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Preface

Justyna Szczupiel

This monograph, titled *Current Research in Nonlinear Analysis and Differential Geometry*, presents a collection of chapters addressing selected topics in functional analysis, operator theory and the theory of differential and integral equations. Although the individual contributions explore diverse mathematical problems and techniques, they are unified by a shared analytical perspective and a focus on qualitative properties of solutions and structural features of function spaces.

The inspiration for this volume arises from the scientific legacy of Professor Józef Banaś, whose influential contributions, particularly in the development and application of measures of noncompactness, have shaped several areas of nonlinear analysis. His pioneering work has left a lasting impact on the study of fixed point theory, integral and differential equations, and the geometry of functional spaces. This monograph honors Professor Banaś on the occasion of his 75th birthday and features contributions from researchers who have been scientifically connected with him throughout his career.

In the opening chapter, M. Mursaleen and E. Savaş provide a concise survey of measures of noncompactness and explore their role in the analysis of infinite systems of differential equations. The focus is placed on applications within classical and tempered sequence spaces, offering both a theoretical overview and a bridge to applied problems in the framework of sequence spaces.

M. Krajewska continues the study of infinite systems of differential equations, specifically investigating their solvability in Banach tempered sequence spaces. By employing analytical tools based on measures of noncompactness and abstract results from the theory of differential equations in Banach spaces, the chapter offers new existence results in this setting.

A. Dubiel examines nonlinear Volterra–Stieltjes integral equations with an emphasis on quadratic cases. The chapter addresses the existence of solutions in the space of bounded, continuous real-valued functions on the positive real half-axis that converge to finite limits at infinity. The methods rely on techniques involving functions of bounded variation, measures of noncompactness, and the Darbo fixed point theorem.

S. Dudek presents an overview of various examples of measures of noncompactness, alongside functionals that do not satisfy the criteria to be considered such measures, despite initial appearances. The discussion is conducted within selected Banach and Fréchet function spaces, highlighting subtle distinctions and methodological caveats in the theory.

R. Nalepa introduces a measure of noncompactness in spaces of functions with tempered increments, establishing its key properties and employing it to study the existence of solutions for a nonlinear quadratic Volterra–Hammerstein integral equation in Hölder spaces. The results extend and complement earlier contributions in the area, highlighting the versatility of noncompactness techniques.

J. Caballero Mena, K. Sadarangani, R. Toledo investigate a Lyapunov-type inequality for a third-order boundary value problem involving nonlocal and integral boundary conditions. An application of the derived inequality yields a lower bound for the eigenvalues associated with the problem, contributing to the spectral theory of higher-order differential equations with complex boundary structures.

In the last chapter, P. Witowicz focuses on affine differential geometry in higher codimensions, in particular surfaces in fourdimensional space. The most important concepts, constructions and examples of theorems are presented.

This volume emphasizes the enduring significance of classical analytical tools—such as compactness, variation, and fixed point principles—in the context of contemporary nonlinear problems. Each chapter is self-contained but collectively contributes to the evolving landscape of nonlinear analysis. The monograph is intended for mathematicians and advanced students interested in modern approaches to analytical questions and aims to support further research and academic instruction in this vibrant field.

Chapter 1

A survey on solvability of infinite systems of differential equations in classical and tempered sequence spaces

Mohammad Mursaleen, Ekrem Savaş

1. Introduction

Measures of noncompactness (MNC) play a pivotal role in various areas of nonlinear analysis, particularly in the study of differential, integral, and integro-differential equations, as well as in optimization theory. The concept was first introduced by Kuratowski in 1930 [20]. Subsequently, other versions were developed, such as the Hausdorff measure of noncompactness (HMNC) by Goldenstein et al. in 1957 [11], and further extensions by Istrăţescu [16], including the inner Hausdorff and Istrăţescu measures.

A significant application of measures of noncompactness emerged with the work of Gabriele Darbo, who in 1955 formulated a fixed point theorem for so-called condensing operators [10]. This result generalizes both the classical Schauder fixed point theorem and a variant of the Banach contraction principle. Beyond its theoretical depth, Darbo's theorem has led to numerous applications across linear and nonlinear analysis.

Recently, the MNC has been effectively applied for systems of differential equations in different sequence spaces, e.g. in l_1 [1], ℓ_p [14, 24, 27], in c_0

and l_1 [23], and in $n(\phi)$ [22]. The problems for solving infinite systems of fractional differential equations of order $1 < \alpha < 2$ in sequence spaces c_0 and ℓ_p have been addressed in [29]. In [4], the double sequence space $m^2(\Delta_v^u, \phi, p)$ has been considered for the solvability of n-order fractional differential equations.

In this chapter, we present a brief survey on solving the infinite systems of differential equations (ISDE) in classical and tempered sequence spaces via MNC. Most recently, Banaś et al. [8] described the method to formulate an open problem concerning the existence of solutions of an infinite system of nonlinear differential (or integral) equations which are obtained during the modelling of the so-called birth-and-death stochastic process.

2. Measures of noncompactness

The notion of quantifying the "measure of noncompactness" of a bounded subset within a metric space originated with Kuratowski. He introduced the function $\alpha(A)$, now known as the Kuratowski measure of noncompactness ([20], [21]), defined as follows:

Definition 1 (p.150, [9])**.** Let $\mathbb{X}$ be a metric space and let $G \subset \mathbb{X}$ be bounded. Kuratowski defined $\alpha(G)$ by:

$$\alpha(G) = \inf\left\{\varepsilon > 0 \,:\, G \subset \cup_{i=1}^{n} S_i \,:\, S_i \subset \mathbb{X},\, \text{diam}(S_i) < \varepsilon,\, n \in \mathbb{N}\right\}.$$

The following theorem illustrates a foundational property of the Kuratowski measure, and serves as a general case of the classical Cantor intersection principle:

Theorem 2 ([20])**.** *Let (F_n) be a decreasing sequence of non-empty, closed, and bounded subsets of a complete metric space $\mathbb{X}$ with $\lim_{n\to\infty} \alpha(F_n) = 0$. Then $\bigcap_{n=1}^{\infty} F_n$ is non-empty and compact.*

This result is particularly instrumental when proving the existence of fixed points by applying MNC. Notably, α satisfies several properties, including subadditivity, positive homogeneity, and the fact that it vanishes on singleton sets (non-singularity).

Theorem 3 (p.154, [9])**.** *Let $\mathbb{X}$ be a normed space. Then:*
If $\mathbb{X}$ is finite-dimensional, $\alpha(B_{\mathbb{X}}) = 0$;
If $\mathbb{X}$ is infinite-dimensional, $\alpha(B_{\mathbb{X}}) = 2$, where $B_{\mathbb{X}}$ denotes the closed unit ball in $\mathbb{X}$.

We now turn to the Hausdorff measure of noncompactness, which is often regarded as the most widely applicable among all such measures. This notion was introduced by Goldenstein and collaborators in 1957 [11].

Definition 4. Let $\mathbb{X}$ be a metric space and let $G \subset \mathbb{X}$ be bounded. The Hausdorff measure of noncompactness (HMNC) $\chi(G)$ is defined by

$$\chi(G) = \inf \left\{ \varepsilon > 0 \, : \, G \subset \cup_{i=1}^{n} B(x_i, r_i) \, : \, x_i \in \mathbb{X}, \, r_i < \varepsilon, \, n \in \mathbb{N} \right\}.$$

The following MNC was given by Istrătescu (p.168, [9]):

Definition 5. Let G be a bounded subset of a complete metric space $\mathbb{X}$. Then, β denotes the Istrătescu measure of noncompactness, also known as the lattice measure of noncompactness, which is defined as

$$\beta(G) = \sup\{\varepsilon > 0 \, : \, G \, \text{has an infinite } \varepsilon\text{-discrete subset}\}.$$

The HMNC satisfies many desirable properties in certain functional spaces. However, it is often not straightforward to construct a useful MNC tailored to a specific space. To overcome this difficulty, an axiomatic definition was introduced by Banaś and Goebel ([5], 1980).

Definition 6. Let $\mathfrak{M}_{\mathbb{E}}$ be a family of all nonempty bounded subsets of a space $\mathbb{E}$. A function $\mu : \mathfrak{M}_{\mathbb{E}} \to \mathbb{R}_+$ is called a measure of noncompactness in a Banach space $\mathbb{E}$ if it fulfills the following properties:

1. The kernel of μ, defined by $ker\mu = \{X \in \mathfrak{M}_{\mathbb{E}} \, : \, \mu(X) = 0\}$ is nonempty and contained within the family of relatively compact subsets of $\mathbb{E}$, i.e., $ker\mu \subset \mathcal{N}_{\mathbb{E}}$.

2. If $P \subset Q$, the $\mu(P) \leqslant \mu(Q)$; that is, μ is monotonic with respect to set inclusion.

3. For any set $X \in \mathfrak{M}_{\mathbb{E}}$, the values of μ are invariant under closure and convex hull operations:
   $\mu(\overline{X}) = \mu(coX) = \mu(X)$.

4. The function μ is convex in the sense that for all $X, Y \in \mathfrak{M}_{\mathbb{E}}$ and $c \in [0, 1]$,

   $\mu(cX + (1-c)Y) \leqslant c\mu(X) + (1-c)\mu(Y)$.

5. For a descending sequence (X_n) of closed subsets in $\mathfrak{M}_{\mathbb{E}}$, $\cap_{n=1}^{\infty} X_n \neq \emptyset$ if $\mu(X_n) \to 0$ $(n \to \infty)$.

3. MNC in sequence spaces

3.1. Classical sequence spaces

Using the preceding result, we can derive explicit formulas for computing the HMNC in certain classical sequence spaces, namely ℓ_p for $(1 \leqslant p < \infty)$, c_0, c. These spaces possess the structure of BK-spaces with AK, and their norms are monotonic. Consequently, as a direct application of Theorem 1 of [11] and ([9], p. 160), we obtain the following:

(a) For a bounded set $G \subset \ell_p$ with $1 \leqslant p < \infty$, we have:

$$\chi(G) = \lim_{n\to\infty} \sup_{(x_i)\in G} \left(\sum_{k\geqslant n} |x_k|^p \right)^{1/p}.$$

(b) If G is a bounded subset of the normed space c_0, the measure of non-compactness is given by:

$$\chi(G) = \lim_{n\to\infty} \sup_{(x_i)\in G} \left(\max_{k\geqslant n} |x_k| \right).$$

(c) For $G \subset c$, the measure $\chi(G)$ satisfies the inequalities:

$$\frac{1}{2}\mu(G) \leqslant \chi(G) \leqslant \mu(G) \tag{1}$$

where the auxiliary function $\mu(G)$ is defined by

$$\mu(G) = \lim_{n\to\infty} \left\{ \sup_{(x_i)\in G} \left(\sup_{k\geqslant n} \left| x_k - \lim_{k\to\infty} x_k \right| \right) \right\}. \tag{2}$$

Although (2) is regular measure of noncompactness, it relies on knowledge of the limit of sequences in the set, which may be impractical in applications. A more operationally useful formulation uses a Cauchy-type condition that avoids explicitly computing sequence limits.

Specifically, for $G \in \mathfrak{M}_c$, we define:

$$\mu_c(G) = \lim_{k\to\infty} \Big\{ \sup_{(x_i)\in G} \Big\{ \sup\{|x_n - x_m| : n, m \geqslant k\} \Big\} \Big\}. \tag{3}$$

This form of the measure, μ_c has been shown in several references (e.g., [5,9]) to be both regular and equivalent to χ in c.

It is important to emphasize that for the space l_∞, there is currently no established formula that characterizes the HMNC χ. Furthermore, even for regular measures in ℓ_∞, explicit formulas are generally unknown (see [3,5,9]). As a result, any measures of noncompactness in this space must be constructed axiomatically. While some useful expressions for such measures have been proposed and employed in the literature (see, e.g., [5,9]), their validity as true measures of noncompactness remains unproven. Specifically, the necessary properties confirming that these expressions define a proper measure of noncompactness in l_∞ are not yet rigorously established, as discussed by Banaś and Krajewska in [6].

3.2. Tempered sequence spaces

As noted in the introduction, analyzing initial value problems (IVPs) for ISDE directly within classical sequence spaces is often impractical or inadequate. To address this limitation, it becomes necessary to extend the framework by employing broader classes of sequence spaces. One such approach involves the use of tempered sequence spaces, a concept introduced by Banaś et. al. [6]. Let $\beta = (\beta_k)$ be a nonincreasing sequence of positive real numbers. This type of sequence is referred to as a tempered sequence. Using this, we define the space

$$\mathbb{V} :\equiv c_0^\beta := \{y = (y_k)_{k=1}^\infty : \beta_k y_k \to 0\}.$$

This space is a linear space over $\mathbb{R}$ or $\mathbb{C}$ and becomes a Banach space when equipped with the norm $\|y\|_{c_0^\beta} = \sup\limits_{k\in\mathbb{N}} \{\beta_k |y_k|\}$.

In a related development, Rabbani et al. [36] introduced a new tempered version of the space ℓ_p denoted by ℓ_p^β, and investigated measures of noncompactness (mnc) on it. This space is defined as:

$$\mathbb{J} :\equiv \ell_p^\beta := \left\{ y = (y_k)_{k=1}^\infty : \sum_{k=1}^\infty \beta_k^p |y_k|^p < \infty \right\}, 1 \leqslant p < \infty.$$

It forms a Banach space under the norm $\|y\|_{\ell_p^\beta} = \left(\sum\limits_{k=1}^\infty \beta_k^p |y_k|^p \right)^{\frac{1}{p}}$.

For the space $(c_0^\beta, \|.\|_{c_0^\beta})$, the Hausdorff measure of noncompactness χ takes the form

$$\chi_{c_0^\beta}(\mathbb{B}^\beta) = \lim_{n\to\infty}\left[\sup_{(y_i)\in\mathbb{B}^\beta}\left\{\left(\sup_{k\geqslant n}\ (\beta_k|y_k|)\right)\right\}\right], \quad \mathbb{B}^\beta \in \mathfrak{M}_{c_0^\beta}. \tag{4}$$

Analogously, in the tempered space ℓ_p^β, the corresponding Hausdorff measure of noncompactness is expressed as:

$$\chi_{\ell_p^\beta}(\mathbb{B}^\beta) = \lim_{n\to\infty}\left[\sup_{(y_i)\in\mathbb{B}^\beta}\left(\sum_{k\geqslant n}\beta_k^p|y_k|^p\right)^{\frac{1}{p}}\right], \quad \mathbb{B}^\alpha \in \mathfrak{M}_{\ell_p^\beta}. \tag{5}$$

These extensions of the classical sequence spaces provide a more flexible setting for analyzing infinite systems of differential equations, particularly when standard normed spaces are too restrictive or fail to capture the desired asymptotic behavior.

3.3. Non-classical tempered sequence space $m^\beta(\psi, p)$

Let $\mathfrak{S}$ be the space of all real sequences.

Assume that $\mathfrak{F}$ is the space of finite subsets of natural numbers. $\forall\ \eta\in\mathfrak{F}$, consider $c(\eta) = (c_n(\eta))$, such that $c_n(\eta) = 1$ if $n\in\eta$ and $c_n(\eta) = 0$, otherwise. Furthermore, given $p\in\mathbb{N}$, by $\mathfrak{F}_\varrho$ we denote the family of all sets $\eta\in\mathfrak{F}$ containing at most ϱ elements and we put

$$\Psi = \left\{\psi = (\psi_r)\in\mathfrak{S}:\ \psi_1 > 0,\ \Delta\psi_r \geqslant 0 \text{ and } \Delta(\frac{\psi_r}{r}) \leqslant 0\ (r\in\mathbb{N})\right\},$$

where $\Delta\psi_r = \psi_r - \psi_{r-1}$ (see [34]).

Let $\psi\in\Psi$. The tempered sequence space $m^\beta(\psi, p)$ is defined as:

$$m^\beta(\psi, p) = \left\{z = (z_k)\in\mathfrak{S}:\ \sup_{\varrho_1\geqslant 1}\sup_{\eta\in\mathfrak{F}_{\varrho_1}}\left(\frac{1}{\psi_{\varrho_1}}\sum_{k\in\eta}|z_k|^p\beta_k^p\right) < \infty\right\},\ 1\leqslant p<\infty,$$

where β is a tempered sequence. For $p = 1$, $m^\beta(\psi, p)$ is reduced to $m^\beta(\psi)$ ([26]).

Theorem 7. *([25])*

(i) Let $\varsigma > 0$ be a fixed integer. Then the sequence space $m^{\beta}(\psi, q)$ is a Banach space by the following norm

$$\|x\|_{m^{\beta}(\psi,q)} = \sum_{i=1}^{\varsigma}|x_i|\beta_i + \sup_{\varrho_1 \geqslant 1}\sup_{\eta\in\mathfrak{F}_{\varrho_1}}\left(\frac{1}{\psi_{\varrho_1}}\sum_{k\in\eta}|x_k|^q\beta_k^q\right)^{\frac{1}{q}}.$$

(ii) Let $\emptyset \neq \mathcal{G} \subseteq m^{\beta}(\psi, q)$ be bounded. Then χ on $m^{\beta}(\psi, q)$ is defined by:

$$\chi(\mathcal{G}) := \lim_{n\to\infty}\left\{\sup_{x\in\mathcal{G}}\left(\sup_{\varrho_1 \geqslant n}\sup_{\eta\in\mathfrak{F}_{\varrho_1}}\left(\frac{1}{\psi_{\varrho_1}}\sum_{k\in\eta}|x_k|^q\beta_k^q\right)^{\frac{1}{q}}\right)\right\}. \tag{6}$$

4. Differential equations

The theory of differential equations has been significantly enriched by using MNC. While Lipschitz continuity is typically sufficient to guarantee existence and uniqueness of solutions in finite-dimensional spaces, while in infinite - dimensional contexts—such as sequence spaces - one often relies on Kamke-type comparison functions (see, for example, [31]).

4.1. Cauchy problem

Let $(\mathbb{E}, \|\cdot\|)$ be a Banach space. Consider the initial value problem:

$$x'(t) = f(t, x) \tag{7}$$

with initial condition

$$x(0) = x_0 \tag{8}$$

where $f : [0, T] \times B(x_0, r) \to \mathbb{E}$ is a given function and $B(x_0, r)$ is a closed ball of radius r centered at $x_0 \in \mathbb{E}$.

We now list several results guaranteeing the existence of a local solution $x(t) \in \mathbb{E}$ for $t \in [0, \delta] \subset [0, T]$, under appropriate assumptions on the function f.

Let μ be a measure of noncompactness and $\mathbb{E}_{\mu} : ker\mu$.

Theorem 8 (p.224, [9])**.** *Assume:*

(i) $f : [0,T] \times B(x_0, r) \to \mathbb{E}$ *is uniformly continuous and bounded, with* $\|f(t,x)\| \leqslant A$.

(ii) f *satisfies the Kamke-type comparison condition:*

$$\mu(x_0 + f(t,X)) \leqslant w(t, \mu(X)) \tag{9}$$

for $t \in [0,T]$ *and* $X \subset B(x_0, r)$, *where* $w(t,u)$ *is a Kamke comparison function,*

(iii) $x_0 \in \mathbb{E}_\mu$, *where* $\mathbb{E}_\mu = \{x \in \mathbb{E} : \mu(\{x\}) = 0\}$

(iv) $\sup\{t + ta(t) : t \in [0,T]\} \leqslant 1$, *where*

$$a(t) = \sup\{\|f(0,x_0) - f(s,x)\| : s \leqslant t, \|x - x_0\| \leqslant As\},$$

(v) $At \leqslant r$.

Then the problem (7)-(8) has at least one solution $x(t) \in \mathbb{E}_\mu$ *for* $t \in [0,T]$.

It is evident from the preceding theorem that one may adopt a modified form of the Kamke comparison function. This function can be shown to fulfill the criteria required of a Kamke-type comparison function. Under this assumption, inequality (9) transforms into

$$\mu(x_0 + f(t,X)) \leqslant p(t)\mu(X)$$

for any set $X \in \mathfrak{M}_{\mathbb{E}}$. Based on this setup, we now state another result concerning the existence of a solution to the Cauchy problem (7)–(8), utilizing the specific form $w(t,u) = p(t)u$ of the comparison function.

Theorem 9 (p.227, [9])**.** *Assume the following conditions hold:*

(i) $f : [0,T] \times B(x_0, r) \to \mathbb{E}$ *is uniformly continuous with* $\|f(t,x)\| \leqslant A$.

(ii) For almost every $t \in [0,T]$ *and all* $x \in B(x_0, r)$, *the inequality*

$$\mu(x_0 + f(t,X)) \leqslant p(t)\mu(X)$$

is satisfied, where $x_0 \in \mathbb{E}_\mu$ *and* $p(t) \in L^1[0,T]$.

Then (7)-(8) has at least one solution $x(t) \in \mathbb{E}_\mu$ *for* $t \in [0,T]$.

Theorem 10. *Let μ be a sublinear measure of noncompactness and $\{x_0\} \in \ker \mu$. If*

(i) f is uniformly continuous on $I \times B(x_0, r)$,

(ii) $\|f(t, x)\| \leqslant A$ and $AT \leqslant r$,

(iii) $\mu(f(t, X)) \leqslant p(t)\mu(X)$, for almost all $t \in I$, $X \subset B(x_0, r)$ where $p \in L^1(I)$,

then the problem (7)-(8) has at least one solution $x(t) \in \mathbb{E}_\mu$ for all $t \in I$.

The following theorem represents a slight modification of Theorem 10, adapted to better suit the needs of our subsequent analysis (see also [7, 9]).

Theorem 11 (Modified Version)**.** *Assume $f : [0, T] \times \mathbb{E} \to \mathbb{E}$ satisfies:*

(i) $\|f(t, x)\| \leqslant P + Q\|x\|$,

(ii) f is uniformly continuous on the set $[0, T_1] \times B(x_0, r)$ with $QT_1 < 1$,

(iii) $r = \frac{(P+Q)T_1\|x_0\|}{1-QT_1}$,

(iv) $\mu(f(t, X)) \leqslant p(t)\mu(X)$, for some sublinear μ with $x_0 \in \mathbb{E}_\mu$.

Then (7)-(8) has a solution $x(t) \in \mathbb{E}_\mu$ for $t \in [0, T_1]$.

Remark 12. When $\mu = \chi$, the uniform continuity assumption on f may be weakened to mere continuity [28]. The same holds for any regular measure of noncompactness equivalent to χ.

4.2. Solvability in classical sequence spaces

We now consider infinite systems of first-order differential equations in classical Banach sequence spaces [7].

Case: Space c_0

Consider

$$x_i' = f_i(t, x_1, x_2, \dots),\ x_i(0) = x_i^0. \tag{10}$$

Assume:

(1) $x_0 = (x_i^0) \in c_0$.

(2) The function $f = (f_1, f_2, \dots)$ maps $I \times c_0$ into c_0 and it is continuous on $I \times c_0$.

(3) There exists an ascending sequence (k_n) of positive integers such that the following inequality holds for each $t \in I$, $x = (x_i) \in c_0$ and $n = 1, 2 \ldots$:

$$|f_n(t, x_1, x_2, \ldots)| \leqslant p_n(t) + q_n(t) \sup\{|x_i| \,:\, i \geqslant k_n\},$$

where $p_i(t) \to 0$ uniformly and $(q_i(t))$ is equibounded on I.

Then, for $Q = \sup_{t \in I} \sup_n q_n(t)$ and $QT_1 < 1$, the system (10) has at least one solution $x = x(t) = (x_i(t)) \in c_0$ on $[0, T_1]$.

Case: Space c

Consider

$$x_n' = \sum_{i=1}^{n} a_i(t) x_i + g_n(t, x_{n+1}, x_{n+2}, \ldots),\ x_n(0) = x_n^0 \tag{11}$$

Assume:

(1) $x_0 = (x_n^0) \in c$,

(2) $g = (g_1, g_2, \ldots) : I \times c \to c$ is uniformly continuous and there exists a sequence (b_n) such that $b_n \to 0$ and $|g_n| \leqslant b_n$,

(3) $a_i = a_i(t)$ are continuous with $\sum_{i=1}^{\infty} |a_i(t)| < \infty$,

(4) There exists (d_i) such that $\sum_{i=1}^{\infty} d_i < \infty$ and $|a_i(t) - a_i(s)| \leqslant d_i w(|t - s|)$, where $w(\varepsilon) \to 0$ as $\varepsilon \to 0$.

Then, for $Q = \sup_{t \in I} \sum_{i=1}^{\infty} |a_i(t)|$ and $QT < 1$, system (11) has a solution $x = x(t) = (x_i(t)) \in c$.

Case: Space ℓ_1

Consider

$$x_i' = f_i(t, x_1, x_2, \ldots),\ x_i(0) = x_i^0, \tag{12}$$

Assume:

(1) $x_0 = (x_n^0) \in \ell_1$,

(2) $f - i : I \times \mathbb{R}^{\infty} \to \mathbb{R}$ maps continuously into ℓ_1,

(3) $|f_i(t, x_1, x_2, \ldots)| \leqslant p_i(t) + q_i(t)|x_i|$ with $\sum_{i=1}^{\infty} p_i(t) < \infty$ and $(q_i(t))$ is equibounded,

(4) $q(t) := \lim_{i\to\infty} \sup q_i(t) \in L^1(I)$.

Then, if $QT < 1$ with $Q = \sup\{q_i(t)\}$, the system (12) has at least one solution $x(t) \in \ell_1$.

It is worth noting that Theorem 17, with appropriate modifications to its assumptions, has also been applied in [9] to study perturbed infinite systems of differential equations in ℓ_1

$$x_i' = a_i(t) + q_i(t, x_1, x_2, \dots),\ x_i(0) = x_i^0.$$

Furthermore, by extending the approach naturally to cover broader settings, existence results can be derived for analogous systems in the space ℓ_p for for $1 \leqslant p < \infty$ (cf. [30]).

We now turn our attention to study ISDE in ℓ_∞. As discussed in Section 3, a direct formula for the Hausdorff measure of noncompactness does not exist in ℓ_∞. To address this, alternative measures such as μ_1, μ_3 (cf. Chapter-5, [9]) are employed to establish existence results for solutions to the system

$$x_i' = a_i(t)x_i + f_i(x_i, x_{i+1}, x_{i+2}, \dots) \tag{13}$$

with

$$x_i(0) = x_i^0, \tag{14}$$

for $t \in I = [0, T]$ and for $i = 1, 2, \dots$.
The following assumptions are made:

(1) The initial values satisfy $\lim_{i\to\infty} x_i^0 = a$ for some real number a.

(2) The functions $a_i : T \to \mathbb{R}$ are continuous and the sequence $(a_i(t))$ converges uniformly on the interval I to the zero function.

(3) There exists a sequence d_i of non-negative reals with $\lim_{i\to\infty} d_i = 0$ and $|f_i(x_i, x_{i+1}, \dots)| \leqslant d_i$ for all $x = (x_1, x_2, \dots) \in \ell_\infty$.

(4) The mapping $f = (f_1, f_2, \dots)$ is continuous from ℓ_∞ into itself.

Theorem 13 (p.247, [9]). *Under assumptions (1)-(4), the initial value problem (13)—(14) admits at least one solution $x = x(t) = (x_i(t))$ with $x(t) \in \ell_\infty$ for all $t \in I_1 = [0, T_1]$, where $T_1 \leqslant T$ and T_1 is determined as in Theorem 9. Moreover,* $\lim_{i\to\infty} x_i(t) = a$ *uniformly w.r.t.* $t \in I_1$.

For related existence results concerning infinite systems of second-order differential equations, we refer the reader to [32].

4.3. Solvability in tempered sequence spaces

This section is devoted to explore ISDE within the context of the tempered sequence spaces.

Note that tempered sequence spaces are not merely an extension of classical sequence spaces. But sometime it is not possible to study the existence of solutions of some differential equations in classical sequence spaces c_0, c, ℓ_∞ and $\ell_p,\ p \geqslant 1$. For example, take the following case of infinite system [12]:

$$x'_k(\tau) = x_k(\tau), \forall\ k \in \mathbb{N},\ \tau \in [0, T] \tag{15}$$

with

$$x_k(0) = k,\ k \in \mathbb{N}. \tag{16}$$

The solution of this system has the form

$$x(\tau) = (x_k(\tau)) = (ke^\tau) = (e^\tau, 2e^\tau, 3e^\tau, ...),$$

but $x(\tau) \notin \ell_p(1 \leqslant p < \infty)\ \forall\ \tau \in [0, T]$.

So we see that ℓ_p does not work to consider solvability of problem (15)-(16).

This provokes us to enlarge the space under consideration and that gives the idea of enhancing the classical sequence spaces to tempered sequence spaces [6].

Solvability in the space c_0^β

We examine the following system with a lower triangular structure:

$$x'_n = \sum_{i=1}^{k_n} a_{nn_i}(t)x_{n_i} + f_n(t, x_1, x_2, \dots) \tag{17}$$

with

$$x_n(0) = x_0^n, \quad \text{for } x = 1, 2, \dots . \tag{18}$$

In this framework, for each fixed $n \in \mathbb{N}$ the index set n_k is such that $1 \leqslant n_1 < n_2 < \cdots < n_{k_n} \leqslant n$. Moreover, there exists an integer $K > 0$ such that $k_n \leqslant K$ for all n. This implies that the linear component of each equation in the system involves only a finite number of terms, with the count uniformly bounded by K. Systems satisfying this structural constraint are referred to as infinite systems while linear parts have constant width.

We proceed under with assuming the following:

(i) For every $n \in \mathbb{N}$ and $i = 1, 2, \ldots, k_n$, the coefficients $a_{nn_i} = a_{nn_i}(t)$ are continuous functions on $I = [0, T]$;

(ii) The coefficients $a_{nn_i}(t)$ are uniformly bounded: there exists $A > 0$ such that $|a_{nn_i}(t)| \leqslant A$ for all $t \in I$;

(iii) the initial data $(x_0^n) \in c_0^\beta$;

(iv) For each n, the function $f_n(t, x)$ is continuous when restricted to $I \times c_0^\beta$;

(v) there exists a nonnegative sequence (p_n) such that $\beta_n p_n \to 0$ as $n \to \infty$, $|f_n(t, x)| \leqslant p_n$ for all $t \in I$, $x \in c_0^\beta$ and $n \in \mathbb{N}$.

Under these conditions, we establish the following existence result:

Theorem 14. *Under the assumptions (i)–(v), the system* (17)–(18) *admits a solution $x(t) = (x_n(t))_{n=1}^\infty$ in the Banach space c_0^β.*

We now turn our attention to a more general form of the semilinear lower diagonal infinite system of differential equations considered earlier. Specifically, we investigate the system

$$x'_n = \sum_{i=1}^{k_n} a_{nn_i}(t) x_{n_i} + f_n(t, x_1, x_2, \ldots) \tag{19}$$

with

$$x_n(0) = x_n^0, \quad \text{for } n = 1, 2, \ldots . \tag{20}$$

In contrast to the previous case, we now drop the condition that the system has a uniformly bounded number of terms in its linear part. The assumptions are revised accordingly:

(ii') the index sequence (n_i) satisfies $n_i \to \infty$ as $i \to \infty$;

(ii") the sum of the magnitudes of the coefficients in each linear component is uniformly bounded: there exists a constant $A > 0$ such that

$$\sum_{i=1}^{k_n} |a_{nn_i}(t)| \leqslant A$$

for all $t \in I = [0, T_1]$, $n \in \mathbb{N}$.

These replace both the constant-width condition and the uniform boundedness of individual coefficients from Theorem 14.
Under these relaxed assumptions, we have the following existence result:

Theorem 15. *Assume that Theorem 14's (i), (ii'), (ii"), and (iii)–(v) are satisfied. There is at least one solution to the initial value problem* (19)–(20) *in* c_0^β *defined on* $I = [0, T_1]$, *where* T_1 *is an integer selected in accordance with Theorem 11.*

Solvability in the space c^β

We now focus on analyzing infinite systems of perturbed diagonal differential equations given by

$$x_n' = a_n(t)x_n + g_n(t, x_1, x_2, \dots) \tag{21}$$

with

$$x_n(0) = x_n^0, \tag{22}$$

for $t \in I = [0, T]$, $n \in \mathbb{N}$.
This class of initial value problems will be studied within the framework of c^β, where $\beta_n > 0$ for all $n \in \mathbb{N}$ and is nonincreasing.

We will examine the initial value problem given by (21)–(22) under the conditions:

(i) $x_0 = (x_n^0) \in c^\beta$;

(ii) the function $g = (g_1, g_2, \dots)$ maps $I \times c^\beta$ into c^β and is continuous on $I \times c^\beta$;

(iii) there exists a sequence (p_n) with $\beta_n p_n \to 0$ as $n \to \infty$, and for every $t \in I$, $x = (x_n) \in c^\beta$ and $n \in \mathbb{N}$,

$$|g_n(t, x_1, x_2, \dots)| \leqslant p_n;$$

(iv) each function $a_n(t)$ is continuous on I and $(a_n(t))$ converges uniformly on I to $a = a(t)$.

It follows from condition (iv) that the family $(a_n(t))$ is equibounded on I. Thus the quantity

$$A = \sup_{t \in I} \{a_n(t)\} < \infty.$$

We are now ready to present our main result for this setting:

Theorem 16. *Let (i)–(iv) hold. If $AT < 1$, then the system* (21)–(22) *admits at least one solution $x(t) = (x_n(t))$ on I with $x(t) \in c^\beta$ for all $t \in I$.*

Solvability in the tempered sequence space ℓ_∞^β

Recall, that $\mu(X)$ for $X \in \mathfrak{M}_{\ell_\infty^\beta}$ is defined as

$$\mu(X) = \limsup_{n \to \infty} diam X_n^\beta,$$

where $X_n^\beta = \{\beta_n x_n : x = (x_i) \in X\}$. This expression can be equivalently restated in the following, more compact form:

$$\mu(X) = \limsup_{n \to \infty} diam \beta_n x_n, \tag{23}$$

where $X_n = \{x_n : x = (x_i) \in X\}$ denotes the projection of X onto the nth coordinate.

We now turn our attention to a perturbed ISDE of the form

$$x_n' = \sum_{j=k_n}^{n} a_{nj}(t) x_j + g_n(t, x_1, x_2, \dots) \tag{24}$$

subject to

$$x_n(0) = x_n^0 \tag{25}$$

for all $n = 1, 2 \dots$ and $t \in I = [0, T]$.
In this context, suppose that the index sequence (k_n) satisfies the conditions $1 \leqslant k_n \leqslant n$ for all $n \in \mathbb{N}$ and $k_n \to \infty$ as $n \to \infty$.

It is worth noting that the systems structured as in (24) have rarely been addressed in the existing literature (see, for instance, [9]).

For later use, we define a mapping $f = f(t, x)$ on the product space $I \times \ell_\infty^\beta$ as follows:

$$f(t, x) = (f_n(t, x))_{n \in \mathbb{N}},$$

where each component function $f_n(t,x)$ is given by

$$f_n(t,x) = f_n(t,x_1,x_2,\dots) = \sum_{j=k_n}^{n} a_{nj}(t)x_j + g_n(t,x_1,x_2,\dots)$$

for all $n = 1,2,\dots$. Additionally, we introduce the operator $g(t,x)$ defined similarly on $I \times \ell_\infty^\beta$, by

$$g(t,x) = (g_n(t,x))_{n\in\mathbb{N}}.$$

We will analyze the initial value problem (24)–(25) under the following set of assumptions:

(i) $x_0 = (x_n^0) \in \ell_\infty^\beta$;

(ii) The mapping $g : I \times \ell_\infty^\beta \to \ell_\infty^\beta$ is uniformly continuous over its domain;

(iii) There exists a sequence (p_n) satisfying $\beta_n p_n \to 0$ as $n \to \infty$, such that

$$|g_n(t,x)| \leqslant p_n$$

for $x = (x_n) \in \ell_\infty^\beta$, $t \in I$;

(iv) Each function $a_{nj} : I \to \mathbb{R}$ (with $j = k_n, k_n+1,\dots,n$) is continuous and nondecreasing on I. Moreover, the family of partial sums

$$A_n(t) = \sum_{j=k_n}^{n} a_{nj}(t), \quad \bar{A}_n(t) = \sum_{j=k_n}^{n} |a_{nj}(t)|$$

is such that $(A_n(t))$ is equicontinuous on I and $(\bar{A}_n(t))$ is uniformly bounded on I.

Due to the boundedness in (iv), we may define

$$A = \sup_{t\in I} \bar{A}_n(t),\ n = 1,2,\dots,$$

which is finite.

Now we conclude the following statement for the system (24)–(25):

Theorem 17. *Suppose conditions (i)–(iv) hold and that $AT < 1$. Then system (24)–(25) admits at least one solution $x(t) = (x_k(t))$ on $I = [0,T]$, such that $x(t) \in \ell_\infty^\beta$ for all $t \in I$.*

Remark 18. Note that in place of the condition in assumption (iv), it is equally permissible to require that these functions are nonincreasing on I.

5. Fractional differential equations

For integer-order derivatives $\mathrm{D}^{\gamma} f(x)$ with $\gamma \in \mathbb{N}$, the evaluation requires knowledge of f only in an arbitrarily small neighborhood of x, indicating that such operators are local. In contrast, for $\gamma \notin \mathbb{N}$, the Riemann–Liouville fractional derivative $\mathrm{D}_a^{\gamma} f(x)$ depends on the behavior of f over the entire interval $[a, x]$, making it a nonlocal (global) operator.
Agarwal et al. [2] contributed to the development of the theory of fractional calculus by focusing on the Riemann–Liouville integral operator. Among various fractional derivatives, the Hilfer fractional derivative—introduced by Hilfer [15]—is particularly notable as it interpolates between the Riemann–Liouville and Caputo derivatives, controlled by a parameter. This flexibility enhances its applicability in modeling physical and engineering phenomena. Jajarmi et al. [17] further investigated the analytical and practical aspects of a regularized ψ-Hilfer derivative, emphasizing its utility in applied problems.

We recall some notions concerning fractinal derivatives.

Riemann–Liouville Fractional Integral [19]:

The left and right-sided integrals of a measurable function w defined on $[a, b]$ are defined by

$$I^{\gamma}_{a^+(b_-)} w(s) = \begin{cases} I^{\gamma}_{a^+} w(s) = \dfrac{1}{\Gamma(\gamma)} \int\limits_a^s (s-r)^{\gamma-1} w(r) dr & \gamma > 0; \\ I^{\gamma}_{b_-} w(s) = \dfrac{1}{\Gamma(\gamma)} \int\limits_s^b (r-s)^{\gamma-1} w(r) dr & \gamma > 0; \\ w(s) & \gamma = 0. \end{cases}$$

Riemann–Liouville Fractional Derivative [19]:

For $w \in AC^k[a, b]$ with $k = [\gamma] + 1$,

$$\mathrm{D}^{\gamma}_{a^+,(b_-)} w(s) = \begin{cases} \mathrm{D}^{\gamma}_{a^+} w(s) = (\frac{d^k}{ds^k}) I^{k-\gamma}_{a^+} w(s) & \gamma > 0; \\ \mathrm{D}^{\gamma}_{(b_-)} w(s) = (-1)^k (\dfrac{d^k}{ds^k}) I^{k-\gamma}_{b_-} w(s) & \gamma > 0; \\ w(s) & \gamma = 0. \end{cases}$$

Caputo Fractional Derivative [19]:

For $k = [\gamma] + 1$, $\mathrm{D} = \frac{d}{ds}$ and $w \in AC^k[a,b]$, The Caputo fractional derivatives of order γ are defined by

$$ {}^c\mathrm{D}^{\gamma}_{a^+,(b_-)}w(s) = \begin{cases} {}^c\mathrm{D}^{\gamma}_{a^+}w(s) = (I^{k-\gamma}_{a^+}\mathrm{D}^k w)(s) & s > a; \\ {}^c\mathrm{D}^{\gamma}_{b_-}w(s) = (-1)^k (I^{k-\gamma}_{b_-}\mathrm{D}^k w)(s) & s < b. \end{cases} $$

Hilfer Fractional Derivative [15, 37]:

Let $k = [\gamma] + 1$, $w \in AC^k[a,b]$. Then the (left and right-sided) Hilfer fractional derivatives of order γ and type η are given by

$$ \mathrm{D}^{\gamma,\eta}_{a^+,(b_-)}w(s) = \begin{cases} \mathrm{D}^{\gamma,\eta}_{a^+}w(s) = (I^{\eta(k-\gamma)}_{a^+} D^{\gamma+\eta(k-\gamma)}_{a^+} w)(s), & s > a; \\ \mathrm{D}^{\gamma,\eta}_{(b_-)}w(s) = (I^{\eta(k-\gamma)}_{b_-} D^{\gamma+\eta(k-\gamma)}_{b_-} w)(s), & s < b. \end{cases} $$

Remark 19.

(i) For $\eta = 0$, Hilfer derivative reduces to the Riemann–Liouville derivative.

(ii) For $\eta = 1$, it corresponds to the Caputo derivative.

5.1. Hilfer fractional differential equations

Consider the system:

$$ \mathrm{D}^{\gamma,\eta}_{a^+}y_k + g_k(s,y) = 0, \quad y_k(a) = y_k(b) = 0, \quad s \in (a,b), \quad k \in \mathbb{N}; \tag{26} $$

where $\gamma \in (1,2]$, $\eta \in [0,1]$, $\mathrm{D}^{\gamma,\eta}_{a^+}$ denotes the Hilfer fractional derivative operator of order γ and type η. The function $g_k(s,y) = c_k(s) + f_k(s, y_1, y_2, y_3, ...)$ where c_k and f_k are continuous on $[a,b]$ for each $k \in \mathbb{N}$.

Recently in [12], the solvability of system (26) has been studied in tempered sequence spaces ℓ^{β}_p and c^{β}_0.

Existence criterion in ℓ^{β}_p

To establish the existence of solutions for the system (26) in ℓ^{β}_p, we impose the following assumptions:

M.1 The functions $g_k : [a,b] \times \mathbb{R}^{\infty} \to \mathbb{R}$ and operator $\mathcal{Q} : [a,b] \times \ell^{\beta}_p \to \ell^{\beta}_p$ are defined by

$$ (\mathcal{Q}y)(s) := (g_1(s,y), g_2(s,y), g_3(s,y), ...), \tag{27} $$

and the family of mappings $((\mathcal{Q}y)(s))_{s\in[a,b]}$ is uniformly equicontinuous $\forall\ y \in \ell_p^\beta$.

M.2 There exist non-negative measurable functions $d_k(s)$ and $h_k(s)$ on $[a,b]$ such that

$$|f_k(s, y_1, y_2, y_3, ...)|^p \leqslant d_k(s) + h_k(s)|y_k(s)|^p,$$

for all $s \in [a,b]$, $k \in \mathbb{N}$ and $y = (y_k) \in \ell_p^\beta$.

M.3 The functions $c_k(s)$, $d_k(s)$ and $h_k(s)$ are continuous on $[a,b]$ for each $k \in \mathbb{N}$. Furthermore, the series $\sum\limits_{k=1}^{\infty} \beta_k^p c_k(s)$ and $\sum\limits_{k=1}^{\infty} \beta_k^p d_k(s)$ converge uniformly on $[a,b]$ and the sequence $\{h_k\}_{k=1}^{\infty}$ is equibounded.

Additional Definitions:
Let us define the constants

(i) $\mathcal{E} := \sup\limits_{s\in[a,b]} \left\{ \sum\limits_{k=1}^{\infty} \beta_k^p \left(c_k(s) + d_k(s)\right) \right\}$,

(ii) $\mathcal{D} := \sup\limits_{k\in\mathbb{N},\ s\in[a,b]} \{h_k(s)\}$.

Theorem 20. *Assume that conditions (M.1)–(M.3) hold. Then the infinite system* (26) *admits at least one solution $y(s) = (y_k(s))$ in the space ℓ_p^β for every $s \in [a,b]$, provided that $\Lambda^p (b-a)^{\frac{p+q}{q}} \mathcal{D} < 1$, $\frac{1}{p} + \frac{1}{q} = 1$. Moreover, the solution satisfies $\{y(s)\} \in \ker \chi$ for each $s \in [a,b]$.*

Existence criterion in c_0^β

We now examine the existence of solutions to the system (26) within the space c_0^β, under the following conditions:

N.1 Let $g_k : [a,b] \times \mathbb{R}^\infty \to \mathbb{R}$ and define the operator
$\mathcal{P} : [a,b] \times c_0^\beta \to c_0^\beta$,

$$(\mathcal{P}y)(s) = (g_1(s,y), g_2(s,y), g_3(s,y), ...).$$

Then, for every $y \in c_0^\beta$, the family $((\mathcal{P}y)(s))_{s\in[a,b]}$ is pointwise equicontinuous in c_0^β.

N.2 There exist non-negative continuous functions $d_k(s)$ and $h_k(s)$ such that

$$|f_k(s, y_1, y_2, y_3, ...)| \leqslant d_k(s) + h_k(s) \sup_{j \geqslant k}\{|y_j|\},$$

for all $s \in [a,b]$, $k \in \mathbb{N}$ and $y = (y_k) \in c_0^\beta$.

N.3 The functions $c_k(s)$, $d_k(s)$ and $h_k(s)$ are continuous on $[a,b]$ for each $k \in \mathbb{N}$, and the sequences $(\beta_k \{c_k(s)\})_{k=1}^\infty$ and $(\beta_k \{d_k(s)\})_{k=1}^\infty$ converge uniformly to 0 on $[a,b]$. Moreover, $\{h_k(s)\}_{k=1}^\infty$ is uniformly bounded on $[a,b]$.

Additional Notation:
Define the constants

(i) $\mathcal{E} := \sup_{s \in [a,b]} \{\alpha_k (c_k(s) + d_k(s))\}$,

(ii) $\mathcal{D} := \sup_{k \in \mathbb{N},\ s \in [a,b]} \{h_k(s)\}$.

Theorem 21. *Suppose the conditions (N.1)–(N.3) are satisfied. Then the system* (26) *possesses at least one solution* $y(s) = (y_k(s)) \in c_0^\beta \ \forall\ s \in [a,b]$, *provided the inequality* $\Lambda(b-a)\mathcal{D} < 1$ *holds. Furthermore, the solution belongs to the kernel of the measure* χ, *i.e.* $\{y(s)\} \in \ker \chi$ *for every* $s \in [a,b]$.

5.2. Langevin fractional differential equations

Consider the generalized Langevin fractional differential equations:

$$ {}^{\mu_n}_{c}\mathrm{D}^{\gamma_n} \left({}^{\mu_n}_{c}\mathrm{D}^{\nu_n} + \xi_n\right) w_n(s) = g_n \left(s, w(s), \varphi({}^{\mu}_{c}\mathrm{D}^{\rho} w(s))\right), \tag{28}$$

for $s \in [0,1]$ and $n = 1, 2, 3....$ with

$$w_n(0) = 0, \quad {}^{\mu_n}_{c}\mathrm{D}^{\nu_n} w_n(0) = 0, \quad w_n(1) = \sigma_n w_n(\lambda_n), \tag{29}$$

where $w(s) = \{w_n(s)\}_{n=1}^\infty \in \ell_p^\beta$, $\varphi({}^{\mu}_{c}\mathrm{D}^{\rho} w(s)) = \{\varphi_n({}^{\mu_n}_{c}\mathrm{D}^{\rho_n} w_n(s))\}_{n=1}^\infty \in \ell_p^\beta, \varphi_n : [0,1] \times \ell_p^\beta \to \ell_p^\beta$ are continuous functions, $g_n : [0,1] \times \ell_p^\beta \times \ell_p^\alpha \to \ell_p^\beta$ are differentiable functions, $0 < \mu_n \leqslant 1,\ 1 < \gamma_n \leqslant 2,\ 0 < \rho_n < \nu_n \leqslant 1,\ {}^{\mu}_{c}\mathrm{D}^{\gamma}$ is the Caputo generalized fractional derivative [18], $\sigma_n \in \mathbb{R},\ 0 < \lambda_n < 1$ and $\sigma_n \lambda_n^{\nu_n - 1} \neq 1\ \forall n \in \mathbb{N}$.

Existence in $\mathbb{E}_p^\beta$

Inzamam et al. [13] examines the solution of an infinite system of Langevin fractional differential equations in tempered sequence space $\mathbb{E}_p^\beta$, where

$$\mathbb{E}_p^\beta = \{y(s) : y \in \mathrm{C}([0,1], \ell_p^\beta) \text{ and } {}_c^\mu\mathrm{D}^\rho y(s) \in \mathrm{C}([0,1], \ell_p^\beta)\},$$

and $\mathrm{C}([0,1], \ell_p^\beta)$ is the space of sequences of continuous functions on $[0,1]$ which are in the space ℓ_p^β.

Assume the following:

N.1 For every n the jointly continuous functions $g_n : [0,1] \times \ell_p^\beta \times \ell_p^\beta \to \ell_p^\beta$ meet the Lipschitz condition with Lipschitz constant K:

$$|g_n(s, y_1, z_1) - g_n(s, y_2, z_2)| \leqslant K\left(|y_1 - y_2| + |z_1 - z_2|\right), \; y_k, z_k \in \ell_p^\beta,$$

$k = 1, 2$.

N.2 There exist $\alpha_i \in \mathbb{R}$, $i \geqslant n$ such that $\sum\limits_{i \geqslant n}^{\infty} \alpha_i x_i(s) = 0, \; \forall s \in [0,1]$, i.e. $\{x_n(s)\}_{n=1}^\infty \in \ell_1^\beta$ is integrable on $[0,1]$,

$$0 < \mathcal{N} = \sum_{n=1}^{\infty} \beta_n \int_0^1 x_n(r)dr = \int_0^1 \sum_{n=1}^{\infty} \beta_n x_n(r)dr.$$

N.3 $|g_n(s, y, z)|^p \leqslant x_n(s) + h_n(s)\big(|y_n(s)|^p + |z_n(s)|^p\big)$ holds for $s \in [0,1]$, $n \in \mathbb{N}$ and $y, z \in \ell_p^\beta$, where $\{h_n(s)\}_{n=1}^\infty$ is equibounded on $[0,1]$ and $\{x_n(s)\}_{n=1}^\infty$ behaves in the same way as in (N.2). In other words, the following inequality holds

$$0 < \mathfrak{M} = \sup_{s \in [0,1]} \sup_{n \in \mathbb{N}} (h_n(s)).$$

N.4 For operators $\mathcal{Q}_n$ described in (27) and $q \geqslant 1$ there exists constant $\mathcal{R}_{\gamma,\rho}$ defined in the following way:

$$\mathcal{R}_{\gamma,\rho} = \sup_{s \in [0,1]} \sup_{n \in \mathbb{N}} \bigg(\int_0^1 |\mathcal{Q}_n(s, r, \gamma_n, \rho_n)|^q dr \bigg).$$

N.5 The functions $\varphi_n : \mathbb{R} \to \mathbb{R}$ are continuous and additive for each $n \in \mathbb{N}$. Thus, it satisfies the equation that follows:

$$\varphi_n(y+z) = \varphi_n(y) + \varphi_n(z),\ y,z \in \mathbb{R},\ n \in \mathbb{N}.$$

Theorem 22. *Put* $\mathrm{G}_\gamma(z) = z^{\frac{1}{p}}(\mathcal{R}_{\gamma,0}^{\frac{p-1}{p}} + \delta\mathcal{R}_{\gamma,\rho}^{\frac{p-1}{p}})$. *The system (28)-(29) has at least one solution in* $\mathbb{E}_p^\beta$ *under conditions* (N.1)–(N.5), *if* $\mathrm{G}_\gamma(\mathfrak{M}) + \mathrm{G}_0(\xi^p) < 1$ *and* $\mu_n(\nu_n - \rho_n) > \frac{1}{p}$ $(p > 1)$.

Remark 23. Solvability of an infinite system of fractional differential equations (ISFDE) with p-Laplacian operator in $\mathbb{E}_p^\beta$ is studied in [33].

5.3. Implicit Hadamard-Caputo fractional differential equation

In this section, we present the solvability of (BVP) for the following implicit Hadamard-Caputo fractional differential equations in $m^\beta(\psi,p)$ (see [26]).

$$\begin{cases} {}_H^C D^\gamma z(\sigma) = f_i(\sigma, z(\sigma)),\ \sigma \in [1,C]\ 0 < \gamma \leqslant 1,\ i \in \mathbb{N} \\ \sum_{o=1}^{m} \alpha_k z(\sigma_o) = \sigma_1, \end{cases} \tag{30}$$

where ${}_H^C D^\gamma$ is the Hadamard-Caputo fractional derivative, $f_i : [1,C] \times \mathbb{R} \to \mathbb{R}$ are maps, for $(i \in \mathbb{N})$, $z_1, a_o \in \mathbb{R}$, $o = 1,2,\ldots,m$ and $1 < \sigma_1 < \sigma_2 <, \ldots < \sigma_m < C$.

The equation (30) is equivalent to the integral equation

$$x(\sigma) = ax_1 - \frac{1}{\Gamma(\gamma)} \int_1^{\sigma_k} (\log\frac{\sigma}{\varrho})^{\gamma-1} f(\sigma, x(\varrho)) \frac{d\varrho}{\varrho} + \\ \frac{1}{\Gamma(\gamma)} \int_1^{\sigma} (\log\frac{\sigma}{\varrho})^{\gamma-1} f(\sigma, x(\varrho)) \frac{d\varrho}{\varrho}.$$

We present an existence result for the implicit Hadamard - Caputo fractional differential equation (30) in the tempered sequence space $C([1,C], m^\beta(\psi,p))$.

Theorem 24. *Let the following condition hold:*
(H) $f = (f_1, f_2, \ldots)$ *continuously maps the set* $[1,C] \times m^\beta(\psi,p)$ *to* $m^\beta(\psi,p)$ *and the family of mappings* $\{f_i(\sigma), z(\sigma)\}_{\sigma \in [1,C]}$ *is equicontinuous at each*

point of $m^\beta(\psi,p)$*, so that*

$$|f_i(\sigma,z(\sigma))|^p \leqslant b_1^p|z_i(\sigma)|^p|u_i(\sigma)|^p,$$

where $b_1>0$*,* $z\in C([1,C],m^\beta(\psi,p))$*,* $p\geqslant 1$*,* u_i *are positive real functions, equibounded in* σ *for any* $\sigma\in[1,C]$ *and* $i=1,2,\dots$.
If $\left[b_1U|ax_1| + \frac{2^{\frac{p+1}{p}} b_1U(\log C)^\gamma \ln C}{\Gamma(\gamma)}\right] < 1$*, then it admits at least one solution* $x\in C([1,C],m^\beta(\psi,p))$*, where* $\sup\limits_{\sigma\in[1,C],i\in\mathbb{N}} u_i(\sigma)\leqslant U$.

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Summary

In this chapter, we present a brief survey on measures of noncompactness and their applications in solving the infinite systems of differential equations in classical sequence spaces and tempered sequence spaces.

Streszczenie

W tym rozdziale przedstawiamy krótki przegląd miar niezwartości oraz ich zastosowań w rozwiązywaniu nieskończonych układów równań różniczkowych w klasycznych przestrzeniach ciągów oraz przestrzeniach ciągów temperowanych.

Chapter 2

Existence of solutions for infinite systems of differential equations in spaces of tempered sequences

Monika Krajewska

1. Introduction

The theory of ordinary differential equations is one of the most significant branches of mathematics, as evidenced by its vast range of applications in describing the behavior of numerous phenomena in nature, engineering, and the surrounding reality. The classical theory of these equations — the theory of ordinary differential equations in finite-dimensional spaces—has, since the 1950s, become almost a closed theory. This status has been thoroughly presented in classical books and monographs such as [11], [16], [19], [29], [30]. In 1950, the well-known French mathematician J. Dieudonné [14] demonstrated through two examples that the classical results of the theory of differential equations in finite-dimensional spaces cease to hold in the case of infinite-dimensional spaces. One need only mention that Peano's classical theorem on the existence of solutions to the Cauchy problem for the differential equation $x' = f(t, x)$ under the assumption of continuity of the right-hand side, is no longer valid when considered in infinite-dimensional spaces. This realization became a stimulus for initiating research on ordinary differential equations in infinite-dimensional spaces. The first significant results in this direction were obtained in the late 1950s and early 1960s (cf. [20], [27], [38]).

These early results still relied on standard tools of analysis. However, over time, new methods from functional analysis, particularly those related to the theory of measures of noncompactness, began to be employed. This allowed for the development of numerous results within the framework of the so-called theory of ordinary differential equations in Banach spaces (i.e., infinite-dimensional spaces). These results were presented, among others, in works [2], [15], [17], [33], [35], and in the book [12] (see also [1], [3], [4], [9], [13], [28], [34]).

It is worth noting that the aforementioned monograph by K. Deimling [12] identified many potential applications of the theory of differential equations in Banach spaces. However, in later works, these applications were rarely developed—mainly due to the difficulties associated with the use of tools and techniques related to measures of noncompactness, which play a key role in these applications.

One of the potential applications indicated in [12] involved infinite systems of ordinary differential equations. Such systems arise naturally as differential equations in sequence Banach spaces and also appear when considering certain problems in the theory of branching processes, in modeling some phenomena in neural network theory, and in polymer dissociation [10], [12], [18], [26], [39]. It is also important to note that some problems considered in mechanics lead to infinite systems of differential equations [29], [37], [39], [40].

Infinite systems of ordinary differential equations also appear in the context of numerical methods for solving certain partial differential equations, such as parabolic-type equations [12]. For example, by applying a semidiscretization process to parabolic PDEs, one obtains an infinite system of ordinary differential equations [12], [36], [37].

It is also worth mentioning that the pioneer of the theory of infinite systems of differential equations was the Kazakh mathematician K.P. Persidskii, who initiated the study of such systems even before the theory of differential equations in Banach spaces began to be intensively developed (cf. [29], [30], [31]). However, these two approaches to the theory of infinite systems of differential equations eventually began to intertwine.

Indeed, as already mentioned, infinite systems of ordinary differential equations can be treated as a special case of differential equations in (sequence) Banach spaces. Therefore, when analyzing such systems, one can employ results from the general theory of differential equations in these spaces (cf. [3], [4], [8], [9], [12], [13], [17], [22], [23], [25], [32], [33]). Naturally, this approach requires a solid command of the methods involving measures of noncompactness, which are essential in this context. To date, only a few works have followed this line of research [3], [8], [9], [12], [24], [25].
On the other hand, infinite systems of differential equations require the application of certain specialized analytical methods tailored to the specific nature of such systems. This direction of research was initiated in particular by the work [8], with the results subsequently discussed in the monograph [9]. The present chapter continues the aforementioned studies. It contains the survey of theorems and examples of infinite systems of ordinary differential equations using the tools of the theory of measures of noncompactness.

2. Selected facts from the theory of measures of noncompactness

Now we present some basic facts concerning the theory of measures of noncompactness (cf. [4]).
Let us assume that E is a real Banach space with norm denoted by $\|\cdot\|_E$ or simply $\|\cdot\|$. Let $\mathbb{R}$ denote the set of real numbers and $\mathbb{R}_+ = [0, \infty)$ the set of non-negative real numbers. The ball in E with center x_0 and radius $r > 0$ is denoted by: $B(x_0, r)$.
If X is a subset of E, then $\overline{X}$ denotes the closure of X, and $\operatorname{Conv} X$ denotes the convex and closed hull of X. The diameter of a bounded set Y is denoted by $\operatorname{diam} Y$.
For subsets $X, Y \subset E$, the operations $X + Y$ and αX are defined as:

$$X + Y = \{x + y : x \in X, y \in Y\}, \quad \alpha X = \{\alpha x : x \in X\}, \quad \alpha \in \mathbb{R}.$$

Let $\mathfrak{M}_E$ denote the family of all nonempty bounded subsets of E, and let $\mathfrak{N}_E$ denote the subfamily of relatively compact sets in E.
Let $X, Y \in \mathfrak{M}_E$. The quantity

$$d(X, Y) = \inf\{r : X \subset B(Y, r)\},$$

where $B(Y, r) = \bigcup_{y \in Y} B(y, r)$, is called the nonsymmetric distance between

X and Y. The Hausdorff distance between X and Y is defined by:

$$D(X,Y) = \max\{d(X,Y), d(Y,X)\}.$$

Note that D is a pseudometric on $\mathfrak{M}_E$, and a metric on the family $\mathfrak{M}_E^c$ of all closed sets in $\mathfrak{M}_E$. The metric space $(\mathfrak{M}_E^c, D)$ is complete if E is a Banach space (see [21]).
Now, we recall the definitions of sequence spaces c_0, c, l_p and l_∞ being **the classical sequence spaces**.
By the space c_0 we mean the set of all real (or complex) sequences $x = (x_n)$ converging to zero and normed by the classical supremum (or maximum) norm:

$$\|x\|_{c_0} = \|(x_n)\|_{c_0} = \sup\{|x_n| : n = 1, 2, \dots\} = \max\{|x_n| : n = 1, 2, \dots\}.$$

Obviously c_0 with this norm creates the Banach space.
Next, denote by c the space of all sequences $x = (x_n)$ converging to a (finite) limit, with the norm

$$\|x\|_c = \|(x_n)\|_c = \sup\{|x_n| : n = 1, 2, \dots\}.$$

The space c with the norm $\|\cdot\|_c$ is a Banach space and c_0 is a closed subspace of c.
If we fix a number p, $p \geqslant 1$, then by l_p we denote the space consisting of all sequences $x = (x_n)$ such that $\sum_{n=1}^{\infty} |x_n|^p < \infty$. If we norm it by

$$\|x\|_{l_p} = \|(x_n)\|_{l_p} = \Big(\sum_{n=1}^{\infty} |x_n|^p\Big)^{1/p}$$

it becomes a Banach space.
Finally, by the symbol l_∞ we denote the space of all bounded sequences $x = (x_n)$ with the supremum norm

$$\|x\|_{l_\infty} = \|(x_n)\|_{l_\infty} = \sup\{|x_n| : n = 1, 2, \dots\}.$$

With this norm, ℓ_∞ is also a Banach space (cf. [13]).
We now move on to discuss basic facts concerning **measures of noncompactness** in Banach spaces (cf. [4]). Let E be a Banach space.

Definition 1. A function $\mu : \mathfrak{M}_E \to \mathbb{R}_+$ is called a measure of noncompactness if the following conditions are satisfied:

(i) The family $\ker\mu = \{X \in \mathfrak{M}_E : \mu(X) = 0\}$ is nonempty and $\ker\mu \subset \mathfrak{N}_E$;

(ii) $X \subset Y \Rightarrow \mu(X) \leqslant \mu(Y)$;

(iii) $\mu(\bar{X}) = \mu(X)$;

(iv) $\mu(\operatorname{Conv} X) = \mu(X)$;

(v) $\mu(\lambda X + (1-\lambda)Y) \leqslant \lambda\mu(X) + (1-\lambda)\mu(Y)$ for $\lambda \in [0,1]$;

(vi) if (X_n) is a sequence of closed sets from $\mathfrak{M}_E$ such that $X_{n+1} \subset X_n$ for $n = 1, 2, \ldots$ and $\lim_{n\to\infty} \mu(X_n) = 0$ then the set $X_\infty = \cap_{n=1}^{\infty} X_n$ is nonempty.

The family $\ker\mu$ from axiom (i) is said to be *the kernel of the measure* μ. Further, let us observe that from axiom (vi) it follows that $\mu(X_\infty) \leqslant \mu(X_n)$ for $n = 1, 2, \ldots$. This yields that $\mu(X_\infty) = 0$. Hence we conclude that the intersection set X_∞ belongs to the kernel $\ker\mu$. This simple fact plays a very essential role in applications.
In the sequel we will also consider measures of noncompactness having some additional properties. Thus, a measure μ is referred to as *sublinear* if it satisfies the following two conditions:

(vii) $\mu(\lambda X) = |\lambda|\mu(X)$, $\lambda \in \mathbb{R}$;

(viii) $\mu(X + Y) \leqslant \mu(X) + \mu(Y)$.

We say that a measure of noncompactness has *maximum property* if

(ix) $\mu(X \cup Y) = \max\{\mu(X), \mu(Y)\}$.

The measure μ is said to be *full* if

(x) $\ker\mu = \mathfrak{N}_E$.

Finally, the measure of noncompactness μ is called *regular* if it is sublinear, full and has maximum property.
One of the earliest and most useful measures of noncompactness is the so-called **Hausdorff measure of noncompactness**, defined by:

$$\chi(X) = \inf\{\varepsilon > 0 : X \text{ has a finite } \varepsilon\text{-net in } E\}.$$

The Hausdorff measure χ is a regular measure of noncompactness. It can be shown that:

$$\chi(X) = D(X, \mathfrak{N}_E),$$

where

$$D(X, \mathfrak{N}_E) = \inf\{D(X, Y) : Y \in \mathfrak{N}_E\}.$$

Moreover, in certain Banach spaces the Hausdorf measure can be expressed by a formula referring to the structure of these spaces.
To present the mentioned formulas let us consider first the space c_0 and let us take an arbitrary nonempty and bounded subset of c_0 i.e., take a set $X \in \mathfrak{M}_{c_0}$. Then we have [4]

$$\chi(X) = \lim_{n\to\infty} \Big\{ \sup_{(x_n)\in X} \Big\{ \sup\{|x_i| : i \geqslant n\} \Big\} \Big\}.$$

Next, if we fix arbitrarily a number p, $p \geqslant 1$, then for $X \in \mathfrak{M}_{l_p}$ we have [4,9]

$$\chi(X) = \lim_{n\to\infty} \Big\{ \sup \Big\{ \Big(\sum_{k=n}^{\infty} |x_n|^p \Big)^{1/p} : x = (x_i) \in X \Big\} \Big\}.$$

In the case of the sequence space c the situation is a bit more complicated. Namely, we do not know a formula for the Hausdorff measure χ in c but we know only a good estimate χ. Indeed, for $X \in \mathfrak{M}_c$ let us define the quantity $\mu(X)$ by the formula

$$\mu(X) = \lim_{n\to\infty} \Big\{ \sup_{(x_k)\in X} \Big\{ \sup\{|x_i - \lim_{k\to\infty} x_k| : i \geqslant n\} \Big\} \Big\}. \tag{1}$$

Then we have the estimate

$$\frac{1}{2}\mu(X) \leqslant \chi(X) \leqslant \mu(X) \tag{2}$$

and this estimate is sharp [4]. It can be shown that measure (1) is regular. Nevertheless, let us pay attention to the fact that the measure μ has only theoretical meaning since the use of formula (1) requires to know limits of sequences belonging to a set X. Therefore, to obtain a more convenient formula we can use the classical Cauchy condition associated with the limit of a sequence, since such an approach does not require the use of the limit of a sequence. Thus, for $X \in \mathfrak{M}_c$ we define the quantity

$$\mu_c(X) = \lim_{k\to\infty} \Big\{ \sup_{(x_i)\in X} \Big\{ \sup\{|x_n - x_m| : n, m \geqslant k\} \Big\} \Big\}. \tag{3}$$

It is worthwhile mentioning that in a few papers and monographs (see [4, 8, 9], for example) we can encounter results asserting that the measure μ_c defined by formula (3) is regular and equivalent to the Hausdorff measure χ in the space c. On the other hand, the proof of this fact was presented only in the work [6].

In the space ℓ_∞, no convenient formula is known for a regular measure of noncompactness. In practice, we often use other expressions. To present the above mentioned formulas let us fix a set $X \in \mathfrak{M}_{l_\infty}$. Next, we define the following three quantities:

$$\mu_1^\infty(X) = \lim_{n\to\infty} \Big\{ \sup_{(x_i)\in X} \{ \sup\{|x_i| : i \geqslant n\} \} \Big\}, \tag{4}$$

$$\mu_2^\infty(X) = \lim_{k\to\infty} \Big\{ \sup_{(x_i)\in X} \{ \sup\{|x_n - x_m| : n, m \geqslant k\} \} \Big\}, \tag{5}$$

$$\mu_3^\infty(X) = \limsup_{n\to\infty} \operatorname{diam} X_n, \tag{6}$$

where $X_n = \{x_n : x = (x_i) \in X\}$ and $\operatorname{diam} X_n = \sup\big\{|x_n - y_n| : x = (x_i),\ y = (y_i) \in X\big\}$. Observe that the formula expressing the quantity μ_1^∞ coincides with the formula for the Hausdorff measure of noncompactness in the space c_0. On the other hand, formula (5) for the quantity μ_2^∞ coincides with formula (3) for the measure of noncompactness μ_c in the sequence space c. Moreover the functions are a sublinear measures of noncompactness, μ_1^∞ and μ_2^∞ have the maximum property but there are not full.

3. Existence theorems for differential equations in Banach spaces

In this section, we recall a few theorems concerning the existence of solutions to the Cauchy initial value problem for differential equations in a Banach space.

Let us assume that E is a given Banach space, and that the function

$$f : I \times B(x_0, r) \to E$$

is given, where $I \subset \mathbb{R}$ is a given interval, and $B(x_0, r) \subset E$ is the ball centered at some point $x_0 \in E$ with radius $r > 0$.
We consider the differential equation:

$$x'(t) = f(t, x(t)) \tag{3}$$

with the initial condition:

$$x(t_0) = x_0. \tag{4}$$

The Cauchy problem for equation (3) with initial condition (4) consists in finding a solution $x = x(t)$ of the differential equation (3), defined on some interval $J \subset I$, which satisfies equation (3) on that interval and also the initial condition (4), where x_0 is a given point in the Banach space E.
Furthermore, let us assume that μ is a given measure of noncompactness in the space E.
The kernel set of the measure of noncompactness μ in E is defined as the set

$$\ker(\mu) = \{x \in E : \{x\} \in \ker \mu\}$$

(cf. [9]). The kernel $\ker(\mu)$ is a closed subset of E. Moreover, the functions are sublinear measures of noncompactness, then $\ker(\mu)$ is a closed linear subspace of E.
We now formulate a theorem which is convenient for applications and which will be used in further considerations.

Theorem 2. *Assume that the function f is uniformly continuous on the set $I \times B(x_0, r)$, and $\|f(t, x)\| \leqslant A$ for all $t \in I$ and $x \in B(x_0, r)$. Let μ be a sublinear measure of noncompactness in E such that $x_0 \in \ker(\mu)$. Assume that for every nonempty bounded set $A \subset E$ and for almost all $t \in I$, the following inequality holds:*

$$\mu(f(t, A)) \leqslant \phi(t)\mu(A) \tag{5}$$

where $\phi : I \to \mathbb{R}_+$ is an integrable function on I. Then the initial value problem (3)–(4) has at least one solution $x(t)$ on the interval I such that $x(t) \in \ker(\mu)$ for all $t \in I$.

We now state a theorem which is a slight modification of Theorem (2) and is more convenient for applications (cf. [8]; [9]).

Theorem 3. *Let* $f : I \times B(x_0, r) \to E$ *be a function such that*

$$\|f(t,x)\| \leqslant \alpha\|x\| + \beta \tag{6}$$

for all $t \in I$*,* $x, y \in B(x_0, r)$*, where* $\alpha, \beta \geqslant 0$ *are constants. Moreover, suppose that* f *is uniformly continuous on* $I \times B(x_0, r)$*, and that*

$$\mu(f(t, X)) \leqslant \phi(t)\mu(X) \tag{7}$$

for all $t \in I$*, where* μ *is a sublinear measure of noncompactness such that* $\ker(\mu) \neq \emptyset$ *and* $\phi : I \to \mathbb{R}_+$ *is an integrable function on* I*. Then the initial value problem (3)–(4) has a solution* $x(t)$ *on the interval* I *such that* $x(t) \in B(x_0, r)$ *for all* $t \in I$*.*

Let's pay attention to some interesting facts. If we take the Banach space to be a sequence space, then a differential equation becomes equivalent to an infinite system of differential equations. This allows for more subtle considerations regarding the solutions of infinite systems of differential equations, where the assumptions for the existence of solutions utilize the structure of the given space.
The results for infinite systems of differential equations in sequence Banach spaces were presented in detail, among others, in the paper [5].
Now, we show that even in rather simple situations the mentioned classical sequence spaces are not sufficient for the location of our investigations.

Example 4. To show the influence of the choice of initial values in a sequence space in which are located solutions of a considered initial value problem for an infinite system of differential equations, let us consider the linear diagonal infinite system of differential equations

$$x_n' = x_n \tag{8}$$

with the initial conditions

$$x_n(0) = n, \quad \text{for } n = 1, 2, \dots . \tag{9}$$

We consider problem (8)–(9) on an interval $I = [0, T]$.
It is easily seen that the solution of (8)-(9) has the form

$$x(t) = (x_n(t)) = (ne^t) = (e^t, 2e^t, 3e^t, \dots).$$

This means that $x(t) \notin l_\infty$ for each $t \in I$. Thus the sequence space l_∞ is not suitable to consider solvability of problem (8)–(9) in this space. Obviously, such a situation appears quite naturally since the initial point $(x_n^o) = (n)$ is not a member of l_∞.

Example 5. Let us consider the infinite system of differential equations

$$x_n' = n\frac{\sqrt{|x_n|}}{\sqrt{|x_n|}+1} \tag{10}$$

for $n = 1, 2, \ldots$, together with initial conditions

$$x_n(0) = 0, \quad \text{for } n = 1, 2, \ldots . \tag{11}$$

Let us fix arbitrarily a natural number n. Then, we can easily calculate that the solution of problem (10)–(11) has the form

$$x_n(t) = \frac{n^2t^2}{2 + nt + 2\sqrt{1+nt}}$$

for $t \in I$. Hence, we obtain the estimate

$$\begin{aligned} x_n(t) &\geqslant \frac{n^2t^2}{2 + nt + 2\sqrt{1 + 2nt + n^2t^2}} \\ &= \frac{n^2t^2}{2 + nt + 2(1+nt)} \geqslant \frac{n^2t^2}{4 + 4nt} \quad = \frac{1}{4}\Big(nt - 1 + \frac{1}{nt+1}\Big) \end{aligned} \tag{12}$$

for $n = 1, 2, \ldots$ and for $t \in I$.

Further, let us represent the solution of (10)–(11) in the form $x(t) = (x_n(t)) = (x_1(t), x_2(t), \ldots)$. Then, from estimate (12) we infer that $x(t) \notin l_\infty$ for any $t > 0$. On the other hand let us notice that the right-hand sides of equations (10) are not bounded. Indeed, we have

$$\frac{n\sqrt{x}}{\sqrt{x}+1} \to n, \quad \text{as } x \to \infty.$$

The above given examples suggest that we have to enlarge the spaces under considerations to ensure that solutions of infinite systems of differential equations starting from a point in such a space remain in the space in question when t runs over some interval I. It seems that a natural way to realize the enlargement is to consider the so-called tempered sequence

spaces. Those spaces can be obtained from classical sequence spaces with help of a tempering sequence. For example, if we take the classical space l_∞ and the tempering sequence $\beta_n = \frac{1}{n}$ $(n = 1, 2, \dots)$ then the new sequence space l_∞^β with $\beta = (\beta_n) = (\frac{1}{n})$ is understood as the space of all sequences (x_n) such that the sequence $(\beta_n x_n) = (\frac{1}{n} x_n)$ is bounded.
It is worthwhile noticing that such an approach enables us to study an essentially larger class of infinite systems of differential equations in comparison with the classical setting.
In this article we discuss some classes of infinite systems of differential equations having solutions in the above mentioned tempered sequence spaces. The results of the chapter generalize several ones obtained up to now in classical sequence spaces (see [5, 8, 9, 12, 13, 24]).
Now, we describ the details concerning tempered sequence spaces. Such sequence spaces can be obtained if we consider the so-called tempered sequence spaces.
To define the mentioned spaces let us fix a real sequence $\beta = (\beta_n)$ such that β_n is positive for $n = 1, 2, \dots$ and the sequence (β_n) is nonincreasing. Such a sequence β will be called the tempering sequence. Next, consider the set X consisting of all real (or complex) sequences $x = (x_n)$ such that $\beta_n x_n \to 0$ as $n \to \infty$. It is easily seen that X forms a linear space over the field of real (or complex) numbers. We will denote this space by the symbol c_0^β.
It is easy to check that c_0^β is a Banach space under the norm

$$\|x\|_{c_0^\beta} = \|(x_n)\|_{c_0^\beta} = \sup\{\beta_n |x_n| : n = 1, 2, \dots\} = \max\{\beta_n |x_n| : n = 1, 2, \dots\}.$$

In a similar way we may consider the space c^β consisting of real (complex) sequences (x_n) such that the sequence $(\beta_n x_n)$ converges to a finite limit. Obviously c^β forms a linear space and it becomes a Banach space if we norm it by the supremum norm

$$\|x\|_{c^\beta} = \|(x_n)\|_{c^\beta} = \sup\{\beta_n |x_n| : n = 1, 2, \dots\}.$$

In the same way we can consider the tempered sequence space l_∞^β of all sequences (x_n) (real or complex) such that the sequence $(\beta_n x_n)$ is bounded. The space l_∞^β is a Banach space under the norm

$$\|x\|_{l_\infty^\beta} = \|(x_n)\|_{l_\infty^\beta} = \sup\{\beta_n |x_n| : n = 1, 2, \dots\}.$$

Let us pay attention to the fact that taking $\beta_n = 1$ for $n = 1, 2, \dots$ we

obtain spaces $c_0^\beta = c_0$, $c^\beta = c$ and $l_\infty^\beta = l_\infty$. Similarly, if the sequence (β_n) is bounded from below by a positive constant m i.e., if $\beta_n \geqslant m > 0$ for $n = 1, 2, \ldots$, then the norms in the tempered sequence spaces c_0^β, c^β and l_∞^β are equivalent to the classical supremum norm in each of the spaces c_0, c and l_∞. Thus, to obtain an essential enlargement of the spaces c_0, c and l_∞ we should assume that the tempering sequence (β_n) converges to zero. In what follows we will impose such a requirement.
The most important fact for our further purposes is the assertion saying that the pairs of the spaces (c_0, c_0^β), (c, c^β) and $(l_\infty, l_\infty^\beta)$ are isometric. Indeed, consider for example the spaces l_∞ and l_∞^β. Next, take the mapping $J : l_\infty^\beta \to l_\infty$ defined in the following way

$$J(x) = J((x_n)) = (\beta_n x_n).$$

Then, for arbitrarily fixed $x, y \in l_\infty^\beta$ we have

$$\begin{aligned}
\|J(x) - J(y)\|_{l_\infty} &= \|J((x_n)) - J((y_n))\|_{l_\infty} \\
&= \|(\beta_n x_n) - (\beta_n y_n)\|_{l_\infty} \\
&= \sup\{|\beta_n x_n - \beta_n y_n| : n = 1, 2, \ldots\} \\
&= \sup\{\beta_n |x_n - y_n| : n = 1, 2, \ldots\} = \|x - y\|_{l_\infty^\beta}.
\end{aligned}$$

This shows that the mapping J is an isometry between the spaces l_∞^β and l_∞. Obviously, the same mapping establishes the isometry between the spaces c^β and c and the spaces c_0^β and c_0, respectively.
The above assertions enable us to define measures of noncompactness in the tempered sequence spaces c_0^β, c^β and l_∞^β. In fact, the Hausdorff measure of noncompactness $\chi(X)$ for $X \in \mathfrak{M}_{c_0^\beta}$ can be expressed in the following way (cf. Section 2):

$$\chi(X) = \lim_{n\to\infty} \Big\{ \sup_{(x_i)\in X} \{ \sup \{\beta_i |x_i| : i \geqslant n\} \} \Big\}. \tag{13}$$

Similarly, the analogue of the measure of noncompactness μ_c defined by formula (3) has the form

$$\mu_{c^\beta}(X) = \lim_{k\to\infty} \Big\{ \sup_{(x_i)\in X} \{ \sup\{|\beta_n x_n - \beta_m x_m| : n, m \geqslant k\} \} \Big\}, \tag{14}$$

where $X \in \mathfrak{M}_{c^\beta}$.

Obviously, in view of the fact that the spaces c and c^β are isometric (by the above mentioned isometry J), on the basis of theorem from [6] we have the estimates

$$\chi(X) \leqslant \mu_{c^\beta}(X) \leqslant 2\chi(X)$$

for each $X \in \mathfrak{M}_{c^\beta}$, where χ denotes the Hausdorff measure of noncompactness in the space c^β.
Now, let us take into account the tempered sequence space l_∞^β. Then, keeping in mind formulas (4) - (6) expressing measures of noncompactness in the space l_∞, we obtain the following formulas for the counterparts of those measures in the space l_∞^β:

$$\mu_1^\beta(X) = \lim_{n\to\infty} \Big\{ \sup_{(x_i)\in X} \{ \sup \{\beta_i |x_i| : i \geqslant n\}\}\Big\}, \tag{15}$$

$$\mu_2^\beta(X) = \lim_{k\to\infty} \Big\{ \sup_{(x_i)\in X} \{ \sup\{|\beta_n x_n - \beta_m x_m| : n, m \geqslant k\}\}\Big\}, \tag{16}$$

$$\mu_3^\beta(X) = \limsup_{n\to\infty} \operatorname{diam} X_n^\beta, \tag{17}$$

where $X \in \mathfrak{M}_{l_\infty^\beta}$. Moreover, X_n^β in (17) is understood in the following way

$$X_n^\beta = \{x_n \beta_n : (x_i) \in X\}.$$

Apart from this $\operatorname{diam} X_n^\beta = \sup\{\beta_n |x_n - y_n| : (x_i), (y_i) \in X\}$.
Further, taking into account theorem proved in [6] we deduce the inequalities

$$\chi(X) \leqslant \mu_2^\beta(X), \tag{18}$$

$$\chi(X) \leqslant \mu_3^\beta(X), \tag{19}$$

$$\mu_2^\beta(X) \leqslant 2\mu_1^\beta(X), \tag{20}$$

$$\mu_3^\beta(X) \leqslant 2\mu_1^\beta(X), \tag{21}$$

where $X \in \mathfrak{M}_{l_\infty^\beta}$ and the symbol χ denotes the Hausdorff measure of noncompactness in the space l_∞^β.
In view of inequalities (18)–(21) it is easily seen that the kernel $\ker \mu_1^\beta$ consists of all sets X belonging to the family $\mathfrak{M}_{l_\infty^\beta}$ such that the sequences $(\beta_n x_n)$ tend to zero at infinity uniformly with respect to the set X i.e., for any $\varepsilon > 0$ there exists a natural number n_0 such that $\beta_n |x_n| \leqslant \varepsilon$ for all $(x_i) \in X$ and for $n \geqslant n_0$.

Similarly, the kernel $\ker \mu_2^\beta$ consists of all sets $X \in \mathfrak{M}_{l_\infty^\beta}$ such that the sequences $(\beta_n x_n)$ tend to finite limits uniformly on the set X. In other words, the sequences $(\beta_n x_n)$ satisfy Cauchy condition uniformly with respect to X. Finally, the kernel $\ker \mu_3^\beta$ consists of all sets X belonging to the family $\mathfrak{M}_{l_\infty^\beta}$ such that the thickness of the bundle formed by sequences $(\beta_n x_n)$, where $(x_i) \in X$, tends to zero at infinity.
Let us also observe that the measures of noncompactness μ_1^β, μ_2^β, μ_3^β are not regular in the space l_∞^β.

4. Infinite systems of differential equations in the tempered sequence space c_o^β

The considerations of this section will be located in the Banach tempered sequence space c_0^β described in Section 3. Thus, we will assume that $\beta = (\beta_n)$ is a sequence with positive terms which is nonincreasing. The space c_0^β consists of all sequences (x_n) such that the sequence $(\beta_n x_n)$ converges to zero. We will consider here only real sequences (x_n). The norm in the space c_0^β is defined by the formula

$$\|x\|_{c_0^\beta} = \|(x_n)\|_{c_0^\beta} = \sup\{\beta_n |x_n| : n = 1, 2, \dots\}.$$

To simplify the notation we will use the symbol $\|\cdot\|$ instead of $\|\cdot\|_{c_0^\beta}$.

4.1. Semilinear lower diagonal infinite systems of differential equations

The object of our study in this subsection will be first semilinear lower diagonal infinite systems of differential equations having the form

$$x_n' = \sum_{i=1}^{k_n} a_{nn_i}(t) x_{n_i} + f_n(t, x_1, x_2, \dots) \tag{22}$$

with the initial value conditions

$$x_n(0) = x_0^n, \quad \text{for } x = 1, 2, \dots . \tag{23}$$

We assume that for any fixed $n \in \mathbb{N}$ the sequence $(n_1, n_2, \dots, n_{k_n})$ is such that $1 \leqslant n_1 < n_2 < \dots < n_{k_n} \leqslant n$. Moreover, the sequence (n_1) tends to

infinity when $n \to \infty$. Apart from that we assume that there exists a natural number K such that $k_n \leqslant K$ for all $n = 1, 2, \ldots$. In other words, this means that any "linear part" of system (22) contains only finite number of nonzero terms and the number of those terms does not exceed K. In what follows infinite systems (22) satisfying the above requirement will be called infinite systems of differential equations with *linear parts of constant width.*
Apart from the requirement concerning the constant width of linear parts we will impose the following assumptions in our study of initial value problem (22)–(23):

(i) The function $a_{nn_i} = a_{nn_i}(t)$ is continuous on a fixed interval $I = [0, T]$ for $n = 1, 2, \ldots$ and for $i = 1, 2, \ldots, k_n$;

(ii) the functions $a_{nn_i}(t)$ are uniformly bounded on the interval I by a positive constant A i.e., $|a_{nn_i}(t)| \leqslant A$ for $t \in I$ and for $n = 1, 2 \ldots$ and for $i = 1, 2, \ldots, k_n$;

(iii) the sequence (x_0^n) belongs to the space c_0^β;

(iv) for each fixed n the function $f_n(t, x_1, x_2, \ldots) = f_n(t, x)$ acts from the set $I \times \mathbb{R}^\infty$ into $\mathbb{R}$. Moreover, the function $f_n : I \times c_0^\beta \to \mathbb{R}$ is continuous on $I \times c_0^\beta$;

(v) there exists a sequence (p_n) of nonnegative terms with the property that $\beta_n p_n \to 0$ as $n \to \infty$ and such that $|f_n(t, x)| \leqslant p_n$ for $t \in I$, $x \in c_0^\beta$ and for $n = 1, 2 \ldots$.

Now, we can formulate our existence result.

Theorem 6. *Assume that the functions involved in system* (22) *having linear parts of constant width K, satisfy conditions (i)–(v). Then initial value problem* (22)–(23) *has at least one solution* $x(t) = (x_n(t)) = ((x_1(t), x_2(t), \ldots)$ *in the sequence space* c_0^β *on the interval* I.

Proof. For arbitrarily fixed $n \in \mathbb{N}$ let us denote

$$g_n(t, x) = g_n(t, x_1, x_2, \ldots) = \sum_{i=1}^{k_n} a_{nn_i}(t) x_{n_i} + f_n(t, x_1, x_2, \ldots),$$

where $t \in I$ and $x = (x_n) \in c_0^\beta$. Then, keeping in mind our assumptions, we obtain

$$\begin{aligned}
\beta_n|g_n(t,x_1,x_2,\dots)| &\leqslant \beta_n \sum_{i=1}^{k_n} |a_{nn_i}(t)||x_{n_i}| + \beta_n|f_n(t,x_1,x_2,\dots)| \\
&\leqslant \beta_n A \sum_{i=1}^{k_n} |x_{n_i}| + \beta_n p_n = A\sum_{i=1}^{k_n} \beta_n |x_{n_i}| + \beta_n p_n \\
&\leqslant A \sum_{i=1}^{k_n} \beta_{n_i}|x_{n_i}| + \beta_n p_n \\
&\leqslant AK \max\{\beta_{n_i}|x_{n_i}| : i = 1,2,\dots,k_n\} + \beta_n p_n \\
&\leqslant AK \sup\{\beta_j |x_j| : j \geqslant n_1\} + \beta_n p_n.
\end{aligned}$$

Hence, replacing n by j and j by i, we can write the above inequality in the form

$$\beta_j|g_j(t,x_1,x_2,\dots)| \leqslant AK \sup\{\beta_i|x_i| : i \geqslant j_1\} + \beta_j p_j. \tag{24}$$

Next, let us notice that from estimate (24) it follows that the following inequality holds

$$\begin{aligned}
\|g(t,x)\| &= \sup\{\beta_j|g_j(t,x_1,x_2,\dots)| : j = 1,2,\dots\} \\
&\leqslant AK \sup_{j\in\mathbb{N}} \big\{ \sup\{\beta_i|x_i| : i \geqslant j_1\}\big\} + \sup\{\beta_j p_j : j = 1,2,\dots\} \qquad (25) \\
&= AK\|x\| + P,
\end{aligned}$$

where the operator $g = g(t,x)$ is defined on the set $I \times c_0^\beta$ in the following way

$$g(t,x) = (g_1(t,x), g_2(t,x), \dots).$$

In view of estimate (25) we see that g transforms the set $I \times c_0^\beta$ into the space c_0^β.

Now, we show that the operator g is continuous on the set $I \times c_0^\beta$. To this end we split the operator g into two terms

$$g(t,x) = (Lx)(t) + f(t,x),$$

where the operators L and f are defined as follows:

$$(Lx)(t) = ((L_1x)(t), (L_2x)(t), \dots)$$

where

$$(L_n x)(t) = \sum_{i=1}^{k_n} a_{nn_i}(t) x_{n_i}$$

$(n = 1, 2, \dots)$, and

$$f(t, x) = (f(t_1, x), f(t_2, x), \dots).$$

First we show that the operator f is continuous on the set $I \times c_0^\beta$. To do this fix arbitrarily a number $\varepsilon > 0$ and a point $x \in c_0^\beta$. According to assumption (v) we can choose a natural number n_0 such that

$$\beta_n p_n \leqslant \frac{\varepsilon}{2} \tag{26}$$

for $n \geqslant n_0$. Next, in view of assumption (iv) we can find a number δ_i $(i = 1, 2, \dots, n_0)$ such that for any $y \in c_0^\beta$ such that $\|x - y\| \leqslant \delta_i$ and for arbitrary $t \in I$ we have

$$|f_i(t, x) - f_i(t, y)| \leqslant \frac{\varepsilon}{\beta_1}.$$

Let us take $\delta = \min\{\delta_1, \delta_2, \dots, \delta_{n_0}\}$. Then, for arbitrary $y \in c_0^\beta$ such that $\|x - y\| \leqslant \delta$ and for $t \in I$ we have

$$|f_i(t, x) - f_i(t, y)| \leqslant \frac{\varepsilon}{\beta_1}. \tag{27}$$

Combining (26) and (27), for $y \in c_0^\beta$ with $\|x - y\| \leqslant \delta$ and for $t \in I$, we obtain

$$\begin{aligned}
\|f(t, x) - f(t, y)\| &= \sup\{\beta_n |f_n(t, x) - f_n(t, y)| : n = 1, 2, \dots\} \\
&= \max\Big\{ \max\{\beta_n |f_n(t, x) - f_n(t, y)| : n = 1, 2, \dots, n_0\}, \\
&\qquad \sup\{\beta_n |f_n(t, x) - f_n(t, y)| : n > n_0\}\Big\} \\
&\leqslant \max\Big\{ \max\{\beta_1 |f_n(t, x) - f_n(t, y)| : n = 1, 2, \dots, n_0\}, \\
&\qquad \sup\{\beta_n [|f_n(t, x)| + |f_n(t, y)|] : n > n_0\}\Big\} \\
&\leqslant \max\Big\{ \beta_1 \Big(\frac{\varepsilon}{\beta_1}\Big), \sup\{2\beta_n p_n : n > n_0\}\Big\} = \varepsilon.
\end{aligned}$$

This shows that the operator f is continuous at an arbitrary point $(t,x) \in I \times c_0^\beta$.
Next, we show that the operator L is continuous on the set $I \times c_0^\beta$. Similarly as before, fix arbitrarily $x \in c_0^\beta$, $t \in I$ and a number $\varepsilon > 0$. Then, for $y \in c_0^\beta$ with $\|x-y\| \leqslant \varepsilon$ and for an arbitrary fixed natural number n, in view of imposed assumptions we obtain

$$\begin{aligned}
&\beta_n|(L_n x)(t) - (L_n y)(t))| \\
&= \beta_n\Big|\sum_{i=1}^{k_n} a_{nn_i}(t)x_{n_i} - \sum_{i=1}^{k_n} a_{nn_i}(t)y_{n_i}\Big| \\
&\leqslant \beta_n \sum_{i=1}^{k_n} |a_{nn_i}(t)||x_{n_i} - y_{n_i}| \\
&\leqslant A\sum_{i=1}^{k_n} \beta_n|x_{n_i} - y_{n_i}| \leqslant A\sum_{i=1}^{k_n} \beta_i|x_{n_i} - y_{n_i}| \\
&\leqslant AK \max\{\beta_i|x_{n_i} - y_{n_i}| : i = 1,2,\dots,k_n\} \\
&\leqslant AK \sup\{\beta_j|x_j - y_j| : j \geqslant n_1\} \\
&\leqslant AK \sup\{\beta_j|x_j - y_j| : j = 1,2,\dots\} = AK\|x-y\| \leqslant AK\varepsilon.
\end{aligned}$$

Hence we deduce that the operator L is continuous on the set $I \times c_0^\beta$. Consequently, as we announced before, we conclude that the operator g is continuous on the set $I \times c_0^\beta$.
In what follows let us take a number T_1 such that $T_1 < T$ and $AKT_1 < 1$. According to assumptions of our theorem take the number $r = \frac{(P+AK)T_1\|x_0\|}{1-AKT_1}$ and consider the ball $B(x_0,r)$. Next, choose an arbitrary subset X of the ball $B(x_0,r)$. Then, for $x \in X$ and $t \in [0,T_1]$, in view of estimate (24), for an arbitrary fixed natural number n, we obtain:

$$\begin{aligned}
&\sup\{\beta_j|g_j(t,x_1,x_2,\dots)| : j \geqslant n\} \\
&\leqslant \sup\{AK\sup\{\beta_i|x_i| : i \geqslant j_1\} : j \geqslant n\} + \sup\{\beta_j p_j : j \geqslant n\} \\
&\leqslant AK\sup\{\sup\{\beta_i|x_i| : i \geqslant n_1\}, \sup\{\beta_i|x_i| : i \geqslant (n+1)_1\}, \\
&\quad \sup\{\beta_i|x_i| : i \geqslant (n+2)_1\}, \dots\} + \sup\{\beta_j p_j : j \geqslant n\}.
\end{aligned}$$

This yields the estimate

$$\begin{aligned}&\sup_{x\in X}\left\{\sup\{\beta_j|g_j(t,x_1,x_2,\dots)|:j\geqslant n\}\right\}\\&\leqslant AK\sup_{x\in X}\left\{\sup\{\sup\{\beta_i|x_i|:i\geqslant j_i\}:j\geqslant n\}+\sup\{\beta_j p_j:j\geqslant n\}\right\}.\end{aligned}$$

Passing with $n\to\infty$ in the above estimate and taking into account that $j_1\to\infty$ as $j\to\infty$, we obtain

$$\chi(g(t,X))\leqslant AK\chi(X),$$

where χ denotes the Hausdorff measure of noncompactness in the space c_0^β expressed with help of formula (13). Finally, in view of the above established facts and Theorem 3, we complete the proof. □

The following example illustrates the result in Theorem 6.

Example 7. Consider the infinite system of differential equations

$$\begin{cases}x_1'=x_1+\frac{\sqrt{|x_1|}}{\sqrt{|x_1|+1}},\\x_2'=x_1+x_2+2\frac{\sqrt{|x_2|}}{\sqrt{|x_2|+1}},\\x_3'=x_2+x_3+3\frac{\sqrt{|x_3|}}{\sqrt{|x_3|+1}},\\\dots\\x_n'=x_{n-1}+x_n+n\frac{\sqrt{|x_n|}}{\sqrt{|x_n|+1}},\\\dots\end{cases}\tag{28}$$

with initial conditions

$$x_n(0)=n\quad\text{for } n=1,2,\dots\,.\tag{29}$$

Observe that (28) is a semilinear lower diagonal infinite system of differential equations with linear parts of constant width $K=2$. Moreover, it is easily seen that system (28) is a particular case of system (23) if we take $a_{nn_i}(t)=1$ for $t\in I$, where we put $I=[0,T_1]$, where $T_1>0$ is a number chosen according to assumptions of Theorem 3. Additionally, $n=1,2\dots$ and $i=1,2$ for $n\geqslant 2$. Hence we see that assumption (i) of Theorem 6 is satisfied. Further, we see that $|a_{nn_i}(t)|\leqslant 1$ for $t\in I$ and $n=1,2,\dots$, $i=1,2$. This means that functions $a_{nn_i}(t)$ satisfy assumption (ii).

In what follows let us take the sequence $\beta_n = \frac{1}{n^2}$ for $n = 1, 2 \dots$. Obviously we see that $x_0 = (x_0^n) = (n) \in c_0^\beta$, where $\beta = (\beta_n) = (\frac{1}{n^2})$. Thus assumption (iii) is satisfied. From the form of system (28) we see that we can take

$$f_n(t, x_1, x_2, \dots) = n\frac{\sqrt{|x_n|}}{\sqrt{|x_n|} + 1}$$

for $n = 1, 2 \dots$. Obviously, the function $f_n = f_n(t, x)$ is continuous on the set $I \times c_0^\beta$. Moreover, we have

$$|f_n(t, x_1, x_2, \dots)| \leqslant n, \quad \text{for } n = 1, 2 \dots .$$

Thus we conclude that the functions f_n satisfy assumptions (iv) and (v) with $p_n = n$ for $n = 1, 2 \dots$.
Finally, on the basis of Theorem 6 we deduce that there exists at least one solution $x(t) = (x_n(t))$ of initial value problem (28)–(29) defined on some interval $I = [0, T_1]$ such that for each $t \in I$ the sequence $(x_n(t))$ belongs to the space c_0^β with $\beta = (\frac{1}{n^2})$. This means that $x_n(t) = o(n^2)$ as $n \to \infty$, for any fixed $t \in [0, T_1]$.
In the sequel we will also consider the semilinear lower diagonal infinite system of differential equations of the form (22) i.e.,

$$x_n' = \sum_{i=1}^{k_n} a_{nn_i}(t)x_{n_i} + f_n(t, x_1, x_2, \dots) \tag{30}$$

with initial value conditions

$$x_n(0) = x_n^0, \quad \text{for } n = 1, 2, \dots . \tag{31}$$

Now, we dispense with the assumption requiring that system (30) has linear parts of constant width. We replace this assumption, as well as assumption (ii), by the following hypotheses:

(ii') The sequence (n_1) tends to ∞ as $n \to \infty$;

(ii") the sequence $\left(\sum_{i=1}^{k_n} |a_{nn_i}(t)|\right)$ is uniformly bounded on the interval $I = [0, T_1]$ i.e., there exists a constant $A > 0$ such that

$$\sum_{i=1}^{k_n} |a_{nn_i}(t)| \leqslant A$$

for each $t \in I$ and for $n = 1, 2, \ldots$.

Then we have the following result.

Theorem 8. *Assume that* (i), (ii'), (ii"), (iii)–(v) *of Theorem 6 are satisfied. Then initial value problem* (30)–(31) *has at least one solution* $x(t) = (x_n(t))$ *in the sequence space* c_0^β *defined on the interval* $I = [0, T_1]$*, where* T_1 *is a number chosen according to Theorem 3.*

The proof of this theorem can be done in a similar way to the proof of the previous theorem. Moreover, it was presented in detail in [5], so we can omit it now.
Next we provide an example showing the applicability of Theorem 8.

Example 9. We consider the lower diagonal infinite system of differential equations. To expose this system in a transparent way we will assume that n is an even natural number, say $n = 2k$. Then, we can present the announced system as follows:

$$\begin{cases} x_1' = x_1 + \frac{x_1}{1+x_1^2}, \\ x_2' = x_1 + tx_2 + 2\frac{x_1+x_2}{1+x_1^2+x_2^2}, \\ x_3' = \frac{t^2}{2!}x_3 + 3\frac{x_2+x_3}{1+x_2^2+x_3^2}, \\ x_4' = \frac{t^2}{2!}x_3 + \frac{t^3}{3!}x_4 + 4\frac{x_3+x_4}{1+x_3^2+x_4^2}, \\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\ x_{n-1}'(= x_{2k-1}') = \frac{t^k}{k!}x_{k+1} + \ldots + \frac{t^{2k-2}}{(2k-2)!}x_{2k-1} \\ \qquad +(2k-1)\frac{x_{2k-2}+x_{2k-1}}{1+x_{2k-2}^2+x_{2k-1}^2}, \\ x_n'(= x_{2k}') = \frac{t^k}{k!}x_{k+1} + \ldots + \frac{t^{2k-2}}{(2k-2)!}x_{2k-1} + \frac{t^{2k-1}}{(2k-1)!}x_{2k} \\ \qquad +2k\frac{x_{2k-1}+x_{2k}}{1+x_{2k-1}^2+x_{2k}^2}, \\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ . \end{cases} \tag{32}$$

We also assume that the following initial conditions are satisfied

$$x_n(0) = n^2 \tag{33}$$

for $n = 1, 2, \ldots$.
Let us observe that initial value problem (32)–(33) is a particular case of problem (30)–(31). To justify this assertion we show that the components involved in (32)–(33) satisfy assumptions of Theorem 8. First of all let us

observe that functions $a_{nn_i}(t)$ appearing in infinite system (32) have the form

$$a_{nn_i}(t) = \frac{t^{n_i-1}}{(n_i-1)!}$$

for $n_i = \frac{n}{2}+1, \frac{n}{2}+2, \ldots, n$ (if n is even) or $n_i = [\frac{n}{2}]+2, [\frac{n}{2}]+3, \ldots, n$ (if n is odd). Obviously the functions $a_{nn_i}(t)$ are continuous on each interval of the form $[0,T]$. Thus, assumption (i) is satisfied.
Since $n_1 = \frac{n}{2}+1$ for n even or $n_1 = [\frac{n}{2}]+2$ for n odd, we see that assumption (ii') is satisfied. To check assumption (ii") observe that we have

$$\sum_{i=1}^{k_n} |a_{nn_i}(t)| = \sum_{i=1}^{n} |a_{nn_i}(t)| \leqslant 1 + t + \frac{t^2}{2!} + \cdots + \frac{t^n}{n!} \leqslant e^t$$

for $t \in [0,T]$. Hence we see that assumption (ii") is satisfied with $A = e^T$.
Further, take the tempering sequence of the form $\beta = (\beta_n) = (\frac{1}{n^3})$. Then the sequence $(x_0^n) = (n^2)$ is a member of the tempered sequence space c_0^β, so assumption (iii) is satisfied. Similarly, it is not hard to verify that the functions f_n, where

$$f_n(t,x) = f_n(t,x_1,x_2,\ldots) = n\frac{x_{n-1}+x_n}{1+x_{n-1}^2+x_n^2}$$

$(n = 2,3\ldots)$ are continuous on the set $I \times c_0^\beta$. Moreover, for each fixed n we obtain

$$|f_n(t,x)| \leqslant n\frac{|x_{n-1}|+|x_n|}{1+x_{n-1}^2+x_n^2} \leqslant n.$$

Thus, we can put $p_n = n$ in assumption (v). Obviously we see that $\beta_n p_n = \frac{1}{n^2} \to 0$ as $n \to \infty$. Thus assumption (v) is satisfied.
Hence, in view of Theorem 8, initial value problem (32)–(33) has at least one solution $x(t) = (x_n(t))$ belonging to the sequence space c_0^β and defined for $t \in I = [0,T_1]$, where T_1 satisfies the inequality $T_1A = T_1e^{T_1} < 1$. We can calculate that $T_1 \leqslant 0.568\ldots$.

Remark 10. Observe that in Example 9 instead of $\beta = (\beta_n) = (1/n^3)$ we can take the tempering sequence of the form $\beta_n = 1/n^{2+\delta}$, where δ is an arbitrary positive number. Similarly, in Example 7 we can take the tempering sequence of the form $\beta_n = 1/n^{1+\delta}$, where $\delta > 0$ is an arbitrary number and $n = 1,2\ldots$.

4.2. Semilinear upper diagonal infinite system of differential equations

In this section we will consider the semilinear upper diagonal infinite system of differential equations which has the form

$$x'_n = \sum_{i=1}^{k_n} a_{nn_i}(t)x_{n_i} + f_n(t, x_1, x_2, ...) \tag{34}$$

with the initial value conditions

$$x_n(0) = x_0^n, \quad \text{for} \quad n = 1, 2, \ldots . \tag{35}$$

We will assume that for any fixed natural number n ($n \in \mathbb{N}$) the sequence $(n_1, n_2, ..., n_{k_n})$ satisfies the inequalities $n \leqslant n_1 < n_2 < \cdots < n_{k_n}$.
As before, we will also assume that there exists a natural number K such that $k_n \leqslant K$ for $n = 1, 2, ...$ Such an assumption means that the linear part of each equation appearing in system (34) contains only finite number of nonzero terms and the number of those term does not exceed K. In the sequel of the chapter infinite systems (34) satisfying the above constraint will be called infinite systems of differential equations with *linear part of constant width.*
In our further investigations of problem (34)-(35), apart from the assumption concerning the constant width of linear parts of equations in (34) we will also impose the following assumptions:

(i) The functions $a_{nn_i} = a_{nn_i}(t)$ are equicoutinuous on the interval $I = [0, T]$ for $n = 1, 2, ...$ and for $i = 1, 2, ..., k_n$.

(ii) The functions $a_{nn_i}(t)$ are uniformly bounded on the interval I by a positive constant A i.e., $|a_{nn_i}(t)| \leqslant A$ for $t \in I$ and for $n = 1, 2, ...,$ $i = 1, 2, ..., k_n$.

(iii) The sequence (x_0^n) is a member of the space c_0^β.

(iv) For every fixed n the function $f_n(t, x_1, x_2, ...) = f_n(t, x)$ acts from the set $I \times \mathbb{R}^\infty$ into $\mathbb{R}$. Moreover, the function $f_n : I \times c_0^\beta \to \mathbb{R}$ is continuous on $I \times c_0^\beta$.

(v) There exists a sequence (p_n) of nonnegative terms such that $\beta_n p_n \to 0$ as $n \to \infty$ and such that $|f_n(t, x)| \leqslant p_n$ for $t \in I$, $x \in c_0^\beta$ and for $n = 1, 2, \ldots$.

(vi) There exists a positive constant M such that $\beta_n/\beta_{n_{k_n}} \leqslant M$ for any $n = 1, 2, \ldots$.

Now, we are in a position to present our existence result concerning the initial value problem (34)-(35).

Theorem 11. *Assume that* (34) *is an infinite sublinear upper diagonal system of differential equations with linear parts of constant width K, satisfying assumptions (i)-(vi). Then, initial value problem* (34)-(35) *has at least one solution $x(t) = (x_n(t)) = (x_1(t), x_2(t), \ldots)$ in the sequence space c_0^β on interval $I_1 = [0, T_1]$, where $T_1 \leqslant T$ and $T_1 < 1/AKM$.*

Proof. To simplify the proof, for arbitrarily fixed $n \in \mathbb{N}$ let us denote

$$g_n(t, x) = g_n(t, x_1, x_2, \ldots) = \sum_{i=1}^{k_n} a_{nn_i}(t) x_{n_i} + f_n(t, x_1, x_2, \ldots),$$

where $t \in I$ and $x = (x_n) \in c_0^\beta$. Then, keeping in mind the imposed assumptions, we obtain:

$$\begin{aligned}
&\beta_n |g_n(t, x_1, x_2, \ldots)| \leqslant \beta_n \sum_{i=1}^{k_n} |a_{nn_i}(t)||x_{n_i}| \\
&+\beta_n |f_n(t, x_1, x_2, \ldots)| \leqslant \beta_n A \sum_{i=1}^{k_n} |x_{n_i}| + \beta_n p_n \\
&= \beta_n A \left[|x_{n_1}| + |x_{n_2}| + \cdots + |x_{n_{k_n}}| \right] + \beta_n p_n \\
&= A \left[\frac{\beta_n}{\beta_{n_1}} \beta_{n_1} |x_{n_1}| + \frac{\beta_n}{\beta_{n_2}} \beta_{n_2} |x_{n_2}| + \cdots + \frac{\beta_n}{\beta_{n_{k_n}}} \beta_{n_{k_n}} |x_{n_{k_n}}| \right] + \beta_n p_n \\
&\leqslant A \frac{\beta_n}{\beta_{n_{k_n}}} \left[\beta_{n_1} |x_{n_1}| + \beta_{n_2} |x_{n_2}| + \cdots + \beta_{n_{k_n}} |x_{n_{k_n}}| \right] + \beta_n p_n \\
&\qquad \leqslant AM \left[\beta_{n_1} |x_{n_1}| + \beta_{n_2} |x_{n_2}| + \cdots + \beta_{n_{k_n}} |x_{n_{k_n}}| \right] + \beta_n p_n \\
&\qquad \leqslant AMK \max \{ \beta_{n_i} |x_{n_i}| : \quad i = 1, 2, \ldots, k_n \} + \beta_n p_n \\
&\qquad \leqslant AMK \sup \{ \beta_j |x_j| : \quad j \geqslant n_1 \} + \beta_n p_n.
\end{aligned}$$

Hence, replacing n by j and j by i, we can rewrite the above inequality in the form

$$\beta_j|g_j(t,x_1,x_2,...)| \leqslant AKM \sup\{\beta_i|x_i| : \quad i \geqslant j_1\} + \beta_j p_j. \tag{36}$$

Now, let us observe that from estimate (36) we derive the following inequality

$$\begin{aligned}||g(t,x)|| &= \sup\{\beta_j|g_j(t,x_1,x_2,...)| : \ j=1,2,...\}\\ &\leqslant AKM \sup_j\{\sup\{\beta_i|x_i| : \quad i \geqslant j_1\}\} + \sup\{\beta_j p_j : \quad j=1,2,...\}\\ &\leqslant AKM||x|| + P,\end{aligned} \tag{37}$$

where the operator $g = g(t,x)$ is defined on the set $I \times c_0^\beta$ in the following way

$$g(t,x) = (g_1(t,x),\ g_2(t,x),...).$$

Moreover, the constant P is defined as

$$P = \sup\{\beta_n p_n : \quad n=1,2,...\}.$$

Obviously, $P < \infty$ in view of assumption (v).
In what follows we show that the operator g acts continuously from the set $I \times c_0^\beta$ into the space c_0^β. To this end we represent the operator g in the form

$$g(t,x) = (Lx)(t) + f(t,x),$$

where the operators L and f are defined as follows:

$$(Lx)(t) = ((L_1x)(t), (L_2x)(t),...),$$

where

$$(L_nx)(t) = \sum_{i=1}^{k_n} a_{nn_i}(t)x_{n_i}$$

$(n=1,2,...)$, and

$$f(t,x) = (f_1(t,x), f_2(t,x),...).$$

At first we show that f is continuous on the set $I \times c_0^\beta$.
To realize this goal fix arbitrarily a number $\varepsilon > 0$ and $x \in c_0^\beta$, $t \in I$. Then, in view of assumption (v) we can find a natural number n_0 such that

$$\beta_n p_n < \frac{\varepsilon}{2} \tag{38}$$

for $n \geqslant n_0$. Further, in virtue of assumption (iv) we can find a number δ_i $(i = 1, 2, ..., n_0)$ such that for any $y \in c_0^\beta$ such that $||x - y|| \leqslant \delta_i$ and for $s \in I$ such that $|t - s| \leqslant \delta_i$ we have

$$|f_i(t, x) - f_i(s, y)| \leqslant \frac{\varepsilon}{\beta_1}.$$

Next, take $\delta = \min\{\delta_1, \delta_2, ..., \delta_n\}$. Then, for arbitrary $y \in c_0^\beta$ such that $||x - y|| \leqslant \delta$ and for $s \in I$ such that $|t - s| \leqslant \delta$, we get

$$|f_i(t, x) - f_i(s, y)| \leqslant \frac{\varepsilon}{\beta_1}. \tag{39}$$

Linking (38) and (39), for $y \in c_0^\beta$ with $||x - y|| \leqslant \delta$ and for $s \in I$ with $|t - s| \leqslant \delta$, we obtain

$$\begin{aligned} ||f(t, x) - f(s, y)|| &= \sup\{\beta_n|f_n(t, x) - f_n(s, y)| : \quad n = 1, 2, ...\} \\ &= \max\{\max\{\beta_n|f_n(t, x) - f_n(s, y)| : \quad n = 1, 2, ..., n_0\}, \\ &\quad \sup\{\beta_n|f_n(t, x) - f_n(s, y)| : \quad n > n_0\}\} \\ &\leqslant \max\{\max\{\beta_n|f_n(t, x) - f_n(s, y)| : \quad n = 1, 2, ..., n_0\}, \\ &\quad \sup\{\beta_n[|f_n(t, x)| + |f_n(s, y)| : \quad n > n_0]\}\} \\ &\leqslant \max\left\{\beta_n \frac{\varepsilon}{\beta_1}, \sup\{2\beta_n p_n : \quad n > n_0\}\right\} \\ &\leqslant \max\left\{\beta_1 \frac{\varepsilon}{\beta_1}, \sup\{2\beta_n p_n : \quad n > n_0\}\right\} = \varepsilon. \end{aligned}$$

This shows that the operator f is continuous on the set $I \times c_0^\beta$.
Now, we show that the operator L is continuous on the set $I \times c_0^\beta$. Similarly as previously, fix arbitrary $x \in c_0^\beta$, $t \in I$ and a number $\varepsilon > 0$. Then, for $y \in c_0^\beta$ with $||x - y|| \leqslant \varepsilon$, for $s \in I$ with $|t - s| \leqslant \varepsilon$ and for an arbitrarily fixed natural number n, in virtue of our assumptions we get

$$\begin{aligned}
&\beta_n|(L_nx)(t)-(L_ny)(s)|\\
=&\beta_n\left|\sum_{i=1}^{k_n}a_{nn_i}(t)x_{n_i}-\sum_{i=1}^{k_n}a_{nn_i}(s)y_{n_i}\right|\\
\leqslant&\beta_n\left|\sum_{i=1}^{k_n}a_{nn_i}(t)x_{n_i}-\sum_{i=1}^{k_n}a_{nn_i}(s)x_{n_i}\right|\\
+&\beta_n\left|\sum_{i=1}^{k_n}a_{nn_i}(s)x_{n_i}-\sum_{i=1}^{k_n}a_{nn_i}(s)y_{n_i}\right|\\
\leqslant&\beta_n\sum_{i=1}^{k_n}|a_{nn_i}(t)-a_{nn_i}(s)||x_{n_i}|+\beta_n\sum_{i=1}^{k_n}|a_{nn_i}(s)||x_{n_i}-y_{n_i}|\\
\leqslant&\beta_n\sum_{i=1}^{k_n}\omega(|t-s|)|x_{n_i}|+\beta_nA\sum_{i=1}^{k_n}|x_{n_i}-y_{n_i}|\\
\leqslant&\omega(\varepsilon)\sum_{i=1}^{k_n}\beta_n|x_{n_i}|+A\sum_{i=1}^{k_n}\beta_n|x_{n_i}-y_{n_i}|,
\end{aligned}$$

where the symbol $\omega=\omega(\varepsilon)$ denotes the common modulus of continuity of the functions $a_{nn_i}(t)$ on the interval I. Such a modulus exists in view of assumption (i). Further, keeping in mind assumption (vi), we obtain

$$\begin{aligned}
&\beta_n|(L_nx)(t)-(L_ny)(s)|\\
\leqslant&\omega(\varepsilon)\sum_{i=1}^{k_n}\frac{\beta_n}{\beta_{n_i}}\beta_{n_i}|x_{n_i}|+A\sum_{i=1}^{k_n}\frac{\beta_n}{\beta_{n_i}}\beta_{n_i}|x_{n_i}-y_{n_i}|\\
\leqslant&\omega(\varepsilon)\sum_{i=1}^{k_n}\frac{\beta_n}{\beta_{n_{k_n}}}\beta_{n_i}|x_{n_i}|+A\sum_{i=1}^{k_n}\frac{\beta_n}{\beta_{n_{k_n}}}\beta_{n_i}|x_{n_i}-y_{n_i}|\\
\leqslant&M\omega(\varepsilon)\sum_{i=1}^{k_n}\beta_{n_i}|x_{n_i}|+AM\sum_{i=1}^{k_n}\beta_{n_i}|x_{n_i}-y_{n_i}|\\
\leqslant&KM\omega(\varepsilon)||x||+AM||x-y||\leqslant KM||x||\omega(\varepsilon)+AM\varepsilon.
\end{aligned}$$

The above obtained estimate allows us to infer that the operator L is continuous on the set $I\times c_0^\beta$. Joining this fact with the continuity of the operator f established before, we deduce that the operator g is continuous on the set

$I \times c_0^\beta$.
Next, let us take a positive number T_1 such that $T_1 \leqslant T$ and $AKMT_1 < 1$. Denote $I_1 = [0, T_1]$. Keeping in mind the above established facts and Theorem (3) let us take the number

$$r = \frac{(P + AKM)T_1||x_0||}{1 - AKMT_1}.$$

Now, consider the ball $B(x_0, r)$ and choose an arbitrary nonempty subset X of $B(x_0, r)$. Then, for a fixed element $x \in X$ and for an arbitrary number $t \in I_1$, in view of estimates (36) and (37), for arbitrarily fixed natural number n, we obtain:

$$\begin{aligned} &\sup\{\beta_j |g_j(t, x_1, x_2, ...)| : \quad j \geqslant n\} \\ &\qquad \leqslant \sup \{AKM \sup \{\beta_i |x_i| : \quad i \geqslant j_1\} : \quad j \geqslant n\} + \sup \{\beta_j p_j : \quad j \geqslant n\} \\ &\qquad \leqslant AKM \sup\{\sup \{\beta_i |x_i| : \quad i \geqslant n_1\}, \quad \sup \{\beta_i |x_i| : \quad i \geqslant (n+1)_1\}, \\ &\qquad\quad \sup\{\beta_i |x_i| : \quad i \geqslant (n+2)_1\}, ...\} + \sup\{\beta_j p_j : \quad j \geqslant n\}. \end{aligned}$$

Consequently, we arrive at the following estimate:

$$\begin{aligned} &\sup_{x \in X}\{\sup\{\beta_j |g_j(t, x_1, x_2, ...)| : \quad j \geqslant n\}\} \\ \leqslant &AKM \sup_{x \in X}\{\sup\{\sup\{\beta_i |x_i| : \quad i \geqslant j_1\} : \quad j \geqslant n\}\} \\ + &\sup\{\beta_j p_j : \quad j \geqslant n\}. \end{aligned}$$

Now, passing with $n \to \infty$ and bearing in mind that $j_1 \to \infty$ as $j \to \infty$, in view of formula expressing the Hausdorff measure of noncompactness in the space c_0^β, we derive the following inequality

$$\chi(g(t, X)) \leqslant AKM\chi(X).$$

Finally, gathering all the above stated facts, in view of Theorem (3) we complete the proof. □

In order to illustrate the result contained in Theorem (11) we consider the following example.

Example 12. Let us take into account the following infinite system of differential equations

$$\begin{cases} x_1' = x_1 + \frac{t}{1+t}x_2 + \frac{x_1}{1+x_1^2}, \\ x_2' = x_2 + \frac{t}{2+t}x_3 + \frac{t}{4+t}x_4 + \frac{2x_2}{1+x_2^2}, \\ x_3' = x_3 + \frac{t}{3+t}x_4 + \frac{t}{6+t}x_6 + \frac{3x}{1+x_3^2}, \\ \dots\dots\dots\dots\dots\dots\dots\dots\dots\dots \\ x_n' = x_n + \frac{t}{n+t}x_{n+1} + \frac{t}{2n+t}x_{2n} + \frac{nx_n}{1+x_n^2}, \\ \dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots , \end{cases} \tag{40}$$

with initial conditions

$$x_n(0) = 2n + 1 \quad \text{for} \quad n = 1, 2, \dots . \tag{41}$$

Observe that (40) is a semilinear upper diagonal infinite system of differential equations with linear parts of constant width $K = 3$. Apart from this it is easily seen that system (40) is a particular case of system (34) if we take $a_{nn}(t) = 1$ for $n \in \mathbb{N}$ and for $t \in I_1 = [0, T_1]$, where $T_1 \leqslant T$ is a number chosen according to assumptions of Theorem (3). Moreover, $a_{nn_i}(t) = \frac{t}{n_i+t}$ for $i = 2$ and $n_2 = n + 1$, while for $i = 3$ we see that $n_3 = 2n$. Obviously $|a_{nn_i}(t)| \leqslant 1$ for $t \in I_1$ and $n = 1, 2, \dots$, $i = 1, 2, 3$. This implies that the functions $a_{nn_i}(t)$ satisfy assumption (ii) of Theorem (11).
On the other hand it is easy to check that the functions $a_{nn_i}(t)$ satisfy the Lipschitz condition with the constant 1 for $n = 1, 2, \dots$ and $i = 1, 2, 3$. Hence we see that assumption (i) is satisfied of our theorem.
In the sequel of our considerations let us take the tempering sequence $\beta_n = \frac{1}{n^2}$ for $n = 1, 2, \dots$. Obviously we see that $x_0 = (x_0^n) = (2n + 1) \in c_0^\beta$, where $\beta = (\beta_n) = \left(\frac{1}{n^2}\right)$. This proves that assumption (iii) is satisfied with the space c_0^β for β stated above.
Further, let us observe that from the form of system (40) it follows that

$$f_n(t, x_1, x_2, \dots) = \frac{nx_n}{1 + x_n^2}$$

for $n = 1, 2, \dots$. It is clear that the function $f_n = f_n(t, x)$ is continuous on the set $I \times c_0^\beta$. Apart from this, we have

$$|f_n(t, x_1, x_2, \dots)| \leqslant \frac{1}{2}n, \quad \text{for} \quad n = 1, 2, \dots .$$

The above established facts allows us to deduce that functions f_n satisfy assumptions (iv) and (v) with $p_n = \frac{1}{2}n$ for $n = 1, 2, ...$ On the other hand we see that

$$\frac{\beta_n}{\beta_{n_{k_n}}} = \frac{\beta_n}{\beta_{2n}} = 4.$$

Thus we see that assumption (vi) is satisfied with $M = 4$.
Finally, in view of Theorem (11) we deduce that there exists at least one solution $x(t) = (x_n(t))$ of initial value problem (40)-(41) defined on some interval $I_1 = [0, T_1]$ such that for any $t \in I_1$ the sequence $(x_n(t))$ belongs to the space c_0^β with $\beta = \left(\frac{1}{n^2}\right)$. Obviously, we can easily calculate that $T_1 < \min\{T, 1/12\}$.

In what follows we provide an analog of Theorem (8) for infinite semilinear upper diagonal system of differential equations.
Namely, we will consider problem (34)-(35) for infinite upper diagonal system of differential equations and we dispense with the assumption requiring that system (34) has linear parts with constant width.
More precisely, we replace that assumption and assumption (ii) by the following hypotheses:

(ii') The sequence $\left(\sum_{i=1}^{k_n} |a_{nn_i}(t)|\right)$ is uniformly bounded on the interval $I_1 = [0, T_1]$, where T_1 is chosen according to Theorem 3. This means that there exists a constant $A > 0$ such that

$$\sum_{i=1}^{k_n} |a_{nn_i}(t)| \leqslant A$$

for any $n = 1, 2, ...$ and for $t \in I_1$.

(ii") The functions $a_{nn_i}(t)$ are nondecreasing on the interval I_1 for $n = 1, 2, ..., i = 1, 2, ..., k_n$ and the sequence $\left(\sum_{i=1}^{k_n} a_{nn_i}(t)\right)$ is equicontinuous on the interval I_1.

Now we formulate the announced result.

Theorem 13. *Under assumptions (i), (iii)-(vi) of Theorem* (11) *and assumptions (ii'), (ii"), initial value problem* (34)-(35) *for infinite upper diagonal system of differential equations has at least one solution* $x(t) = (x_n(t))$ *in the sequence space* c_0^β *defined on the interval* $I_1 = [0, T_1]$.

The proof of this theorem can be done in a similar way to the proof of the previous theorem. Moreover, it was presented in detail in [7], so we can omit it now.

Remark 14. Observe that the proof of Theorem (13) remains the same if in assumption (ii”) instead of the requirement that the functions $a_{nn_i}(t)$ are nondecreasing on the interval I_1 we impose that the functions $a_{nn_i}(t)$ are nonincreasing on I_1.

The below given example illustrates the result contained in Theorem (13).

Example 15. Let us consider the upper diagonal infinite system of differential equations of the form:

$$\begin{cases} x_1' = x_1 + \frac{x_1}{1+x_1^2}, \\ x_2' = tx_2 + 2\frac{x_1+x_2}{1+x_1^2+x_2^2}, \\ x_3' = \frac{t^2}{2!}x_3 + \frac{t^3}{3!}x_4 + 3\frac{x_2+x_3}{1+x_2^2+x_3^2}, \\ x_4' = \frac{t^3}{3!}x_4 + \frac{t^4}{4!}x_5 + 4\frac{x_3+x_4}{1+x_3^2+x_4^2}, \\ x_5' = \frac{t^4}{4!}x_5 + \frac{t^5}{5!}x_6 + \frac{t^6}{6!}x_7 + 5\frac{x_4+x_5}{1+x_4^2+x_5^2}, \\ \cdots\cdots\cdots\cdots\cdots\cdots \\ x_{n-1}'(=x_{2k-1}') = \frac{t^{2k-2}}{(2k-2)!}x_{2k-1} + \cdots + \frac{t^{3k-3}}{(3k-3)!}x_{3k-3} \\ \qquad\qquad +(2k-1)\frac{x_{2k-2}+x_{2k-1}}{1+x_{2k-2}^2+x_{2k-1}^2}, \\ x_n'(=x_{2k}') = \frac{t^{2k-1}}{(2k-1)!}x_{2k} + \cdots + \frac{t^{3k-2}}{(3k-2)!}x_{3k-1} \\ \qquad\qquad +2k\frac{x_{2k-1}+x_{2k}}{1+x_{2k-1}^2+x_{2k}^2}, \\ \cdots\cdots\cdots\cdots\cdots\cdots . \end{cases} \tag{42}$$

Moreover, we will also assume that the following initial conditions are satisfied

$$x_n(0) = n^2 \tag{43}$$

for $n = 1, 2, \ldots$.

Let us notice that initial value problem (42)-(43) is a particular case of problem (34)-(35) considered in Theorem (13). Indeed, as we stated above, infinite system of differential equations (42) is upper diagonal and the functions $a_{nn_i}(t)$ appeared in (42) have the form

$$a_{nn_i}(t) = \frac{t^{n_i-1}}{(n_i-1)!}$$

for $n_i = n, n+1, ..., \frac{3}{2}n - 1$ (if n is even) or $n_i = n, n+1, ..., 3\left[\frac{n}{2}\right]$ (if n is odd). Observe that the functions $a_{nn_i}(t)$ are equicontinuous on the interval $I_1 = [0, T_1]$ since we see that

$$a'_{nn_i}(t) = \frac{t^{n_i-2}}{(n_i-2)!} \leqslant e^{T_1}$$

for $n = 1, 2, ..., i = 1, 2, ..., k_n$ and $t \in I_1$. Thus, assumption (i) is satisfied. Further, let us note that

$$\sum_{i=1}^{k_n} |a_{nn_i}(t)| \leqslant 1 + t + \frac{t^2}{2!} + \cdots + \frac{t^n}{n!} \leqslant e^t$$

for $t \in I_1$. This shows that assumption (ii') is satisfied with $A = e^{T_1}$. Moreover, it is obvious that the functions $a_{nn_i}(t)$ are nondecreasing on the interval I_1. Further, for any fixed n we have:

$$\sum_{i=1}^{k_n} a_{nn_i}(t) = 1 + t + \frac{t^2}{2!} + \cdots + \frac{t^{k_n-1}}{(k_n-1)!}.$$

This implies that

$$\left(\sum_{i=1}^{k_n} a_{nn_i}(t)\right)' \leqslant e^{T_1}.$$

Hence we infer that the functions $\sum_{i=1}^{k_n} a_{nn_i}(t)$ satisfy the Lipschitz condition with the constant $L = e^{T_1}$. From this we derive that the sequence $\left(\sum_{i=1}^{k_n} a_{nn_i}(t)\right)$ is equicontinuous on the interval I_1. Thus assumption (ii") is satisfied.

In what follows let us take the tempering sequence having the form $\beta = (\beta_n) = \left(\frac{1}{n^3}\right)$. Then the initial sequence $(n_0^n) = (n^2)$ belongs to the tempered function space c_0^β and this assertion implies that assumption (iii) is satisfied. Similarly as in the earlier mentioned paper [7] it is not difficult to verify that the functions

$$f_n(t, x) = f_n(t, x_1, x_2, ...) = n\frac{x_{n-1} + x_n}{1 + x_{n-1}^2 + x_n^2}$$

$(n = 2, 3, ...)$ are continuous on the set $I_1 \times c_0^\beta$. Additionally, we see that $|f_n(t, x)| \leqslant n$. Thus we infer that assumptions (iv) and (v) are satisfied with $p_n = n$, since $\beta_n p_n = \frac{1}{n^2} \to 0$.
Finally, taking into account that $n_{k_n} = \frac{3}{2}n - 1$ for n even and $n_{k_n} = 3\left[\frac{n}{2}\right]$ for n odd, we get

$$\frac{\beta_n}{\beta_{n_{k_n}}} = \left(\frac{\frac{3}{2}n - 1}{n}\right)^3 = \left(\frac{3}{2} - \frac{1}{n}\right)^3 \leqslant \frac{27}{8}$$

if n is even. On the other hand, for n odd we have

$$\frac{\beta_n}{\beta_{n_{k_n}}} = \left(\frac{3\left[\frac{n}{2}\right]}{n}\right)^3 \leqslant \left(\frac{\frac{3}{2}n}{n}\right)^3 = \frac{27}{8}.$$

Thus we see that there is is satisfied assumption (vi) with $M = 27/8$.
From the above argumentations and Theorem (13) we conclude that problem (42)-(43) has at least one solution $x(t) = (x_n(t))$ in the space c_0^β defined on a suitable interval I_1.

5. Infinite systems of differential equations in the tempered sequence space c^β

In this section we study the existence of solutions of a perturbed diagonal infinite system of differential equations in the sequence space c^β. Consider the infinite perturbed diagonal systems of differential equations of the form

$$x_n' = a_n(t)x_n + g_n(t, x_1, x_2, \dots) \tag{44}$$

with the initial conditions

$$x_n(0) = x_n^0, \tag{45}$$

for $n = 1, 2, \dots$ and $t \in I = [0, T]$. Problem (44)–(45) will be investigated in the sequence space c^β, where $\beta = (\beta_n)$ is a tempering sequence i.e., the sequence (β_n) is nonincreasing and has positive terms.
Infinite systems of differential equations (44)–(45) contain, as particular cases, the systems considered in the theory of neural sets (cf. [12, pp. 86-87], and [26]). Let us also mention that system (44)–(45) was studied in [8]. The existence result concerning initial value problem (44)–(45) which we are going to present here, will generalize essentially results obtained in the

above quoted papers [8, 26] and the monograph [12]. In our considerations we will utilize the measure of noncompactness μ_2^β in the space c^β defined by formula (14).
Initial value problem (44)–(45) will be studied under the following assumptions.

(i) $x_0 = (x_n^0) \in c^\beta$;

(ii) the mapping $g = (g_1, g_2, \dots)$ acts from the set $I \times c^\beta$ into c^β and is continuous on $I \times c^\beta$;

(iii) There exists a sequence (p_n) with $\beta_n p_n \to 0$ as $n \to \infty$ such that

$$|g_n(t, x_1, x_2, \dots)| \leqslant p_n$$

for $t \in I$, $x = (x_n) \in c^\beta$ and for $n = 1, 2, \dots$.

(iv) The functions $a_n(t)$ are continuous on I and the sequence $(a_n(t))$ converges uniformly on I (to a function $a = a(t)$).

Notice that in view of the imposed assumptions the sequence $(a_n(t))$ is equibounded on I. This implies that the constant

$$A = \sup\{a_n(t) : t \in I, n = 1, 2, \dots\}$$

is finite.
Now, we can formulate our result.

Theorem 16. *Let assumptions (i)–(iv) be satisfied. If $AT < 1$ then initial value problem* (44)–(45) *has a solution $x(t) = (x_n(t))$ on the interval I such that $x(t) \in c^\beta$ for each $t \in I$.*

Proof. At the beginning, for $t \in I$ and $x = (x_n) \in c^\beta$ let us denote

$$f_n(t, x) = a_n(t)x_n + g_n(t, x), f(t, x) = (f_1(t, x), f_2(t, x), \dots),$$

where n is an arbitrarily fixed natural number. Further, fix arbitrary natural numbers m, n. Without loss of generality we can assume that $m < n$. Then, we obtain

$$
\begin{aligned}
&|\beta_n f_n(t,x) - \beta_m f_m(t,x)| \\
&\leqslant |\beta_n a_n(t)x_n - \beta_m a_m(t)x_m| + |\beta_n g_n(t,x) - \beta_m g_m(t,x)| \\
&\leqslant |\beta_n a_n(t)x_n - \beta_m a_n(t)x_m| + |\beta_m a_n(t)x_m - \beta_m a_m(t)x_m| \\
&\quad + \beta_n |g_n(t,x)| + \beta_m |g_m(t,x)| \\
&\leqslant |a_n(t)||\beta_n x_n - \beta_m x_m| + \beta_m |x_m||a_n(t) - a_m(t)| + \beta_n p_n + \beta_m p_m.
\end{aligned} \tag{46}
$$

In view of the imposed assumptions we deduce that $(\beta_k x_k)$ is a Cauchy sequence. The same statement is also valid for the function sequence $(a_k(t))$. Moreover, we see that $\beta_n p_n \to 0$ as $n \to \infty$. Taking into account the above established facts, from estimate (46) we deduce that $(\beta_n f_n(t,x))$ is a Cauchy sequence. This yields that $(f_n(t,x)) \subset c^\beta$.
Next, observe that for arbitrary $n \in \mathbb{N}, t \in I$ and for a fixed $x \in c^\beta$, we have

$$
|\beta_n f_n(t,x)| \leqslant |\beta_n a_n(t)x_n| + |\beta_n g_n(t,x)| \leqslant |a_n(t)|\beta_n|x_n| + \beta_n p_n \leqslant A\|x\| + P, \tag{47}
$$

where $P = \sup\{\beta_n p_n : n = 1, 2 \ldots\}$ and the symbol $\|\cdot\|$ denotes the norm in the space c^β (cf. Section 4). Obviously $P < \infty$. From estimate (47) we deduce the following one

$$
\|f(t,x)\| \leqslant A\|x\| + P. \tag{48}
$$

Now, we consider the mapping $f(t,x)$ on the set $I \times B(x_0, r)$, where r is taken according to Theorem 3, i.e.,

$$
r = \frac{(A+P)T\|x_0\|}{1 - AT}.
$$

To prove the continuity of the mapping $f(t,x)$ let us fix arbitrarily $t \in I$ and $x \in B(x_0, r)$. Next, choose arbitrary $s \in I$ and $y \in B(x_0, r)$. Then, in view of the imposed assumptions, we obtain

$$
\begin{aligned}
&\|f(t,x) - f(s,y)\| \\
&= \sup\Big\{|\beta_n f_n(t,x) - \beta_n f_n(s,y)| : n = 1, 2, \ldots\Big\} \\
&\leqslant \sup\Big\{\beta_n |a_n(t)x_n - a_n(s)y_n| : n = 1, 2, \ldots\Big\}
\end{aligned}
$$

$$
\begin{aligned}
&+\sup\Big\{\beta_n|g_n(t,x)-g_n(s,y)|:n=1,2,\dots\Big\}\\
\leqslant\ &\sup\Big\{\beta_n\Big[|a_n(t)x_n-a_n(s)x_n|+|a_n(s)x_n-a_n(s)y_n|\Big]:n=1,2,\dots\Big\}\\
&+\sup\Big\{\beta_n|g_n(t,x)-g_n(s,y)|:n=1,2,\dots\Big\}\\
\leqslant\ &\sup\Big\{\beta_n|x_n||a_n(t)-a_n(s)|:n=1,2,\dots\Big\}\\
&+\sup\Big\{|a_n(s)|\beta_n|x_n-y_n|:n=1,2,\dots\Big\}\\
&+\sup\Big\{\beta_n|g_n(t,x)-g_n(s,y)|:n=1,2,\dots\Big\}\\
\leqslant\ &(\|x_0\|+r)\sup\Big\{|a_n(t)-a_n(s)|:n=1,2,\dots\Big\}\\
&+A\|x-y\|+\|g(t,x)-g(s,y)\|.
\end{aligned}
$$

Hence, keeping in mind the fact that the sequence $(a_n(t))$ is equicontinuous on the interval I and the mapping g is continuous at the point (t,x) we conclude that the mapping f is continuous at (t,x). In view of the arbitrariness of t and x this yields that f is continuous on the set $I\times B(x_0,r)$.
Now, let us take a nonempty subset X of the ball $B(x_0,r)$. Fix $t\in I$ and $x=(x_n)\in X$. Then, in view of (46), for arbitrarily fixed natural numbers m,n we obtain

$$
\begin{aligned}
&|\beta_n f_n(t,x)-\beta_m f_m(t,x)|\\
&\leqslant|a_n(t)||\beta_n x_n-\beta_m x_m|+\|x\||a_n(t)-a_m(t)|+\beta_n p_n+\beta_m p_m.
\end{aligned}
$$

Hence, taking into account the imposed assumptions, we derive the estimate

$$
\mu_2^\beta(f(t,X))\leqslant a(t)\mu_2^\beta(X), \tag{49}
$$

where (as we mentioned above) μ_2^β is the measure of noncompactness in the space c^β defined by formula (14). Finally, linking estimates (48) and (49), in view of Theorem 3 we conclude that problem (44)–(45) has at least one solution in the space c^β. The proof is complete. □

Now we given an example illustrating our considerations.

Example 17. Consider the perturbed diagonal infinite system of differential equations

$$
x_n'=(n\sin\frac{t}{n})x_n+\arctan(x_n+x_{n+1}) \tag{50}
$$

with the initial conditions of the form

$$x_n(0) = n + 1 \tag{51}$$

for $n = 1, 2, \ldots$ and for $t \in I = [0, T]$, where T is a fixed positive number such that $T \leqslant \frac{\pi}{2}$. The value of T will be estimated precisely later.
Observe that initial value problem (50)–(51) is a special case of problem (44) - (45) if we put $a_n(t) = n \sin \frac{t}{n}$, $g_n(t, x_1, x_2, \ldots) = \arctan(x_n + x_{n+1})$ and if we accept the tempering sequence $\beta = (\beta_n) = (\frac{1}{n})$. We show briefly that in such a case infinite system (50) with initial conditions (51) satisfies assumptions of Theorem 16. To this end observe that the function sequence $(a_n(t))$ consists of functions continuous on the interval I and it is uniformly convergent on I to the function $a(t) = t,\ \ t \in I$. Thus the sequence $(a_n(t))$ satisfies assumption (iv).
Further, we have

$$|g_n(t, x_1, x_2, \ldots)| = |\arctan(x_n + x_{n+1})| \leqslant \frac{\pi}{2}$$

for $n = 1, 2 \ldots$. Thus, taking $p_n = \frac{\pi}{2}$ we see that assumption (iii) is satisfied. Similarly we verify assumption (i).
To check assumption (ii) let us fix arbitrarily $x, y \in c^\beta$, $x = (x_k)$, $y = (y_k)$. Then, for a fixed $n \in \mathbb{N}$ we obtain:

$$\begin{aligned} &\beta_n |g_n(t, x_1, x_2, \ldots) - g_n(t, y_1, y_2, \ldots)| \\ &= \frac{1}{n} |\arctan(x_n + x_{n+1}) - \arctan(y_n + y_{n+1})| \\ &\leqslant \frac{1}{n} |x_n + x_{n+1} - y_n - y_{n+1}| \leqslant \frac{1}{n}|x_n - y_n| + \frac{1}{n}|x_{n+1} - y_{n+1}| \\ &\leqslant \frac{n+1}{n} \Big(\frac{1}{n+1}|x_n - y_n| + \frac{1}{n+1}|x_{n+1} - y_{n+1}| \Big) \\ &\leqslant 2 \Big(\frac{1}{n}|x_n - y_n| + \frac{1}{n+1}|x_{n+1} - y_{n+1}| \Big). \end{aligned} \tag{52}$$

Next, in view of (52), for arbitrarily fixed $t, s \in I$ and $x, y \in c^\beta$, we obtain

$$\begin{aligned}
&\|g(t,x)-g(s,y)\| \\
&= \sup\Big\{\frac{1}{n}|g_n(t,x)-g_n(s,y)| : n=1,2,\dots\Big\} \\
&\leqslant \sup\Big\{2\big(\frac{1}{n}|x_n-y_n| + \frac{1}{n+1}|x_{n+1}-y_{n+1}|\big) : n=1,2\dots\Big\} \\
&\leqslant 2\sup\Big\{\frac{1}{n}|x_n-y_n| : n=1,2\dots\Big\} + 2\sup\Big\{\frac{1}{n+1}|x_{n+1}-y_{n+1}| : n=1,2\dots\Big\} \\
&\leqslant 4\|x-y\|,
\end{aligned}$$

where the symbol $\|\cdot\|$ denotes the norm in the space c^β. Thus we showed that the mapping g is continuous on the set $I \times c^\beta$ (even Lipschitz continuous). This means that the mapping g satisfies assumption (ii).
Finally, let us observe that using standard methods of mathematical analysis, we obtain

$$\begin{aligned}
A &= \sup\{|a_n(t)| : t \in [0,T], n=1,2,\dots\} \\
&= \sup\{n \sin\frac{t}{n} : t \in [0,T], n=1,2,\dots\} \leqslant \frac{\pi}{2}.
\end{aligned}$$

Thus, if we take $T < \frac{2}{\pi}$, then applying Theorem 16 we deduce that initial value problem (50)–(51) has at least one solution $x(t) = (x_n(t))$ such that $(x_n(t)) \in c^\beta$ for any $t \in [0,T]$.

6. Infinite systems of differential equations in the tempered sequence space l_∞^β

In this section we will work in the space l_∞^β described in detail in Section 3. We will assume here that the tempering sequence $\beta = (\beta_n)$ consists of positive terms and is nonincreasing. We will utilize the measure of noncompactness μ_3^β defined on the family $\mathfrak{M}_{l_\infty^\beta}$ by formula (17). For simplicity, that measure will be denoted by μ. Recall, that for $X \in \mathfrak{M}_{l_\infty^\beta}$ we put

$$\mu(X) = \limsup_{n\to\infty} \operatorname{diam} X_n^\beta,$$

where $X_n^\beta = \{\beta_n x_n : x = (x_i) \in X\}$. Equivalently, this formula can be written in a more convenient way

$$\mu(X) = \limsup_{n\to\infty} \operatorname{diam} \beta_n X_n, \tag{53}$$

where $X_n = \{x_n : x = (x_i) \in X\}$. We refer to Section 3 for the properties of the measure μ.
In what follows we will investigate the following perturbed semilinear lower diagonal infinite system of differential equations

$$x_n' = \sum_{j=k_n}^{n} a_{nj}(t)x_j + g_n(t, x_1, x_2, \dots) \tag{54}$$

with the initial conditions

$$x_n(0) = x_n^0 \tag{55}$$

for $n = 1, 2\dots$ and $t \in I = [0, T]$.
Throughout this section we will assume that the sequence (k_n) appearing in (54) is such that $1 \leqslant k_n \leqslant n$ for $n = 1, 2, \dots$ and $k_n \to \infty$ as $n \to \infty$.
It is worthwhile mentioning that infinite systems of differential equations having form (54) were up to now considered very seldom (cf. [5, 9]).
For further purposes we denote by $f = f(t, x)$ the mapping defined on the set $I \times l_\infty^\beta$ in the following way

$$f(t, x) = (f_1(t, x), f_2(t, x), \dots),$$

where

$$f_n(t, x) = f_n(t, x_1, x_2, \dots) = \sum_{j=k_n}^{n} a_{nj}(t)x_j + g_n(t, x_1, x_2, \dots)$$

for $n = 1, 2, \dots$. Moreover, we will also define the mapping $g(t, x)$ on the set $I \times l_\infty^\beta$ by putting

$$g(t, x) = (g_1(t, x), g_2(t, x), \dots).$$

Now, we formulate assumptions under which problem (54)–(55) will be studied. Namely, we will impose the following hypotheses.

(i) $x_0 = (x_n^0) \in l_\infty^\beta$;

(ii) the mapping g acts from the set $I \times l_\infty^\beta$ into l_∞^β and is uniformly continuous on $I \times l_\infty^\beta$;

(iii) there exists a sequence (p_n) with $\beta_n p_n \to 0$ as $n \to \infty$ and such that

$$|g_n(t, x_1, x_2, \dots)| \leqslant p_n$$

for $t \in I, x = (x_n) \in l_\infty^\beta$ and $n = 1, 2 \dots$;

(iv) The functions $a_{nj} : I \to \mathbb{R}$ $(j = k_n, k_n + 1, \dots, n,\ n = 1, 2 \dots)$ are continuous and nondecreasing on I. Moreover, we assume that the function sequence $(A_n(t))$ is equicontinuous on the interval I and the sequence $(\bar{A}_n(t))$ is uniformly bounded on I, where

$$A_n(t) = \sum_{j=k_n}^{n} a_{nj}(t), \quad \bar{A}_n(t) = \sum_{j=k_n}^{n} |a_{nj}(t)|$$

for $n = 1, 2, \dots$.

Keeping in mind assumption (iv), for further purposes we can define the constant

$$A = \sup\{\bar{A}_n(t) : t \in I, n = 1, 2, \dots\}.$$

In view of assumptions (iv) we see that $A < \infty$.
Now, we can formulate the following result concerning initial value problem (54)–(55).

Theorem 18. *Assume that conditions* (i)–(iv) *are satisfied and* $AT < 1$. *Then initial value problem* (54)–(55) *has at least one solution* $x(t) = (x_k(t))$ *on the interval* $I = [0, T]$ *such that* $x(t) \in l_\infty^\beta$ *for* $t \in I$.

We omit the details of the proof (cf. [6]).

Remark 19. Observe that instead of the requirement imposed in assumption (iv) that the functions a_{nj} $(j = k_n, k_n + 1, \dots, n;\ n = 1, 2, \dots)$ are nondecreasing on I, we can assume that those functions are nonincreasing on I.

The next example shows the applicability of the result in Theorem 18.

Example 20. Consider the semilinear lower diagonal perturbed infinite system of differential equations

$$x_n' = \sum_{j=k_n}^{n} t^{n+j} x_j + \sin(x_n + x_{n+1} + x_{n+2}) \tag{56}$$

with the initial conditions

$$x_n(0) = n, \tag{57}$$

for $n = 1, 2\dots$ and for $t \in I = [0, T]$, where $T < 1$. Moreover, we assume that (k_n) is a nondecreasing sequence of natural numbers such that $1 \leqslant k_n \leqslant n$ and $k_n \to \infty$ as $n \to \infty$.

Observe that (56)–(57) is a special case of (54)–(55), where $a_{nj}(t) = t^{n+j}$ for $j = k_n, k_n + 1, \dots, n$ and for $n = 1, 2, \dots$. Apart from this, the function g_n has the form

$$g_n(t, x_1, x_2, \dots) = \sin(x_n + x_{n+1} + x_{n+2})$$

for $n = 1, 2, \dots$. It is easily seen that infinite system (56) with initial conditions (57) satisfies assumptions of Theorem 18 if we take the tempering sequence (β_n) of the form $\beta_n = \frac{1}{n}$ for $n = 1, 2, \dots$. Indeed, we obviously see that $(x_0^n) = (n) \in l_\infty^\beta$. This means that assumption (i) is satisfied.

Now, take an arbitrary element $x = (x_k) \in l_\infty^\beta$ and a number $t \in I$. Then, for a fixed natural number n we obtain

$$\begin{aligned}
\beta_n |g_n(t, x_1, x_2, \dots)| &= \frac{1}{n} |\sin(x_n + x_{n+1} + x_{n+2})| \\
&\leqslant \frac{1}{n} \Big(|x_n| + |x_{n+1}| + |x_{n+2}| \Big) \\
&= \frac{n+2}{n} \Big(\frac{1}{n+2} |x_n| + \frac{1}{n+2} |x_{n+1}| + \frac{1}{n+2} |x_{n+2}| \Big) \\
&\leqslant 3 \Big(\frac{1}{n} |x_n| + \frac{1}{n+1} |x_{n+1}| + \frac{1}{n+2} |x_{n+2}| \Big) \leqslant 3\|x\|,
\end{aligned}$$

where the symbol $\|\cdot\|$ denotes the norm in the space l_∞^β. Hence we obtain

$$\|g(t, x)\| \leqslant 3\|x\|$$

which shows that g acts from the set $I \times l_\infty^\beta$ into l_∞^β.

Further, if we fix arbitrarily $n \in \mathbb{N}$, $x = (x_1, x_2, \dots) \in l_\infty^\beta$, $y = (y_1, y_2, \dots) \in$

l_∞^β and $t, s \in I$, then we obtain

$$\begin{aligned}
&\beta_n|g_n(t,x_1,x_2,\dots) - g_n(s,y_1,y_2,\dots)| \\
&= \frac{1}{n}|\sin(x_n + x_{n+1} + x_{n+2}) - \sin(y_n + y_{n+1} + y_{n+2})| \\
&\leqslant \frac{1}{n}\Big(|x_n - y_n| + |x_{n+1} - y_{n+1}| + |x_{n+2} - y_{n+2}|\Big) \\
&= \frac{n+2}{n}\Big(\frac{1}{n+2}|x_n - y_n| + \frac{1}{n+2}|x_{n+1} - y_{n+1}| + \frac{1}{n+2}|x_{n+2} - y_{n+2}|\Big) \\
&\leqslant 3\Big(\frac{1}{n}|x_n - y_n| + \frac{1}{n+1}|x_{n+1} - y_{n+1}| + \frac{1}{n+2}|x_{n+2} - y_{n+2}|\Big) \\
&\leqslant 3\|x - y\|.
\end{aligned}$$

Hence we derive the estimate

$$\|g(t,x) - g(s,y)\| \leqslant 3\|x - y\|$$

which shows that the mapping g is uniformly continuous on the set $I \times l_\infty^\beta$. Thus the mapping g satisfies assumption (ii) of Theorem 18.
Now, we have

$$|g_n(t,x_1,x_2,\dots)| = |\sin(x_n + x_{n+1} + x_{n+2})| \leqslant 1$$

which shows that assumption (iii) is satisfied with $p_n = 1$ for, $n = 1, 2\dots$. To show that assumption (iv) is satisfied let us notice that we have

$$A_n(t) = \bar{A_n}(t) = t^{n+k_n}\frac{1 - t^{n-k_n+1}}{1-t}$$

for $t \in I = [0, T]$ and for $n = 1, 2, \dots$. Using the standard methods of analysis it is not hard to show that the sequence $(A_n(t))$ is equicontinuous on the interval I. Moreover, we have the estimate

$$A_n(t) = \bar{A_n}(t) \leqslant A \leqslant \frac{1}{1-T}$$

for any $t \in I$. Summing up we see that initial value problem (56)–(57) satisfies the assumptions in Theorem 18. Therefore the infinite system (56) with initial value conditions (57) has at least one solution in the space l_∞^β.

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Summary

The goal of the chapter is to investigate the existence of solutions for infinite systems of differential equations. We look for solutions in Banach tempered sequence spaces, using techniques associated with measures of noncompactness, and results from differential equations in abstract Banach spaces.

Streszczenie

Celem rozdziału jest zbadanie istnienia rozwiązań dla nieskończonych układów równań różniczkowych. Poszukujemy rozwiązań w przestrzeniach Banacha ciągów temperowanych, wykorzystując techniki związane z miarami niezwartości oraz wyniki dotyczące równań różniczkowych w abstrakcyjnych przestrzeniach Banacha.

Chapter 3

Solvability of Volterra-Stieltjes integral equations in the class of functions converging at infinity

Agnieszka Dubiel

1. Introduction

Differential equations have wide-ranging applications, among others, in astronomy, physics, economics, biology, mechanics, engineering, and electrochemistry. They also play a key role in describing diffraction theory, control theory, transport theory, kinetic theory of gases, radiative transfer, and queuing theory [2, 10, 11, 13–16, 18–20]. Many integral equations can be regarded as special cases of Volterra-Stieltjes, Hammerstein-Stieltjes, or Urysohn-Stieltjes integral equations. Investigating these types of integral equations is significantly simpler and allows for results that can be applied to a broader class of equations.

We study quadratic Volterra-Stieltjes integral equations in the space of real functions that are continuous on the positive half-axis and have finite limits at infinity. We present conditions that ensure the existence of solutions to these quadratic integral equations in this space, applying techniques based on measures of noncompactness and Darbo fixed point theorem.

This chapter is of an overview and is based on the works listed in the bibliography. In [6], we studied similar integral equation, but there we used methods based on compact integral operators and the Schauder fixed point principle. However, these tools are insufficient for the analysis of quadratic Volterra-Stieltjes integral equations, which is why we instead use techniques related to the theory of measures of noncompactness and Darbo fixed point theorem [12]. The following considerations allow us to extend the results obtained in [6] and [3], and improve some of the results presented in [4].

2. Notation, definitions and auxiliary facts

In this chapter, we will present some basic facts and notations that will be helpful in further considerations. The symbol $BC(\mathbb{R}_+)$ denotes the space consisting of all real-valued functions that are defined, continuous, and bounded on $\mathbb{R}_+$. In the space $BC(\mathbb{R}_+)$, we define classical norm

$$||x||_\infty = \sup\{|x(t)| : t \in \mathbb{R}_+\}.$$

The space $BC(\mathbb{R}_+)$ with the above norm is a Banach space. In the following considerations, for the sake of simplicity, we will use the symbol $||x||$ instead of $||x||_\infty$, unless this leads to ambiguity.

Let E be a real Banach space with a norm $||\cdot||$, then the symbol $B(x, r)$ denotes the closed ball with center at point x and radius r. The ball $B(\theta, r)$, where θ is the zero vector in E, is denoted by B_r. For any subset X of the Banach space E we denote by $\overline{X}$ the closure of the set X, while $\text{Conv}X$ denotes the closed convex hull of the set X. We will also use the standard notations $X + Y$ and λX to describe basic algebraic operations on subsets of space E.

We will now recall some fundamental properties related to the variation of a function (cf. [1]). Let us assume that x is a real function defined on a fixed interval $[a, b]$. We denote the variation of the function f on the interval $[a, b]$ by $\bigvee_a^b f$. If $\bigvee_a^b f < \infty$, we call the function f a *function of bounded variation* on the interval $[a, b]$. The set of all real functions of bounded variation on the interval $[a, b]$ is denoted by $BV([a, b])$. If we consider a function of two variables $u(t, s) = u : [a, b] \times [c, d] \to \mathbb{R}$, then we denote the variation of the function $t \mapsto u(t, s)$ on the interval $[p, q] \subset [a, b]$ by $\bigvee_{t=p}^q u(t, s)$. Similarly, we define $\bigvee_{s=p}^q u(t, s)$.

The Riemann-Stieltjes integral is an essential part of further considerations.

We will now present some facts related to this concept [1]. Let f and g be real-valued functions defined on the interval $[a, b]$. The *Riemann-Stieltjes integral* (or simply the *Stieltjes integral*) of the function f with respect to the function g is denoted by

$$\int_a^b f(x)dg(x).$$

An important fact is that if the function f is continuous and the function g has bounded variation, then the Riemann-Stieltjes integral exists [1].

We will now state two very important lemmas (cf. [1]) that we will frequently use.

Lemma 1. *If the function f is Stieltjes integrable on the interval $[a, b]$ with respect to a function g, of bounded variation, then the following inequality holds*

$$\left|\int_a^b f(x)dg(x)\right| \leqslant \int_a^b |f(x)|d\Big(\bigvee_a^x g\Big).$$

Lemma 2. *Let f_1 and f_2 be Stieltjes integrable functions on the interval $[a, b]$ with respect to a nondecreasing function g such that $f_1(x) \leqslant f_2(x)$ for $x \in [a, b]$. Then*

$$\int_a^b f_1(x)dg(x) \leqslant \int_a^b f_2(x)dg(x).$$

Due to the form of the integral equations we consider in the following chapters, we will focus on Stieltjes integral of the form

$$\int_a^b x(t, s)d_s g(t, s),$$

where the functions $g, x : [a, b]\times[c, d] \to \mathbb{R}$ satisfy the necessary assumptions ensuring the existence of such integrals; for example, the function $x(t, s) = x$ is continuous on $[a, b] \times [c, d]$ and the function $s \mapsto g(t, s)$ has bounded variation on $[c, d]$ for each fixed $t \in [a, b]$. The symbol d_s denotes integration with respect to the variable s.

In our considerations, we will use a condition that is equivalent to the existence of a finite limit of a function at infinity. This condition is well known as the Cauchy condition at infinity. To precisely formulate this condition, let us assume that $x : \mathbb{R}_+ \to \mathbb{R}$. We will say that the function $x = x(t)$ *satisfies the Cauchy condition at infinity* if

$$\forall_{\varepsilon>0}\ \exists_{T>0}\ \forall_{t,s\geqslant T}\ \ \big|x(t)-x(s)\big| < \varepsilon.$$

As we mentioned earlier, the finite limit $\lim_{t\to\infty} x(t)$ exists if and only if the function x satisfies the Cauchy condition at infinity.

Let us now recall the concept of modulus of continuity. At first, let us fix a nonempty and bounded subset X of the space $BC(\mathbb{R}_+)$. Next, choose $\varepsilon > 0$ and $T > 0$. Take $x \in X$ and define the *modulus of continuity* $\omega^T(x,\varepsilon)$ *of the function* x on the interval $[0,T]$ by

$$\omega^T(x,\varepsilon) = \sup\big\{\big|x(t)-x(s)\big| : t,s \in [0,T], |t-s| \leqslant \varepsilon\big\}.$$

We now define the quantity $\varepsilon \mapsto \omega^T(X,\varepsilon)$, called the *modulus of continuity of the set* X, as follows:

$$\omega^T(X,\varepsilon) = \sup\big\{\omega^T(x,\varepsilon) : x \in X\big\}.$$

Next, we define the following quantities [8, 9]:

$$\omega_0^T(X) = \lim_{\varepsilon\to 0} \omega^T(X,\varepsilon),$$
$$\omega_0(X) = \lim_{T\to\infty} \omega_0^T(X).$$

Let us now consider the quantities $b(X)$ and $\gamma = \gamma(X)$ defined as follows:

$$b(X) = \lim_{T\to\infty} \Big\{ \sup_{x\in X} \big\{ \sup\big\{\big|x(t)-x(s)\big| : t,s \geqslant T\big\}\big\}\Big\}.$$

$$\gamma(X) = \omega_0(X) + b(X). \tag{1}$$

Now we introduce the definition of the measure of noncompactness, which will allow us to say a bit more about the properties of the quantity γ, described by the formula (1).

Both well-known measures of noncompactness (the Kuratowski measure and the Hausdorff measure of noncompactness) face significant challenges when attempting to express these measures using formulas that reflect the structure of the Banach spaces in which they are defined. In the case of the Kuratowski measure of noncompactness, no such formulas are known for any Banach space [8]. The situation is slightly better for the Hausdorff measure of noncompactness, as for specific Banach spaces, formulas can be provided, such as for the sequence spaces c_0, l^p, and the function space $C([a,b])$ [8]. However, in some Banach spaces, it is possible to estimate the Hausdorff (or Kuratowski) measure of noncompactness using formulas in a way that these measures become equivalent to set functions defined by those formulas. This holds, for example, for the sequence space c and the Lebesgue space $L^p(a,b)$ [8]. In the cases mentioned above, it is essential to understand compactness criteria related to the structure of the respective Banach space (such as the Arzelá-Ascoli criterion for $C([a,b])$ or the Riesz and Kolmogorov criteria for $L^p(a,b)$.

In many Banach spaces, such criteria are not known, and we only have certain sufficient conditions for relative compactness in these spaces. A typical example is the sequence space l^∞ or the function space $BC(\mathbb{R}_+)$ mentioned earlier. To construct measures of noncompactness for this type of Banach space, one can use the axiomatic definition of a measure of noncompactness introduced in 1980 by Józef Banaś and Kazimierz Goebel [8]. This kind of measure of noncompactness is convenient to use and not overly general. By employing a measures of noncompactness that satisfy the axioms of this definition, it becomes possible to characterize solutions of various operator equations studied using noncompactness techniques.

Let us now present the axiomatic definition of a measure of noncompactness. To this end, we define a Banach space E. We denote by $\mathfrak{M}_E$ the family of all nonempty and bounded subsets of the space E, and by $\mathfrak{N}_E$ the subfamily of $\mathfrak{M}_E$, consisting of all relatively compact sets.

Definition 3. A function $\mu : \mathfrak{M}_E \to \mathbb{R}_+$ is called *the measure of noncompactness* in the space E if it satisfies the following conditions:

1° The family $\ker \mu = \{X \in \mathfrak{M}_E : \mu(X) = 0\}$ is nonempty and $\ker \mu \subset \mathfrak{N}_E$.

2° $X \subset Y \Rightarrow \mu(X) \leqslant \mu(Y)$.

3° $\mu(\overline{X}) = \mu(X)$.

4° $\mu(\text{Conv}X) = \mu(X)$.

5° $\mu(\lambda X + (1-\lambda)Y) \leqslant \lambda\mu(X) + (1-\lambda)\mu(Y)$ for $\lambda \in [0,1]$.

6° If (X_n) is a sequence of closed sets from $\mathfrak{M}_E$ such that $X_{n+1} \subset X_n$ for $n = 1, 2, \dots$ and $\lim_{n\to\infty} \mu(X_n) = 0$, then the set $X_\infty = \bigcap_{n=1}^{\infty} X_n$ is nonempty.

The family $\ker \mu$ appeared in axiom 1° is called *the kernel of the measure* μ. If $\ker \mu = \mathfrak{N}_E$ then the measure of noncompactness μ is called *full*. It is worth noting that the set X_∞ from axiom 6° is an element of the kernel $\ker \mu$. This directly follows from inequality $\mu(X_\infty) \leqslant \mu(X_n)$ for any $n = 1, 2, \dots$, which implies that $\mu(X_\infty) = 0$, meaning that $X_\infty \in \ker \mu$. This simple observation plays a key role in the applications of the technique associated with measures of noncompactness.

Let us recall [8] that the measure of noncompactness μ is called *sublinear* if it additionally satisfies the following conditions:

7° $\mu(\lambda X) = |\lambda|\mu(X)$ for $\lambda \in \mathbb{R}$.

8° $\mu(X + Y) \leqslant \mu(X) + \mu(Y)$.

If it satisfies the condition

9° $\mu(X \cup Y) = \max\{\mu(X), \mu(Y)\}$

then it is referred to as *the measure with maximum property.*

A full and sublinear measure of noncompactness μ which has the maximum property is called *regular* [8].

It can be shown that quantity γ described by Eq. (1) is a measure of noncompactness in the space $BC(\mathbb{R}_+)$ (cf. [9]) which is sublinear and has the maximum property. However, the measure γ is not full. We can show that the kernel $\ker \mu$ consists of all bounded subsets X of the space $BC(\mathbb{R}_+)$ such that functions from X are locally equicontinuous on $\mathbb{R}_+$ and approach limits at infinity at the same rate. In other words, the functions in X converge to limits at infinity uniformly with respect to the set X.

Let us pay attention to the fact that measures of noncompactness are very useful in several applications [1, 8]. In particular, the following fixed point theorem, known as the Darbo-type fixed point theorem, plays a crucial role in these applications [12].

Theorem 4. *Let Ω be a nonempty, bounded, closed and convex subset of a Banach space E. Assume that $T : \Omega \to \Omega$ is a continuous operator and there exists a constant $k \in [0, 1)$ such that $\mu(TX) \leqslant k\mu(X)$ for any nonempty subset X of Ω, where μ is a measure of noncompactness in E. Then T has at least one fixed point in the set Ω.*

It can be shown that the set $Fix\, T$ of fixed points of the operator T belonging to Ω is a member of the kernel $\ker \mu$. These facts allow us to describe the solutions of the operator equations under consideration (cf. [8]). The transformation T defined in the theorem is referred to as a *Darbo contraction.*

3. Main results

The main object of the study in this chapter is quadratic Volterra-Stieltjes integral equation having the form

$$x(t) = a(t) + u(t, x(t)) \int_0^t f(t, s, x(s)) d_s K(t, s), \tag{2}$$

where $t \in \mathbb{R}_+$. We will consider Eq. (2) in the space $BC(\mathbb{R}_+)$.
Our goal is to show that integral equation (2) has at least one solution in the space $BC(\mathbb{R}_+)$ that converges at infinity, clearly to a finite limit [7]. For the purposes of what follows, we denote by Δ the triangle $\Delta = \{(t, s) : 0 \leqslant s \leqslant t\}$.

Now, we state the assumptions under which we will investigate the solvability of Eq. (2). Specifically, we impose the following conditions.

(i) The function $a = a(t)$ is a member of the space $BC(\mathbb{R}_+)$ and there exists the limit $\lim_{t\to\infty} a(t)$.

(ii) $f : \Delta \times \mathbb{R} \to \mathbb{R}$ is a continuous function and there exists a continuous and nondecreasing function $\phi : \mathbb{R}_+ \to \mathbb{R}_+$ such that

$$|f(t, s, x)| \leqslant \phi(|x|)$$

for all $(t, s) \in \Delta$ and $x \in \mathbb{R}$. Moreover, we assume that the function f is uniformly continuous on each set of the form $\Delta \times [-r, r]$, for arbitrary $r > 0$.

(iii) $K(t,s) = K : \Delta \to \mathbb{R}$ is a continuous function on the triangle Δ and $K(t,0) = 0$ for each $t \geqslant 0$.

(iv) For any fixed $t > 0$ the function $s \mapsto K(t,s)$ has a bounded variation on the interval $[0,t]$ and the function $t \mapsto \bigvee_{s=0}^{t} K(t,s)$ is bounded on $\mathbb{R}_+$.

(v) For every $\varepsilon > 0$ there exists $\delta > 0$ such that for all $t_1, t_2 \in \mathbb{R}_+$, $t_1 < t_2$, $t_2 - t_1 \leqslant \delta$, the following inequality holds

$$\bigvee_{s=0}^{t_1} \left[K(t_2,s) - K(t_1,s)\right] \leqslant \varepsilon.$$

(vi) The following equalities hold:

$$\lim_{T\to\infty} \left\{ \sup \left[\bigvee_{\tau=s}^{t} K(t,\tau) : T \leqslant s < t \right] \right\} = 0,$$

$$\lim_{T\to\infty} \left\{ \sup \left[\bigvee_{\tau=0}^{s} \left[K(t,\tau) - K(s,\tau)\right] : T \leqslant s < t \right] \right\} = 0,$$

$$\lim_{T\to\infty} \left\{ \sup \left[\left|v(t,\tau,y) - v(s,\tau,y)\right| : t,s \geqslant T, \right.\right.$$
$$\left.\left. \tau \in \mathbb{R}_+, \tau \leqslant s, \tau \leqslant t, y \in [-r,r] \right] \right\} = 0,$$

for each fixed $r > 0$.

(vii) The function $u(t,x) = u : \mathbb{R}_+ \times \mathbb{R} \to \mathbb{R}$ is continuous and there exists a function $k(r) = k : \mathbb{R}_+ \to \mathbb{R}_+$ which is nondecreasing and continuous on $\mathbb{R}_+$ with the property $k(0) = 0$ and such that for each $r > 0$ the following inequality is satisfied

$$\left|u(t,x) - u(t,y)\right| \leqslant k(r)|x - y|$$

for all $x, y \in [-r,r]$ and for any $t \in \mathbb{R}_+$.

(viii) The function $t \mapsto u(t,x)$ satisfies the Cauchy condition at infinity uniformly with respect to the variable x belonging to any bounded interval i.e., the following condition is satisfied

$$\forall_{r>0}\ \forall_{\varepsilon>0}\ \exists_{T>0}\ \forall_{t,s\geqslant T}\ \forall_{x\in[-r,r]}\ \ \big|u(t,x)-u(s,x)\big| \leqslant \varepsilon.$$

(ix) There exists a number r_0 satisfying the inequality

$$||a|| + \overline{K}(rk(r)+\overline{F})\phi(r) \leqslant r,$$

such that

$$\overline{K}k(r_0)\phi(r_0) < 1,$$

where

$$\overline{K} = \sup\left\{\bigvee_{s=0}^{t} K(t,s) : t \in \mathbb{R}_+\right\},\ \overline{K} < \infty.$$

Now we are prepared to formulate the main result of the chapter concerning the solvability of Eq. (2).

Theorem 5. *Under the assumptions* (i) – (vi), (vii), (viii) *and* (ix), *there exists at least one solution $x = x(t)$ of Eq. (2) in the space $BC(\mathbb{R}_+)$ converging to a finite limit at infinity.*

Proof sketch. We will outline the proof of the above theorem. Details can be found in the paper [7]. For further purposes let us consider the operators U, F, Q defined on the space $BC(\mathbb{R}_+)$ in the following way:

$$(Ux)(t) = u\big(t, x(t)\big),$$

$$(Fx)(t) = \int_0^t f\big(t, s, x(s)\big) d_s K(t,s),$$

$$(Qx)(t) = a(t) + (Ux)(t)(Fx)(t),$$

for $t \in \mathbb{R}_+$. Obviously Eq. (2) can be written in the following form

$$x(t) = (Qx)(t).$$

Steps of the proof:

(I) We show that the function Qx is continuous on the set $\mathbb{R}_+$.

To show that $Qx \in C(\mathbb{R}_+)$, one must use the properties of the Lebesgue integral and the variation of the function, Lemma 1, Lemma 2, and assumptions (i), (ii), (iv), (v), (viii).

(II) We show that the function Qx is bounded on the set $\mathbb{R}_+$.

To show that $Qx \in B(\mathbb{R}_+)$, one must use Lemma 1 and Lemma 2.

(III) Using the previous step, we show that $Q(B_{r_0}) \subset B_{r_0}$.

Since the function Qx is continuous and bounded, the operator Q transforms the space $BC(\mathbb{R}^+)$ into itself. Using assumption (ix), we infer that there exists a number $r_0 > 0$ such that Q transforms the ball B_{r_0} into itself.

(IV) We show that the operator Q is continuous on the ball B_{r_0}.

By combining the estimates obtained in the previous steps, we will infer that the operator Q is continuous on the B_{r_0} and continuously transforms the ball B_{r_0} into itself.

(V) We prove that Q is a Darbo contraction with respect to the measure of noncompactness γ.

We obtain the following estimate:

$$\gamma(QX) \leqslant \overline{K}\phi(r_0)k(r_0)\gamma(X),$$

where γ is the measure of noncompactness in the space $BC(\mathbb{R}_+)$. Taking into account the above estimate, the second part of assumption (ix), and applying Theorem 4, we conclude that there exists at least one solution $x = x(t)$ of equation (2) in the space $BC(\mathbb{R}_+)$. This solution converges to a finite limit at infinity.

□

4. Special cases of the quadratic Volterra-Stieltjes integral equation

The results presented in this chapter generalize those obtained in [6] and improve some of the results from [4].

Let us consider the following Volterra-Stieltjes integral equation (cf. [6]):

$$x(t) = a(t) + \int_0^t f(t,s,x(s))d_sK(t,s). \tag{3}$$

Equation (3) is a special case of the integral equation (2). However, the solutions to this integral equation can be obtained without using the technique of measures of noncompactness [6], using instead on the classical Schauder fixed point theorem and a sufficient condition for the relative compactness of a set in the space $BC(\mathbb{R}_+)$ [5, 6, 8].

Let us assume that:

(x) There exists a positive number r_0^1 satisfying the inequality

$$||a|| + \overline{K}\phi(r) \leqslant r,$$

where ϕ is the nondecreasing function defined in (ii) and $\overline{K}$ was defined in (ix).

Theorem 6. *Under the assumptions* (i) – (vi) *and* (x), *the integral equation (3) has at least one solution $x = x(t)$ in the space $BC(\mathbb{R}_+)$ such that $||x|| \leqslant r_0^1$ for some $r_0^1 > 0$ and for which $\lim_{t\to\infty} x(t)$ exists and is finite.*

Proof sketch. We will outline the proof of the above Theorem. Details can be found in the paper [6]. For further purposes let us consider the operator F defined on the space $BC(\mathbb{R}_+)$ in the following way:

$$(Fx)(t) = \int_0^t f(t,s,x(s))d_sK(t,s), \tag{4}$$

for $t \in \mathbb{R}_+$. Obviously Eq. (3) can be written in the form

$$x(t) = (Fx)(t).$$

Steps of the proof:

(I) We show that the function Fx is continuous on the set $\mathbb{R}_+$.

(II) We show that the function Fx is bounded on the set $\mathbb{R}_+$.

(III) Using the previous step, we show that $F(B_{r_0}) \subset B_{r_0}$.

(IV) We show that the operator Q is continuous on the ball B_{r_0}.

(V) By virtue of Schauder's theorem, we show that the operator F has a fixed point x belonging to the ball $B^1_{r_0}$. The function $x = x(t)$ is a solution to the Volterra-Stieltjes integral equation (3).

□

We will now present the definitions and theorems used in the steps of the above proof.

Let us assume, to begin with, that M is a nonempty and compact subset of a metric space $(\mathcal{X}, d)$. Let $C(M)$ denote the set of all real-valued functions defined and continuous on M. If the set $C(M)$ is equipped with the supremum metric (the supremum norm), then it becomes a complete metric space ($C(M)$ is a Banach space). Let $A \subset C(M)$. We will say that the functions in the set A are *equicontinuous* if the following condition is satisfied:

$$\forall_{\varepsilon>0}\ \exists_{\delta>0}\ \forall_{x\in A}\ \forall_{t,s\in M}\quad \Big[d(t,s) \leqslant \delta \Rightarrow \big|x(t)-x(s)\big| \leqslant \varepsilon\Big].$$

Now we will present theorem, which provides a sufficient condition for the relative compactness of a set in the space $BC(\mathbb{R}_+)$ [5, 8].

Theorem 7. *Let X be a nonempty, bounded subset of the Banach space $BC(\mathbb{R}_+)$. Assume that the functions in X are locally equicontinuous on $\mathbb{R}_+$, meaning that for each $T > 0$, the functions in X are equicontinuous on the interval $[0, T]$. Additionally, we assume that the following condition is true:*

$$\forall_{\varepsilon>0}\ \exists_{T>0}\ \forall_{x\in X}\ \forall_{t,s\in[T,\infty)}\quad \big|x(t)-x(s)\big| \leqslant \varepsilon.$$

Then the set X is relatively compact in the space $BC(\mathbb{R}_+)$.

Note that the second condition imposed in the above theorem implies that the functions in X satisfy the Cauchy condition at infinity uniformly with respect to the set X. We conclude from this that these functions converge at infinity to finite limits at a "uniform rate" [5, 8].

It is worth noting that the well-known Arzelà-Ascoli criterion for relative compactness does not apply in the space $BC(\mathbb{R}_+)$. Moreover, no definitive criterion (i.e., necessary and sufficient condition) for relative compactness is known in this space.

Let the set C be a nonempty, closed, and convex subset of the Banach space E. Recall that a *compact operator* is a continuous mapping $T : C \to C$ such that the set $T(C)$ is relatively compact in E.

Theorem 8 (Schauder's Fixed Point Theorem)**.** *If K is a nonempty, closed, and convex subset of the Banach space E, then any compact operator $T : K \to K$ has a fixed point.*

The proof of Theorem 6, without using techniques related to measures of noncompactness, is much simpler. This situation was possible because the integral equation (3) is not a quadratic integral equation. It appears that such an approach cannot be applied to the integral equation (2), as in this case, we are compelled to support our findings with the technique of measures of noncompactness. For these reasons, the results concerning integral equations (3) and (2) should be treated as independent of each other.

It is worth noting that the quadratic integral equation (2) also encompasses many other interesting, special cases, depending on the choice of the functions $g = g(t, s)$ and $u(t, x(t))$.

First, let us recall some important facts regarding the conversion of the Stieltjes integral into the Riemann integral. As mentioned earlier, if the function f is continuous and the function g has bounded variation, then the Riemann-Stieltjes integral $\int_a^b f(x)dg(x)$ exists. Now, we recall other theorems that allow the reduction of the Stieltjes integral to the Riemann integral [1, 17].

Theorem 9. *Let $g \in C^1([a, b])$ and let the function f be Riemann integrable. Then f is Stieltjes integrable with respect to the function g and we have*

$$\int_a^b f(x)dg(x) = \int_a^b f(t)g'(t)dt,$$

where the integral on the right hand side is understood as the Riemann integral.

Let $AC([a,b])$ denote the set of uniformly continuous functions on the interval $[a,b]$.

Theorem 10. *If $f \in C([a,b])$ and $g \in AC([a,b])$, then the function f is integrable in the Riemann-Stieltjes sense with respect to the function g and the following equality holds:*

$$\int_a^b f(x)dg(x) = \int_a^b f(x)g'(x)dx,$$

where g' denotes the derivative of the function g, which exists almost everywhere on the interval $[a,b]$, and the integral on the right hand side is understood as a Lebesgue integral.

Using the above theorems, we can show that the equations below are special cases of the quadratic Volterra-Stieltjes integral equation (2).
An important special case of the general integral equation (2) is the *Volterra-Chandrasekhar* integral equation, which arises in the following form:

$$x(t) = a(t) + v(t, x(t)) \int_0^t \frac{f(t, s, x(s))}{t+s} ds, \tag{4}$$

where $t \in \mathbb{R}_+$. The integral equation (4) is a generalization of the well-known Chandrasekhar equation describing the propagation of radiation. Indeed, if we take the function $g(t,s) = g : \Delta \to \mathbb{R}$ defined as

$$g(t,s) = \begin{cases} t \ln \frac{t+s}{t} & 0 < s \leqslant t \\ 0 & t = 0 \end{cases},$$

then the Volterra-Stieltjes integral equation of the form (2) is a generalization of the aforementioned integral equation [3]. In this case, we have

$$d_s g(t,s) = \Big(\frac{\partial}{\partial s} g(t,s)\Big) ds = \frac{t}{t+s} ds.$$

Another interesting case of equation (2), considered in this chapter is the fractional-order *Volterra-Liouville* integral equation of the form

$$x(t) = a(t) + \frac{v(t, x(t))}{\Gamma(\alpha)} \int_0^t \frac{f(t, s, x(s))}{(t-s)^{1-\alpha}} ds, \tag{5}$$

where $t \in \mathbb{R}_+$ and Γ denotes the Euler gamma function. Additionally, we assume that $\alpha \in (0, 1]$. If we take the function $g(t, s) = g : \Delta \to \mathbb{R}$ as

$$g(t,s) = \frac{1}{\alpha}\big[t^\alpha - (t-s)^\alpha\big],$$

then we obtain

$$d_s g(t,s) = \frac{1}{(t-s)^{1-\alpha}} ds.$$

This shows that the integral equation (5) is a special case of the integral equation

$$x(t) = a(t) + \frac{v(t, x(t))}{\Gamma(\alpha)} \int_0^t f(t, s, x(s)) d_s g(t,s), \tag{6}$$

which is a slight generalization of the integral equation (2) (cf. [3]) considered in this chapter.

The *Erdélyi-Kober* integral equation is another example of an equation that is a special case of the quadratic Volterra-Stieltjes integral equation (cf. [3]). It takes the form:

$$x(t) = a(t) + \frac{1}{\Gamma(\alpha)} \int_0^t \frac{s^{\gamma m} f(t, s, x(s))}{(t^m - s^m)^{1-\alpha}} ds, \tag{7}$$

where $m, \gamma, \alpha \in \mathbb{R}_+, \alpha \in (0, 1]$ and $t \in \mathbb{R}_+$. Additionally, the symbol Γ represents the Euler gamma function. To justify the above statement, it suffices to take $g(t, s) = g : \Delta \to \mathbb{R}$ of the form

$$g(t,s) = t^{\alpha m} - (t^m - s^m)^\alpha.$$

Then we have

$$d_s g(t,s) = \frac{\alpha m s^{m-1}}{(t^m - s^m)^{1-\alpha}} ds.$$

Another very interesting example is the *Volterra-Wiener-Hopf* integral equation (cf. [11]), which has the following form:

$$x(t) = a(t) + \int_0^t k(t-s) f(t, s, x(s)) ds, \tag{8}$$

where $t \in \mathbb{R}_+$. To show that equation (8) can be treated as a special case of the Volterra-Stieltjes integral equation, let us take the function $g(t, s) =$

$g : \Delta \to \mathbb{R}_+$ in the form

$$g(t,s) = \int_0^s k(t-z)dz.$$

Then we get

$$d_s g(t,s) = \frac{\partial}{\partial s}\Big(\int_0^s k(t-z)dz\Big)ds = k(t-s)ds.$$

This shows that the integral equation (8) is a special case of the integral equation (2).

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Summary

The aim of the chapter is to present results on the solvability for a class of nonlinear Volterra-Stieltjes integral equations, particularly quadratic ones. We are interested in finding solutions within the space of real functions that are continuous, bounded on the positive real half-axis $\mathbb{R}_+$ and converge to finite limits at infinity. Our investigation utilizes techniques related to functions of bounded variation, measures of noncompactness and the Darbo fixed point theorem.

Streszczenie

Celem niniejszego rozdziału jest przedstawienie wyników dotyczących istnienia rozwiązań dla pewnej klasy nieliniowych równań całkowych Volterry-Stieltjesa, zwłaszcza równań kwadratowych. Poszukujemy rozwiązań w przestrzeni rzeczywistych funkcji ciągłych i ograniczonych na dodatniej półosi rzeczywistej $\mathbb{R}_+$, które mają skończoną granicę w nieskończoności. W naszych rozważaniach wykorzystujemy narzędzia związane z funkcjami o ograniczonej wariacji, technikę miar niezwartości oraz twierdzenie Darbo o punkcie stałym.

Chapter 4

On some examples and counterexamples in the theory of measures of noncompactness

Szymon Dudek

1. Introduction

Measures of noncompactness are very useful tools in the theory of operator equations in the Banach spaces and Fréchet spaces ([4, 7, 11, 16, 17]). The aim of this chapter is to provide some examples and counterexamples in the theory of measures of noncompactness. We are going to systematize and consolidate topics that appear in a scattered and sporadic manner in the literature or are entirely overlooked. The chapter may be useful for researchers beginning their work with the theory of measures of noncompactness.

2. Measures of noncompactness in the Banach spaces

In this section we collect some notations, basic definitions and facts which will be used further on.

We will denote by $\mathbb{R}$ the set of real numbers and put $\mathbb{R}_+ = [0, \infty)$. Let $(E, \|\cdot\|)$ be a Banach space. If $X \subset E$, we use $\overline{X}$ and $\mathrm{Conv}X$ to denote the closure and convex closure of X, respectively. Next, let us denote by

$\mathfrak{M}_E$ the family of all nonempty and bounded subsets of E and by $\mathfrak{N}_E$ its subfamily consisting all relatively compact sets. We use the following axiomatic definition of the measure of noncompactness given in [5].

Definition 1. A mapping $\mu : \mathfrak{M}_E \to \mathbb{R}_+$ will be called *a measure of noncompactness* in E if it satisfies the following conditions:

1° The family $\ker \mu = \{X \in \mathfrak{M}_E : \mu(X) = 0\}$ is nonempty and $\ker \mu \subset \mathfrak{N}_E$.

2° $X \subset Y \Rightarrow \mu(X) \leqslant \mu(Y)$.

3° $\mu(\overline{X}) = \mu(\text{Conv}X) = \mu(X)$.

4° If (X_n) is a sequence of closed sets from $\mathfrak{M}_E$ such that $X_{n+1} \subset X_n$ for $n = 1, 2, ...$ and if $\lim_{n\to\infty} \mu(X_n) = 0$, then the set $X_\infty = \bigcap_{n=1}^{\infty} X_n$ is nonempty.

The family $\ker \mu$ described in 1° is said to be *the kernel of the measure of noncompactness* μ.
The first measure of noncompactness α_E was introduced by K. Kuratowski in 1930 ([15]). This function is defined by

$$\alpha_E(X) := \inf\{\varepsilon > 0 : X \subset \bigcup_{i=1}^{n} S_i,\ S_i \subset E,\ \text{diam}(S_i) < \varepsilon\ (i = 1, \ldots, n),\ n \in \mathbb{N}\},$$

for $X \in \mathfrak{M}_E$.
By β_E we denote the so-called Hausdorff measure of noncompactness, defined on bounded subsets of the space E. It is given by the formula

$$\beta_E(X) := \inf\{r > 0 : X \text{ has a finite } r-\text{net in } E\}.$$

This measure is often used because, in many classical infinite-dimensional Banach spaces, it can be expressed in a more transparent form (see [5, 6]). This, in turn, simplifies the calculation of the specific value of $\beta_E(X)$, in contrast to the Kuratowski's measure of noncompactness. The measure β_E was not introduced by Hausdorff but by Goldenštein and co-authors in 1957 ([13]).
In the next part of this chapter we will use the concept of r-separated set defined as follows

Definition 2. A subset X of a metric space (M, d) is called $r-$*separated* if $d(x, y) \geqslant r$ for each $x, y \in X$, $x \neq y$.

3. The space $C([0,T])$

Based on Definition 1, we can construct many measures of noncompactness in various Banach spaces ([1, 3, 4, 11]). To do this, it is needed to know the criterion of relative compactness in the given space. In the space $C([0,T])$ it is well-known Arzelá-Ascoli theorem.

Theorem 3. *A nonempty subset $X \subset C([0,T])$ is relatively compact if and only if the following conditions are satisfied:*

1° *X is bounded,*

2° *X is equicontinuous, i.e.*

$$\underset{\varepsilon>0}{\forall}\ \underset{\delta>0}{\exists}\ \underset{x\in X}{\forall}\ \underset{t,s\in[0,T]}{\forall}\ |t-s| \leqslant \delta \implies |x(t)-x(s)| \leqslant \varepsilon.$$

Based on Theorem 3, we can define a certain measure of noncompactness. To do this, let us assume that $X \in \mathfrak{M}_{C([0,T])}$. Let us choose arbitrary $\varepsilon > 0$ and $T > 0$. For $x \in X$ by $\omega^T(x,\varepsilon)$ we denote the so called *modulus of continuity* of the function x on the interval $[0,T]$, i.e.

$$\omega^T(x,\varepsilon) := \sup\{|x(t)-x(s)| : t,s \in [0,T],\ |t-s| \leqslant \varepsilon\}.$$

Next, let us put

$$\omega^T(X,\varepsilon) := \sup\{\omega^T(x,\varepsilon) : x \in X\},$$

$$\omega_0^T(X) := \lim_{\varepsilon\to 0} \omega^T(X,\varepsilon).$$

The above defined function ω_0^T is the measure of noncompactness in the space $C([0,T])$ (see [5]). Moreover $\beta_{C([0,T])}(X) = \frac{1}{2}\omega_0^T(X)$ for all $X \in \mathfrak{M}_{C([0,T])}$.

4. The space $BC(\mathbb{R}_+)$

It might seem that by taking the limit $T \to \infty$, we could naturally extend the measure ω_0^T into the space $BC(\mathbb{R}_+)$ consisting of all bounded and continuous functions on $\mathbb{R}_+$. Then we would obtain the following function

$$\omega_0^\infty(X) := \lim_{T\to\infty} \omega_0^T(X).$$

In other words

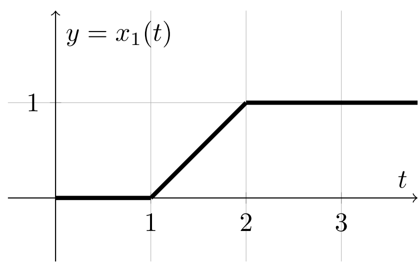
$$\omega_0^\infty(X) := \lim_{T\to\infty} \lim_{\varepsilon\to 0} \sup_{x\in X} \sup\{|x(t)-x(s)| : t,s\in[0,T],\ |t-s|\leqslant\varepsilon\}.$$

However, it is not sufficient to obtain a measure of noncompactness. The following counterexample will demonstrate that condition 1° from Definition 1 is not satisfied (i.e. $\ker\omega_0^\infty \not\subset \mathfrak{N}_{BC(\mathbb{R}_+)}$).

Example 4. Let $X = \{x_n\}_{n\in\mathbb{N}} \subset BC(\mathbb{R}_+)$ be the sequence of functions defined by the formula

$$x_n(t) := \begin{cases} 0 & \text{for } t\in[0,n) \\ t-n & \text{for } t\in[n,n+1) \\ 1 & \text{for } t\in[n+1,\infty) \end{cases}.$$

Obviously the set X is bounded ($X \in \mathfrak{M}_{BC(\mathbb{R}_+)}$). The graph of the first function of this sequence looks as follows



Then, for any $n,m\in\mathbb{N}$ ($n\neq m$), we have $\|x_n - x_m\|_\infty = 1$. Therefore, the set X is $1-$separated and contains infinitely many elements, so it is not relatively compact ($X \notin \mathfrak{N}_{BC(\mathbb{R}_+)}$).
We will show that $X \in \ker\omega_0^\infty$. Let us notice that

$$\underset{T>0}{\forall}\ \underset{n\in\mathbb{N}}{\forall}\ \underset{\varepsilon>0}{\forall}\quad 0\leqslant \omega^T(x_n,\varepsilon)\leqslant\varepsilon.$$

Taking the supremum over all $x_n\in X$ we obtain

$$0\leqslant\omega^T(X,\varepsilon)\leqslant\varepsilon.$$

Thus, if $\varepsilon\to 0$ then we get $\omega_0^T(X) = 0$ and consequently $\omega_0^\infty(X) = 0$. Finally, we have $X\in\ker\omega_0^\infty$ and $X\notin\mathfrak{N}_{BC(\mathbb{R}_+)}$ which contradicts the

condition 1° from Definition 1. Therefore, the function ω_0^∞ is not a measure of noncompactness in the space $BC(\mathbb{R}_+)$.
Nevertheless, the function ω_0^∞ can still be used to construct a measure of noncompactness in the space $BC(\mathbb{R}_+)$. It can be achieved by adding to ω_0^∞ at least one function that provides some additional information about the behavior of the set $X \in BC(\mathbb{R}_+)$ at infinity. For example, the function

$$\mu(X) := \omega_0^\infty(X) + \limsup_{t\to\infty} \operatorname{diam} X(t), \tag{1}$$

where $X(t) = \{x(t) : x \in X\}$, is the measure of noncompactness in the space $BC(\mathbb{R}_+)$ (see [1]). The kernel $\ker\mu$ of this measure consists of all sets X, that are locally equicontinuous on $\mathbb{R}_+$ and the thickness of the bundle $X(t)$ formed by the functions from X tends to zero if $t \to \infty$.
In the past, many other measures of noncompactness in the space $BC(\mathbb{R}_+)$ have been introduced. For example, the function defined by the formula

$$\mu_1(X) := \omega_0^\infty(X) + \lim_{T\to\infty} \Big\{ \sup_{x\in X} \big\{ \sup\{|x(t)| : t \geqslant T\}\big\}\Big\}, \quad X \in \mathfrak{M}_{BC}(\mathbb{R}_+)$$

is the measure of noncompactness in the space $BC(\mathbb{R}_+)$ ([7]). The kernel $\ker\mu_1$ of the measure μ_1 consists of all sets $X \in \mathfrak{M}_{BC}(\mathbb{R}_+)$ such that functions from X are locally equicontinuous on $\mathbb{R}_+$ and vanish uniformly at infinity, i.e. for any $\varepsilon > 0$ there exists $T > 0$ such that $|x(t)| \leqslant \varepsilon$ for all $x \in X$ and for any $t \geqslant T$.

5. Measures of noncompactness in the Fréchet spaces

Definition 5. ([8]) Let F is a linear space over $\mathbb{K}$, a *seminorm* is a function $p : F \to [0, \infty)$ having the properties:

(a) $p(x + y) \leqslant p(x) + p(y)$ for all $x, y \in F$.

(b) $p(\alpha x) = |\alpha|\, p(x)$ for all $\alpha \in \mathbb{K}$ and $x \in F$.

It follows from (b) that $p(0) = 0$.

In what follows, we will use the concept of a sequence of seminorms. For notational convenience, we will write $\{||\cdot||_n\}_{n\in\mathbb{N}}$ instead of $\{p_n(\cdot)\}_{n\in\mathbb{N}}$.

Definition 6. Let there be a sequence of seminorms $\{||\cdot||_n\}_{n\in\mathbb{N}}$ defined in the linear space F over the field of real numbers. If the following conditions hold:

1° $\forall_{n\in\mathbb{N}}\ \|x\|_n = 0 \Rightarrow x = \theta$,

2° $\forall_{n\in\mathbb{N}}\ \exists_{x\in F}\ \|x\|_n > 0$,

then $(F, \{||\cdot||_n\}_{n\in\mathbb{N}})$ is called a *countably normed space.* If additionally

3° the space F is complete with the metric defined by

$$d(x,y) := \sup\Big\{2^{-n}\frac{||x-y||_n}{1+||x-y||_n}\ :\ n\in\mathbb{N}\Big\},\quad x,y\in F, \tag{2}$$

then $(F, \{||\cdot||_n\}_{n\in\mathbb{N}})$ is called *the real Fréchet space.*

Note that due to the metric (2), the entire space F is bounded. Nevertheless, we will say that the subset $X \subset F$ is *bounded* if

$$\sup\{\|x\|_n : x\in X\} < \infty, \quad \text{for } n = 1,2,\ldots\ .$$

In the case of Fréchet space F it is more convenient to introduce the sequence of functions $\{\mu_n\}$ playing the role of measure of noncompactness (often called a family of measures of noncompactness). This idea was introduced in [16].
Let us provide the appropriate notions. The family of all nonempty and bounded subsets of F will be denoted by $\mathfrak{M}_F$ while its subfamily consisting of all relatively compact sets is denoted by $\mathfrak{N}_F$.

Definition 7. ([16]) A family of functions $\{\mu_n\}_{n\in\mathbb{N}}$, where $\mu_n : \mathfrak{M}_F \to [0,\infty)$, is said to be a *measure of noncompactness* in a real Fréchet space F if it satisfies the following conditions:

1° The family $\ker\{\mu_n\} := \{X \in \mathfrak{M}_F : \mu_n(X) = 0 \text{ for } n\in\mathbb{N}\}$ is nonempty and $\ker\{\mu_n\} \subset \mathfrak{N}_F$.

2° $X \subset Y \Rightarrow \mu_n(X) \leqslant \mu_n(Y)$ for $n\in\mathbb{N}$.

3° $\mu_n(\mathrm{Conv}X) = \mu_n(X)$ for $n\in\mathbb{N}$.

4° If $\{X_i\}$ is a sequence of closed sets from $\mathfrak{M}_F$ such that $X_{i+1}\subset X_i$ $(i = 1,2,...)$ and if $\lim\limits_{i\to\infty}\mu_n(X_i) = 0$ for each $n\in\mathbb{N}$, then the intersection set $X_\infty := \bigcap\limits_{i=1}^{\infty} X_i$ is nonempty.

6. The space $C(\mathbb{R}_+)$

In this section we will consider the space $C(\mathbb{R}_+)$ consisting of all continuous functions on $\mathbb{R}_+$. This space equipped with the family of seminorms

$$||x||_n := \sup\{|x(t)| : t \in [0,n]\}, \quad n \in \mathbb{N}$$

becomes the Fréchet space with the metric

$$d_C(x,y) := \sup\Big\{2^{-n}\frac{||x-y||_n}{1+||x-y||_n} : \ n \in \mathbb{N}\Big\}, \quad x,y \in C(\mathbb{R}_+). \tag{3}$$

Let us recall the compactness criterion for the subsets of the space $C(\mathbb{R}_+)$ (see [16]).

Theorem 8. *A set $X \subset C(\mathbb{R}_+)$ is relatively compact if and only if the set $X|_{[0,T]}$ is equicontinuous and uniformly bounded for each $T > 0$.*

In the Fréchet space $C(\mathbb{R}_+)$, we can introduce, similarly to the space $C([0,T])$, a family of functions $\omega_0^n : \mathfrak{M}_{C(\mathbb{R}_+)} \to [0,\infty)$ given by the formula

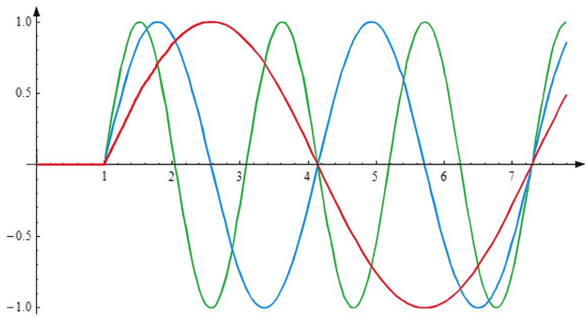
$$\omega_0^n(X) := \lim_{\varepsilon\to 0}\left\{\sup_{x\in X}\left\{\sup\{|x(t)-x(s)| : \ t,s \in [0,n], \ |t-s| \leqslant \varepsilon\}\right\}\right\}. \tag{4}$$

In [16], it was shown that this family satisfies conditions $1^\circ - 4^\circ$ from Definition 7, and thus it is a measure of noncompactness in the Fréchet space $C(\mathbb{R}_+)$. It also turns out that it is full, i.e. $\ker\{\omega_0^n\} = \mathfrak{N}_{C(\mathbb{R}_+)}$.
It is worth mentioning that the particular elements of the family $\{\mu_n\}_{n\in\mathbb{N}}$ appearing in Definition 7 do not necessarily need to be measures of noncompactness in the sense of Definition 1. We will illustrate this with the following counterexample.

Example 9. In the space $C(\mathbb{R}_+)$, consider the sequence of functions $\{x_n\}$ given by the formula

$$x_n(t) := \begin{cases} 0 & \text{for } t \in [0,1] \\ \sin n(t-1) & \text{for } t \in (1,+\infty) \end{cases}.$$

The first few terms are shown in the figure below



If we take $X = \{x_n : n \in \mathbb{N}\}$ and $\{\mu_n\} = \{\omega_0^n\}$, then $\mu_1(X) = \omega_0^1(X) = 0$, which implies $X \in \ker \mu_1$. However, based on Theorem 8, it can be shown that this is not a relatively compact set, thus $\ker \mu_1 \not\subset \mathfrak{N}_{C(\mathbb{R}_+)}$.

7. The space $C(\mathbb{R}_+, E)$

Let us consider the Fréchet space $C(\mathbb{R}_+, E)$ consisting of all functions defined and continuous on $\mathbb{R}_+$ with values in the Banach space E. The space $C(\mathbb{R}_+, E)$ is equipped with the family of seminorms $(||\cdot||_n)_{n\in N}$ given by the formula

$$||x||_n := \sup\{||x(t)||_E : t \in [0, n]\}, \quad x \in C(\mathbb{R}_+, E), \ n \in \mathbb{N}.$$

We have the following criterion in this space

Theorem 10. *A set $X \subset C(\mathbb{R}_+)$ is relatively compact if and only if the set $X|_{[0,T]}$ is equicontinuous for each $T > 0$ and $X(t)$ is a relatively compact set in E for each $t \in \mathbb{R}_+$.*

Now, we introduce a family of functions $\{\mu_n\}_{n\in\mathbb{N}}$ in the Fréchet space $C(\mathbb{R}_+, E)$ defined by the formula

$$\mu_n(X) := \omega_0^n(X) + \overline{\alpha}_n(X), \quad X \in \mathfrak{M}_{C(\mathbb{R}_+,E)}, \ n \in \mathbb{N}, \tag{5}$$

where

$$\overline{\alpha}_n(X) := \sup\{\alpha_E(X(\tau)) : \tau \in [0, n]\},$$

and ω_0^n is defined similarly to formula (4), namely

$$\omega_0^n(X) := \lim_{\varepsilon\to 0} \left\{ \sup_{x\in X} \left\{ \sup\{||x(t) - x(s)||_E : \ t, s \in [0, n], \ |t - s| \leqslant \varepsilon\}\right\}\right\}.$$

The family of mappings $\{\mu_n\}_{n\in\mathbb{N}}$ defined by (5) constitutes a measure of noncompactness in the Fréchet space $C(\mathbb{R}_+, E)$.
Let us notice that the two components of the measure μ_n correspond to the first and the second condition in Theorem 10 respectively. Therefore, if E is a finite-dimensional space, then the second component of the measure defined by formula (5) equals zero. On the other hand, if E is an infinite-dimensional Banach space, the family $\{\omega_0^n\}$ defined in (4) does not constitute a measure of non-compactness alone, and an additional component—such as $\overline{\alpha}_n$ is necessary. This follows from Kottman's theorem, which was proven in 1975 in [14].

Theorem 11 ([14]). *In every infinite-dimensional Banach space E, there exists an infinite sequence $\{e_n\} \subset B_E(\theta, 1)$ that is 1-separated.*

Let the sequence $\{e_n\} \subset B_E(\theta, 1)$ be 1-separated, meaning that for any $n,\ m \in \mathbb{N}$ $(n \neq m)$, we have $\|e_n - e_m\|_E \geqslant 1$. Thus, taking the sequence $\{x_n\} \subset C(\mathbb{R}_+, E)$ defined by $x_n(t) := e_n$ $(t \in \mathbb{R}_+)$, we obtain that for any $n,\ m \in \mathbb{N}$ $(n \neq m)$ and any $k \in \mathbb{N}$, we have

$$\frac{\|x_n - x_m\|_k}{1+\|x_n - x_m\|_k} \geqslant \frac{\|x_n(0) - x_m(0)\|_E}{1+\|x_n(0) - x_m(0)\|_E} \geqslant \frac{\|e_n - e_m\|_E}{1+\|e_n - e_m\|_E} \geqslant \frac{1}{1+1} = \frac{1}{2},$$

since the function $t \mapsto \frac{t}{1+t}$ is increasing on $\mathbb{R}_+$.
In conclusion, based on (3), we derive

$$\underset{\substack{n,\ m \in \mathbb{N},\\ n \neq m}}{\forall} \quad d_C(x_n, x_m) \geqslant \frac{1}{4}.$$

Thus, the sequence $\{x_n\}$ is $\frac{1}{4}$-separated, so its elements do not form a relatively compact set. On the other hand, we have $\omega_0^n(\{x_i\}) = 0$ for $n \in \mathbb{N}$ (because x_i are constant functions for each $i \in \mathbb{N}$), so the measure $\{\mu_n\} = \{\omega_0^n\}$ does not satisfy condition 1° from Definition 7. Hence, $\ker\{\omega_0^n\} \not\subset \mathfrak{N}_{C(\mathbb{R}_+,E)}$.

8. The space $BC(\mathbb{R}_+ \times \mathbb{R}_+)$

In recent years there have been several articles [9, 10, 18] with some error. In the above mentioned papers the space $BC(\mathbb{R}_+ \times \mathbb{R}_+)$ consisting of bounded and continuous functions defined on $\mathbb{R}_+ \times \mathbb{R}_+$, $(\mathbb{R}_+ = [0, +\infty))$ with supremum norm $\|\cdot\|_\infty$ was considered. The authors used the function

$\mu : \mathfrak{M}_{BC(\mathbb{R}_+\times\mathbb{R}_+)} \to [0,+\infty)$ defined on the bounded subsets $X \subset BC(\mathbb{R}_+ \times \mathbb{R}_+)$ and given by the formula

$$\mu(X) := \omega_0(X) + \limsup_{t,s\to\infty} \operatorname{diam} X(t,s), \tag{6}$$

where

$$\begin{aligned}\omega^T(x,\varepsilon) \;&:=\; \sup\Big\{|x(t_1,s_1)-x(t_2,s_2)| : t_1,t_2,s_1,s_2 \in [0,T],\\ &\qquad |t_1-t_2| \leqslant \varepsilon, |s_1-s_2| \leqslant \varepsilon\Big\},\end{aligned}$$

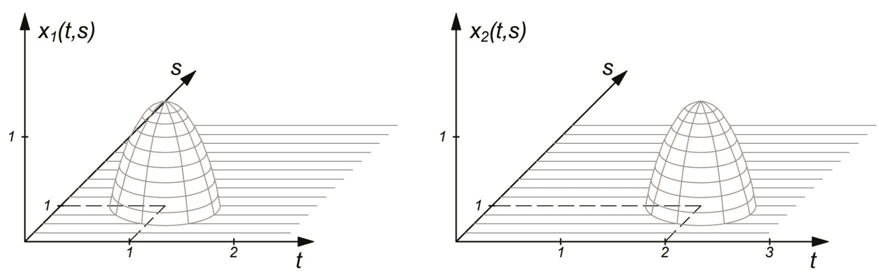
$$\begin{gathered}\omega^T(X,\varepsilon) := \sup\left\{\omega^T(x,\varepsilon) : x\in X\right\},\\ \omega_0^T(X) := \lim_{\varepsilon\to 0}\omega^T(X,\varepsilon),\\ \omega_0(X) := \lim_{T\to\infty}\omega_0^T(X),\\ \operatorname{diam} X(t,s) := \sup\{|x(t,s)-y(t,s)| : x,y\in X\}\end{gathered} \tag{7}$$

for $x \in X,\ \varepsilon > 0$ and $T > 0$.
The authors used the technique of measures of noncompactness and fixed point theorems for the function μ defined by (6). Nevertheless, the function μ is not the measure of noncompactness because it does not satisfy the condition 1° from the Definition 1. We show this in the example given below.

Example 12. ([12]) Let us consider the sequence of functions $\{x_n\} \subset BC(\mathbb{R}_+ \times \mathbb{R}_+)$ defined by the formula

$$x_n(t,s) := \begin{cases} -4\left[(t-n)^2+(s-1)^2\right]+1 & \text{for } (t-n)^2+(s-1)^2 \leqslant \frac{1}{4},\\ 0 & \text{for } (t-n)^2+(s-1)^2 > \frac{1}{4}.\end{cases}$$

The graphs of the first and second elements of this sequence have the form



Denote by X the set of elements of the sequence $\{x_n\}$, i.e. $X := \{x_n : n \in \mathbb{N}\}$. Obviously $\omega_0(X) = 0$. Moreover, for $T > 2$ and $t, s \geqslant T$ we have $\operatorname{diam} X(t,s) = 0$ and therefore $\mu(X) = 0$. On the other hand, the set X is $1-$separated in the space $BC(\mathbb{R}_+ \times \mathbb{R}_+)$ because $\|x_n - x_m\|_\infty = 1$ for $n \neq m$. This fact implies that X is not a relatively compact set.

In the paper [12] we have proposed a correct version of the measure $\mu_a : \mathfrak{M}_{BC(\mathbb{R}_+\times\mathbb{R}_+)} \to [0, +\infty)$ of type (6) given by th formula

$$\mu_a(X) := \omega_0(X) + \lim_{T\to\infty} \sup\{|x(t,s) - y(t,s)| : x, y \in X,\ \max\{t,s\} \geqslant T\}. \tag{8}$$

The function (8) is a measure of noncompactness in the space $BC(\mathbb{R}_+\times\mathbb{R}_+)$ but it is not a full measure (see [12]). In other words $\ker \mu_a \neq \mathfrak{N}_{BC(\mathbb{R}_+\times\mathbb{R}_+)}$. For example, if we consider the set $X = \{x, y\} \subset BC(\mathbb{R}_+ \times \mathbb{R}_+)$ where $x(t,s) = \sin(t^2 + s^2)$ and $y(t,s) = -\sin(t^2 + s^2)$, the second term of the function μ_a equals 2. Therefore, X is relatively compact (as a finite element set) in $BC(\mathbb{R}_+ \times \mathbb{R}_+)$ but $X \notin \ker \mu_a$. The kernel of the measure μ_a consists of all sets $X \in \mathfrak{M}_{BC(\mathbb{R}_+\times\mathbb{R}_+)}$ such that functions belonging to X are locally equicontinuous on $\mathbb{R}_+ \times \mathbb{R}_+$ and the thickness of the bundle $X(t,s) = \sup\{|x(t,s) - y(t,s)| : x, y \in X\}$ formed by functions from X, tends to zero when $\max\{t,s\} \to \infty$.

We also provided examples of two other important measures of noncompactness $BC(\mathbb{R}_+ \times \mathbb{R}_+)$ expressed by the formulas

$$\mu_b(X) := \omega_0(X) + b_\infty(X), \tag{9}$$

$$\mu_c(X) := \omega_0(X) + c_\infty(X), \tag{10}$$

where

$$b_\infty(X) := \lim_{T\to\infty} \sup\Big\{|x(t_1,s_1) - x(t_2,s_2)| : x \in X,\ \max\{t_1,s_1\} \geqslant T,\ \max\{t_2,s_2\} \geqslant T\Big\},$$

$$c_\infty(X) := \lim_{T\to\infty} \sup\{|x(t,s)| : x \in X,\ \max\{t,s\} \geqslant T\},$$

for $X \in \mathfrak{M}_{BC(\mathbb{R}_+\times\mathbb{R}_+)}$.

The measures μ_b and μ_c have also small kernels. For example, if we assume that X is the singleton $\{g\}$ where $g(t,s) = \sin t$, we will have $\mu_b(X) = b_\infty(X) = 2$ and $\mu_c(X) = c_\infty(X) = 1$. In spite of this imperfection, these measures could be valuable if we investigate the asymptotic behavior of solutions of some class of integral equations. Equivalents of these measures in the space $BC(\mathbb{R}_+)$ have often been considered in the literature (see [1,2]).

9. The space $C(\mathbb{R}_+ \times \mathbb{R}_+)$

Let us notice that the measure (4) in the space $C(\mathbb{R}_+)$ had simpler formula than the measure (1) in the space $BC(\mathbb{R}_+)$. A similar situation occurs for the spaces $C(\mathbb{R}_+ \times \mathbb{R}_+)$ and $BC(\mathbb{R}_+ \times \mathbb{R}_+)$. In the space $C(\mathbb{R}_+ \times \mathbb{R}_+)$, consisting of all continuous functions on $\mathbb{R}_+ \times \mathbb{R}_+$, we can introduce a measure $\{\omega_0^n\}_{n=1}^{\infty}$, where $\omega_0^n : \mathfrak{M}_{C(\mathbb{R}_+ \times \mathbb{R}_+)} \to [0, +\infty)$ is defined by the formula (7) (with $T = n$), i.e.

$$\omega_0^n(X) := \lim_{\varepsilon \to 0} \omega^n(X, \varepsilon),$$

where

$$\omega^n(X, \varepsilon) := \sup \{\omega^n(x, \varepsilon) : x \in X\},$$

$$\begin{aligned} \omega^n(x, \varepsilon) := \sup\{&|x(t_1, s_1) - x(t_2, s_2)| : t_1, t_2, s_1, s_2 \in [0, n], \\ &|t_1 - t_2| \leqslant \varepsilon, |s_1 - s_2| \leqslant \varepsilon\}. \end{aligned}$$

Note that $\ker\{\omega_0^n\} = \mathfrak{N}_{C(\mathbb{R}_+ \times \mathbb{R}_+)}$. This implies that the measure $\{\omega_0^n\}$ perfectly describes relatively compact sets in the Fréchet space $C(\mathbb{R}_+ \times \mathbb{R}_+)$ (in contrast to the above mentioned measures in the space $BC(\mathbb{R}_+ \times \mathbb{R}_+)$).

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Summary

The main aim of this chapter is to present some examples of measures of noncompactness and functions that are not such measures, even though it might seem otherwise. Our considerations will be conducted within selected functional Banach and Fréchet spaces. The chapter is an overview.

Streszczenie

Głównym celem niniejszego rozdziału jest przedstawienie wybranych przykładów miar niezwartości oraz funkcji, które takimi miarami nie są, choć mogłoby się wydawać inaczej. Nasze rozważania będą prowadzone w wybranych przestrzeniach funkcyjnych Banacha i Frécheta. Rozdział ma charakter przeglądowy.

Chapter 5

Existence results for quadratic Volterra–Hammerstein integral equations in Hölder spaces

Rafał Nalepa

1. Introduction

This chapter establishes the existence of solutions to a nonlinear quadratic Volterra–Hammerstein integral equation in the Hölder space. To achieve this, we employ a specially constructed measure of noncompactness, building on the criterion for relative compactness in function spaces with increments tempered by a modulus of continuity [7]. Specifically, we utilize the measure introduced in [8], defined for the Hölder space where our equation is studied. The utility of Hölder spaces for integral equations is well established, with numerous authors employing them in diverse contexts [11, 13].
This measure proves to be a powerful tool for nonlinear integral equations. We demonstrate its efficacy by obtaining an existence result for quadratic Hammerstein equations in Hölder spaces, significantly extending prior solvability theorems [1–3]. Our approach, based on Darbo's fixed point theorem combined with this sublinear measure, offers greater flexibility than methods reliant directly on Schauder's principle and compactness criteria [7].

Moreover, this chapter extends the applications in [8] by providing new existence results for nonlinear quadratic Volterra–Hammerstein equations in Hölder spaces through the developed measure.

2. Preliminaries

This section introduces essential concepts for our study. We begin with spaces of functions whose increments are tempered by a given modulus of continuity, focusing on their key properties and the characterization of compactness within these spaces. Next, we provide the axiomatic definition of measures of noncompactness [4,5,9] and state Darbo's fixed point theorem [10]. Finally, we define a specific measure of noncompactness adapted to the space $C_\omega(M)$ and establish its fundamental properties, which will be instrumental in the subsequent analysis.

2.1. The space of functions with tempered increments

Let (M,d) be a compact metric space and let $\omega : \mathbb{R}_+ \to \mathbb{R}_+$ be a modulus of continuity, that is, a nondecreasing function with $\omega(0) = 0$, $\omega(\varepsilon) > 0$ for $\varepsilon > 0$, and (typically) continuous at 0, i.e. $\omega(\varepsilon) \to 0$ as $\varepsilon \to 0$.
The space $C_\omega(M)$ consists of all real functions $x : M \to \mathbb{R}$ for which there exists $k_x > 0$ such that

$$|x(u) - x(v)| \leqslant k_x\, \omega(d(u,v)),$$

for all $u, v \in M$. Equivalently, $x \in C_\omega(M)$ if and only if

$$\sup_{u \neq v} \frac{|x(u) - x(v)|}{\omega(d(u,v))} < \infty.$$

In the space $C_\omega(M)$ we define the norm

$$\|x\| = |x(u_0)| + \sup_{u \neq v} \frac{|x(u) - x(v)|}{\omega(d(u,v))},$$

where $u_0 \in M$ is fixed. With this norm, $C_\omega(M)$ is a Banach space.

Example 1. For the modulus $\omega_H(\varepsilon) = \varepsilon^\alpha$ with $0 < \alpha \leqslant 1$, the space $H_\alpha(M) := C_{\omega_H}(M)$ consists of all Hölder continuous functions of order α, that is, all functions $x : M \to \mathbb{R}$ satisfying

$$|x(u) - x(v)| \leqslant H_x \, (d(u,v))^{\alpha},$$

for all $u, v \in M$. The corresponding norm in the space $H_\alpha(M)$ is

$$||x||_\alpha = |x(u_0)| + \sup_{u \neq v} \frac{|x(u) - x(v)|}{(d(u,v))^\alpha},$$

where $u_0 \in M$ is a fixed element.

A comprehensive treatment of the properties of function spaces with growths tempered by a modulus of continuity, including a detailed analysis of functions satisfying the Hölder condition, can be found in [7].
We now present a fundamental criterion for relative compactness in the space $C_\omega(M)$, which – together with its proof – can be found in [7].

Theorem 2. *A bounded subset $X \subset C_\omega(M)$ is relatively compact if its elements are equicontinuous with respect to ω: for every $\varepsilon > 0$, there exists $\delta > 0$ such that*

$$\frac{|x(u) - x(v)|}{\omega(d(u,v))} \leqslant \varepsilon$$

for all $x \in X$, $u, v \in M$, $u \neq v$, with $d(u,v) \leqslant \delta$.

Corollary 3. *Let $\omega_1, \omega_2 : \mathbb{R}_+ \to \mathbb{R}_+$ be moduli of continuity such that*

$$\lim_{\varepsilon \to 0} \frac{\omega_2(\varepsilon)}{\omega_1(\varepsilon)} = 0,$$

i.e., $\omega_2(\varepsilon) = o(\omega_1(\varepsilon))$ as $\varepsilon \to 0$. Then any bounded set $X \subset C_{\omega_2}(M)$ is relatively compact in $C_{\omega_1}(M)$.

This result takes a particularly useful and concrete form in the special case of Hölder spaces.

Example 4. Consider two Hölder-type moduli of continuity: $\omega_1(\varepsilon) = \varepsilon^\alpha$ and $\omega_2(\varepsilon) = \varepsilon^\gamma$ with $0 < \alpha < \gamma \leqslant 1$. Then

$$\lim_{\varepsilon \to 0} \frac{\omega_2(\varepsilon)}{\omega_1(\varepsilon)} = \lim_{\varepsilon \to 0} \varepsilon^{\gamma - \alpha} = 0,$$

since $\gamma - \alpha > 0$. Therefore, by the preceding corollary, any bounded set in $C_{\omega_2}(M) = H_\gamma(M)$ is relatively compact in $C_{\omega_1}(M) = H_\alpha(M)$.

This implies that if a set $A \subset H_\gamma(M)$ is bounded, i.e., there exists a constant $K > 0$ such that

$$|x(u) - x(v)| \leqslant K(d(u,v))^\gamma$$

for all $x \in A$ and $u, v \in M$, then A is relatively compact in the Hölder space $H_\alpha(M)$.

Remark 5. The condition in Theorem 2 is sufficient but not necessary for relative compactness. For example, consider a singleton $x \in C_\omega(M)$ satisfying $|x(u) - x(v)| = \omega(d(u,v))$ for all $u, v \in M$. This set is compact but fails to satisfy the equicontinuity condition.

2.2. Measures of noncompactness and Darbo's fixed point theorem

Let E be a real Banach space, and let $\mathfrak{M}_E$ denote the family of all nonempty bounded subsets of E. Denote by $\mathfrak{N}_E$ the subfamily of relatively compact sets.

Definition 6. A function $\mu : \mathfrak{M}_E \to \mathbb{R}_+$ is called a *measure of noncompactness* in E if the following hold:

1. The kernel $\ker\mu = \{X \in \mathfrak{M}_E : \mu(X) = 0\}$ is nonempty and satisfies $\ker\mu \subset \mathfrak{N}_E$.
2. If $X \subset Y$, then $\mu(X) \leqslant \mu(Y)$.
3. $\mu(\overline{X}) = \mu(X)$.
4. $\mu(\operatorname{Conv} X) = \mu(X)$.
5. $\mu(\lambda X + (1-\lambda)Y) \leqslant \lambda\mu(X) + (1-\lambda)\mu(Y)$ for all $\lambda \in [0,1]$.
6. If (X_n) is a decreasing sequence of closed sets in $\mathfrak{M}_E$ and $\lim_{n\to\infty} \mu(X_n) = 0$, then $\bigcap_{n=1}^{\infty} X_n \neq \emptyset$.

The set $\ker\mu$ is called the *kernel* of the measure μ.

Definition 7. Let $\mu : \mathfrak{M}_E \to \mathbb{R}_+$ be a measure of noncompactness on a Banach space E. We say that:

1. μ is *sublinear* if for all $X, Y \subset E$,

$$\mu(X + Y) \leqslant \mu(X) + \mu(Y).$$

2. μ is *homogeneous* if for all $X \subset E$ and $\lambda \in \mathbb{R}$,

$$\mu(\lambda X) = |\lambda|\mu(X).$$

3. μ has the *maximum property* if for all $X, Y \subset E$,

$$\mu(X \cup Y) = \max\{\mu(X), \mu(Y)\}.$$

4. μ is *full* if $\ker \mu = \mathfrak{N}_E$.

5. μ is *regular* if it is sublinear, has the maximum property, and is full.

Theorem 8 (Darbo's fixed point theorem)**.** *Let μ be a measure of noncompactness in E and let $\Omega \subset E$ be a nonempty, bounded, closed, and convex subset. If $F : \Omega \to \Omega$ is continuous and there exists $k \in [0, 1)$ such that*

$$\mu(FX) \leqslant k\mu(X)$$

for every nonempty $X \subset \Omega$, then F has at least one fixed point in Ω.

2.3. A measure of noncompactness in $C_\omega(M)$

Using the compactness criterion from Theorem 2, we construct a measure of noncompactness specifically adapted to the space $C_\omega(M)$. For any nonempty bounded subset $X \subset C_\omega(M)$ and $\varepsilon > 0$, define

$$\beta(x, \varepsilon) = \sup\left\{\frac{|x(u) - x(v)|}{\omega(d(u, v))} : u, v \in M,\ u \neq v,\ d(u, v) \leqslant \varepsilon\right\}$$

and

$$\beta(X, \varepsilon) = \sup_{x \in X} \beta(x, \varepsilon).$$

Since $\varepsilon \mapsto \beta(X, \varepsilon)$ is nondecreasing, the limit

$$\beta_0(X) = \lim_{\varepsilon \to 0} \beta(X, \varepsilon)$$

exists.

Theorem 9. *The function $\beta_0 : \mathfrak{M}_{C_\omega(M)} \to \mathbb{R}_+$ is a regular measure of noncompactness in $C_\omega(M)$, i.e., it is sublinear, homogeneous, has the maximum property, and is full.*

Remark 10. The measure β_0 quantifies the "local non-equicontinuity" of a function set. It vanishes precisely when the set is relatively compact, as characterized by Theorem 2.

3. Main result

We now consider the quadratic Volterra–Hammerstein type integral equation of the form

$$x(t) = p(t) + x(t) \int_0^t k(t,\tau) g(\tau, x(\tau))\, d\tau, \tag{1}$$

where $t \in I = [0,1]$.
Let us note that equations of a similar type, although studied in other function spaces or in a different research context, were considered in the works [1–3, 6, 14, 15].
We will show that equation (1) has at least one solution in the space $H_\alpha(I)$, where α is a fixed number from the interval $(0,1)$.

We will consider equation (1) under the following assumptions:

(i) There exists a number γ, with $\alpha < \gamma \leqslant 1$, such that $p \in H_\gamma(I)$. This means that there exists a constant $P_\gamma > 0$ such that the following inequality is satisfied:

$$|p(t) - p(s)| \leqslant P_\gamma |t-s|^\gamma, \quad \text{for all } t, s \in I.$$

(ii) The function $k : I \times I \to \mathbb{R}$ is continuous and there exists a constant $k_\gamma > 0$ such that for all $t, s, \tau \in I$ and for the number γ from assumption (i) the following inequality holds:

$$|k(t,\tau) - k(s,\tau)| \leqslant k_\gamma |t-s|^\gamma.$$

(iii) The function $g : I \times \mathbb{R} \to \mathbb{R}$ is continuous, and there exists a nondecreasing function $Q : \mathbb{R}_+ \to \mathbb{R}_+$ such that $Q(r) \to 0$ as $r \to 0$, and the following inequality is satisfied:

$$|g(\tau, x) - g(\tau, y)| \leqslant Q(|x-y|),$$

for all $\tau \in I$ and $x, y \in \mathbb{R}$.

(*iv*) There exists a constant $r_0 > 0$ which is a solution to the inequality

$$|p(0)| + P_\gamma + r(2\overline{K} + k_\gamma)(Q(r) + \overline{G}) \leqslant r$$

and such that

$$\overline{K}(Q(r_0) + \overline{G}) < 1.$$

Remark 11. Under assumptions *(ii)* and *(iii)*, the constants $\overline{G}$ and $\overline{K}$, defined by

$$\begin{aligned} \overline{G} &= \sup\{\, |g(\tau,0)| : \tau \in I \,\}, \\ \overline{K} &= \sup\{\, |k(t,s)| : t,s \in I \,\} \end{aligned}$$

are finite.

Theorem 12. *Under assumptions* (*i*)–(*iv*), *equation* (1) *has at least one solution in the space* $H_\alpha(I)$, *where* $0 < \alpha < 1$.

Proof of Theoerm 12. Let $x \in H_\alpha(I)$ be an arbitrary function. We consider the operator Ω defined as follows

$$(\Omega x)(t) = p(t) + x(t)\int_0^t k(t,\tau)g(\tau,x(\tau))\,d\tau$$

for $t \in I$. We will show that $\Omega x \in H_\alpha(I)$.
To this end, let us choose arbitrary $t, s \in I$. Without loss of generality, we may assume that $s < t$. Then, according to the adopted assumptions, we have

$$\begin{aligned}
&|(\Omega x)(t) - (\Omega x)(s)| \leqslant |p(t) - p(s)| \\
&+ \left| x(t)\int_0^t k(t,\tau)g(\tau,x(\tau))\,d\tau - x(s)\int_0^s k(s,\tau)g(\tau,x(\tau))\,d\tau \right| \\
&\leqslant P_\gamma |t-s|^\gamma + \left| x(t)\int_0^t k(t,\tau)g(\tau,x(\tau))\,d\tau \right. \\
&\left. -x(s)\int_0^t k(t,\tau)g(\tau,x(\tau))\,d\tau \right| \\
&+ \left| x(s)\int_0^t k(t,\tau)g(\tau,x(\tau))\,d\tau - x(s)\int_0^s k(s,\tau)g(\tau,x(\tau))\,d\tau \right|
\end{aligned}$$

$$
\begin{aligned}
&\leqslant P_\gamma |t-s|^\gamma + |x(t)-x(s)| \int_0^t |k(t,\tau)|\,|g(\tau,x(\tau))|\,d\tau \\
&+ |x(s)| \int_0^s |k(t,\tau)-k(s,\tau)||g(\tau,x(\tau))|\,d\tau \\
&+ |x(s)| \int_s^t |k(t,\tau)||g(\tau,x(\tau))|\,d\tau \leqslant P_\gamma |t-s|^\gamma \\
&+ |x(t)-x(s)| \overline{K} \left(Q(||x||_\infty) + \overline{G} \right) t \\
&+ |x(s)| k_\gamma |t-s|^\gamma \left(Q(||x||_\infty) + \overline{G} \right) s \\
&+ |x(s)| \overline{K} \left(Q(||x||_\infty) + \overline{G} \right) |t-s| \\
&\leqslant P_\gamma |t-s|^\gamma + |x(t)-x(s)| \overline{K} \left(Q(||x||_\infty) + \overline{G} \right) \\
&+ ||x||_\infty k_\gamma |t-s|^\gamma \left(Q(||x||_\infty) + \overline{G} \right) \\
&+ ||x||_\infty \overline{K} \left(Q(||x||_\infty) + \overline{G} \right) |t-s|.
\end{aligned}
$$

From the above estimation, it follows that the following inequality holds.

$$
\begin{aligned}
&\frac{|(\Omega x)(t) - (\Omega x)(s)|}{|t-s|^\alpha} \leqslant P_\gamma |t-s|^{\gamma-\alpha} \\
&+ \frac{|x(t)-x(s)|}{|t-s|^\alpha} \overline{K} \left(Q(||x||_\infty) + \overline{G} \right) \\
&+ ||x||_\infty k_\gamma \left(Q(||x||_\infty) + \overline{G} \right) |t-s|^{\gamma-\alpha} \\
&+ ||x||_\infty \overline{K} \left(Q(||x||_\infty) + \overline{G} \right) |t-s|^{1-\alpha}
\end{aligned}
\tag{2}
$$

for $t, s \in I$ and $t \neq s$. Taking into account the fact that

$$
(\Omega x)(0) = p(0),
$$

and based on (2), for any $t, s \in I$, $t \neq s$, we obtain the estimate

$$
\begin{aligned}
&|(\Omega x)(0)| + \frac{|(\Omega x)(t) - (\Omega x)(s)|}{|t-s|^\alpha} \leqslant |p(0)| + P_\gamma \\
&+ \overline{K} \left(Q(||x||_\infty) + \overline{G} \right) \frac{|x(t)-x(s)|}{|t-s|^\alpha}
\end{aligned}
$$

$$
\begin{aligned}
&+ ||x||_\infty k_\gamma \left(Q(||x||_\infty) + \overline{G}\right) + ||x||_\infty \overline{K} \left(Q(||x||_\infty) + \overline{G}\right) \\
&\leqslant |p(0)| + P_\gamma + ||x||_\alpha \overline{K} \left(Q(||x||_\infty) + \overline{G}\right) \\
&+ ||x||_\alpha \left(\overline{K} + k_\gamma\right) \left(Q(||x||_\infty) + \overline{G}\right) \\
&\leqslant |p(0)| + P_\gamma + ||x||_\alpha \left(2\overline{K} + k_\gamma\right) \left(Q(||x||_\infty) + \overline{G}\right).
\end{aligned}
$$

Therefore, we obtain that

$$
||\Omega x||_\alpha \leqslant |p(0)| + P_\gamma + ||x||_\alpha \left(2\overline{K} + k_\gamma\right) \left(Q(||x||_\infty) + \overline{G}\right). \tag{3}
$$

The inequality (3) implies that the operator Ω maps the space $H_\alpha(I)$ into itself. Moreover, from the first inequality in assumption (iv), we deduce that the operator Ω maps the ball B_{r_0} into itself, where r_0 is specifically selected to satisfy the conditions stated in (iv).
We will now show that the operator Ω is continuous on the ball B_{r_0}. To this end, fix an arbitrary function $x \in B_{r_0}$ and a number $\varepsilon > 0$. Let $\delta_1 > 0$ be such that

$$
|Q(t)| \leqslant \frac{\varepsilon}{2} \frac{1}{(2\overline{K} + k_\gamma) r_0},
$$

for $t \leqslant \delta_1$. Next, fix $y \in B_{r_0}$ such that $||x - y|| \leqslant \delta$, where

$$
\delta \leqslant \min\left\{\delta_1, \frac{\varepsilon}{2(2\overline{K} + k_\gamma)(Q(r_0) + \overline{G})}\right\}.
$$

At this stage of the proof, we apply the assumptions as stated in Theorem 12. Therefore, for all $t, s \in [0,1]$ with $t \neq s$, it follows that

$$
\begin{aligned}
&\frac{|[(\Omega x)(t) - (\Omega y)(t)] - [(\Omega x)(s) - (\Omega y)(s)]|}{|t - s|^\alpha} \\
&= \Bigg|\left[x(t) \int_0^t k(t,\tau) g(\tau, x(\tau))\, d\tau - y(t) \int_0^t k(t,\tau) g(\tau, y(\tau))\, d\tau\right] \\
&- \left[x(s) \int_0^s k(s,\tau) g(\tau, x(\tau))\, d\tau - y(s) \int_0^s k(s,\tau) g(\tau, y(\tau))\, d\tau\right]\Bigg| \, |t - s|^{-\alpha}
\end{aligned}
$$

$$
\begin{aligned}
&\leqslant \Bigg| \Bigg[x(t) \int_0^t k(t,\tau) g(\tau, x(\tau))\, d\tau - y(t) \int_0^t k(t,\tau) g(\tau, x(\tau))\, d\tau \Bigg] \\
&+ \Bigg[y(t) \int_0^t k(t,\tau) g(\tau, x(\tau))\, d\tau - y(t) \int_0^t k(t,\tau) g(\tau, y(\tau))\, d\tau \Bigg] \\
&- \Bigg[x(s) \int_0^s k(s,\tau) g(\tau, x(\tau))\, d\tau - y(s) \int_0^s k(s,\tau) g(\tau, x(\tau))\, d\tau \Bigg] \\
&- \Bigg[y(s) \int_0^s k(s,\tau) g(\tau, x(\tau))\, d\tau - y(s) \int_0^s k(s,\tau) g(\tau, y(\tau))\, d\tau \Bigg] \Bigg| \, |t-s|^{-\alpha} \\
&\leqslant \Bigg| [x(t) - y(t)] \int_0^t k(t,\tau) g(\tau, x(\tau))\, d\tau \\
&+ y(t) \int_0^t k(t,\tau) \left[g(\tau, x(\tau)) - g(\tau, y(\tau)) \right] \, d\tau \\
&- [x(s) - y(s)] \int_0^s k(s,\tau) g(\tau, x(\tau))\, d\tau \\
&-y(s) \int_0^s k(s,\tau) \left[g(\tau, x(\tau)) - g(\tau, y(\tau)) \right] \, d\tau \Bigg| \, |t-s|^{-\alpha} \\
&\leqslant \Bigg| [x(t) - y(t)] \int_0^t k(t,\tau) g(\tau, x(\tau))\, d\tau - [x(s) - y(s)] \int_0^t k(t,\tau) g(\tau, x(\tau))\, d\tau \\
&+ [x(s) - y(s)] \int_0^t k(t,\tau) g(\tau, x(\tau))\, d\tau \\
&+ y(t) \int_0^t k(t,\tau) \left[g(\tau, x(\tau)) - g(\tau, y(\tau)) \right] \, d\tau \\
&- [x(s) - y(s)] \int_0^s k(s,\tau) g(\tau, x(\tau))\, d\tau \\
&-y(s) \int_0^s k(s,\tau) \left[g(\tau, x(\tau)) - g(\tau, y(\tau)) \right] \, d\tau \Bigg| \, |t-s|^{-\alpha} \\
&\leqslant \left| [x(t) - y(t)] - [x(s) - y(s)] \right| \frac{1}{|t-s|^\alpha} \int_0^t |k(t,\tau)| |g(\tau, x(\tau))| \, d\tau \\
&+ |x(s) - y(s)| \frac{1}{|t-s|^\alpha} \int_0^s |k(t,\tau) - k(s,\tau)| \, |g(\tau, x(\tau))| \, d\tau \\
&+ |x(s) - y(s)| \frac{1}{|t-s|^\alpha} \int_s^t |k(t,\tau)| |g(\tau, x(\tau))| \, d\tau \\
&+ \Bigg| y(t) \int_0^t k(t,\tau) \left[g(\tau, x(\tau)) - g(\tau, y(\tau)) \right] \, d\tau \\
&-y(s) \int_0^s k(s,\tau) \left[g(\tau, x(\tau)) - g(\tau, y(\tau)) \right] \, d\tau \Bigg| \frac{1}{|t-s|^\alpha}
\end{aligned}
$$

$$\leqslant \frac{|[x(t)-y(t)]-[x(s)-y(s)]|}{|t-s|^{\alpha}}\overline{K}\left(Q(||x||_{\infty})+\overline{G}\right)t$$
$$+|[x(s)-y(s)]-[x(0)-y(0)]$$
$$+[x(0)-y(0)]|k_{\gamma}\frac{|t-s|^{\gamma}}{|t-s|^{\alpha}}\left(Q(||x||_{\infty})+\overline{G}\right)s$$
$$+|[x(s)-y(s)]-[x(0)-y(0)]+[x(0)-y(0)]|\,\frac{|t-s|}{|t-s|^{\alpha}}\overline{K}\left(Q(||x||_{\infty})+\overline{G}\right)$$
$$+\left|y(t)\int_0^t k(t,\tau)\left[g(\tau,x(\tau))-g(\tau,y(\tau))\right]\,d\tau\right.$$
$$\left.-y(s)\int_0^t k(t,\tau)\left[g(\tau,x(\tau))-g(\tau,y(\tau))\right]\,d\tau\right|\frac{1}{|t-s|^{\alpha}}$$
$$+\left|y(s)\int_0^t k(t,\tau)\left[g(\tau,x(\tau))-g(\tau,y(\tau))\right]\,d\tau\right.$$
$$\left.-y(s)\int_0^s k(s,\tau)\left[g(\tau,x(\tau))-g(\tau,y(\tau))\right]\,d\tau\right|\frac{1}{|t-s|^{\alpha}}$$
$$\leqslant ||x-y||_{\alpha}\overline{K}\left(Q(||x||_{\infty})+\overline{G}\right)$$
$$+\left[\sup\{|[x(t)-y(t)]-[x(s)-y(s)]|:t,s\in I,\ t\neq s\}\right.$$
$$\left.+|x(0)-y(0)|\right]k_{\gamma}|t-s|^{\gamma-\alpha}\left(Q(||x||_{\infty})+\overline{G}\right)$$
$$+\left[\sup\{|[x(t)-y(t)]-[x(s)-y(s)]|:t,s\in I,\ t\neq s\}\right.$$
$$\left.+|x(0)-y(0)|\right]\overline{K}\left(Q(||x||_{\infty})+\overline{G}\right)|t-s|^{1-\alpha}$$
$$+\frac{|y(t)-y(s)|}{|t-s|^{\alpha}}\int_0^t|k(t,\tau)|Q(|x(\tau)-y(\tau)|)\,d\tau$$
$$+|y(s)|\frac{1}{|t-s|^{\alpha}}\int_0^s|k(t,\tau)-k(s,\tau)|Q(|x(\tau)-y(\tau)|)\,d\tau$$
$$+|y(s)|\frac{1}{|t-s|^{\alpha}}\int_s^t|k(t,\tau)|Q(|x(\tau)-y(\tau)|)\,d\tau$$
$$\leqslant ||x-y||_{\alpha}\overline{K}\left(Q(||x||_{\infty})+\overline{G}\right)$$
$$+\left[|x(0)-y(0)|+\sup\left\{\frac{|[x(t)-y(t)]-[x(s)-y(s)]|}{|t-s|^{\alpha}}:t,s\in I,\ t\neq s\right\}\right.$$
$$\left.\cdot\sup\{|t-s|^{\alpha}:t,s\in I\}\right]k_{\gamma}\left(Q(||x||_{\infty})+\overline{G}\right)$$
$$+\left[|x(0)-y(0)|+\sup\left\{\frac{|[x(t)-y(t)]-[x(s)-y(s)]|}{|t-s|^{\alpha}}:t,s\in I,\ t\neq s\right\}\right.$$
$$\left.\cdot\sup\{|t-s|^{\alpha}:t,s\in I\}\right]\overline{K}\left(Q(||x||_{\infty})+\overline{G}\right)+||y||_{\alpha}\overline{K}Q(||x-y||_{\infty})t$$

$$
\begin{aligned}
&+ ||y||_\infty k_\gamma |t-s|^{\gamma-\alpha} Q(||x-y||_\infty) s + ||y||_\infty \overline{K} Q(||x-y||_\infty)|t-s|^{1-\alpha} \\
&\leqslant ||x-y||_\alpha \overline{K} \left(Q(||x||_\alpha) + \overline{G}\right) + ||x-y||_\alpha k_\gamma \left(Q(||x||_\alpha) + \overline{G}\right) \\
&+ ||x-y||_\alpha \overline{K} \left(Q(||x||_\alpha) + \overline{G}\right) + ||y||_\alpha \overline{K} Q(||x-y||_\alpha) \\
&+ ||y||_\alpha k_\gamma Q(||x-y||_\alpha) + ||y||_\alpha \overline{K} Q(||x-y||_\alpha) \\
&\leqslant \delta \overline{K} \left(Q(r_0) + \overline{G}\right) + \delta k_\gamma \left(Q(r_0) + \overline{G}\right) + \delta \overline{K} \left(Q(r_0) + \overline{G}\right) \\
&+ r_0 \overline{K} Q(\delta) + r_0 k_\gamma Q(\delta) + r_0 \overline{K} Q(\delta) \\
&= \left(2\overline{K} + k_\gamma\right) \left(Q(r_0) + \overline{G}\right) \delta + \left(2\overline{K} + k_\gamma\right) r_0 Q(\delta). \qquad (4)
\end{aligned}
$$

It is clear from the definition of the operator Ω that

$$
|(\Omega x)(0) - (\Omega y)(0)| = 0. \qquad (5)
$$

On the basis of (4) and (5), we obtain the following estimate

$$
||\Omega x - \Omega y||_\alpha \leqslant \left(2\overline{K} + k_\gamma\right) \left(Q(r_0) + \overline{G}\right) \delta + \left(2\overline{K} + k_\gamma\right) r_0 Q(\delta).
$$

From the above estimate and condition (iii), it follows that the operator Ω is continuous on the ball B_{r_0}.

Now, let us fix an arbitrary nonempty subset X of the ball B_{r_0}. Take $\varepsilon > 0$ and any $x \in X$. By (2), for any $t, s \in I$ such that $|t-s| \leqslant \varepsilon$, the following inequality holds

$$
\begin{aligned}
\frac{|(\Omega x)(t) - (\Omega x)(s)|}{|t-s|^\alpha} &\leqslant P_\gamma \varepsilon^{\gamma-\alpha} + \overline{K} \left(Q(r_0) + \overline{G}\right) \beta(x, \varepsilon) \\
&+ r_0 k_\gamma \left(Q(r_0) + \overline{G}\right) \varepsilon^{\gamma-\alpha} + r_0 \overline{K} \left(Q(r_0) + \overline{G}\right) \varepsilon^{1-\alpha}.
\end{aligned}
$$

Therefore, the following estimate is valid

$$
\begin{aligned}
\beta(\Omega x, \varepsilon) \leqslant\; & P_\gamma \varepsilon^{\gamma-\alpha} + \overline{K} \left(Q(r_0) + \overline{G}\right) \beta(x, \varepsilon) \\
&+ r_0 \overline{K} \left(Q(r_0) + \overline{G}\right) \varepsilon^{1-\alpha} + r_0 k_\gamma \left(Q(r_0) + \overline{G}\right) \varepsilon^{\gamma-\alpha}.
\end{aligned}
$$

This means that

$$
\beta_0(\Omega X) \leqslant \overline{K} \left(Q(r_0) + \overline{G}\right) \beta_0(X).
$$

Therefore, based on the second inequality in condition (iv) and Darbo's theorem, we conclude that equation (1) has at least one solution in the space $H_\alpha(I)$ belonging to the ball B_{r_0}. □

4. Application

Let us consider the following quadratic Volterra–Hammerstein type integral equation

$$x(t) = t^2 + x(t) \int_0^t \frac{1}{40} \cdot \frac{1+\tau^2}{1+t^2+\tau^2} \cdot \sqrt[3]{1+\tau^2 x^2(\tau)}\, d\tau, \quad t \in I = [0,1]. \quad (6)$$

Observe that equation (6) is a particular case of equation (1) with the following choices of functions

$$\begin{aligned} p(t) &:= t^2, \\ k(t,\tau) &:= \frac{1}{40} \cdot \frac{1+\tau^2}{1+t^2+\tau^2}, \\ g(\tau, x) &:= \sqrt[3]{1+\tau^2 x^2}. \end{aligned}$$

We will now verify that the assumptions of Theorem 12 hold.

Assumption (i): The function $p(t) = t^2$ belongs to the Hölder space $H_1(I)$, as it satisfies the Lipschitz condition

$$|p(t) - p(s)| = |t^2 - s^2| = |t-s|\,|t+s| \leqslant 2|t-s|,$$

for all $t, s \in I = [0,1]$. We may take $\gamma = 1$ and $P_\gamma = 2$.

Assumption (ii): The kernel $k(t,\tau)$ is continuous on $I \times I$, and for any fixed $\tau \in I$, the mapping $t \mapsto k(t,\tau)$ satisfies the Lipschitz condition with respect to t. Indeed, one can compute the partial derivative

$$\left|\frac{\partial k(t,\tau)}{\partial t}\right| = \left|\frac{-2t(1+\tau^2)}{40(1+t^2+\tau^2)^2}\right| \leqslant \frac{2(1+\tau^2)}{40(1+t^2+\tau^2)^2} \leqslant \frac{2}{40} = \frac{1}{20},$$

for all $t, \tau \in I$. Thus, by the mean value theorem, for all $t, s, \tau \in I$, we have

$$|k(t,\tau) - k(s,\tau)| \leqslant k_\gamma |t-s|,$$

where we may take

$$k_\gamma = \frac{1}{20}.$$

Moreover, the constant

$$\overline{K} = \sup_{t,\tau \in I} |k(t,\tau)| \leqslant \frac{1}{40}.$$

Assumption (iii): The nonlinearity $g(\tau, x) = (1 + \tau^2 x^2)^{1/3}$ is continuous on $I \times \mathbb{R}$. To estimate the difference $|g(\tau, x) - g(\tau, y)|$, we apply the inequality (cf. [12])

$$|(a + u^q)^{1/p} - (a + v^q)^{1/p}| \leqslant |u - v|^{q/p}, \quad \text{for } 1 \leqslant q < p,$$

with $a = 1$, $u = \tau|x|$, $v = \tau|y|$, $p = 3$, and $q = 2$. Then,

$$\begin{aligned} |g(\tau, x) - g(\tau, y)| &= \left|(1 + \tau^2 x^2)^{1/3} - (1 + \tau^2 y^2)^{1/3}\right| \\ &\leqslant |\tau|x| - \tau|y||^{2/3} \leqslant \tau^{2/3}|x - y|^{2/3} \leqslant |x - y|^{2/3}, \end{aligned}$$

so condition (iii) is satisfied with $Q(r) = r^{2/3}$. Moreover,

$$\overline{G} = \sup_{\tau \in I} |g(\tau, 0)| = 1.$$

Assumption (iv): We now check that there exists a number $r_0 > 0$ satisfying the inequalities

$$|p(0)| + P_\gamma + r_0(2\overline{K} + k_\gamma)(Q(r_0) + \overline{G}) \leqslant r_0 \quad \text{and} \quad \overline{K}(Q(r_0) + \overline{G}) < 1.$$

Substituting the values $p(0) = 0$, $P_\gamma = 2$, $\overline{K} = \frac{1}{40}$, $k_\gamma = \frac{1}{20}$, and $Q(r) = r^{2/3}$, the first inequality becomes

$$2 + r_0(2 \cdot \tfrac{1}{40} + \tfrac{1}{20})(r_0^{2/3} + 1) = 2 + r_0 \cdot \tfrac{1}{10}(r_0^{2/3} + 1).$$

Evaluating this expression for $r_0 = 17$

$$2 + \frac{17}{10}(17^{2/3} + 1) \approx 14.94.$$

The second condition is also satisfied:

$$\overline{K}(Q(17) + \overline{G}) = \frac{1}{40}(6.571 + 1) \approx 0.189 < 1.$$

Therefore, all the assumptions of Theorem 12 are satisfied, and we conclude that equation (6) has at least one solution in the Hölder space $H_\alpha(I)$ for any $\alpha \in (0,1)$.

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Summary

Based on a criterion for relative compactness in spaces of functions with tempered increments, we present a measure of noncompactness by giving its definition and fundamental properties. The main focus of this chapter is the proof of existence of solutions to a nonlinear quadratic Volterra–Hammerstein integral equation in the Hölder space, utilizing this measure. The results complement and extend previously established facts in this area.

Streszczenie

Na podstawie kryterium relatywnej zwartości w przestrzeni funkcji o utemperowanych przyrostach przedstawiamy miarę niezwartości, podając jej formułę oraz podstawowe własności. Głównym celem tego rozdziału jest dowód istnienia rozwiązań nieliniowego kwadratowego równania całkowego typu Volterry–Hammersteina w przestrzeni funkcji spełniajacych warunek Höldera, wykorzystujący tę miarę. Wyniki uzupełniają i rozszerzają wcześniej ustalone fakty w tej dziedzinie.

Chapter 6

A Lyapunov-type inequality for a third-order boundary value problem with nonlocal conditions of integral type

Josefa Caballero Mena, Kishin Sadarangani, Rayco Toledo

1. Introduction and preliminaries

The well-known Lyapunov inequality [1] says that if $h \in C([a,b])$ and $x(t)$ is a nontrivial solution of the following boundary value problem with Dirichlet conditions

$$\begin{cases} x''(t) + h(t)x(t) = 0, & a \leqslant t \leqslant b, \\ x(a) = x(b) = 0, \end{cases}$$

then

$$\int_a^b |h(s)|ds > \frac{4}{b-a}, \tag{1}$$

where the constant 4 is sharp, this is, it cannot be replaced by a larger number.

A great number of extensions and generalizations of Lyapunov's inequality for other classes of boundary value problems have appeared in the literature (see for example [2–8], among others and the references therein).

Particularly in [4], the authors obtained a Lyapunov-type inequality for the following third-order boundary value problem

$$\begin{cases} x'''(t) + h(t)x(t) = 0, & a \leqslant t \leqslant b, \\ x(a) = x(b) = x(c) = 0, & \text{with } a < c < b. \end{cases}$$

Particularly, they proved that if the above boundary value problem has a solution $x(t)$ such that $x(t) \neq 0$ for any $t \in (a, c) \cup (c, b)$, then

$$\int_a^b |h(t)|dt > \frac{4}{(b-a)^2}.$$

More recently, in [7] the authors considered the same above mentioned third-order boundary value problem and they obtained that

$$\int_a^b |h(t)|dt > \frac{8}{(b-a)^2}.$$

In this chapter, our main purpose is to obtain a Lyapunov-type inequality associated to the following third-order boundary value problem

$$\begin{cases} x'''(t) + h(t)x(t) = 0, & a \leqslant t \leqslant b, \\ x(a) = x(b) = \int_a^b x(\xi)\ d\xi = 0, \end{cases} \tag{2}$$

where $h : [a, b] \to \mathbb{R}$ is a continuous function.
Before presenting the following section, we need the following result which appears in [9].

Proposition 1 ([9]). *If $h : [a, b] \to \mathbb{R}$ is a continuous function, then Problem (2) has a unique solution given by*

$$x(t) = \int_a^b G(t, s)h(s)ds,$$

where

$$G(t,s) = \begin{cases} \frac{(a-s)^2(b-t)\left(-(b-a)^2+(t-a)(2(b-s)+(b-a))\right)}{2(b-a)^3}, & a \leqslant s \leqslant t \leqslant b, \\ \\ \frac{(b-s)^2(t-a)\left((b-a)^2-(b-t)(2(s-a)+(b-a))\right)}{2(b-a)^3}, & a \leqslant t \leqslant s \leqslant b. \end{cases} \tag{3}$$

2. Main results

Our starting point in this section is the following estimate of the Green's function appearing in Proposition 1.

Proposition 2. *The Green's function $G(t,s)$ satisfies*

$$|G(t,s)| \leqslant 2(b-a)^2, \quad \textit{for any } t,s \in [a,b].$$

Proof.

- If $a \leqslant s \leqslant t \leqslant b$

$$\begin{aligned}
|G(t,s)| &= \frac{1}{2(b-a)^3}\Big|(a-s)^2(b-t)\Big|\Big|-(b-a)^2+(t-a)(2(b-s)+(b-a))\Big| \\
&\leqslant \frac{1}{2(b-a)^3}(a-b)^2(b-a)((b-a)^2+(b-a)(2(b-a)+(b-a))) \\
&= \frac{1}{2}\cdot 4(a-b)^2 = 2(b-a)^2.
\end{aligned}$$

- If $a \leqslant t \leqslant s \leqslant b$

$$\begin{aligned}
|G(t,s)| &\leqslant \frac{1}{2(b-a)^3}(a-b)^2(b-a)((b-a)^2+(b-a)(2(b-a)+(b-a))) \\
&= \frac{1}{2}\cdot 4(a-b)^2 = 2(b-a)^2.
\end{aligned}$$

This finishes the proof. □

Theorem 3 (Lyapunov-type inequality)**.** *If Problem (2), where $h : [a,b] \to \mathbb{R}$ is a continuous function, has a nontrivial solution, then*

$$\frac{1}{2(b-a)} \leqslant \int_a^b |h(s)|ds.$$

Proof. By Proposition 1, we know that the solution $x(t)$ to Problem (2) satisfies the following integral equation

$$x(t) = \int_a^b G(t,s)h(s)x(s)ds.$$

From this, we get

$$\begin{aligned} |x(t)| &\leqslant \int_a^b |G(t,s)||h(s)||x(s)|ds \\ &\leqslant \|x\|_\infty \left(\int_a^b |G(t,s)||h(s)|ds \right) \\ &\leqslant 2(b-a)^2 \|x\|_\infty \left(\int_a^b |h(s)|ds \right), \end{aligned}$$

where we have used Proposition 2.
Therefore, we infer

$$\|x\|_\infty \leqslant 2(b-a)^2 \|x\|_\infty \left(\int_a^b |h(s)|ds \right)$$

and, taking into account that x is a nontrivial solution, i.e. $\|x\|_\infty \neq 0$, we obtain

$$\frac{1}{2(b-a)^2} \leqslant \int_a^b |h(s)|ds,$$

and this gives us the desired result. □

Next, we apply this result to the eigenvalue problem corresponding to our Problem (2).

Corollary 4. *Suppose that λ is an eigenvalue of the problem*

$$\begin{cases} x'''(t) + \lambda x(t) = 0, & t \in [a,b], \\ x(a) = x(b) = \int_a^b x(\xi)d\xi = 0, \end{cases} \tag{4}$$

then

$$\frac{1}{2(b-a)^3} \leqslant |\lambda|.$$

Proof. This problem is a particular case of Problem (2) with $h(t) = \lambda$. Since λ is an eigenvalue, we can find a nontrivial solution x_λ to Problem (4). Then, applying Theorem 3, we obtain

$$\frac{1}{2(b-a)^2} \leqslant |\lambda|(b-a),$$

or equivalently,

$$\frac{1}{2(b-a)^3} \leqslant |\lambda|.$$

This finishes the proof. □

In [9], the authors obtained the following estimation of the Green's function $G(t,s)$ defined in (3).

Proposition 5 ([9])**.** *The Green's function satisfies*

$$\int_a^b |G(t,s)|ds \leqslant \frac{5}{96}(b-a)^3, \quad \textit{for all } t \in [a,b].$$

By using this estimate we have the following result.

Lemma 6. *Suppose that Problem (2), where $h : [a,b] \to \mathbb{R}$ is a continuous function, has a nontrivial solution, then*

$$\frac{96}{5(b-a)^3} \leqslant \|h\|_\infty.$$

Proof. By Proposition 1, we know that the solution to Problem (2) is given by

$$x(t) = \int_a^b G(t,s)h(s)x(s)ds.$$

Hence, for any $t \in [a,b]$, we have that

$$\begin{aligned} |x(t)| &\leqslant \int_a^b |G(t,s)||h(s)||x(s)|ds \leqslant \|x\|_\infty \|h\|_\infty \left(\int_a^b |G(t,s)|ds \right) \\ &\leqslant \frac{5}{96}(b-a)^3 \|x\|_\infty \|h\|_\infty, \end{aligned}$$

where we have applied Proposition 5.
Therefore,

$$\|x\|_\infty \leqslant \frac{5}{96}(b-a)^3 \|x\|_\infty \|h\|_\infty$$

and, since $\|x\|_\infty \neq 0$, we infer that

$$\frac{96}{5(b-a)^3} \leqslant \|h\|_\infty.$$

□

Lemma 6 gives us a new lower bound of the eigenvalues of Problem (4), which we present in the following Corollary.

Corollary 7. *Suppose that λ is an eigenvalue to Problem (4) then*

$$\frac{96}{5(b-a)^3} \leqslant |\lambda|.$$

Proof. By using the same reasoning that the proof of Theorem 3 with $h(t) = \lambda$ and applying Proposition 5, we obtain the desired result. □

Remark 8. Notice that Corollary 7 improves the bound of the eigenvalues obtained in Corollary 4.

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Summary

In this chapter, we derive a Lyapunov-type inequality for a third-order boundary value problem with nonlocal and integral boundary conditions. As an application of our result, we get a lower bound for the eigenvalues associated to the corresponding boundary value problem.

Streszczenie

W tym rozdziale wyprowadzamy nierówność typu Lapunowa dla zagadnienia brzegowego trzeciego rzędu z nielokalnymi i całkowymi warunkami brzegowymi. Jako zastosowanie naszego wyniku, otrzymujemy dolne oszacowanie dla wartości własnych związanych z odpowiadającym zagadnieniem brzegowym.

Chapter 7

Some aspects of affine differential geometry in higher codimensions

Paweł Witowicz

1. Introduction

Affine differential geometry, in general, investigates properties of immersions of submanifolds into flat affine spaces which are invariant under action of affine groups on the affine spaces. In particular, in equiaffine geometry the group of affine transformations which preserve the volume is considered. The manifolds are studied with affine connections which come from the flat connection of the ambient space by a projections along so called transversal spaces. A transversal space is a generalization of the normal complement of a vector subspace. A connection ∇ is called equiaffine if there exists a parallel volume form θ on M, that is $\nabla\theta = 0$. For comparison, in Riemannian geometry of submanifolds the transversal bundle is the normal bundle. This geometry studies objects and quantities of submanifolds which are invariant under action of isometry groups of the ambient space. Such quantities include lengths and angles. Affine geometry concerns such properties as volumes or shapes without preserving proportions and metric quantities. In this short chapter we recall some research work done in affine geometry in the case of codimension greater than one. We purposely omit an issue of centroaffine geometry. We concentrate here on canonical constructions of transversal bundles, important in affine geometry especially in higher codimensions. We also underline classification results of manifolds having particular invariant properties.

2. Affine immersions

In affine geometry of higher codimensions we consider an immersion $f : M^n \to \mathbb{R}^{n+p}$ of an n-dimensional manifold M^n into the affine space $\mathbb{R}^{n+p}$ for $p > 1$. The affine space $\mathbb{R}^{n+p}$ is endowed with the standard flat affine connection D. In order to do differential geometry we need an affine connection on M^n which we can obtain after choosing a transversal bundle on M^n. A vector bundle σ over M^n is called transversal if for every $x \in M^n$ $\mathbb{R}^{n+p} = f_*(T_xM^n) \oplus \sigma_x$. A given transversal bundle induces an affine connection ∇ on M^n such that $f_*(\nabla_X Y)$ is a tangent part of $D_X f_* Y$ in the above direct sum where X and Y are vector fields tangent to M^n. If a bundle σ is given, f is called an affine immersion with the induced connection ∇ (see [10]). The transversal part of $D_X f_* Y$ is called the affine fundamental form or the second fundamental form and we will denote it by $h(X, Y)$. Mapping $(X, Y) \mapsto h(X, Y)$ is a bilinear form on M^n with its values in σ. In the rest of the chapter we will omit pushforwarding of vector fields by f because of the local character of considerations. Thus we have

$$D_X Y = \nabla_X Y + h(X, Y). \tag{1}$$

One of the main requirements of a transversal bundle is its independence of a basis of vector fields in TM^n. Such canonical construction should also be equiaffine in the sense that there is a volume form on M^n which is parallel with respect to the induced connection. The immersion itself should fulfill certain conditions of regularity, called also nondegeneracy, at least to avoid a reduction of codimension. The nondegeneracy engages only the second fundamental form and cannot depend on a transversal bundle. We then define transversal connection $\nabla^\perp$ in the following way: for a transversal vector field ξ and a tangent vector X, $\nabla^\perp_X \xi$ is the transversal part of $D_X \xi$ whereas the tangent part is $-S_\xi X$ where S_ξ is called the shape operator with respect to ξ. S_ξ is an endomorphism of TM^n. We will write

$$D_X \xi = -S_\xi X + \nabla^\perp_X \xi. \tag{2}$$

Then we can define the cubic form C: $C(X,Y,Z)=\nabla^{\perp}_X h(Y,Z)-h(\nabla_X Y,Z)-h(Y,\nabla_X Z)$. When a local basis $\xi_1,...,\xi_p$ is fixed, we define bilinear forms h^i, one forms θ^i_j, $(1,1)$ tensors S_i and trilinear forms C^i for $i,j=1,...,p$ such that:

$$h(X,Y)=\sum_i h^i(X,Y)\xi_i, \tag{3}$$

$$D_X\xi_i=-S_iX+\nabla^{\perp}_X\xi_i, \tag{4}$$

$$\nabla^{\perp}_X\xi_j=\sum_i \tau^i_j(X)\xi_i, \tag{5}$$

$$C(X,Y,Z)=\sum_i C^i(X,Y,Z)\xi_i \tag{6}$$

for arbitrary tangent vector fields X,Y,Z.
The following classical equations are fulfilled (see [10], [11]). They are called the Gauss (7), Codazzi ((8), (9)) and Ricci (10) equations, satisfied for all i,j.:

$$R(X,Y)Z=S_{h(Y,Z)}X-S_{h(X,Z)}Y, \tag{7}$$

$$(\nabla_X h^i)(Y,Z)+\sum_j \tau^i_j h^j(Y,Z) \quad \text{are symmetric in } X,Y,Z, \tag{8}$$

$$(\nabla_X S_i)Y-(\nabla_Y S_i)X=\sum_j(-\tau^j_i(Y)S_jX+\tau^j_i(X)S_jY), \tag{9}$$

$$h^i(X,S_jY)-h^i(Y,S_jX)=d\tau^i_j(X,Y)+\sum_k(\tau^k_j(Y)\tau^i_k(X)-\tau^k_j(X)\tau^i_k(Y)). \tag{10}$$

3. Surfaces in $\mathbb{R}^4$

3.1. Nomizu-Vrancken theory

Before 1992 two canonical transversal bundles were well-known for surfaces in $\mathbb{R}^4$. The first one was defined by Burstin and Mayer in 1927 in [1] and the second one by Klingenberg in 1951 in [5] and [6]. Both constructions assume the same notion of regularity (or nondegeneracy) of a symmetric bilinear form. We recall this definition. Let $u=(X_1,X_2)$ be a pair of local, linearly independent tangent vector fields on the surface M^2 and Det denotes the usual determinant in $\mathbb{R}^4$. Then a bilinear form, defined locally on M^2 is given by

$$G_u(Y,Z) = \mathrm{Det}(X_1, X_2, D_Y X_1, D_Z X_2) + \mathrm{Det}(X_1, X_2, D_Z X_1, D_Y X_2). \quad (11)$$

The property that G_u is nondegenerate does not depend on the frame u so the nondegeneracy of G_u defines well a nondegeneracy of the surface. This property also does not depend on a transversal bundle. Then a bilinear symmetric form g_u is defined by

$$g_u(Y,Z) = \frac{G_u(Y,Z)}{(det_u G_u)^{\frac{1}{3}}}, \quad (12)$$

where $det_u G_u = G_u(X_1,X_1)G_u(X_2,X_2) - (G_u(X_1,X_2))^2$. In fact, g_u does not depend on u so it will be denoted by g. It is commonly called the affine metric.

In [11] Nomizu and Vrancken gave detailed constructions of Burstin-Mayer and Klingenberg transversal bundles and proved that they are not always equiaffine on nondegenerate surfaces - the volume element ω_g determined by the affine metric is not always parallel with respect to the induced connection ∇, that is $\nabla\omega_g$ is not equal to zero.

In the same article they defined a new, equiaffine transversal bundle. To develop their theory they needed - for a tangent g-orthonormal frame X_1, X_2 in an initial arbitrary transversal bundle - a specially chosen transversal frame ξ_1, ξ_2 such that

$$\mathrm{Det}(X_1, X_2, \xi_1, \xi_2) = 1 \quad (13)$$

and bilinear forms h^1 and h^2, applied to the tangent frame have the following matrices:

$$h^1 = \begin{bmatrix} 1 & 0 \\ 0 & \pm 1 \end{bmatrix}, \quad h^2 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}. \quad (14)$$

The sign in the matrix of h^1 is negative for definite affine metric and positive for indefinite one. Such transversal basis exists and is unique.

Next, a metric $g^\perp$ in transversal bundle is defined so that the frame ξ_1, ξ_2 be orthogonal. The Nomizu-Vrancken bundle is equiaffine and unique provided that a natural symmetry condition $\nabla^\perp g^\perp = 0$ holds. It exists for all nondegenerate surfaces. It is often called the affine normal bundle. The article [11] contains also a classification result: if the cubic forms C^1 and C^2 vanish on M^2 then the surface is locally either a product of two planar

curves or a complex paraboloid.
In the article [4] another advantage of Nomizu-Vrancken bundle is shown. A variation of area functional is studied for perturbation of a surface in affine normal directions. The result obtained there is that the area of the surface is extremal under such variations with compact support if and only if the trace of the shape operator S_ξ vanishes for all ξ contained in the affine normal bundle. This property justifies the choice of the affine normal bundle as a right one. We mention here that a consequence of vanishing of the traces of all shape operators is that the affine mean curvature vector H vanishes. This vector is defined using a fixed $g^\perp$-orthonormal frame ξ_1, ξ_2 by

$$H = \frac{1}{2}((trace\ S_{\xi_1})\xi_1 + (trace\ S_{\xi_2})\xi_2) \tag{15}$$

but is independent of the frame.
We refer to a few research articles which characterize surfaces in $\mathbb{R}^4$ equipped with Nomizu-Vrancken transversal bundle satisfying various properties.
In [18] Vrancken showed that the property that a surface in $\mathbb{R}^4$ has planar geodesic with respect to the induced connection does not depend on the choice of transversal bundle among three bundles mentioned above. There are two surfaces, up to an affine transformation, which have this property. They have the following parametrizations:

$$x(u,v) = (u, v, \frac{1}{2}(u^2 - v^2), uv), \tag{16}$$

$$y(u,v) = (u, u^2, v, v^2). \tag{17}$$

Surfaces with flat induced connection with flat normal connection are described in [9]. They include any complex curve and any product of two planar curves.
Surfaces with indefinite affine metric which are affine maximal and affine harmonic are classified in [2]. They can be parametrized as

$$x(u,v) = (u, \frac{1}{2}u^2, P_1(u) + v, P_2(u) + \frac{1}{2}v^2) \tag{18}$$

where P_1 and P_2 are arbitrary smooth functions.
Umbilical immersions are those which have each shape operator S_ξ proportional to the identity. Such surfaces are also called affine spheres. As above mentioned articles show, there is rich family of affine spheres in comparison with the Euclidean case. Umbilical surfaces with Nomizu-Vrancken

transversal bundle are studied in [7] and [15]. In [7] we find

Theorem 1. *Let M be an affine umbilical definite surface in $\mathbb{R}^4$ such that $\nabla^\perp g^\perp = 0$. Then M is equivalent to an open part of either*

$$x(u,v) = (u, v, \frac{1}{2}(u^2 - v^2), uv), \quad or \tag{19}$$

$$x(u,v) = (\frac{3}{4}vu^{\frac{4}{3}} + \frac{1}{9}v^3, u^{\frac{4}{3}} + \frac{4}{9}v^2, v, \frac{3}{4}u^2). \tag{20}$$

There is also the following characterization.

Theorem 2. *Let M be an affine definite umbilical surface in $\mathbb{R}^4$. If M has constant curvature with respect to the affine metric then M is affine equivalent to an open part of the complex paraboloid*

$$x(u,v) = (u, v, \frac{1}{2}(u^2 - v^2), uv). \tag{21}$$

It is also proved there that an affine definite umbilical surface in $\mathbb{R}^4$ cannot have the affine mean curvature vector of constant non-zero length with respect to $g^\perp$.
In [16] Vrancken characterized the above mentioned complex paraboloid as the only surface in $\mathbb{R}^4$ which is both affine extremal and Euclidean minimal.

3.2. Burstin-Mayer bundle

The Burstin-Mayer construction of an affine metric as well as a transversal bundle was recalled and appreciated in [22]. It was also applied to finding surfaces which have planar geodesics with respect to the affine metric ([27] with later erratum).
There is also another approach to surfaces in $\mathbb{R}^4$ presented by Wang in [23] and [24], where equiaffine structures depend on a given connection in the tangent bundle of a surface. Thus the Burstin-Mayer bundle and Klingenberg bundle are equiaffine in this theory. However, there are no further research papers developing this idea.

3.3. Locally strictly convex surfaces in $\mathbb{R}^4$

A submanifold M^n of $\mathbb{R}^{n+p}$ is called locally strictly convex at a point $p \in M^n$ if there is a hyperplane π_p which is tangent to M^n at p, having a non-degenerate contact of order one with M^n and there is an open

neighborhood U of p in M^n such that $U \setminus \{p\}$ lies on one side of π_p. We call such π_p a non-singular support hyperplane of U. The contact of order one at p means that the vector of the second derivative of any regular curve on M^n passing through p is transversal to π_p at p.
A theory leading to a construction of a suitable affine metric and equiaffine transversal bundle in the case of locally strictly convex surfaces in $\mathbb{R}^4$ was formulated by Nuño-Ballesteros and Sánchez in [13]. Some basic details are also explained in [28]. A need for a new approach follows from a few observations. One of them is the fact that the Burstin-Mayer affine metric is indefinite for strictly convex surfaces but if such a surface is contained in another strictly convex hypersurface N of $\mathbb{R}^4$, the affine metric of N is definite. A theory describing the affine metric and the affine normal for hypersurfaces is presented in [10]. Moreover, if a surface M is contained in a hypersurface N of $\mathbb{R}^4$, the affine normal plane to M does not always contain the affine normal vector to N.
We sketch briefly the theory contained in [13]. We start from an arbitrary transversal bundle. One can observe that a given non-singular support hyperplane can be spanned by a vector in transversal plane and the tangent plane. Such a vector can be extended to a global vector field ξ having the same property in every point. In the next step we fix a point p of M and a local tangent frame $\mathfrak{u} = \{X_1, X_2\}$ around it. Then there exists a local transversal field ξ in a neighbourhood U of p such that X_1, X_2 and ξ span a non-singular hyperplanes supporting locally M in every point of U. Moreover, ξ can be chosen in such a way that a bilinear form

$$G_{\mathfrak{u}}(Y, Z) = [X_1, X_2, D_Y Z, \xi] \tag{22}$$

is positive definite. Such a field ξ is called a metric field and is not unique. However it can be extended on M if M is orientable. The following symmetric bilinear form g_ξ turns out to be independent of the frame $\mathfrak{u}$ up to sign:

$$g_\xi(Y, Z) = \frac{G_{\mathfrak{u}}(Y, Z)}{(det_{\mathfrak{u}} G_{\mathfrak{u}})^{1/4}}. \tag{23}$$

We will abbreviate the notation writing g instead of g_ξ. Analogically to Nomizu-Vrancken construction, a special transversal frame is needed which is described below.

Theorem 3. *[[13], Theorem 3.7] Let $M \subset \mathbb{R}^4$ be a locally strictly convex surface and ξ a metric field. Let $\mathfrak{u} = \{X_1, X_2\}$ be a local orthonormal tangent* *frame of* g_ξ *and let* σ *be an arbitrary transversal plane bundle. Then there exists a unique local frame* $\{\xi_1, \xi_2\}$ *of* σ*, such that* $Det(X_1, X_2, \xi_1, \xi_2) = 1$, $h^1(X_1, X_1) = 0$, $\xi + \xi_1$ *is tangent to* M, $h^2(X_1, X_1) = 1$, $h^2(X_1, X_2) = 0$ *and* $h^2(X_2, X_2) = 1$ *(see* (3)*).*

The transversal frame described above is applied to definition of a metric $g^\perp$ in the transversal bundle. It is uniquely determined by:

$$g^\perp(\xi_1, \xi_1) = 1, \tag{24}$$

$$g^\perp(\xi_1, \xi_2) = -\frac{1}{2} h_1(X_2, X_2), \tag{25}$$

$$g^\perp(\xi_2, \xi_2) = 4h_1(X_1, X_2)^2 + \frac{5}{4} h_1(X_2, X_2)^2. \tag{26}$$

It does not depend on the choice of g-othonormal frame X_1, X_2. The uniqueness of transversal frames is possible when the following conditions are fulfilled.

Definition 4. Let $\mathfrak{u} = \{X_1, X_2\}$ be a g-orthonormal tangent frame. An equiaffine plane bundle σ is symmetric, if

$$\begin{gathered}(\nabla g)(X_2, X_1, X_1) - (\nabla g)(X_1, X_2, X_1) = 0, \\ (\nabla g)(X_1, X_2, X_2) - (\nabla g)(X_2, X_1, X_2) = 0\end{gathered} \tag{27}$$

and antisymmetric, if

$$\begin{aligned}&(\nabla g)(X_2, X_1, X_1) + (\nabla g)(X_1, X_2, X_1) = 0, \\ &(\nabla g)(X_1, X_2, X_2) + (\nabla g)(X_2, X_1, X_2) = 0.\end{aligned} \tag{28}$$

Here an equiaffine bundle means that the induced connection is compatible with the volume element ω_g, that is $\nabla \omega_g = 0$. The following statement guarantees the uniqueness of these transversal bundles. A point is an inflection where the second fundamental forms h^1 and h^2 are colinear.

Theorem 5 ([13], Theorems 5.4 and 5.5)**.** *Let* $M \subset \mathbb{R}^4$ *be a locally strictly convex surface and* ξ *be a metric field. If* $p \in M$ *is not an inflection, then there exists a unique antisymmetric equiaffine plane bundle and a unique symmetric equiaffine plane bundle defined on a neighborhood of* p*.*

An advantage of the above theory is shown in the following theorem.

Theorem 6. *Let $M \subset \mathbb{R}^4$ be a locally strictly convex surface immersed in a hyperquadric N. Then the affine normal field to N belongs to both the antisymmetric and symmetric equiaffine plane bundles of M, with respect to the Blaschke metric $\mathfrak{G}$ restricted to M.*

Here the affine normal field is the Blaschke transversal field (presented for example in [10]). There is a classification result concerning the above theory:

Theorem 7 ([28])**.** *Let M be a surface contained in a locally strictly convex hyperquadric N in $\mathbb{R}^4$. Let ξ be a metric field on M such that the metric g_ξ coincides with the Blaschke metric on N restricted to M. Let σ be the equiaffine antisymmetric transversal bundle with respect to g_ξ. Assume that the following conditions are satisfied: M has no inflection points; $\nabla^\perp g_\xi^\perp = 0$, where $\nabla^\perp$ is the normal connection induced by σ and $g_\xi^\perp$ is a metric in σ associated with ξ and the cubic form K vanishes on M. Then, M is locally affinely equivalent to an open part of a Clifford torus, a product of an ellipse and a hyperbola or a product of an ellipse and a parabola.*

Another transversal bundle A for locally strictly convex surfaces in $\mathbb{R}^4$ was constructed by Nuño-Ballesteros, Saia and Sánchez in [12]. It is symmetric in the sense of (27) but not always equiaffine. If X_1, X_2 is g-orthonormal tangent frame and ξ_1, ξ_2 - a transversal frame obtained in Theorem 3, then a family of affine distance functions $\Delta : \mathbb{R}^4 \times M \to \mathbb{R}$ is defined by:

$$\Delta(x,p) := \mathrm{Det}(X_1, X_2, \xi_1, p - x). \tag{29}$$

It has the following property.

Theorem 8. *For given $x \in \mathbb{R}^4$ the affine distance function $p \mapsto \Delta(x,p)$ has a singularity in p_0 if and only if $x - p_0$ belongs to the affine normal plane A_{p_0}.*

4. Degenerate surfaces in $\mathbb{R}^4$

We only highlight here an important classification result for degenerate affine metric, obtained by Scharlach and Vrancken in [20]. The authors considered those surfaces which have rank one affine metric (defined by (12)) and are linearly full (i.e. the set of all vectors $h(X,Y)$ fills a plane for X, Y in a tangent plane so the codimension cannot be reduced). They obtained the following

Theorem 9. *Every 1-degenerate parallel affine surface immersion x in $\mathbb{R}^4$ is a ruled surface and can be locally parametrized either by*

$$x(u,v) = \gamma'(u) + v\gamma(u) \quad or \tag{30}$$

$$x(u,v) = (\epsilon\gamma(u) + \gamma''(u)) + v\gamma'(u), \epsilon = \pm 1, \quad or \tag{31}$$

$$x(u,v) = \alpha(u) + v\beta(u),\ \beta'' = -\beta, \tag{32}$$

where α, β and γ are curves in $\mathbb{R}^4$ such that $\mathrm{Det}(\gamma, \gamma', \gamma'', \gamma''') \neq 0$ *and* $\mathrm{Det}(\alpha'', \alpha', \beta', \beta) \neq 0$.

Moreover, a transversal bundle is specified in each of the three above cases because the property that the immersion is parallel ($\nabla h = 0$) depends on the transversal bundle. The details are explained in [20].

5. Affine spheres

An umbilical submanifold M^n of $\mathbb{R}^{n+p}$ (that is, every shape operator S_ξ is proportional to the identity) is a proper affine sphere if for every point $x \in M^n$ there is a transversal vector ξ at x such that $S_\xi \neq 0$. It is an improper affine sphere if for every transversal vector ξ, $S_\xi = 0$.
There is a nice geometric interpretation of affine spheres, valid for all codimensions and arbitrary transversal bundles, expressed in the following theorem was proved by Klingenberg in [5] .

Theorem 10. *An affine sphere M^n in $\mathbb{R}^{n+p}$ is proper if all transversal p-dimensional spaces intersect along a $(p-1)$-dimensional affine subspace of the ambient space. An affine sphere is improper if all the transversal planes are parallel.*

6. Other dimensions and codimensions

A canonical equiaffine transversal bundle for surfaces in $\mathbb{R}^5$ was constructed in [3] by Decruyenare, Dillen, Verstraelen and Vrancken. There is a classification result in this setting presented in [8] where Magid and Vrancken obtained three surfaces in $\mathbb{R}^5$ having vanishing cubic form, depended on the rank of the Ricci tensor *Ric* of the induced connection ∇. The Ricci tensor $Ric(X,Y)$ is the trace of a mapping $V \mapsto R(Y,V)X$ where X, Y, V are tangent vectors and R is the curvature tensor of ∇. For vanishing Ricci tensor the result is

Theorem 11. *Let $f : M^2 \to \mathbb{R}^5$ be a nondegenerate affine surface immersed in $\mathbb{R}^5$ with zero cubic form and zero Ricci tensor. Then, locally, the image of M^2 is an affine motion of*

$$(x_1, x_2, \frac{1}{2}x_1^2, x_1x_2, \frac{1}{2}x_2^2).$$

Surfaces in $\mathbb{R}^5$ are a special case of n-dimensional manifolds and their immersions in $\mathbb{R}^{n+p}$ with $p = \frac{n(n+1)}{2}$. Such a number p is maximal in the sense that it is the maximal dimension of the vector space generated by values of the second fundamental form h at any point. A general theory for such immersions, including the theory of surfaces in $\mathbb{R}^5$ was invented by Sasaki and presented in [14]. An immersion $M^n \to \mathbb{R}^{n+p}$ for $p = \frac{n(n+1)}{2}$ is nondegenerate if the second fundamental form h generates p-dimensional vector space $\mathcal{N}_x$ at every point $x \in M^n$. This form h defines a map $\alpha_x : S^2(T_xM^n) \to \mathcal{N}_x$ and a cubic form defines a map $C_x : S^2(T_xM^n) \to \mathcal{N}_x$ where $S^k(T_xM^n)$ is the symmetric tensor product. Since the immersion is nondegenerate, α_x is a bijection. Then a mapping χ_x is defined by

$$\chi_x := \mathrm{tr}(\alpha_x^{-1} \circ C_x) \circ \alpha_x^{-1} \tag{33}$$

where "tr" denotes the contraction. Then it is shown that there is the unique transversal bundle with fibers $\mathcal{N}_x$ such that $\xi_x = 0$ for every $x \in M^n$. Such a bundle is called the affine normal bundle. Moreover the obtained immersion is equaffine - there exists a volume form which is parallel with respect to the induced connection. An algebraic basis for ideas of this work is a notion of an inverse set of a set of matrices, invented by Weise in 1930's and elaborated in [21]. Sasaki also showed that a volume functional engaging the above mentioned volume form is critical over all variations in affine transversal directions if and only if the trace of the shape operator S_ξ vanishes for all transversal ξ. Finally, he obtained a classification result:

Theorem 12. *Let $f : M^n \to \mathbb{R}^{n+p}$, $p = \frac{n(n+1)}{2}$, be a nondegenerate immersion and assume that the cubic form relative to the affine normal bundle vanishes everywhere. Then the immersion is locally projectively equivalent to the Veronese immersion given by*

$$f(x^1, ..., x^n) = (x^1, ..., x^n, (x^1)^2, x^1x^2, x^1x^3, ..., x^1x^n, (x^2)^2, x^2x^3, ..., (x^n)^2). \tag{34}$$

The last result was developed by Vrancken in [19] where all immersions

with vanishing cubic form are explicitly listed. The list include so called generalized Veronese immersions.
In [29] it was shown that in the case of maximal codimension the following three conditions for a nondegenerate immersion are equivalent: the immersion is parallel, all the geodesics with respect to the induced connection are planar and the affine normal sections of M^n are planar. Thus the work of Vrancken provides the complete classification of affine immersions with planar geodesics in the maximal codimension. The most general case was elaborated by Wiehe in [25]. His notion of nondegeneracy of an immersion $f : M^n \to \mathbb{R}^{n+p}$ for $1 \leqslant p \leqslant \frac{n(n+1)}{2}$, named a regularity, is a modification of the approach of Weise from [21]. We sketch the idea of the regularity. For a fixed a point $x \in M^n$ let $(X_1, ..., X_n)$ be a local tangent frame in its neighborhood and $(\xi_1, ...\xi_p)$ a local transversal frame. Then $h(X_i, X_j) = \sum_\rho h^\rho_{ij}\xi_\rho$ defines bilinear forms $h^1, ..., h^p$ acting on $T_xM^n \times T_xM^n$ locally. For $m, n \in \mathbb{N}$ and $i_1, ..., i_n, j_1, ..., j_n \in \{1, ..., m\}$ define

$$\delta^{i_1 \dots i_n}_{j_1 \dots j_n} = \begin{cases} 1 & \text{if } (i_1, ..., i_n) \text{ is an even permutation of } (j_1, ..., j_n), \\ -1 & \text{if } (i_1, ..., i_n) \text{ is an odd permutation of } (j_1, ..., j_n), \\ 0 & \text{otherwise.} \end{cases}$$

In shortened notation $\delta^{i_1 \dots i_n} = \delta^{i_1 \dots i_n}_{1 \dots n}$ and $\delta_{j_1 \dots j_n} = \delta^{1 \dots n}_{j_1 \dots j_n}$. A determinant of a $(0, q)$-tensor a on an n-dimensional vector space V is defined in the following way (we use the Einstein summation convention):

$$\det(a) = \delta^{r_{21} \dots r_{2n}} \cdots \delta^{r_{q1} \dots r_{qn}} a_{1 r_{21} \dots r_{q1}} \cdots a_{n r_{2n} \dots r_{qn}}.$$

We also define the components of the classical adjoint $adj(a)$ of a:

$$adj(a)^{j_1 \dots j_q} = \frac{1}{(n-1)!} \delta^{j_1 r_{12} \dots r_{1n}} \cdots \delta^{j_q r_{q2} \dots r_{qn}} a_{r_{12} \dots r_{q2}} \cdots a_{r_{1n} \dots r_{qn}}$$

A $(0, 2p)$-tensor $\tilde{h}$ is locally defined by $\tilde{h}_{i_1 j_1 \dots i_p j_p} = \sum_{\beta \in S_p} \operatorname{sgn}(\beta) h^{\beta(1)}_{i_1 j_1} ... h^{\beta(p)}_{i_p j_p}$ where the sum is over all permutations. Then we take a determinant of $\tilde{h}$. The immersion is said to be regular if this determinant never vanishes. The notion of regularity does not depend of a choice of the transversal bundle and local tangent frame. In [25] Wiehe introduces a unique transversal equiaffine bundle on M^n using so called pseudo-inverses H^{ij}_ρ $(i, j = 1, ..., n, \rho = 1, ..., p)$

of second fundamental forms h^ρ. They satisfy the conditions

$$H^{ir}_\gamma h^\gamma_{rj} = p\delta^i_j, \tag{35}$$

$$H^{ij}_\rho h^\gamma_{ij} = n\delta^\gamma_\rho. \tag{36}$$

The transversal bundle defined by Wiehe is determined in the above mentioned local bases by the following system of equations:

$$H^{ij}_\rho C^\rho_{kij} = 0 \tag{37}$$

for every $k = 1, ..., n$ where C^ρ_{kij} are the components of the cubic forms with respect to the frame $X_1, ..., X_n$.
In the same paper it is also proved that vanishing of all the shape operators is equivalent to the fact that the immersion is a critical point of a volume functional under affine normal variations.
We note that Wiehe applied his construction to study of quadrics in [26].

We remark here that the transversal bundle constructed by Wiehe is a generalization of codimensional one case as well as the case of surfaces in $\mathbb{R}^4$. It is mentioned in [25] that the Wiehe notion of regularity in the case of the maximal codimension is the same as in Sasaki's work [14]. However, it is not clear if in this case both transversal bundles coincide.

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Summary

Century long developments of affine differential geometry in higher codimensions are briefly described, particularly concerning surfaces in four-dimensional space. The most important concepts and constructions are described and exemplary theorems, particularly of classification character, presented.

Streszczenie

Przedstawiony został skrótowo rozwój afinicznej geometrii różniczkowej wyższych kowymiarów na przestrzeni prawie wieku, ze szczególnym uwzględnieniem powierzchni w przestrzeni czterowymiarowej. Opisane są najważniejsze pojęcia i konstrukcje tej geometrii wraz z przykładami twierdzeń, szczególnie o charakterze klasyfikacyjnym.

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