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PREFACE

Agnieszka Chlebowicz, Tomasz Zając

*This book is dedicated to Professor Józef Banaś, an outstanding mathematician
and the creator of the Rzeszów school of nonlinear analysis,
on the occasion of his 75th birthday.*

This multi-author monograph consists of six independent chapters and presents selected results in functional analysis (Chapters 1–5) as well as complex analysis (Chapter 6). Some of its authors, to a greater or lesser extent, are or have been associated with Professor Józef Banaś. The text is presented in a self-contained manner and provides references to enable the interested reader to pursue further studies.

In the first chapter, "*Existence of Diametrically Complete Sets with Empty Interior in Reflexive Banach Spaces*", M. Walczyk proves that in every infinite-dimensional and reflexive Banach space there exists an equivalent norm under which one can find a diametrically complete set with empty interior.

In the next chapter, "*New Retraction Constant for the Class of Separable Banach Spaces Containing an Isometric Copy of c_0* ", D. Kapitan shows the interesting result relative to the constant of the Lipschitzian retraction from the closed unit ball onto its boundary in infinite-dimensional Banach space. He proves that optimal retraction constant of the separable Banach space containing an isometric copy of c_0 does not exceed the value $4(1 + \sqrt{2})^2$. This fact complements a well-known and important theorem stating that in every infinite-dimensional Banach space there exists a Lipschitz retraction from the closed unit ball onto the unit sphere.

M. Malec in the chapter *"Isometries Between Subspaces of Codimension k of the Space $C([1, \omega k])$ "* recalls the result from the paper by E. Casini, E. Miglierina and Ł. Piasecki published in 2024. This result says that all ℓ_1 -preduals X such that the ℓ_1 standard basis has a finite amount of $\sigma(\ell_1, X)$ -cluster points are located among subspaces of the spaces $C([1, \omega k])$ of codimension k ($C([1, \omega k])$ is the space of continuous functions on the ordinal interval $[1, \omega k]$ equipped with the order topology).

In the chapter *"On a Certain Renorming of ℓ_2 "*, B. Piątek examines the renormings of the Hilbert space ℓ_2 of square-summable sequences and gives a positive answer to the question of whether a certain renorming of the space ℓ_2 has the fixed point property.

In the chapter *"Selected Geometric Properties of Interpolation Spaces"* J. Markowicz considers a general discrete and abstract method of interpolation based on a k -functional to obtain an interpolation space. She examines which among the selected properties of Banach spaces are preserved when passing to an interpolation space.

Agnieszka Wiśniowska-Wajnryb in the last chapter *"Majorization of Derivatives for Ma-Minda Type of Convex Functions"* introduces the problem of majorization of derivatives and determines the radius of majorization of derivatives for majorants from general class of Ma-Minda convex functions.

The editors thank the authors of the chapters for their cooperation throughout the entire editorial process. The editors also thank the reviewers for their comments and suggestions, which have contributed to improving the quality of the presented monograph. The editors would also like to thank the staff of the Publishing House of Rzeszów University of Technology for their help in the timely publication of this book. Finally, the editors extend their sincere thanks to our colleagues from the Department of Nonlinear Analysis at Rzeszów University of Technology, Agnieszka Dubiel, Rafał Nalepa, and Szymon Dudek, who have made a significant contribution to the completion of this publishing project.

Chapter 1

EXISTENCE OF DIAMETRICALLY COMPLETE SETS WITH EMPTY INTERIOR IN REFLEXIVE BANACH SPACES

Mariola Walczyk

1. Introduction

The definition of a diametrically complete set was introduced in 1911 by E. Meissner ([37]). Observe that some infinite-dimensional Banach spaces are known to contain diametrically complete sets with empty interior. In [38] J. P. Moreno, P. L. Papini and R. R. Phelps gave an example of a diametrically set with empty interior in c_0 and an example of a subset of $C([0, 1])$ which is diametrically complete and is contained in a hyperplane. In [35] E. Maluta and P. L. Papini gave a sufficient condition for the existence of diametrically complete sets with empty interior in infinite-dimensional and reflexive Banach spaces. They also exhibited the example of such a set in the Banach space $E_{\sqrt{2}}$. In 2017, Maluta extended these results by presenting a reflexive LUR Banach space which contains a diametrically complete set with empty interior ([34]). Recent studies on diametrically complete sets have concentrated on three principal research directions:

- characterizations of diametrically complete sets in classical Banach spaces and characterizations of families of convex sets associated with them,
- the determination of the smallest, with respect to set inclusion, diametrically complete set containing a given set,
- the investigation of the existence of diametrically complete sets with empty interior.

In this chapter we consider the third topic. This is a survey of results of establishing the existence, under a suitable equivalent renorming, of diametrically complete sets with empty interior in infinite-dimensional, reflexive Banach spaces. Since in Section 5 we introduce a new norm $\|\cdot\|_{\alpha,\beta,p,\mathcal{F}}$ we get a few new results and proofs.

In Section 2 we introduce notations and recall definitions and results from the geometry of Banach spaces, which will be applied later in the chapter.

Section 3 is devoted to recalling the definition and basic properties of diametrically complete sets.

The theorem proved in Section 4, concerning the existence of an equivalent norm with the Opial and Kadec–Klee properties in every separable Banach space, will be one of the main tools in solving our central problem.

In Section 5, we construct the new Day-type norm $\|\cdot\|_{\alpha,\beta,p,\mathcal{F}}$, which will be used to define the norm in the proof of our fundamental Theorem 47.

In Section 6, we prove the existence of a diametrically complete set with empty interior in suitably renormed (by the Day-type norm) infinite-dimensional, separable and reflexive Banach spaces (Theorem 48).

In Section 7, we demonstrate how our problem, initially posed in infinite-dimensional reflexive Banach spaces, can be reduced, using results of A. R. Lovaglia ([33]) and S. L. Troyanski ([47]), to the case of infinite-dimensional, separable, reflexive Banach spaces. In this way, we completely resolve the problem of the existence of diametrically complete sets with empty interior in reflexive Banach spaces.

2. Basic notions and facts

This section is devoted to collect notations, definitions and some facts from the geometry of Banach spaces.

In our chapter, we will consider Banach spaces over the field of real numbers $\mathbb{R}$. If necessary, instead of the norm $\|\cdot\|$ in the Banach space X , we will write $\|\cdot\|_X$.

Throughout the chapter, we assume that Γ is an infinite set, and by $c_0(\Gamma)$ we denote the Banach space consisting of functions $u : \Gamma \rightarrow \mathbb{R}$, $u = \{u^\gamma\}_{\gamma \in \Gamma}$ such that for every $\epsilon > 0$, the set $\{\gamma \in \Gamma : |u^\gamma| \geq \epsilon\}$ is finite. In the space $c_0(\Gamma)$, the classical supremum norm is denoted by $\|\cdot\|_{c_0(\Gamma)}$, i.e.

$$\|u\|_{c_0(\Gamma)} := \sup_{\gamma \in \Gamma} |u^\gamma|$$

for $u = \{u^\gamma\}_{\gamma \in \Gamma} \in c_0(\Gamma)$. When $\Gamma = \mathbb{N}$, we usually write c_0 instead of $c_0(\mathbb{N})$. The support of a function u , i.e. the set $\{\gamma \in \Gamma : u^\gamma \neq 0\}$ will be denoted by $N(u)$.

If we fix any number $1 \leq p < +\infty$, then the space consisting of all functions $u \in c_0(\Gamma)$ such that $\sum_{\gamma \in N(u)} |u^\gamma|^p < \infty$ is denoted by the symbol $\ell^p(\Gamma)$. The norm in this space is given by the formula

$$\|u\|_{\ell^p} := \left(\sum_{\gamma \in N(u)} |u^\gamma|^p \right)^{\frac{1}{p}}$$

for $u = \{u^\gamma\}_{\gamma \in \Gamma} \in \ell^p(\Gamma) \setminus \{0\}$ and $\|0\|_{\ell^p} = 0$. The space $\ell^p(\mathbb{N})$ is traditionally denoted by ℓ^p . In further considerations, for $1 < p < \infty$, when unnecessary to specify, we will write $\|\cdot\|_p$ instead of the symbol $\|\cdot\|_{\ell^p}$.

We denote by $C([0, 1], \mathbb{R})$ the space of all functions $f : [0, 1] \rightarrow \mathbb{R}$ that are continuous on the interval $[0, 1]$. In the space $C([0, 1], \mathbb{R})$ we have the classical norm

$$\|f\|_C := \max\{|f(t)| : t \in [0, 1]\}$$

for $f \in C([0, 1], \mathbb{R})$.

In the chapter $\ell^\infty := \{u = (u^n)_{n=1}^\infty : \sup_{n \in \mathbb{N}} |u^n| < \infty\}$ with the standard norm $\|u\|_\infty := \sup_{n \in \mathbb{N}} |u^n|$ for every $u = (u^n)_{n=1}^\infty \in \ell^\infty$.

We now recall some definitions and theorems from the geometry of Banach spaces, which will be used in this chapter.

Definition 1 ([48]). For a Banach space $(X, \|\cdot\|_X)$ and a fixed element

$z \in X$ with $\|z\|_X = 1$, the function $\delta_z : [0, 2] \rightarrow [0, 2]$ defined by

$$\delta_z(\epsilon) := \inf \left\{ 1 - \frac{1}{2} \|x + y\|_X : \|x\|_X \leq 1, \|y\|_X \leq 1, x - y = \epsilon z \right\}$$

is called the modulus of convexity of $(X, \|\cdot\|_X)$ in the direction z .

If $\delta_z(\epsilon) > 0$ for all $\epsilon > 0$, then $(X, \|\cdot\|_X)$ is said to be uniformly convex in the direction z .

If $\delta_z(\epsilon) > 0$ for all $\epsilon > 0$ and all $z \in X$ with $\|z\| = 1$, then $(X, \|\cdot\|_X)$ is said to be uniformly convex in every direction.

In [48] V. Zizler proved the following results.

Proposition 2. *Suppose that $(X, \|\cdot\|_X)$ and $(Y, \|\cdot\|_Y)$ are Banach spaces, T is a linear continuous one-to-one mapping of X into Y . Assume that $(Y, \|\cdot\|_Y)$ is uniformly convex in every direction $Tz/\|Tz\|_Y$, $z \in X$ and $\|z\|_X = 1$. Then X has an equivalent norm $\| \cdot \|_X$ given by*

$$\| \|x\| \|_X = \sqrt{\|x\|_X^2 + \|Tx\|_Y^2}$$

for $x \in X$, which is uniformly convex in every direction.

Corollary 3. *Assume that a Banach space $(X, \|\cdot\|_X)$ has a bounded sequence of functionals $\{f_i^*\}_i$ in $(X^*, \|\cdot\|_{X^*})$ which separates the points in $(X, \|\cdot\|_X)$ and that an equivalent norm in X is given by*

$$\| \|x\| \|_X = \sqrt{\|x\|_X^2 + \sum_{i=1}^{\infty} \left(\frac{f_i^*(x)}{2^i} \right)^2}$$

for $x \in X$. Then $(X, \| \cdot \|_X)$ is uniformly convex in every direction.

We also introduce the notion of locally uniformly convex space, that will be crucial in our later considerations.

Definition 4 ([33]). We say that a Banach space $(X, \|\cdot\|_X)$ is *locally uniformly convex* (LUR) if for each $x \in X$, every sequence $\{x_n\}_n$ with $\lim_n \|x_n\|_X = \|x\|_X$ and $\lim_n \|x + x_n\|_X = 2\|x\|_X$ tends strongly to x . In this case we also say that the norm $\|\cdot\|_X$ is LUR.

A. R. Lovaglia proved the following theorem, which plays an important role in the proof of the main result of this chapter, namely Theorem 51.

Theorem 5 ([33]). *Let the Banach spaces $(X_1, \|\cdot\|_1)$ and $(X_2, \|\cdot\|_2)$ be locally uniformly convex (LUR). If on $X := X_1 \times X_2$ the norm $\|\cdot\|$ is defined by*

$$\|x\| = \|(x^1, x^2)\| := \sqrt{\|x^1\|_1^2 + \|x^2\|_2^2}$$

for $x = (x^1, x^2) \in X$, then $(X, \|\cdot\|)$ is also LUR.

In the proof of Theorem 51, we will also apply Troyansky's theorem, which we present in a weaker form that is nevertheless sufficient for our purposes.

Theorem 6 ([47]). *Every reflexive Banach space admits an equivalent locally uniformly convex norm.*

The following properties of Banach spaces are related to weakly convergent sequences.

Definition 7 ([42], [21]). A Banach space $(X, \|\cdot\|_X)$ has the *Opial property* if for each weakly null sequence $\{x_n\}_n$ and each point $x \neq 0$ in X , we have

$$\limsup_{n \rightarrow \infty} \|x_n\|_X < \limsup_{n \rightarrow \infty} \|x_n - x\|_X.$$

A Banach space $(X, \|\cdot\|_X)$ is said to have the *non-strict Opial property* if for each weakly null sequence $\{x_n\}_n$ and each point $x \in X$, we have

$$\limsup_{n \rightarrow \infty} \|x_n\|_X \leq \limsup_{n \rightarrow \infty} \|x_n - x\|_X.$$

In 1982 D. van Dulst proved the following result.

Theorem 8 ([18]). *Every infinite-dimensional and separable Banach space $(X, \|\cdot\|_X)$ admits an equivalent norm $\|\cdot\|_{X,1}$ so that $(X, \|\cdot\|_{X,1})$ has the Opial property.*

We will also recall the theorem used in the proof of D. van Dulst's theorem, which will be one of the fundamental tools for the construction of the norm in section 4 (Schauder basis – see Definition 17).

Theorem 9 ([5]). *Let $(X, \|\cdot\|_X)$ be a Banach space with a Schauder basis $\{e_i\}_i$ and let $\mathcal{P} = \{P_n\}_{n \geq 0}$ be the sequence of the natural projections associated with this basis, i.e., $P_0 = 0$ and $P_n x = P_n(\sum_{i=1}^{\infty} a^i e_i) = \sum_{i=1}^n a^i e_i$ for each $x = \sum_{i=1}^{\infty} a^i e_i \in X$. Then the norm $\|\cdot\|_{\mathcal{P}}$ defined on X by*

$$\|x\|_{\mathcal{P}} = \sup_{k=0,1,\dots} \|x - P_k x\|_X$$

for each $x \in X$ is equivalent to the norm $\|\cdot\|_X$ and the Banach space $(X, \|\cdot\|_X)$ has the non-strict Opial property.

We will also need a norm having the Kadec-Klee property. This property is also related to the behavior of weakly convergent sequences.

Definition 10 ([30], [31]). Let $(X, \|\cdot\|_X)$ be a Banach space. We say that $(X, \|\cdot\|_X)$ has the *Kadec-Klee property* with respect to the weak topology (the Kadec-Klee property for short) if each sequence $\{x_n\}_n$ with $\lim_n \|x_n\|_X = 1$, which converges weakly to a point x with $\|x\|_X = 1$, tends strongly to x .

The theorem stated below shows the relationship between local uniform convexity and the Kadec-Klee property.

Theorem 11 ([16]). Let $(X, \|\cdot\|_X)$ be a Banach space. If $(X, \|\cdot\|_X)$ is locally uniformly convex, then $(X, \|\cdot\|_X)$ has the Kadec-Klee property with respect to the weak topology.

In this chapter, we will study diametrically complete sets (definition of such sets will be given in section 3) that have empty interior. To formulate one of the key properties of a diametrically complete set with empty interior, we will need the notion of a diametrical set and normal structure ([4], [21], [35]).

Definition 12 ([4]). Let $(X, \|\cdot\|)$ be a Banach space. For a nonempty, bounded and convex set $C \subset X$, the number

$$r_{\|\cdot\|}(C, C) := \inf\{\sup\{\|x - x'\| : x' \in C\} : x \in C\}$$

is called the *Chebyshev self-radius* of C .

Definition 13 ([4]). Let $(X, \|\cdot\|)$ be a Banach space and let C be a nonempty, bounded and convex subset of X . We say that the set C is *diametral* if $r_{\|\cdot\|}(C, C) = \text{diam}_{\|\cdot\|}(C)$.

Definition 14. A Banach space $(X, \|\cdot\|_X)$ is said to have *normal structure* if it does not contain any diametral set, that is, if $r_{\|\cdot\|_X}(C, C) < \text{diam}_{\|\cdot\|_X}(C)$ for each nonempty, non-singleton, bounded and convex set $C \subset X$.

In [4] M. S. Brodskii and D. P. Mil'man not only introduced the notion of normal structure but they also characterized it in the terms of diametral sequences.

Definition 15 ([4]). Let $(X, \|\cdot\|)$ be a Banach space. A bounded and not eventually constant sequence $\{x_n\}$ in $(X, \|\cdot\|)$ is said to be *diametral* if

$$\lim_n \text{dist}_{\|\cdot\|}(x_{n+1}, \text{conv}\{x_1, \dots, x_n\}) = \text{diam}_{\|\cdot\|}\{x_1, x_2, \dots\}.$$

Theorem 16 ([4]). A bounded and convex C of a Banach space $(X, \|\cdot\|)$ has normal structure if and only if it does not contain a diametral sequence.

We also recall the notion of a Schauder basis (see [32] and [44]).

Definition 17. Let $(X, \|\cdot\|_X)$ be a Banach space. A sequence $\{e_i\}_i$ in X is called a *Schauder basis* of X if for each $x \in X$, there exists a unique sequence of scalars $\{x^i\}_i$ such that $x = \sum_{i=1}^{\infty} x^i e_i$. A basis $\{e_i\}_i$ is called *normalized* if $\|e_i\| = 1$ for all i .

Definition 18. Assume that $(X, \|\cdot\|)$ is a Banach space with a Schauder basis $\{e_i\}_i$. The functionals e_j^* defined as follows

$$e_j^*\left(\sum_{i=1}^{\infty} x^i e_i\right) := x^j \quad \text{for every } j \in \mathbb{N}$$

are called *biorthogonal functionals* associated with the basis $\{e_i\}_i$.

Remark 19. Throughout this chapter we assume that for each Schauder basis $\{e_i\}_i$ we consider, there exist constants $0 < \tilde{m} \leq \tilde{M} < \infty$ such that $\tilde{m} \leq \|e_i\|_X \leq \tilde{M}$ for each $i \in \mathbb{N}$. It then follows that for the biorthogonal functionals $\{e_i^*\}_i$ associated with the Schauder basis $\{e_i\}_i$, there also exist constants $0 < \tilde{m}_1 \leq \tilde{M}_1 < \infty$ such that $\tilde{m}_1 \leq \|e_i^*\|_{X^*} \leq \tilde{M}_1$ for each $i \in \mathbb{N}$. In addition, we have $\lim_i e_i^*(x) = 0$ for each $x \in X$.

Theorem 20 ([25]). Assume that $\{e_i\}_i$ is a Schauder basis in the Banach space $(X, \|\cdot\|_X)$ with coefficients $e_i^*(x) = a^i$ being linear functionals. Then we have

- (a) The functionals e_i^* , $i = 1, 2, \dots$ are continuous on X ,
- (b) $\sup_n \|P_n\|_{XX} < \infty$, where $P_n x = \sum_{i=1}^n e_i^*(x) e_i$ for $x = \sum_{i=1}^{\infty} a^i e_i \in X$, $n = 1, 2, \dots$ and $\|\cdot\|_{XX}$ is the operator norm associated with the norm $\|\cdot\|_X$,

(c) The norm $\|\cdot\|_0$, defined by the formula

$$\|x\|_0 := \sup_n \|P_n x\|_{XX}$$

for $x \in X$, is equivalent to the norm $\|\cdot\|$.

At this point we recall three important theorems which play an essential role in our considerations in Sections 5 and 6.

Theorem 21 ([44]). *The Banach space $(C([0, 1], \mathbb{R}), \|\cdot\|_C)$ has a Schauder basis.*

The second theorem is due to W. B. Johnson and H. P. Rosenthal.

Theorem 22 ([27]). *Every infinite-dimensional Banach space $(X, \|\cdot\|_X)$ has an infinite-dimensional quotient space $(X/Y, \|\cdot\|_{X/Y})$ with a Schauder basis.*

The third theorem exhibits connections between a Schauder basis in a Banach space and a Schauder basis in a quotient space.

Theorem 23 ([44]). *Let $(X, \|\cdot\|_X)$ be a Banach space with a Schauder basis $\{e_i\}_i$. Let $\{i_n\}_n$ be a finite or infinite increasing sequence of natural numbers and let $\{j_m\}_m$ be the infinite and increasing sequence of natural numbers complementary to $\{i_n\}_n$ in $\mathbb{N}$. If Y is the closed linear span of the sequence $\{e_{i_n}\}_n$ in X and ι is the canonical mapping of X onto the quotient space X/Y with the canonical norm $\|\cdot\|_{X/Y}$, then $\{\iota(e_{j_m})\}_m$ is a Schauder basis in the Banach space $(X/Y, \|\cdot\|_{X/Y})$.*

In the proof of Theorem 51, we will also use of the following results.

Theorem 24 ([2]). *Let Y be a linear, closed, and separable subspace of a reflexive Banach space $(X, \|\cdot\|)$. Then there exists a linear, closed, and separable subspace Z of X containing Y , as well as a linear and bounded projection P from X onto Z with the operator norm $\|P\|_{XZ} = 1$.*

Corollary 25 ([2]). *Let $(X, \|\cdot\|)$ be a non-separable and reflexive Banach space. Then there exists an infinite-dimensional, linear, closed, and separable subspace Z of X , as well as a linear projection P from X onto Z with the operator norm $\|P\|_{XZ} = 1$.*

3. Diametrically complete sets

In this section, we recall the definition of a diametrically complete set ([37]). This definition is closely related to the notion of the diameter of a set. We will provide the fundamental relationship between a diametrically complete set and a diametral set.

Definition 26 ([37]). Let $(X, \|\cdot\|_X)$ be a Banach space and let C be a non-singleton and bounded subset of X . We say that C is a *diametrically complete* set in X if

$$\text{diam}_{\|\cdot\|_X}(C \cup \{x\}) > \text{diam}_{\|\cdot\|_X}(C)$$

for each $x \in X \setminus C$.

In the case of finite-dimensional Banach spaces $(X, \|\cdot\|)$, every diametrically complete set $C \subset X$ has nonempty interior. In the case of infinite-dimensional spaces, this statement is not true. There exist diametrically complete sets with empty interior in infinite-dimensional spaces. A simple example of such a set in c_0 was given by J. P. Moreno, P. L. Papini, and R. R. Phelps in [38].

Example 27. Consider the set $C \subset c_0$ defined in the following way:

$$C = \{x = \{x^i\}_{i \in \mathbb{N}} \in c_0 : 0 \leq x^i \leq 1\}.$$

We will show that the set C has empty interior in $(c_0, \|\cdot\|_\infty)$. Take any $x = \{x^i\}_i \in C$ and define

$$y_n = \{y_n^i\} = x - (x^n + \frac{1}{n})e_n$$

for any $n \in \mathbb{N}$, where e_n are the standard basis vectors of the space c_0 . Then

$$\|y_n - x\|_\infty = |x^n + \frac{1}{n}| > 0,$$

but on the other hand we have $y_n^n = -\frac{1}{n}$, so $y_n \notin C$, which means that x is not an interior point of C . Observe that the set C is diametrically complete. Indeed, if $y = \{y^i\} \notin C$, then there exists $i_0 \in \mathbb{N}$ such that $y^{i_0} \in (-\infty, 0) \cup (1, +\infty)$. Taking $x^0 = \{x_i^0\}$, where $x_i^0 = 0$ for $i \neq i_0$ and

$x_{i_0}^0 = 1$ (respectively), we obtain

$$\text{diam}_{\|\cdot\|_{c_0}}(C \cup \{y\}) > 1 = \text{diam}_{\|\cdot\|_{c_0}} C.$$

The following theorem, which is due to J. P. Moreno, P. L. Papini and R. R. Phelps, exhibits a connection between the diametral property of a set and the emptiness of the interior of a diametrically complete set.

Theorem 28 ([38]). *Let $(X, \|\cdot\|_X)$ be an infinite-dimensional Banach space and let $C \subset X$ be diametrically complete. If the interior of C is empty, then C is diametral.*

It was observed by E. Maluta and P. L. Papini that the converse implication is not true. They presented the following example of a diametral set with empty interior that is not diametrically complete ([35])

Example 29. Let $X = E_{\sqrt{2}}$, $E_{\sqrt{2}} = (\ell^2, |\cdot|_{\sqrt{2}})$, where $|x|_{\sqrt{2}} = \max\{\|x\|_2, \sqrt{2}\|x\|_\infty\}$ for $x \in \ell^2$. Consider the set $C = \text{conv}\{e_1, e_2\}$, where $\{e_i\}_i$ denotes the standard basis of ℓ^2 . Then the set C is diametral in $E_{\sqrt{2}}$ and $\text{diam}_{|\cdot|_{\sqrt{2}}} C = \sqrt{2}$, but it is not diametrically complete, because for $y = (\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}, 0, \dots) \notin C$ we have $\text{diam}_{|\cdot|_{\sqrt{2}}}(C \cup \{y\}) = \sqrt{2}$. Clearly, $\text{int } C = \emptyset$.

Thus, the existence of a diametral set with empty interior in an infinite-dimensional Banach space is a necessary condition for the existence of a diametrically complete set with empty interior, though not a sufficient one. To establish sufficiency, we use the following result of E. Maluta and P. L. Papini.

Theorem 30 ([35]). *Each infinite-dimensional and reflexive Banach space $(X, \|\cdot\|_X)$ which has the non-strict Opial property and lacks normal structure contains diametrically complete sets the interior of which is empty.*

Remark 31. A Banach space satisfying the thesis of the above theorem cannot be uniformly convex, uniformly convex in every direction, or reflexive with the Opial property, since each of these properties entails normal structure. Nevertheless, such a space can be locally uniformly convex ([35]).

4. Construction of a norm having both the Kadec-Klee and the Opial properties

In this section we will prove the theorem about the existence of an equivalent norm with the Opial and the Kadec-Klee properties in every infinite-

dimensional, separable Banach space. This theorem will be one of the main tools for solving the problem of the existence a diametrically complete set with empty interior in reflexive spaces.

First we recall the notion of universality.

Definition 32 ([26]). A Banach space $(Y, \|\cdot\|_Y)$ is said to be *universal* for the class of separable Banach spaces if every separable Banach space $(X, \|\cdot\|_X)$ is isometrically isomorphic to a subspace Y_1 of Y , that is, there exists a linear and norm-preserving isomorphism $T: X \rightarrow Y_1$ with $T(X) = Y_1$.

There are two important examples of universal Banach spaces for the class of separable Banach spaces ([26]).

Theorem 33. *The space $C([0, 1], \mathbb{R})$ furnished with the standard max-norm $\|\cdot\|_C$ is universal for the class of separable Banach spaces.*

Theorem 34. *The Banach space ℓ^∞ of all bounded real sequences equipped with the standard sup-norm $\|\cdot\|_\infty$ is universal for the class of separable Banach spaces.*

Remark 35. If $(X, \|\cdot\|_X)$ and $(Y, \|\cdot\|_Y)$ are Banach spaces and $T: X \rightarrow Y$ is a linear isometric embedding of X into Y , then we identify $T(X)$ with X and write $X \subset Y$.

The following theorem, due to M. I. Kadec and V. L. Klee, turns out to be crucial in our subsequent considerations.

Theorem 36 ([30], [31]). *Let $(Y, \|\cdot\|_Y)$ be an infinite-dimensional Banach space, Y^* its dual space with the standard norm $\|\cdot\|_{Y^*}$, and let X be a closed and separable subspace of $(Y^*, \|\cdot\|_{Y^*})$. Then there exist an equivalent norm $\|\cdot\|_{Y,1}$ on Y and a norm $\|\cdot\|_{Y^*,1}$ on Y^* induced by $\|\cdot\|_{Y,1}$ such that if $\{y_i\}_i$ is any sequence in Y^* which converges weakly* to $\tilde{y} \in X$ and if $\lim_i \|y_i\|_{Y^*,1} = \|\tilde{y}\|_{Y^*,1}$, then $\lim_i \|y_i - \tilde{y}\|_{Y^*,1} = \lim_i \|y_i - \tilde{y}\|_{Y^*} = 0$.*

The next two theorems will demonstrate the existence of an equivalent norm having both the Kadec-Klee property and the Opial property in infinite-dimensional and separable Banach spaces. The Kadec-Klee theorem (Theorem 36), the universal spaces $C([0, 1], \mathbb{R})$, ℓ^∞ , and the method presented in [18] in the proof of D. van Dulst's Theorem will be the fundamental tools in constructing the new norm. Our theorems are even a bit stronger than we need since we show that the infinite-dimensional and separable Banach spaces with suitable chosen equivalent norms are also uniformly convex

in every direction. These theorems were proved in [9], but for convenience of the reader we also recall their proofs.

Theorem 37. *The Banach space $C([0, 1], \mathbb{R})$ endowed with the standard max-norm $\|\cdot\|_C$ has an equivalent norm $\|\cdot\|_{C,1}$ such that $(C([0, 1], \mathbb{R}), \|\cdot\|_{C,1})$ has both the Kadec-Klee and the Opial properties and is uniformly convex in every direction.*

Proof. For the Banach space $(C([0, 1], \mathbb{R}), \|\cdot\|_C)$ we cannot directly apply Theorem 36 since there does not exist a predual space for it. Therefore, we use the space ℓ^∞ (in ℓ^∞ we have the standard norm $\|\cdot\|_\infty$), which has the predual space and is universal for the class of separable Banach spaces.

Consider $Y = \ell^1$ and $\ell^\infty = (\ell^1)^* = Y^*$ with the standard norms. By the universality of $(\ell^\infty, \|\cdot\|_\infty)$ for the class of separable Banach spaces, we have $X = C([0, 1], \mathbb{R}) \subset \ell^\infty$ (see the notation introduced in Remark 35). By Theorem 36, applied to $Y = \ell^1$ and $Y^* = \ell^\infty$, there exists an equivalent norm $\|\cdot\|_{\infty,1}$ on ℓ^∞ such that if $\{y_m\}_m$ is any sequence in ℓ^∞ that converges weakly* to $\tilde{y} \in C([0, 1], \mathbb{R})$, and if $\lim_m \|y_m\|_{\infty,1} = \|\tilde{y}\|_{\infty,1}$, then $\lim_m \|y_m - \tilde{y}\|_{\infty,1} = \lim_m \|y_m - \tilde{y}\|_\infty = 0$. Hence in $C([0, 1], \mathbb{R})$ we obtain the Kadec-Klee property of the norm $\|\cdot\|_{\infty,1}$ with respect to the weak topology. It is obvious that in $C([0, 1], \mathbb{R})$ the norm $\|\cdot\|_{\infty,1}$ is equivalent to the norm $\|\cdot\|_C$.

Now let $\{g_i\}_i$ be any Schauder basis in $(C([0, 1], \mathbb{R}), \|\cdot\|_C)$ with $\|g_i\|_C = 1$ for $i = 1, 2, \dots$ (see Theorem 21) and let $\{g_i^*\}_i$ be the sequence of biorthogonal functionals associated with this basis. We know that the sequence $\{g_i^*\}_i$ is bounded (see Remark 19). Let $\{P_n\}_{n=0}^\infty$ be the sequence of projections in $C([0, 1], \mathbb{R})$, which are defined in the following way: $P_0 := 0$, $P_n h := \sum_{i=1}^n g_i^*(h) g_i$ for $n = 1, 2, \dots$ and $h \in C([0, 1], \mathbb{R})$. The sequence $\{\|P_n\|_{CC}\}_{n=0}^\infty$ of the operator norms of these projections with respect to the norm $\|\cdot\|_C$ in $C([0, 1], \mathbb{R})$ is also bounded (see Theorem 20). At this point we introduce a new norm on $C([0, 1], \mathbb{R})$ as follows:

$$\|h\|_{C,1} := \sqrt{\|h\|_{\mathcal{P},\infty,1}^2 + \|h\|_2^2},$$

where

$$\|h\|_{\mathcal{P},\infty,1} := \sup_{n=0,1,2,\dots} \|h - P_n h\|_{\infty,1}$$

and

$$\|h\|_2 := \sqrt{\sum_{i=1}^{\infty} \left(\frac{g_i^*(h)}{2^i} \right)^2}$$

for $h \in C([0, 1], \mathbb{R})$. Then we have

- The norm $\|\cdot\|_{C,1}$ is equivalent to the norm $\|\cdot\|_{\infty,1}$ in $C([0, 1], \mathbb{R})$. Indeed, consider $h \in C([0, 1], \mathbb{R})$. Since

$$\|h\|_{\mathcal{P},\infty,1} = \sup_{n=0,1,2,\dots} \|h - P_n h\|_{\infty,1} \geq \|h\|_{\infty,1}$$

so we have $\|h\|_{C,1} \geq \|h\|_{\infty,1}$ for $h \in C([0, 1], \mathbb{R})$. Moreover, there exist constants $\tilde{K}, \tilde{M} > 0$ such that $\|h\|_{\mathcal{P},\infty,1} \leq \tilde{K} \|h\|_{\infty,1}$ and $\|h\|_2 \leq \tilde{M} \|h\|_{\infty,1}$. Hence, we obtain

$$\|h\|_{C,1} \leq \sqrt{(\tilde{K}^2 + \tilde{M}^2) \|h\|_{\infty,1}^2} \leq \sqrt{(\tilde{K} + \tilde{M})^2 \|h\|_{\infty,1}^2} = (\tilde{K} + \tilde{M}) \|h\|_{\infty,1}$$

for $h \in C([0, 1], \mathbb{R})$.

Thus, the inequalities

$$\|h\|_{\infty,1} \leq \|h\|_{C,1} \leq (\tilde{K} + \tilde{M}) \|h\|_{\infty,1}$$

hold for every $h \in C([0, 1], \mathbb{R})$. It follows immediately that the norm $\|\cdot\|_{C,1}$ is equivalent to the maximum norm $\|\cdot\|$ on $C([0, 1], \mathbb{R})$, since the norm $\|\cdot\|_{C,1}$ is equivalent to the norm $\|\cdot\|_{\infty,1}$, and the norm $\|\cdot\|_{\infty,1}$ is equivalent to the norm $\|\cdot\|_C$.

- The Banach space $(C([0, 1], \mathbb{R}), \|\cdot\|_{C,1})$ is uniformly convex in every direction by Corollary 3.
- The Banach space $(C([0, 1], \mathbb{R}), \|\cdot\|_{C,1})$ has the Opial property. For each weakly null sequence $\{h_m\}_m$ and $h \in C([0, 1], \mathbb{R}) \setminus \{0\}$ we get

$$\lim_m \sum_{i=1}^{\infty} \left(\frac{g_i^*(h_m)}{2^i} \right)^2 = 0$$

and

$$\lim_m \sum_{i=1}^{\infty} \left(\frac{g_i^*(h_m - h)}{2^i} \right)^2 = \sum_{i=1}^{\infty} \left(\frac{g_i^*(h)}{2^i} \right)^2.$$

Therefore applying the non-strict Opial property of the norm $\|\cdot\|_{\mathcal{P},\infty,1}$ (see Theorem 9) we obtain

$$\begin{aligned}
 \limsup_m \|h_m - h\|_{C,1} &= \limsup_m \sqrt{\|h_m - h\|_{\mathcal{P},\infty,1}^2 + \|h_m - h\|_2^2} \\
 &= \limsup_m \sqrt{\|h_m - h\|_{\mathcal{P},\infty,1}^2 + \sum_{i=1}^{\infty} \left(\frac{g_i^*(h_m - h)}{2^i} \right)^2} \\
 &= \limsup_m \sqrt{\|h_m - h\|_{\mathcal{P},\infty,1}^2 + \sum_{i=1}^{\infty} \left(\frac{g_i^*(h)}{2^i} \right)^2} > \limsup_m \|h_m - h\|_{\mathcal{P},\infty,1} \\
 &\geq \limsup_m \|h_m\|_{\mathcal{P},\infty,1} = \limsup_m \sqrt{\|h_m\|_{\mathcal{P},\infty,1}^2 + \sum_{i=1}^{\infty} \left(\frac{g_i^*(h_m)}{2^i} \right)^2} \\
 &= \limsup_m \|h_m\|_{C,1}.
 \end{aligned}$$

We will now show that the Banach space $(C([0,1], \mathbb{R}), \|\cdot\|_{C,1})$ has the Kadec-Klee property. Assume that a sequence $\{\tilde{h}_m\}_m \subset C([0,1], \mathbb{R})$ converges weakly to $\tilde{h} \in C([0,1], \mathbb{R})$ and $\lim_m \|\tilde{h}_m\|_{C,1} = \|\tilde{h}\|_{C,1} = 1$. Then

$$\begin{aligned}
 \|\tilde{h}\|_{\mathcal{P},\infty,1} &> 0, \\
 \|\tilde{h}\|_2 &> 0
 \end{aligned}$$

and

$$0 < \lim_m \|\tilde{h}_m\|_2 = \|\tilde{h}\|_2 = \beta < 1.$$

Therefore, the limit $\lim_m \|\tilde{h}_m\|_{\mathcal{P},\infty,1}$ and we have

$$0 < \lim_m \|\tilde{h}_m\|_{\mathcal{P},\infty,1} = \|\tilde{h}\|_{\mathcal{P},\infty,1} = \sqrt{1 - \beta^2} = \gamma < 1.$$

Since $\lim_n P_n \tilde{h} = \tilde{h}$, we get

$$\gamma = \|\tilde{h}\|_{\mathcal{P},\infty,1} = \|\tilde{h} - P_{\bar{n}} \tilde{h}\|_{\infty,1}$$

for some $0 \leq \bar{n} < \infty$. Observe now that

$$\lim_m P_{\bar{n}} \tilde{h}_m = P_{\bar{n}} \tilde{h} \tag{1}$$

in the norm $\|\cdot\|_C$ (so also in the norms $\|\cdot\|_{\infty,1}$ and $\|\cdot\|_{C,1}$) and this implies that the sequence $\{\tilde{h}_m - P_{\tilde{n}}\tilde{h}_m\}_m$ tends weakly in $C([0,1],\mathbb{R})$ to $\tilde{h} - P_{\tilde{n}}\tilde{h}$. Therefore

$$\begin{aligned}\gamma = \|\tilde{h}\|_{\mathcal{P},\infty,1} &= \|\tilde{h} - P_{\tilde{n}}\tilde{h}\|_{\infty,1} \leq \liminf_m \|\tilde{h}_m - P_{\tilde{n}}\tilde{h}_m\|_{\infty,1} \\ &\leq \limsup_m \|\tilde{h}_m - P_{\tilde{n}}\tilde{h}_m\|_{\infty,1} \leq \lim_m \|\tilde{h}_m\|_{\mathcal{P},\infty,1} = \gamma\end{aligned}$$

and this means that

$$\lim_m \|\tilde{h}_m - P_{\tilde{n}}\tilde{h}_m\|_{\infty,1} = \|\tilde{h} - P_{\tilde{n}}\tilde{h}\|_{\infty,1} = \gamma.$$

Using the Kadec-Klee property of the norm $\|\cdot\|_{\infty,1}$ in $C([0,1],\mathbb{R})$, we get

$$\lim_m (\tilde{h}_m - P_{\tilde{n}}\tilde{h}_m) = \tilde{h} - P_{\tilde{n}}\tilde{h}$$

in $(C([0,1],\mathbb{R}), \|\cdot\|_{\infty,1})$. Since the norms $\|\cdot\|_{\infty,1}$ and $\|\cdot\|_{C,1}$ are equivalent we also have the same convergence

$$\lim_m (\tilde{h}_m - P_{\tilde{n}}\tilde{h}_m) = \tilde{h} - P_{\tilde{n}}\tilde{h}$$

in the norm $\|\cdot\|_{C,1}$. But we have the convergence $\lim_m P_{\tilde{n}}\tilde{h}_m = P_{\tilde{n}}\tilde{h}$ in the norm $\|\cdot\|_{C,1}$ (equality (1)) and hence we obtain

$$\tilde{h} = \lim_m (\tilde{h}_m - P_{\tilde{n}}\tilde{h}_m) + P_{\tilde{n}}\tilde{h} = \lim_m (\tilde{h}_m - P_{\tilde{n}}\tilde{h}_m) + \lim_m P_{\tilde{n}}\tilde{h}_m = \lim_m \tilde{h}_m$$

in the norm $\|\cdot\|_{C,1}$, as required.

□

Remark 38. Since at a pinch we can multiply the norm $\|\cdot\|_{C,1}$ by a suitably chosen constant greater than 1 we assume that we have

$$\|\cdot\|_C \leq \|\cdot\|_{C,1} \leq L_{C,1} \|\cdot\|_C$$

in $C([0,1],\mathbb{R})$.

Remark 39. Note that Theorem 37 is also true if we replace the Banach space $(C([0,1],\mathbb{R}), \|\cdot\|_C)$ by any Banach space $(X, \|\cdot\|_X)$ with a Schauder basis ([5]).

Basing on Theorem 37 and universality of the space $(C([0, 1], \mathbb{R}), \|\cdot\|_C)$ for the class of separable Banach spaces, we obtain the following theorem.

Theorem 40. *Every infinite-dimensional, separable Banach space $(X, \|\cdot\|_X)$ admits an equivalent norm $\|\cdot\|_{X,1}$ such that $(X, \|\cdot\|_{X,1})$ has both the Kadec-Klee and the Opial properties, and*

$$\|\cdot\|_X \leq \|\cdot\|_{X,1} \leq L_{C,1} \|\cdot\|_X.$$

In addition, $(X, \|\cdot\|_{X,1})$ is uniformly convex in every direction.

Proof. By Theorem 37, the Banach space $C([0, 1], \mathbb{R})$ endowed with the standard max-norm $\|\cdot\|_C$ has an equivalent norm $\|\cdot\|_{C,1}$ such that the space $(C([0, 1], \mathbb{R}), \|\cdot\|_{C,1})$ has both the Kadec-Klee and the Opial properties, and

$$\|\cdot\|_C \leq \|\cdot\|_{C,1} \leq L_{C,1} \|\cdot\|_C.$$

The space $(C([0, 1], \mathbb{R}), \|\cdot\|_{C,1})$ is also uniformly convex in every direction. By the universality of $(C([0, 1], \mathbb{R}), \|\cdot\|_C)$ for the class of separable Banach spaces, we have $X \subset C([0, 1], \mathbb{R})$ (see the notation introduced in Remark 35 above). Now it is sufficient to take the restriction of the norm $\|\cdot\|_{C,1}$ to X in order to obtain the desired norm $\|\cdot\|_{X,1}$. $\square$

5. Construction of the equivalent norm $\|\cdot\|_{\alpha,\beta,p,\mathcal{F}}$

In this section we present a generalization of the Day norm ([13]) in c_0 to renorming of separable Banach spaces $(X, \|\cdot\|_X)$.

We begin by introducing the following notations. Let $u = \{u^\gamma\}_{\gamma \in \Gamma \in c_0(\Gamma)}$. We denote the support of u by $N(u)$. Then the sequence of indices $\{\tau(j, u)\}_j$ is defined as follows:

1. if the support $N(u)$ of u is infinite, then $N(u)$ is enumerated as $\{\tau(j, u)\}_j$ in such a way that $|u^{\tau(j, u)}| \geq |u^{\tau(j+1, u)}|$ for $j \in \mathbb{N}$,
2. if the support $N(u) = \{\tilde{\gamma}\}$ is a singleton, then we set $\tau(1, u) = \tilde{\gamma}$ and extend $\tau(\cdot, u)$ to all of $\mathbb{N}$ so that $\tau(\cdot, u) : \mathbb{N} \rightarrow \Gamma$ is an injection,
3. if the support $N(u)$ of u is finite and consists of $k(u) \geq 2$ elements, then $N(u)$ is enumerated as $\{\tau(j, u)\}_{j=1}^{k(u)}$ in such a way that $|t^{\tau(j, u)}| \geq |t^{\tau(j+1, u)}|$ for $j \in \{1, \dots, k(u) - 1\}$ and we extend $\tau(\cdot, u)$ to all of $\mathbb{N}$ so that $\tau(\cdot, u) : \mathbb{N} \rightarrow \Gamma$ is an injection,

4. if $u = 0$, then $\tau(\cdot, u) : \mathbb{N} \rightarrow \Gamma$ is defined to be an arbitrarily chosen injection.

For each $u = \{u^\gamma\}_{\gamma \in \Gamma} \in c_0(\Gamma)$ we assume that $\tilde{N}(u) := \{\tau(j, u) \in \Gamma : j \in \mathbb{N}\}$.

Inspired by the Day norm in the space $c_0(\Gamma)$ ([13]), we now introduce its generalization $|||\cdot|||_{\beta,p}$. Fix $1 < p < \infty$ and take a strictly decreasing sequence of positive terms $\beta = \{\beta_j\}_j$ such that the series $\sum_{j=1}^{\infty} \beta_j^p$ is convergent. For $u = \{u^i\}_{i \in \Gamma} \in c_0(\Gamma) \setminus \{0\}$ we define $D_{\beta,p}(u) = \{D_{\beta,p}^i(u)\}_{i \in \Gamma} \in \ell^p(\Gamma)$ by

$$D_{\beta,p}^i(u) := \begin{cases} \beta_j u^{\tau(j,u)}, & \text{if } i = \tau(j, u) \text{ for some } j \in \mathbb{N}, \\ 0, & \text{otherwise.} \end{cases}$$

and set

$$|||u|||_{\beta,p} := \|D_{\beta,p}(u)\|_p = \left(\sum_{j=1}^{\infty} |\beta_j u^{\tau(j,u)}|^p \right)^{1/p}$$

for $u \in c_0(\Gamma) \setminus \{0\}$ and $|||0|||_{\beta,p} := \|D_{\beta,p}(0)\|_p = 0$.

We recall an elementary inequality ([10], [43]), which turns out to play an essential role in the proofs of our next results.

Lemma 41. *Assume that*

1. $s = \{s^j\}_j$ is a positive decreasing sequence,
2. $t = \{t^j\}_j \in c_0$,
3. $t^j \geq 0$ for each $j \in \mathbb{N}$,
4. The function $g : \mathbb{N} \rightarrow \mathbb{N}$ is injective.

Then

$$\sum_{j=1}^{\infty} s^j \cdot t^{g(j)} \leq \sum_{j=1}^{\infty} s^j \cdot t^{\tau(j,t)}.$$

Theorem 42 ([6]). *For each $1 < p < \infty$ the function $|||\cdot|||_{\beta,p}$ is a norm in the space $c_0(\Gamma)$.*

Proof. For any $\alpha \in \mathbb{R} \setminus \{0\}$, $u = \{u^i\}_i \in c_0(\Gamma) \setminus \{0\}$ we have

$$\begin{aligned} |||\alpha u|||_{\beta,p} &= \|D(\alpha u)\|_p = \left(\sum_{j=1}^{\infty} |\beta_j(\alpha u)^{\tau(j,\alpha u)}|^p \right)^{\frac{1}{p}} \\ &= \left(|\alpha|^p \sum_{j=1}^{\infty} |\beta_j u^{\tau(j,u)}|^p \right)^{\frac{1}{p}} = |\alpha| \cdot |||u|||_{\beta,p}. \end{aligned}$$

Then, based on the conclusion 41, we obtain

$$\begin{aligned} |||u+v|||_{\beta,p} &= \|D_{\beta,p}(u+v)\|_p \\ &= \left(\sum_{j=1}^{\infty} |\beta_j(u+v)^{\tau(j,u+v)}|^p \right)^{\frac{1}{p}} \\ &\leq \left(\sum_{j=1}^{\infty} |\beta_j u^{\tau(j,u+v)}|^p \right)^{\frac{1}{p}} + \left(\sum_{j=1}^{\infty} |\beta_j v^{\tau(j,u+v)}|^p \right)^{\frac{1}{p}} \\ &\leq \left(\sum_{j=1}^{\infty} |\beta_j u^{\tau(j,u)}|^p \right)^{\frac{1}{p}} + \left(\sum_{j=1}^{\infty} |\beta_j v^{\tau(j,v)}|^p \right)^{\frac{1}{p}} \\ &= |||u|||_{\beta,p} + |||v|||_{\beta,p} \end{aligned}$$

dla $u = \{u^i\}_i, \{v^i\}_i \in c_0(\Gamma) \setminus \{0\}$ i $u \neq v$. □

It is easy to observe that

$$\beta_1 \|u\|_{c_0(\Gamma)} \leq |||u|||_{\beta,p} \leq \left(\sum_{j=1}^{\infty} \beta_j^p \right)^{\frac{1}{p}} \|u\|_{c_0(\Gamma)}$$

for each $u \in c_0(\Gamma)$.

In the case $\{\beta_j\}_j = \{\frac{1}{2^j}\}_j$ and $p = 2$, we obtain the classical Day norm $||| \cdot |||$ ([13]).

Theorem 43. *If we additionally assume that there exist a constant $L > 1$*

and a strictly increasing sequence of natural numbers $\{j_n\}_n$ such that

$$\sum_{j=j_n+1}^{\infty} \beta_j^p \leq L \beta_{j_n+1}^p$$

for each $n \in \mathbb{N}$, then the Banach space $(c_0(\Gamma), ||| \cdot |||_{\beta,p})$ is locally uniformly convex.

The above theorem is a generalization of J. Rainwater's theorem that the space $c_0(\Gamma)$ with the Day norm $||| \cdot |||$ is a locally uniformly convex ([43]). Since the assumptions about the norm $||| \cdot |||_{\beta,p}$ in theorem 43 are more general than those in [43], the proof of Theorem 43 is significantly more complicated compared to the proof of J. Rainwater's theorem and very long (see also [6], [7]). Therefore we omit it.

Now we will show how we can introduce an equivalent norm $\| \cdot \|_{\alpha,\beta,p,\mathcal{F}}$ in separable spaces, which is related to the generalized Day norm and which we will use in the proof of Theorem 47. This theorem is the key tool in the proof of Theorem 48 on the existence of a diametrically complete set with empty interior in infinite-dimensional, separable and reflexive Banach spaces. We define the Day-type norm $\| \cdot \|_{\alpha,\beta,p}$ in the following way — it is a modification of the method by M. A. Smith ([45]).

Assume that $(X, \| \cdot \|_X)$ is a separable Banach space. Fix $\alpha \in (0, 1)$, and let $\mathcal{F} = \{f_k^*\}_k$ be a sequence of nonzero functionals in X^* . Assume that the sequence $\mathcal{F} = \{f_k^*\}_k$ separates points in X and that $\lim_{k \rightarrow \infty} f_k^*(x) = 0$ for each $x \in X$. By the Banach–Steinhaus theorem ([3]), the sequence $\mathcal{F} = \{f_k^*\}_k$ is bounded in X^* , i.e. $\|f_k^*\|_{X^*} \leq \bar{K}$ for each $k \in \mathbb{N}$, whereby we can assume that $1 \leq \bar{K} \in \mathbb{R}$. To each $x \in X$, we now assign a sequence of the form

$$u(x) = \{u^i(x)\}_i = \{\alpha \|x\|_X, f_1^*(x), f_2^*(x), f_2^*(x), \dots, f_k^*(x), \dots, f_k^*(x), \dots\} \in c_0.$$

Here we repeat the k -th coordinate of $\mathcal{F}(x)$ exactly k times. We fix $1 < p < \infty$ and take a strictly decreasing sequence $\beta = \{\beta_j\}_j$ of positive terms such that $\sum_{j=1}^{\infty} \beta_j^p$ converges. Next, using the generalized Day norm, we define a norm in X as follows:

$$\|x\|_{\alpha,\beta,p,\mathcal{F}} = |||u(x)|||_{\beta,p} = \|D_{\beta,p}(u(x))\|_p,$$

where $\| \cdot \|_p$ is the standard norm in the space ℓ^p . Since $\|f_k^*\|_{X^*} \leq \bar{K}$ and $\|f_k^*(x)\|_{X^*} \leq \|f_k^*\|_{X^*} \cdot \|x\|_X$ for $k = 1, 2, \dots$, $x \in X$ we immediately obtain

the inequality

$$\begin{aligned} \|x\|_{\alpha,\beta,p,\mathcal{F}} &= |||\{\alpha\|x\|_X, f_1^*(x), f_2^*(x), f_2^*(x), \dots, f_k^*(x), \dots, f_k^*(x), \dots\}|||_{\beta,p} \\ &\leq \left(\sum_{j=1}^{\infty} \beta_j^p \bar{K}^p \|x\|_X^p \right)^{\frac{1}{p}} = \bar{K} \left(\sum_{j=1}^{\infty} \beta_j^p \right)^{\frac{1}{p}} \|x\|_X. \end{aligned}$$

Hence we have

$$\alpha\beta_1\|x\|_X \leq \|x\|_{\alpha,\beta,p,\mathcal{F}} \leq \bar{K} \left(\sum_{j=1}^{\infty} \beta_j^p \right)^{\frac{1}{p}} \|x\|_X \quad (2)$$

for each $x \in X$. This means that the norm $\|x\|_{\alpha,\beta,p,\mathcal{F}}$ is equivalent to the norm $\|\cdot\|_X$.

Now we will show that the function $\|\cdot\|_{\alpha,\beta,p,\mathcal{F}}$ is a norm. It is enough to verify that it satisfies the triangle inequality. This follows from the definition of the function $\|\cdot\|_{\alpha,\beta,p,\mathcal{F}}$ and the following auxiliary lemma.

Lemma 44. *Under the above assumptions and notations we have*

$$\|u(x+y)\|_{\beta,p} \leq \|u(x) + u(y)\|_{\beta,p} \leq \|u(x)\|_{\beta,p} + \|u(y)\|_{\beta,p}.$$

for every $x, y \in X$.

Proof. For $x, y \in X$ we have

$$\begin{aligned} u(x+y) &= \{u^i(x+y)\}_i \\ &= \{\alpha\|x+y\|_X, f_1^*(x+y), f_2^*(x+y), f_2^*(x+y), \dots, f_k^*(x+y), \dots, f_k^*(x+y), \dots\} \end{aligned}$$

and

$$\begin{aligned} u(x) + u(y) &= \{u^i(x) + u^i(y)\}_i \\ &= \{\alpha\|x\|_X + \alpha\|y\|_X, f_1^*(x+y), f_2^*(x+y), f_2^*(x+y), \dots, f_k^*(x+y), \dots, f_k^*(x+y), \dots\}. \end{aligned}$$

Thus we obtain

$$|u^i(x)| + |u^i(y)| \geq |u^i(x+y)|$$

$i = 1, 2, \dots$

Next, using the inequality given in Lemma 41 and Theorem 42, we get

$$\begin{aligned}
 |||u(x+y)|||_{\beta,p} &= \left(\sum_{j=1}^{\infty} \beta_j^p |u^{\tau(j,u(x+y))}(x+y)|^p \right)^{\frac{1}{p}} \\
 &\leq \left(\sum_{j=1}^{\infty} \beta_j^p \left(|u^{\tau(j,u(x+y))}(x)| + |u^{\tau(j,u(x+y))}(y)| \right)^p \right)^{\frac{1}{p}} \\
 &\leq \left(\sum_{j=1}^{\infty} \beta_j^p \left(|u^{\tau(j,u(x)+u(y))}(x)| + |u^{\tau(j,u(x)+u(y))}(y)| \right)^p \right)^{\frac{1}{p}} \\
 &= |||u(x) + u(y)|||_{\beta,p} \\
 &\leq |||u(x)|||_{\beta,p} + |||u(y)|||_{\beta,p}.
 \end{aligned}$$

□

By modifying the proof of Maluta's theorem (Theorem 3.1 in [34]), and the proof of Theorem 5.4 in [9] we are able to prove the theorem on the transfer of the weak Opial property from the space $(X, \|\cdot\|_X)$ to the space $(X, \|\cdot\|_{\alpha,\beta,p,\mathcal{F}})$.

Theorem 45. *Let $(X, \|\cdot\|_X)$ be a separable Banach space. Under the above assumptions and notations, if a Banach space $(X, \|\cdot\|_X)$ has the non-strict Opial property, then so does the Banach space $(X, \|\cdot\|_{\alpha,\beta,p,\mathcal{F}})$.*

Proof. Assume that a sequence $\{x_n\}_n \subset X$ tends weakly to $0 \in X$ and that $x \in X \setminus \{0\}$. Without any loss of generality we may assume that the limits $\lim_n \|x_n\|_X$ and $\lim_n \|x_n - x\|_X$ exist. Take $0 < \epsilon < 1$. By the non-strict Opial property of the Banach space $(X, \|\cdot\|_X)$, we have

$$\lim_n \|x_n\|_X \leq \lim_n \|x_n - x\|_X$$

and therefore there exists $\tilde{n}_0 \in \mathbb{N}$ such that

$$\|x_n\|_X < \lim_n \|x_n\|_X + \frac{\epsilon}{2} \leq \|x_n - x\|_X + \epsilon \quad (3)$$

for each $\tilde{n}_0 < n \in \mathbb{N}$. Since $\lim_k f_k^*(x) = 0$, there exists $\tilde{k} \in \mathbb{N}$ such that

$$|f_k^*(x)| < \epsilon$$

for each $\tilde{k} < k \in \mathbb{N}$. Therefore

$$\begin{aligned} |f_k^*(x_n)| &\leq |f_k^*(x_n) - f_k^*(x)| + |f_k^*(x)| \\ &< |f_k^*(x_n) - f_k^*(x)| + \epsilon \end{aligned}$$

for each $\tilde{k} < k \in \mathbb{N}$ and all $n \in \mathbb{N}$.

Now for each $1 \leq k \leq \tilde{k}$, we have either $f_k^*(x) = 0$ or $f_k^*(x) \neq 0$. In the second case, setting $\eta_k := \min\{\epsilon, \frac{1}{2}|f_k^*(x)|\}$ and taking into account the weak convergence of $\{x_n\}$ to 0, we find $\tilde{n}_k \in \mathbb{N}$ such that

$$|f_k^*(x_n)| < \eta_k$$

for $\tilde{n}_k < n \in \mathbb{N}$ and hence we obtain

$$\begin{aligned} |f_k^*(x_n) - f_k^*(x)| &\geq |f_k^*(x)| - |f_k^*(x_n)| \\ &> |f_k^*(x)| - \eta_k \geq \frac{1}{2}|f_k^*(x)| \geq \eta_k > |f_k^*(x_n)| \end{aligned}$$

for $\tilde{n}_k < n$. It is obvious that in the first case we have

$$|f_k^*(x_n)| \leq |f_k^*(x_n) - f_k^*(x)|$$

and then we set $\tilde{n}_k := 1$. This implies that

$$|f_k^*(x_n)| \leq |f_k^*(x_n) - f_k^*(x)|$$

for each $1 \leq k \leq \tilde{k}$ and all $\max\{\tilde{n}_1, \dots, \tilde{n}_{\tilde{k}}\} < n \in \mathbb{N}$.

Combining all the above inequalities, we get

$$|f_k^*(x_n)| \leq |f_k^*(x_n) - f_k^*(x)| + \epsilon = |f_k^*(x_n - x)| + \epsilon \quad (4)$$

for each $k \in \mathbb{N}$ and for all $\max\{\tilde{n}_1, \dots, \tilde{n}_{\tilde{k}}\} < n \in \mathbb{N}$. Therefore, from the inequalities (3), (4) and from the defining the function $u(\cdot)$, it follows that

$$|u^i(x_n)| \leq |u^i(x_n - x)| + \epsilon, \quad (5)$$

for each $i \in \mathbb{N}$ and for all $\max\{\tilde{n}_0, \tilde{n}_1, \dots, \tilde{n}_{\tilde{k}}\} < n \in \mathbb{N}$.

Next, take $\max\{\tilde{n}_0, \tilde{n}_1, \dots, \tilde{n}_{\tilde{k}}\} < n \in \mathbb{N}$. Assuming in the Lemma 41 $\{s^j\}_j = \{\beta_j^p\}_j$, $\{t^j\}_j = \{|u^{\tau(j, u(x_n - x))}(x_n - x)|^p\}_j$, $\{g(j)\}_j = \{\tau(j, u(x_n))\}_j$

and using the inequality (5) we obtain

$$\begin{aligned}
 \|x_n\|_{\alpha,\beta,p,\mathcal{F}} &= |||u(x_n)|||_{\beta,p} \|D_{\beta,p}(u(x_n))\|_p \\
 &= \left(\sum_{j=1}^{\infty} \beta_j^p |u^{\tau(j,u(x_n))}(x_n)|^p \right)^{\frac{1}{p}} \\
 &\leq \left(\sum_{j=1}^{\infty} \beta_j^p (|u^{\tau(j,u(x_n))}(x_n - x)| + \epsilon)^p \right)^{\frac{1}{p}} \\
 &\leq \left(\sum_{j=1}^{\infty} \beta_j^p |u^{\tau(j,u(x_n))}(x_n - x)|^p \right)^{\frac{1}{p}} + \epsilon \left(\sum_{j=1}^{\infty} \beta_j^p \right)^{\frac{1}{p}} \\
 &\leq \left(\sum_{j=1}^{\infty} \beta_j^p |u^{\tau(j,u(x_n-x))}(x_n - x)|^p \right)^{\frac{1}{p}} + \epsilon \left(\sum_{j=1}^{\infty} \beta_j^p \right)^{\frac{1}{p}} \\
 &= \|D_{\beta,p}(u(x_n - x))\|_p + \epsilon \left(\sum_{j=1}^{\infty} \beta_j^p \right)^{\frac{1}{p}} \\
 &= |||u(x_n - x)|||_{\beta,p} + \epsilon \left(\sum_{j=1}^{\infty} \beta_j^p \right)^{\frac{1}{p}} \\
 &= \|x_n - x\|_{\alpha,\beta,p,\mathcal{F}} + \epsilon \left(\sum_{j=1}^{\infty} \beta_j^p \right)^{\frac{1}{p}}.
 \end{aligned}$$

Finally, by letting n tend to $+\infty$, we get

$$\limsup_n \|x_n\|_{\alpha,\beta,p,\mathcal{F}} \leq \limsup_n \|x_n - x\|_{\alpha,\beta,p,\mathcal{F}} + \epsilon \left(\sum_{j=1}^{\infty} \beta_j^p \right)^{\frac{1}{p}}.$$

Since $0 < \epsilon < 1$ is arbitrary, we conclude that

$$\limsup_n \|x_n\|_{\alpha,\beta,p,\mathcal{F}} \leq \limsup_n \|x_n - x\|_{\alpha,\beta,p,\mathcal{F}},$$

as required. □

Assuming additionally that the space $(X, \|\cdot\|_X)$ has the Kadec-Klee property and that for a sequence $\beta = \{\beta_j\}_j$ there exist a constant $L > 1$ and a strictly increasing sequence of natural numbers $\{j_n\}_n$, such that $\sum_{j=j_n+1}^{\infty} \beta_j^p \leq L\beta_{j_n+1}^p$ for each $n \in \mathbb{N}$ and applying Theorem 43 and M. A. Smith's method ([45]) (see also the proof of Theorem 6.3 in [9]), we obtain the following theorem.

Theorem 46. *Let $(X, \|\cdot\|_X)$ be an infinite-dimensional, reflexive and separable Banach space. If $(X, \|\cdot\|_X)$ has the Kadec-Klee property, then the Banach space $(X, \|\cdot\|_{\alpha, \beta, p, \mathcal{F}})$ is LUR.*

Proof. Let $x \in X$ and $\{x_n\}_n \subset X$ be given such that $\|x\|_{\alpha, \beta, p, \mathcal{F}} = 1$, $\lim_n \|x_n\|_{\alpha, \beta, p, \mathcal{F}} = 1$ and $\lim_n \|x + x_n\|_{\alpha, \beta, p, \mathcal{F}} = 2$. So we have $|||u(x)|||_{\beta, p} = 1$, $\lim_n |||u(x_n)|||_{\beta, p} = 1$, $\lim_n |||u(x + x_n)|||_{\beta, p} = 2$ and by Lemma 44 we get

$$\begin{aligned} 2 &= |||u(x)|||_{\beta, p} + \lim_n |||u(x_n)|||_{\beta, p} \geq \limsup_n |||u(x) + u(x_n)|||_{\beta, p} \\ &\geq \liminf_n |||u(x) + u(x_n)|||_{\beta, p} \geq \lim_n |||u(x + x_n)|||_{\beta, p} = 2, \end{aligned}$$

and this means that $\lim_n |||u(x) + u(x_n)|||_{\beta, p} = 2$. Now applying the local uniform convexity of $(c_0, |||\cdot|||_{\beta, p})$ (Theorem 43), we immediately obtain the strong convergence of the sequence $\{u(x_n)\}_n$ to $u(x)$ in the norm $|||\cdot|||_{\beta, p}$. But we have

$$\beta_1 \|u(x) - u(x_n)\|_{c_0} \leq |||u(x) - u(x_n)|||_{\beta, p} \xrightarrow{n} 0$$

and

$$\begin{aligned} &\|u(x) - u(x_n)\|_{c_0} = \\ &= \max\{\alpha|\|x\|_X - \|x_n\|_X|, |f_1^*(x) - f_1^*(x_n)|, |f_2^*(x) - f_2^*(x_n)|, \dots\}. \end{aligned}$$

This implies that $\lim_n \|x_n\|_X = \|x\|_X$ and $\lim_n f_k^*(x_n) = f_k^*(x)$ for $k = 1, 2, \dots$. By assumption, the sequence of functionals $\mathcal{F} = \{f_k^*\}_k \subset X^*$ separates the points in $(X, \|\cdot\|_X)$ and therefore the reflexivity of the space $(X, \|\cdot\|_X)$ and the convergence $\lim_n f_k^*(x_n) = f_k^*(x)$ for every $k = 1, 2, \dots$ imply weak convergence of the sequence $\{x_n\}_n$ to x . Finally, it follows from the Kadec-Klee property of the space $\|\cdot\|_X$, weak convergence of the sequence $\{x_n\}_n$ to x and the equality $\lim_n \|x_n\|_X = \|x\|_X$ that $\lim_n x_n = x$ in $(X, \|\cdot\|_X)$. The norms $\|\cdot\|_X$ and $\|\cdot\|_{\alpha, \beta, p, \mathcal{F}}$ are equivalent, which completes the proof of the theorem. $\square$

6. Existence of a diametrically complete set with empty interior in a reflexive and separable Banach space

In this section, we will prove that, after an appropriate renorming, in every infinite-dimensional, separable and reflexive Banach space there exists a diametrically complete set with empty interior. We will show, namely, that there exists an equivalent norm that satisfies the assumptions of Theorem 30. The construction of this norm is the following (see also the proof of Theorem 3.1 in [9]).

If the Banach space $(X, \|\cdot\|_X)$ is infinite-dimensional and separable, then from Theorem 40 it follows that this space has an equivalent norm $\|\cdot\|_{X,1}$ such that the space $(X, \|\cdot\|_{X,1})$ has both the Kadec-Klee property and the Opial property. So without any loss of generality we can assume that the norm $\|\cdot\|_X$ has both these properties. By Theorem 22, there exists a closed subspace Y of the Banach space $(X, \|\cdot\|_X)$ such that the quotient space X/Y with the canonical norm $\|\cdot\|_{X/Y}$ has a Schauder basis. We may also assume that $\dim Y \geq 1$ (see Theorem 23). For this quotient space X/Y , there exists the standard embedding $\iota : X \rightarrow X/Y$. Let $\{\hat{z}_m\}_m$ be a normalized basis in $(X/Y, \|\cdot\|_{X/Y})$ and let $\{\hat{z}_m^*\}$ be the sequence of biorthogonal functionals associated with this basis. Then there is a constant $\tilde{K}$ such that $\|\hat{z}_m^*\|_{(X/Y)^*} \leq \tilde{K}$ for all $m \in \mathbb{N}$. Next we choose a normalized sequence of functionals $\{\tilde{f}_r^*\}_r$ in $(X^*, \|\cdot\|_{X^*})$ which separates the points in $(X, \|\cdot\|_X)$. We also fix $0 < \alpha \leq \frac{1}{2}$ and for each $x \in X$, we define the sequence

$$\hat{\mathcal{F}}(x) = \{\hat{f}_k^*(x)\}_k = \left\{ \frac{\alpha}{2} \tilde{f}_1^*(x), \hat{z}_1^*(\iota(x)), \frac{\alpha}{2} \tilde{f}_2^*(x), \hat{z}_2^*(\iota(x)), \dots \right\} \in c_0.$$

Then the sequence $\hat{\mathcal{F}} = \{\hat{f}_k^*\}_k$ is bounded in X^* , that is, $\|\hat{f}_k^*\|_{X^*} \leq \tilde{K}$ for each $k \in \mathbb{N}$. Using this sequence, we construct a norm $\|\cdot\|_{\alpha, \beta, p, \hat{\mathcal{F}}}$ as in Section 5, where $1 < p < \infty$, $0 < \alpha < 1$ i $\beta = \{\beta_j\}_j$ is a strictly decreasing sequence of positive terms such that the series $\sum_{j=1}^{\infty} \beta_j^p$ is convergent. This norm is equivalent to the norm $\|\cdot\|_X$.

Modifying the method used by M. A. Smith and B. Turett ([46]), we are able to prove the following theorem.

Theorem 47. *Assume that*

1. $(X, \|\cdot\|_X)$ *is an infinite-dimensional, reflexive and separable Banach*

space,

2. Y is a closed subspace of $(X, \|\cdot\|_X)$ such that $\dim Y \geq 1$ and the quotient space X/Y with the canonical norm $\|\cdot\|_{X/Y}$ has a Schauder basis,
3. $\iota : X \rightarrow X/Y$ is the standard embedding,
4. $\{\hat{z}_m\}_m$ is a normalized Schauder basis in $(X/Y, \|\cdot\|_{X/Y})$ and $\{\hat{z}_m^*\}$ is the sequence of biorthogonal functionals associated with this basis,
5. $\{\tilde{f}_r^*\}_r$ is a normalized sequence of functionals in $(X^*, \|\cdot\|_{X^*})$, which separates the points in $(X, \|\cdot\|_X)$,
6. $0 < \alpha \leq \frac{1}{2}$,
7. $1 < p < \infty$,
8. $\beta = \{\beta_j\}_j$ is a strictly decreasing sequence of positive terms $\beta = \{\beta_j\}_j$ such that the series $\sum_{j=1}^{\infty} \beta_j^p$ is convergent,
9. $\hat{\mathcal{F}}(x) = \{\hat{f}_k^*(x)\}_k = \{\frac{\alpha}{2}\tilde{f}_1^*(x), \hat{z}_1^*(\iota(x)), \frac{\alpha}{2^2}\tilde{f}_2^*(x), \hat{z}_2^*(\iota(x)), \dots\} \in c_0$.

Then the space $(X, \|\cdot\|_{\alpha, \beta, p, \hat{\mathcal{F}}})$ lacks normal structure.

Proof. By the reflexivity of $(X, \|\cdot\|_X)$, there exists a sequence $\{z_m\}_m \subset X$ such that $z_m \in \hat{z}_m$ and $\|z_m\|_X = 1$ for each $m \in \mathbb{N}$. Then for $m_2 > m_1$, we have

$$\begin{aligned}
 \hat{\mathcal{F}}(z_{m_2} - z_{m_1}) &= \{\hat{f}_k^*(z_{m_2} - z_{m_1})\}_k \\
 &= \left\{ \frac{\alpha}{2} \tilde{f}_1^*(z_{m_2} - z_{m_1}), \hat{z}_1^*(\iota(z_{m_2} - z_{m_1})), \frac{\alpha}{2^2} \tilde{f}_2^*(z_{m_2} - z_{m_1}), \hat{z}_2^*(\iota(z_{m_2} - z_{m_1})), \dots \right\} \\
 &= \left\{ \frac{\alpha}{2} \tilde{f}_1^*(z_{m_2} - z_{m_1}), 0, \frac{\alpha}{2^2} \tilde{f}_2^*(z_{m_2} - z_{m_1}), 0, \dots, \right. \\
 &\quad \frac{\alpha}{2^{m_1-1}} \tilde{f}_{m_1-1}^*(z_{m_2} - z_{m_1}), 0, \frac{\alpha}{2^{m_1}} \tilde{f}_{m_1}^*(z_{m_2} - z_{m_1}), -1, \frac{\alpha}{2^{m_1+1}} \tilde{f}_{m_1+1}^*(z_{m_2} - z_{m_1}), \\
 &\quad 0, \dots, \frac{\alpha}{2^{m_2-1}} \tilde{f}_{m_2-1}^*(z_{m_2} - z_{m_1}), 0, \frac{\alpha}{2^{m_2}} \tilde{f}_{m_2}^*(z_{m_2} - z_{m_1}), \\
 &\quad \left. 1, \frac{\alpha}{2^{m_2+1}} \tilde{f}_{m_2+1}^*(z_{m_2} - z_{m_1}), 0, \dots \right\}
 \end{aligned}$$

and

$$\frac{\alpha}{2^r} |\tilde{f}_r^*(z_{m_2} - z_{m_1})| \leq \frac{\alpha}{2^r} 2 \leq \frac{1}{2}$$

for all $r \in \mathbb{N}$. Next, we also have

$$\alpha \|z_{m_2} - z_{m_1}\|_X \leq 2\alpha \leq 1.$$

and therefore

$$\left(\sum_{j=1}^{2m_1+2m_2} \beta_j^p \right)^{\frac{1}{p}} \leq \|z_{m_2} - z_{m_1}\|_{\alpha, \beta, p, \hat{\mathcal{F}}} \leq \left(\sum_{j=1}^{\infty} \beta_j^p \right)^{\frac{1}{p}},$$

This means that $\text{diam}_{\|\cdot\|_{\alpha, \beta, p, \hat{\mathcal{F}}}} \{z_m\}_m = \left(\sum_{j=1}^{\infty} \beta_j^p \right)^{\frac{1}{p}}$.

Now we compute $\lim_m \text{dist}_{\|\cdot\|_{\alpha, \beta, p, \hat{\mathcal{F}}}}(z_{m+1}, \text{conv}\{z_1, \dots, z_m\})$. To this end, suppose $a_1 + \dots + a_m = 1$, where $0 \leq a_k \leq 1$ for $k = 1, \dots, m$. Then we have

$$\begin{aligned} \hat{\mathcal{F}}(z_{m+1} - \sum_{k=1}^m a_k z_k) &= \{\hat{f}_k^*(z_{m+1} - \sum_{k=1}^m a_k z_k)\}_k \\ &= \left\{ \frac{\alpha}{2} \tilde{f}_1^*(z_{m+1} - \sum_{k=1}^m a_k z_k), -a_1, \frac{\alpha}{2^2} \tilde{f}_2^*(z_{m+1} - \sum_{k=1}^m a_k z_k), -a_2, \dots, \right. \\ &\quad \frac{\alpha}{2^m} \tilde{f}_m^*(z_{m+1} - \sum_{k=1}^m a_k z_k), -a_m, \frac{\alpha}{2^{m+1}} \tilde{f}_{m+1}^*(z_{m+1} - \sum_{k=1}^m a_k z_k), 1, \\ &\quad \left. \frac{\alpha}{2^{m+2}} \tilde{f}_{m+2}^*(z_{m+1} - \sum_{k=1}^m a_k z_k), 0, \dots \right\} \end{aligned}$$

and

$$\frac{\alpha}{2^r} |\tilde{f}_r^*(z_{m+1} - \sum_{k=1}^m a_k z_k)| \leq \frac{\alpha}{2^r} 2 \leq \frac{1}{2}$$

for all $r \in \mathbb{N}$. Next, we also have

$$\alpha \|z_{m+1} - \sum_{k=1}^m a_k z_k\|_X \leq 2\alpha \leq 1.$$

Hence we obtain

$$\left(\sum_{j=1}^{2m+2} \beta_j^p \right)^{\frac{1}{p}} \leq \|z_{m+1} - \sum_{k=1}^m a_k z_k\|_{\alpha, \beta, p, \hat{\mathcal{F}}} \leq \text{diam}_{\|\cdot\|_{\alpha, \beta, p, \hat{\mathcal{F}}}} \{z_m\}_m = \left(\sum_{j=1}^{\infty} \beta_j^p \right)^{\frac{1}{p}}.$$

This means that $\lim_m \text{dist}_{\|\cdot\|_{\alpha,\beta,p,\hat{\mathcal{F}}}}(z_{m+1}, \text{conv}\{z_1, \dots, z_m\}) = \left(\sum_{j=1}^{\infty} \beta_j^p\right)^{\frac{1}{p}}$. Hence the sequence $\{z_m\}_m \subset X$ is diametral in $(X, \|\cdot\|_{\alpha,\beta,p,\hat{\mathcal{F}}})$ and applying Theorem 16, we complete the proof. $\square$

Now we can prove the theorem which turns out to be crucial in the proof of the main result of our chapter.

Theorem 48. *(see also [9]) Each infinite-dimensional, reflexive and separable Banach space $(X, \|\cdot\|_X)$ has an equivalent norm $\|\cdot\|_0$ such that $(X, \|\cdot\|_0)$ is LUR and contains a diametrically complete set the interior of which is empty.*

Proof. As noted at the beginning of this chapter, we may assume without loss of generality that the Banach space $(X, \|\cdot\|_X)$ has both the Kadec-Klee property and the Opial property. Consider the norm $\|\cdot\|_0 = \|\cdot\|_{\alpha,\beta,p,\hat{\mathcal{F}}}$ where the norm $\|\cdot\|_{\alpha,\beta,p,\hat{\mathcal{F}}}$ was introduced earlier in this chapter, with $0 < \alpha \leq \frac{1}{2}$, $1 < p < \infty$ and a strictly decreasing sequence $\beta = \{\beta_j\}$ of positive terms such that the series $\sum_{j=1}^{\infty} \beta_j^p$ converges. Assume further that there exists a constant $L > 1$ and a strictly increasing sequence of natural numbers $\{j_n\}_n$ such that $\sum_{j=j_n+1}^{\infty} \beta_j^p \leq L\beta_{j_n+1}^p$ for every $n \in \mathbb{N}$. By Theorem 45, the space $(X, \|\cdot\|_0)$ has the non-strict Opial property and by Theorem 47 this space lacks normal structure. Therefore, we can apply Theorem 30 to the space $(X, \|\cdot\|_0)$, from which it follows that this space contains a diametrically complete set with empty interior. Finally, by Theorem 46, we conclude that the space $(X, \|\cdot\|_0)$ is locally uniformly convex. $\square$

7. Existence of a diametrically complete set with empty interior in a reflexive Banach space

In this section, we will show how the problem of existence of a diametrically complete set with empty interior in any infinite-dimensional reflexive Banach space can be reduced to the same problem in an infinite-dimensional, separable and reflexive Banach space.

We begin with the auxiliary theorem.

Theorem 49. *Let $(X_1, \|\cdot\|_1)$ and $(X_2, \|\cdot\|_2)$ be infinite-dimensional Banach spaces, and let $X = X_1 \times X_2$ be endowed with the norm*

$$\|x\| := \sqrt{\|x^1\|_1^2 + \|x^2\|_2^2},$$

where $x = (x^1, x^2) \in X$. If $(X_1, \|\cdot\|_1)$ contains a diametrically complete set the interior of which is empty, then $(X, \|\cdot\|)$ also contains a diametrically complete set the interior of which is empty.

Proof. Let $C_1 \subset X_1$ be a diametrically complete set in $(X_1, \|\cdot\|_1)$ such that the interior of C_1 in $(X_1, \|\cdot\|_1)$ is empty. Set $C := C_1 \times \{0\} \subset X$. It is obvious that the interior of C is empty in $(X, \|\cdot\|)$ and that $\text{diam}_{\|\cdot\|} C = \text{diam}_{\|\cdot\|_1} C_1$. To show that C is diametrically complete, we take $x = (x^1, x^2) \in X \setminus C$, and consider the following two cases.

Case 1. $x^1 \notin C_1$. Since the set C_1 is diametrically complete in $(X_1, \|\cdot\|_1)$, we have

$$\begin{aligned} \text{diam}_{\|\cdot\|}(C \cup \{x\}) &\geq \sup_{(\tilde{x}^1, 0) \in C} \|(\tilde{x}^1, 0) - (x^1, x^2)\| = \sup_{\tilde{x}^1 \in C_1} \sqrt{\|\tilde{x}^1 - x^1\|_1^2 + \|x^2\|_2^2} \\ &\geq \sup_{\tilde{x}^1 \in C_1} \|\tilde{x}^1 - x^1\|_1 > \text{diam}_{\|\cdot\|_1} C_1 = \text{diam}_{\|\cdot\|} C. \end{aligned}$$

Case 2. $x^1 \in C_1$. Then, from the assumption that $x = (x^1, x^2) \notin C = C_1 \times \{0\}$ we have $x^2 \neq 0$. Since the set C_1 is diametrically complete with empty interior, C_1 is diametral, and therefore we obtain

$$\begin{aligned} \text{diam}_{\|\cdot\|}(C \cup \{x\}) &\geq \sup_{\tilde{x}^1 \in C_1} \sqrt{\|\tilde{x}^1 - x^1\|_1^2 + \|x^2\|_2^2} > \\ &> \sup_{\tilde{x}^1 \in C_1} \|\tilde{x}^1 - x^1\|_1 = \text{diam}_{\|\cdot\|_1} C_1 = \text{diam}_{\|\cdot\|} C. \end{aligned}$$

This completes the proof. □

Remark 50. Theorem 49 is also valid if $(X_2, \|\cdot\|_2)$ is a finite-dimensional Banach space.

Now we can extend Theorem 48 to all infinite-dimensional and reflexive Banach spaces.

Theorem 51. *Each infinite-dimensional reflexive Banach space $(X, \|\cdot\|)$ has an equivalent norm $\|\cdot\|_0$ such that $(X, \|\cdot\|_0)$ is LUR and contains a diametrically complete set the interior of which is empty.*

Proof. By theorem 48 it follows that our theorem is true for separable spaces. Assume that $(X, \|\cdot\|)$ is an infinite-dimensional, non-separable and reflexive Banach space. By Theorem 25 there exists an infinite-dimensional,

linear, close and separable subspace X_1 of X , and a linear projection P of X onto X_1 with $\|P\|_{XX_1} = 1$. Hence $X := X_1 \oplus X_2$, where $X_2 := (I - P)(X)$ and I is the identity operator on X .

It then follows from Theorem 48 that $(X_1, \|\cdot\|)$, as an infinite-dimensional, separable and reflexive Banach space, has an equivalent norm $\|\cdot\|_1$ such that $(X_1, \|\cdot\|_1)$ is LUR and contains a diametrically complete set C_1 the interior of which is empty. Moreover, by Troyanski's theorem (Theorem 6), the Banach space $(X_2, \|\cdot\|)$ admits an equivalent locally uniformly convex norm $\|\cdot\|_2$. So we can define on X an equivalent norm $\|\cdot\|_0$ by setting

$$\|x\|_0 := \sqrt{\|x^1\|_1^2 + \|x^2\|_2^2}$$

for $x = x^1 \oplus x^2 \in X$, where $x^1 = Px \in X_1$ and $x^2 = (I - P)x \in X_2$. Theorem 5 implies that $(X, \|\cdot\|_0)$ is LUR and by Theorem 49 we get that $C = C_1 \times \{0\} \subset X$ is a diametrically complete set the interior of which is empty. $\square$

As a conclusion from the theorems 49, 51 and remark 50 we get the following generalization of Theorem 51.

Theorem 52. *Let $(X_1, \|\cdot\|_1)$ and $(X_2, \|\cdot\|_2)$ be Banach spaces and let $X = X_1 \times X_2$ be endowed with the norm*

$$\|x\| := \sqrt{\|x^1\|_1^2 + \|x^2\|_2^2},$$

for $x = (x^1, x^2) \in X$. If $(X_1, \|\cdot\|_1)$ is an infinite-dimensional and reflexive space, then $(X, \|\cdot\|)$ has an equivalent norm $\|\cdot\|_0$ such that $(X, \|\cdot\|_0)$ contains a diametrically complete set the interior of which is empty.

Remark 53. Observe also that the condition given in Theorem 30 for the existence of diametrically complete sets with empty interior is only sufficient, but not necessary, as the following example shows. In this example we use a result of E. Maluta ([34]).

Example 54. In [34] E. Maluta has shown that ℓ^2 furnished with the Day type norm $\|\cdot\|_L$ ([46]) is LUR and contains diametrically complete sets with empty interior. We consider, in addition, the Banach space $(L^p([0, 1], \mathbb{R}))$, with the standard norm $\|\cdot\|_p$, $1 < p < \infty$, $p \neq 2$, and we set

$$X := \ell^2 \times L^p([0, 1], \mathbb{R})$$

with the norm

$$|||(x^1, x^2)||| := \sqrt{\|x^1\|_L^2 + \|x^2\|_p^2}.$$

The reflexive Banach space $(X, |||\cdot|||)$ is LUR (see Theorem 5) and does not have the non-strict Opial property, but by Theorem 49 it does contain diametrically complete set with empty interior.

Finally, we present an interesting and pertinent example of a certain nonreflexive Banach space.

Example 55. It is not known whether every reflexive Banach space $(X, \|\cdot\|)$ can be renormed so as to have the non-strict Opial property. This statement is false in general as the Banach space $\ell^\infty(\Gamma)$ with an uncountable Γ shows ([15]). In addition, $\ell^\infty(\Gamma)$ with an uncountable Γ cannot be equivalently renormed so as to be LUR or even strictly convex ([14]). Hence for uncountable Γ , the space

$$X := \ell^2 \times \ell^\infty(\Gamma)$$

with the norm

$$\|(x^1, x^2)\| := \sqrt{\|x^1\|_L^2 + \|x^2\|_\infty^2},$$

admits no norm with the non-strict Opial property and no strictly convex norm, but by E. Maluta's result ([34]) and Theorem 52 it does contain diametrically complete set with empty interior.

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Summary

In this chapter we prove that every infinite-dimensional and reflexive Banach space $(X, \|\cdot\|)$ admits an equivalent norm $\|\cdot\|_0$ such that $(X, \|\cdot\|_0)$ contains a diametrically complete set with empty interior. First we will show the existence of such a norm for each separable reflexive Banach space. Then we reduce the problem of existence a diametrically complete set with empty interior in any infinite-dimensional and reflexive Banach space to the separable case.

Streszczenie

W tej pracy dowodzimy, że każda nieskończenie wymiarowa i refleksywna przestrzeń Banacha $(X, \|\cdot\|)$ dopuszcza normę równoważną $\|\cdot\|_0$ taką, że $(X, \|\cdot\|_0)$ zawiera zbiór diametralnie zupełny o pustym wnętrzu. Najpierw wykażemy istnienie takiej normy dla każdej ośrodkowej refleksywnej przestrzeni Banacha. Następnie redukujemy problem istnienia zbioru diametralnie zupełnego o pustym wnętrzu w dowolnej nieskończenie wymiarowej i refleksywnej przestrzeni Banacha do przypadku ośrodkowego.

Chapter 2

NEW RETRACTION CONSTANT FOR THE CLASS OF SEPARABLE BANACH SPACES CONTAINING AN ISOMETRIC COPY OF c_0

Dawid Kapitan

1. Introduction

Let X denote an infinite-dimensional real Banach space with the closed unit ball B_X . Since the work of Lin and Sternfeld [9] it is known that for any $k > 1$ there exists a map $T : B_X \rightarrow B_X$ such that T is k -lipschitzian (i.e., for every $x, y \in B_X$, $\|Tx - Ty\| \leq k\|x - y\|$) and

$$d(T) := \inf \{\|x - Tx\| : x \in B_X\} > 0.$$

It is natural to attempt to provide a uniform estimate of minimal displacement for the entire class of Lipschitz self-mappings of the closed unit ball. For this purpose, Goebel [4] introduced the characteristic

$$\psi_X(k) = \sup \{d(T) \mid T : B_X \rightarrow B_X, T \in \mathcal{L}(k)\}, \quad k \geq 1.$$

This paper is a part of D. Kapitan's master's thesis, which received a distinction in the Polish Mathematical Society's competition in honour of Józef Marcinkiewicz (1910–1940).

Here, $\mathcal{L}(k)$ denotes the class of k -lipschitzian maps. It is known that in any Banach space X we have

$$\Psi_X(k) \leq 1 - \frac{1}{k},$$

and for some spaces the above estimate is exact, i.e., there are spaces for which $\Psi_X(k) = 1 - 1/k$ for every $k \geq 1$. We call such spaces *extremal*. Moreover, X is *strictly extremal*, if for any $k \geq 1$ there is $T : B_X \rightarrow B_X$ such that $T \in \mathcal{L}(k)$ and $\|x - Tx\| > 1 - 1/k$ for any $x \in B_X$. A broad class of such spaces is provided by the following, recent result:

Theorem 1. ([7]) *If a separable Banach space X contains an isomorphic (resp. isometric) copy of the space c_0 , then X as well as all subspaces of X of finite codimension are extremal (resp. strictly extremal).*

Here, c_0 denotes the Banach space of all sequences converging to 0 endowed with its standard supremum norm.

The problem of evaluating $\Psi_X(k)$ is strictly connected with the so-called *optimal retraction problem*. Recall that in every infinite-dimensional Banach space X there exists a lipschitzian retraction $R : B_X \rightarrow S_X$ from the closed unit ball onto its boundary (the fact that R is a retraction means that $Rx = x$ for every $x \in S_X$). This naturally raises the question of the optimal retraction constant

$$k_0(X) = \inf \{k : \text{there exists a } k\text{-lipschitzian retraction } R : B_X \rightarrow S_X\}.$$

Over the years, numerous authors have employed various techniques to obtain estimates of the optimal retraction constant in specific Banach spaces. Still, the exact value of $k_0(X)$ remains unknown for any Banach space. The universal bound from below is $k_0(X) \geq 3$ and for Hilbert space H , $k_0(H) > 4.5$ (see Chapter 21 in [5]). Considerably more work has been devoted to deriving upper bounds for the retraction constant. The best one so far was obtained in the space $BC_0(\mathbb{R})$ of all bounded and continuous functions vanishing at 0 provided with the standard supremum norm. The construction from [10] shows that $k_0(BC_0(\mathbb{R})) \leq 6.828$.

In what follows, we will be concerned with general estimates of $k_0(X)$ for some classes of Banach spaces. It has been known for a long time that good estimates of function $\Psi_X(k)$ can provide reasonable bounds for the optimal retraction constant. In the extremal case the following estimate from [1] is known:

Theorem 2. *If a Banach space X is extremal, then $k_0(X) \leq 30.84$.*

Better estimate of $k_0(X)$ was later obtained in [6] under the additional assumption that space X is *cut-invariant*. Let us clarify the meaning of this.

Let K be an infinite set and let $B(K)$ denote the space of all bounded functions $x : K \rightarrow \mathbb{R}$ provided with the norm $\|x\| = \sup \{|x(t)| : t \in K\}$. Call subspace $X \subset B(K)$ cut-invariant, if for any $x \in X$, $Qx := \alpha \circ x \in X$, where α is a cut function defined by the formula

$$\alpha(t) = \begin{cases} -1 & \text{if } t \leq -1, \\ t & \text{if } t \in [-1, 1], \\ 1 & \text{if } t \geq 1. \end{cases}$$

Clearly, Q is a nonexpansive retraction from X onto its closed unit ball B_X . Moreover, Q generates the family of retractions $Q_r : X \rightarrow rB_X$ defined by

$$Q_r x = \begin{cases} 0 & \text{if } r = 0, \\ rQ\left(\frac{1}{r}x\right) & \text{if } r > 0. \end{cases}$$

It is easy to check that for any $r_1, r_2 \geq 0$ and $x, y \in X$ the following inequality holds:

$$\|Q_{r_1}x - Q_{r_2}y\| \leq \max \{\|x - y\|, |r_1 - r_2|\}.$$

The class of cut-invariant spaces include many classical spaces such as c_0 , c , $C[0, 1]$, $BC(\mathbb{R})$ and some of their subspaces.

The aforementioned result from [6] concerning the optimal retraction constant in cut-invariant spaces states that:

Theorem 3. *If a Banach space X is extremal and cut-invariant, then $k_0(X) \leq 4(1 + \sqrt{2})^2 \approx 23.31$.*

The above estimate was significantly improved in [11]. The main result of the quoted paper states that $k_0(X) \leq 4(2 + \sqrt{3}) \approx 14.92$ for any extremal and cut-invariant Banach space X .

Note, however, that there is a wide collection of Banach spaces which are extremal but not cut-invariant and the best retraction constant for this class is provided by Theorem 2. Examples of such spaces include Banach spaces $C^n[0, 1]$ of differentiable functions provided with all kinds of standard norms. Much more examples can be found among the hyperplanes of

classical Banach spaces, e.g.,

$$X_1 = \left\{ x \in c : \lim_{i \rightarrow \infty} x_i = \frac{1}{2}x_1 \right\}, \quad X_2 = \left\{ x \in C[0, 1] : \int_0^1 x(t)dt = 0 \right\}.$$

Note that spaces $C^n[0, 1]$ contain a copy of c_0 , either isomorphic or isometric, depending on the norm. Spaces X_1, X_2 contain an isometric copy of c_0 . Extremality of all the above mentioned spaces follows from Theorem 1.

In the next section we will provide a general estimate of $k_0(X)$ for the class of separable Banach spaces containing an isometric copy of c_0 which is better than the one provided by Theorem 2.

The most current state of knowledge about the optimal retraction problem and related topics can be found in the book [3], which is the second edition of [12].

2. Main result

Inspired by the construction from [6] we will now prove the following result.

Theorem 4. *If X is a separable Banach space which contains an isometric copy of c_0 , then $k_0(X) \leq 4(1 + \sqrt{2})^2$.*

In the proof of this fact we will make use of the map $T : B_{c_0} \rightarrow B_{c_0}$ from Example 20.2 in [5] defined by the formula

$$Tx = (1, \alpha(kx_1), \alpha(kx_2), \alpha(kx_3), \dots), \quad k > 1 \text{ fixed},$$

where α is the cut function defined in the previous section. It is easily verified that $T \in \mathcal{L}(k)$ and for any $x \in B_{c_0}$, $\|x - Tx\| > 1 - 1/k$. Also, recall that if a separable Banach space contains an isometric copy of c_0 , then there is another copy of c_0 which is 1-complemented (see e.g. [7] for details).

Proof of Theorem 4. If X contains an isometric copy of c_0 , then there exists another subspace $Z \subset X$ such that Z is isometric to c_0 and there is a linear projection P from X onto Z such that $\|P\| = 1$. Let $\phi : Z \rightarrow c_0$ denote the isometry and let $T : B_Z \rightarrow B_Z$ be k -lipschizian map such that $\|x - Tx\| > 1 - 1/k$ for any $x \in B_Z$. Now, consider $T_1 : 2B_X \rightarrow B_Z \subset B_X$

defined by the formula

$$T_1x = \begin{cases} TPx, & \|x\| \leq 1, \\ T\hat{Q}Px, & 1 < \|x\| \leq 2 - \frac{1}{k}, \\ \hat{Q}_{k(2-\|x\|)}T\hat{Q}Px, & 2 - \frac{1}{k} < \|x\| \leq 2, \end{cases}$$

where

$$\hat{Q} = \phi^{-1} \circ Q \circ \phi, \quad \hat{Q}_{k(2-\|x\|)} = \phi^{-1} \circ Q_{k(2-\|x\|)} \circ \phi,$$

are retractions from the previous section transferred to the space Z . We leave to the reader to check that $T_1 \in \mathcal{L}(k)$. Moreover, $\|x - T_1x\| > 1 - \frac{1}{k}$ for every $x \in 2B_X$. Indeed, if $\|x\| \leq 1$, then

$$\|x - T_1x\| = \|x - TPx\| \geq \|Px - PTPx\| = \|Px - TPx\| > 1 - \frac{1}{k}.$$

If $1 < \|x\| \leq 2 - \frac{1}{k}$, then

$$\begin{aligned} \|x - T_1x\| &= \|x - T\hat{Q}Px\| \geq \|Px - PT\hat{Q}Px\| = \|Px - T\hat{Q}Px\| \\ &\geq \|\hat{Q}Px - \hat{Q}T\hat{Q}Px\| = \|\hat{Q}Px - T\hat{Q}Px\| \\ &> 1 - \frac{1}{k}. \end{aligned}$$

Finally, if $2 - \frac{1}{k} < \|x\| \leq 2$, then $\|T_1x\| \leq k(2 - \|x\|)$. Consequently,

$$\|x - T_1x\| \geq \|x\| - k(2 - \|x\|) > 1 - \frac{1}{k}.$$

For $x \in B_X$, put $T_2x = \frac{1}{2}T_1(2x)$. Then $T_2 : B_X \rightarrow B_X$ is k -lipschitzian, $T_2(S_X) = \{0\}$ and for any $x \in B_X$,

$$\|x - T_2x\| = \|x - \frac{1}{2}T_1(2x)\| = \frac{1}{2}\|2x - T_1(2x)\| > \frac{1}{2}\left(1 - \frac{1}{k}\right).$$

Let P_{rad} denote the radial projection from X onto B_X and recall that $P_{rad} \in \mathcal{L}(2)$. Now, define the retraction $R : B_X \rightarrow S_X$ by the formula

$$Rx = P_{rad}\left(\frac{x - T_2x}{\frac{1}{2}(1 - \frac{1}{k})}\right).$$

Then, for every $x, y \in B_X$,

$$\begin{aligned} \|Rx - Ry\| &\leq \frac{2}{\frac{1}{2}(1 - \frac{1}{k})} \|x - T_2x - y + T_2y\| \\ &\leq \frac{4k}{k-1} (\|x - y\| + \|T_2x - T_2y\|) \\ &\leq 4 \frac{k(k+1)}{k-1}. \end{aligned}$$

Since $k > 1$ was arbitrary, we get

$$k_0(X) \leq 4 \min_{k>1} \frac{k(k+1)}{k-1} = 4(1 + \sqrt{2})^2 \approx 23.31.$$

□

Let us finish our work with some concluding remarks.

Remark 5. The spaces $C^n[0, 1]$ considered with the norm

$$\|x\| = \sum_{i=0}^{n-1} |x^{(i)}(0)| + \max_{t \in [0,1]} |x^{(n)}(t)|$$

contain an isometric copy of c_0 . Hence, $k_0(C^n[0, 1], \|\cdot\|) \leq 23.31$. So far, the best estimate was 30.84.

Remark 6. Recall that a Banach space X is called an L_1 -predual (or a Lindenstrauss space) if its dual space X^* is isometric to Lebesgue space $L_1(\mu)$ for some measure μ . A large collection of such spaces is presented in Section 4 in [8]. Another very interesting class of ℓ_1 -preduals was investigated in [2], namely, the class of hyperplanes $W_f = \ker f$ in the space c of convergent sequences, where $f \in \ell_1 = c^*$, $\|f\| = 1$ and $|f_{j_0}| \geq 1/2$ for some $j_0 \in \mathbb{N}$. Hyperplanes in c are a great source of extremal spaces which fail to be cut-invariant.

The old result of Zippin [13] states that every infinite-dimensional L_1 -predual contains an isometric copy of c_0 . In the light of the Theorem 4, $k_0(X) \leq 23.31$ for any separable L_1 -predual X . Again, the best constant for this class of spaces was 30.84.

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Summary

In every infinite-dimensional Banach space there exists a lipschitzian retraction from the closed unit ball onto its boundary. In this chapter we prove that if a separable Banach space contains an isometric copy of c_0 , then the optimal retraction constant of the space does not exceed $4(1 + \sqrt{2})^2$. In particular, this provides a new universal retraction constant in the class of separable L_1 -preduals.

Streszczenie

W każdej nieskończonej wymiarowej przestrzeni Banacha istnieje retrakcja lipschitzowska z domkniętej kuli jednostkowej na jej brzeg. W tym rozdziale dowodzimy, że jeśli ośrodkowa przestrzeń Banacha zawiera izometryczną kopię przestrzeni c_0 , to optymalna stała retrakcji w tej przestrzeni nie przekracza $4(1 + \sqrt{2})^2$. W szczególności, otrzymujemy w ten sposób nową uniwersalną stałą retrakcji w klasie ośrodkowych przestrzeni L_1 -predualnych.

Chapter 3

ISOMETRIES BETWEEN SUBSPACES OF CODIMENSION k OF THE SPACE $C([1, \omega k])$

Marek Malec

1. Hyperplanes in the space c

We begin by introducing some necessary notation. Let X and Y be Banach spaces. Then B_X and S_X stand for the closed unit ball of X and the unit sphere of X , respectively. If $A \subset X$, then $\text{ext}A$ is the set of all extreme points of A . By $X = Y$ we mean that X is isometrically isomorphic to Y . The dual of X is denoted by X^* and $\sigma(X^*, X)$ stands for the weak* topology on X^* induced by X . If $A \subset X^*$, then by $\overline{A}^*$ we mean the $\sigma(X^*, X)$ -closure of A and by A' the set of all $\sigma(X^*, X)$ -cluster points of A ; that is,

$$A' = \left\{ x^* \in X^* : x^* \in \overline{(A \setminus \{x^*\})}^* \right\}.$$

Let us recall now that c^* can be identified with ℓ_1 via duality map $\phi : \ell_1 \rightarrow c^*$ given by the formula

$$\phi(x^*)(x) = x^*(1) \lim_{j \rightarrow \infty} x(j) + \sum_{j=1}^{\infty} x^*(j+1)x(j),$$

where $x^* = (x^*(1), x^*(2), \dots) \in \ell_1$ and $x = (x(1), x(2), \dots) \in c$. For any $x^* \in B_{\ell_1}$ we define the hyperplane W_{x^*} of the space c by

$$W_{x^*} = \left\{ x \in c : \lim_{j \rightarrow \infty} x(j) = \sum_{j=1}^{\infty} x^*(j)x(j) \right\}.$$

Hyperplanes W_{x^*} were intensively studied over the last 10 years. The research was initialized by E. Casini, E. Miglierina and Ł. Piasecki in [2], where they determined the basic properties of W_{x^*} hyperplanes.

Theorem 1 (E. Casini, E. Miglierina, Ł. Piasecki [2]). *Let $x^* \in B_{\ell_1}$. Then $W_{x^*}^* = \ell_1$ and the duality is given by the map $\phi : \ell_1 \rightarrow W_{x^*}^*$ defined by*

$$\phi(y^*)(x) = \sum_{j=1}^{\infty} y^*(j)x(j),$$

where $y^* \in \ell_1$ and $x \in W_{x^*}$. Moreover, if $(e_j^*)_{j \in \mathbb{N}}$ denotes the standard basis in ℓ_1 , then

$$e_j^* \xrightarrow{j \rightarrow \infty} x^*.$$

Remark 2. In [2] authors relied on a different definition which uses kernels of functionals on c instead of weak*-limits to identify a particular hyperplane. The definitions are equivalent and their relation can be described in a simple and direct manner:

$$W_{x^*} = W_f = \ker f = \left\{ x \in c : f(1) \lim_{j \rightarrow \infty} x(j) + \sum_{j=1}^{\infty} f(j+1)x(j) = 0 \right\},$$

where

$$f = \left(\frac{1}{1 + \|x^*\|}, \frac{-x^*(1)}{1 + \|x^*\|}, \frac{-x^*(2)}{1 + \|x^*\|}, \dots \right) \in S_{c^*}.$$

For a suitable vector $x^* \in B_{\ell_1}$, the hyperplane W_{x^*} can be isometric to the space c itself or to the space c_0 of sequences convergent to 0.

Remark 3 (E. Casini, E. Miglierina, Ł. Piasecki [2]). Let $x^* \in B_{\ell_1}$. Then

- $W_{x^*} = c$ if and only if $|x^*(j)| = 1$ for some $j \in \mathbb{N}$,

- $W_{x^*} = c_0$ if and only if $x^* = (0, 0, \dots)$.

Now, we get to the point.

Theorem 4 (E. Casini, E. Miglierina, Ł. Piasecki [2]). *Let X be an ℓ_1 -predual such that $e_j^* \xrightarrow{j \rightarrow \infty} x^*$. Then $X = W_{x^*}$.*

A somewhat different variant of this result was provided 3 years later by the last author.

Theorem 5 (Ł. Piasecki [3]). *Let X be an ℓ_1 -predual for which $(\text{ext } B_{\ell_1})' = \{\pm x^*\}$; that is, $(\text{ext } B_{\ell_1})'$ is either a singleton or consists of two elements. Then there exists $y^* \in B_{\ell_1}$ such that $X = W_{y^*}$.*

In other words, if $(\text{ext } B_{\ell_1})' = \{\pm x^*\}$, a hyperplane is (up-to-isometry) uniquely determined by the choice of $x^* \in B_{\ell_1}$. The question arises, what happens when there are more weak*-cluster points; for instance, $(\text{ext } B_{\ell_1})' = \{\pm x_1^*, \pm x_2^*\}$?

2. Subspaces of codimension k in $C([1, \omega k])$

As uncovered by E. Casini, M. Miglierina and Ł. Piasecki [1], all ℓ_1 -preduals X such that the ℓ_1 standard basis has a finite amount of $\sigma(\ell_1, X)$ -cluster points are located among subspaces of the spaces $C([1, \omega k])$ of codimension k , where $C([1, \omega k])$ is the space of continuous functions on the ordinal interval $[1, \omega k]$ equipped with the order topology and ω denotes the first infinite ordinal. In order to accurately quote the said result, we need to introduce appropriate notation. For every $k \in \mathbb{N}$ we put

$$\Omega_0^k = [1, \omega k] \setminus \{\omega, \omega 2, \dots, \omega k\}.$$

From now on, we identify ordinals $\omega(i-1) + j$ with (i, j) and ordinals ωi with $(i, 0)$ for $i, j \in \mathbb{N}$. We also note that the ordinal interval $[1, \omega)$ is homeomorphic to the set of natural numbers $\mathbb{N}$ endowed with the discrete topology. For that reason, we can use ordinals $1 \leq n < \omega$ interchangeably with $n \in \mathbb{N}$.

It is well known that the dual of the space $C([1, \omega k])$ is isometric to $\ell_1([1, \omega k])$ and the duality is given by the linear map $\phi : \ell_1([1, \omega k]) \rightarrow$

$C([1, \omega k])^*$ defined by

$$\phi(y^*)(f) = \sum_{i=1}^k \left(\sum_{j=1}^{\infty} f(i, j) y^*(i, j) \right) + \sum_{i=1}^k f(i, 0) y^*(i, 0)$$

for every $y^* \in \ell_1([1, \omega k])$ and every $f \in C([1, \omega k])$.

The standard basis of $\ell_1([1, \omega k])$ is denoted by $(e_{i,j}^*)$, so for $f \in C([1, \omega k])$

$$e_{i,j}^*(f) = f(i, j),$$

where $i = 1, \dots, k$ and $j \in \mathbb{N} \cup \{0\}$. Therefore, an element $y^* \in \ell_1([1, \omega k])$ can be written as

$$y^* = \sum_{i=1}^k \left(\sum_{j=0}^{\infty} y^*(i, j) e_{i,j}^* \right)$$

and, similarly, each vector $z^* \in \ell_1(\Omega_0^k)$ can be written as

$$z^* = \sum_{i=1}^k \left(\sum_{j=1}^{\infty} z^*(i, j) e_{i,j}^* \right).$$

For arbitrarily chosen elements $x_1^*, \dots, x_k^* \in B_{\ell_1(\Omega_0^k)}$, we define the subspace $W_{x_1^*, \dots, x_k^*}$ of $C([1, \omega k])$ as

$$W_{x_1^*, \dots, x_k^*} = \left\{ f \in C([1, \omega k]) : \quad \forall_{s=1, \dots, k} \quad f(s, 0) = \sum_{i=1}^k \left(\sum_{j=1}^{\infty} f(i, j) x_s^*(i, j) \right) \right\}.$$

Remark 6. In the special case when $k = 1$, we obviously get the already reviewed class of ℓ_1 -predual hyperplanes in c .

We can recall the aforesaid results now.

Theorem 7 (E. Casini, E. Miglierina, Ł. Piasecki [1]). *The dual of the space $W_{x_1^*, \dots, x_k^*}$ is isometric to ℓ_1 (more explicitly, to $\ell_1(\Omega_0^k)$) and the duality is given by the linear map $\phi : \ell_1(\Omega_0^k) \rightarrow W_{x_1^*, \dots, x_k^*}^*$ defined by*

$$\phi(z^*)(f) = \sum_{i=1}^k \left(\sum_{j=1}^{\infty} f(i, j) z^*(i, j) \right)$$

for every $z^* \in \ell_1(\Omega_0^k)$ and every $f \in W_{x_1^*, \dots, x_k^*}$. Moreover, for $s = 1, \dots, k$,

$$e_{s,j}^* \xrightarrow[j \rightarrow \infty]{\sigma(\ell_1(\Omega_0^k), W_{x_1^*, \dots, x_k^*})} x_s^*.$$

Theorem 8 (E. Casini, E. Miglierina, Ł. Piasecki [1]). *Let X be a Banach space such that $X^* = \ell_1(\Omega_0^k)$ and it holds that*

$$e_{s,j}^* \xrightarrow[j \rightarrow \infty]{\sigma(\ell_1(\Omega_0^k), X)} x_s^*$$

for each $s = 1, \dots, k$. Then $X = W_{x_1^*, \dots, x_k^*}$.

Corollary 9 (E. Casini, E. Miglierina, Ł. Piasecki [1]). *Let $x_1^*, \dots, x_k^* \in B_{\ell_1}$. Then there exists a closed subspace W of $C([1, \omega k])$ of codimension k such that*

- (a) $W^* = \ell_1$,
- (b) $x_1^*, \dots, x_k^*$ are the $\sigma(\ell_1, W)$ -cluster points of the ℓ_1 standard basis.

We also recall a well-known fact that if $T : \ell_1 \rightarrow \ell_1$ is an isometric isomorphism, then it takes the form

$$T(e_j^*) = \varepsilon(j)e_{\pi(j)}^*,$$

where $(e_j^*)_{j \in \mathbb{N}}$ is the ℓ_1 standard basis, $\varepsilon : \mathbb{N} \rightarrow \{-1, 1\}$ and $\pi : \mathbb{N} \rightarrow \mathbb{N}$ is a permutation.

In [1] the authors presented the following example.

Example 10 (E. Casini, E. Miglierina, Ł. Piasecki [1]). Take $x_1^*, x_2^* \in B_{\ell_1(\Omega_0^2)}$ defined by

$$x_1^*(i, j) = \begin{cases} \frac{1}{2^{2j-1}} & \text{for } i = 1 \text{ and } j \in \mathbb{N} \\ \frac{1}{2^{2j}} & \text{for } i = 2 \text{ and } j \in \mathbb{N} \end{cases}$$

and $x_2^*(i, j) = 0$ for $i = 1, 2$ and $j \in \mathbb{N}$. Then

$$W_{x_1^*, x_2^*} = \left\{ f \in C([1, \omega 2]) : f(1, 0) = \sum_{j=1}^{\infty} \frac{f(1, j)}{2^{2j-1}} + \sum_{j=1}^{\infty} \frac{f(2, j)}{2^{2j}}, f(2, 0) = 0 \right\}.$$

By Theorem 7 we have $W_{x_1^*, x_2^*} = \ell_1(\Omega_0^2)$ and

$$\begin{aligned} e_{1,j}^* &\xrightarrow{j \rightarrow \infty} x_1^* = \left(\frac{1}{2}, \frac{1}{8}, \frac{1}{32}, \dots; \frac{1}{4}, \frac{1}{16}, \frac{1}{64}, \dots \right), \\ e_{2,j}^* &\xrightarrow{j \rightarrow \infty} x_2^* = (0, 0, 0, \dots; 0, 0, 0, \dots). \end{aligned}$$

On the other hand, we get

$$\begin{aligned} W_{x_2^*, x_1^*} &= \left\{ f \in C([1, \omega 2]) : f(1, 0) = 0, f(2, 0) = \sum_{j=1}^{\infty} \frac{f(1, j)}{2^{2j-1}} + \sum_{j=1}^{\infty} \frac{f(2, j)}{2^{2j}} \right\}, \\ e_{1,j}^* &\xrightarrow{j \rightarrow \infty} x_2^* = (0, 0, 0, \dots; 0, 0, 0, \dots), \\ e_{2,j}^* &\xrightarrow{j \rightarrow \infty} x_1^* = \left(\frac{1}{2}, \frac{1}{8}, \frac{1}{32}, \dots; \frac{1}{4}, \frac{1}{16}, \frac{1}{64}, \dots \right). \end{aligned}$$

Clearly,

$$\left(\text{ext } B_{W_{x_1^*, x_2^*}^*} \right)' = \left(\text{ext } B_{W_{x_2^*, x_1^*}^*} \right)'.$$

But is it true that $W_{x_1^*, x_2^*} = W_{x_2^*, x_1^*}$? Suppose that $T : W_{x_1^*, x_2^*} \rightarrow W_{x_2^*, x_1^*}$ is an isometric isomorphism. Then its adjoint T^* is a weak*-continuous isometry from $W_{x_2^*, x_1^*}^* = \ell_1(\Omega_0^2)$ onto $W_{x_1^*, x_2^*}^* = \ell_1(\Omega_0^2)$. By weak*-continuity we obtain

$$T^*(e_{2,j}^*) \xrightarrow{j \rightarrow \infty} T^*(x_1^*),$$

and since T^* is an isometry, we must have $T^*(x_1^*) = \pm x_1^*$. W.L.O.G. we may assume that $T^*(x_1^*) = x_1^*$. But this means that

$$T^*(e_{2,j}^*) = e_{2,j}^* \xrightarrow{j \rightarrow \infty} x_2^*,$$

which is a contradiction.

Consequently, the set $(\text{ext } B_{\ell_1})'$ itself does not give us enough information to identify a specific ℓ_1 -predual. As we will see in the next example, there is even more to that.

Example 11. Let X be an ℓ_1 -predual such that $(\text{ext } B_{\ell_1})' = \{\pm x_1^*, \pm x_2^*\}$,

where $x_1^* = \left(\frac{1}{2}, \frac{1}{4}, \frac{1}{8}, \dots\right)$ and $x_2^* = (0, 0, \dots)$. Consider the following rearrangements of x_1^* into $\ell_1(\Omega_0^2)$:

$$\begin{aligned} y_1^* &= \left(\frac{1}{2}, \frac{1}{8}, \frac{1}{32}, \frac{1}{128}, \dots; \frac{1}{4}, \frac{1}{16}, \frac{1}{64}, \frac{1}{256}, \dots\right), \\ y_2^* &= \left(\frac{1}{2}, \frac{1}{4}, \frac{1}{8}, \frac{1}{32}, \frac{1}{64}, \frac{1}{128}, \dots; \frac{1}{16}, \frac{1}{256}, \dots\right), \dots, \\ y_n^* &= \left(\frac{1}{2}, \frac{1}{4}, \dots, \frac{1}{2^{2^n-1}}, \frac{1}{2^{2^n+1}}, \frac{1}{2^{2^n+2}}, \dots; \frac{1}{2^{2^n}}, \frac{1}{2^{2 \cdot 2^n}}, \frac{1}{2^{3 \cdot 2^n}}, \dots\right), \dots \end{aligned}$$

Namely, in each step we remove odd terms of the sequence $(y_n^*(2, j))_{j \in \mathbb{N}}$ and insert them into the sequence $(y_{n+1}^*(1, j))_{j \in \mathbb{N}}$. Every rearrangement of x_2^* is denoted as $y^* = (0, 0, \dots; 0, 0, \dots)$. These rearrangements correspond to isometries from ℓ_1 onto $\ell_1(\Omega_0^2)$, and each of them determines which subsequence of $\text{ext}B_{\ell_1}$ is $\sigma(\ell_1, X)$ -convergent to a given element of $(\text{ext}B_{\ell_1})'$ with regard to the definition of $W_{x_1^*, \dots, x_k^*}$ spaces.

Analogically as in Example 10, assume that $T^* : W_{y_2^*, y^*}^* \rightarrow W_{y_1^*, y^*}^*$ is a weak*-continuous isometric isomorphism. W.L.O.G. we get

$$T^*(e_{1,j}^*) \xrightarrow[j \rightarrow \infty]{\sigma(\ell_1(\Omega_0^2), W_{y_1^*, y^*})} T^*(y_2^*) = y_1^*.$$

Thus,

$$T^*(e_{1,3j-1}^*) = e_{2,2j-1}^* \xrightarrow[j \rightarrow \infty]{\sigma(\ell_1(\Omega_0^2), W_{y_1^*, y^*})} y^*,$$

another contradiction. By conducting the same reasoning for other $n \in \mathbb{N}$, we conclude that $(W_{y_n^*, y^*})_{n \in \mathbb{N}}$ is a sequence of pairwise non-isometric spaces such that x_1^*, x_2^* are the $\sigma(\ell_1, W_{y_n^*, y^*})$ -cluster points of the ℓ_1 standard basis for each $n \in \mathbb{N}$.

As we can see, the environment is much more rich when the ℓ_1 standard basis is not convergent. Already when $k = 2$, we have infinite possibilities. Therefore, to precisely identify the structure of an ℓ_1 -predual we not only need the set of weak*-cluster points of the ℓ_1 standard basis, but we also need to know which subsequence of the basis is weak*-convergent to a given cluster point.

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Summary

We look into the class of spaces of codimension k of spaces $C([1, \omega k])$, which serves as an isometric model repository for all ℓ_1 -preduals X such that the standard basis in ℓ_1 has a finite amount of weak*-cluster points. In the profoundly studied case of $k = 1$, the set $(\text{ext} B_{\ell_1})'$ of weak*-cluster points of the extreme points of the closed unit ball of ℓ_1 is enough to uniquely determine a specific ℓ_1 -predual. However, this is no longer true once $k \geq 2$. Indeed, already when $k = 2$, there may be infinitely many non-isometric ℓ_1 -preduals, sharing the same set $(\text{ext} B_{\ell_1})'$.

Streszczenie

Badamy klasę podprzestrzeni kowymiaru k przestrzeni $C([1, \omega k])$, która służy jako repozytorium izometrycznych modeli dla wszystkich przestrzeni ℓ_1 -predualnych X , dla których standardowa baza w ℓ_1 ma skończoną ilość słabych*-punktów skupienia. W dogłębnie przestudiowanym przypadku $k = 1$, zbiór $(\text{ext} B_{\ell_1})'$ słabych*-punktów skupienia punktów ekstremalnych domkniętej kuli jednostkowej w ℓ_1 wystarcza do jednoznacznego określenia konkretnej przestrzeni ℓ_1 -predualnej. Jednakże, nie jest to już prawdą gdy $k \geq 2$. W istocie, już gdy $k = 2$, może być nieskończenie wiele nieizometrycznych przestrzeni ℓ_1 -predualnych, dzielących ten sam zbiór $(\text{ext} B_{\ell_1})'$.

Chapter 4

ON A CERTAIN RENORMING OF l_2

Bożena Piątek

1. Introduction

Let (M, d) be a complete metric space and $T: M \rightarrow M$. Clearly, T is Lipschitzian if there is a $k > 0$ such that

$$d(Tx, Ty) \leq k \cdot d(x, y)$$

and $\bar{x}$ is called a fixed point if $\bar{x} = T(\bar{x})$.

The best-known result concerning the existence of fixed points is the Banach contraction principle, which says that each contraction, i.e. a Lipschitz mapping with $k < 1$, has exactly one fixed point.

An entirely different situation occurs if k is equal to 1. Then, we say that T is nonexpansive and the existence of fixed points cannot be guaranteed if M is a closed convex subset of a Banach (or Hilbert) space (see [12]). Furthermore, the more restrictive conditions imposed on a set, such as weak compactness, do not have to improve the result in the case of Banach spaces (research background can be found in [9]). The same can happen if the norm in a Hilbert space is changed. One of the best-known questions in this field is the following:

Question 1. Let $\|\cdot\|$ be a norm in l_2 equivalent to $\|\cdot\|_2$, i.e. there are $a, b > 0$ such that

$$a \cdot \|x\|_2 \leq \|x\| \leq b \cdot \|x\|_2, \quad x \in l_2.$$

Does each nonexpansive self-mapping T of a closed bounded convex set have a fixed point?

If a Banach space has the property discussed in the question, we say that the space has the fixed point property. Obviously, in a closed bounded convex subset of a Banach space a nonexpansive self-mapping T has an approximate fixed point sequence, i.e. a sequence (x_n) for which $\|x_n - T(x_n)\| \rightarrow 0$.

The main goal of this chapter is to investigate the existence of fixed points in a certain renorming of l_2 , which does not satisfy any of the well-known conditions guaranteeing the fixed point property.

2. Background of l_2 renormings

As usually, l_2 denotes the Hilbert space of square-summable sequences with the natural basis $e_n = (0, \dots, 0, 1, 0, \dots)$. In 2021, G. Dutta and P. Veeramani introduced the following renorming of l_2 :

$$X_q = (l_2, |\cdot|) \quad \text{and} \quad |x| := \max\{\|x\|_2, q(x)\},$$

where

$$q(x) = \beta \sup \left\{ \frac{|x(1) + \dots + x(n)|}{a_n} \right\}$$

and (a_n) is a sequence of nonzero numbers. In particular, they asked whether the space X_q has the fixed point property for $a_n = n$. The present author's affirmative answer to this question can be found in [11]. Moreover, in the same paper, it was shown that the nonexpansive self-mappings also have fixed points under a much weaker assumption that $a_n = O(\sqrt{n})$. Note that the condition $a_n = O(\sqrt{n})$ is necessary because it guarantees that the norm in X_q is equivalent to the norm $\|\cdot\|_2$ (see [11, Lemma 2.1]).

However, the proof of the existence of fixed points for nonexpansive self-mappings is not direct. It is based on the fact that the space X_q satisfies the WORTH property. Let us recall that a Banach space X has the WORTH property if

$$\lim_n \|\|x_n + x\| - \|x_n - x\|\| = 0$$

for all weakly null sequences (x_n) and $x \in X$ (see [3, 13]).

The fact that X_q has the WORTH property follows from the following lemma (see [11, Lemma 4.2]):

Lemma 2. (*Piątek*) *For each x and (x_n) such that $x_n \xrightarrow{w} 0$ the following equality holds:*

$$\lim_n q(x + x_n) = \max \left\{ q(x), \lim_n q(x_n) \right\}$$

as long as both limits exist.

Note that the same lemma can be applied to show that the space X_q also satisfies some more conditions that we will recall in the sequel.

In [5], J. Garcia-Falset introduced a coefficient $R(X)$ in the following way

$$R(X) = \sup \{ \liminf_n \|x + x_n\| \},$$

where the supremum is taken over all weakly null sequences in B_X and over all vectors $x \in B_X$.

In the same paper, it was proved that a reflexive Banach space X has the fixed point property provided the space satisfies the weak Opial condition and $R(X) < 2$. In [6], the same author showed that the Opial property is not necessary to obtain the fixed point result for a reflexive Banach space X for which $R(X) < 2$.

In [1], the coefficient $R(X)$ was generalized to $M(X)$. Namely, let

$$R(a, X) = \sup \{ \liminf_n \|x + x_n\| \}$$

and now the supremum is taken over all $x \in X$ with $\|x\| \leq a$ and all weakly null sequences with $D[(x_n)] \leq 1$. Recall that

$$D[(x_n)] = \limsup_n \left(\limsup_m \|x_n - x_m\| \right)$$

as long as the sequence (x_n) is bounded. Then,

$$M(X) = \sup \left\{ \frac{1+a}{R(a, X)} : a \geq 0 \right\}.$$

And a reflexive Banach space has the fixed point property provided $M(X) > 1$ (see [1, Theorem 2.2]).

Let us recall that the space $l_{2,1}$ is defined as l_2 with the norm given by

$$\|x\| = \|x^+\|_2 + \|x^-\|_2,$$

where $x^+(n) = \max\{x(n), 0\}$ and $x^-(n) = \max\{-x(n), 0\}$. Hence, the coefficient $R(l_{2,1}) = 2$, but $M(l_{2,1}) \geq \sqrt{2}$ (see [1, Remark 2]).

Now, we will show that

$$R(X_q) = \sqrt{2} \tag{1}$$

as long as the sequence (a_n) is such that $a_n = \sqrt{n}$. Simultaneously, we can just assume that $a_n = O(\sqrt{n})$.

Example 3. Let $\beta > 2$ and x be an arbitrary element of the unit ball. Take any weakly null sequence (x_n) such that $|x_n| \leq 1$, $n \in \mathbb{N}$. We will estimate the norm $|x + x_n|$. We consider two cases:

Case I. Let $|x + x_n| = \|x + x_n\|_2$ for infinitely many n . Due to the well-known equality for l_2 space,

$$\limsup_n \|x + x_n\|_2^2 = \|x\|_2^2 + \limsup_n \|x_n\|_2^2 \leq 2.$$

Case II. If $|x + x_n| = q(x + x_n)$ for infinitely many n , then there is a sequence of natural numbers $(k(n))$ for which $\limsup_n q(x + x_n) = \lim_n q(x + x_{k(n)})$. Moreover, for each subsequence $(x_{l(n)})$ of $(x_{k(n)})$ such that the limit $\lim_n q(x_{l(n)})$ exists, the following inequality holds:

$$\lim_n q(x + x_{l(n)}) = \max \left\{ q(x), \lim_n q(x_{l(n)}) \right\} \leq 1.$$

Finally, we get $\limsup_n |x + x_n| \leq \sqrt{2}$, which completes the proof of (1).

Next, we focus on the renorming introduced in [8] (also see [4, 7]) by E. Llorens Fuster and A. Jimenez Melado. This result is worth emphasising as the space does not satisfy any of the best-known properties that guarantee the fixed point property. Let $X = (l_2, \|\cdot\|)$ with

$$\|x\| := \max \left\{ \frac{1}{3} \|x\|_2, \mathcal{S}(x), \mathcal{M}(x) \right\}$$

and

$$\mathcal{S}(x) = \sup_{n \geq 2} |x(1) + x(n) + x(n+1) + x(n+2)|,$$

$$\mathcal{M}(x) = \sup_{i,j \in \mathbb{N}} |x(2i-1)| + |x(2j)|.$$

The proof of the fact that $M(X) = 1$ can be deduced from the more general result. Namely, the space does not satisfy a more general condition, the Prus-Szczepanik property (see [8], where the reader can find a much longer list of the well-known properties that, on the one hand, guarantee the existence of a fixed point, but, at the same time, do not hold in the space X).

However, in the same paper the authors proposed a method that allows us to show that each nonexpansive self-mapping T has a fixed point. For the reader's convenience, let us introduce the main idea of the proof. Let $p: X \rightarrow \mathbb{R}$ be a semi-norm and assume that there is $k \in \mathbb{N}$ such that for all $x_1, \dots, x_k$ in l_2 with pairwise disjoint supports, we have

$$p(z) \leq \max\{p(z - x_1), \dots, p(z - x_k)\}, \quad z \in l_2. \quad (2)$$

If $T: C \rightarrow C$ is a nonexpansive self-mapping of a weakly-compact and convex subset C of the space $(l_2, \|\cdot\|)$, where $\|x\| = \max\{\|x\|_2, p(x)\}$ and p satisfies the aforementioned condition, then the minimal weakly compact convex and T -invariant set $K \subset C$ must be a singleton. Clearly, in the case of the semi-norm $p(x) = \max\{\mathcal{S}(x), \mathcal{M}(x)\}$, the condition holds. Note that this idea can also be applied, for instance, to show the existence of fixed points in the case of James' or van Dulst's renorming of l_2 . Unfortunately, although the norm of the space X_q can be described as a maximum of the norm of l_2 and the semi-norm q , it does not satisfy the condition (2) – see [2, Theorem 31] – as long as the sequence (a_n) is (strictly) increasing.

3. Main results

The results described in the previous sections motivate us to ask the following question:

Question 4. *Let us define the norm*

$$|x| = \max \left\{ \frac{1}{3} \|x\|_2, \mathcal{S}(x), \mathcal{M}(x), q(x) \right\}, \quad (3)$$

where the sequence (a_n) is such that $a_n = O(\sqrt{n})$. Does the space $X = (l_2, |\cdot|)$ have the fixed point property?

The main goal of this chapter is to give an affirmative answer to this question. Before we proceed to the proof, let us show that our renorming does not satisfy some of the aforementioned conditions. First, we will prove that the space X does not satisfy the WORTH property. To simplify our considerations, we assume that $a_n = \sqrt{n}$ in a similar way as it was done in [11].

Example 5. Let us choose $x = \frac{1}{\beta}e_1$ and $x_n = e_{2M+2n-1}$, where M is so large that $a_{2M+k} > 2\beta$ for $k > 0$, where $\beta > 2$. Hence, we get

$$\|x \pm x_n\|_2^2 = 1 + 1/\beta^2.$$

Similarly,

$$q(x \pm x_n) = \mathcal{M}(x \pm x_n) = 1.$$

And finally,

$$\mathcal{S}(x + x_n) = 1 + \frac{1}{\beta}, \quad \text{but} \quad \mathcal{S}(x - x_n) = 1.$$

These lead to

$$|x + x_n| = \frac{\beta + 1}{\beta} > \frac{\sqrt{\beta^2 + 1}}{\beta} = |x - x_n|$$

for all natural n and the space X does not have the WORTH property.

Now, we will see that $R(X) = 2$ while $M(X) = 1$:

Example 6. Let us choose an odd number N such that $a_n > \beta$ for all $n \geq N$ and let $x = e_N$. We take $x_m = e_{N+4m-1}$ as a weakly null sequence. Therefore, $|x| = \mathcal{M}(x) = 1$ and $|x_m| = \mathcal{M}(x_m) = 1$ for all natural m . Simultaneously,

$$\mathcal{M}(x + x_m) = 2, \quad m \in \mathbb{N},$$

from which it follows that $R(X) = 2$ and the fixed point property cannot be deduced from the aforementioned result of Garcia Falset. In the same way, one can show that the coefficient $M(X)$ is equal to 1.

Our next step is to show the fixed point property of X . To do this, we apply the ultra-power method. Let Y be a Banach space and let $[Y]$ denote the quotient space $l_\infty(Y)/c_0(Y)$ endowed with the quotient norm $\|[y_n]\| = \limsup_n \|y_n\|$, where $[y_n]$ denotes the equivalent class of a sequence (y_n) in $l_\infty(Y)$. Then, identifying each point $x \in Y$ with the equivalent class of the sequence $(x, x, \dots)$, we can treat Y as a subset of $[Y]$. In the same way, for each $M \subset Y$ we denote the set $\{[y_n] : y_n \in M, n \in \mathbb{N}\}$ by $[M]$. If M is a weakly compact and convex subset of Y , then $[M]$ is a closed bounded and convex subset of $[Y]$. Furthermore, for any bounded mapping T one can consider $[T]$ given by $[T]([y_n]) = [T(y_n)]$ and for each approximate fixed point sequence (z_n) the element $[z_n]$ is a fixed point of $[T]$. The following lemma due to Lin is an ultra-power counterpart of the well-known Goebel–Karlovitiz result (see [6, 10]):

Lemma 7. (*Lin*): *Let K be a minimal weakly compact convex subset for a nonexpansive mapping T . If $[W]$ is any nonempty closed convex subset of $[K]$ which is invariant under $[T]$, then $\sup\{\|[w_n] - x\| : [w_n] \in [W]\} = \text{diam}(K)$ for every $x \in K$.*

The aforementioned lemma will be a key tool in the proof of our main theorem:

Theorem 8. *The space $X = (l_2, |\cdot|)$ with the norm given by (3) has the fixed point property.*

Proof. Let C be a weakly compact and convex subset of l_2 and $T: C \rightarrow C$ be a nonexpansive mapping. According to the Zorn's lemma, the set C contains a weakly compact and convex subset K , which is minimal with respect to T . That means $T(K) \subset K$ and no strictly smaller weakly compact and convex subset of K is T -invariant, which yields $\bar{co}T(K) = K$. If $\text{diam}(K) = 0$, then K is a singleton and it contains a fixed point. Otherwise, we assume $\text{diam}(K) > 0$. Without loss of generality, we may assume that $\text{diam}(K) = 1$ and, moreover, there exists a weakly null approximate fixed point sequence (x_n) in K .

Therefore, there is a subsequence $(x_{k(n)})$ and a sequence (y_n) in l_2 for which

1. $\text{supp } y_i \cap \text{supp } y_j = \emptyset$ if $i \neq j$;
2. $\limsup_n \|x_{k(n)} - y_n\| = 0$.

Condition 2 guarantees that $[y_n] = [x_{k(n)}] \in [K]$, where $[K]$ is the set described in the preceding part of the section.

In the following, we define $z_n^i = y_{3n+i}$, $i = 1, 2, 3$.

Let us define

$$[W] := \left\{ [u_n] \in [K] : |[z_n^i] - [u_n]| \leq \frac{3}{4}, i = 1, 2, 3; D([u_n]) \leq \frac{3}{4} \right\}.$$

The set $[W]$ is not empty, because it contains $[v_n]$, when $v_n = \frac{z_n^1 + z_n^2 + z_n^3}{4}$. Moreover, the set is closed, convex and $[T]$ -invariant.

Let us take any $[u_n] \in [W]$ and consider the following three estimates:

1. Since Condition 1 guarantees that z_n^1, z_n^2, z_n^3 have pairwise disjoint supports, we get

$$\sum_{i=1}^3 \|z_n^i - u_n\|_2^2 = 2\|u_n\|_2^2 + \|u_n - (z_n^1 + z_n^2 + z_n^3)\|_2^2.$$

Hence,

$$2\|u_n\|_2^2 \leq \sum_{i=1}^3 \|z_n^i - u_n\|_2^2 \leq \sum_{i=1}^3 \left(3 \cdot |z_n^i - u_n| \right)^2,$$

which leads to

$$\limsup \|u_n\|_2 \leq \sqrt{\frac{27}{2} \cdot \left(\frac{3}{4}\right)^2} = \frac{9\sqrt{6}}{8}.$$

2. The semi-norm $p(z) = \max\{\mathcal{M}(z), \mathcal{S}(z)\}$ satisfies the condition (2), so,

$$\begin{aligned} \limsup_n p(u_n) &\leq \limsup_n \max\{p(z_n^1 - u_n), p(z_n^2 - u_n), p(z_n^3 - u_n)\} \\ &\leq \max_{i \in \{1,2,3\}} (\limsup_n p(z_n^i - u_n)) \\ &\leq \max_{i \in \{1,2,3\}} \limsup_n |z_n^i - u_n| \leq \frac{3}{4}. \end{aligned}$$

3. For the semi-norm q we begin with similar patterns as in Case II

of Example 3. Indeed, let $(u_{l(n)})$ be such a subsequence of (u_n) that

$$\limsup_n q(u_n) = \lim_n q(u_{l(n)}) = \max\{q(u), \lim q(u_{l(n)} - u)\}$$

and $(u_{l(n)})$ is weakly convergent to u . From the weak lower semicontinuity of the norm it follows that

$$\begin{aligned} \lim_n q(u_{l(n)} - u) &\leq \limsup_n |u_{l(n)} - u| \\ &\leq \limsup_n (\limsup_m |u_{l(n)} - u_{l(m)}|) \\ &= D[(u_{l(n)})] \leq D[(u_n)] \leq \frac{3}{4}. \end{aligned}$$

Simultaneously, the sequence $(u_{l(n)} - z_n^1)$ converges weakly to u . Therefore,

$$q(u) \leq |u - 0| \leq \limsup_n |u_{l(n)} - z_{l(n)}^1| \leq \frac{3}{4}.$$

These inequalities lead to the following estimation:

$$\limsup_n q(u_n) \leq \frac{3}{4}.$$

Note that the proofs of the first two cases go with the same patterns as in [8].

Finally, we get

$$\begin{aligned} |[u_n]| &= \limsup_n |u_n| = \limsup_n \max \left\{ \frac{1}{3} \|u_n\|_2, p(u_n), q(u_n) \right\} \\ &\leq \max \left\{ \limsup_n \frac{1}{3} \|u_n\|_2, \limsup_n p(u_n), \limsup_n q(u_n) \right\} \\ &\leq \max \left\{ \frac{1}{3} \cdot \frac{9\sqrt{6}}{8}, \frac{3}{4}, \frac{3}{4} \right\} = \frac{3\sqrt{6}}{8} < 1. \end{aligned}$$

Since $[u_n]$ was arbitrarily chosen, this fact contradicts Lemma 7.

□

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Summary

Motivated by [A. Jimenez Melado, E. Llorens-Fuster, A class of renormings of l_2 with the fixed point property, J. Nonlinear Convex Anal. 14 (2013), no. 2, 351-362] and [G. Dutta, P. Veeramani, A short survey on open problems in metric fixed point theory and some related results for nonexpansive mappings, J. Anal. 29 (2021), 369-381], in [B. Piątek, On some renormings of l_2 , J. Math. Anal. Appl. 547 (2) (2025), Article ID 129383 9 pages], we asked whether a certain renorming of l_2 has the fixed point property. The main goal of this chapter is to give an affirmative answer to this question. Moreover, we analyze whether our space satisfies some conditions, which guarantee the existence of fixed points.

Streszczenie

Zainspirowani pracami [A. Jimenez Melado, E. Llorens-Fuster, A class of renormings of l_2 with the fixed point property, J. Nonlinear Convex Anal. 14 (2013), no. 2, 351-362] oraz [G. Dutta, P. Veeramani, A short survey on open problems in metric fixed point theory and some related results for nonexpansive mappings, J. Anal. 29 (2021), 369-381], w artykule [B. Piątek, On some renormings of l_2 , J. Math. Anal. Appl. 547 (2) (2025), Article ID 129383 9 pages] postawiliśmy pytanie, czy pewne przenormowanie przestrzeni l_2 ma własność punktu stałego. Głównym celem tego rozdziału jest udzielenie pozytywnej odpowiedzi na to pytanie. Ponadto analizujemy, czy nasza przestrzeń spełnia pewne warunki gwarantujące istnienie punktów stałych.

Chapter 5

SELECTED GEOMETRIC PROPERTIES OF INTERPOLATION SPACES

Joanna Markowicz

1. Preliminaries

The theory of interpolation spaces is a branch of functional analysis with many applications in other areas of analysis such as theory of differential partial equations, theory of approximation and numerical analysis. The paper is dedicated to an abstract discrete method of interpolation and to selected geometric properties of interpolation spaces constructed with the method. In [5] the discrete method of interpolation was used to find the factorization of weakly compact operators through reflexive spaces. Using that method, Davis [4] proved that every uniformly convex space with an unconditional base is isomorphic to a complementing subspace of a uniformly convex space with a symmetric base (see also theorem 3.b.2 w [13]). In the paper, we use a general discrete method of interpolation which for a given interpolation couple $\mathbf{X} = (X_0, X_1)$ of Banach spaces and for a space E with an unconditional base with an unconditional constant equals 1 leads to a construction of an interpolation space $K_{p,\theta}(\mathbf{X}, E)$.

Definition 1. Let X_0 and X_1 be Banach spaces. The couple $\mathbf{X} = (X_0, X_1)$ is called an *interpolation couple* if both spaces X_0 and X_1 and linearly and continuously embedded in a linear and topological space V .

Let $\mathbf{X} = (X_0, X_1)$ be an interpolation space. The sum $X_0 + X_1$ is defined as the space $X_0 + X_1 = \{x \in V : x = x^0 + x^1, x^0 \in X_0, x^1 \in X_1\}$ with the norm

$$\|x\|_{X_0+X_1} = \inf\{\|x^0\|_{X_0} + \|x^1\|_{X_1} : x = x^0 + x^1, x^0 \in X_0, x^1 \in X_1\}. \quad (1)$$

The intersection $X_0 \cap X_1$ is considered with the norm

$$\|x\|_{X_0 \cap X_1} = \max\{\|x\|_{X_0}, \|x\|_{X_1}\}.$$

Both $X_0 + X_1$ and $X_0 \cap X_1$ are Banach spaces.

For a given interpolation couple $\mathbf{X} = (X_0, X_1)$ we have a k -functional $k(t, x, \mathbf{X})$ defined for all $x \in X_0 + X_1$ and for all $t > 0$ in the following way

$$k(t, x, \mathbf{X}) = \inf\{\|x^0\|_{X_0} + t\|x^1\|_{X_1} : x = x^0 + x^1, x^0 \in X_0, x^1 \in X_1\}.$$

This functional was introduced by J. Peetre. For all $x \in X_0 + X_1$ the function $k(\cdot, x, \mathbf{X})$ is concave and for every given $t > 0$, the k -functional $k(t, \cdot, \mathbf{X})$ is a norm in the space $X_0 + X_1$ which is equivalent to (1). It follows from the following inequalities

$$\min(1, t)\|x\|_{X_0+X_1} \leq k(t, x, \mathbf{X}) \leq \max(1, t)\|x\|_{X_0+X_1}$$

for all $t > 0$ and for all $x \in X_0 + X_1$.

We shall consider the space $X_0 + X_1$ with the norm

$$\|x\|_p = k_p(x, a, b) = \inf \left\{ \left(a^p \|x^0\|_{X_0}^p + b^p \|x^1\|_{X_1}^p \right)^{\frac{1}{p}} \right\}, \quad (2)$$

where $x = x^0 + x^1, x^0 \in X_0, x^1 \in X_1$ and $p \in [1, \infty)$ with $a, b > 0$. It is obvious that the norm is equivalent to the norm given in the formula (1). We denote the space $X_0 + X_1$ with the norm (2) by $\Sigma_p(\mathbf{X}, a, b)$.

The generalization of the definition of the functional $k(t, x, \mathbf{X})$ was considered for instance in [12] and [14, p. 220].

Definition 2. We say that an interpolation couple $\mathbf{X} = (X_0, X_1)$ is *p-exact* if the infimum in (2) is attained, i.e. for every $x \in X_0 + X_1$ there exist $x^0 \in X_0$ and $x^1 \in X_1$ such that

$$\|x\|_p = \left(a^p \|x^0\|_{X_0}^p + b^p \|x^1\|_{X_1}^p \right)^{\frac{1}{p}}. \quad (3)$$

Let us assume that τ is an admissible topology in $\Sigma_p(\mathbf{X}, a, b)$. We say that an interpolation couple $\mathbf{X} = (X_0, X_1)$ is τ -closed if unit balls of the spaces X_0 and X_1 are continuously τ -closed in $\Sigma_p(\mathbf{X}, a, b)$ and if at least one of them is continuously τ -compact. The following proposition gives conditions for an interpolation couple $\mathbf{X}$ to be p -exact.

Proposition 3. *Let $\mathbf{X} = (X_0, X_1)$ be an interpolation couple and let $p \in [1, \infty)$. If $\mathbf{X}$ is τ -closed or both spaces X_0, X_1 are reflexive, then the couple $\mathbf{X}$ is p -exact.*

We shall use the interpolation method that is based on spaces with unconditional bases. Moreover, we shall consider bases for which the set of indices is the set of integers.

Let $p \in [1, \infty)$ and let $\mathbf{X} = (X_0, X_1)$ be an interpolation space. Let E be a Banach space with a normalized, unconditional basis $(e_i)_{i \in \mathbb{Z}}$ whose the unconditional constant equals 1. Let us assume that $(a_i)_{i \in \mathbb{Z}}$ and $(b_i)_{i \in \mathbb{Z}}$ are sequences of positive numbers for which $\sum_{i \in \mathbb{Z}} \min\{a_i, b_i\} < \infty$.

Definition 4. *The interpolation space $K_p(\mathbf{X}, E, (a_i), (b_i))$ is defined as a space of all elements $x \in X_0 + X_1$ such that the series $\sum_{i \in \mathbb{Z}} k_p(x, a_i, b_i) e_i$ converges in E . The space $K_p(\mathbf{X}, E, (a_i), (b_i))$ is considered with the norm*

$$\|x\| = \left\| \sum_{i \in \mathbb{Z}} k_p(x, a_i, b_i) e_i \right\|_E. \quad (4)$$

For the norm (4) there exists also the following formula

$$\|x\| = \inf \left\| \sum_{i \in \mathbb{Z}} \left(a_i^p \|x^0(i)\|_{X_0}^p + b_i^p \|x^1(i)\|_{X_1}^p \right)^{\frac{1}{p}} e_i \right\|_E, \quad (5)$$

where the infimum is taken over all decompositions $x = x^0(i) + x^1(i)$ where $x^0(i) \in X_0, x^1(i) \in X_1$ for every $i \in \mathbb{Z}$ such that the series

$$\sum_{i \in \mathbb{Z}} \left(a_i^p \|x^0(i)\|_{X_0}^p + b_i^p \|x^1(i)\|_{X_1}^p \right)^{\frac{1}{p}} e_i$$

converges in E .

The space $K_p(\mathbf{X}, E, (a_i), (b_i))$ with the norm given by the formula (4) is a Banach space. Indeed, the spaces $\Sigma_p(\mathbf{X}, a, b)$ are Banach spaces and the interpolation space $K_p(\mathbf{X}, E, (a_i), (b_i))$ is a direct sum of those spaces.

We have

$$2^{-\frac{1}{q}} \min\{a_1, b_1\} \|x\|_{X_0+X_1} \leq \|x\|$$

for every $x \in K_p(\mathbf{X}, E, (a_i), (b_i))$ where $\frac{1}{q} + \frac{1}{p} = 1$. Moreover,

$$\|x\| \leq \sum_{i \in \mathbb{Z}} \min\{a_i, b_i\} \|x\|_{X_0 \cap X_1}$$

for every $x \in X_0 \cap X_1$. The inequalities show that the following embeddings

$$X_0 \cap X_1 \subset K_p(\mathbf{X}, E, (a_i), (b_i)) \subset X_0 + X_1$$

are continuous.

The name *interpolation space* for the space $K_p(\mathbf{X}, E, (a_i), (b_i))$ comes from the theorem of interpolation of operators.

Theorem 5. [14, p. 219] Let $\mathbf{X} = (X_0, X_1)$ and $\mathbf{Y} = (Y_0, Y_1)$ be two interpolation couples and let $T : X_0 + X_1 \rightarrow Y_0 + Y_1$ be a linear map acting from X_0 to Y_0 as a bounded operator with the norm $\|T\|_0$ and from X_1 to Y_1 as a bounded operator with the norm $\|T\|_1$. Then the operator T maps the space $K_p(\mathbf{X}, E, \{a_i\}, \{b_i\})$ in the space $K_p(\mathbf{Y}, E, \{a_i\}, \{b_i\})$ and its norm on those space is less or equal to $\max\{\|T\|_0, \|T\|_1\}$.

In the special case when $a_i = e^{\theta i}$ and $b_i = e^{(\theta-1)i}$ for every $i \in \mathbb{Z}$ where $\theta \in (0, 1)$, we denote the interpolation space $K_p(\mathbf{X}, E, (e^{\theta i}), (e^{(\theta-1)i}))$ by $K_{p,\theta}(\mathbf{X}, E)$ and its norm given by the formula (4) we denote by $\|\cdot\|_{p,\theta}$.

We consider bases $(e_i)_{i \in \mathbb{Z}}$ of E which satisfy an additional assumption: there exists a constant $M > 0$ such that

$$\left\| \sum_{i \in \mathbb{Z}} \alpha_i e_{i+k} \right\|_E \leq M \left\| \sum_{i \in \mathbb{Z}} \alpha_i e_i \right\|_E \quad (6)$$

for every $k \in \mathbb{Z}$. In particular, symmetric bases, the standard base of the space $l^p(\mathbb{Z})$ or standard bases of Orlicz sequence spaces and Lorentz sequence spaces satisfy (6) (see [13], p. 115).

The interpolation space is a special case of a direct sum. The special property of the norm of the considered interpolation space describes next proposition. The proof of the result is based on the method of the proof of the lemma 2.g.13 from [14] and the property itself is very useful in many other proofs.

Proposition 6. [18] Let $\mathbf{X} = (X_0, X_1)$ be an interpolation space and let $\theta \in (0, 1)$, $p \in [1, \infty)$. Next, let E be a Banach space with a normalized unconditional base $(e_i)_{i \in \mathbb{Z}}$ with the unconditional constant equals 1, which satisfies (6). Then the inequality

$$\|x\|_{p,\theta} \leq C \left\| \sum_{i \in \mathbb{Z}} e^{\theta i} \|x^0(i)\|_{X_0} e_i \right\|_E^{1-\theta} \left\| \sum_{i \in \mathbb{Z}} e^{(\theta-1)i} \|x^1(i)\|_{X_1} e_i \right\|_E^{\theta}, \quad (7)$$

where $C = (1 + e^{1-\theta})M$ holds for every $x \in K_{p,\theta}(\mathbf{X}, E)$ and for every decomposition $x = x^0(i) + x^1(i)$ where $x^0(i) \in X_0$, $x^1(i) \in X_1$, $i \in \mathbb{Z}$ such that the series from the right-hand side converge.

Proof. Let $x \in K_{p,\theta}(\mathbf{X}, E)$ and let $x = x^0(i) + x^1(i)$ be a decomposition satisfying the assumptions of the above proposition. Put

$$A_0 = \left\| \sum_{i \in \mathbb{Z}} e^{\theta i} \|x^0(i)\|_{X_0} e_i \right\|_E \quad \text{and} \quad A_1 = \left\| \sum_{i \in \mathbb{Z}} e^{(\theta-1)i} \|x^1(i)\|_{X_1} e_i \right\|_E.$$

Our assumption guarantees that $A_0, A_1 > 0$. We can find $k \in \mathbb{Z}$ such that

$$e^k \leq \frac{A_1}{A_0} < e^{k+1}.$$

For every $i \in \mathbb{Z}$ we have the decomposition $x = x^0(i-k) + x^1(i-k)$. Thus

$$\begin{aligned} \|x\|_{p,\theta} &\leq \left\| \sum_{i \in \mathbb{Z}} \left(e^{\theta i p} \|x^0(i-k)\|_{X_0}^p + e^{(\theta-1)i p} \|x^1(i-k)\|_{X_1}^p \right)^{\frac{1}{p}} e_i \right\|_E \\ &\leq \left\| \sum_{i \in \mathbb{Z}} e^{\theta i} \|x^0(i-k)\|_{X_0} e_i \right\|_E + \left\| \sum_{i \in \mathbb{Z}} e^{(\theta-1)i} \|x^1(i-k)\|_{X_1} e_i \right\|_E \\ &= \left\| \sum_{i \in \mathbb{Z}} e^{\theta(i+k)} \|x^0(i)\|_{X_0} e_{i+k} \right\|_E + \left\| \sum_{i \in \mathbb{Z}} e^{(\theta-1)(i+k)} \|x^1(i)\|_{X_1} e_{i+k} \right\|_E \\ &\leq M(e^{\theta k} A_0 + e^{(\theta-1)k} A_1) \leq C A_0^{1-\theta} A_1^{\theta}. \end{aligned}$$

□

In some assumptions of theorems of the paper there appears a notion of

uniform monotonicity. Let us recall its definition.

Definition 7. We say that a Banach lattice X is *uniformly monotone* if for every $\varepsilon \in (0, 1)$ there exists $\delta \in (0, 1)$ such that if $x, y \in X$, $0 \leq y \leq x$, $\|x\| = 1$ and $\|y\| \geq \varepsilon$, then $\|x - y\| \leq 1 - \delta$.

Uniform monotonicity can be described with the help of the following function called modulus of monotonicity.

Definition 8. The *modulus of monotonicity* $\delta_{m,X} : [0, 1] \rightarrow [0, 1]$ of a Banach lattice is defined as

$$\delta_{m,X}(\varepsilon) = \inf\{1 - \|x - y\| : x, y \in X, 0 \leq y \leq x, \|x\| \leq 1, \|y\| \geq \varepsilon\}. \quad (8)$$

The lattice X is uniformly monotone if and only if $\delta_{m,X}(\varepsilon) > 0$ for every $\varepsilon > 0$.

Example 9. [8] Let $1 \leq p < \infty$ and X be the lattice $L^p([0, 1])$ or l^p . Then

$$\delta_{m,X}(\varepsilon) = 1 - (1 - \varepsilon^p)^{\frac{1}{p}}$$

for every $\varepsilon \in [0, 1]$

2. Opial properties in interpolation spaces

Opial property and uniform Opial property were introduced respectively in [20] and [21]. Both those properties have many applications in metric fixed point theory (see [16]). Other application was given in [6]. In the paper we consider Opial properties with respect to a weak topology, as it was considered originally in literature. In this section we present conditions for an interpolation couple $\mathbf{X} = (X_0, X_1)$ that guarantee that the sum $\Sigma_p(\mathbf{X}, a, b) = X_0 + X_1$ has nonstrict Opial property or Opial property. Next, we give sufficient conditions for an interpolation space $K_{p,\theta}(\mathbf{X}, E)$ to have Opial property. Moreover, we give an estimation of the modulus connected to uniform Opial property $s_{K_{p,\theta}}$ with the help of the modulus of monotonicity $\delta_{m,E}$ of the lattice E and moduli s_{X_0} and s_{X_1} of the spaces X_0 and X_1 . Thanks to that estimation we can conclude the conditions under which an interpolation space $K_{p,\theta}(\mathbf{X}, E)$ has uniform Opial property.

By $\mathcal{N}_1(\tau)$ we shall denote a set of all sequences (x_n) convergent to 0 with respect to the topology τ for which $\|x_n\| \geq 1$ for all n . The condition

$\mathcal{N}_1(\tau) = \emptyset$ defines spaces for which sequence convergence with respect to τ is equivalent to convergent in norm. If $\tau = w$ then such property is called *Schur property*. All finitely dimensional spaces have Schur property. Moreover, the space l^1 has that property (see [16]).

Definition 10. We say that a Banach space X has *nonstrict Opial property with respect to a topology τ* in X if

$$\liminf_{n \rightarrow \infty} \|x_n - x\| \leq \liminf_{n \rightarrow \infty} \|x_n\|$$

for every bounded sequence (x_n) such that $x = \tau\text{-}\lim_{n \rightarrow \infty} x_n$ w X .

Definition 11. We say that a Banach space X has *Opial property with respect to a topology τ* in X if

$$\liminf_{n \rightarrow \infty} \|x_n - x\| < \liminf_{n \rightarrow \infty} \|x_n\|$$

for every bounded sequence (x_n) in X such that $x = \tau\text{-}\lim_{n \rightarrow \infty} x_n$ in X where $x \neq 0$.

In the above definitions *liminf* can be replaced by *limsup*.

Definition 12. We say that a Banach space X has *uniform Opial property with respect to a topology τ* in X if for every $c > 0$ there exists $r > 0$ such that the inequality

$$1 + r \leq \liminf_{n \rightarrow \infty} \|x_n + x\|$$

is satisfied for every bounded sequence $(x_n) \in \mathcal{N}_1(\tau)$ in X and for every $x \in X$ for which $\|x\| \geq c$.

Uniform Opial property with respect to τ implies Opial property with respect to τ .

In addition, we assume that spaces for which $\mathcal{N}_1(\tau) = \emptyset$ have uniform Opial property with respect to the topology τ . We have the following examples.

Example 13.

1. The spaces l^p have uniform Opial property for every $p \in (1, \infty)$.
2. The spaces $L^p([0, 1])$ for $p \in (1, \infty)$, $p \neq 2$ do not have nonstrict Opial property with respect to the topology $\tau = w$.

3. Let Ω be a space with a σ finite measure μ . Then the spaces $L^p(\Omega)$ for $p \in [1, \infty)$ have uniform Opial property with respect to the topology locally convergent in measure (see [7]).

Uniform Opial property can be also describe with help of a function $s_{X,\tau}$ defined in the following way.

Definition 14. Let X be a Banach space. The modulus $s_{X,\tau}$ is defined as

$$s_{X,\tau}(c) = \inf \left\{ 1 - \liminf_{n \rightarrow \infty} \|x_n - x\| \right\},$$

where $c \in [0, 1]$ and infimum is taken over all bounded sequences (x_n) in X such that $\liminf_{n \rightarrow \infty} \|x_n\| \leq 1$ and $\tau\text{-}\lim_{n \rightarrow \infty} x_n = x$ where $\|x\| \geq c$.

A space X has uniform Opial property with respect to a topology τ if and only if $s_{X,\tau}(c) > 0$ for every $c \in (0, 1]$. Furthermore, a space X have nonstrict Opial property with respect to a topology τ if and only if $s_{X,\tau}(c) \geq 0$ for every $c \in [0, 1]$.

Theorem 15 ([17]). *If X has nonstrict Opial property with respect to a topology τ and $c > 0$ then in the definition of $s_{X,\tau}(c)$ the condition $\|y\| \geq c$ can be replaced by $\|y\| = c$.*

Theorem 16 ([18]). *Let $p \in [1, \infty)$, $a, b > 0$ and let $\mathbf{X} = (X_0, X_1)$ be an interpolation couple. Moreover, let $\Sigma_p(\mathbf{X}, a, b)$ be a space with a norm given by (3). Furthermore, assume that the spaces X_0 and X_1 are reflexive.*

- 1) *If both spaces X_0 and X_1 have nonstrict Opial property, then $\Sigma_p(\mathbf{X}, a, b)$ has nonstrict Opial property.*
- 2) *If both spaces X_0 and X_1 have Opial property, then $\Sigma_p(\mathbf{X}, a, b)$ has Opial property.*

The following theorem gives conditions for an interpolation space $K_{p,\theta}(\mathbf{X}, E)$ to have Opial property.

Theorem 17 ([18]). *Let E be a real Banach space with a normalized unconditional base $(e_i)_{i \in \mathbb{Z}}$ with an unconditional constant equals 1, which satisfies (6). Let us assume that $p \in [1, \infty)$, $\theta \in (0, 1)$ and $\mathbf{X} = (X_0, X_1)$ is an interpolation couple such that X_0 and X_1 are reflexive. If E is uniformly monotone, both spaces X_0 and X_1 have nonstrict Opial property and at least one of X_0 and X_1 have Opial property, then the interpolation space $K_{p,\theta}(\mathbf{X}, E)$ have Opial property.*

Next theorem presents an estimation of the modulus $s_{K_{p,\theta}}$ connected with Opial property in $K_{p,\theta}(\mathbf{X}, E)$.

Theorem 18 ([18]). *Assume that E is a real Banach space with a normalized unconditional base $(e_i)_{i \in \mathbb{Z}}$ with an unconditional constant equals 1, which satisfies (6). Let us assume that $p \in [1, \infty)$, $\theta \in (0, 1)$ and $\mathbf{X} = (X_0, X_1)$ is an interpolation couple of reflexive Banach spaces that have nonstrict Opial property. Then*

$$s_{K_{p,\theta}}(t) \geq \delta_{m,E}(\max\{c_0 s_{X_0}(c_0), c_1 s_{X_1}(c_1)\}), \quad (9)$$

where

$$c_0 = \frac{1}{2} \left(\frac{t}{C} \right)^{\frac{1}{1-\theta}}, \quad c_1 = \frac{1}{2} \left(\frac{t}{C} \right)^{\frac{1}{\theta}} \quad (10)$$

and C is the constant from jest (7). Consequently, if E is uniformly monotone and X_0 or X_1 have uniform Opial property, then $K_{p,\theta}(\mathbf{X}, E)$ has uniform Opial property.

3. Uniform convexity in interpolation spaces

Uniform convexity is another classical geometric properties of Banach spaces. It was introduced by J. A. Clarkson in [2] and states that if two points in the unit ball are sufficiently far apart, then their midpoint lies well inside the unit ball. This property can be described with the help of a function called modulus of convexity.

Definition 19. Let X be a Banach space. We define the modulus of convexity of X as a function $\delta_X : [0, 2] \rightarrow [0, 1]$ defined as

$$\delta_X(\varepsilon) = \inf \left\{ 1 - \left\| \frac{x+y}{2} \right\| : x, y \in B(X), \|x-y\| \geq \varepsilon \right\}. \quad (11)$$

A space X is *uniformly convex* if $\delta_X(\varepsilon) > 0$ for every $\varepsilon > 0$.

In the above definition of δ_X , the condition $x, y \in B(X)$ can be replaced by $x, y \in S(X)$, and the condition $\|x-y\| \geq \varepsilon$ can be replaced by $\|x-y\| = \varepsilon$ (see [14]).

Example 20 ([11]). The spaces L^p i l^p for $p \in (1, \infty)$ are uniformly convex. Let $X = L^p$ or $X = l^p$. If $p \geq 2$, then the modulus of convexity of X is

given by the formula

$$\delta_X(\varepsilon) = 1 - \left(1 - \left(\frac{\varepsilon}{2}\right)^p\right)^{1/p}.$$

If $1 < p \leq 2$, then the modulus is given by the implicit formula

$$\left(1 - \delta_X(\varepsilon) + \frac{\varepsilon}{2}\right)^p + \left|1 - \delta_X(\varepsilon) - \frac{\varepsilon}{2}\right|^p = 2$$

for every $\varepsilon \in [0, 2]$.

The main result regarding uniform convexity in interpolation spaces was proved by Beauzamy in [1] (see also theorem 2.g.21 in [14]). The Beauzamy's theorem refers to the real interpolation method introduced by Lions and Peetre in [15]. The respective theorem for the complex interpolation method was proved by Cwikel and Reisner in [3].

The observation of uniform convexity in the space $\Sigma_p(\mathbf{X}, a, b)$ is the following. The space $\Sigma_p(\mathbf{X}, a, b)$ can be considered as a quotient space of the direct sum $(Y_0 \oplus Y_1)_E$, where E is the plane $\mathbb{R}^2$ with l^p -norm and Y_0, Y_1 are isometric to X_0 and X_1 , respectively. If both X_0 and X_1 are uniformly convex, then $\Sigma_p(\mathbf{X}, a, b)$ is uniformly convex and its modulus of convexity does not depend on the coefficients a, b . From the fact that the interpolation space $K_p(\mathbf{X}, E, (a_i), (b_i))$ is a direct sum of $\Sigma_p(\mathbf{X}, a_i, b_i)$ we obtain that $K_p(\mathbf{X}, E, (a_i), (b_i))$ is uniformly convex.

The following theorem about uniform convexity of $K_{p,\theta}(\mathbf{X}, E)$ says that if one of the spaces from the interpolation couple $\mathbf{X} = (X_0, X_1)$ is uniformly convex, then the interpolation space $K_{p,\theta}(\mathbf{X}, E)$ is also uniformly convex.

Theorem 21 ([19]). *Let $\mathbf{X} = (X_0, X_1)$ be an interpolation couple for which at least one of the spaces X_0, X_1 is uniformly convex. Furthermore, let us assume that E is a uniformly convex Banach space with a normalized unconditional base $(e_i)_{i \in \mathbb{Z}}$ with an unconditional constant equals 1, which satisfies (6). Then the space $K_{p,\theta}(\mathbf{X}, E)$ is uniformly convex.*

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Summary

In theory of interpolation spaces we can put a natural question whether a given property of a Banach space is saved under passing to an interpolation space. In literature many interpolation methods were examined. The answer to this problem is determined by the given method of interpolation. In the paper we consider a general discrete and abstract method of interpolation based of a k -functional to obtain an interpolation space. We examine Opial properties and uniform convexity in such spaces. Namely, we give results about non strict Opial property and Opial property in the space $\Sigma_p(\mathbf{X}, a, b) = X_0 + X_1$ and about Opial property in $K_{p,\theta}(\mathbf{X}, E)$. Moreover, the estimation of the modulus $s_{K_{p,\theta}}$ connected to uniform Opial property for the interpolation space $K_{p,\theta}(\mathbf{X}, E)$ is presented. We also give condition for the interpolation space $K_{p,\theta}(\mathbf{X}, E)$ to be uniformly convex.

Streszczenie

W teorii przestrzeni interpolacyjnych naturalnym jest pytanie, czy dana własność przestrzeni Banacha jest zachowywana przy przejściu do przestrzeni interpolacyjnej. W literaturze badano wiele metod interpolacji. Odpowiedź na to pytanie zależy od konkretnej metody interpolacji. W tym rozdziale rozważamy ogólną dyskretną i abstrakcyjną metodę interpolacji opartą na k -funkcjonalach do konstrukcji przestrzeni interpolacyjnej. Badamy własności Opiała oraz jednostajną wypukłość w takich przestrzeniach. Przedstawiamy wyniki dotyczące nieostrej własności Opiała oraz własności Opiała w przestrzeni $\Sigma_p(\mathbf{X}, a, b) = X_0 + X_1$, a także własności Opiała w $K_{p,\theta}(\mathbf{X}, E)$. Ponadto, podajemy oszacowanie modułu $s_{K_{p,\theta}}$ związanego z jednostajną własnością Opiała dla przestrzeni interpolacyjnej $K_{p,\theta}(\mathbf{X}, E)$. Przedstawiamy również warunek jednostajnej wypukłości przestrzeni interpolacyjnej $K_{p,\theta}(\mathbf{X}, E)$.

Chapter 6

MAJORIZATION OF DERIVATIVES FOR MA-MINDA TYPE OF CONVEX FUNCTIONS

Agnieszka Wiśniowska-Wajnryb

1. Introduction

Denote by U_r the disk with center at 0 and the radius r in the complex plane $\mathbb{C}$ and by $U = U_1$ the unit disk, that is

$$U_r = \{z \in \mathbb{C} : |z| < r\}, \quad 0 < r \leq 1.$$

Let f and F be two functions holomorphic in the disk U_r . We say that a function f is subordinate to F in U_r and write $f \prec F$ in U_r , if there exists a holomorphic function ω such that $|\omega(z)| \leq |z|$ and $f(z) = F(\omega(z))$ for $z \in U_r$. Hence $f \prec F$ in U implies $f(U) \subset F(U)$. If F is univalent in U then

$$f \prec F \text{ in } U \Leftrightarrow [f(0) = F(0) \text{ and } f(U_r) \subset F(U_r) \text{ for } 0 < r \leq 1].$$

We say that a function f is majorized by F in U_r and write $f \ll F$ in U_r , if $|f(z)| \leq |F(z)|$ for every $z \in U_r$. Hence $f \ll F$ in U_r if there exists a function ψ holomorphic in U_r such that $|\psi(z)| \leq 1$ for $z \in U_r$ and $f(z) = \psi(z) \cdot F(z)$ for every $z \in U_r$.

Let $\mathcal{H}$ denote the class of all holomorphic functions in U and let $\mathcal{S}$

denote the class of functions $F \in \mathcal{H}$ that are univalent in U and normalized by $F(0) = F'(0) - 1 = 0$.

M. Biernacki [1] investigated the relation between subordination $f \prec F$ in U and majorization of derivatives $f' \ll F'$ in some smaller disk U_{r_0} , if the functions f, F are univalent in U . This problem was also considered by G.M. Goluzin [5] p. 330 and Shah Tao-shing [16].

Many authors (see f.e. [2], [3], [4], [8], [10], [13]) investigated the relation between majorization $f \ll F$ in U and majorization $f' \ll F'$ in some smaller disk U_{r_0} if $f \in \mathcal{H}$ and $F \in \mathcal{F}$, where $\mathcal{F}$ is a certain subclass of $\mathcal{S}$. The greatest number $r_0 = r_0(\mathcal{H}, \mathcal{F})$, for which the implication

$$f \ll F \text{ in } U \Rightarrow f' \ll F' \text{ in } U_{r_0}$$

holds true for every pair of functions $f \in \mathcal{H}, F \in \mathcal{F}$, is called the radius of majorization of derivatives.

Z. Lewandowski in [10] (see also T.H. MacGregor [13]) proved that if F is univalent or starlike in U then

$$f \ll F \text{ in } U \Rightarrow f' \ll F' \text{ in } U_{2-\sqrt{3}}.$$

T.H. MacGregor [13] (see also Z. Bogucki and J. Zderkiewicz [3]) proved that if F is convex in U then

$$f \ll F \text{ in } U \Rightarrow f' \ll F' \text{ in } U_{1/3}.$$

Thus

$$r_0(\mathcal{H}, \mathcal{ST}) = r_0(\mathcal{H}, \mathcal{S}) = 2 - \sqrt{3},$$

$$r_0(\mathcal{H}, \mathcal{CV}) = \frac{1}{3},$$

where $\mathcal{ST}$ and $\mathcal{CV}$ denote the usual classes of starlike and convex functions in U , respectively.

The problem of majorization of derivatives was generalized by J. Janowski and J. Stankiewicz [8] in the following way. To determine the smallest number $T(r) = T(r, \mathcal{H}, \mathcal{F})$, $r \in [0, 1)$ such that for every pair of functions $f \in \mathcal{H}, F \in \mathcal{F}$ the implication

$$f \ll F \text{ in } U \Rightarrow |f'(z)| \leq T(r) \cdot |F'(z)|$$

holds true for $|z| = r < 1$.

Some results for this generalized problem were obtained by J. Janowski and J. Stankiewicz [8] and f. e. by F. Bogowski and Cz. Bucka [2], A. Wiśniowska-Wajnryb [18].

The purpose of the present paper is to find the radius $r_0(\mathcal{H}, \mathcal{CV}(\varphi))$, where $\mathcal{CV}(\varphi)$ denotes a general class of Ma-Minda convex functions. Here φ is a given holomorphic univalent function such that $\varphi(0) = 1$, $\varphi'(0) > 0$ and $\varphi(U)$ is a convex domain symmetric with respect to the real axis and contained in the right half-plane. The assumption $F \in \mathcal{CV}(\varphi)$ means that majorant F satisfies the following condition

$$1 + \frac{zf''(z)}{f'(z)} \prec \varphi(z) \text{ in } U.$$

2. Univalent majorants

We start by proving the general theorem which allows us to determine the radius $r_0(\mathcal{H}, \mathcal{F})$, if $\mathcal{F}$ is any subclass of the class $\mathcal{S}$. We give a unified approach to solve majorization problems for all subclasses of $\mathcal{S}$ which were investigated before one by one by many authors (see f.e. [2], [3], [4], [7], [8], [10], [13], [17]). We apply some modification of the method used in [8].

For any subclass $\mathcal{F}$ of the class $\mathcal{S}$ we define

$$m = m(r, \mathcal{F}) = \min \left\{ \left| \frac{zF'(z)}{F(z)} \right| : F \in \mathcal{F}, |z| = r \right\}.$$

Theorem 1. *Let $F \in \mathcal{F} \subset \mathcal{S}$ and let $f \in \mathcal{H}$ and $f(z) = a_1z + a_2z^2 + \dots$, $a_1 \neq 0$. If $f \ll F$ in U then $f' \ll F'$ in the disk U_{r_0} , where r_0 is the unique root in the interval $(0, 1)$ of the equation*

$$m = \frac{2r}{1 - r^2}.$$

Proof. Let $f \ll F$ in U . Then there exists a function $\psi \in \Omega_0$ such that

$$f(z) = \psi(z) \cdot F(z) \text{ for } z \in U, \tag{1}$$

where

$$\Omega_0 = \{\psi \in \mathcal{H} : \psi(z) = a_0 + a_1z + \dots, a_0 \neq 0 \text{ and } |\psi(z)| \leq 1 \text{ for } z \in U\}.$$

Differentiating (1) we get

$$f'(z) = \psi'(z) \cdot F(z) + \psi(z) \cdot F'(z),$$

hence

$$\left| \frac{f'(z)}{F'(z)} \right| = \left| \psi'(z) \frac{F(z)}{F'(z)} + \psi(z) \right|. \quad (2)$$

It is known (see [5]) that if $\psi \in \Omega_0$, then

$$|\psi'(z)| \leq \frac{1 - |\psi(z)|^2}{1 - |z|^2} \quad \text{for } z \in U. \quad (3)$$

Moreover if $F \in \mathcal{F}$, then

$$\left| \frac{zF'(z)}{F(z)} \right| \geq m \quad \text{for } |z| = r < 1. \quad (4)$$

Using (3) and (4) we obtain from (2)

$$\left| \frac{f'(z)}{F'(z)} \right| \leq \left| \frac{F(z)}{F'(z)} \right| \frac{1 - |\psi(z)|^2}{1 - |z|^2} + |\psi(z)| \leq \frac{r}{m} \frac{1 - |\psi(z)|^2}{1 - |z|^2} + |\psi(z)|,$$

hence

$$\left| \frac{f'(z)}{F'(z)} \right| \leq \frac{-r}{m(1 - r^2)} |\psi(z)|^2 + |\psi(z)| + \frac{r}{m(1 - r^2)}. \quad (5)$$

For a fixed r the right hand side of (5) is a function P of variable $u = |\psi(z)|$

$$P(u) = \frac{-r}{m(1 - r^2)} u^2 + u + \frac{r}{m(1 - r^2)}, \quad \text{where } u \in [0, 1].$$

We have

$$|f'(z)| \leq P(u) |F'(z)|$$

and hence $f' \ll F'$ in the disk U_{r_0} , where

$$r_0 = \sup_{r \in (0,1)} \left\{ \max_{u \in [0,1]} P(u) \leq 1 \right\}.$$

The graph of the function P is a parabola with its arms pointing downwards

and $P(1) = 1$. Hence

$$\max_{u \in [0,1]} P(u) \leq 1 \Leftrightarrow \frac{m(1-r^2)}{2r} \geq 1,$$

this means when abscissa of the vertex of the parabola is greater than or equal to 1.

Thus

$$f \ll F \text{ in } U \Rightarrow f' \ll F' \text{ in } U_{r_0},$$

where r_0 is the unique root in the interval $(0, 1)$ of the equation

$$m = m(r, \mathcal{F}) = \frac{2r}{1-r^2}.$$

Example 2. If $F \in \mathcal{F}$, where $\mathcal{F} = \mathcal{S}$ or $\mathcal{F} = \mathcal{ST}$, then (see f.e. [5])

$$\frac{1-r}{1+r} \leq \left| \frac{zF'(z)}{F(z)} \right| \leq \frac{1+r}{1-r}, \quad |z| = r.$$

Thus

$$m(r, \mathcal{S}) = m(r, \mathcal{ST}) = \frac{1-r}{1+r}$$

and $r_0 \in (0, 1)$ is the solution of the equation

$$\frac{1-r}{1+r} = \frac{2r}{1-r^2} \Leftrightarrow r^2 - 4r + 1 = 0.$$

Hence we get the result of Z. Lewandowski [10]

$$r_0(\mathcal{H}, \mathcal{ST}) = r_0(\mathcal{H}, \mathcal{S}) = 2 - \sqrt{3}.$$

Example 3. If $F \in \mathcal{CV}$, then

$$\frac{1}{1+r} \leq \left| \frac{zF'(z)}{F(z)} \right| \leq \frac{1}{1-r}, \quad |z| = r.$$

Thus

$$m(r, \mathcal{CV}) = \frac{1}{1+r}$$

and $r_0 \in (0, 1)$ is the solution of the equation

$$\frac{1}{1+r} = \frac{2r}{1-r^2}.$$

Hence we get the result of T.H. MacGregor [13]

$$r_0(\mathcal{H}, \mathcal{CV}) = \frac{1}{3}.$$

3. Starlike and convex majorants

Let φ be a holomorphic univalent function which satisfies the conditions: $\operatorname{Re} \varphi(z) > 0$ for $z \in U$, $\varphi(U)$ is symmetric with respect to the real axis and starlike with respect to $\varphi(0) = 1$, and $\varphi'(0) > 0$.

By $\mathcal{CV}(\varphi)$ we denote the subclass of convex functions defined by

$$1 + \frac{zf''(z)}{f'(z)} \prec \varphi(z) \text{ in } U$$

and by $\mathcal{ST}(\varphi)$ we denote the subclass of starlike functions defined by

$$\frac{zf'(z)}{f(z)} \prec \varphi(z) \text{ in } U.$$

These classes were introduced and investigated by W. Ma and D. Minda in [11]. When φ varies, $\mathcal{CV}(\varphi)$ and $\mathcal{ST}(\varphi)$ generate a number of known subclasses of convex and starlike functions, respectively.

In particular

$$\mathcal{CV} = \mathcal{CV}(\varphi_0), \quad \mathcal{ST} = \mathcal{ST}(\varphi_0), \quad \text{where } \varphi_0(z) = \frac{1+z}{1-z}, \quad z \in U.$$

When

$$\varphi_1(z) = \frac{1 + (1-2\alpha)z}{1-z}, \quad 0 \leq \alpha < 1,$$

then

$$\varphi_1(U) = \{w = u + iv : \operatorname{Re} w > \alpha\}$$

and the classes $\mathcal{CV}(\varphi_1)$ and $\mathcal{ST}(\varphi_1)$ coincide with the classes $\mathcal{CV}(\alpha)$ and $\mathcal{ST}(\alpha)$ consisting of all functions that are convex of order α and starlike of order α , respectively.

For

$$\varphi_2(z) = \left(\frac{1+z}{1-z} \right)^\alpha, \quad 0 < \alpha \leq 1,$$

we have

$$\varphi_2(U) = \{w = u + iv : |\arg w| < \alpha\pi/2\}$$

and $\mathcal{CV}(\varphi_2)$ and $\mathcal{ST}(\varphi_2)$ consist of strongly convex functions of order α and of strongly starlike functions of order α , respectively.

If $\mathcal{UCV}$ denote the class of uniformly convex functions introduced by A.W. Goodman in [6], then (see [12], [15])

$$\mathcal{UCV} = \mathcal{CV}(\varphi_3), \quad \text{where } \varphi_3(z) = 1 + \frac{2}{\pi^2} \left(\log \frac{1+\sqrt{z}}{1-\sqrt{z}} \right)^2, \quad z \in U.$$

Let's get back to the problem of majorization of derivatives. If the majorant belongs to Ma-Minda class of starlike functions then as an immediate consequence of Theorem 1 we obtain

Corollary 4. *Let $F \in \mathcal{ST}(\varphi)$ and let $f \in \mathcal{H}$ and $f(z) = a_1z + a_2z^2 + \dots, a_1 \neq 0$. If $f \ll F$ in U , then $f' \ll F'$ in the disk U_{r_0} , where r_0 is the unique root in the interval $(0, 1)$ of the equation*

$$\min_{|z|=r} |\varphi(z)| = \frac{2r}{1-r^2}.$$

If additionally $\varphi(U)$ is convex, then $\min_{|z|=r} |\varphi(z)| = \varphi(-r)$ and so r_0 is the unique root in the interval $(0, 1)$ of the equation

$$\varphi(-r) = \frac{2r}{1-r^2}.$$

Remark 5. The results of Corollary 4 were recently proved in [4] (see Theorem 2 and Remark 1).

Example 6. If $F \in \mathcal{ST}(\alpha)$, then

$$\frac{zF'(z)}{F(z)} \prec \varphi_1(z) = \frac{1+(1-2\alpha)z}{1-z} \quad \text{in } U.$$

Thus

$$m(r, \mathcal{ST}(\alpha)) = \varphi_1(-r) = \frac{1-(1-2\alpha)r}{1+r}$$

and $r_0 \in (0, 1)$ is a solution of the equation

$$\frac{1 - (1 - 2\alpha)r}{1 + r} = \frac{2r}{1 - r^2}.$$

Hence

$$r_0(\mathcal{H}, \mathcal{ST}(\alpha)) = \frac{2 - \alpha - \sqrt{3 - 2\alpha + \alpha^2}}{1 - 2\alpha} \quad \text{for } \alpha \neq \frac{1}{2}$$

and

$$r_0(\mathcal{H}, \mathcal{ST}(1/2)) = \frac{1}{3}.$$

Example 7. If $F \in \mathcal{ST}_\alpha$, where $\mathcal{ST}_\alpha$ denotes the class of strongly starlike functions of order α , then

$$\frac{zF'(z)}{F(z)} \prec \varphi_2(z) = \left(\frac{1+z}{1-z}\right)^\alpha \quad \text{in } U.$$

Thus

$$m(r, \mathcal{ST}_\alpha) = \varphi_2(-r) = \left(\frac{1-r}{1+r}\right)^\alpha$$

and $r_0 \in (0, 1)$ is a solution of the equation

$$\left(\frac{1-r}{1+r}\right)^\alpha = \frac{2r}{1-r^2}.$$

Example 8. Let $\mathcal{S}_p$ denote the subclass of starlike functions corresponding to $\mathcal{UCV}$, introduced and investigated by F. Rønning in [15], defined by the relation

$$f \in \mathcal{UCV} \Leftrightarrow zf'(z) \in \mathcal{S}_p.$$

If $F \in \mathcal{S}_p$, then

$$\frac{zF'(z)}{F(z)} \prec \varphi_3(z) = 1 + \frac{2}{\pi^2} \left(\log \frac{1 + \sqrt{z}}{1 - \sqrt{z}} \right)^2 \quad \text{in } U.$$

Since

$$\varphi_3(U) = \{w = u + iv : v^2 < 2u - 1\} = \{w : \operatorname{Re} w > |w - 1|\}$$

we get

$$m(r, \mathcal{S}_p) = \varphi_3(-r) = 1 + \frac{2}{\pi^2} \left(\log \frac{1 + \sqrt{-r}}{1 - \sqrt{-r}} \right)^2 = 1 - \frac{2}{\pi^2} \left(\arccos \frac{1-r}{1+r} \right)^2$$

and $r_0 \in (0, 1)$ is a solution of the equation

$$1 - \frac{2}{\pi^2} \left(\arccos \frac{1-r}{1+r} \right)^2 = \frac{2r}{1-r^2}.$$

It is more difficult to determine the radius of majorization of derivatives if majorant belongs to the class $\mathcal{CV}(\varphi)$. We will find it explicitly on the assumption of the convexity of region $\varphi(U)$.

Define k_φ by the conditions $k_\varphi(0) = k'_\varphi(0) - 1 = 0$ and

$$1 + \frac{zk''_\varphi(z)}{k'_\varphi(z)} = \varphi(z), \quad z \in U.$$

It is clear that k_φ belongs to $CV(\varphi)$ and

$$\log k'_\varphi(z) = \int_0^z \frac{\varphi(t) - 1}{t} dt.$$

From the assumption that $\varphi(U)$ is symmetric with respect to the real axis it follows that φ has real coefficients. Thus the same is true for functions

$$k'_\varphi(z) = \exp \left\{ \int_0^z \frac{\varphi(t) - 1}{t} dt \right\}$$

and

$$k_\varphi(z) = \int_0^z k'_\varphi(t) dt.$$

We need the following lemma [14]

Lemma 9. *Let h be analytic in U , $h(0) = 1$ and let function q with $q(0) = 1$ satisfy the differential equation*

$$q(z) + \frac{zq'(z)}{q(z)} = h(z), \quad z \in U.$$

If we suppose that h is convex in U and $\operatorname{Re} q(z) > 0, z \in U$, then q is

univalent and is given by

$$q(z) = \frac{zk'(z)}{k(z)}, \quad k(z) = \int_0^z \frac{g(t)}{t} dt \quad \text{and} \quad g(z) = z \exp \left(\int_0^z \frac{h(t) - 1}{t} dt \right).$$

Moreover, if f is analytic in U , $f(0) = 0$ and $1 + zf''(z)/f'(z) \prec h(z)$ in U , then $zf'(z)/f(z) \prec q(z)$ in U . The result is sharp and the extremal function is $f = k$.

Theorem 10. Let $F \in \mathcal{CV}(\varphi)$ and let $f \in \mathcal{H}$ and $f(z) = a_1z + a_2z^2 + \dots$, $a_1 \neq 0$. Assume additionally that $\varphi(U)$ is convex. If $f \ll F$ in U , then $f' \ll F'$ in the disk U_{r_0} , where r_0 is the unique root in the interval $(0, 1)$ of the equation

$$\frac{-rk'_\varphi(-r)}{k_\varphi(-r)} = \frac{2r}{1 - r^2}.$$

Proof. Let $F \in \mathcal{CV}(\varphi)$ and $q(z) = zk'_\varphi(z)/k_\varphi(z)$. Then

$$q(z) + \frac{zq'(z)}{q(z)} = \varphi(z), \quad z \in U.$$

Moreover φ is convex in U and since $k_\varphi \in \mathcal{CV}(\varphi) \subset \mathcal{ST}$ we have $\operatorname{Re} q(z) > 0$, $z \in U$. Thus the conditions of Lemma 9 are satisfied, so we get

$$\frac{zF'(z)}{F(z)} \prec q(z) = \frac{zk'_\varphi(z)}{k_\varphi(z)} \quad \text{in } U.$$

Thus

$$\min_{|z|=r} \left| \frac{zF'(z)}{F(z)} \right| = \min_{|z|=r} \left| \frac{zk'_\varphi(z)}{k_\varphi(z)} \right| = \min_{|z|=r} |q(z)|.$$

Hence by Theorem 1 if $f \in \mathcal{H}$ and $f \ll F$ in U , then $f' \ll F'$ in the disk U_{r_0} , where r_0 is the unique root in the interval $(0, 1)$ of the equation

$$\min_{|z|=r} \left| \frac{zk'_\varphi(z)}{k_\varphi(z)} \right| = \frac{2r}{1 - r^2}.$$

Now we determine $\min_{|z|=r} |q(z)|$. Let $z = re^{i\theta}$. It suffices to consider a case

$\theta \in [0, \pi]$, because of $q(\bar{z}) = \overline{q(z)}$. Hence we have

$$\begin{aligned} \min_{|z|=r} |q(z)| &= \min_{\theta \in [0, \pi]} |q(re^{i\theta})| = \min_{\theta \in [0, \pi]} \left| \frac{re^{i\theta} k'_\varphi(re^{i\theta})}{k_\varphi(re^{i\theta})} \right| = \min_{\theta \in [0, \pi]} \left| \frac{re^{i\theta} k'_\varphi(re^{i\theta})}{\int_0^{re^{i\theta}} k'_\varphi(t) dt} \right| \\ &= \min_{\theta \in [0, \pi]} \left| \frac{re^{i\theta} k'_\varphi(re^{i\theta})}{\int_0^r k'_\varphi(xe^{i\theta}) e^{i\theta} dx} \right| = \min_{\theta \in [0, \pi]} \frac{r}{\left| \int_0^r \frac{k'_\varphi(xe^{i\theta}) dx}{k'_\varphi(re^{i\theta})} \right|}, \end{aligned} \quad (6)$$

where

$$k'_\varphi(z) = \exp \left\{ \int_0^z \frac{\varphi(t) - 1}{t} dt \right\}.$$

We show that

$$\min_{\theta \in [0, \pi]} \frac{r}{\left| \int_0^r \frac{k'_\varphi(xe^{i\theta}) dx}{k'_\varphi(re^{i\theta})} \right|} = \frac{-rk'_\varphi(-r)}{k'_\varphi(-r)}. \quad (7)$$

In order to get (7) we prove that

$$\max_{\theta \in [0, \pi]} \left| \int_0^r \frac{k'_\varphi(xe^{i\theta})}{k'_\varphi(re^{i\theta})} dx \right| = \int_0^r \frac{k'_\varphi(-x)}{k'_\varphi(-r)} dx = \frac{-k_\varphi(-r)}{k'_\varphi(-r)}. \quad (8)$$

In view of the inequality

$$\left| \int_0^r \frac{k'_\varphi(xe^{i\theta})}{k'_\varphi(re^{i\theta})} dx \right| \leq \int_0^r \left| \frac{k'_\varphi(xe^{i\theta})}{k'_\varphi(re^{i\theta})} \right| dx$$

it suffices to show that

$$\left| \frac{k'_\varphi(xe^{i\theta})}{k'_\varphi(re^{i\theta})} \right| \leq \frac{k'_\varphi(-x)}{k'_\varphi(-r)} \quad (9)$$

for $\theta \in [0, \pi]$ and $x \in [0, r]$. We have

$$\left| \frac{k'_\varphi(xe^{i\theta})}{k'_\varphi(re^{i\theta})} \right| = \frac{\exp[\operatorname{Re} \log k'_\varphi(xe^{i\theta})]}{\exp[\operatorname{Re} \log k'_\varphi(re^{i\theta})]} = \exp[\operatorname{Re} \log k'_\varphi(xe^{i\theta}) - \operatorname{Re} \log k'_\varphi(re^{i\theta})]$$

and in order to get (9) we must prove that

$$\operatorname{Re} \log k'_\varphi(xe^{i\theta}) - \operatorname{Re} \log k'_\varphi(re^{i\theta}) \leq \log k'_\varphi(-x) - \log k'_\varphi(-r) \quad (10)$$

for $\theta \in [0, \pi]$ and $x \in [0, r]$. Let

$$G(\theta) = \operatorname{Re} \log k'_\varphi(xe^{i\theta}) - \operatorname{Re} \log k'_\varphi(re^{i\theta}), \quad \theta \in [0, \pi].$$

Then

$$\begin{aligned} G'(\theta) &= \operatorname{Re} \frac{ixe^{i\theta}k''_\varphi(xe^{i\theta})}{k'_\varphi(xe^{i\theta})} - \operatorname{Re} \frac{ire^{i\theta}k''_\varphi(re^{i\theta})}{k'_\varphi(re^{i\theta})} \\ &= \operatorname{Re}[i(\varphi(xe^{i\theta}) - 1)] - \operatorname{Re}[i(\varphi(re^{i\theta}) - 1)] \\ &= -\operatorname{Im}(\varphi(xe^{i\theta}) - 1) + \operatorname{Im}(\varphi(re^{i\theta}) - 1) \\ &= \operatorname{Im} \varphi(re^{i\theta}) - \operatorname{Im} \varphi(xe^{i\theta}). \end{aligned}$$

If $\theta = 0$ or $\theta = \pi$ then $\operatorname{Im} \varphi(re^{i\theta}) = \operatorname{Im} \varphi(xe^{i\theta}) = 0$ and so $G'(\theta) = 0$.

We show that $\operatorname{Im} \varphi(re^{i\theta}) > \operatorname{Im} \varphi(xe^{i\theta})$ for fixed $\theta \in (0, \pi)$ which implies $G'(\theta) > 0$ for $\theta \in (0, \pi)$. Therefore $G(\theta) \leq G(\pi)$ for $\theta \in [0, \pi]$ which is equivalent to (10).

For fixed $\theta \in (0, \pi)$ let $H(x) = \operatorname{Im} \varphi(xe^{i\theta})$ and $0 \leq x \leq r < 1$. We have

$$H(0) = \operatorname{Im} \varphi(0) = 0 \quad \text{and} \quad H(r) = \operatorname{Im} \varphi(re^{i\theta}) > 0.$$

Moreover

$$H'(x) = \operatorname{Im}[\varphi'(xe^{i\theta})e^{i\theta}] = \frac{1}{x} \operatorname{Im} \left(-i \frac{d}{d\theta} \varphi(xe^{i\theta}) \right) = -\frac{1}{x} \operatorname{Re} \frac{d}{d\theta} \varphi(xe^{i\theta}).$$

As we noted before the function φ has real coefficients which implies that it maps $(-r, r)$ one-to-one into the real axis. The assumption $\varphi'(0) > 0$ implies that φ is increasing on $(-r, r)$. Thus $\varphi(-x) < \varphi(0) = 1 < \varphi(x)$. From the convexity of the function φ it follows that $\varphi(|z| = x)$ is a convex curve and so $\operatorname{Re} \varphi(xe^{i\theta})$ is a decreasing function of the variable θ for $\theta \in [0, \pi]$. Hence

$$H'(x) = -\frac{1}{x} \frac{d}{d\theta} \operatorname{Re} \varphi(xe^{i\theta}) > 0$$

which means that (10) and thereby (9), (8), (7) hold true.

Example 11. If $F \in \mathcal{CV}(\alpha)$, then

$$1 + \frac{zF''(z)}{F'(z)} \prec \varphi_1(z) = \frac{1 + (1 - 2\alpha)z}{1 - z} \quad \text{in } U.$$

Thus

$$m(r, \mathcal{CV}(\alpha)) = \frac{-rk'_{\varphi_1}(-r)}{k_{\varphi_1}(-r)}$$

and $r_0 \in (0, 1)$ is the solution of the equation

$$\frac{-rk'_{\varphi_1}(-r)}{k_{\varphi_1}(-r)} = \frac{2r}{1-r^2}, \quad (11)$$

where

$$k'_{\varphi_1}(z) = \exp \left\{ \int_0^z \frac{\varphi_1(t) - 1}{t} dt \right\} = \exp \left\{ \int_0^z \frac{2(1-\alpha)}{1-t} dt \right\}, \quad 0 \leq \alpha < 1.$$

Thus

$$k'_{\varphi_1}(z) = \frac{1}{(1-z)^{2(1-\alpha)}} \quad \text{and} \quad k_{\varphi_1}(z) = \int_0^z \frac{1}{(1-t)^{2(1-\alpha)}} dt.$$

Hence

$$k_{\varphi_1}(z) = \frac{1}{1-2\alpha} \left[\frac{1}{(1-z)^{1-2\alpha}} - 1 \right] \quad \text{for } \alpha \neq 1/2$$

and

$$k_{\varphi_1}(z) = \int_0^z \frac{1}{1-t} dt = \log \frac{1}{1-z} \quad \text{for } \alpha = 1/2.$$

Therefore from (11) we have that if $f \ll F$ in U and $F \in \mathcal{CV}(\alpha)$, $\alpha \in [0, 1)$, then $f' \ll F'$ in the disk U_{r_0} , where $r_0 \in (0, 1)$ is the solution of the equation

$$(1-2\alpha)r + 2(1+r)^{1-2\alpha} = 3-2\alpha \quad \text{for } \alpha \neq 1/2,$$

and

$$(1+r)^2 = e^{1-r} \quad \text{for } \alpha = 1/2.$$

Note that for $\alpha = 0$ we get

$$r_0(\mathcal{H}, \mathcal{CV}) = \frac{1}{3}.$$

Example 12. Let $F \in \mathcal{UCV} = \mathcal{CV}(\varphi_3)$. Then

$$1 + \frac{zF''(z)}{F'(z)} \prec \varphi_3(z) = 1 + \frac{2}{\pi^2} \left(\log \frac{1+\sqrt{z}}{1-\sqrt{z}} \right)^2 \quad \text{in } U.$$

In view of Theorem 10

$$m(r, \mathcal{UCV}) = \frac{-rk'_{\varphi_3}(-r)}{k_{\varphi_3}(-r)}$$

and $r_0 \in (0, 1)$ is the solution of the equation

$$\frac{-rk'_{\varphi_3}(-r)}{k_{\varphi_3}(-r)} = \frac{2r}{1-r^2},$$

where

$$k'_{\varphi_3}(z) = \exp \left\{ \int_0^z \frac{\varphi_3(t) - 1}{t} dt \right\} = \exp \left\{ \frac{2}{\pi^2} \int_0^z \frac{\left(\log \frac{1+\sqrt{t}}{1-\sqrt{t}} \right)^2}{t} dt \right\}.$$

Example 13. Let F be in the class of k -uniformly convex functions $k\text{-}\mathcal{UCV}$, $k \geq 0$, introduced by S. Kanas and A. Wiśniowska [9]. A function $f \in \mathcal{S}$ is k -uniformly convex in U , if the image of every circular arc $\gamma \subset U$ with center ζ , where $|\zeta| \leq k$, is convex. Note that $0\text{-}\mathcal{UCV} = \mathcal{CV}$ and for $k = 1$ we get Goodman's definition of the class $\mathcal{UCV}$ so $1\text{-}\mathcal{UCV} = \mathcal{UCV}$. It was shown in [9] that

$$f \in k\text{-}\mathcal{UCV} \Leftrightarrow \operatorname{Re} \left\{ 1 + \frac{zf''(z)}{f'(z)} \right\} \geq k \left| \frac{zf''(z)}{f'(z)} \right| \quad \text{for } z \in U, |\zeta| \leq k.$$

For $k \in (0, 1)$ we get [9]

$$f \in k\text{-}\mathcal{UCV} \Leftrightarrow 1 + \frac{zf''(z)}{f'(z)} \prec \varphi_4(z) \quad \text{in } U,$$

where

$$\varphi_4(z) = \frac{1}{1-k^2} \cosh \left\{ \left(\frac{2}{\pi} \arccos k \right) \log \frac{1+\sqrt{z}}{1-\sqrt{z}} \right\} - \frac{k^2}{1-k^2}, \quad z \in U.$$

Hence if $F \in k\text{-}\mathcal{UCV}$ with $k \in (0, 1)$ then

$$m(r, k\text{-}\mathcal{UCV}) = \frac{-rk'_{\varphi_4}(-r)}{k_{\varphi_4}(-r)}$$

and $r_0 \in (0, 1)$ is the solution of the equation

$$\frac{-rk'_{\varphi_4}(-r)}{k_{\varphi_4}(-r)} = \frac{2r}{1-r^2}, \quad (12)$$

where

$$\begin{aligned} k'_{\varphi_4}(z) &= \exp \left\{ \int_0^z \frac{\varphi_4(t) - 1}{t} dt \right\} \\ &= \exp \left\{ \frac{2}{1-k^2} \int_0^z \frac{\sinh^2 \left\{ \left(\frac{\arccos k}{\pi} \right) \log \frac{1+\sqrt{t}}{1-\sqrt{t}} \right\}}{t} dt \right\}. \end{aligned}$$

It is very difficult to solve explicitly the differential equation (12) for arbitrary $k \in (0, 1)$, except the case $k = \sqrt{2}/2$. For $k = \sqrt{2}/2$ the computation is easy and after some calculations we get

$$\begin{aligned} \varphi_4(z) &= 2 \cosh \left\{ \frac{1}{2} \log \frac{1+\sqrt{z}}{1-\sqrt{z}} \right\} - 1 = \frac{2}{\sqrt{1-z}} - 1, \\ k'_{\varphi_4}(z) &= \exp \left\{ 2 \int_0^z \frac{\frac{1}{\sqrt{1-t}} - 1}{t} dt \right\} = \frac{16}{(1+\sqrt{1-z})^4}, \end{aligned}$$

hence

$$k_{\varphi_4}(z) = \frac{16}{3} \frac{1+3\sqrt{1-z}}{(1+\sqrt{1-z})^3} - \frac{8}{3},$$

and

$$\frac{zk'_{\varphi_4}(z)}{k_{\varphi_4}(z)} = \frac{6}{(1-z) + 4\sqrt{1-z} + 1}.$$

Therefore from (12) we have that if $f \ll F$ in U and $F \in (\sqrt{2}/2)\text{-}\mathcal{UCV}$, then $f' \ll F'$ in the disk U_{r_0} , where $r_0 \in (0, 1)$ is the solution of the equation

$$\frac{3}{2+r+4\sqrt{1+r}} = \frac{r}{1-r^2}.$$

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Summary

We prove a general result which allows us to find the radius of majorization of derivatives if majorant belongs to any subclass of the class $\mathcal{S}$. The aim of this paper is to determinate the radius of majorization of derivatives for majorants from Ma-Minda classes of convex functions.

Streszczenie

Dowodzimy ogólny rezultat pozwalający znaleźć promień majoryzacji pochodnych, gdy majoranta należy do dowolnej podklasy klasy $\mathcal{S}$. Celem pracy jest wyznaczenie promienia majoryzacji pochodnych dla majorant z klasy funkcji wypukłych typu Ma-Minda.

ADVANCES IN FUNCTIONAL AND COMPLEX ANALYSIS

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