$$

$$\square \text{ from (4) and (6)}. \tag{11}$$

We have an empty disjunct so the conclusion is true.

3. Fuzzy models of associative knowledge representation

In many interpretation and diagnostics tasks the unreliable knowledge and facts are used. Unreliability comes up as a consequence of absence of a part of the data or doubt in correctness of these data. Representing such knowledge using the notation 1 – true, and 2 – false isn't always possible, so we use the numbers from the interval [0;1].

The first method, which use indetermination, was suggested in the MYCIN [5] expert system, which was used for choosing the treatment course of the inflectional diseases. In this method we operate with the coefficients of confidence which lay in [-1;1], and the proper method of formula combination is used. The similar approach was presented in one of the HOMEOPAT system versions, as a way in making up the diagnosis and drugs prescription.

The general formulation of the problem is following: let $X = \{X_1, X_2, ..., X_k\}$ is some set of the objects and $C = \{C_1, C_2, ..., C_n\}$ is some set of their characteristics and relationship between these objects is ($\infty : \infty$), so each element of the X can have many corresponding objects from C set, and vice-versa. The relationship between the elements of these sets can be displayed as a diagram called semantic network. The arrows, connecting nodes $X_i$and C_j, mean that object X_i has a characteristic C_j.

Such a diagram can be represented as fuzzy subset R of the Cartesian product $X \times C$, which is defined like:

$$R = \int_{X\times C} \mu\ R(x,c)/(x,c).$$

Accordingly, the object X_i with a set of characteristics C can be represented as:

$$R = \int_C \mu\ R(X_i,c)/(X_i,c).$$

If we consider two fuzzy sets

$$R = \int_C \mu\ 1/(X_i,c)$$

and

$$S = \int_C \mu\ (X_i,c)/(X_i,c),$$

which define the object X_i and some of its approximation, taking into account the coefficient of proximity, such as fuzzy sum

$$\beta = \int_C\ (1 - \mu\ (X_i,c))^{1/2},$$

it is possible, using the current fuzzy characteristics of the object, to find the approximation with a minimal coefficient of proximity. Consequently we can find the most plausible object (or objects).

In addition it can be shown, the more characteristics is found and the more measures of independence of these characteristics are, the more precise of the result will be. Another way, which uses the definition of linguistic variable was represented in [6].

The linguistic variable is a set $< \beta, T, U, G, M >$, where β is the name of the variable, T is the set of its values which are the names of fuzzy variables and the domain of definition of each value from T is the set X. Set T is called basic term-set of the linguistic variable; U – the set of syntax rules, which generate terms using quantificators; G – syntax procedure, which operate with the elements of term-set T, as a result to generate new terms (values); M – semantic procedure, which transform every new value, which is created by procedure G, to a new fuzzy variable, so the procedure generate a fuzzy set.

Logically-linguistic method of system description is based on the fact that the system functionality is described using the simple language in terms of linguistic variables. The input and output parameters of the system are considered as linguistic variables and the qualitative description of the process is represented by a set of claims in the following form:

L_1: if $< a_{11} >$ and/or $< a_{12} >$ and/or ... and/or $< a_{1m} >$, then $< b_{11} >$ and/or ... and/or $< b_{1n} >$,

L_2: if $< a_{21} >$ and/or $< a_{22} >$ and/or ... and/or $< a_{2m} >$, then $< b_{21} >$ and/or ... and/or $< b_{2n} >$,

. . . .

L_k: if $< a_{k1} >$ and/or $< a_{k2} >$ and/or ... and/or $< a_{km} >$, then $< b_{k1} >$ and/or ... and/or $< b_{kn} >$,

where $< a_{ij} >$,$i = 1, 2, ..., k$; $j = 1, 2, ..., m$ are complex fuzzy claims, which are defined on the values of input linguistic variables, and $< b_{ij} >$, $i = 1, 2, ..., k$; $j = 1, 2, ..., n$ are fuzzy claims, defined on the values of output linguistic variables. Such a set is called an fuzzy knowledge base.

Using the rules of conversion of conjunctive and disjunctive forms, the description of the system can be displayed in the following way:

$$\begin{array}{ll} L_1: & \text{if } < A_1 >, \text{ then } < B_1 >, \\ L_2: & \text{if } < A_2 >, \text{ then } < B_2 >, \\ \ldots\ldots\ldots\ldots & \\ L_k: & \text{if } < A_k >, \text{ then } < B_k >, \end{array}$$

where $A_1, A_2, ..., A_k$ – fuzzy sets, which are defined on the Cartesian product X universal sets of input linguistic variables; $B_1, B_2, \ldots, B_k$ – fuzzy sets, which are defined on the Cartesian product Y universal sets of output linguistic variables.

The set of implications $\{L_1, L_2, ..., L_k\}$ displays the functional interconnection of input and output variables and is base for building the general indistinct relation R, which is defined on the Cartesian product XxY of input and output variable sets.

Output of the system is based on the compositional rule of Zade, which is formulated in the following way: if a fuzzy set A is defined on the set X , then compositional rule $B = A \circ R$,where R is an fuzzy relationship, which declares an fuzzy implication, defines an fuzzy set B on the set Y with a membership function:

Figure 1: Fuzzy logical output.

$$\mu_B(y) \;=\; \bigcup_{x \in X} [\mu_A(x) \bigcap \mu_R(x, y)]$$

So, in this case, the compositional rule defines a law for fuzzy model system functionality.

Functional scheme of the process of fuzzy output in a simplified form is displayed in the Figure 1. As it is shown on the scheme the execution of the first stage of output which is called phasification is done by a phasificator. The machine of fuzzy logical output is responsible for a procedure of fuzzy output, which form the second stage of the output basing on the given fuzzy knowledge base (rule set) and the composition stage. Dephasificator executes the last stage of fuzzy output – dephasification.

4. Diagnostics bases on neural networks

As a rule, as a consequence of the cerebrum study and mechanisms of its functioning there have been created new computer models, namely artificial neural networks (NN). The tasks of the office automation processes based upon the research in the sphere of the artificial intelligence (AI) are of current importance to present day. NN permit to solve applications such as pattern recognition, modeling, fast data conversion (parallel computational processes), identifications, management, and expert systems creation [7]

Theoretically, NN can solve a wide frame of tasks in the specific data domain. (as it is the human brain model prototype), but it is still not practically possible to

Figure 2: General view of neuron.

create the integrated universal NN for the specific data domain at present, since there is no integrated construction algorithm (functioning) of the NN. The moment to date the specific structure NN and with the defined learning algorithms are used for the solution of the concrete group of tasks out of the fixed data domain.

As it is well-known, each neuron has a number of qualitative characteristics, such as condition (excited or dormant), input and output connections. The one-way only connections, mated with the inputs of the other neurons are called synapses and the output connection of the given neuron from which the signal (actuating or dormancy) comes on the synapses of the other neurons are called axon. The neuron overview is presented on figure 2. Per se, the functioning of every neuron is relatively simple. As a rule, the set of the $X = [x_1, x_2, x_3, \ldots x_n]$ signals come to the neuron input. Each of the signals may be the output of the other neuron or source. Every input signal is multiplied on the corresponding angular coefficient $W = [w_1, w_2, w_3, \ldots w_n]$. It complies with the force of the synapse of the biological neuron. The products of the $w_i x_i$ are summarized and come on the adding element. To initialize the networks, the input x_0 ($x_0 = +1$) and the weighting factors of the synaptic ties w0 are specially entered. The neuron condition in the current moment is defined as the weighted total of its inputs:

$$S = \sum_{i=1}^{n} x_i w_i + x_0 w_0$$

The neuron output is the output of its condition:

$$Y = F(S)$$

The F is the function of activation. It is monotonous, contiguously differentiable on the interval either (-1,1), or (0,+1).

In the multilayer neuronal networks (MNN) the basic elements outputs of each layer come to the inputs of all the basic elements of the next layer. The activation function $F(S)$ is chosen as being the same for all the neurons of the network. In [7] the MNN it is determined in such a symbol form $N^K_{n_0,n_1,\ldots,n_R}$, where K is the number of the layers in the network, n_0 is the number of the network inputs; $n_i (i = 1, K-1)$ - the number of the basic elements in the - interlayers, n_K - the number of the basic elements in the output layer and simultaneously the number of the outputs $q_1, ..., q_{n_K}$ of the MNN. The intermediary a layer has n_a neurons. There are no connections between the basic elements in the layer. The layer basic elements outputs come to the neurons inputs of only the next (+1) layer. The output for any neuron is determined as being $q_i^{(a)} = f(\sum_{j=1}^{n_{a-1}} w_{i,j}^{(a)} q_1^{(a-1)} + w_{i,0}^a q_0^{(a-1)}) = f(s_i^{(a)})$.

5. Specialized NN in HOMEOPATH system

The scientists have been into the development of the mathematical methods solutions of medicinal tasks for many years already. The effectiveness of the similar mathematical methods may be followed by the set of the medical diagnostics systems, developed in the last time. The general trait of these systems is their dependence on the specific methods of the group data processing, poorly applied to the unit objects and, also, on the features of the medical information [8].

The neuronal networks (NN) are easy-to-use instruments of the information models presentation. In the general case, the network receives some input signal from the outer world and passes it through itself with the conversion in each of the neurons. Hence, the signal processing is being made in the process of its passage through the network connections, the result of which is the specific output signal. For the purposes of the neuronal network designing in the system of HOMEOPATH there has been chosen the mostly spread structure of the neuronal networks – multilayer one. This structure imports that every neuron of the arbitrary layer is connected with all the outputs and inputs (axons) of the preceding layer or with all the NN inputs in the case of the first layer. In other words, the network has the following structure of the layers: the input, the intermediate (latent) and output. Such neuronal networks are also called fully connected [7].

For the solution of the diagnostics task in the system of HOMEOPATH the NN of the following architecture is being used (Figure 3).

The task of the habituation of the MNN in classical form could be presented as following. Let there is specified some series of x^* input data. It is requested to find such solution x, which can be used to classify the newly presented input data. The criterion $R(x, x^*)$ determines the quality of solution. The variety of solutions x is determined by the choice of the weighting factors $w_i^{(a)}$ adjustment algorithm. Such definition of the problem allow to build the training process which comes to the receipt of the best solution out of the series of possible ones. In other words, the MNN training is the process of data x^* accumulation and, concurrently, the process of the choice of solution x. The NN of the HOMEOPATH system uses the algorithm of the inverse distribution the gist of which is in the distribution of the error signals from the NN outputs to its inputs in the direction back to the direct signals distribution in the usual mode of operation (identification regime). In other words, using the technologies of the series tuning of neurons starting with the last output layer and finishing with the tuning of the first layer elements. The NN habituation may be done the necessary number of times. For the habituation we use so called δ rule, which lies in the realization of the training strategy with the "teacher". Let us label as y^* the required neuron output, y is the real output. The error of the training is calculated

Figure 3: Architecture neuron network.

according to the following formula $\delta = y^* - y$ in the algorithm of the gradient descent (the weighted factor) $w_i(k+1) = w_i(k) - \gamma\delta x_i$, $\gamma > 0$, where γ is the "strengthening of the algorithm" factor, x_i is the i input of the neuron synaptic connection.

Conclusions

The specification of the data domain of the system allows a transition from the semantic network model to the logical representation in order to the further implementation of resolutions method for discovering the true claims of semantic network.

What relates to the indistinct models of knowledge representation, the approach, which is suggested in the paper, is rather effective in solving the problems of diagnostics on the models, which can be described by binary semantic networks. Naturally, such networks could be generalized to the more complex cases with the corresponding modification of fuzzy arithmetic.

The main advantage of linguistic model is universality. It doesn't much matter, what kind of data are in the input – concrete numerical values or some kind of indeterminacy, which is described by an fuzzy set. However such universality leads to a greater complexity, we have to use the space with an $m \times n$ dimensions. That is why the general fuzzy model can be simplified by reducing the knowledge base implication, as well as the procedure for teaching the model.

The difficulties of using neural networks for diagnosing are connected with the complexity of the present databases. The further studies in this way are the search of a model with an effective learning procedure. Some results of these studies are already published in [9].

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Received 10.06.2009

J o u r n a l of
Mathematics
and Applications
No 32, pp 85-90 (2010)

On certain functions with positive real part

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ABSTRACT: The aim of the paper is to find certain conditions on the complex-valued functions $A, B : U \to \mathbb{C}$ defined in the open unit disc $U = \{z \in \mathbb{C} : \ |z| < 1\}$ such that the differential inequality

$$\begin{aligned}\text{Re } [A(z)p^2(z) + B(z)p(z) + \alpha(zp'(z) - a)^3 \\ -3a\beta\left(zp'(z) - \frac{b}{2}\right)^2 + 3a^2\gamma(zp'(z)) + \delta] > 0\end{aligned}$$

implies Re $p(z) > 0$, where $p \in \mathcal{H}[1, n]$, $a, b \in \mathbb{R}_+$, $\alpha, \beta, \gamma \in \mathbb{C}$

AMS Subject Classification: *30C80*
Key Words and Phrases: *holomorphic function, differential inequality, functions with positive real par*

1. Introduction and preliminaries

We denote by $\mathcal{H}[U]$ the class of holomorphic functions in the open unit disc. For $a \in \mathbb{C}$ and $n \in \mathbb{N}^*$ we let

$$\mathcal{H}[a, n] = \{f \in \mathcal{H}[U], \quad f(z) = a + a_n z^n + a_{n+1} z^{n+1} + ..., \ z \in U\}$$

and

$$\mathcal{A}_n = \{f \in \mathcal{H}[U], \quad f(z) = z + a_{n+1} z^{n+1} + a_{n+2} z^{n+2} + ..., \ z \in U\}$$

with $\mathcal{A}_1 = \mathcal{A}$.

In order to prove the new results we shall use the following lemma, which is a particular form of Theorem 2.3.i [3, p.35].

Lemma 1.1. [3] *Let* $\psi : \mathbb{C}^2 \times U \to \mathbb{C}$ *a function which satisfies*

$$\text{Re } \psi(\rho i, \sigma; z) \leq 0, \tag{1.1}$$

where $\rho, \sigma \in \mathbb{R}$, $\sigma \le -\frac{n}{2}(1+\rho^2)$, $z \in U$ *and* $n \ge 1$.

If $p \in \mathcal{H}[1,n]$ *and*

$$\mathrm{Re}\ \psi\left(p(z), zp'(z); z\right) > 0 \tag{1.2}$$

then

$$\mathrm{Re}\ p(z) > 0.$$

2. Main results

Following the work done in [4] we obtain the next theorem.

Theorem 2.1. *Let* $a, b \in \mathbb{R}_+$, $\alpha, \beta, \gamma \in \mathbb{C}$, $\mathrm{Re}\ \alpha \ge 0$, $\alpha+\beta \in \mathbb{R}_+$, $\alpha a+\beta b+\gamma a \in \mathbb{R}_+$,

$$\delta < \left(\frac{n^3}{8} + a^3\right) \mathrm{Re}\ \alpha + \frac{3an^2}{4}(\alpha+\beta) + \frac{3an}{2}(\alpha a+\beta b+\gamma a) + \frac{3ab^2}{4} \mathrm{Re}\ \beta$$

and n *be a positive integer. Suppose that the functions* $A, B : U \to \mathbb{C}$ *satisfy*

(i) $\mathrm{Re}\ A(z) > -\dfrac{3n^3}{8}\mathrm{Re}\ \alpha - \dfrac{3an^2}{2}(\alpha+\beta) - \dfrac{3an}{2}(\alpha a+\beta b+\gamma a)$;

$$(ii)\quad \mathrm{Im}^2 B(z) \le 4\left[\frac{3n^3}{8}\mathrm{Re}\ \alpha + \frac{3an^2}{2}(\alpha+\beta) + \frac{3an}{2}(\alpha a+\beta b+\gamma a) + \mathrm{Re}\ A(z)\right] \cdot$$
$$\cdot \left[\left(\frac{n^3}{8}+a^3\right)\mathrm{Re}\ \alpha + \frac{3an^2}{4}(\alpha+\beta) + \frac{3an}{2}(\alpha a+\beta b+\gamma a) + \frac{3ab^2}{4}\mathrm{Re}\ \beta - \delta\right]. \tag{2.1}$$

If $p \in \mathcal{H}[1,n]$ *and*

$$\begin{aligned} &\mathrm{Re}\ [A(z)p^2(z) + B(z)p(z) + \alpha(zp'(z)-a)^3 \\ &-3a\beta\left(zp'(z)-\frac{b}{2}\right)^2 + 3a^2\gamma(zp'(z)) + \delta] > 0 \end{aligned} \tag{2.2}$$

then

$$\mathrm{Re}\ p(z) > 0.$$

Proof. We let $\psi : \mathbb{C}^2 \times U \to \mathbb{C}$ be defined by

$$\begin{aligned} \psi(p(z), zp'(z); z) &= A(z)p^2(z) + B(z)p(z) + \alpha(zp'(z)-a)^3 \\ &\quad -3a\beta\left(zp'(z)-\frac{b}{2}\right)^2 + 3a^2\gamma(zp'(z)) + \delta]. \end{aligned} \tag{2.3}$$

From (2.2) we get

$$\mathrm{Re}\ \psi(p(z), zp'(z); z) > 0, \quad z \in U. \tag{2.4}$$

For $\sigma, \rho \in \mathbb{R}$ satisfying $\sigma \le -\frac{n}{2}(1+\rho^2)$, hence

$$-\sigma^2 \le -\frac{n^2}{4}(1+\rho^2)^2,\ \sigma^3 \le -\frac{n^3}{8}(1+\rho^2)^3$$

and $z \in U$, by using (2.1) we obtain

$$\begin{aligned}
&\mathrm{Re}\ \psi(\rho i, \sigma; z)\\
&= \mathrm{Re}\ [A(z)(\rho i)^2 + B(z)\rho i + \alpha(\sigma - a)^3 - 3a\beta\left(\sigma - \frac{b}{2}\right)^2 + 3a^2\gamma\sigma + \delta]\\
&= -\rho^2 \mathrm{Re}\ A(z) - \rho \mathrm{Im}\ B(z) + (\sigma^3 - a^3)\mathrm{Re}\ \alpha - 3a(\alpha+\beta)\sigma^2\\
&\quad +3a\left(\alpha a + \beta b + \gamma a\right)\sigma - \frac{3ab^2}{4}\mathrm{Re}\ \beta + \delta\\
&\le -\rho^2 \mathrm{Re}\, A(z) - \rho \mathrm{Im}\ B(z) - \frac{n^3}{8}(1+\rho^2)^3 \mathrm{Re}\ \alpha - a^3 \mathrm{Re}\ \alpha - \frac{3an^2}{4}(\alpha+\beta)(1+\rho^2)^2\\
&\quad -\frac{3an}{2}\left(\alpha a + \beta b\gamma a\right)(1+\rho^2) - \frac{3ab^2}{4}\mathrm{Re}\ \beta + \delta\\
&= -\frac{n^3}{8}\rho^6 \mathrm{Re}\ \alpha - \left[\frac{3n^3}{8}\mathrm{Re}\ \alpha + \frac{3an^2}{4}(\alpha+\beta)\right]\rho^4\\
&\quad -\left[\left(\frac{3n^3}{8}\mathrm{Re}\ \alpha + \frac{3an^2}{2}(\alpha+\beta) + \frac{3an}{2}\left(\alpha a + \beta b + \gamma a\right) + \mathrm{Re}\ A(z)\right)\rho^2 + \right.\\
&\quad +\rho \mathrm{Im}\ B(z) + \left(\frac{n^3}{8} + a^3\right)\mathrm{Re}\ \alpha + \frac{3an^2}{4}\left(\alpha+\beta\right)\\
&\quad \left. +\frac{3an}{2}\left(\alpha a + \beta b + \gamma a\right) + \frac{3ab^2}{4}\mathrm{Re}\ \beta - \delta\right] \le 0.
\end{aligned}$$

By using Lemma 1.1 we have Re $p(z) > 0$. □

Remark 2.1. For $a = 1$ similar results were obtained in [1], for $b = 0$ the result were obtained earlier by the author in [5] and for $a = 1$ and $b = 0$ we reobtain a result from [2].

Taking $\beta = \gamma = \bar{\alpha}$ in the Theorem 2.1, we have

Corollary 2.1. *Let* $a, b \in \mathbb{R}_+$, $\alpha \in \mathbb{C}$, Re $\alpha \ge 0$,

$$\delta < \left(\frac{n^3}{8} + a^3 + \frac{3an^2}{2} + \frac{3an}{2}(2a+b) + \frac{3ab^2}{4}\right)\cdot \mathrm{Re}\ \alpha$$

and n *be a positive integer. Suppose that the functions* $A, B : U \to \mathbb{C}$ *satisfy*

$$\begin{aligned}
&(i) \quad \mathrm{Re}\ A(z) > \left[-\frac{3n^3}{8} - 3an^2 - \frac{3an}{2}(2a+b)\right]\mathrm{Re}\ \alpha;\\
&(ii) \quad \mathrm{Im}^2 B(z) \le 4\cdot\left[\left(\frac{3n^3}{8} + 3an^2 - \frac{3an}{2}(2a+b)\right)\cdot \mathrm{Re}\ \alpha + \mathrm{Re}\ A(z)\right]\cdot\\
&\qquad \cdot\left[\left(\frac{n^3}{8} + a^3 + \frac{3an^2}{2} + \frac{3an}{2}(2a+b) + \frac{3ab^2}{4}\right)\mathrm{Re}\ \alpha - \delta\right].
\end{aligned} \tag{2.5}$$

If $p \in \mathcal{H}[1,n]$ *and*

$$\begin{aligned} \text{Re } [A(z)p^2(z) + B(z)p(z) + \alpha(zp'(z) - a)^3 \\ -3a\bar{\alpha}\left(zp'(z) - \frac{b}{2}\right)^2 + 3a^2\bar{\alpha}(zp'(z)) + \delta] > 0 \end{aligned} \tag{2.6}$$

then

$$\text{Re } p(z) > 0.$$

Taking $\alpha + \beta = \alpha a + \beta b + \gamma a = \alpha + \gamma = 1$ in the Theorem 2.1, we obtain

Corollary 2.2. *Let* $a, b \in \mathbb{R}_+$, $\alpha \in \mathbb{C}$, $\text{Re } \alpha \geq 0$,

$$\delta < \left(\frac{n^3}{8} + a^3\right) \text{Re } \alpha + \frac{3an^2}{4} + \frac{3an}{2} + \frac{3ab^2}{4}(1 - \alpha)$$

and n *be a positive integer. Suppose that the functions* $A, B : U \to \mathbb{C}$ *satisfy*

$$\begin{aligned} &(i) \qquad \text{Re } A(z) > -\frac{3n^3}{8}\text{Re } \alpha - \frac{3an^2}{2} - \frac{3an}{2}; \\ &(ii) \qquad \text{Im }^2 B(z) \leq\ 4 \cdot \left[\frac{3n^3}{8}\text{Re } \alpha + \frac{3an^2}{2} + \frac{3an}{2} + \text{Re } A(z)\right] \cdot \\ &\qquad \cdot \left[\left(\frac{n^3}{8} + a^3\right) \text{Re } \alpha + \frac{3an^2}{4} + \frac{3an}{2} + \frac{3ab^2}{4}(1 - \alpha) - \delta\right]. \end{aligned} \tag{2.7}$$

If $p \in \mathcal{H}[1,n]$ *and*

$$\begin{aligned} \text{Re } [A(z)p^2(z) + B(z)p(z) + \alpha(zp'(z) - a)^3 - 3a(1 - \alpha)\left(zp'(z) - \frac{b}{2}\right)^2 \\ +3a^2(1 - \alpha)(zp'(z)) + \delta\big] > 0 \end{aligned} \tag{2.8}$$

then

$$\text{Re } p(z) > 0.$$

Taking $\alpha = 0$ in the Theorem 2.1, we have

Corollary 2.3. *Let* $a, b \in \mathbb{R}_+$, $\beta, \gamma > 0$, $\delta < \dfrac{3an^2}{4}\beta + \dfrac{3an}{2}(\beta b + a\gamma) + \dfrac{3an^2}{4}$ *and* n *be a positive integer. Suppose that the functions* $A, B : U \to \mathbb{C}$ *satisfy*

$$\begin{aligned} &(i) \qquad \text{Re } A(z) > -\frac{3an^2}{2}\beta - \frac{3an}{2}(\beta b + a\gamma); \\ &(ii) \qquad \text{Im }^2 B(z) \leq 4\left(\frac{3an^2}{2}\beta + \frac{3an}{2}(\beta b + a\gamma) + \text{Re } A(z)\right) \cdot \\ &\qquad \cdot \left(\frac{3an^2}{4}\beta + \frac{3an}{2}(\beta b + a\gamma) - \delta\right). \end{aligned} \tag{2.9}$$

If $p \in \mathcal{H}[1,n]$ *and*

$$\text{Re}\,[A(z)p^2(z) + B(z)p(z) - 3a\beta\left(zp'(z) - \frac{b}{2}\right)^2 + 3a^2\gamma(zp'(z)) + \delta] > 0 \tag{2.10}$$

then

$$\text{Re } p(z) > 0.$$

Taking $\beta = \gamma = 0$ in the Theorem 2.1, we obtain

Corollary 2.4. *Let $a, b \in \mathbb{R}_+$, $\alpha > 0$, $\delta < \left(\frac{n^3}{8} + a^3 + \frac{3an^2}{4} + \frac{3a^2n}{2}\right) \cdot \alpha$ and n be a positive integer. Suppose that the functions $A, B : U \to \mathbb{C}$ satisfy*

$$\begin{aligned} &(i) \qquad \text{Re } A(z) > -\left(\frac{3n^3}{8} - \frac{3an^2}{2} - \frac{3a^2n}{2}\right) \cdot \alpha; \\ &(ii) \qquad \text{Im }^2 B(z) \le 4\left[\left(\frac{3n^3}{8} + \frac{3an^2}{2} + \frac{3a^2n}{2}\right)\alpha + \text{Re } A(z)\right] \cdot \\ &\qquad\qquad \cdot \left[\left(\frac{n^3}{8} + a^3 + \frac{3an^2}{4} + \frac{3a^2n}{2}\right)\alpha - \delta\right]. \end{aligned} \tag{2.11}$$

If $p \in \mathcal{H}[1, n]$ and

$$\text{Re } [A(z)p^2(z) + B(z)p(z) + \alpha(zp'(z) - a)^3 + \delta] > 0 \tag{2.12}$$

then

$$\text{Re } p(z) > 0.$$

Letting $n = 1$, $a = 1$, $b = 1$, $\alpha = 2 + i$, $\delta = 15$, $A(z) = 1 - z$ and $B(z) = 1 + 2z$ in Corollary 2.1, we have

Example 2.1. *If $p \in \mathcal{H}[1, 1]$ and*

$$\begin{aligned} &\text{Re}\,[(1 - z)p^2(z) + (1 + 2z)p(z) + (2 + i)(zp'(z) - 1)^3 \\ &\quad -3(2 - i)\left(zp'(z) - \frac{1}{2}\right)^2 + 3(2 - i)(zp'(z)) + \delta] > 0 \end{aligned} \tag{2.13}$$

then

$$\text{Re } p(z) > 0.$$

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Received 29.01.2009, Revisted 25.09.2009

J o u r n a l of
Mathematics
and Applications

No 32, pp 91-94 (2010)

An interpolation of polynomials with application to some Volterra operator pencils

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Submitted by: *Jaroslav Zemánek*

ABSTRACT: We study the behaviour of differences of the Laguerre polynomials on $L^2(0,1)$ space. The main result yields

$$||L_m(V)^n - L_m(V)^{n+1}|| \leq \frac{\text{const}}{\sqrt{n}} \text{ as } n \to \infty$$

for natural fixed m

AMS Subject Classification: *47A30, 47G10*
Key Words and Phrases: *Volterra operator, Laguerre polynomials*

1. Introduction

An operator A is power-bounded if

$$\sup_{n \geq 0} ||A^n|| < \infty.$$

Denote by V the classical Volterra operator

$$(Vf)(x) = \int_0^x f(s)ds, \quad \text{on } L^p(0,1), \ 1 \leq p \leq \infty$$

and by $L_m(V)$ the Laguerre polynomials generated by the Volterra operator

$$L_m(V) = \sum_{k=0}^{m} \binom{m}{k} (-1)^k \frac{V^k}{k!} \ , (m \geq 1),$$

[1]The first author supported by the European Community Program "Operator theory methods for differential equations" (TODEQ)

respectively.

In 1987, Allan [1, p. 15] recorded the observation made by Pedersen that $I-V$ is similar to $(I+V)^{-1}$, namely

$$S^{-1}(I-V)S=(I+V)^{-1}$$

where $(Sf)(t)=e^t f(t)$.

By Halmos [3, Problem 150] we know that $||(I+V)^{-1}||=1$ on $L^2(0,1)$. Hence $I-V$ is a power-bounded on $L^2(0,1)$. In 1945, Hille [4, Theorem 11] proved that $||(I-V)^n||_1=O(n^{\frac{1}{4}})$.

It was proved in [6, p. 770] that

$$||(I-V)^{n+1}-(I-V)^n||_p \leq n^{-\frac{1}{2}+|\frac{1}{4}-\frac{1}{2p}|}$$

and this estimation is sharp.

By [9, Theorem 7] and [2, Theorem 9.1] we know that

$$\lim_{n\to\infty}||L_m(V)^n-L_m(V)^{n+1}||=0$$

for natural fixed m.

We shall find an upper estimate of the differences of consecutive powers of the Laguerre polynomials on $L_2(0,1)$.

In the proof of Theorem 2, we shall use the following interpolation result, which seems to have an independent interest.

2. The results

Theorem 1. Let $p(z)=(z-a_1)(z-a_2)...(z-a_n)$ be a polynomial with the positive roots $a_1<a_2<...<a_n$. Then there is a polynomial $q(z)$, also with positive roots only, such that

$$p(z)=(1-\omega)+\omega q(z)$$

for some $0<\omega<1$.

Proof. Observe that $p(z)$ is real for all real z. By the Rolle theorem, the derivative $p^{'}(z)$ has roots $c_1,c_2,...$ such that $a_1<c_1<a_2<c_2<...<a_{m-1}<c_{m-1}<a_m$. Hence $p(c_k)\neq 0$ for $k=1,2,...m-1$. Let

$$0<\epsilon<\min_{k\in 1,2,\ldots,m-1}(|p(c_k)|,1).$$

Then the new polynomial

$$q(z)=\frac{p(z)-\epsilon}{1-\epsilon}$$

will have all the roots positive again, and with $\omega=1-\epsilon\in(0,1)$, we shall have $p(z)=\epsilon+(1-\epsilon)q(z)=(1-\omega)+\omega q(z)$.

Theorem 2. The following estimate holds for the Laguerre polynomials generated by the Volterra operator

$$||L_m(V)^n - L_m(V)^{n+1}|| \leq \frac{\text{const}}{\sqrt{n}}$$

on $L^2(0,1)$ space for natural fixed m.

Proof. Recall that the zeros of the Laguerre polynomials $L_m(\cdot)$ are real, positive and simple (see [5], p. 84). By Theorem 2, there exists a polynomial $Q_m(V)$ such that

$$L_m(V) = (1-\omega)I + \omega Q_m(V)$$

for some $0 < \omega < 1$.

Now the the Nevanlinna theorem [7, Theorem 4.5.3] yields

$$||L_m(V)^n - L_m(V)^{n+1}|| \leq \frac{\text{const}}{\sqrt{n}}$$

which gets the claim.

Question. What about the lower estimation of $||L_m(V)^n - L_m(V)^{n+1}||$ on $L^2(0,1)$ space ?

Acknowledgment. We are grateful to Professors Yuri Lyubich and Jaroslav Zemánek for useful discussions.

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Received 17.09.2009

J o u r n a l of
Mathematics
and Applications
No 32, pp 95-101 (2010)

The harmonic center of a trilateral and the Apollonius point of a triangle

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Submitted by: *Leopold Koczan*

ABSTRACT: In the paper, we observe thet the harmonic center of trilateral $D(z_1, z_2, z_3)$, where D is a disc on $\overline{\mathbb{C}}$ and the Apollonius point of a triangle $\Delta(z_1, z_2, z_3)$ are the same point on $\overline{\mathbb{C}}$. This simply fact has some interesting observations

AMS Subject Classification: *30C85, 30C75*
Key Words and Phrases: *The harmonic center of a trilateral, the Apollonius point of a triangle, Möbius transformation*

1. Introduction

A harmonic center of trilateral was introduced by Wilczek K. in [6] and was used to study distortion properties of quasihomographies of a Jordan curve, see [5], quasiconformal mapping on the unit disc, see [9]. Using the notion of a harmonic center was introduce the harmonic reflection, see [7], a harmonic quadrilaterals, see [8] and some extension operator, see [7], [4].

We will denote by G a Jordan domain on $\overline{\mathbb{C}}$, $G \subset \overline{\mathbb{C}}$ and by Γ its boundary positve oriented with respect to G, $\Gamma = \partial G \subset \overline{\mathbb{C}}$.

Let α will be any arc of Γ, $\alpha \subset \Gamma$. By $\omega(z; \alpha, \Gamma)$ or equivalent $\omega(z; \alpha, G)$ we will denote the harmonic measure of the arc α in a domain G at the point $z \in G$, shortly: the harmonic measure of the arc α at the point z.

We recall some geometrical properties of the harmonic measure:

1. The harmonic measure is a probability measure on a curve Γ.
2. The harmonic measure is a conformal invariant, that means if $F : G \to G'$ are conformal mapping than:

$$\omega(z; \alpha, \Gamma) = \omega(F(z); F(\alpha), F(\Gamma))$$

3. If $G = \mathbb{U} := \{z \in \overline{\mathbb{C}}; \operatorname{Im} z \geq 0\}$, $A, B \in \mathbb{R}$, $\overline{AB}$ are the segment on real line $\overline{\mathbb{R}}$ and φ is the oriented angle of view of the segment $\overline{AB}$ from the point $z \in U$ than
$$\omega(z; \overline{AB}, \mathbb{U}) = \frac{1}{\pi} \arg \frac{z-A}{z-B} = \frac{1}{\pi}\varphi \tag{1}$$
4. A constatnt value line $L_{\mathbb{U}}\left(\overline{AB}, \lambda\right) = \left\{z \in \mathbb{U} : \omega(z; \overline{AB}, U) = \lambda\right\}, \lambda \in [0,1]$ is a circle intersect the real line for an angle $\pi\lambda$ and containing points A, B.
5. If $G = \mathbb{D} := \{z \in \overline{\mathbb{C}}; |z| \leq 1\}$, $z_1, z_2 \in \mathbb{T} = \{z : |z| = 1\}$, $\widehat{z_1 z_2}$ are the segment on unit circle $\mathbb{T}$ and φ is the angle of view of the segment $\widehat{z_1 z_2}$ from the point $z \in U$ and ϑ is the angle of view the arc $\widehat{z_1 z_2}$ from any point $z^* \in \mathbb{T}$ than
$$\omega(z; \widehat{z_1 z_2}, \mathbb{T}) = \frac{1}{\pi} \arg \frac{z-z_2}{z-z_1} : \frac{1-z_2}{1-z_1} = \frac{1}{\pi}(\varphi - \vartheta). \tag{2}$$
6. A constatnt value line $L_{\mathbb{D}}(\widehat{z_1 z_2}, \lambda) = \{z \in \mathbb{D} : \omega(z; \widehat{z_1 z_2}, \mathbb{D}) = \lambda\}, \lambda \in [0,1]$ is a circle intersect the unit circle for an anglr $\pi\lambda$ and containing points z_1, z_2.
7. If $G = \mathbb{D}^* := \{z \in \overline{\mathbb{C}}; |z| \geq 1\}$, $z_1, z_2 \in \mathbb{T}^* = \{z : |z| = 1\}$, $\widehat{z_1 z_2}$ are the segment on unit circle $\mathbb{T}$ and φ^* is the angle of view of the segment $\widehat{z_2 z_1}$ from the point $z \in \mathbb{D}^*$ and ϑ is the angle of view the arc $\widehat{z_1 z_2}$ from any point $z^* \in \mathbb{T}$ than
$$\omega(z; \widehat{z_2 z_1}, \mathbb{T}^*) = \frac{1}{\pi}(\varphi^* + \vartheta) \mod 1. \tag{3}$$

2. The harmonic center of a trilateral

A configuration $\Gamma(z_1, z_2, z_3)$ formed by Jordan curve $\Gamma \subset \overline{\mathbb{C}}$ and triple points z_1, z_2, z_3 ordered according to the oriented curve Γ are named by a trilateral.

By a conjugate trilateral we understand $\Gamma^*(z_1, z_2, z_3)$, where $\Gamma^* = \partial G^*$ and $G^* = \overline{\mathbb{C}} \setminus G$. The curve Γ^* has oposit orientation to Γ.

We recall that (by Riemmnann Theorem) every two trilaterals are conformal equivalent, that means any trilateral can be conformally mapped onto another trilateral.

Definition 2.1 [5] *By a harmonic center of a trilateral* $\Gamma(z_1, z_2, z_3)$ *we colled a point* $s \in G$, $s = s_\Gamma(z_1, z_2, z_4)$ *such that the harmonic measure of the arcs* $\widehat{z_1 z_2}$, $\widehat{z_2 z_3}$, $\widehat{z_3 z_1}$ *are the same and equal to* $1/3$
$$\omega(s; \widehat{z_1 z_2}, \Gamma) = \omega(s; \widehat{z_2 z_3}, \Gamma) = \omega(s; \widehat{z_3 z_1}, \Gamma)$$

Geometrically the harmonic center s of the trilateral is the point of intersection of three constant value lines of the harmonic measure, respectivelly of the arcs $\widehat{z_1 z_2}$, $\widehat{z_2 z_3}$, $\widehat{z_3 z_1}$ for value $\lambda = 1/3$.

Fact 2.1 *The harmonic center of a trilateral is a conformal invariant, it means that if* $F : G \to G'$ *is conformal mapping (on domain* $G \subset \overline{\mathbb{C}}$*) and* $D(z_1, z_2, z_3)$ *is arbitrary trilateral asociatted with some domain* $D \subset \overline{G}$ *than*
$$s_D(z_1, z_2, z_3) = s_{F(D)}(F(z_1), F(z_2), F(z_3)).$$

Examples 2.1 We can construct (using (1) or (2)) the harmonic center of some trilaterals associate with the real line or the unit circle.

1. Let $A, B, C \in \mathbb{R}$ be fixed and ordered points on the real line. Using (1) we can calculate that
$$s_{\mathbb{R}} = (A, B, C) = -\frac{BCp_1 + CAp_3 + ABp_2}{Ap_1 + Bp_2 + Cp_3}.$$
Note that the both angles of view of the sides $\overline{AB}$ and $\overline{BC}$ are equal to $\pi/3$.

2. Let $\mathbb{T}(p_1, p_2, p_3)$ be a trilateral, where p_k, $k = 1, 2, 3$ are cube roots of 1. Using (2) we can deduced that $s_{\mathbb{T}}(p_1, p_2, p_3) = 0$.

3. Let $z_1, z_2, z_3 \in \mathbb{T}$ be fixed and ordered points on the unit circle. Using (2) and Fact 2.1 we can calculate that
$$s_{\mathbb{T}} = (z_1, z_2, z_3) = -\frac{z_2z_3p_1 + z_3z_1p_3 + z_1z_2p_2}{z_1p_1 + z_2p_2 + z_3p_3}. \tag{4}$$

Fact 2.2 [7] *The harmonic centers* $s = s_{\mathbb{T}}(z_1, z_2, z_3)$ *and* $s^* = s^*_{\mathbb{T}}(z_1, z_2, z_3)$ *of arbitrary trilateral (associated with unit circle) and its conjugate trilateral are symmetric points with respect to the unit circle* $\mathbb{T}$*. Exactly:* $s^* = 1/\overline{s}$*.*

3. The Apollonius point of a triangle

The Apollonius circles is the well known and old concept, but the Apollonius point of triangle was introduced by H. Haruki and T.M. Rassian in [2] in 1996. H. Haruki and T.M. Rassian used this new concept to some characterisation of Möbius transformation from the standpoin of conformal mapping.

The notion of Apollnius point was the basis to constructed Apollonius quadruple and another, see [2]. [3].

We start at the well known concept Apollonius circle.

Definition 3.1 (Apollonius circle) *The locus of points* $P \in \overline{\mathbb{C}}$ *such that the ratio of its distanas from two fixed complex points* $A, B \in \overline{\mathbb{C}}$ *is a constans value is called the Apollonius circle.*

For every Apollonius circle the points A *and* B *are symetric with respect this circle.*

We will be write $\mathcal{A}(A, B; k)$ for the Apollonius circle designated by the points A, B and ratio $k > 0$.

Theorem 3.1 (Apollonius Theorem) [2], [1] *The Apollonius circle is:*

(i) *the perpendicular bisector of the segment* $\overline{AB}$ *if* $k = 1$,

(ii) *the circle on diameter* $\overline{IE}$*, where the points* I*,* E *lie on straight line* AB *and* $|A - I| = k \cdot |I - B|$ *and* $|E - A| = k \cdot |E - B|$ *if* $k \neq 1$.

Definition 3.2 *Let $\Delta\, ABC \subset \overline{\mathbb{C}}$ be an arbitrary triangle and $L \in \overline{\mathbb{C}}$. We denote lengths of respectifly segments*

$$a = |B - C|, \quad b = |C - A|, \quad c = |A - B|,$$
$$x = |L - A|, \quad y = |L - B|, \quad z = |L - C|.$$

If

$$ax = by = cz \tag{5}$$

holds, then the point L is said to be an Apollonius point of a triangle $\Delta\, ABC$.

It is easy to observe

Fact 3.1 *Let $\Delta\, ABC$ be a triangle. If it is not equilateral triangle than exist exactly two Apollonius pints L_1, L_2 of triangle $\Delta\, ABC$. Points L_1 and L_2 are intersection of three Apollonius circles $\mathcal{A}\left(A, B, \frac{a}{b}\right)$, $\mathcal{A}\left(B, C, \frac{b}{c}\right)$, $\mathcal{A}\left(C, A, \frac{c}{a}\right)$. If $\Delta\, ABC$ is equilateral triangle then L_1 is a centroid of this triangle and we can say $L_2 = \infty$.*

Examples 3.1 [2] Let $\Delta\, ABC$ be a triangle on $\overline{\mathbb{C}}$ and L_1 and L_2 be a Apollonius points of the triangle $\Delta\, ABC$

1. If $\Delta\, ABC$ is a equilateral triangle then L_1 is a centroid of triangle $\Delta\, ABC$.
2. If $\Delta\, ABC$ is an isoscales triangle, with equal angles $\pi/6$ at the ends of its base BC than L_1 is a midpoint of segment BC and the point L_2 is the third point of eqiulateral triangle $\Delta\, BCL_2$.
3. If $\Delta\, ABC$ is an isoscales triangle, with equal angles $\pi/12$ at the ends of its base BC than L_1 is a symetric point of A with respect a segment BC and L_2 is the third point of isoscales right-angled triangle $\Delta\, BCL_2$.
4. If $\Delta\, ABC$ is a triangle on the complex plane such that

   $$\alpha = \angle BAC = 90^\circ, \quad \beta = \angle CBA = 60^\circ, \quad \gamma = \angle ACB = 30^\circ$$

   then the point L_1 is a point inside $\Delta\, ABC$ such that

   $$\delta_A = \angle BL_1C = 150^\circ, \quad \delta_B = \angle CL_1A = 120^\circ, \quad \delta_C = \angle AL_1B = 90^\circ$$

   and the second Apollonius point L_2 is the symmetric point of C with respect to the side AB. Note that

   $$\xi_A = \angle CL_2B = 330^\circ, \quad \xi_B = \angle AL_2C = 0^\circ, \quad \xi_C = \angle BL_2A = 30^\circ,$$

In the paper [2] authors investigated some properties of mapping $w = f(z)$ on a domian $R \subset \overline{\mathbb{C}}$ expressed by Apollonius points and Apollonius quadrilaterals. From this we deduce

Theorem 3.2 *Let $w = f(z)$ be the Möbius transformation on $\overline{\mathbb{C}}$, two points $L_1, L_2 \in \overline{\mathbb{C}}$ and any triangle $\Delta\, ABC \subset \overline{\mathbb{C}}$. Denote*

$$A' = f(A), \quad B' = f(B), \quad C' = f(C), \quad L_1' = f(L_1), \quad L_2' = f(L_2).$$

If L_1 and L_2 are the Apollonius points of triangle $\Delta\, ABC$ than L_1' and L_2' are the Apollonius points of triangle $\Delta\, A'B'C'$.

4. Main results

First we observe (using Examples 2.1 and Example 3.1) that the harmonic center and the Applonius point can be the same point, for example.

1. If $T_1 = \mathbb{T}(p_1, p_2, p_3)$, where $p_1 = 1, p_2 = \frac{1}{2}(-1 + i\sqrt{3}), p_3 = \frac{1}{2}(-1 - i\sqrt{3})$ are cube roots of 1, we have $s = 0$ and the Apollonius point of an eqilateral triangle $\Delta\, p_1p_2p_3$ is the point $L_1 = 0$.

   That means $s = L_1$.

2. If $T_2 = \mathbb{T}(1, z_2, z_3)$, where $z_2 = e^{i\pi/3} = \frac{1}{2}(1 + i\sqrt{3})$, $z_3 = e^{-i\pi/3} = \frac{1}{2}(1 - i\sqrt{3})$ we can calculate from (2) that $s = \frac{1}{2}$ and $s^* = 2$.

   The triangle $\Delta\, z_1z_2z_3$ is a isosceles triangle with equal angles 30° then from Example 3.1 we have $L_1 = \frac{1}{2}$ and $L_2 = 2$.

   That means $s = L_1$ and $s^* = L_2$.

3. If $T_3 = \mathbb{T}(1, z_2, z_3)$, where $z_2 = e^{i\pi/6} = \frac{1}{2}(\sqrt{3} + i)$, $z_3 = e^{-i\pi/6} = \frac{1}{2}(\sqrt{3} - i)$ we can calculate from (2) that $s = \sqrt{3} - 1$ and $s^* = \frac{1+\sqrt{3}}{2}$.

   The triangle $\Delta\, z_1z_2z_3$ is a isosceles triangle with equal angles 30° then from Example 3.1 we have $L_1 = \sqrt{3} - 1$ and $L_2 = \frac{1+\sqrt{3}}{2}$.

   That means $s = L_1$ and $s^* = L_2$.

We can generalize above observation and obtain

Theorem 4.1 *Let $\Gamma \subset \overline{\mathbb{C}}$ be any circle on $\overline{\mathbb{C}}$. For arbitrary three points $z_1, z_2, z_3 \in \Gamma$ ordered according to the positive orientation to Γ we have*

$$s_\Gamma(z_1, z_2, z_3) = L_1, \qquad s_{\Gamma^*}(z_1, z_2, z_3) = L_2 \tag{6}$$

where L_1 and L_2 are the Apollonius points of the triangle $\Delta z_1z_2z_3$.

Proof. For a point $z_1, z_2, z_3 \in \mathbb{T}$ we denote

$$a = |z_2 - z_3|, \quad b = |z_3 - z_1|, \quad c = |z_1 - z_2|, \quad x = |s - z_1|, \quad y = |s - z_2|, \quad z = |s - z_3|.$$

We can calculate that the point s designated from (4) satisfies condition (5) for Apollonius point of triangle $\Delta\, z_1z_2z_3$.

By analogy we can check that the point $s^* = \frac{1}{s}$ wchich is the harmonic center of conjugate trilateral also satisfies condition (5).

The proof of Theorem 4.1 follows from the above observations, Theorem 3.2 and Fact 2.1.

Fact 4.1 *Let $\Delta\, ABC$ be arbitrary triangle on $\mathbb{C}$ and L be a point on $\mathbb{C}$. Denote*

$$\alpha = \angle BAC, \quad \beta = \angle CBA, \quad \gamma = \angle ACB,$$

and

$$\delta_A = \angle BL_1C, \quad \delta_B = \angle CL_1A, \quad \delta_C = \angle AL_1B,$$

and

$$\xi_A = \angle CL_2B, \quad \xi_B = \angle AL_2C, \quad \xi_C = \angle BL_2A,$$

If

$$\delta_A - \alpha = \delta_B - \beta = \delta_C - \gamma = \frac{\pi}{3}$$

and

$$\xi_A + \alpha = \xi_B + \beta = \xi_C + \gamma = \frac{\pi}{3} \mod \pi$$

holds than L_1 *and* L_2 *are two Apollonius points of the triangle* $\Delta\, ABC$.

A ilustration of this fact is a poin 4 of Examples 3.1.

Proof. It is corollary of Theorem 4.1 and property (2) and (3) of the harmonic measure.

Fact 4.2 *Let* $A, B, C \in \mathbb{T}$, *and* L_1 *and* L_2 *be two Apollonius points of triangle* $\Delta\, ABC$, *then*

$$L_1 = \frac{BCp_1 + CAp_3 + ABp_2}{Ap_1 + Bp_2 + Cp_3}, \quad L_2 = \frac{1}{\overline{L_1}}.$$

The point L_1 *is the point of intersection of three circles intersect the unit circle for angle* $\pi/3$ *and containing, respectivle, points* z_1, z_2, *or* z_2, z_3, *or* z_3, z_1.

Proof. It is corollary of Theorem 4.1 and condition property (4) of the harmonic center of the trilateral $\mathbb{T}(A, B, C)$

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Received 7.01.2010

J o u r n a l of
Mathematics
and Applications

No 32, pp 103-107 (2010)

Notes on the ideal-based zero-divisor graph

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ABSTRACT: Let I be an ideal of a commutative ring R. Denote by $S(I)$ the set $\{x \in R \mid xy \in I$ for some $y \in R \setminus I\}$. The zero-divisor graph of R with respect to I is an undirected graph, denoted by $\Gamma_I(R)$, with vertices $S(I) \setminus I$ where distinct vertices x and y are adjacent if and only if $xy \in I$. In this paper we study the diameter and the girth of $\Gamma_I(R)$, when the prime ideals of R contained in $S(I)$ are linearly ordered

AMS Subject Classification: *13A05, 05C75*
Key Words and Phrases: *Ideal-based zero-divisor graph, Primal ideal*

1. Introduction

The idea of associating a graph with the zero-divisors of a commutative ring was introduced by Beck in 1988, where the author talked about the colorings of such graphs. By the definition he gave, every element of the ring R was a vertex in the graph, and two vertices x, y were adjacent if and only if $xy = 0$ ([3]). We adopt the approach used by D. F. Anderson and P. S. Livingston ([1]) and consider only non-zero zero-divisors as vertices of the graph. The zero-divisor graph of a commutative ring has been studied extensively by several authors (see, for example, [3, 1, 6, 7, 8]).

Redmond [9] (see also [4, 7]) introduced the definition of the zero-divisor graph with respect to an ideal. Let I be an ideal of a ring R. The zero-divisor graph of R with respect to I is an undirected graph, denoted by $\Gamma_I(R)$, with vertices $\{x \in R\backslash I : xy \in I$ for some $y \in R\backslash I\}$ where distinct vertices x and y are adjacent if and only if $xy \in I$. Therefore, if $I = 0$ then $\Gamma_I(R) = \Gamma(R)$.

For the sake of completeness, we state some definitions and notations used throughout. We will use R to denote a commutative ring with identity. We use $Z(R)$ to denote the set of zero-divisors of R; we use $Z(R)^*$ to denote the set of non-zero zero-divisors of R. By the zero-divisor graph of R, denoted $\Gamma(R)$, we mean the graph whose vertices are the non-zero zero-divisors of R, and for distinct $x, y \in Z(R)^*$, there is an edge connecting x and y if and only if $xy = 0$. A graph is said to be connected if there exists a path between any two distinct vertices. For two distinct vertices a and b in a graph G, the distance between a and b, denoted $\mathrm{d}(a,b)$, is the length of the shortest path connecting a and b, if such a path exists; otherwise, $\mathrm{d}(a,b)=\infty$. The diameter of a connected graph is the supremum of the distances between vertices. We will use the notation $\mathrm{diam}(G)$ to denote the diameter of the graph of G. A complete graph is a graph in which every pair of distinct vertices is connected by an edge. The girth of a graph G, denoted $\mathrm{gr}(G)$, is the length of a shortest cycle in G, provided G contains a cycle; otherwise, $\mathrm{gr}(G)=\infty$. It is shown in [9] that, $\Gamma_I(R)$ is connected with $diam(\Gamma_I(R))\leq 3$ and if $\Gamma_I(R)$ contains a cycle, then the $gr(\Gamma_I(R))$ is $3,4$ or ∞.

Let I be an ideal of R. I is called a radical ideal if $I=\sqrt{I}$, where $\sqrt{I}$ is the set of all elements of a of R with $a^n\in I$ for some positive integer n. I is called quasi-primary if $\sqrt{I}$ is a prime ideal of R. Also we denote by $\overline{a}$, the coset $a+I$ in R/I. We say that R is a chained ring if the ideals of R are linearly ordered by inclusion. It is easy to see that R is a chained ring if and only if eithet $x|y$ or $y|x$ for all $x,y\in R$.

We recall from [5], that an element $a\in R$ is called prime to an ideal I of R if $ra\in I$ (where $r\in R$) implies that $r\in I$. Denote by $S(I)$ the set of all elements of R that are not prime to I. A proper ideal I of R is said to be primal if $S(I)$ forms an ideal; this ideal is always a prime ideal, called the adjoint ideal P of I. In this case we also say that I is a P-primal ideal of R.

2. Results

Theorem 2.1 *let I be an ideal of R, and let x,y be distinct elements of $\sqrt{I}\backslash I$ with $xy\notin I$. Then:*

(1) The ideal (x,y) is not prime to I.

(2) If I is a primary ideal, then $diam(\Gamma_I(R))\leq 2$.

Proof.

(1) Since $x\in\sqrt{I}$, there exists a least positive integer n such that $x^ny\in I$. As $xy\notin I$, $n\geq 2$. Let m be the least positive integer such that $x^{n-1}y^m\in I$. Since $x^{n-1}y\notin I$, $m\geq 2$. Then $x^{n-1}y^{m-1}\in(I:_R(x,y))\backslash I$. This means that the ideal (x,y) is not prime to I.

(2) If I is primary, then $S(I)=\sqrt{I}$. Choose two distinct vertices x,y in $\Gamma_I(R)$. If $xy\in I$, then $d(x,y)=1$. So assume that $xy\notin I$. Then $x,y\in S(I)\backslash I=\sqrt{I}\backslash I$. As in the proof of (1), we may find a path $x-x^{n-1}y^{m-1}-y$ from x to y in $\Gamma_I(R)$. Hence $d(x,y)=2$. Thus $diam(\Gamma_I(R))\leq 2$.

□

Lemma 2.1 *Let I be an ideal of R. If $x\in\sqrt{I}\backslash I$ and $y\in S(I)\backslash I$, then $d(x,y)\leq 2$ in $\Gamma_I(R)$.*

Proof. We may assume that x, y are distinct with $xy \notin I$. Since y is not prime to I, and $xy \notin I$, there is $z \in S(I)\backslash(I \cup \{x\})$ such that $yz \in I$. Since $x \in \sqrt{I}$, there exists a least positive integer n for which $x^n z \in I$. Then $x - x^{n-1}z - y$ is a path of length 2 from x to y in $\Gamma_I(R)$, that is $d(x, y) \leq 2$. □

Lemma 2.2 *Let I be an ideal of R. Assume that $x \in S(I)\backslash\sqrt{I}$ and $y \in S(I)\backslash I$ are such that $\overline{x}|\overline{zy}^n$ in R/I for some integer $n \geq 1$ and $z \in R\backslash S(I)$. Then $d(x, y) \leq 2$ in $\Gamma_I(R)$.*

Proof. We may assume that x and y are distinct with $xy \notin I$. Since $x \in S(I)\backslash\sqrt{I}$, there exists a $w \in S(I)\backslash(I \cup \{x, y\})$ such that $wx \in I$. Since $\overline{x}|\overline{zy}^n$ and since z is prime to I, we have $y^n w \in I$. Let k be the least positive integer such that $y^k w \in I$. Then $x - y^{k-1}w - y$ is a path of length 2 from x to y in $\Gamma_I(R)$. Thus $d(x, y) \leq 2$. □

Theorem 2.2 *Let I be an ideal of R. Then the prime ideals of R contained in $S(I)$ are linearly ordered if and only if for all $x, y \in S(I)$, there is an integer $n = n(x, y) \geq 1$ and an element $z \in R\backslash S(I)$ such that either $\overline{x}|\overline{zy}^n$ or $\overline{y}|\overline{zx}^n$ in R/I.*

Proof. It is easy to see that $Z(R/I) = S(I)/I$. So the prime ideals of R contained in $S(I)$ are linearly ordered if and only if the prime ideals of R/I contained in $Z(R/I)$ are linearly ordered, if and only if the prime ideals of $T(R/I)$ are linearly ordered, if and only if there is an integer $n = n(x, y) \geq 1$ such that either $\overline{x}|\overline{y}^n$ or $\overline{y}|\overline{x}^n$ in $T(R/I)$ (see [2, Theorem 1]). The assertion now easily follows. □

Theorem 2.3 *Let I be an ideal of R with $S(I)^2 \nsubseteq I$ such that the prime ideal of R contained in $S(I)$ are linearly ordered. Then $diam(\Gamma_I(R)) = 2$.*

Proof. If I is a radical ideal, then I is the intersection of all minimal prime ideals containing I. But $S(I)$ is the union of all prime ideals containing I. Consequently as the prime ideals contained in $S(I)$ are linearly ordered, I is a prime ideal. So $I = \sqrt{I} = S(I)$, and so $S(I)^2 \subseteq I$, a contradiction. Therefore I is not a radical ideal. Furthermore, $\Gamma_I(R)$ is not complete by hypothesis. Thus $diam(\Gamma_I(R)) > 1$. As $S(I)^2 \nsubseteq I$, there exist distinct elements $x, y \in \Gamma_I(R)$ such that $xy \notin I$. If $x, y \in \sqrt{I}$, then $d(x, y) = 2$ by Theorem 2.1. If $x \in \sqrt{I}$ and $y \in S(I)\backslash\sqrt{I}$, then $d(x, y) = 2$ by Lemma 2.1. So assume that $x, y \in S(I)\backslash\sqrt{I}$. By Theorem 2.2, there exists an integer $n \geq 1$ and an element $z \in R\backslash S(I)$ such that either $\overline{x}|\overline{zy}^n$ or $\overline{y}|\overline{zx}^n$ in R/I. In any case, $d(x, y) = 2$ by Lemma 2.2. Therefore $diam(\Gamma_I(R)) \leq$ and so $diam(\Gamma_I(R)) = 2$. □

Let I be an ideal of R. Set

$$N_I(R) = \{x \in R | x^2 \in I\}$$

Clearly $I \subseteq N_I(R) \subseteq \sqrt{I}$.

Lemma 2.3 *Let I be an ideal of R such that R/I is a chained ring. Assume that $x, y \in R$. Then*

(1) If $xy \in I$, then either $x \in N_I(R)$ or $y \in N_I(R)$.

(2) If $x, y \in N_I(R)$, then $xy \in I$.

(3) If $x, y \in S(I)\backslash N_I(R)$, then $xy \notin I$.

(4) If $x \in S(I)\backslash I$, then $xz \in I$ for some $z \in N_I(R)\backslash I$.

(5) If $x_1, ..., x_n \in S(I)\backslash I$, then there is a $y \in N_I(R)\backslash I$ such that $x_i y \in I$ for every $1 \leq i \leq n$.

(6) $N_I(R)$ is an ideal of R.

Proof.

(1) Since I is a chained ideal, we may assume that $\overline{x}|\overline{y}$ in R/I. So there exist $z \in R$ and $z \in I$ with $y = xz + a$. Then $y^2 = xyz + ay \in I$ shows that $y \in N_I(R)$.

(2) As in part (1), there exist $z \in R$ and $a \in I$ with $y = xz + a$. Now $xy = x^2 z + ax \in I$.

(3) This is a direct consequence of part (1).

(4) If $x \in N_I(R)$, we can put $y = x$. So assume that $x \in S(I)\backslash N_I(R)$. Since $x \in S(I)$, there exists $y \in R\backslash I$ with $xy \in I$. So $y \in N_I(R)$ by part (3).

(5) Since R/I is a divided ring, there is an integer $1 \leq j \leq n$ such that $\overline{x_j}|\overline{x_i}$ for all $1 \leq i \leq n$. On the other hand, by part (4), there exists a $y \in N_I(R)\backslash I$ with $x_j y \imath I$. So $x_i y \in I$ for all $1 \leq i \leq n$.

(6) It follows from part (2) above.

□

Theorem 2.4 *Let I be an ideal of R such that R/I is a chained ring. Then $diam(\Gamma_I(R)) \leq 2$*

Proof. There is nothing to prove if $|S(I)\backslash I| < 2$. So assume that $|S(I)\backslash I| \geq 2$. Pick two distinct vertices $x, y \in S(I)\backslash I$. If $x, y \in N_I(R)$, then $xy \in I$ by Lemma 2.3(2), and hence $d(x, y) = 1$. If $x \in N_I(R)$ and $y \notin N_I(R)$, then $yz \in I$ for some $z \in N_I(R)\backslash I$ by Lemma 2.3(4). So, by Lemma 2.3(2), $xz \in I$. Hence $d(x, y) \leq 2$. Finally assume that $x, y \in R\backslash N_I(R)$. Then, there exists a $z \in N_I(R)\backslash I$ with $xz, yz \in I$ by Lemma 2.3(5). Therefore $d(x, y) \leq 2$, and hence $dam(\Gamma_I(R)) \leq 2$. □

Finally we give a result about the girth of the graph $\Gamma_I(R)$.

Theorem 2.5 *Let I be a quasi-primary ideal of R with $\sqrt{I} \subsetneq S(I)$ and $|\sqrt{I}\backslash I| \geq 2$. Then $gr(\Gamma_I(R)) = 3$ or ∞.*

Proof. Pick an element $z \in S(I)\backslash\sqrt{I}$. It is easy to see that there exists $w \in \sqrt{I}\backslash I$ with $zw \in I$. If $w^2 \notin I$, and if m is the least positive integer such that $w^m \in I$. Then $w^{m-1} \neq w$, and hence $z - w - w^{m-1} - z$ is a triangle in $\Gamma_I(R)$. So assume that $w^2 \in I$. Choose an element $d \in \sqrt{I}\backslash\{w\}$. If $wd \notin I$, then $z - w - wd - z$ is a triangle in $\Gamma_I(R)$. Now assume that $wd \in I$. If $zd \in I$, then $z - w - d - z$ is a triangle in $\Gamma_I(R)$. So assume that $zd \notin I$. If $w = zd$, then $zd^2 = wd \in I$. Thus $w - z^2 - d - w$ is a triangle in $\Gamma_I(R)$. So we may assume that w and zd are distinct. As $d \in \sqrt{I}$, there exists a least positive integer such that $zd^n \in I$. Then $n > 1$ since $zd \notin I$. Assume that $n > 2$. Clearly $zd^{n-1} \neq d$. If w and zd^{n-1} are distinct, then we have a triangle $w - zd^{n-1} - d - w$ in $\Gamma_I(R)$. If $w = zd^{n-1}$, then $z^2d^{n-1} = zw \in I$. Moreover d^{n-1} and w are distinct since otherwise $zd^{n-1} = zw \in I$, which contradicts the minimality of n. Hence $w - z^2 - d^{n-1} - w$ is a triangle in $\Gamma_I(R)$. Now assume that $n = 2$. Clearly $zd^2 \in I$. If $zd \neq d$, then $w - zd - d - w$ is a triangle in $\Gamma_I(R)$. So assume that $zd = d$. Hence $d^2 = zd^2 \in I$. Since $zw \in I$ and $zd \notin I$, we have $w + d \notin I$. Thus w, d and $w + d$ are all distinct. Since w^2, d^2 and wd all belong to I, $w - w + d - d - w$ is a triangle in $\Gamma_I(R)$. Consequently $gr(\Gamma_I(R)) = 3$ □

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Received 26.01.2009, *Revisted 28.01.2009*

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