Fekete-Szegő Problems for Certain Class of Analytic Functions Associated with Quasi-Subordination

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ABSTRACT: In this paper, we determine the coefficient estimates and the Fekete-Szegő inequalities for $\mathcal{M}_q^\alpha(\gamma, \lambda, \phi)$, the class of analytic and univalent functions associated with quasi-subordination.

1.   Introduction and preliminaries

Let $\mathcal{A}$ be the class of analytic functions defined on the unit disc $\mathbb{U} = \{z \in \mathbb{C} : |z| < 1\}$ with the normalized conditions $f(0) = 0 = f'(0) - 1$. Let $\mathcal{S}$ be the class of all functions $f \in \mathcal{A}$ which are univalent in $\mathbb{U}$. So $f(z) \in \mathcal{S}$ has the form

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n, \quad z \in \mathbb{U}. \tag{1.1}$$

Definition 1.1. For two analytic functions f and g, the function $f(z)$ is subordinate to $g(z)$, written as $f \prec g$, if there exists a Schwarz' function $w(z)$, analytic in $\mathbb{U}$, with $w(0) = 0$, $|w(z)| < 1$, $z \in \mathbb{U}$, such that

$$f(z) = g(w(z)), \quad z \in \mathbb{U}. \tag{1.2}$$

In particular, if the function g is univalent in $\mathbb{U}$, then $f \prec g$ if

$$f(0) = g(0) \quad \text{and} \quad f(\mathbb{U}) \subset g(\mathbb{U}).$$

Let $\phi(z)$ be an analytic and univalent function in $\mathbb{U}$ with $\mathrm{Re} f(z) > 0$, $\phi(0) = 1$ and $\phi'(0) > 0$, which maps the unit disk $\mathbb{U}$ on to a region starlike with respect to 1 and symmetric with respect to real axis. So $\phi(z)$ has the form

$$\phi(z) = 1 + B_1 z + B_2 z^2 + \cdots, \tag{1.3}$$

where all coefficients are real and $B_1 > 0$. Let $h(z)$ be an analytic function in $\mathbb{U}$ and $|h(z)| \leq 1$, such that

$$h(z) = c_0 + c_1 z + c_2 z^2 + \cdots. \tag{1.4}$$

In 1970, Robertson [19] introduced the concept of quasi-subordination as follows:

Definition 1.2. The function f is said to be quasi-subordinate to g, written as

$$f(z) \prec_q g(z), \tag{1.5}$$

if there exist analytic functions h and w, with $|h(z)| \leq 1$, $w(0) = 0$ and $|w| < 1$, such that $\frac{f(z)}{h(z)}$ is analytic in $\mathbb{U}$ and

$$\frac{f(z)}{h(z)} \prec g(z), \quad z \in \mathbb{U}. \tag{1.6}$$

Also the above expression is equivalent to

$$f(z) = h(z)g(w(z)), \quad z \in \mathbb{U}. \tag{1.7}$$

Observe that if $h(z) \equiv 1$, then $f(z) = g(w(z))$, so $f(z) \prec g(z)$ in $\mathbb{U}$. Also if $w(z) = z$, then $f(z) = h(z)g(z)$ and it is said to f is majorized by g and written as $f(z) \ll g(z)$ in $\mathbb{U}$. Hence it is obvious that quasi-subordination is a generalization of subordination and majorization (see [19]).

In [15], Ma and Minda gave unified representation of various subclasses of starlike and convex functions by using subordination. They introduced the classes $\mathcal{S}^*(\phi)$ and $\mathcal{C}(\phi)$ of analytic functions $f \in \mathcal{A}$, that satisfy the conditions $\frac{zf'(z)}{f(z)} \prec \phi(z)$ and $1 + \frac{zf''(z)}{f'(z)} \prec \phi(z)$ respectively, which includes several well-known subclasses. In particular, if $\phi(z) = \frac{1+Az}{1+Bz}$, $(-1 \leq B < A \leq 1)$, the class $\mathcal{S}^*(\phi)$ reduces to the class $\mathcal{S}^*[A, B]$, introduced by Janowski [10]. Also for the choice of $\phi(z) = \frac{1+(1-2\alpha)z}{1-z}$ where $(0 \leq \alpha < 1)$, the class $\mathcal{S}^*(\phi)$ becomes the class of starlike functions of order α.

Motivated by Ma and Minda, Mohd and Darus [14], introduced two classes $\mathcal{S}_q^*(\phi)$ and $\mathcal{C}_q(\phi)$ of analytic functions $f(z) \in \mathcal{A}$, that satisfying the conditions $\frac{zf'(z)}{f(z)} - 1 \prec_q \phi(z) - 1$ and $\frac{zf''(z)}{f'(z)} \prec_q \phi(z) - 1$ respectively, which are analogous to $\mathcal{S}^*(\phi)$ and $\mathcal{C}(\phi)$. They also introduced $\mathcal{M}_q(\alpha, \phi)$ be the class of functions $f(z) \in \mathcal{A}$, that satisfying the condition $(1 - \alpha)\frac{zf'(z)}{f(z)} + \alpha\left(1 + \frac{zf''(z)}{f'(z)}\right) - 1 \prec_q \phi(z) - 1$, where $0 \leq \alpha \leq 1$ [14]. This class is analogous of the well-known class of α-convex functions [16].

Recently, El-Ashwah and Kanas [6], introduced and studied the following subclasses by using quasi-subordination:

$$\mathcal{S}_q^*(\gamma, \phi) = \left\{ f \in \mathcal{A} : \frac{1}{\gamma}\left(\frac{zf'(z)}{f(z)} - 1\right) \prec_q \phi(z) - 1; \; z \in \mathbb{U}, \; 0 \neq \gamma \in \mathbb{C} \right\},$$

and

$$\mathcal{C}_q(\gamma, \phi) = \left\{ f \in \mathcal{A} : \frac{1}{\gamma}\frac{zf''(z)}{f'(z)} \prec_q \phi(z) - 1; \; z \in \mathbb{U}, \; 0 \neq \gamma \in \mathbb{C} \right\}.$$

For $h(z) = 1$, the classes $\mathcal{S}_q^*(\gamma, \phi) = \mathcal{S}^*(\gamma, \phi)$ and $\mathcal{C}_q(\gamma, \phi) = \mathcal{C}(\gamma, \phi)$, were introduced and studied in [18]. For $\gamma = 1$, the classes $\mathcal{S}_q^*(\gamma, \phi)$ and $\mathcal{C}_q(\gamma, \phi)$, reduce to $\mathcal{S}_q^*(\phi)$ and $\mathcal{C}_q(\phi)$, respectively studied in [14].

Motivated by El-Ashwah and Kanas, we introduce the following subclass of $\mathcal{A}$:

Definition 1.3. For $0 \neq \gamma \in \mathbb{C}$, $\alpha \geq 0$ and $0 \leq \lambda \leq 1$, the class $\mathcal{M}_q^\alpha(\gamma, \lambda, \phi)$ is defined by

$$\mathcal{M}_q^\alpha(\gamma, \lambda, \phi) = \left\{ f \in \mathcal{A} : \frac{1}{\gamma} \left[(1 - \alpha) \frac{z\mathcal{F}_\lambda'(z)}{\mathcal{F}_\lambda(z)} + \alpha \left(1 + \frac{z\mathcal{F}_\lambda''(z)}{\mathcal{F}_\lambda'(z)} \right) - 1 \right] \prec_q \right.$$
$$\left. \phi(z) - 1, \ z \in \mathbb{U} \right\}, \tag{1.8}$$

where

$$\mathcal{F}_\lambda(z) = (1 - \lambda)f(z) + \lambda z f'(z) = z + \sum_{n=2}^\infty \{1 + (n-1)\lambda\} a_n z^n. \tag{1.9}$$

For special choices of α, λ, γ and ϕ, the class $\mathcal{M}_q^\alpha(\gamma, \lambda, \phi)$ unifies the following known classes.

(i) For $0 \neq \gamma \in \mathbb{C}$, $\lambda = 0$ and $\alpha = 0$, the class $\mathcal{M}_q^\alpha(\gamma, \lambda, \phi)$ reduces to $\mathcal{S}_q^*(\gamma, \phi)$ studied in [6].

(ii) For $0 \leq \alpha \leq 1$, $\gamma = 1$ and $\lambda = 0$, $\mathcal{M}_q^\alpha(\gamma, \lambda, \phi)$ reduce to $\mathcal{M}_q^*(\alpha, \phi)$ which was introduced and studied by Mohd and Darus in [14]. In particular, $\alpha = 0$ and $\alpha = 1$ the class $\mathcal{M}_q^*(\alpha, \phi)$ reduce to $\mathcal{S}_q^*(\phi)$ and $\mathcal{C}_q(\phi)$ respectively, which were also studied in [14].

(iii) For $0 \leq \alpha \leq 1$, $\gamma = 1$, $\lambda = 0$ and $h(z) \equiv 1$, the class $\mathcal{M}_q^*(\alpha, \phi)$ reduces to the well-known class of α-convex functions [16].

In 1933, Fekete and Szegő proved that, for $f \in \mathcal{S}$ given by (1.1)

$$|a_3 - \mu a_2^2| \leq \begin{cases} 3 - 4\mu, & if \quad \mu \leq 0, \\[2mm] 1 + 2e^{\frac{-2}{1-\mu}}, & if \quad 0 \leq \mu < 1, \\[2mm] 4 - 3\mu, & if \quad \mu \geq 1, \end{cases} \tag{1.10}$$

and the result is sharp. The problem of finding the sharp bounds for the non-linear functional $|a_3 - \mu a_2^2|$ of many compact family of functions is popularly known as the Fekete-Szegő problem. Several known authors at different times obtained the sharp bound of the Fekete-Szegő functional $|a_3 - \mu a_2^2|$ for various subclasses of $\mathcal{S}$ (see [5, 6, 7, 22, 23]). In this paper, we determine the coefficient estimates and the Fekete-Szegő inequality of the functions in the class $\mathcal{M}_q^\alpha(\gamma, \lambda, \phi)$.

Let Ω be the class of the functions of the form:

$$w(z) = w_1 z + w_2 z^2 + \cdots, \tag{1.11}$$

is analytic in the unit disk $\mathbb{U}$ and satisfy the condition $|w(z)| < 1$.

We need the following lemma to prove our main result.

Lemma 1.1. ([*11*], *p*.10) *If $w \in \Omega$, then for any complex number μ*

$$|w_1| \leq 1, \quad |w_2 - \mu w_1^2| \leq 1 + (|\mu| - 1)|w_1|^2 \leq \max\{1, |\mu|\}.$$

The result is sharp for the functions $w(z) = z$ when $|\mu| \geq 1$ and for $w(z) = z^2$ when $|\mu| < 1$.

2. Main result

Throughout this paper, we assume that the functions $\phi(z), h(z)$ and $w(z)$ defined by (1.3), (1.4) and (1.11), respectively.

Theorem 2.1. *Let $0 \neq \gamma \in \mathbb{C}$, $\alpha \geq 0$ and $0 \leq \lambda \leq 1$. If $f \in \mathcal{A}$ of the form (1.1) belongs to the class $\mathcal{M}_q^\alpha(\gamma, \lambda, \phi)$, then*

$$|a_2| \leq \frac{|\gamma| B_1}{(1 + \alpha)(1 + \lambda)}, \tag{2.12}$$

$$|a_3| \leq \frac{|\gamma| B_1}{2(1 + 2\alpha)(1 + 2\lambda)}\left[1 + \max\left\{1, \left(\frac{(1 + 3\alpha)|\gamma|}{(1 + \alpha)^2} B_1 + \frac{|B_2|}{B_1}\right)\right\}\right], \tag{2.13}$$

and for any complex number μ,

$$|a_3 - \mu a_2^2| \leq \frac{|\gamma| B_1}{2(1 + 2\alpha)(1 + 2\lambda)}\left[1 + \max\left\{1, \left(|Q| B_1 + \frac{|B_2|}{B_1}\right)\right\}\right], \tag{2.14}$$

where

$$Q = \frac{2\mu(1 + 2\alpha)(1 + 2\lambda) - (1 + \lambda)^2(1 + 3\alpha)}{(1 + \alpha)^2(1 + \lambda)^2}. \tag{2.15}$$

The result is sharp.

Proof. Let $f \in \mathcal{M}_q^\alpha(\gamma, \lambda, \phi)$. Then by Definition 1.3,

$$\frac{1}{\gamma}\left[(1 - \alpha)\frac{z\mathcal{F}_\lambda'(z)}{\mathcal{F}_\lambda(z)} + \alpha\left(1 + \frac{z\mathcal{F}_\lambda''(z)}{\mathcal{F}_\lambda'(z)}\right) - 1\right] = h(z)(\phi(w(z)) - 1), \tag{2.16}$$

where $F_\lambda(z)$ defined by (1.9).

Using the series expansion of $\mathcal{F}_\lambda(z)$, $\mathcal{F}_\lambda'(z)$ and $\mathcal{F}_\lambda''(z)$ from (1.9), we get

$$\frac{1}{\gamma}\left[(1 - \alpha)\frac{z\mathcal{F}_\lambda'(z)}{\mathcal{F}_\lambda(z)} + \alpha\left(1 + \frac{z\mathcal{F}_\lambda''(z)}{\mathcal{F}_\lambda'(z)}\right) - 1\right] = \frac{1}{\gamma}\big[(1 + \alpha)(1 + \lambda)a_2 z$$
$$+ \{2(1 + 2\alpha)(1 + 2\lambda)a_3 - (1 + 3\alpha)(1 + \lambda)^2 a_2^2\}z^2 + \cdots\big]. \tag{2.17}$$

Also

$$\phi(w(z)) - 1 = B_1 w_1 z + (B_1 w_2 + B_2 w_1^2)z^2 + \cdots, \tag{2.18}$$

and

$$h(z)(\phi(w(z)) - 1) = B_1\, c_0\, w_1 z + [B_1 c_1 w_1 + c_0(B_1 w_2 + B_2 w_1^2)]z^2 + \cdots. \qquad (2.19)$$

Making use of (2.17), (2.18) and (2.19) in (2.16), and equating the coefficients of z and z^2 in the resulting equation, we get

$$a_2 = \frac{\gamma B_1 c_0}{(1+\alpha)(1+\lambda)}, \qquad (2.20)$$

and

$$a_3 = \frac{\gamma}{2(1+2\alpha)(1+2\lambda)}\left[(B_1 c_1 w_1 + B_1 c_0 w_2) + c_0\left(B_2 + \frac{(1+3\alpha)\gamma}{(1+\alpha)^2}B_1^2 c_0\right)w_1^2\right]. \qquad (2.21)$$

Thus, for any complex number μ, we have

$$a_3 - \mu a_2^2 = \frac{\gamma B_1}{2(1+2\alpha)(1+2\lambda)}\left[c_1 w_1 + c_0\left(w_2 + \frac{B_2}{B_1}w_1^2\right) - QB_1 c_0^2 w_1^2\right], \qquad (2.22)$$

where Q is given by (2.15).
Since $h(z)$ is analytic and bounded in $\mathbb{U}$, hence by ([17], p. 172), we have

$$|c_0| \leq 1 \quad \text{and} \quad |c_n| = 1 - |c_0|^2 \leq 1 \text{ for } n > 0. \qquad (2.23)$$

By using this fact and $|w_1| \leq 1$, we get from (2.20), (2.21), (2.22) and (2.23) we obtain

$$|a_2| \leq \frac{|\gamma|B_1}{(1+\alpha)(1+\lambda)}, \qquad (2.24)$$

$$|a_3| \leq \frac{|\gamma|B_1}{2(1+2\alpha)(1+2\lambda)}\left\{1 + \left|w_2 - \left(-\frac{(1+3\alpha)\gamma}{(1+\alpha)^2}B_1 c_0 - \frac{B_2}{B_1}\right)w_1^2\right|\right\}, \qquad (2.25)$$

and

$$|a_3 - \mu a_2^2| \leq \frac{|\gamma|B_1}{2(1+2\alpha)(1+2\lambda)}\left[1 + \left|w_2 - \left(QB_1 c_0 - \frac{B_2}{B_1}\right)w_1^2\right|\right]. \qquad (2.26)$$

Case-I: If $c_0 = 0$, then (2.22) gives

$$|a_3 - \mu a_2^2| \leq \frac{|\gamma|B_1}{2(1+2\alpha)(1+2\lambda)}. \qquad (2.27)$$

Case-II: If $c_0 \neq 0$, then by applying the Lemma 1.1 to

$$\left|w_2 - \left(QB_1 c_0 - \frac{B_2}{B_1}\right)w_1^2\right|, \qquad (2.28)$$

we get from (2.26)

$$|a_3 - \mu a_2^2| \leq \frac{|\gamma|B_1}{2(1+2\alpha)(1+2\lambda)}\left[1 + \max\left\{1, \left(|Q|B_1 + \frac{|B_2|}{B_1}\right)\right\}\right]. \qquad (2.29)$$

The required result (2.14) follows from (2.27) and (2.29). In a similar manner we can prove the required assertion (2.13). The result is sharp for the function $f(z)$ given by

$$\frac{1}{\gamma}\left[(1-\alpha)\frac{z\mathcal{F}'_\lambda(z)}{\mathcal{F}_\lambda(z)}+\alpha\left(1+\frac{z\mathcal{F}''_\lambda(z)}{\mathcal{F}'_\lambda(z)}\right)-1\right]=\phi(z)-1,$$

or

$$\frac{1}{\gamma}\left[(1-\alpha)\frac{z\mathcal{F}'_\lambda(z)}{\mathcal{F}_\lambda(z)}+\alpha\left(1+\frac{z\mathcal{F}''_\lambda(z)}{\mathcal{F}'_\lambda(z)}\right)-1\right]=\phi(z^2)-1.$$

This completes the proof of Theorem 2.1. □

Putting $\gamma=1$, $\alpha=0$ and $\lambda=0$ in Theorem 2.1, we get the following sharp results for the class $\mathcal{S}^*_q(\phi)$.

Corollary 2.1. *Let $f\in\mathcal{A}$ of the form* (1.1) *belongs to the class $\mathcal{S}^*_q(\phi)$, then*

$$|a_2|\leq B_1$$

and for any complex number μ,

$$|a_3-\mu a_2^2|\leq\frac{B_1}{2}\left[1+\max\left\{1,|1-2\mu|B_1+\frac{|B_2|}{B_1}\right\}\right].$$

The result is sharp.

Putting $\gamma=1$, $\alpha=0$ and $\lambda=1$ in Theorem 2.1, we get the following sharp results for the class $\mathcal{C}_q(\phi)$.

Corollary 2.2. *Let $f\in\mathcal{A}$ of the form* (1.1) *belongs to the class $\mathcal{C}_q(\phi)$, then*

$$|a_2|\leq\frac{B_1}{2}$$

and for any complex number μ,

$$|a_3-\mu a_2^2|\leq\frac{B_1}{6}\left[1+\max\left\{1,\left(\left|1-\frac{3\mu}{2}\right|B_1+\frac{|B_2|}{B_1}\right)\right\}\right].$$

The result is sharp.

Remark 2.1. The Corollary 2.1 and Corollary 2.2 are due to the results obtained by Mohd and Darus [14].

The next theorem gives the result based on majorization.

Theorem 2.2. *Let $0\neq\gamma\in\mathbb{C}$, $\alpha\geq0$ and $0\leq\lambda\leq1$. If $f\in\mathcal{A}$ of the form* (1.1) *satisfies*

$$\frac{1}{\gamma}\left[(1-\alpha)\frac{z\mathcal{F}'_\lambda(z)}{\mathcal{F}_\lambda(z)}+\alpha\left(1+\frac{z\mathcal{F}''_\lambda(z)}{\mathcal{F}'_\lambda(z)}\right)-1\right]\ll(\phi(z)-1),\ z\in\mathbb{U}, \qquad (2.30)$$

then

$$|a_2| \le \frac{|\gamma|B_1}{(1+\alpha)(1+\lambda)}, \tag{2.31}$$

$$|a_3| \le \frac{|\gamma|B_1}{2(1+2\alpha)(1+2\lambda)} \left[1 + \frac{(1+3\alpha)|\gamma|}{(1+\alpha)^2} B_1 + \frac{|B_2|}{B_1} \right] \tag{2.32}$$

and for any complex number μ,

$$|a_3 - \mu a_2^2| \le \frac{|\gamma|B_1}{2(1+2\alpha)(1+2\lambda)} \left[1 + |Q|B_1 + \frac{|B_2|}{B_1} \right], \tag{2.33}$$

where Q is given by (2.15). The result is sharp.

Proof. Let us assume that (2.30) holds. Then from the definition of majorization, there exists an analytic function $h(z)$ such that

$$\frac{1}{\gamma} \left[(1-\alpha) \frac{z\mathcal{F}_\lambda'(z)}{\mathcal{F}_\lambda(z)} + \alpha \left(1 + \frac{z\mathcal{F}_\lambda''(z)}{\mathcal{F}_\lambda'(z)} \right) - 1 \right] = h(z)(\phi(z) - 1). \tag{2.34}$$

Following similar steps as in the Theorem 2.1, and by setting $w(z) = z$, that is, for $w_1 = 1$, $w_n = 0$, $n \ge 2$, we obtain

$$a_2 = \frac{\gamma B_1 c_0}{(1+\alpha)(1+\lambda)},$$

which gives on use of the fact $c_n \le 1$, for $n > 0$,

$$|a_2| \le \frac{|\gamma|B_1}{(1+\alpha)(1+\lambda)},$$

$$a_3 = \frac{\gamma}{2(1+2\alpha)(1+2\lambda)} \left[B_1 c_1 + c_0 \left(B_2 + \frac{(1+3\alpha)\gamma}{(1+\alpha)^2} B_1^2 c_0 \right) \right]. \tag{2.35}$$

Thus for any complex number μ, we have

$$a_3 - \mu a_2^2 = \frac{\gamma B_1}{2(1+2\alpha)(1+2\lambda)} \left[c_1 + c_0 \left(\frac{B_2}{B_1} \right) - Q B_1 c_0^2 \right]. \tag{2.36}$$

Following similar steps in Theorem2.1 we get the following from (2.36):
for $c_0 = 0$,

$$|a_3 - \mu a_2^2| \le \frac{|\gamma|B_1}{2(1+2\alpha)(1+2\lambda)}, \tag{2.37}$$

and for $c_0 \ne 0$

$$|a_3 - \mu a_2^2| \le \frac{|\gamma|B_1}{2(1+2\alpha)(1+2\lambda)} \left[1 + \frac{|B_2|}{B_1} + |Q|B_1 \right]. \tag{2.38}$$

Thus, the assertion (2.33) of Theorem 2.2 follows from (2.37) and (2.38). Following the above steps we can prove the assertion (2.32) of Theorem 2.2. The result is sharp for the function

$$\frac{1}{\gamma}\left[(1-\alpha)\frac{z\mathcal{F}_\lambda'(z)}{\mathcal{F}_\lambda(z)} + \alpha\left(1+\frac{z\mathcal{F}_\lambda''(z)}{\mathcal{F}_\lambda'(z)}\right) - 1\right] = \phi(z) - 1, \quad z \in \mathbb{U},$$

which completes the proof of the Theorem 2.2. □

For $h(z) = 1$, that is, for $c_0 = 1$ and $c_n = 0, n \geq 1$, we have the following theorem:

Theorem 2.3. *Let $0 \neq \gamma \in \mathbb{C}$, $\alpha \geq 0$ and $0 \leq \lambda \leq 1$. If $f \in \mathcal{A}$ of the form (1.1) belongs $\mathcal{M}^\alpha(\gamma, \lambda, \phi)$, then*

$$|a_2| \leq \frac{|\gamma|B_1}{(1+\alpha)(1+\lambda)}, \tag{2.39}$$

$$|a_3| \leq \frac{|\gamma|B_1}{2(1+2\alpha)(1+2\lambda)}\left[\max\left\{1,\left(\frac{(1+3\alpha)|\gamma|}{(1+\alpha)^2}B_1 + \frac{|B_2|}{B_1}\right)\right\}\right] \tag{2.40}$$

and for any complex number μ,

$$|a_3 - \mu a_2^2| \leq \frac{|\gamma|B_1}{2(1+2\alpha)(1+2\lambda)}\left[\max\left\{1,\left(|Q|B_1 + \frac{|B_2|}{B_1}\right)\right\}\right], \tag{2.41}$$

where Q is given by (2.15). The result is sharp.

Proof. Proof is similar to Theorem 2.1. □

Remark 2.2. For $\gamma = 1$ and $\lambda = 0$, the Theorem 2.3 due to the result in [14] and [2] for $k = 1$.

Conclusion: In this paper we have introduced a new subclass of univalent functions and obtained sharp coefficient estimates.

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