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# Quaternions and Cauchy Classical Theory of Elasticity

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Abstract

***Developed by French mathematician Augustin-Louis Cauchy, the classical theory of elasticity is the starting point to show the value and the physical reality of quaternions. The classical balance equations for the isotropic, elastic crystal, demonstrate the usefulness of quaternions. The family of wave equations and the diffusion equation are a straightforward consequence of the quaternion representation of the Cauchy model of the elastic solid. Using the quaternion algebra, we present the derivation of the quaternion form of the multiple wave equations. The fundamental consequences of all derived equations and relations for physics, chemistry, and future prospects are presented.***

Keywords

*wave mechanics, quaternions, Cauchy model, elastic solid, Schrödinger equation*

## 1. Introduction

Hamilton tried for 10 years to create an analog of the complex numbers and finally in 1843, while on a walk with his wife, he realized that three distinct imaginary units are necessary. He carved a new idea on the Broom Bridge in Dublin, which at present is immortalized by a commemorative plaque [1]. This simplified, trivial, and unfortunately very common opinion tells that quaternions were invented as an extension to the complex numbers. The genuine, unquestionable Hamilton motivation was a very fundamental physics of solids and liquids.

The Cauchy classical theory of elasticity was already developed in 1822 [2]. The Navier–Stokes equations that are central to fluid mechanics were formulated as well. In 1828, Poisson [3] studied the elementary waves (the longitudinal and transverse). The purpose and beauty of Hamilton quaternions was immediately recognized. In 1869, James Clerk Maxwell wrote [4]: "The invention of the calculus of quaternions is a step towards the knowledge of quantities related to space which can only be compared for its importance, with the invention of triple coordinates by Descartes. The ideas of this calculus ¼ are fitted to be of the greatest in all parts of science." However, despite this, all Maxwell attempts to reformulate electromagnetism using quaternions were unsuccessful.

In 1885, Neumann [5] gave the proof of the uniqueness of solutions of three fundamental boundary-initial value problems for finite elastic, compressible solid. A rigorous completeness proof of the Cauchy theory was given by Duhem [6]. Regrettably, the Euler and Navier–Stokes equations that are used to quantify the motion of a fluid are not yet solved for an unknown velocity vector even in incompressible fluids. Since understanding the Navier–Stokes equations is considered to be the first step to the perception of the elusive phenomenon of turbulence, the Clay Mathematics Institute in 2000 made this problem one of its seven Millennium Prize problems in mathematics and offered 1 million US dollars prize to the first person providing a solution for a specific statement of the problem. Contrary to the Navier–Stokes equations, the Cauchy theory of ideal elastic solid is well-founded and allows for the advanced analysis of the various phenomena. Advanced examination of the Cauchy theory and to the same degree, the majority of physical problems cannot be reduced to vectorial models. The vector product does not permit the formulation of algebra, for example, the division operation is not defined [7].

Compared with the calculus of vectors, the quaternions have slipped into the realm of obscurity and at present, they are used practically only in computation of rotation in every computer graphics film studio. In the same way, the calculus of imaginary numbers by many is considered as an effective tool, by no means the physical reality. Not many scientists expect that they will find a deeper understanding of the physics by restating basic principles in terms of quaternion algebra. In this article, we will discuss that the Hamilton algebra of quaternions, $\mathbb{H}$, and Hamilton concept of the four-dimensional space allow us to work out many problems, and they are the remedy that can be looked

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at as analogous to the widely accepted four-dimensional time–space continuum.

The algebra of quaternions owns all laws of algebra with unique properties. The essential here are (1) the multiplication of the quaternions that are not commutative and (2) they allow quantifying twists and compression. Later, we will discuss that quaternions are a physical reality. Not only helpful and convenient, but the quaternions also allow entering and understanding the processes in continua, for example, the wave mechanics.

The original arguments to implement the classical mechanics equations in the field of wave mechanics were given by Kleinert [8]. The Kleinert concept combined with the Cauchy model of elastic solid has been analyzed with the arbitrary assumption of the complex potential field [9]. Recently, the Cauchy theory was rigorously combined with the quaternion algebra [10], and such representation of the Cauchy equation of motion produced the Klein–Gordon wave equation [11]. In this work, the fundamental new results, explicitly the deformations and the family of waves in elastic solid, are presented.

In the following sections, we will present the rigorous derivation of the quaternion representation of the Cauchy deformation field (Section 2), the essentials of quaternion algebra (Section 3), and the quaternion representation that allows considering multiple forms of waves and standing waves in ideal elastic solid (Section 4). The final result is the vast possibility of waveforms in the elastic continuum (Section 5).

## 2. The Cauchy Deformation Field Transformed to Quattro Group

The Cauchy model of the elastic solid is a mathematical idealization of isotropic elastic material [12, 13]. We consider a case of an ideal face-centered cubic (FCC) structure (Poisson number $\nu$ = 0.25). The small deformation limit judges constant[1]: the density *r*, the Young modulus, and the transverse wave velocity $c=\sqrt{0.4Y/\rho}=\text{const.}$ Solid is a closed system of the constant volume $\Omega\subset\mathbb{R}^3$ and to make the problem easier we do not consider external fields.

Equation of motion relates displacement **u** with compression (div**u**) and twist (rot **u**)

$$\frac{\partial^2\mathbf{u}}{\partial t^2} = 3\frac{0.4Y}{\rho}\operatorname{grad}\operatorname{div}\mathbf{u} - \frac{0.4Y}{\rho}\operatorname{rot}\operatorname{rot}\mathbf{u}\,. \qquad (1)$$

From Eq. (1), the energy per mass unit in the deformation field follows [14]

1 Relations among the elastic constants: $K=Y/(3-6\nu)=Y/1.5$ and $Y=2(1+\nu)G=2.5\times G$.

$$e=\frac{\rho_e}{\rho}=\frac{1}{2}\dot{\mathbf{u}}\circ\dot{\mathbf{u}}+\frac{3}{2}\frac{0.4Y}{\rho}(\operatorname{div}\mathbf{u})^2+\frac{1}{2}\frac{0.4Y}{\rho}\operatorname{rot}\mathbf{u}\circ\operatorname{rot}\mathbf{u}\,, \qquad (2)$$

where $\dot{\mathbf{u}}=\partial\mathbf{u}/\partial t$ and $\circ$ denotes scalar multiplication (scalar inner product) in $\mathbb{R}^3$. Shortly

$$\frac{\partial^2\mathbf{u}}{\partial t^2}=3c^2\operatorname{grad}\operatorname{div}\mathbf{u}-c^2\operatorname{rot}\operatorname{rot}\mathbf{u}\,, \qquad (3)$$

$$e=\frac{1}{2}\dot{\mathbf{u}}\circ\dot{\mathbf{u}}+\frac{3}{2}c^2(\operatorname{div}\mathbf{u})^2+\frac{1}{2}c^2\operatorname{rot}\mathbf{u}\circ\operatorname{rot}\mathbf{u}\,. \qquad (4)$$

Equation (3) and relation (4) imply the Euler–Lagrange differential equation $\frac{\partial e}{\partial\mathbf{u}}-\frac{d}{dt}\left(\frac{\partial e}{\partial\dot{\mathbf{u}}}\right)=0$. It means that one can derive the vector equation of motion from the scalar relation of energy conservation and vice versa. The scalar equation (4) and the vector equation (3) (*Quattro group*) rule the deformation in the ideal elastic continua. By the Helmholtz decomposition theorem, every deformation can be expressed by the compression and twist, and if **u** belongs to $C^3$ class of functions then $\mathbf{u}=\mathbf{u}_0+\mathbf{u}_\phi$, where $\operatorname{rot}\mathbf{u}_0=0$ and $\operatorname{div}\mathbf{u}_\phi=0$.

Upon acting on Eq. (3) by divergence and rotation operators, we decompose it and get the transverse and the longitudinal wave equations in the usual form $\mathbf{u}_{tt}=k\,\mathbf{u}_{xx}$:

$$\begin{aligned}&\operatorname{div}\left(\frac{\partial^2\mathbf{u}}{\partial t^2}=3c^2\operatorname{grad}\operatorname{div}\mathbf{u}-c^2\operatorname{rot}\operatorname{rot}\mathbf{u}\right)\ \Rightarrow\ \frac{\partial^2\sigma_0}{\partial t^2}=3c^2\Delta\sigma_0,\\ &\operatorname{rot}\left(\frac{\partial^2\mathbf{u}}{\partial t^2}=3c^2\operatorname{grad}\operatorname{div}\mathbf{u}-c^2\operatorname{rot}\operatorname{rot}\mathbf{u}\right)\ \overset{\Delta\mathbf{u}=\nabla(\nabla\cdot\mathbf{u})-\nabla\times(\nabla\times\mathbf{u})}{\Rightarrow}\ \frac{\partial^2\hat{\sigma}}{\partial t^2}=c^2\Delta\hat{\sigma},\end{aligned} \qquad (5)$$

where $\sigma_0=\operatorname{div}\mathbf{u}_0=(\sigma_0,0,0,0)$ and $\hat{\sigma}=\operatorname{rot}\mathbf{u}_\phi=(0,\sigma_1,\sigma_2,\sigma_3)$ also $\hat{\sigma}=\sigma_1 i+\sigma_2 j+\sigma_3 k$.

The Cauchy theory combined with the Helmholtz decomposition theorem results in four second-order scalar differential equations (5) and implies the transverse and longitudinal waves in the Cauchy elastic solid. Relation (4) takes the form

$$e=1/2\,\dot{\mathbf{u}}\circ\dot{\mathbf{u}}+3/2\,c^2\sigma_0^2+1/2\,c^2\hat{\sigma}\circ\hat{\sigma}\,. \qquad (6)$$

Decomposition (Eq. 5) results in four equations: $\mathbf{u}=\mathbf{u}_0+\mathbf{u}_\phi\Rightarrow\sigma=\sigma_0+\hat{\sigma}$ and implies the existence of deformation field $\sigma$ that represents the twist and compression fields as a superposition of a real (scalar compression $\sigma_0$) and imaginary (twist vector, $\hat{\sigma}$) field parts at each point

$$\sigma=\sigma_0+\hat{\sigma}\quad\in\mathbb{H}\quad\text{and}\quad\sigma^*=\sigma_0-\hat{\sigma}\quad\in\mathbb{H}\,, \qquad (7)$$

where the following constraint holds

$$\operatorname{div}\hat{\sigma}=\operatorname{div}\operatorname{rot}\mathbf{u}_\phi=0. \qquad (8)$$

The foundation of this *Quattro grouping* in Eq. (7) is the Hamilton quaternion algebra $\mathbb{H}$ (Section 3). On acting on the deformation field *s*, it allows more advanced exploration of its structure, properties, multiple waves, and so on (Sections 4 and 5).

## 3. *Quaternions*: Essentials of the Quaternion Algebra

We show the basic definitions and formulas of the quaternion numbers and functions, and they are limited to those used in this work [15]. Let $\mathbb{R}^4$ be the four-dimensional Euclidean vector space with the orthonormal basis $\{e_0, e_1, e_2, e_3\}$: $e_0 = (1,0,0,0)$, $e_1 = (0,1,0,0)$, $e_2 = (0,0,1,0)$, and $e_3 = (0,0,0,1)$ and with the three-dimensional vector subspace $P = \operatorname{span}\{e_1, e_2, e_3\}$. In practice, the following algebraic notation is used: $e_0 = 1,\ e_1 = i,\ e_2 = j,$ and $e_3 = k$. Thus, an arbitrary quaternion *q*, that is, $q \in H := \square \otimes P$, can be written in terms of its basic components

$$q = (q_0, q_1, q_2, q_3) = q_0 + q_1 i + q_2 j + q_3 k \tag{9}$$

and in the form of the ordered pair of a scalar and vector: $q = q_0 + \hat{q} = [q_0, \vec{q}]$ as already postulated in Eq. (7). The imaginary units obey the following relations:

$$i^2 = j^2 = k^2 = -1, \quad ij = -ji = k, \quad jk = -kj = i, \qquad ki = -ik = j. \tag{10}$$

The multiplication is non-commutative. A conjugate quaternion is defined as

$$q^* = q_0 - q_1 i - q_2 j - q_3 k = q_0 - \hat{q}. \tag{11}$$

From Eqs. (9)–(11), we obtain the results: $q \cdot q^* = q^* \cdot q = \sum_{i=0}^{3} q_i^2$. We will use here the Cauchy–Riemann operator *D* acting on the quaternion-valued functions *q*. Under the constraint in Eq. (7), *D* equals

$$Dq = \operatorname{grad} q_0 + \operatorname{rot} \hat{q}, \quad \text{where} \quad q = q_0 + \hat{q}. \tag{12}$$

Note that $DDq = \Delta q$ and hence *D* corresponds physically to the gradient in $\mathbb{R}^3$. The exponent function has its trigonometric representation: $e^q = e^{q_0}\left(\cos|\hat{q}| + \hat{q}/|\hat{q}| \sin|\hat{q}|\right)$.

## 4. Quaternion Representation of Wave Mechanics

Adding equations in Eq. (5) and from constraint (8), we get quaternion form of the motion equation

$$\frac{1}{c^2}\frac{\partial^2 \sigma}{\partial t^2} = \Delta\sigma + 2\Delta\sigma_0 \quad \text{where} \quad \sigma = \sigma_0 + \hat{\sigma}. \tag{13}$$

Since $\dot{\mathbf{u}} \circ \dot{\mathbf{u}} = \hat{\dot{u}} \circ \hat{\dot{u}} = -\hat{\dot{u}} \cdot \hat{\dot{u}} = \hat{\dot{u}} \cdot \hat{\dot{u}}^*$, where $\hat{\dot{u}} = \dot{u}_1 i + \dot{u}_2 j + \dot{u}_3 k$ and $\dot{\mathbf{u}} = (\dot{u}_1, \dot{u}_2, \dot{u}_3)$, the overall energy of the deformation field, the formula (6), becomes the quaternion form

$$e = -\tfrac{1}{2}\hat{\dot{\mathbf{u}}} \circ \hat{\dot{\mathbf{u}}} + \tfrac{1}{2}c^2\, \sigma \cdot \sigma^* + c^2 \sigma_0^2, \quad \text{where} \quad \hat{\dot{\mathbf{u}}} = \dot{u}_1 \mathbf{i} + \dot{u}_2 \mathbf{j} + \dot{u}_3 \mathbf{k}. \tag{14}$$

It was already shown that splitting Eq. (13) results in: (1) the non-linear quaternion wave and (2) the Poisson equation [11]. In this work, we show that on splitting Eq. (13) into the system of the wave and Poisson equations, the multiple non-linear forms of the wave equation follow, that is, the quaternion motion equation generates the family of the non-linear waves. To do so, we consider the wave showing the energy $E_n = \text{const}$. Subsequently, Eq. (13) can be written as a multisystem:

$$\begin{cases} \dfrac{1}{c^2}\dfrac{\partial^2 \sigma}{\partial t^2} - (n+1)\Delta\sigma + k_n^2 \sigma \cdot \sigma^* = 0, \\ (n-2)\Delta\sigma_0 + n\,\Delta\hat{\phi} - k_n^2 \sigma \cdot \sigma^* = 0, \end{cases} \quad \text{for} \quad n = 0,1,2,\ldots \tag{15}$$

where $k_n = 1/\lambda_n$ and $\lambda_n = f(E_n)$ denotes the wavelength. By adding equations in Eq. (15), the momentum balance is expressed again by a single partial differential equation (13). System (15) is a hyperbolic–elliptic quaternion representation of a wave equation (13) and has the solution of the form:

$$\sigma = \sigma_0 + \hat{\phi} = \sigma_0 + \phi_1 \mathbf{i} + \phi_2 \mathbf{j} + \phi_3 \mathbf{k} \in H. \tag{16}$$

The second equation in Eq. (15) is the Poisson equation, and it describes compression potential and is a function of energy density [11]. Equation (15) must obey constraint (8) and require boundary conditions for a solution.

## 5. Summary: The Quasi-Stationary Waves in the Cauchy Elastic Solid

The aim of our work is to show the usefulness and ontology of the quaternions. On combining the Navier–Cauchy model of the elastic solid with the quaternion algebra $\mathbb{Q}$, we presented the approach that allowed the self-consistent interpretation of the deformations and the wave phenomena.

The analysis of system (15) in a case when *n* = 0 shows that wave equation is the quaternion Klein–Gordon type equation [16] and Poisson equation

$$\begin{cases} \left(\dfrac{1}{c^2}\dfrac{\partial^2}{\partial t^2} - \Delta + k_n^2 \sigma^*\right) \cdot \sigma = 0, \\ 2\Delta\sigma_0 = -k_n^2 \sigma \cdot \sigma^* = 0. \end{cases} \tag{17}$$

For more details, the particulars related to the processes and the physical constants at the Planck scale were already published in Danielewski and Sapa [11].

The quasi-stationary wave means that the wave energy is constant in an arbitrary volume W and can be treated, for example, as the particle. Such a wave and its space evolution might be analyzed based on energy formulae (14). It has been shown that one can relate the velocity $\dot{\mathbf{u}}$ with the gradient of the deformation $D\sigma$, Eq. (12). Consequently, the energy functional follows [10], and it was shown that there exists a multiplayer $\lambda \neq 0$ such that $\psi$ minimizes the resulting functional

$$Q[\psi] = \int_{\Omega} \left[ \frac{\hbar^2}{2m} (D\psi) \cdot (D\psi)^* + V(x)\, \psi \cdot \psi^* + \lambda \left( \frac{1}{|\Omega|} - \psi \cdot \psi^* \right) \right] \mathrm{d}x \quad (18)$$

and $\psi$ satisfies the differential equation

$$-\frac{\hbar^2}{2m} \Delta\psi + V(x)\, \psi = \lambda\psi \,. \quad (19)$$

A constant factor on the right-hand side can be considered as extra energy of the wave in the presence of the *external* field $V = V(x)$. For $E = \lambda$, Eq. (19) is the time-independent Schrödinger-type quaternion equation satisfied by the wave in the ground state of the energy $E$

$$-\frac{\hbar^2}{2m} \Delta\psi + V(x)\, \psi = E\psi \,. \quad (20)$$

It has to be satisfied together with the condition equivalent to the constraint in Eq. (8)

$$\mathrm{div}\,\hat{\psi} = 0 \quad \text{where} \quad \psi = \psi_0 + \hat{\psi} \,. \quad (21)$$

Thus, the quantum space can be regarded as an analog to the elastic solid [9–11].

The analysis of the multisystem in Eq. (15) with constraint (8) derived in this work requires that the boundary conditions are formulated in the quaternion form. An example of such condition was already presented in the case of a closed system [11].

Quaternions are much more comfortable than vectors in most cases and have huge advantages in the calculation of twist (and rotations). We demonstrated that energy-momentum and conservation in the elastic Navier–Cauchy continuum implies a quaternion form of the multiple wave equations and that quaternions can be regarded as the most concise form of physical reality. Our derivation provides new evidence that there is a rigorously defined mathematical connection between classical and wave mechanics. All the obtained results support the physical reality of quaternions and allow for the interpretation of wave mechanics.

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