Selected Topics in OPERATOR THEORY and TOPOLOGY

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Preface

Szymon Dudek

This monograph, titled *Selected Topics in Operator Theory and Topology*, brings together a set of chapters devoted to selected problems in the fields of mathematical and functional analysis, operator theory, topology, and the theory of differential and integral equations. Despite the diversity of topics and techniques, all contributions are united by a common analytical perspective and a focus on qualitative properties of solutions and structural characteristics of function spaces.

The inspiration and intellectual impetus behind this volume stems from the scientific achievements and research legacy of Professor Józef Banaś, whose work has significantly shaped the field of nonlinear analysis. In particular, his pioneering contributions to the theory and applications of measures of noncompactness have had a profound influence on generations of mathematicians working in fixed point theory, integral and differential equations, and functional spaces. This year, we have the honor of celebrating Professor Banaś's 75th birthday. This monograph contains results of authors who are or have been scientifically associated with Professor Banaś.

In the opening chapter, A. Martinón provides a subtle issue in set theory and general topology, focusing on distinctions between operations on classes of sets versus operations on their elements. Through the analysis of intersections and unions in the lattice of all topologies defined on a fixed set, the chapter clarifies notational and conceptual ambiguities that have implications for the formal structure of topological systems.

J. Szczupiel focuses on the variation of functions defined on the half-axis $\mathbb{R}_+$. The chapter introduces a definition of variation suitable for this setting,

explores its equivalent formulations, and studies the asymptotic behavior of such functions. Additionally, the Riemann–Stieltjes integral on unbounded intervals is introduced and discussed, with illustrative examples emphasizing its practical use.

J. Madej investigates infinite systems of quadratic integral equations of Urysohn and Hammerstein type on the half-line $\mathbb{R}_+$. By employing the technique of measures of noncompactness, the chapter offers existence results and fixed-point theorems, underlining the central role of presented analytical approach in the study of nonlinear integral systems.

A. Chlebowicz presents a review of classical and contemporary results on the construction of measures of noncompactness in spaces of real- and Banach-space-valued functions. The chapter discusses classical constructions in spaces such as $C([a,b],\mathbb{R})$, $BC(\mathbb{R}_+,\mathbb{R})$, and $C([a,b],E)$, and then extends the analysis to the space $BC(\mathbb{R}_+,E)$, highlighting how earlier results may serve as a methodological foundation for modern generalizations.

L. Olszowy offers a concise survey of measures of weak noncompactness in the spaces $L^1[0,a]$, $L^1(\mathbb{R}_+)$, and the Fréchet space $L^1_{\mathrm{loc}}(\mathbb{R}_+)$. The chapter provides both a synthesis of known results and a clear exposition of the functional-analytic techniques relevant to weak compactness in the spaces of integrable functions.

D. Cruz-Uribe focuses on the matrix weighted norm inequalities and matrix weights, with particular emphasis on convex body sparse domination for singular integrals, Rubio de Francia extrapolation, and Jones factorization.

The volume as a whole demonstrates the continuing relevance of classical concepts—such as compactness, variation, and integral representations—in contemporary analysis. Each chapter, while self-contained, contributes to a broader narrative about the refinement and extension of foundational tools, and the interplay between abstract theory and concrete applications. The monograph is intended for readers interested in the rigorous treatment of modern analytical problems, and it offers a resource for both further research and academic teaching.

The editor would also like to express his sincere gratitude to the reviewers for their valuable comments, which have greatly contributed to the improvement of this monograph. The support and assistance of colleagues from the Department of Nonlinear Analysis at Rzeszów University of Technology have proved invaluable, with special acknowledgement to Dr Rafał Nalepa and Dr Tomasz Zając for their contributions. The editor extends his heartfelt thanks to them.

Chapter 1

A note about notation: operations with classes of sets

Antonio Martinón

1. Introduction

In this note we will use the term *class* to refer to a set of sets.

Given the classes of sets $\mathcal{F}$ and $\mathcal{G}$, we write

$$\mathcal{F} \cap \mathcal{G} := \{h : h \in \mathcal{F} \text{ and } h \in \mathcal{G}\} \text{ and } \mathcal{F} \Cap \mathcal{G} := \{f \cap g : f \in \mathcal{F} \text{ and } g \in \mathcal{G}\} .$$

The use of the usual notation would lead us to write $\mathcal{F} \cap \mathcal{G}$ instead of $\mathcal{F} \Cap \mathcal{G}$, thus causing confusion. Analogously, we write

$$\mathcal{F} \cup \mathcal{G} := \{h : h \in \mathcal{F} \text{ or } h \in \mathcal{G}\} \text{ and } \mathcal{F} \Cup \mathcal{G} := \{f \cup g : f \in \mathcal{F} \text{ and } g \in \mathcal{G}\} ,$$

with different notation so as not to confuse these two classes.

To introduce us to the matter of this note, in the first section we refer, as a model example, to classes of subsets of $\mathbb{R}$ and operations between them. In the second section we consider the lattice of topologies on a set to illustrate the ideas of the first section.

2. Operations on the class of all the subsets of $\mathbb{R}$

In the set $\mathbb{R}$ of all the real numbers we consider the sum of two non-empty sets A, B, called the Minkowski sum:

$$A + B := \{a + b : a \in A \text{ and } b \in B\} . \tag{1}$$

For example, the sum of intervals satisfies $[a, b] + [c, d] = [a + c, b + d]$.

Now we consider the class $2^{\mathbb{R}} \setminus \{\emptyset\}$ of all the non-empty subsets of $\mathbb{R}$ and the sum defined by (1). Given two subclasses $\mathcal{F}$ and $\mathcal{G}$ of $2^{\mathbb{R}} \setminus \{\emptyset\}$, we have that

$$\mathcal{F} + \mathcal{G} := \{f + g : f \in \mathcal{F} \text{ and } g \in \mathcal{G}\} . \tag{2}$$

As in $2^{\mathbb{R}}$ are defined the set operations $\cap$ and $\cup$, these operations are transferred to subclasses of $2^{\mathbb{R}}$. However, a difficulty now arises with the notation.

On the one hand, the intersection of classes is denoted by

$$\mathcal{F} \cap \mathcal{G} := \{h : h \in \mathcal{F} \text{ and } h \in \mathcal{G}\} .$$

But, if we use the notation (2) for the operation $\cap$, we would obtain the class $\{f \cap g : f \in \mathcal{F} \text{ and } g \in \mathcal{G}\}$, which is different from $\mathcal{F} \cap \mathcal{G}$. Therefore, we need to use a new notation:

$$\mathcal{F} \barwedge \mathcal{G} := \{f \cap g : f \in \mathcal{F} \text{ and } g \in \mathcal{G}\} .$$

Analogously, we have

$$\mathcal{F} \cup \mathcal{G} := \{h : h \in \mathcal{F} \text{ or } h \in \mathcal{G}\} \text{ and } \mathcal{F} \veebar \mathcal{G} := \{f \cup g : f \in \mathcal{F} \text{ and } g \in \mathcal{G}\} .$$

We can also consider the complementation operation on $2^{\mathbb{R}}$: $A \subset \mathbb{R} \longrightarrow A^c := \mathbb{R} \setminus A$. Then, for a class $\mathcal{F}$ of subsets of $\mathbb{R}$, we must distinguish between

$$\mathcal{F}^c = \{A \in 2^X : A \notin \mathcal{F}\} \text{ and } \mathcal{F}^{cc} = \{A^c : A \in \mathcal{F}\} .$$

Of course, what has been said for two classes can be extended in an obvious way to a family of classes: $\cap_{i \in I} \mathcal{F}_i$, $\barwedge_{i \in I} \mathcal{F}_i$, $\cup_{i \in I} \mathcal{F}_i$ and $\veebar_{i \in I} \mathcal{F}_i$, for any family $(\mathcal{F}_i)_{i \in I}$ of subclasses of $2^{\mathbb{R}}$.

The ideas discussed so far for subclasses of $2^{\mathbb{R}}$ can be applied to any class of sets.

3. Operations with topologies in a set

In this section X is a set and $Top(X)$ is the class of topologies defined on X, which is ordered by inclusion, so that we obtain a complete lattice. The book by Willard [1] is a general reference for this section.

In the following, τ and π will be topologies on X. If $\tau \subset \pi$ it is said that τ is weaker than π and π is stronger than τ.

We have

$$\tau \cap \pi = \{U \subset X : U \in \tau \text{ and } U \in \pi\} ,$$

which is a topology. In fact, it is the infimum topology $\tau \wedge \pi$ of the topologies τ and π in the lattice $Top(X)$; that is, $\tau \cap \pi$ is the strongest topology that is weaker that τ and π. Clearly, a very different class is

$$\tau \Cap \pi := \{U \cap V : U \in \tau \text{ and } V \in \pi\} .$$

On the other hand,

$$\tau \cup \pi = \{U \subset E : U \in \tau \text{ or } U \in \pi\} ,$$

is not necessarily a topology. It is clear that any topology that is stronger than τ and π must contain $\tau \cup \pi$, and the weakest of them all is the supremum $\tau \vee \pi$ in the lattice $Top(X)$. Then $\tau \cup \pi$ is a subbase for the topology $\tau \vee \pi$. A basis is formed by the finite intersections of sets of $\tau \cup \pi$, so that an open set of the basis is of the form

$$U_1 \cap \cdots \cap U_h \cap V_1 \cap \cdots \cap V_k \quad (U_1, ..., U_h \in \tau \ , \ V_1, ..., V_k \in \pi) .$$

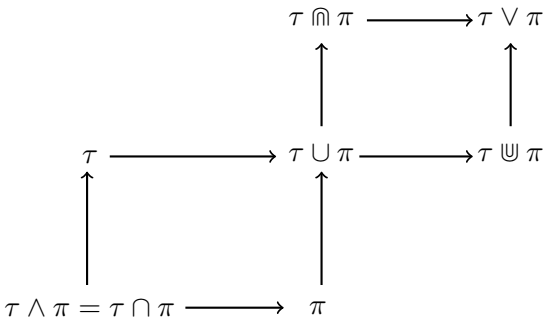
Hence an open set of that base has the form $U \cap V$, with $U \in \tau$ and $V \in \pi$. Consequently,

$$\tau \Cap \pi \text{ is basis for the topology } \tau \vee \pi .$$

We can also consider the class

$$\tau \Cup \pi = \{U \cup V : U \in \tau \text{ and } V \in \pi\} .$$

To recap, the subclasses τ and π of 2^X belonging to $Top(X)$ give rise to a lot of subclasses, which are related as follows, where "$\longrightarrow$" means "$\subseteq$":



The classes $\tau \Cup \pi$ and $\tau \Cap \pi$ are not comparable. Indeed, let us consider in $X = \{1, 2, 3\}$ the topologies $\tau = \{\emptyset, \{1\}, \{2, 3\}, X\}$ and $\pi = \{\emptyset, \{1, 2\}, \{3\}, X\}$. The set $\{1, 3\}$ belongs to $\tau \Cup \pi$, but not to $\tau \Cap \pi$. On the other hand, the set $\{2\}$ belongs to $\tau \Cap \pi$, but not to $\tau \Cup \pi$.

References

[1] Stephen Willard, *General Topology*, Addison-Wesley Publishing Company, 1970.

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Summary

Given the classes of sets $\mathcal{F}$ and $\mathcal{G}$, it is necessary to distinguish between $\{h : h \in \mathcal{F} \text{ and } h \in \mathcal{G}\}$ and $\{f \cap g : f \in \mathcal{F} \text{ and } g \in \mathcal{G}\}$. The first class is always represented by $\mathcal{F} \cap \mathcal{G}$, but the same notation would correspond to the second class if the usual criteria were used. A similar situation arises with the union of sets. These considerations are illustrated in the lattice of all the topologies defined on a set.

Streszczenie

Dla klas zbiorów $\mathcal{F}$ i $\mathcal{G}$ konieczne jest rozróżnienie pomiędzy $\{h : h \in \mathcal{F} \text{ oraz } h \in \mathcal{G}\}$ a $\{f \cap g : f \in \mathcal{F} \text{ i } g \in \mathcal{G}\}$. Pierwsza z tych klas jest zawsze reprezentowana przez $\mathcal{F} \cap \mathcal{G}$, lecz to samo oznaczenie odpowiadałoby drugiej klasie, gdyby użyć zwykłych kryteriów. Podobna sytuacja pojawia się przy sumie zbiorów. Uwagi te ilustruje krata wszystkich topologii określonych na pewnym zbiorze.

Chapter 2

Functions of bounded variation on real half-axis

Justyna Szczupiel

1. Introduction

The foundations of the theory of real functions have wide applications in other mathematics areas. The concept of a variation of a function on a bounded interval, introduced by the French mathematician C. Jordan, leads us to considerations of properties of this type of functions. They form very important and useful structure. Functions of bounded variation are also very significant in the Riemann-Stieltjes integral theory. This concept is very useful and has wide applications in probability theory.
The concept of the variation of a function can be extended to an unbounded interval. In particular, we will consider the real half-axis. The main purpose of this work is to introduce two equivalent definitions of the variation of functions on $\mathbb{R}_+$. We will also present very important properties of functions of bounded variation which will lead us to describe the structure formed by these functions. In the last part, we will extend the concept of the Riemann-Stieltjes integral to the mentioned set $\mathbb{R}_+$.

2. Variation of a function on the real half-axis

2.1. Definitions and basic facts

At the beginning, we introduce the concept of the variation of a function on the real half axis. For this purpose we recall, among other things, the concept of partition of an interval which reduces to a sequence.

Definition 1. Let $f : \mathbb{R}_+ \to \mathbb{R}$ be a function. Let us consider a set $P = \{t_0, t_1, ...\}$ such that $0 = t_0 < t_1 < t_2 < ...$, i.e., a sequence (t_n) is strictly increasing and $\lim\limits_{n\to\infty} t_n = \infty$. We say that $P = (t_n)$ is a partition of $\mathbb{R}_+$. The quantity defined by the formula

$$\overset{\infty}{\underset{0}{V}}(f, P) = \sum_{j=1}^{\infty} |f(t_j) - f(t_{j-1})|$$

is called *the variation of the function f with respect to the partition P on $\mathbb{R}_+$*. Now we denote by $\mathcal{P}(\mathbb{R}_+)$ a set of all partitions of $\mathbb{R}_+$. *The variation of the function f on $\mathbb{R}_+$* is defined as the following quantity:

$$\overset{\infty}{\underset{0}{V}} f = \sup_{P\in\mathcal{P}(\mathbb{R}_+)} \overset{\infty}{\underset{0}{V}}(f, P).$$

If $\overset{\infty}{\underset{0}{V}} f < +\infty$ then we say that the function f is the function of bounded variation on $\mathbb{R}_+$. A family of all functions of bounded variation on $\mathbb{R}_+$ is denoted by $BV(\mathbb{R}_+)$.
We are going to present an equivalent definition of the variation of function f on $\mathbb{R}_+$, which is associated with the concept of the variation of a function on a bounded interval.

Theorem 2. *A function $f : \mathbb{R}_+ \to \mathbb{R}$ has bounded variation on $\mathbb{R}_+$ if and only if it is of bounded variation on every compact subset $[0, A]$ of $\mathbb{R}_+$ and the variations $\overset{A}{\underset{0}{V}} f$ are equibounded. Then*

$$\overset{\infty}{\underset{0}{V}} f = \sup_{A>0} \overset{A}{\underset{0}{V}} f = \lim_{A\to\infty} \overset{A}{\underset{0}{V}} f.$$

Proof. ($\geqslant$)
Let us fix $A > 0$ and a partition P of the interval $[0, A]$, $P = \{t_0, t_1, ..., t_m\}$. Then

$$\overset{A}{\underset{0}{V}}(f, P) = \sum_{j=1}^{m} |f(t_j) - f(t_{j-1})|.$$

We take a sequence $\{t_m, t_{m+1}, t_{m+2}, ...\}$ which is strictly increasing and $t_n \to \infty$ for $n \geqslant m+1$. Let us consider a partition $\{t_0, t_1, ..., t_m, t_{m+1}, ...\} = \overline{P}$ of $\mathbb{R}_+$. Then

$$\overset{A}{\underset{0}{V}}(f, P) = \sum_{j=1}^{m} |f(t_j) - f(t_{j-1})| \leqslant \sum_{j=1}^{\infty} |f(t_j) - f(t_{j-1})| = \overset{\infty}{\underset{0}{V}}(f, \overline{P}) \leqslant \overset{\infty}{\underset{0}{V}} f.$$

Since $A > 0$ is an arbitrary fixed positive number so

$$\underset{A>0}{\forall} \overset{A}{\underset{0}{V}} f \leqslant \overset{\infty}{\underset{0}{V}} f, \quad \text{i.e.,} \quad \lim_{A\to\infty} \overset{A}{\underset{0}{V}} f \leqslant \overset{\infty}{\underset{0}{V}} f.$$

($\leqslant$)

Let us fix $\varepsilon > 0$. We take a partition $P = \{t_0, t_1, ...\}$ of $\mathbb{R}_+$ such that

$$\overset{\infty}{\underset{0}{V}} f - \frac{\varepsilon}{2} \leqslant \overset{\infty}{\underset{0}{V}}(f, P). \tag{1}$$

Obviously

$$\overset{\infty}{\underset{0}{V}}(f, P) = \sum_{j=1}^{\infty} |f(t_j) - f(t_{j-1})|.$$

We find an element $t_n \in P$ such that

$$\overset{\infty}{\underset{0}{V}}(f, P) - \frac{\varepsilon}{2} = \sum_{j=1}^{\infty} |f(t_j) - f(t_{j-1})| - \frac{\varepsilon}{2} \leqslant \sum_{j=1}^{n} |f(t_j) - f(t_{j-1})|. \tag{2}$$

Subtracting in (1) $\frac{\varepsilon}{2}$ and combining it with the inequality (2), we have:

$$\overset{\infty}{\underset{0}{V}} f - \varepsilon \leqslant \overset{\infty}{\underset{0}{V}}(f, P) - \frac{\varepsilon}{2} \leqslant \sum_{j=1}^{n} |f(t_j) - f(t_{j-1})| \leqslant \overset{t_n}{\underset{0}{V}} f \leqslant \lim_{n\to\infty} \overset{t_n}{\underset{0}{V}} f = \lim_{A\to\infty} \overset{A}{\underset{0}{V}} f.$$

As $\varepsilon > 0$ we can take any positive number, so

$$\overset{\infty}{\underset{0}{V}} f \leqslant \lim_{A\to\infty} \overset{A}{\underset{0}{V}} f$$

and finally

$$\overset{\infty}{\underset{0}{V}} f = \lim_{A\to\infty} \overset{A}{\underset{0}{V}} f.$$

Obviously, the limit above is also equal to the supremum of all variations of the function f on intervals $[0, A]$ (for $A > 0$). □

2.2. Properties of variation of function on $\mathbb{R}_+$

The variation of a function on a given interval has many interesting and useful properties. We begin by introducing the most important ones.

Theorem 3 ([1,2,6])**.** *Let $f : \mathbb{R}_+ \to \mathbb{R}$ be a function. Then*

1. $\overset{\infty}{\underset{0}{V}}(\alpha) f = |\alpha| \cdot \overset{\infty}{\underset{0}{V}} f, \quad \alpha \in \mathbb{R}$
2. $\overset{\infty}{\underset{0}{V}}(f+g) \leqslant \overset{\infty}{\underset{0}{V}} f + \overset{\infty}{\underset{0}{V}} g, \quad g : \mathbb{R}_+ \to \mathbb{R}$
3. $\overset{b}{\underset{a}{V}} f = \overset{c}{\underset{a}{V}} f + \overset{b}{\underset{c}{V}} f, \quad a, b, c \in \mathbb{R}_+, \quad a < c < b$
4. $|f(x) - f(y)| \leqslant \overset{y}{\underset{x}{V}} f, \quad x, y \in \mathbb{R}_+, \quad x < y$
5. $P \subset Q \implies \overset{b}{\underset{a}{V}}(f, P) \leqslant \overset{b}{\underset{a}{V}}(f, Q)$ *for* $P, Q \in \mathcal{P}([a, b])$, $a, b \in \mathbb{R}_+$, $a < b$.

Proof. 1. Let us fix $P = \{t_0, t_1, ...\} \in \mathcal{P}(\mathbb{R}_+)$. Then

$$\overset{\infty}{\underset{0}{V}}(\alpha f, P) = \sum_{j=1}^{\infty} |(\alpha f)(t_j) - (\alpha f)(t_{j-1})| = |\alpha| \sum_{j=1}^{\infty} |f(t_j) - f(t_{j-1})| =$$

$$= |\alpha| \overset{\infty}{\underset{0}{V}}(f, P).$$

Taking the supremum of all partitions of $\mathbb{R}_+$ we have

$$\overset{\infty}{\underset{0}{V}} \alpha f = |\alpha| \overset{\infty}{\underset{0}{V}} f.$$

2. Let us fix $P = \{t_0, t_1, ...\} \in \mathcal{P}(\mathbb{R}_+)$. Then

$$\overset{\infty}{V}_0(f+g, P) = \sum_{j=1}^{\infty} |(f+g)(t_j) - (f+g)(t_{j-1})| \leqslant \sum_{j=1}^{\infty} |f(t_j) - f(t_{j-1})| +$$

$$+ \sum_{j=1}^{\infty} |g(t_j) - g(t_{j-1})| = \overset{\infty}{V}_0(f, P) + \overset{\infty}{V}_0(g, P).$$

Taking the supremum of all partitions of $\mathbb{R}_+$ we have

$$\overset{\infty}{V}_0(f+g) \leqslant \overset{\infty}{V}_0 f + \overset{\infty}{V}_0 g.$$

3. $(\geqslant)$
Let us fix $P_1 = \{x_0, x_1, ..., x_m\} \in \mathcal{P}([a, c])$, $P_2 = \{y_0, y_1, ..., y_n\} \in \mathcal{P}([c, b])$, i.e., $a = x_0 < x_1 < ... < x_m = c = y_0 < y_1 < ... < y_n = b$. Then

$$\sum_{j=1}^{m} |f(x_j) - f(x_{j-1})| + \sum_{j=1}^{n} |f(y_j) - f(y_{j-1})| \leqslant \overset{b}{V}_a f.$$

Taking the supremum of all partitions of $[a, c]$ and $[c, b]$, we have

$$\overset{c}{V}_a f + \overset{b}{V}_c f \leqslant \overset{b}{V}_a f.$$

$(\leqslant)$
Let us fix $P = \{z_0, z_1, ..., z_n\} \in \mathcal{P}([a, b])$ such that $z_{k-1} \leqslant c \leqslant z_k$, $k \in \{1, 2, ..., n\}$. Then

$$\sum_{j=1}^{n} |f(z_j) - f(z_{j-1})| \leqslant |f(z_1) - f(z_0)| + ... + |f(c) - f(z_{k-1})| +$$

$$+ |f(z_k) - f(c)| + ... + |f(z_n) - f(z_{n-1})| \leqslant \overset{c}{V}_a f + \overset{b}{V}_c f.$$

Taking the supremum of all partitions of $[a, b]$, we have

$$\overset{b}{V}_a f \leqslant \overset{c}{V}_a f + \overset{b}{V}_c f.$$

4. Let us consider a partition $P_0 = \{x, y\} \in \mathcal{P}([x, y])$. Then

$$|f(y) - f(x)| = \overset{y}{\underset{x}{V}}(f, P_0) \leqslant \sup_{P \in \mathcal{P}([x,y])} \overset{y}{\underset{x}{V}}(f, P) \leqslant \overset{y}{\underset{x}{V}} f.$$

5. Let $P = \{t_0, t_1, ..., t_m\}$ and $Q = P \cup \{c\}$ be partitions of $[a, b]$ such that there exists $i \in \{1, 2, ..., m\}$ such that $t_{i-1} < c < t_i$. Then

$$\overset{b}{\underset{a}{V}}(f, P) = \sum_{j=1}^{m} |f(t_j) - f(t_{j-1})| = \sum_{j=1}^{i-1} |f(t_j) - f(t_{j-1})| + |f(t_i) - f(t_{i-1})| +$$

$$+ \sum_{j=i+1}^{m} |f(t_j) - f(t_{j-1})| \leqslant \sum_{j=1}^{i-1} |f(t_j) - f(t_{j-1})| + |f(t_i) - f(c)| +$$

$$+ |f(c) - f(t_{i-1})| + \sum_{j=i+1}^{m} |f(t_j) - f(t_{j-1})| = \overset{b}{\underset{a}{V}}(f, Q),$$

so $\overset{b}{\underset{a}{V}}(f, P) \leqslant \overset{b}{\underset{a}{V}}(f, Q)$ for the partition Q which was created by adding one element to the partition P. Now using the principle of mathematical induction it is easy to show that the inequality $\overset{b}{\underset{a}{V}}(f, P) \leqslant \overset{b}{\underset{a}{V}}(f, Q)$ is true for all partitions $P, Q \in \mathcal{P}([a, b])$ such that $P \subset Q$.

□

Theorem 4 ([4]). *Let $f : \mathbb{R}_+ \to \mathbb{R}$ be a function. Then $\overset{\infty}{\underset{0}{V}} f = 0 \iff f$ is a constant function on $\mathbb{R}_+$.*

Proof. ($\Rightarrow$) Assume that $\overset{\infty}{\underset{0}{V}} f = 0$. More precisely, $\sup_{P \in \mathcal{P}(\mathbb{R}_+)} \overset{\infty}{\underset{0}{V}}(f, P) = 0$, i.e., for any partition $P = \{t_0, t_1, ...\} \in \mathcal{P}(\mathbb{R}_+)$

$$\sum_{j=1}^{\infty} |f(t_j) - f(t_{j-1})| = 0,$$

what implies

$$\underset{j \in \mathbb{N}}{\forall} |f(t_j) - f(t_{j-1})| = 0,$$

thus f is a constant function.
($\Leftarrow$) Let f be a constant function, i.e., $f(x) = a$ for $x \in \mathbb{R}_+$, $a \in \mathbb{R}$. We take a partition $P = \{t_0, t_1, ...\}$ of $\mathbb{R}_+$. Then

$$\overset{\infty}{\underset{0}{V}}(f, P) = \sum_{j=1}^{\infty} |f(t_j) - f(t_{j-1})| = \sum_{j=1}^{\infty} |a - a| = 0,$$

$$\overset{\infty}{\underset{0}{V}} f = \sup_{P \in \mathcal{P}(\mathbb{R}_+)} \overset{\infty}{\underset{0}{V}}(f, P) = 0.$$

□

Lemma 5. *Let $f : \mathbb{R}_+ \to \mathbb{R}$ be a function, $A > 0$. Then*

$$\overset{A}{\underset{0}{V}} f + \overset{\infty}{\underset{A}{V}} f = \overset{\infty}{\underset{0}{V}} f.$$

Proof. Let us fix $B > A$, then

$$\overset{A}{\underset{0}{V}} f + \overset{B}{\underset{A}{V}} f = \overset{B}{\underset{0}{V}} f.$$

Moreover,

$$\overset{\infty}{\underset{0}{V}} f = \lim_{B \to \infty} \overset{B}{\underset{0}{V}} f = \lim_{B \to \infty} (\overset{A}{\underset{0}{V}} f + \overset{B}{\underset{A}{V}} f) = \overset{A}{\underset{0}{V}} f + \overset{\infty}{\underset{A}{V}} f.$$

□

Theorem 6 ([5,6])**.** *Let $f : \mathbb{R}_+ \to \mathbb{R}$ be a function and $f \in BV(\mathbb{R}_+)$. Then $f \in B(\mathbb{R}_+)$.*

Proof. Let us fix $x > 0$. Then

$$|f(x)| = |f(x) - f(0) + f(0)| \leqslant |f(x) - f(0)| + |f(0)| \leqslant$$

$$\leqslant \overset{x}{\underset{0}{V}} f + |f(0)| \leqslant \overset{\infty}{\underset{0}{V}} f + |f(0)| < \infty.$$

□

Theorem 7 ([3])**.** *Let $f : \mathbb{R}_+ \to \mathbb{R}$ be a function and $f \in BV(\mathbb{R}_+)$. Then $\lim_{x \to \infty} f(x) < \infty$.*

Proof. Let us fix $\varepsilon > 0$ and $A > 0$ such that

$$\overset{\infty}{\underset{0}{V}} f - \overset{A}{\underset{0}{V}} f = \overset{\infty}{\underset{A}{V}} f \leqslant \varepsilon.$$

Thus for any given $B, D > A$, such that $B < D$ also we have:

$$\overset{D}{\underset{B}{V}} f \leqslant \varepsilon.$$

Moreover,

$$|f(D) - f(B)| \leqslant \overset{D}{\underset{B}{V}} f \leqslant \varepsilon,$$

i.e., function f satisfies the Cauchy's condition, so there exists the finite limit of f in infinity ($\lim\limits_{x\to\infty} f(x) \in \mathbb{R}$).

□

Remark 8. The converse theorem is not true, i.e.,

$$\lim_{x\to\infty} f(x) < \infty \nRightarrow f \in BV(\mathbb{R}_+).$$

For example, for function $f : \mathbb{R}_+ \to \mathbb{R}$ given by the formula

$$f(x) = \begin{cases} \frac{1}{x} & \text{for } x \in \mathbb{N} \\ 0 & \text{for } x \notin \mathbb{N} \end{cases}$$

we have: $\lim\limits_{x\to\infty} f(x) = 0 < \infty$ but $\overset{\infty}{\underset{0}{V}} f = 2 \sum\limits_{n=1}^{\infty} \frac{1}{n} = \infty$.

Remark 9. The definition described in Theorem 2. is also presented in [3] but in a different way.

Functions which play a very important role in the theory of functions of bounded variation are the monotonic functions.

Theorem 10 ([4])**.** *Let $f : \mathbb{R}_+ \to \mathbb{R}$ be a monotonic function, $A > 0$. Then* $\overset{\infty}{\underset{0}{V}} f = \lim\limits_{A\to\infty} |f(A) - f(0)|$.

Proof. Without loss of generality, assume that f is a nondecreasing function. We take a partition $P \in \mathcal{P}(\mathbb{R}_+)$, $P = \{t_0, t_1, ...\}$ such that $t_0 = 0$. Then

$$\overset{\infty}{\underset{0}{V}}(f, P) = \sum_{j=1}^{\infty} |f(t_j) - f(t_{j-1})| = \sum_{j=1}^{\infty} (f(t_j) - f(t_{j-1})) = \lim_{n\to\infty} (f(t_n) - f(0)),$$

thus

$$\sup_{P\in\mathcal{P}(\mathbb{R}_+)} \overset{\infty}{\underset{0}{V}}(f,P) = \overset{\infty}{\underset{0}{V}} f = \lim_{A\to\infty}(f(A)-f(0)).$$

Similarly for a nonincreasing function we have:

$$\overset{\infty}{\underset{0}{V}} f = \lim_{A\to\infty}(f(0)-f(A)).$$

□

Remark 11. There is no direct relationship between the continuity of a function and its bounded variation.

For example, for function $f:\mathbb{R}_+\to\mathbb{R}$ given by the formula

$$f(x)=\begin{cases} x & \text{for } x\in[0,1)\\ 2 & \text{for } x\in[1,\infty)\end{cases}$$

we have: $\overset{\infty}{\underset{0}{V}} f = 2 \leqslant \infty$, i.e., $f\in BV(\mathbb{R}_+)$ and f is discontinuous at $x=1$. On the other hand, for function $f:\mathbb{R}_+\to\mathbb{R}$ given by the formula

$$f(x)=\begin{cases} 0 & \text{for } x=0\\ x\sin(\frac{1}{x}) & \text{for } x\in(0,1]\\ \sin 1 & \text{for } x\in(1,\infty)\end{cases}$$

we have: $\overset{\infty}{\underset{0}{V}} f = \infty$, i.e., $f\notin BV(\mathbb{R}_+)$ and f is continuous at $\mathbb{R}_+$.

The next properties will give us more information about the structure of the set $BV(\mathbb{R}_+)$.

Theorem 12 ([5])**.** *Let $f,g:\mathbb{R}_+\to\mathbb{R}$ be functions such that $f,g\in BV(\mathbb{R}_+)$. Then $f\cdot g\in BV(\mathbb{R}_+)$.*

Proof. Let us take a partition $P=\{t_0,t_1,...\}$ of $\mathbb{R}_+$, then:

$$\overset{\infty}{\underset{0}{V}}(f\cdot g,P) = \sum_{i=1}^{\infty}|f(t_i)\cdot g(t_i)-f(t_{i-1})\cdot g(t_{i-1})| =$$

$$= \sum_{i=1}^{\infty}|f(t_i)\cdot g(t_i)-f(t_{i-1})\cdot g(t_i)+f(t_{i-1})\cdot g(t_i)-f(t_{i-1})\cdot g(t_{i-1})|$$

$$\leqslant \sum_{i=1}^{\infty} |(f(t_i) - f(t_{i-1}))| \cdot |g(t_i)| + \sum_{i=1}^{\infty} |(g(t_i) - g(t_{i-1}))| \cdot |f(t_{i-1})| < \infty.$$

Taking the supremum of all partitions of $\mathbb{R}_+$, we get that $\overset{\infty}{\underset{0}{V}}(f \cdot g) < \infty$.

□

Moreover, basing on the inequality above we get:

$$\overset{\infty}{\underset{0}{V}}(f \cdot g) \leqslant \sup_{x \in \mathbb{R}_+} |g(x)| \cdot \overset{\infty}{\underset{0}{V}} f + \sup_{x \in R_+} |f(x)| + \overset{\infty}{\underset{0}{V}} g.$$

Theorem 13 ([5])**.** *Let $f : \mathbb{R}_+ \to \mathbb{R}$ be a function and assume that there exists $\alpha > 0$ such that $|f(x)| \geqslant \alpha$ for all $x \in \mathbb{R}_+$. Then $\frac{1}{f} \in BV(\mathbb{R}_+)$.*

Proof. For any partition $P = \{t_0, t_1, ...\}$ of $\mathbb{R}_+$, then:

$$\overset{\infty}{\underset{0}{V}}(\frac{1}{f}, P) = \sum_{i=1}^{\infty} |\frac{1}{f(t_i)} - \frac{1}{f(t_{i-1})}| = \sum_{i=1}^{\infty} \frac{|f(t_{i-1}) - f(t_i)|}{|f(t_i) \cdot f(t_{i-1})|} \leqslant$$

$$\leqslant \frac{1}{\alpha^2} \cdot \sum_{i=1}^{\infty} |f(t_i) - f(t_{i-1})| = \frac{1}{\alpha^2} \cdot \overset{\infty}{\underset{0}{V}}(f, P) \leqslant \frac{1}{\alpha^2} \cdot \overset{\infty}{\underset{0}{V}} f < \infty.$$

□

Let us observe that the set $BV(\mathbb{R}_+)$, with the classical operations on functions, is a linear space over the field $\mathbb{R}$ and also forms an algebra.

Theorem 14 ([2])**.** *The following formula*

$$||f||_{BV} = |f(0)| + \overset{\infty}{\underset{0}{V}} f$$

defines a norm in the space $BV(\mathbb{R}_+)$. Moreover, this norm is complete in $BV(\mathbb{R}_+)$, i.e., $BV(\mathbb{R}_+)$ is a Banach space.

Proof. At the begining we show that $||\cdot||_{BV}$ is a norm. Obviously $||f||_{BV} \geqslant 0$ for $f \in BV(\mathbb{R}+)$.

1. $||f||_{BV} = 0 \iff (|f(0)| = 0 \wedge \overset{\infty}{\underset{0}{V}} f = 0) \iff (f(0) = 0 \wedge$ f is a constant function) $\iff f \equiv 0$

2. $||\lambda f||_{BV} = |\lambda f(0)| + \overset{\infty}{\underset{0}{V}}(\lambda f) = |\lambda| \cdot |f(0)| + |\lambda| \overset{\infty}{\underset{0}{V}} f = |\lambda|(|f(0)| + \overset{\infty}{\underset{0}{V}} f) =$
$= |\lambda| \cdot ||f|||_{BV}$

3. $||f+g||_{BV} = |f(0)+g(0)| + \overset{\infty}{\underset{0}{V}}(f+g) \leqslant |f(0)| + |g(0)| + \overset{\infty}{\underset{0}{V}} f + \overset{\infty}{\underset{0}{V}} g =$
$= ||f||_{BV} + ||g||_{BV}$

In the next step we will show that this norm is complete. Let $(f_n) \subset BV(\mathbb{R}_+)$ be a Cauchy's sequence, i.e.,

$$\underset{\varepsilon>0}{\forall} \underset{n_0 \in \mathbb{N}}{\exists} \underset{m,n \in \mathbb{N} \wedge m,n \geqslant n_0}{\forall} ||f_n - f_m||_{BV} = |f_n(0) - f_m(0)| + \overset{\infty}{\underset{0}{V}}(f_n - f_m) \leqslant \frac{\varepsilon}{2}$$

Thus $|f_n(0) - f_m(0)| \leqslant \frac{\varepsilon}{2}$ and as $m \to \infty$ we get $|f_n(0) - f(0)| \leqslant \frac{\varepsilon}{2}$, $n \geqslant n_0$ so $(f_n(0))$ is a Cauchy's sequence which is convergent to $f(0)$.

Moreover, $\overset{\infty}{\underset{0}{V}}(f_n - f_m) \leqslant \frac{\varepsilon}{2}$ and for $x > 0$ $\overset{x}{\underset{0}{V}}(f_n - f_m) \leqslant \frac{\varepsilon}{2}$. We take a partition $P = \{0, x\}$ of the interval $[0, x]$ and we have:

$$|f_n(x) - f_m(x)| - |f_n(0) - f_m(0)| \leqslant |(f_n(x) - f_m(x)) - (f_n(0) - f_m(0))| =$$

$$= \overset{x}{\underset{0}{V}}(f_n - f_m, P) \leqslant \overset{x}{\underset{0}{V}}(f_n - f_m) \leqslant \frac{\varepsilon}{2}$$

and $|f_n(x) - f_m(x)| \leqslant |f_n(0) - f_m(0)| + \frac{\varepsilon}{2} \leqslant \varepsilon$. Thus for every $x \geqslant 0$ (f_n) is a Cauchy's sequence, convergent to $f(x)$. In other words, the sequence (f_n) is pointwise convergent to f.

In the last part we prove the convergence in the sense of the norm. (f_n) is a Cauchy's sequence so $||f_n||_{BV} \leqslant M$, in particular $\overset{\infty}{\underset{0}{V}} f_n \leqslant M$, thus variations of functions f_n, $n \in \mathbb{N}$ are equibounded and also $f \in BV(\mathbb{R}_+)$. Obviously $\overset{\infty}{\underset{0}{V}}(f_n - f_m) \leqslant \frac{\varepsilon}{2}$ and as $m \to \infty$

$$\overset{\infty}{\underset{0}{V}}(f_n - f) \leqslant \liminf_{m \to \infty} \overset{\infty}{\underset{0}{V}}(f_n - f_m) \leqslant \frac{\varepsilon}{2}$$

so $||f_n - f||_{BV} \leqslant \varepsilon$, i. e. the Cauchy's sequence (f_n) is convergent in the sense of the norm to the function f belonging to the space $BV(\mathbb{R}_+)$ and space $BV(\mathbb{R}_+)$ is a complete space. In other words, $BV(\mathbb{R}_+)$ is a Banach space.

□

Theorem 15 ([2])**.** *The norm defined by the formula*

$$||f||^1_{BV} = ||f||_\infty + \overset{\infty}{\underset{0}{V}} f$$

is equivalent to the norm $||f||_{BV}$*.*

Proof. Obviously $||f||^1_{BV}$ is a norm. Let $f : \mathbb{R}_+ \to \mathbb{R}$ be a function such that $f \in BV(\mathbb{R}_+)$. Then

$$||f||_{BV} = |f(0)| + \overset{\infty}{\underset{0}{V}} f \leqslant ||f||_\infty + \overset{\infty}{\underset{0}{V}} f = ||f||^1_{BV}.$$

Moreover, $||f||_\infty \leqslant |f(0)| + \overset{\infty}{\underset{0}{V}} f$, thus

$$||f||^1_{BV} = ||f||_\infty + \overset{\infty}{\underset{0}{V}} f \leqslant |f(0)| + 2\overset{\infty}{\underset{0}{V}} f \leqslant 2|f(0)| + 2\overset{\infty}{\underset{0}{V}} f = 2||f||_{BV},$$

$$\text{i.e., } \frac{1}{2}||f||^1_{BV} \leqslant ||f||_{BV} \leqslant ||f||^1_{BV}.$$

□

Theorem 16 ([2])**.** *The space* $BV(\mathbb{R}_+)$ *forms with the norm* $||\cdot||_{BV}$ *a Banach algebra such that* $||f \cdot g||_{BV} \leqslant 4||f||_{BV} \cdot ||g||_{BV}$ *and with norm* $||\cdot||^1_{BV}$ *forms a normalized Banach algebra* ($||f \cdot g||^1_{BV} \leqslant ||f||^1_{BV} \cdot ||g||^1_{BV}$)*.*

Proof. Let us fix functions $f, g : \mathbb{R}_+ \to \mathbb{R}$, $f, g \in BV(\mathbb{R}_+)$. We have:

$$||f \cdot g||^1_{BV} = ||f \cdot g||_\infty + \overset{\infty}{\underset{0}{V}}(f \cdot g) \leqslant ||f||_\infty \cdot ||g||_\infty + ||f||_\infty \overset{\infty}{\underset{0}{V}} g + ||g||_\infty \overset{\infty}{\underset{0}{V}} f$$

$$\leqslant ||f||_\infty \cdot ||g||_\infty + ||f||_\infty \overset{\infty}{\underset{0}{V}} g + ||g||_\infty \overset{\infty}{\underset{0}{V}} f \leqslant + \overset{\infty}{\underset{0}{V}} f \cdot \overset{\infty}{\underset{0}{V}} g =$$

$$= ||g||_\infty(||f||_\infty + \overset{\infty}{\underset{0}{V}} f) + \overset{\infty}{\underset{0}{V}} g(||f||_\infty + \overset{\infty}{\underset{0}{V}} f) = (||f||_\infty + \overset{\infty}{\underset{0}{V}} f)(||g||_\infty + \overset{\infty}{\underset{0}{V}} g) =$$

$$= ||f||^1_{BV} \cdot ||g||^1_{BV}$$

and

$$||f \cdot g||_{BV} \leqslant ||f \cdot g||^1_{BV} \leqslant ||f||^1_{BV} \cdot ||g||^1_{BV}$$

$$\leqslant 2||f||_{BV} \cdot 2||g||_{BV} = 4||f||_{BV} \cdot ||g||_{BV}.$$

□

One of the most important properties of a function of bounded variation on a bounded interval is t it can be expressed as the difference of two increasing functions. We will now extend this property to $\mathbb{R}_+$.

Theorem 17 ([6])**.** *(Jordan's decomposition theorem) Let $f : \mathbb{R}_+ \to \mathbb{R}$ be a function. Then $f \in BV(\mathbb{R}_+)$ if and only if there exist increasing and bounded functions $g, h : \mathbb{R}_+ \to \mathbb{R}$ such that $f = g - h$.*

Proof. ($\Leftarrow$)
Assume that $f(x) = g(x) - h(x)$ for $x \in \mathbb{R}_+$ and functions g, h are increasing and bounded. From Theorem 10. we know that $g, h \in BV(\mathbb{R}_+)$. Moreover, $\overset{\infty}{\underset{0}{V}} g = \lim\limits_{A\to\infty} (g(A) - g(0))$, $\overset{\infty}{\underset{0}{V}} h = \lim\limits_{A\to\infty} (h(A) - h(0))$.
$BV(\mathbb{R}_+)$ is a linear space so $f \in BV(\mathbb{R}_+)$ as a linear combination of functions from $BV(\mathbb{R}_+)$.
($\Rightarrow$)
Let $f : \mathbb{R}_+ \to \mathbb{R}$ be a function, $f \in BV(\mathbb{R}_+)$. Let us consider a function $V_f(x) = \overset{x}{\underset{0}{V}} f$, $x \in \mathbb{R}_+$. The function V_f is nondecreasing because for $x_1, x_2 > 0$, $x_1 < x_2$ we have

$$V_f(x_2) - V_f(x_1) = \overset{x_2}{\underset{0}{V}} f - \overset{x_1}{\underset{0}{V}} f = \overset{x_2}{\underset{x_1}{V}} f \geqslant 0.$$

Let us put $g(x) = V_f(x)$ and next $h(x) = V_f(x) - f(x)$. The function h is also nondecreasing because for $x_1, x_2 > 0$, $x_1 < x_2$ we have

$$h(x_2) - h(x_1) = \overset{x_2}{\underset{0}{V}} f - f(x_2) - \overset{x_1}{\underset{0}{V}} f + f(x_1) = \overset{x_2}{\underset{x_1}{V}} f - (f(x_2) - f(x_1)) \geqslant 0.$$

Obviously $f(x) = g(x) - h(x)$ and functions g and h satisfy the assumptions of the theorem. □

Remark 18 ([6])**.** The decomposition $f(x) = g(x) - h(x)$ described in the proof of the Jordan's decomposition theorem is not the only one, e. g. we can add to the functions g and h any nondecreasing function ϕ, then $f(x) = (g + \phi)(x) - (h + \phi)(x)$.

We will now present the so-called Jordan's canonical decomposition:

$$g(x) = \frac{\overset{x}{\underset{0}{V}} f + f(x)}{2} = \frac{V_f(x) + f(x)}{2}, \quad h(x) = \frac{\overset{x}{\underset{0}{V}} f - f(x)}{2} = \frac{V_f(x) - f(x)}{2}.$$

Obviously, $f(x) = g(x) - h(x)$ and for $x_1, x_2 \in \mathbb{R}_+$, $x_1 < x_2$ we have

$$g(x_2) - g(x_1) = \frac{\overset{x_2}{\underset{0}{V}} f + f(x_2) - \overset{x_1}{\underset{0}{V}} f - f(x_1)}{2} = \frac{\overset{x_2}{\underset{x_1}{V}} f + f(x_2) - f(x_1)}{2} =$$

$$= \frac{\overset{x_2}{\underset{x_1}{V}} f - (f(x_1) - f(x_2))}{2} \geqslant 0,$$

$$h(x_2) - h(x_1) = \frac{\overset{x_2}{\underset{0}{V}} f - f(x_2) - \overset{x_1}{\underset{0}{V}} f + f(x_1)}{2} = \frac{\overset{x_2}{\underset{x_1}{V}} f - (f(x_2) - f(x_1))}{2} \geqslant 0,$$

so functions g and h are nondecreasing.

Moreover, those functions are as weakly as possible increasing, i.e., for any increasing functions $\overline{g}, \overline{h} : \mathbb{R}_+ \to \mathbb{R}$ such that $f(x) = \overline{g}(x) - \overline{h}(x)$ and for $x, y \in \mathbb{R}_+$, $x < y$ we have:

$$g(y) - g(x) \leqslant \overline{g}(y) - \overline{g}(x),$$
$$h(y) - h(x) \leqslant \overline{h}(y) - \overline{h}(x).$$

Obviously,

$$\overset{y}{\underset{x}{V}} f \leqslant \overset{y}{\underset{x}{V}} \overline{g} + \overset{y}{\underset{x}{V}} \overline{h} = \overline{g}(y) - \overline{g}(x) + \overline{h}(y) - \overline{h}(x),$$

so we have

$$g(y) - g(x) = \frac{\overset{y}{\underset{x}{V}} f + f(y) - f(x)}{2} \leqslant \overline{g}(y) - \overline{g}(x),$$

$$h(y) - h(x) = \frac{\overset{y}{\underset{x}{V}} f - (f(y) - f(x))}{2} \leqslant \overline{h}(y) - \overline{h}(x).$$

Theorem 19 ([2])**.** *(The majorant principle) Let $f : \mathbb{R}_+ \to \mathbb{R}$ be a function. Then $f \in BV(\mathbb{R}_+)$ if and only if there exists an increasing and bounded function $g : \mathbb{R}_+ \to \mathbb{R}$ such that $|f(y) - f(x)| \leqslant g(y) - g(x)$ for $x, y \in \mathbb{R}_+$, $x < y$.*

Proof. ($\Leftarrow$)
Assume that there exists an increasing and bounded function $g : \mathbb{R}_+ \to \mathbb{R}$ such that $|f(y) - f(x)| \leqslant g(y) - g(x)$ for any $x, y \in \mathbb{R}_+$, $x < y$. Then for a partition $P = \{t_0, t_1, t_2, ...\}$ of $\mathbb{R}_+$ we have

$$\overset{\infty}{\underset{0}{V}}(f,P)=\sum_{i=1}^{\infty}|f(t_i)-f(t_{i-1})|\leqslant\sum_{i=1}^{\infty}(g(t_i)-g(t_{i-1}))=\lim_{A\to\infty}(g(A)-g(0)).$$

Taking the supremum of all partitions we get

$$\overset{\infty}{\underset{0}{V}}f\leqslant\lim_{A\to\infty}(g(A)-g(0))<\infty\implies f\in BV(\mathbb{R}_+).$$

($\Rightarrow$)
Assume that $f\in BV(\mathbb{R}_+)$. Define the function $g:\mathbb{R}_+\to\mathbb{R}$, $g(x)=V_f(x)=\overset{x}{\underset{0}{V}}f$ (g is increasing and bounded). Then for $x,y\in\mathbb{R}_+$, $x<y$ we get:

$$|f(y)-f(x)|\leqslant\overset{y}{\underset{x}{V}}f=\overset{y}{\underset{0}{V}}f-\overset{x}{\underset{0}{V}}f=g(y)-g(x).$$

□

3. The Riemann-Stieltjes integral

In this part of the work, we focus on the concept of the Riemann-Stieltjes integral. It was developed mainly thanks to the knowledge about the variation of a function on a bounded interval and the Jordan's decomposition theorem. This theory is very useful for example in probability calculus. Extending the concept of variation to functions defined on the unbounded interval $\mathbb{R}_+$ allows us to consider the Riemann-Stieltjes integral on $\mathbb{R}_+$.

3.1. Definitions and basic facts

We will begin with the definition using the lower and upper Riemann-Stieltjes sums [2, 7, 8].

Definition 20. Let $f,\alpha:\mathbb{R}_+\to\mathbb{R}$ be functions such that α is increasing, f is bounded and $P=\{t_0,t_1,t_2,...\}\in\mathcal{P}(\mathbb{R}_+)$. We denote: $m_j=\inf\{f(x):x\in[t_{j-1},t_j]\}$, $M_j=\sup\{f(x):x\in[t_{j-1},t_j]\}$, $j\in\mathbb{N}$. The quantity

$$L_\alpha(f,P,\mathbb{R}_+)=\sum_{j=1}^{\infty}m_j(\alpha(t_j)-\alpha(t_{j-1}))$$

is called the lower Riemann-Stieltjes sum of function f with respect to P and

α and

$$U_\alpha(f, P, \mathbb{R}_+) = \sum_{j=1}^{\infty} M_j(\alpha(t_j) - \alpha(t_{j-1}))$$

is called the upper Riemann-Stieltjes sum of function f with respect to P and α.

Definition 21. Let $f, \alpha : \mathbb{R}_+ \to \mathbb{R}$ be functions such that α is increasing and f is bounded. Function f is integrable in the Riemann-Stieltjes sense (RS-integrable) with respect to α when for all $\varepsilon > 0$ there exists partition P of $\mathbb{R}_+$ such that

$$U_\alpha(f, P, \mathbb{R}_+) - L_\alpha(f, P, \mathbb{R}_+) \leqslant \varepsilon.$$

The set of all functions which are RS-integrable with respect to α on $\mathbb{R}_+$ we denote by $RS_\alpha(\mathbb{R}_+)$.
We have the following relationship:

$$f \in RS_\alpha(\mathbb{R}_+) \iff \inf\{U_\alpha(f, P, \mathbb{R}_+) : P \in \mathcal{P}(\mathbb{R}_+)\} =$$

$$= \sup\{L_\alpha(f, P, \mathbb{R}_+) : P \in \mathcal{P}(\mathbb{R}_+)\}$$

This common quantity we denote by

$$\int_0^\infty f(x)d\alpha(x) = \int_0^\infty f d\alpha$$

and call the Riemann-Stieltjes integral (or the RS-integral) of function f with respect to α on $\mathbb{R}_+$.

Remark 22. For $\alpha(x) = x$ the RS-integral is simply the Riemann integral, then we denote: $f \in RS(\mathbb{R}_+)$ or $f \in R(\mathbb{R}_+)$.

We will now present an equivalent definition of the Riemann-Stieltjes integral using sample points.

Definition 23. Let $f, \alpha : \mathbb{R}_+ \to \mathbb{R}$ be functions such that α is increasing, f is bounded and $P = \{t_0, t_1, t_2, ...\} \in \mathcal{P}(\mathbb{R}_+)$. For the partition P we take a set $\Pi = \{\tau_1, \tau_2, ...\}$ such that $0 = t_0 \leqslant \tau_1 \leqslant t_1 \leqslant \tau_2 \leqslant t_2 ...$, i.e., $\tau_i \in [t_{i-1}, t_i]$, $i \in \mathbb{N}$. The sum

$$S_\alpha(f, P, \Pi, \mathbb{R}_+) = \sum_{j=1}^{\infty} f(\tau_j)(\alpha(t_j) - \alpha(t_{j-1}))$$

is called the Riemann-Stieltjes sum. Subsequently we denote by $\mu(P)$ the diameter of the partition P, i.e., $\mu(P) = \max_{i\in\mathbb{N}}(t_i - t_{i-1})$.
Let us put $A := \lim_{\mu(P)\to 0} S_\alpha(f, P, \Pi, \mathbb{R}_+)$, i.e.,

$$\forall_{\varepsilon>0}\ \exists_{\delta>0}\ |S_\alpha(f, P, \Pi, \mathbb{R}_+) - A| \leqslant \varepsilon$$

for all partitions P of $\mathbb{R}_+$ such that $\mu(P) \leqslant \delta$ and sets Π connected with those partitions.

Theorem 24. *If the limit denoted by A exists then $f \in RS_\alpha(\mathbb{R}_+)$ and* $\lim_{\mu(P)\to 0} S_\alpha(f, P, \Pi, \mathbb{R}_+) = \int_0^\infty f(x)d\alpha(x)$.

3.2. Properties of Riemann-Stieltjes integral on $\mathbb{R}_+$

In the next part, we will present some of the most important basic properties of the Riemann-Stieltjes integral, which are very useful in calculations [2, 7, 8].

Theorem 25. *Let $\alpha, \beta : \mathbb{R}_+ \to \mathbb{R}$ be increasing functions, $\lambda \in \mathbb{R}$, $f, g \in RS_\alpha(\mathbb{R}_+)$. Then:*

1. $f + g \in RS_\alpha(\mathbb{R}_+)$ *and* $\int_0^\infty (f+g)d\alpha = \int_0^\infty f d\alpha + \int_0^\infty g d\alpha$
2. $f \in RS_\alpha(\mathbb{R}_+) \cap RS_\beta(\mathbb{R}_+) \implies f \in RS_{\alpha+\beta}(\mathbb{R}_+)$ *and* $\int_0^\infty f d(\alpha+\beta) = \int_0^\infty f d\alpha + \int_0^\infty f d\beta$
3. $\int_0^\infty (\lambda f)d\alpha = \lambda \int_0^\infty f d\alpha$
4. $\int_0^\infty f d(\lambda\alpha) = \lambda \int_0^\infty f d\alpha$, *for* $\lambda > 0$
5. $f(x) \leqslant g(x)$ *for* $x \in \mathbb{R}_+ \implies \int_0^\infty f d\alpha \leqslant \int_0^\infty g d\alpha$
6. $f \in RS_\alpha([0, a])$, $f \in RS_\alpha([a, \infty))$ *and* $\int_0^\infty f(x)d\alpha(x) = \int_0^a f(x)d\alpha(x) + \int_a^\infty f(x)d\alpha(x)$, *for* $a > 0$

Proof. (6.)
Let us fix $\varepsilon > 0$. Since $f \in RS_\alpha(\mathbb{R}_+)$, we find $\delta > 0$ such that

$$|S_\alpha(f, P_1, \Pi_1, [a, b]) - S_\alpha(f, P_2, \Pi_2, [a, b])| \leqslant \varepsilon$$

for $P_1, P_2 \in \mathcal{P}(\mathbb{R}_+)$, $\mu(P_1), \mu(P_2) \leqslant \delta$ and sets Π_1, Π_2 connected with the partitions P_1 and P_2.
Now we fix $a > 0$ and take partitions $P_1', P_2' \in \mathcal{P}([0, a])$ such that $\mu(P_1')$, $\mu(P_2') \leqslant \delta$ and sets of sample points Π_1', Π_2' connected with those partitions. We also take a partition $P'' \in \mathcal{P}([a, \infty))$ such that $\mu(P'') \leqslant \delta$ and a set of sample points Π'' connected with P''.
Then $P_1 = P_1' \cup P''$, $P_2 = P_2' \cup P''$, $P_1, P_2 \in \mathcal{P}(\mathbb{R}_+)$, $\mu(P_1), \mu(P_2) \leqslant \delta$ and

$$\begin{aligned} &|S_\alpha(f, P_1', \Pi_1', [0, a]) - S_\alpha(f, P_2', \Pi_2', [0, a])| \leqslant \\ &\leqslant |S_\alpha(f, P_1, \Pi_1, \mathbb{R}_+) - S_\alpha(f, P_2, \Pi_2, \mathbb{R}_+)| \leqslant \varepsilon, \end{aligned}$$

thus $f \in RS_\alpha([0, a])$.

Similarly, for $P' \in \mathcal{P}([0, a])$ and $P_1'', P_2'' \in \mathcal{P}([a, \infty))$ we could show that $f \in RS_\alpha([a, \infty))$.

For any partitions $P' \in \mathcal{P}([0, a])$, $P'' \in \mathcal{P}([a, \infty))$ we have $P = P' \cup P'' \in \mathcal{P}(\mathbb{R}_+)$ and $\mu(P) \leqslant \max\{\mu(P'), \mu(P'')\}$. Then

$$S_\alpha(f, P, \Pi, \mathbb{R}_+) = S_\alpha(f, P', \Pi', [0, a]) + S_\alpha(f, P'', \Pi'', [a, \infty))$$

and as $\mu(P) \to 0$, we get the thesis.

□

At the end, we will go to the connection between the Riemann-Stieltjes integral on $\mathbb{R}_+$ and the Riemann-Stieltjes integrals on bounded intervals.

Theorem 26. *Let $f : \mathbb{R}_+ \to \mathbb{R}$ be a function such that $f \in RS_\alpha(\mathbb{R}_+)$, $\alpha : \mathbb{R}_+ \to \mathbb{R}$, α-increasing, f-bounded. Then*

$$\int_0^\infty f(x) d\alpha(x) = \lim_{A \to \infty} \int_0^A f(x) d\alpha(x).$$

In the literature, the concepts of variation of a function and the Riemann-Stieltjes integral are mainly considered on bounded intervals. It is also possible to extend these concepts to unbounded intervals, such as $\mathbb{R}_+$, which opens up an interesting direction for further research.

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Summary

The aim of the chapter is to present some functions concerning the concept of the variation of functions on the real half-axis $\mathbb{R}_+$. We are formulating the announced definition and also some equivalent version of that definition. Moreover, we provide a few properties of functions of bounded variation on the half-axis associated mainly with the behaviour of those functions at infinity. Apart from this we are going also to introduce the definition of the Riemann-Stieltjes integral on an unbounded interval. Some properties of such an integral will be also provided. Our investigations will be illustrated by suitable examples.

Streszczenie

Celem rozdziału jest przedstawienie pewnych funkcji związanych z pojęciem wariacji funkcji na dodatniej półprostej rzeczywistej $\mathbb{R}_+$. Formułujemy zapowiedzianą definicję oraz jej równoważną wersję. Ponadto, podajemy kilka własności funkcji o ograniczonej wariacji na półprostej, związanych głównie z zachowaniem tych funkcji w nieskończoności. Poza tym wprowadzamy również definicję całki Riemanna-Stieltjesa na przedziale nieograniczonym. Przedstawiamy także pewne własności takiej całki. Nasze rozważania zilustrowane zostały odpowiednimi przykładami.

Chapter 3

Measures of noncompactness in the study of the solvability of infinite systems of integral equations on the real half-axis

Justyna Madej

1. Introduction

Integral equations play a significant role both in the theory of linear operators [19, 21, 23] and in the nonlinear analysis [21, 24, 29]. Moreover, integral equations represent effective mathematical tools used in modeling processes and phenomena encountered in many fields such as mechanics, thermodynamics, astronomy, hydrodynamics, mathematical physics, economics and biology [14–20,31]. The theory of integral equations is still being developed, especially investigations concerning infinite systems of integral equations. It is worthwhile mentioning that the results on solutions of infinite systems of integral equations defined on an unbounded interval are quite new [1, 5, 7, 8, 10, 12, 13, 26, 28, 30], especially papers of Banaś and Madej [2, 7, 8] in which we considered infinite systems of integral equations with integral taken as improper one defined on the real half-axis $\mathbb{R}_+$. As far as we know the above quoted papers are the first ones of such a type. Thus

these results are highly nontrivial and original. It turns out that tools that have been successfully used in the study of solutions of infinite systems of integral equations are closely related to the technique of measures of noncompactness.

In the chapter we are going to present results on solutions of infinite systems of quadratic integral equations of Hammerstein and Urysohn type defined on the real half-axis $\mathbb{R}_+$. It is worthwhile mentioning that since the considered systems of integral equations are infinite systems, this implies that solutions of these systems are function sequences which are assumed to satisfy some pre-imposed conditions at infinity. More precisely, we look for solutions of the studied systems that are asymptotically stable. To this end, we impose appropriate assumptions on the considered infinite systems of integral equations, which guarantee not only the existence of solutions, but also that these solutions satisfy the aforementioned condition. In the proofs of the existence theorems there are used suitably constructed to these purposes measures of noncompactness. The technique of measures of noncompactness allows us to apply the classical Schauder fixed point theorem or the well-known Darbo fixed point principle. We apply the axiomatic approach to the concept of measures of noncomactness, because it allows us to construct measures with desired properties and to study spaces in which the conditions for relative compactness of sets are not known.

2. Notations, Definitions and Auxiliary Facts

This section is devoted to collect notations and definitions that will be utilized in our further investigations.

By the symbol $\mathbb{R}$ we denote the set of real numbers. The set of natural numbers will be denoted by $\mathbb{N}$. Moreover, we put $\mathbb{R}_+ = [0, +\infty)$.

Further, let E be a given Banach space with the norm $||\cdot||_E$ and the zero element θ. By the symbols $B(x, r)$ and B_r we denote the closed ball with radius r, centered at x and θ, respectively.

Let X be a subset of the space E. Then the closure of X and the closed convex hull of X are denoted by $\overline{X}$ and $\mathrm{Conv}X$. Moreover, we use standard notations $X + Y$ and λX for classical algebraic operations on subsets of E.

Next, we assume that $\mathfrak{M}_E$ denotes the family of all nonempty and bounded subsets of E while $\mathfrak{N}_E$ stands for its subfamily consisting of all relatively compact sets. The most important tool used in our considerations is the concept of the measure of noncompactness. The first measure

of noncompactness was defined by Kuratowski in 1930 by the following formula [27]

$$\alpha(X) = \inf\left\{\varepsilon > 0 :\ X \subset \bigcup_{i=1}^{n} X_i :\ X_i \subset M,\ \operatorname{diam}(X_i) < \varepsilon,\ i = 1, 2, \ldots, n\right\}$$

for $X \in \mathfrak{M}_E$ and $n \in \mathbb{N}$.

Unfortunately, despite its seemingly simple definition, the Kuratowski measure of noncompactness turns out to be difficult in applications. While the equality $\alpha(B_1) = 0$ in a finite-dimensional space is an immediate consequence of some properties of this measure [6], computing the Kuratowski measure of noncompactness for the unit ball in an infinite-dimensional space required considerable skills. A formal proof of the equality $\alpha(B_1) = 2$ in the case of infinite-dimensional space was established only after forty years [24].

It turns out that more useful and convenient measure of noncompactness is the so-called Hausdorff (or ball) measure of noncompactness defined for any $X \in \mathfrak{M}_E$ in the following way [6]

$$\chi(X) = \inf\{\varepsilon > 0 :\ X \text{ has a finite } \varepsilon\text{-net in } E\}.$$

It is worthwhile mentioning that in some Banach spaces such as c_0, l_p $(1 \leqslant p < \infty)$, $C([a, b])$ we know formulas expressing χ in connection with the structure of these spaces [6]. On the other hand, in some Banach spaces, for example in the space $L^p(a, b)$ or c, we can give formulas for measures of noncompactness which are equivalent to χ. Finally, in a lot of Banach spaces such as l^∞ we are not able to construct formulas expressing the Hausdorff measures of noncompactness χ or measures equivalent to χ. In this situation we need to restrict ourselves to an axiomatic definition of the measure of noncompactness. It turns out, that this approach to the concept of measure of noncompactness is the most useful in many applications. Thus we accept the following definition [6].

Definition 1. A function $\mu : \mathfrak{M}_E \to \mathbb{R}_+$ is said to be a measure of noncompactness in the space E if it satisfies the following conditions:

(i) The family $\ker \mu = \{X \in \mathfrak{M}_E : \mu(X) = 0\}$ is nonempty and $\ker \mu \subset \mathfrak{N}_E$.

(ii) $X \subset Y \Rightarrow \mu(X) \leqslant \mu(Y)$.

(iii) $\mu\left(\overline{X}\right) = \mu(X)$.

(iv) $\mu(\mathrm{Conv}X) = \mu(X)$.

(v) $\mu(\lambda X + (1-\lambda)Y) \leqslant \lambda\mu(X) + (1-\lambda)\mu(Y)$ for $\lambda \in [0,1]$.

(vi) If (X_n) is a sequence of closed sets from $\mathfrak{M}_E$ such that $X_{n+1} \subset X_n$ for $n = 1, 2, \ldots$ and $\lim_{n\to\infty} \mu(X_n) = 0$, then the set $X_\infty = \bigcap_{n=1}^{\infty} X_n$ is nonempty.

The family $\ker\mu$ from axiom (i) is called the kernel of the measure μ. If $\ker\mu = \mathfrak{N}_E$ then the measure of noncompactness μ is said to be *full*.

If μ is a measure of noncompactness in the Banach space E, then μ is called *sublinear* [6] if it satisfies the following conditions:

(vii) $\mu(X+Y) \leqslant \mu(X) + \mu(Y)$.

(viii) $\mu(\lambda X) = |\lambda|\mu(X)$, for $\lambda \in \mathbb{R}$.

If μ satisfies the condition

(ix) $\mu(X \cup Y) = \max\{\mu(X), \mu(Y)\}$

then it is called *the measure of noncompactness with the maximum property*. If μ is a full and sublinear measure of noncompactness having the maximum property then it is called *regular* [6].

Remark 2. It can be proved that the set X_∞ from axiom (vi) is an element of the kernel $\ker\mu$. This fact allows us to deduce some useful properties of solutions of considered operator equations.

Now we recall the Darbo fixed point theorem which utilizes the concept of the measure of noncompactness.

Theorem 3. *Let Ω be a nonempty, bounded, closed and convex subset of a Banach space E and let μ be a measure of noncompactness in the space E. Assume that $Q : \Omega \to \Omega$ is a continuous operator and there exists $k \in [0,1)$ such that*

$$\mu(QX) \leqslant k\mu(X)$$

for any nonempty set $X \subset \Omega$.
Then Q has at least one fixed point in the set Ω.

Remark 4. It can be proved [6] that the set Fix$Q$ consisting of fixed points of the operator Q is an element of the kernel $\ker\mu$ of the measure of noncompactness μ. This fact is often used to chracterize solutions of the considered operator equation.

The above presented Darbo fixed point theorem is a generalization of the well-known Schauder fixed point principle. Let us recall this important theorem.

Theorem 5. *Let Ω be a nonempty, convex and compact subset of a Banach space E. Then every continuous operator $Q : \Omega \to \Omega$ has a fixed point.*

Now we are going to present Banach spaces in which our considerations will be conducted.

Let us notice that since we consider quadratic equations (i.e., having the form of a superposition operator multiplied by an integral operator), this requires us to choose for the study appropriate function spaces that are closed under pointwise multiplication — that is, Banach algebras. Thus we are going to work in the Banach space $BC(\mathbb{R}_+, E)$ consisting of functions $x : \mathbb{R}_+ \to E$ which are continuous and bounded on $\mathbb{R}_+$. This space is endowed with the standard supremum norm

$$||x||_{BC(\mathbb{R}_+,E)} = \sup\{||x(t)||_E : \ t \in \mathbb{R}_+\}$$

for $x \in BC(\mathbb{R}_+, E)$.

Now, taking as E the Banach space l_∞ consisting of real bounded sequences we obtain the space $BC(\mathbb{R}_+, l_\infty)$. For convenience we will denote this space as BC_∞.

Thus, we consider the space BC_∞ consisting of functions $x : \mathbb{R}_+ \to l_\infty$ which are bounded and continuous on $\mathbb{R}_+$. More precisely, any $x \in BC_\infty$ is a function sequence having the following form

$$x(t) = (x_n(t)) = (x_1(t), x_2(t), \dots)$$

for $t \in \mathbb{R}_+$, where for any fixed $t \in \mathbb{R}_+$ the sequence $(x_n(t))$ is an element of the space l_∞. The norm of any function $x \in BC_\infty$ is defined by the following equality

$$||x||_\infty = \sup\{||x(t)||_{l_\infty} : \ t \in \mathbb{R}_+\} = \sup_{t\in\mathbb{R}_+}\{\sup\{|x_n(t)| : \ n \in \mathbb{N}\}\}.$$

In what follows we are going to present the formula for the measure of noncompactness μ_c^3 in the space BC_∞ [5]. To this aim let us fix a set $X \in \mathfrak{M}_{BC_\infty}$. For arbitrary $\varepsilon > 0$ and $x = x(t) = (x_n(t)) \in X$ we define

$$\omega^\infty(x,\varepsilon) = \sup\{\|x(t) - x(s)\|_{l_\infty} : \ t,s \in \mathbb{R}_+, \ |t-s| \leqslant \varepsilon\}$$
$$= \sup\{\sup\{|x_n(t) - x_n(s)| : \ n = 1,2,\dots\} : \ t,s \in \mathbb{R}_+, \ |t-s| \leqslant \varepsilon\}.$$

Next, we obtain

$$\omega^\infty(X,\varepsilon) = \sup_{x\in X}\left\{\sup\left\{\sup_{n\in\mathbb{N}}|x_n(t) - x_n(s)| : \ t,s \in \mathbb{R}_+, \ |t-s| \leqslant \varepsilon\right\}\right\}.$$

Finally, we have

$$\omega_0^\infty(X) = \lim_{\varepsilon\to 0}\omega^\infty(X,\varepsilon)$$
$$= \lim_{\varepsilon\to 0}\left\{\sup_{x\in X}\left\{\sup\left\{\sup_{n\in\mathbb{N}}|x_n(t) - x_n(s)| : \ t,s \in \mathbb{R}_+, \ |t-s| \leqslant \varepsilon\right\}\right\}\right\}. \tag{1}$$

Next, to define the second term $\overline{\mu}_\infty$ of the measure of noncompactness μ_c^3 we will use a formula of the measure of noncompactness in the space l^∞. Namely, for any $X \in \mathfrak{M}_{l_\infty}$ we define the function μ_3 in the following way [6]

$$\mu_3(X) = \limsup_{n\to\infty} \operatorname{diam} X_n,$$

where

$$X_n = \{x_n : \ x = (x_i) \in X\}$$

and

$$\operatorname{diam} X_n = \sup\{|x_n - y_n| : \ x = (x_i), \ y = (y_i) \in X\}.$$

Then, for arbitrarily fixed $T > 0$ we obtain

$$\overline{\mu}_T^3(X) = \sup\{\mu_3(X(t)) : \ t \in [0,T]\}$$
$$= \sup_{t\in[0,T]}\left\{\limsup_{n\to\infty}\{\sup\{|x_n(t) - y_n(t)| : \ x = (x_n(t)), \ y = (y_n(t)) \in X\}\}\right\}. \tag{2}$$

Then we derive the following formula

$$\overline{\mu}^3_\infty(X) = \lim_{T\to\infty} \overline{\mu}^3_T(X)$$
$$= \lim_{T\to\infty}\left\{ \sup_{t\in[0,T]}\left\{ \limsup_{n\to\infty}\{\sup\{|x_n(t)-y_n(t)| : x=(x_n(t)), y=(y_n(t))\in X\}\}\right\}\right\}. \tag{3}$$

Now we are going to define the last component of the measure of noncompactness μ^3_c, i.e., the function c. It is given by the following formula

$$c(X) = \limsup_{t\to\infty} \operatorname{diam} X(t)$$
$$= \limsup_{t\to\infty}\left\{ \sup\left\{ \sup_{n\in\mathbb{N}} |x_n(t)-y_n(t)| :\ x=(x_n(t)),\ y=(y_n(t))\in X\right\}\right\}. \tag{4}$$

Now, adding formulas (1), (3) and (4) we obtain the formula expressing the measure of noncompactness in the space BC_∞

$$\mu^3_c(X) = \omega^\infty_0(X) + \overline{\mu}^3_\infty(X) + c(X). \tag{5}$$

Let us note that the first component ω^∞_0 investigates the equicontinuity of functions belonging to the set X while the second one, i.e., the function $\overline{\mu}^3_\infty$ investigates the compactness of the cross-section $X(t)$ of the set X in the space l_∞. Moreover, the last component analyzes the behaviour of the functions from the set X at infinity.

It is worthwhile mentioning that each component of the measure of noncompactness μ^3_c is sufficient. Omitting any of the components will cause that this measure of noncompactness will not satisfy condition (i) from Definition 1, i.e., $\ker\mu \not\subseteq \mathfrak{N}_{BC_\infty}$. It can be shown with help of appropriate examples.

This measure is sublinear but does not have a maximum property. Moreover, μ^3_c is not full.

Let us notice that we can similarly construct other measures of noncompactness in the space BC_∞ [5]. This is possible by changing the last component, i.e., the function c for the another one. Moreover, we can choose another measure on noncompactness from the space l^∞ in the formula (2). Applications of some of these measures of noncompactness in the space BC_∞ for studying infinite systems of integral equations can be found, among others, in [4, 10, 13].

The following remark will be very useful to characterize solutions of considered operator equations.

Remark 6. The kernel $\ker\mu_c^3$ of the measure of noncompactness μ_c^3 is the family consisting of all nonempty and bounded subsets X of the space BC_∞ such that functions belonging to X are equiuniformly continuous on $\mathbb{R}_+$ and all cross-sections $X(t)$ of X are sets in l_∞ such that the thickness $X(t)$ tends to zero as $n \to \infty$, uniformly with respect to $t \in \mathbb{R}_+$. Moreover, the thickness of the bundle formed by graphs of functions from X tends to zero at infinity.

3. Solvability of infinite system of quadratic Hammerstein integral equations

This section is devoted to present results on the existence of solutions of the infinite system of the quadratic integral equations of Hammerstein type. Our considerations will be located in the Banach space $BC_\infty = BC(\mathbb{R}_+, l_\infty)$ described above.

Thus, let us consider the following infinite system of integral equations

$$x_n(t) = a_n(t) + f_n(t, x_1(t), x_2(t), \dots) \int_0^\infty g_n(t,\tau) h_n(\tau, x_1(\tau), x_2(\tau), \dots) d\tau, \tag{6}$$

for $t \in \mathbb{R}_+$ and $n = 1, 2, \dots$.

We assume that the infinite system of integral equations (6) satisfies the following conditions.

(i) The sequence $(a_n(t))$ is an element of the space BC_∞. Moreover, the functions $a_n = a_n(t)$ are equicontinuous on $\mathbb{R}_+$.

(ii) The functions $g_n(t,\tau) = g_n : \mathbb{R}_+^2 \to \mathbb{R}$ are continuous on the set $\mathbb{R}_+^2$ for $n = 1, 2, \dots$. Apart from this the functions $t \to g_n(t,\tau)$ are equicontinuous on the set $\mathbb{R}_+$ uniformly with respect to $\tau \in \mathbb{R}_+$, i.e., the following condition is satisfied

$$\underset{\varepsilon>0}{\forall} \underset{\delta>0}{\exists} \underset{n\in\mathbb{N}}{\forall} \underset{\tau\in\mathbb{R}_+}{\forall} \underset{t_1,t_2\in\mathbb{R}_+}{\forall} \left[|t_2 - t_1| \leqslant \delta \Rightarrow |g_n(t_2,\tau) - g_n(t_1,\tau)| \leqslant \varepsilon\right].$$

(iii) For any $n \in \mathbb{N}$ and for each $t \in \mathbb{R}_+$ the improper integral

$$\int_0^\infty |g_n(t,\tau)| d\tau$$

is convergent. Moreover, the integrals $\int_0^\infty |g_n(t,\tau)|d\tau$ are for any $n = 1, 2, \dots$ and for each $t \in \mathbb{R}_+$.

(iv) The sequence $(g_n(t,\tau))$ is equibounded on $\mathbb{R}_+^2$, i.e., there exists a constant $G_2 > 0$ such that $|g_n(t,\tau)| \leqslant G_2$ for $t, \tau \in \mathbb{R}_+$ and $n = 1, 2, \dots$.

(v) The functions f_n are defined on the set $\mathbb{R}_+ \times \mathbb{R}^\infty$ and take real values for $n = 1, 2, \dots$. Moreover, the functions $t \to f_n(t, x_1, x_2, \dots)$ are equicontinuous on $\mathbb{R}_+$ uniformly with respect to $x = (x_n) \in l_\infty$, i.e., the following condition is satisfied

$$\begin{aligned}&\underset{\varepsilon>0}{\forall}\ \underset{\delta>0}{\exists}\ \underset{(x_i)\in l_\infty}{\forall}\ \underset{n\in\mathbb{N}}{\forall}\ \underset{t_1,t_2\in\mathbb{R}_+}{\forall} \big[|t_2 - t_1| \leqslant \delta \Rightarrow \\ &|f_n(t_2, x_1, x_2, \dots) - f_n(t_1, x_1, x_2, \dots)| \leqslant \varepsilon\big].\end{aligned}$$

Additionally we assume that functions f_n are equibounded, i.e., there exists a constant $\phi > 0$ such that

$$|f_n(t, x_1, x_2, \dots)| \leqslant \phi$$

for $t \in \mathbb{R}_+$ and $n \in \mathbb{N}$.

(vi) There exists a function $k : \mathbb{R}_+ \to \mathbb{R}_+$ which is nondecreasing on $\mathbb{R}_+$, continuous at 0 and such that $k(0) = 0$. Moreover, the following condition is satisfied

$$|f_n(t, x_1, x_2, \dots) - f_n(t, y_1, y_2, \dots)| \leqslant k(r) \sup\{|x_i - y_i| : \ i \geqslant n\}$$

for any $r > 0$, for $x = (x_i)$, $y = (y_i) \in l_\infty$ such that $\|x\|_{l_\infty} \leqslant r$, $\|y\|_{l_\infty} \leqslant r$ and for all $t \in \mathbb{R}_+$ and $n = 1, 2, \dots$.

(vii) The sequence of functions $\left(\overline{f}_n\right)$, where $\overline{f}_n(t) = |f_n(t, 0, 0, \dots)|$ is an element of the space BC_∞.

(viii) The functions h_n are defined on the set $\mathbb{R}_+ \times \mathbb{R}^\infty$ and take real values for $n = 1, 2, \dots$. Moreover, there exists a function $m : \mathbb{R}_+ \to \mathbb{R}_+$, which is nondecreasing on $\mathbb{R}_+$, continuous at $r = 0$, $m(0) = 0$ and such that the following condition is satisfied

$$|h_n(t, x_1, x_2, \dots) - h_n(t, y_1, y_2, \dots)| \leqslant m(r) \sup\{|x_i - y_i| : \ i \geqslant n\}$$

for any $r > 0$, for $x = (x_i)$, $y = (y_i) \in l_\infty$ such that $\|x\|_{l_\infty} \leqslant r$, $\|y\|_{l_\infty} \leqslant r$ and for all $t \in \mathbb{R}_+$ and $n = 1, 2, \dots$.

(ix) The operator h defined on the set $\mathbb{R}_+ \times l_\infty$ by the formula

$$(hx)(t) = (h_n(t, x)) = (h_1(t, x), h_2(t, x), \dots)$$

is bounded, i.e., there exists a positive constant $\overline{h}$ such that

$$\|(hx)(t)\|_{l_\infty} \leqslant \overline{h}$$

for any $x \in l_\infty$ and for each $t \in \mathbb{R}_+$.

(x) For any $n \in \mathbb{N}$ and for each function $x = x(t) = (x_i(t)) \in BC_\infty$ the improper integral

$$\int_0^\infty |h_n(s, x(s))|ds = \int_0^\infty |h_n(s, x_1(s), x_2(s), \dots)|ds$$

is convergent. Moreover, the integrals $\int_0^\infty |h_n(s, x(s))|ds$ are equibounded for $n \in \mathbb{N}$ and for each $x = x(t) \in BC_\infty$.

(xi) There exists a positive number r_0 which satisfies the inequality

$$A + \overline{F}G_1\overline{h} + G_1\overline{h}rk(r) \leqslant r$$

and such that

$$G_1\overline{h}k(r_0) + \phi G_1 m(r_0) < 1,$$

where

$$A = \sup\{\sup\{|a_n(t)| : \ n = 1, 2, \dots\} : \ t \in \mathbb{R}_+\},$$

$$G_1 = \sup\left\{\int_0^\infty |g_n(t, \tau)|d\tau : \ n = 1, 2, \dots, \ t \in \mathbb{R}_+\right\},$$

$$\overline{H} = \sup\left\{\int_0^\infty |h_n(s, x(s))|ds : \ x \in BC_\infty, \ n = 1, 2, \dots\right\},$$

$$\overline{F} = \sup\left\{\overline{f}_n(t) : \ t \in \mathbb{R}_+, \ n = 1, 2, \dots\right\}$$

are finite constants.

Now we are in a position to present the main existence result concerning the system (6).

Theorem 7. *Under assumptions (i)-(xi) the infinite system of integral equations* (6) *has at least one solution* $x = (x_n(t))$ *in the space* BC_∞.

In the proof of the above presented theorem we use the Darbo fixed point principle (see Theorem 3) and the measure of noncompactness μ_c^3 in the space BC_∞ expressed by the formula (5). At the beginning we prove, that an operator Q generated by the considered system transforms the space BC_∞ into itself.

Next, based on the first part of assumption (xi) we deduce that the operator Q transforms the ball B_{r_0} into itself. Further on, we prove that the operator Q is continuous on the set B_{r_0}.

The key part of the proof is to show that the operator Q satisfies the inequality from the Darbo fixed point theorem with respect to the measure of noncompactness μ_c^3, i.e., that Q is a μ_c^3-contraction. To realize this goal we study the behaviour of the operator Q with respect to the components of the measure of noncompactness μ_c^3, i.e., the functions ω_0^∞, $\overline{\mu}_\infty^3$ and c defined by formulas (1), (3) and (4). We obtain the following estimates

$$\omega_0^\infty(QX) \leqslant G_1\overline{h}k(r_0)\omega_0^\infty(X),$$

$$\overline{\mu}_\infty^3(QX) \leqslant G_1\overline{h}k(r_0)\overline{\mu}_\infty^3(X) + \phi G_1 m(r_0)\overline{\mu}_\infty^3(X),$$

$$c(QX) \leqslant G_1\overline{h}k(r_0)c(X) + \phi G_1 m(r_0)c(X)$$

which are satisfied for any nonempty set $X \subset B_{r_0}$. Gathering the above inequalities we have

$$\mu_c^3(QX) \leqslant [G_1\overline{h}k(r_0) + \phi G_1 m(r_0)]\mu_c^3(X).$$

From the second part of assumption (xi) we infer that

$$G_1\overline{h}k(r_0) + \phi G_1 m(r_0) < 1.$$

Finally, taking into account Theorem 3 we infer that there exists at least one element $x = x(t) = (x_n(t)) \in BC_\infty$ being a fixed point of the operator Q in the ball B_{r_0}. Obviously the function x is a solution of the infinite system of integral equations (6).

Let us note that Remark 4 implies that the set of solutions of the system (6) is an element of the kernel $\ker\mu_c^3$. Moreover, in view of Remark 6 we conclude that the thickness of the bundle formed by graphs of those solutions

tends to zero at infinity, i.e., the following condition is satisfied

$$\underset{\varepsilon>0}{\forall}\ \underset{T>0}{\exists}\ \underset{x,y\in \mathrm{Fix}Q}{\forall}\ \underset{t\geqslant T}{\forall} ||x(t)-y(t)||_{l_\infty} \leqslant \varepsilon.$$

It is easy to notice that this property can be treated as the characterization of the set of solutions of (6) in terms of the concept of asymptotic stability. To show this fact it is enough to adopt the following definition of the asymptotic stability (see [11, 25]).

Definition 8. Let Ω be a nonempty subset of the space BC_∞ and $Q : \Omega \to BC_\infty$. Consider the operator equation having the form

$$x(t) = (Qx)(t), \quad t \in \mathbb{R}_+. \tag{7}$$

We say that solutions of equation (7) are asymptotically stable if there exists a ball $B(x_0, r)$ in the space BC_∞ with $B(x_0, r) \cap \Omega \neq \emptyset$ such that for every $\varepsilon > 0$ there exists a number $T > 0$ with the property

$$||x(t) - y(t)||_{l_\infty} \leqslant \varepsilon$$

for all solutions x, y of equation (7) such that $x, y \in B(x_0, r) \cap \Omega$ and for $t \geqslant T$.

It is enough to notice that the ball B_{r_0} from the proof of Theorem 7 can be treated as the set $B(x_0, r)$ as well as the set Ω from Definition 8.

Now, in view of Definition 8 we can formulate Theorem 7 using the concept of asymptotic stability. More precisely, we have the following version of the mentioned theorem.

Theorem 9. *Under assumptions (i)-(xi) the infinite system of integral equations* (6) *has at least one solution in the space BC_∞. Moreover, those solutions are asymptotically stable.*

Now we are going to present an example illustrating results obtained in this section.

Example. We consider the following infinite system of integral equations

$$x_n(t) = \cos\left(\frac{n^2t+1}{t+n^2}\right) + \left(\frac{2x_n(t)}{x_n^2(t)+n} + \frac{x_{n+1}^2(t)}{2x_{n+1}^2(t)+n}\right)\int_0^\infty \frac{1}{\beta n + t^2+\tau^2}\frac{\arctan(n+x_n^2(\tau))}{\gamma+2n+\tau^2}d\tau, \tag{8}$$

where $t \in \mathbb{R}_+$ and $n = 1, 2, \dots$. Moreover, β and γ are positive constants.

It is easy to prove that the system (8) satisfies assumptions (i)-(xi). Taking, for example, $\beta = 16,\ \gamma = 9$ we obtain, that the considered system has at least one solution in the ball $B_3 = B(\theta, 3)$. Additionally, taking into account Theorem 9 we infer that those solutions are asymptotically stable.

4. Asymptotically stable solutions of infinite system of quadratic Urysohn integral equations

In this section we are going to investigate the existence of solutions of the infinite system of the quadratic Urysohn integral equations having the following form

$$x_n(t) = a_n(t) + f_n(t, x_1(t), x_2(t), \dots)\int_0^\infty u_n(t, \tau, x_1(\tau), x_2(\tau), \dots)d\tau, \tag{9}$$

where $t \in \mathbb{R}_+$ and $n = 1, 2, \dots$.

It is well-known that the integral equation of Urysohn type is one of the most general nonlinear integral equation which contains as a particular case, the integral equation of Hemmerstein type. This implies among others, that the infinite system of integral equations of Urysohn type creates the generalization of the majority of infinite systems of integral equations. Therefore, the result presented in this section extends and generalizes the previous section. Thus we need to adopt stronger assumptions that are more difficult to verify in specific cases.

Namely, we investigate the infinite system of integral equations (9) assuming the below formulated hypotheses.

(i) The sequence $(a_n(t))$ is an element of the space BC_∞. Moreover, the functions $a_n = a_n(t)$ are equicontinuous on the interval $\mathbb{R}_+$.

(ii) The functions f_n are defined on the set $\mathbb{R}_+ \times \mathbb{R}^\infty$ and take real values for $n = 1, 2, \dots$. Additionally, the functions $t \to f_n(t, x_1, x_2, \dots)$ are

equicontinuous on $\mathbb{R}_+$ uniformly with respect to $x = (x_n) \in l_\infty$, i.e., the following condition is satisfied

$$\underset{\varepsilon>0}{\forall}\ \underset{\delta>0}{\exists}\ \underset{(x_i)\in l_\infty}{\forall}\ \underset{n\in\mathbb{N}}{\forall}\ \underset{t_1,t_2\in\mathbb{R}_+}{\forall} [|t_2 - t_1| \leqslant \delta \Rightarrow |f_n(t_2, x_1, x_2, \dots) - f_n(t_1, x_1, x_2, \dots)| \leqslant \varepsilon] .$$

Moreover, the functions f_n are equibounded, i.e., there exists a constant $\phi \geqslant 0$ such that

$$|f_n(t, x_1, x_2, \dots)| \leqslant \phi$$

for $t \in \mathbb{R}_+$, $(x_n) \in l_\infty$ and for $n = 1, 2, \dots$.

(iii) There exists a function $p : \mathbb{R}_+ \to \mathbb{R}_+$ which is nondecreasing on $\mathbb{R}_+$ and continuous at 0. Apart from this the following condition is satisfied

$$|f_n(t, x_1, x_2, \dots) - f_n(t, y_1, y_2, \dots)| \leqslant p(r) \sup\{|x_i - y_i| : \ i \geqslant n\}$$

for any $r > 0$, for $x = (x_i),\ y = (y_i) \in l_\infty$ such that $\|x\|_{l_\infty} \leqslant r$, $\|y\|_{l_\infty} \leqslant r$ and for any $t \in \mathbb{R}_+$ and $n = 1, 2, \dots$.

(iv) The sequence of functions $\left(\overline{f}_n\right)$, where $\overline{f}_n(t) = |f_n(t, 0, 0, \dots)|$, is an element of the space BC_∞.

(v) For each $n \in \mathbb{N}$ the function u_n acts from the set $\mathbb{R}_+ \times \mathbb{R}_+ \times l_\infty$ into $\mathbb{R}$ and the family of functions $\{u_n(t, \tau, x_1, x_2, \dots)\}$ is equicontinuous for $t \in \mathbb{R}_+$ uniformly with respect to $\tau \in \mathbb{R}_+$ and $x = (x_i) \in l_\infty$, i.e., the following condition is satisfied

$$\underset{\varepsilon>0}{\forall}\ \underset{\delta>0}{\exists}\ \underset{t,s,\tau\in\mathbb{R}_+}{\forall}\ \underset{x=(x_i)\in l_\infty}{\forall}\ \underset{n\in\mathbb{N}}{\forall} [|t - s| \leqslant \delta \Rightarrow |u_n(t, \tau, x_1, x_2, \dots) - u_n(s, \tau, x_1, x_2, \dots)| \leqslant \varepsilon] .$$

(vi) For any $n \in \mathbb{N}$ there exist a continuous function $g_n : \mathbb{R}_+ \times \mathbb{R}_+ \to \mathbb{R}_+$ and a continuous and nondecreasing function $h : \mathbb{R}_+ \to \mathbb{R}_+$ such that

$$|u_n(t, \tau, x_1, x_2, \dots)| \leqslant g_n(t, \tau) h(||(x_i)||_{l_\infty})$$

for all $t,\ \tau \in \mathbb{R}_+$ and $(x_i) \in l_\infty$.

(vii) For every $t \in \mathbb{R}_+$ and $n \in \mathbb{N}$ the function $\tau \to g_n(t,\tau)$ is integrable on $\mathbb{R}_+$ and the functions $t \to \int_0^\infty g_n(t,\tau)d\tau$ are equibounded on $\mathbb{R}_+$.

(viii) There exists a function $q : \mathbb{R}_+ \to \mathbb{R}_+$ which is nondecreasing on $\mathbb{R}_+$, continuous at 0 and such that $q(0) = 0$. Moreover, the following condition is satisfied

$$\begin{aligned}&|u_n(t,\tau,x_1,x_2,\dots) - u_n(t,\tau,y_1,y_2,\dots)|\\ &\leqslant q(r)g_n(t,\tau)\sup\{|x_i - y_i| :\ i \geqslant n\}\end{aligned}$$

for any $r > 0$ and for $x = (x_i),\ y = (y_i) \in l_\infty$ such that $||x||_{l_\infty} \leqslant r$, $||y||_{l_\infty} \leqslant r$ and for all $t,\ \tau \in \mathbb{R}_+,\ n = 1,2,\dots$.

(ix) The equality

$$\lim_{T\to\infty}\left\{\sup\left\{\int_T^\infty g_n(t,\tau)d\tau :\ t \in \mathbb{R}_+\right\}\right\} = 0$$

holds uniformly with respect to $n \in \mathbb{N}$.

(x) There exists a positive solution r_0 of the inequality

$$A + \phi Gh(r) \leqslant r$$

such that

$$Gh(r_0)p(r_0) + \left(\overline{F} + r_0p(r_0)\right)Gq(r_0) < 1,$$

where

$$A = \sup\{\sup\{|a_n(t)| :\ n = 1,2,\dots\} :\ t \in \mathbb{R}_+\},$$

$$\overline{F} = \sup\left\{\overline{f}_n(t) :\ t \in \mathbb{R}_+,\ n = 1,2,\dots\right\},$$

$$G = \sup\left\{\int_0^\infty g_n(t,\tau)d\tau :\ n = 1,2,\dots,\ t \in \mathbb{R}_+\right\}$$

are finite constants.

Now we can formulate the following existence theorem.

Theorem 10. *Under assumptions (i)-(x) the infinite system of integral equations* (9) *has at least one solution* $x = x(t) = (x_n(t))$ *in the space* BC_∞. *Moreover, solutions of this system are such that the thickness of the bundle formed by graphs of functions creating those solutions tends to zero at infinity.*

Proof of Theorem 10 is constructed similarly to the proof described in the previous section. The main tool used in the proof is the Darbo fixed point theorem as well as the measure of noncompactness μ_c^3 in the space BC_∞.

On the beginning we prove that the operator Q generated by the infinite system of integral equations (9) transforms the space BC_∞ into itself. Next, from the first part of assumption (x) we deduce that there exists a ball B_{r_0} which is transformed into itself by the operator Q. Moreover, we get that Q is continuous on the set B_{r_0}.

The second part of the proof is dedicated to estimate the measure of noncompactness μ_c^3. More precisely, for any nonempty subset X of the ball B_{r_0} we get

$$\mu_c^3(QX) \leqslant G\left[h(r_0)p(r_0) + \overline{F}q(r_0) + r_0p(r_0)q(r_0)\right]\mu_c^3(X). \tag{10}$$

The above inequality is a consequence of the followieng estimates

$$\omega_0^\infty(QX) \leqslant Gh(r_0)p(r_0)\omega_0^\infty(X),$$

$$\overline{\mu}_\infty^3(QX) \leqslant G\left[h(r_0)p(r_0) + r_0p(r_0)q(r_0) + \overline{F}q(r_0)\right]\overline{\mu}_\infty^3(X),$$

$$c(QX) \leqslant \left[Gh(r_0)p(r_0) + Gr_0p(r_0)q(r_0) + G\overline{F}q(r_0)\right]c(X),$$

and formula (5). Applying the second part of assumption (x) in the estimate (10) we obtain that the operator Q is a μ_c^3-contraction. Using the Darbo fixed point principle we deduce that the infinite system of the quadratic Urysohn integral equations (9) has at least one solution in the space BC_∞ belonging to the ball B_{r_0}.

Next, from Remark 4 and the description of the kernel $\ker\mu_c^3$ contained in Remark 6 we obtain a characterization of the solutions concerning their behavior at infinity. More precisely, the thickness of the bundle formed by graphs of functions creating those solutions tends to zero at infinity. Gathering this property with Definition 8 we obtain the following theorem.

Theorem 11. *Under assumptions (i)-(x) the infinite system of integral equations* (9) *has at least one solution* $x = x(t) = (x_n(t))$ *in the space* BC_∞. *Furthermore, solutions of the system* (9) *are asymptotically stable.*

Now we are going to present an example illustrating the applicability of the result contained in Theorem 11.

Example. We consider the following infinite system of integral equations of Urysohn type

$$x_n(t) = \sin\left(\frac{n^2t+1}{t+n^2}\right) + \left(\frac{2x_n(t)}{x_n^2(t)+n} + \frac{x_{n+1}^2(t)}{2x_{n+1}^2(t)+n}\right) \times \int_0^\infty \left(\frac{1}{36n+t^2+\tau^2} \cdot \frac{\arctan(n+x_n^2(\tau))}{38+2n+\tau^2} + \frac{1}{38n+t^2+\tau^2} \cdot \frac{\arctan(n+1+x_n^2(\tau))}{36+2n+\tau^2}\right) d\tau, \tag{11}$$

for $t \in \mathbb{R}_+$ and $n = 1, 2, \dots$.

It can be shown that the above infinite system of integral equations (11) satisfies assumptions (i)-(x). Thus, in view of Theorem 11 we infer that the system (11) has at least one solution in the space BC_∞ belonging to the ball $B_{1.1} = B(\theta, 1.1)$. Moreover, those solutions are asymptotically stable.

5. Conclusions

It is well-known that quadratic integral equations create rather complicated and hard forms of integral equations, especially if we investigate infinite systems of integral equations with the improper integral defined on the half-axis $\mathbb{R}_+$. Thus the study of such systems is more general than a lot of ones conducted in other works. It is worthwhile mentioning that another interesting result in this direction was obtained in [7]. In this paper authors considered infinite system of the quadratic integral equations of Urysohn type (9) in the space $BC(\mathbb{R}_+, c_0)$ consisting of bounded and continuous functions defined on the real half-axis with values in the space c_0 of real sequences converging to zero. By applying a suitable measure of noncompactness and utilizing the properties of its kernel solutions vanishing at infinity were derived. Let us note that the change of the measure of noncompactness or the Banach space allows us to obtain solutions of the same system but having another properties.

It is well known that infinite systems of integral equations are important in many applications. Various real world events generate their described models with help of infinite systems of those equations. For example, infinite

systems of integral equations appear during modeling the so-called birth and death stochastic process (see [9]). In the mentioned paper authors raised an open problem concerning the solvability of an infinite system of Volterra integral equations associated with birth and death stochastic process. This problem was successfully solved in an appropriate Fréchet space [22].

As we mentioned in the introduction, the theory of infinite systems of integral equations is still being developed. Current states and some open problems concerning infinite systems of integral equations can be found in [3]. Undoubtedly, the advanced technique of nonlinear analysis depending on measures of noncompactness enables us to simplify the very extensive considerations in such a study and has a significant impact on the development of the theory of infinite systems of integral equations.

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Summary

The aim of the chapter is to present results on the solvability of infinite systems of quadratic integral equations of Urysohn and Hammerstein type, which are considered on the real half-axis $\mathbb{R}_+ = [0, +\infty)$. We consider infinite systems of integral equations with the integral taken as improper one. The main tool used in our considerations is the technique of measures of noncompactness which turns to be very useful in the fixed points and operator equations theory. We would like to emphasize the importance of measures of noncompactness in the proofs of the presented existence theorems.

Streszczenie

Celem rozdziału jest przedstawienie wyników dotyczących rozwiązywalności nieskończonych układów kwadratowych równań całkowych typu Urysohna i Hammersteina, rozważanych na dodatniej półprostej rzeczywistej $\mathbb{R}_+ = [0, +\infty)$. Badamy nieskończone układy równań całkowych, w których całki są rozumiane jako całki niewłaściwe. Głównym narzędziem stosowanym w naszych rozważaniach jest technika miar niezwartości, która okazuje się niezwykle użyteczna w teorii punktów stałych i równań operatorowych. Chcemy również podkreślić znaczenie miar niezwartości w dowodach przedstawionych twierdzeń o istnieniu rozwiązań.

Chapter 4

Measures of noncompactness in some function spaces

Agnieszka Chlebowicz

1. Notion of a measure of noncompactness

Let us start with the notion of the measure of noncompactness. At the beginning we need to introduce a few symbols and notations. Denote by $\mathbb{R}_+$ the interval $[0, \infty)$. Next assume that E is a Banach space with a norm $\|\cdot\|_E$ and denote by $X + Y$, λX $(\lambda \in \mathbb{R})$ the algebraic operations on the sets $X, Y \subset E$, moreover let $\overline{X}$ denotes the closure of the set X and $\operatorname{Conv} X$ be the closed convex hull of X. We will consider the family $\mathfrak{M}_E$ of all nonempty and bounded subsets of the space E and the family $\mathfrak{N}_E$ of all relatively compact subsets of E (of course $\mathfrak{N}_E \subset \mathfrak{M}_E$). Now, we recall the definition which was given in the monograph [3].

Definition 1. A function $\mu : \mathfrak{M}_E \to \mathbb{R}_+$ is said to be a measure of noncompactness in the space E if it satisfies the following conditions:

(i) The family $\ker \mu = \{X \in \mathfrak{M}_E : \mu(X) = 0\}$ is nonempty and $\ker \mu \subset \mathfrak{N}_E$.

(ii) $X \subset Y \Rightarrow \mu(X) \leqslant \mu(Y)$.

(iii) $\mu(\overline{X}) = \mu(X)$.

(iv) $\mu(\mathrm{Conv}X) = \mu(X)$.

(v) $\mu(\lambda X + (1-\lambda)Y) \leqslant \lambda\,\mu(X) + (1-\lambda)\,\mu(Y)$ for $\lambda \in [0,1]$.

(vi) If (X_n) is a sequence of closed sets from $\mathfrak{M}_E$ such that $X_{n+1} \subset X_n$ for $n = 1, 2, \ldots$ and if $\lim_{n\to\infty} \mu(X_n) = 0$ then the set $X_\infty = \bigcap_{n=1}^{\infty} X_n$ is nonempty.

The set $\ker \mu$ defined in (i) is called the kernel of the measure of noncompactness μ. We can observe that the intersection set X_∞ appearing in (vi) is an element of the family $\ker \mu$ ([3]).

Further, let us assume that μ is a measure of noncompactness in the space E. We say that μ is sublinear ([3]) if it satisfies the following additional conditions:

(vii) $\mu(X + Y) \leqslant \mu(X) + \mu(Y)$.

(viii) $\mu(\lambda X) = |\lambda|\,\mu(X)$ for $\lambda \in \mathbb{R}$.

Moreover, we say that a measure of noncompactness μ has maximum property if

(ix) $\mu(X \cup Y) = \max\{\mu(X), \mu(Y)\}$.

If the measure of noncompactness μ is such that

(x) $\ker \mu = \mathfrak{N}_E$

then it is called full.
A sublinear measure of noncompactness which is full and has maximum property is said to be regular measure of noncompactness ([3]).

Let us notice that the first measure of noncompactness was defined by Kuratowski ([8]) in the following way

$$\alpha(X) = \inf\{\varepsilon > 0 : X \text{ can be covered by a finite family of sets } X_1, X_2, \ldots, X_m \text{ such that diam } X_i \leqslant \varepsilon \text{ for } i = 1, 2, \ldots, m\}.$$

The measure $\alpha(X)$ is called the Kuratowski measure of noncompactness. It can be shown that the Kuratowski measure of noncompactness is regular ([3]).

Another important measure of noncompactness, called the Hausdorff measure of noncompactness, was defined as follows ([6, 7]):

$$\chi(X) = \inf\{\varepsilon > 0 : X \text{ has a finite } \varepsilon - \text{net in } E\}.$$

It can be shown that the Hausdorff measure of noncompactness χ is regular and it is equivalent to the Kuratowski measure $\alpha(X)$. More precisely, for an arbitrary set $X \in \mathfrak{M}_E$ the following inequalities hold ([3]):

$$\chi(X) \leqslant \alpha(X) \leqslant 2\chi(X).$$

2. Classical measures of noncompactness in spaces of continuous functions

Our goal in this section is to collect known results about measures of noncompactness in the spaces $C([a,b],\mathbb{R})$, $BC(\mathbb{R}_+,\mathbb{R})$ and $C([a,b],E)$, where E is an infinite dimensional Banach space. These results were examined in the papers [1], [3], [4], [5] and [7]. So let us start with the simplest among these spaces. Therefore we consider the classical function Banach space $C([a,b],\mathbb{R})$ defined as follows

$$C([a,b],\mathbb{R}) = \{x : [a,b] \mapsto \mathbb{R} : x \text{ is continuous}\}$$

with the maximum norm

$$\|x\| = \max\{|x(t)| : t \in [a,b]\}.$$

It is well known that in this space we can characterize the relative compact sets using the Arzelá-Ascoli theorem:

Theorem 2. *The set $X \in \mathfrak{M}_{C([a,b],\mathbb{R})}$ is relatively compact if and only if all functions from X are equicontinuous.*

The condition for the set $X \in \mathfrak{M}_{C([a,b],\mathbb{R})}$ to be relatively compact can also be written using the modulus of continuity of the set. To describe this concept we fix $\varepsilon > 0$ and consider the usual modulus of continuity of the function $x \in X$

$$\omega(x,\varepsilon) = \sup\{|x(t) - x(s)| : t, s \in [a,b],\ |t-s| \leqslant \varepsilon\}.$$

Because $X \in \mathfrak{M}_{C([a,b],\mathbb{R})}$ we can take

$$\omega(X,\varepsilon) = \sup\{\omega(x,\varepsilon), x \in X\}$$

and further, since the function $\varepsilon \to \omega(X,\varepsilon)$ is nondecreasing and nonnegative for $\varepsilon > 0$ we can define the limit

$$\omega_0(X) = \lim_{\varepsilon \to 0} \omega(X,\varepsilon).$$

Now we can write the theorem equivalent to theorem 2:

Theorem 3. *The set $X \in \mathfrak{M}_{C([a,b],\mathbb{R})}$ is relatively compact if and only if $\omega_0(X) = 0$.*

In [7] the authors proved that the function ω_0 is a measure of noncompactnes:

Theorem 4. *The function $\mu : \mathfrak{M}_{C([a,b],\mathbb{R})} \to \mathbb{R}_+$ given by formula*

$$\mu(X) = \omega_0(X)$$

is a measure of noncompactness in the space $C([a,b],\mathbb{R})$. Moreover, the following equality occurs:

$$\chi(X) = \frac{1}{2}\omega_0(X).$$

The measure $\mu(X) = \omega_0(X)$ is a regular measure of noncompactness, which means that $\mathfrak{N}_{C([a,b],\mathbb{R})} = \ker \mu$. We point out that in this space we can also define some nonregular measures of noncompactness (see [3]).

Now we will describe the more complicated case, where we take an unbounded set $\mathbb{R}_+$ instead of the compact interval $[a,b]$. It means that we consider the Banach space $BC(\mathbb{R}_+,\mathbb{R})$ defined as

$$BC(\mathbb{R}_+,\mathbb{R}) = \{x : \mathbb{R}_+ \to \mathbb{R} : x \text{ is bounded and continuous}\}$$

and normed by

$$\|x\| = \sup\{|x(t)| : t \in \mathbb{R}_+\}.$$

Our goal is to construct measures of noncompactness in this space. But unfortunately, the Arzéla-Ascoli theorem does not apply in $BC(\mathbb{R}_+,\mathbb{R})$ (see [3]) and we do not know the convenient criteria for relative compactness in this

space. Therefore we can not obtain any simple formulas for the regular measure of noncompactness and we have to use only sufficient conditions for relative compactness (see [3], [4]). This makes us think only about nonregular measures of noncompactness. But it turns out that we can construct quite convenient such a measures. It follows from the Arzéla-Ascoli theorem that if $X \subset \mathfrak{M}_{BC(\mathbb{R}_+,\mathbb{R})}$ is compact in the space $BC(\mathbb{R}_+,\mathbb{R})$ then all functions $x \in X$ are equicontinuous on each interval $[0,a]$. Of course this condition is also sufficient for compactness with respect to the almost uniform convergence topology. However, for compactness in our space (with respect to our norm topology) we have to provide some additional conditions which will control the behaviour of functions from X at infinity.

In this end let us take a nonempty and bounded subset X of the space $BC(\mathbb{R}_+,\mathbb{R})$. Fix numbers $\varepsilon > 0$, $T > 0$ and a function $x \in X$. Next, consider the modulus of continuity of the function x on the interval $[0,T]$:

$$\omega^T(x,\varepsilon) = \sup\{|x(t) - x(s)| : t,s \in [0,T], |t-s| \leqslant \varepsilon\}.$$

Further, let us define

$$\omega^T(X,\varepsilon) = \sup\{\omega^T(x,\varepsilon) : x \in X\}.$$

Taking into account the fact that the function $\varepsilon \to \omega^T(X,\varepsilon)$ is nondecreasing, we infer that there exists a finite limit $\lim_{\varepsilon\to 0}\omega^T(X,\varepsilon)$. Thus, we can put

$$\omega_0^T(X) = \lim_{\varepsilon\to 0}\omega^T(X,\varepsilon).$$

Further, we define the quantity $\omega_0(X)$ in the following way

$$\omega_0(X) = \lim_{T\to\infty}\omega_0^T(X).$$

We can observe that $\omega_0(X)$ is not a measure of noncompactness in the space $BC(\mathbb{R}_+,\mathbb{R})$ in the sense of definition 1. To show this we consider (cf. [4]) the set $X = \{x_n : n \in \mathbb{N}\}$, where x_n is the function from $BC(\mathbb{R}_+,\mathbb{R})$ defined as follows

$$x_n(t) = \begin{cases} \sin((t+n-1)\pi), & t \in [n-1,n], \\ 0, & \text{otherwise.} \end{cases}$$

Notice that $X \in \mathfrak{M}_{BC(\mathbb{R}_+,\mathbb{R})}$. It is easily seen that $\omega_0(X) = 0$ but X is not relatively compact in the space $BC(\mathbb{R}_+,\mathbb{R})$ since $\|x_n - x_m\| = 1$ for $n \neq m$, $n,m = 1,2,\ldots$

We see that we have to write, except $\omega_0(X)$, the second component of the measure of noncompactness which will analyse the behaviour of functions from the set X at infinity. To the end denote $X(t) := \{x(t) : x \in X\}$, recall also that the symbol $\mathrm{diam}X(t)$ denotes the diameter of the set $X(t)$.

Now, we consider the functions $a(X)$, $b(X)$ and $c(X)$ defined on the family $\mathfrak{M}_{BC(\mathbb{R}_+,\mathbb{R})}$ in the following way:

$$a(X) = \lim_{T\to\infty}\left\{\sup_{x\in X}\{\sup\{|x(t)| : t \geqslant T\}\}\right\},$$

$$b(X) = \lim_{T\to\infty}\left\{\sup_{x\in X}\{\sup\{|x(t) - x(s)| : t, s \geqslant T\}\}\right\},$$

$$c(X) = \limsup_{t\to\infty}\mathrm{diam}X(t).$$

It turns out that each of these functions examines the behaviour of the bundle of functions from the set X at infinity. In that case let us define on the family $\mathfrak{M}_{BC(\mathbb{R}_+,\mathbb{R})}$ the quantities μ_a, μ_b, μ_c by putting

$$\begin{aligned}\mu_a(X) &= \omega_0(X) + a(X),\\ \mu_b(X) &= \omega_0(X) + b(X),\\ \mu_c(X) &= \omega_0(X) + c(X).\end{aligned}$$

Then, we can formulate the following theorem (cf. [1]).

Theorem 5. *The functions μ_a, μ_b, μ_c are measures of noncompactness in the space $BC(\mathbb{R}_+,\mathbb{R})$. Moreover, for any $X \in \mathfrak{M}_{BC(\mathbb{R}_+,\mathbb{R})}$ the following inequalities are satisfied:*

$$\begin{aligned}\chi(X) &\leqslant \mu_b(X),\\ \chi(X) &\leqslant \mu_c(X),\\ \mu_b(X) &\leqslant 2\mu_a(X),\\ \mu_c(X) &\leqslant 2\mu_a(X).\end{aligned}$$

Therefore, we have described how the measures of noncompactness in the space $BC(\mathbb{R}_+,\mathbb{R})$ look. We can prove that measures of noncompactness μ_a and μ_b are sublinear and have the maximum property. The measure μ_c is also sublinear but has not the maximum property. All of these measures are not regular. To see that it is enough to describe their kernels. Thus, let us

observe that the kernel $\ker \mu_a$ consists of all bounded subsets X of the space $BC(\mathbb{R}_+, \mathbb{R})$ such that functions from X are locally equicontinuous and tend to zero at infinity with the same rate. Similarly, the kernel $\ker \mu_b$ contains of bounded sets X such that functions from X are locally equicontinuous and tend to limits at infinity with the same rate, that means, function from X tend to limits uniformly with respect to the set X. Finally, the kernel $\ker \mu_c$ consists of all bounded sets X such that functions from X are locally equicontinuous and the thickness of the bundle formed by graphs of functions from X tends to zero at infinity. Let us also notice that $\ker \mu_a \subset \ker \mu_b$ and $\ker \mu_a \subset \ker \mu_c$ but there are no inclusions between $\ker \mu_b$ and $\ker \mu_c$.

Now let us consider the third case of function Banach space - the space $C([a, b], E)$, where E is an infinite dimensional Banach space with the norm $\|\cdot\|_E$ (if E has a finite dimension, this situation comes down to the one described at the beginning of this section for the space $C([a, b], \mathbb{R})$). We can define this space in the following way

$$C([a, b], E) = \{x : [a, b] \mapsto E : x \text{ is continuous}\}$$

with the maximum norm

$$\|x\| = \max\{\|x(t)\|_E : t \in [a, b]\}.$$

Let us denote $X(t) = \{x(t) : x \in X\}$ for any $X \in \mathfrak{M}_{C([a,b],E)}$. The following generalized form of the Arzéla-Ascoli theorem characterizes the compactness in the space $C([a, b], E)$ (see [3]):

Theorem 6. *The set $X \in \mathfrak{M}_{C([a,b],E)}$ is relatively compact if and only if all functions from X are equicontinuous and for each $t \in [a, b]$ the set $X(t)$ is relatively compact in E.*

Therefore, now it is clear that to construct the measure of noncompactness in $C([a, b], E)$ we have to consult two terms - the first which measures how far the functions from X are from being equicontinuous and the second one concerned with the noncompactness of the bundle of functions formed by the elements of the set X at any point of the interval $[a, b]$. So we start with the definition of the first component. We follow the construction of this term for the the real function and write the modulus of continuity of

the function x:

$$\omega(x,\varepsilon) = \sup\{\|x(t) - x(s)\|_E : t, s \in [a,b],\ |t-s| \leqslant \varepsilon\}.$$

Next we define

$$\omega(X,\varepsilon) = \sup\{\omega(x,\varepsilon), x \in X\}$$

and

$$\omega_0(X) = \lim_{\varepsilon \to 0} \omega(X,\varepsilon).$$

The above limit exists and is finite since the function $\varepsilon \to \omega(X,\varepsilon)$ is non-decreasing and nonnegative for $\varepsilon > 0$. This way we have obtained the first component of the measure of noncompactness in $C([a,b],E)$. Next, denote by μ any measure of noncompactness in the space E, then we can define the second term of the measure of noncompactness in $C([a,b],E)$:

$$\mu^*(X) = \sup\{\mu(X(t)) : t \in [a,b]\}.$$

Now we obtain theorem (see [3]) about the formula for the measure of noncompactness in the space $C([a,b],E)$.

Theorem 7. *The function $\overline{\mu} : \mathfrak{M}_{C([a,b],E)} \to \mathbb{R}_+$ given by formula*

$$\overline{\mu}(X) = \omega_0(X) + \mu^*(X)$$

is a measure of noncompactness in the space $C([a,b],E)$.

We can observe that the kernel $\ker \overline{\mu}$ consists of all nonempty and bounded subsets X of the space $C([a,b],E)$ such that functions from X are equicontinuous on $[a,b]$ and $X(t) \in \ker \mu$ for $t \in [a,b]$.

3. The construction of a measures of noncompactness in the space $BC(\mathbb{R}_+,E)$

In this section we will present the theorem from the paper [2], which complements the results described above concerning measures of noncompactness in the spaces $C([a,b],\mathbb{R})$, $BC(\mathbb{R}_+,\mathbb{R})$ and $C([a,b],E)$.

Assume that E is a given infinite dimensional Banach space with the norm $\|\cdot\|_E$ (but our investigations have also sense if E is a finite dimensional Banach space). Let μ be a measure of noncompactness defined in the space

E. Next, let us consider the Banach space $BC(\mathbb{R}_+, E)$ defined as follows:

$$BC(\mathbb{R}_+, E) = \{x : \mathbb{R}_+ \to E : x \text{ is bounded and continuous}\}$$

with the supremum norm

$$\|x\|_\infty = \sup\{\|x(t)\|_E : t \in \mathbb{R}_+\}.$$

Now we will construct some measures of noncompactness in defined above Banach space $BC(\mathbb{R}_+, E)$. To this end take an arbitrary and bounded set $X \subset BC(\mathbb{R}_+, E)$, arbitrarily fixed function $x \in X$ and $\varepsilon > 0$. Analysing the structures measures of noncompactness from the section 2, we can conclude that the measure in the space $BC(\mathbb{R}_+, E)$ should be a sum of three expressions:

- the first component should reflect the behaviour of functions belonging to the set X,
- the second component must be concerned with the noncompactness of the bundle of functions formed by the elements of the set X at any point of $\mathbb{R}_+$,
- the third component should analyse the behaviour of functions from the set X at infinity.

Let us begin our construction of measure of noncompactness in $BC(\mathbb{R}_+, E)$. Firstly, we will deal with the first term of this measure. We define the quantity $\omega^\infty(x, \varepsilon)$ as the modulus of continuity of the function x on the set $\mathbb{R}_+$:

$$\omega^\infty(x, \varepsilon) = \sup\{\|x(t) - x(s)\|_E : t, s \in \mathbb{R}_+,\ |t - s| \leqslant \varepsilon\}. \tag{1}$$

We can observe that $\lim_{\varepsilon \to 0} \omega^\infty(x, \varepsilon) = 0$ if and only if the function $x = x(t)$ is uniformly continuous on the interval $\mathbb{R}_+$. Further, taking into account (1), for $X \in \mathfrak{M}_{BC(\mathbb{R}_+, E)}$ we define:

$$\begin{aligned} \omega^\infty(X, \varepsilon) &= \sup\{\omega^\infty(x, \varepsilon) : x \in X\}, \\ \omega_0^\infty(X) &= \lim_{\varepsilon \to 0} \omega^\infty(X, \varepsilon). \end{aligned} \tag{2}$$

It is easily seen that $\omega_0^\infty(X) = 0$ if and only if functions from the set X are equicontinuous on $\mathbb{R}_+$, or equivalently, functions from X are equiuniformly

continuous on $\mathbb{R}_+$. The term $\omega_0^\infty(X)$ described above is the first component of the measure of noncompactness in $BC(\mathbb{R}_+, E)$.

Now we will introduce the second component of the measure of noncompactness in the space $BC(\mathbb{R}_+, E)$. In what follows, let us consider the function $\overline{\mu}_\infty$ defined on the family $\mathfrak{M}_{BC(\mathbb{R}_+,E)}$ by the formula

$$\overline{\mu}_\infty(X) = \lim_{T\to\infty} \overline{\mu}_T(X), \tag{3}$$

where

$$\overline{\mu}_T(X) = \sup\{\mu(X(t)) : t \in [0,T]\}.$$

The term $\mu_\infty(X)$ described above in (3) is the second component of the measure of noncompactness in $BC(\mathbb{R}_+, E)$. Observe that the existence of the limit in (3) is a consequence of the fact that the function $T \to \overline{\mu}_T(X)$ is nondecreasing and bounded from above on $\mathbb{R}_+$. Indeed, since the set X is a bounded subset in the space $BC(\mathbb{R}_+, E)$ then there exists a constant $c > 0$ such that

$$\sup\{\|x(t)\|_E : t \in \mathbb{R}_+\} \leqslant c$$

for any $x \in X$. Thus, fixing arbitrarily $t \in \mathbb{R}_+$ we conclude that

$$\sup\{\|x(t)\|_E : x \in X\} \leqslant c.$$

This implies that the measures of noncompactness $\mu(X(t))$ are bounded from above for $t \in \mathbb{R}_+$.

Now let us provide the three possible terms for the third component of the measure of noncompactness in $BC(\mathbb{R}_+, E)$. To this end, for $T > 0$, let us put

$$a_T(X) = \sup_{x\in X}\Big\{\sup\{\|x(t)\|_E : t \geqslant T\}\Big\}.$$

Let us mention that the function $T \to a_T(X)$ is nonincreasing (and bounded) on $\mathbb{R}_+$. Thus there exists a finite limit

$$a_\infty(X) = \lim_{T\to\infty} a_T(X). \tag{4}$$

In what follows let us take into account other quantities connected with measuring the behaviour of functions from the set X at infinity. Namely,

for $T > 0$ let us put:

$$b_T(X) = \sup_{x \in X} \Big\{ \sup \{ \|x(t) - x(s)\|_E : t, s \geqslant T \} \Big\},$$

$$b_\infty(X) = \lim_{T \to \infty} b_T(X). \tag{5}$$

Next, for $t \in \mathbb{R}_+$ let us define

$$\operatorname{diam} X(t) = \sup \{ \|x(t) - y(t)\|_E : x, y \in X \}$$

and

$$c(X) = \limsup_{t \to \infty} \operatorname{diam} X(t). \tag{6}$$

Finally, linking (2) – (6), we can define the following quantities:

$$\mu_a(X) = \omega_0^\infty(X) + \overline{\mu}_\infty(X) + a_\infty(X), \tag{7}$$

$$\mu_b(X) = \omega_0^\infty(X) + \overline{\mu}_\infty(X) + b_\infty(X), \tag{8}$$

$$\mu_c(X) = \omega_0^\infty(X) + \overline{\mu}_\infty(X) + c(X). \tag{9}$$

Then, we are prepared to formulate our main result on the form of measures of noncompactness in the space $BC(\mathbb{R}_+, E)$.

Theorem 8. *Let μ be a measure of noncompactness in the Banach space E. Then, the functions μ_a, μ_b and μ_c defined by formulas* (7) – (9) *respectively, are measures of noncompactness in the space $BC(\mathbb{R}_+, E)$ and the following inequalities hold*

$$\mu_b(X) \leqslant 2\mu_a(X),$$
$$\mu_c(X) \leqslant 2\mu_a(X)$$

for an arbitrary set $X \in \mathfrak{M}_{BC(\mathbb{R}_+, E)}$.

The proof of this theorem can be found in [2].

Finally we will describe the kernels of constructed measures of noncompactness μ_a, μ_b and μ_c defined by formulas (7), (8), (9), respectively.

We can observe that the kernel $\ker \mu_a$ of the measure μ_a consists of

all bounded subsets X of the space $BC(\mathbb{R}_+, E)$ satisfying the condition that functions from X are uniformly continuous and equicontinuous on $BC(\mathbb{R}_+, E)$ (in other words, we may say that those functions are equiuniformly continuous on $BC(\mathbb{R}_+, E)$) and tend to zero at infinity with the same rate. Apart from this, all cross-sections $X(t)$ of the set X belong to the kernel of the measure of noncompactness μ in the Banach space E. In particular, the mentioned cross-sections $X(t)$ are relatively compact in E.
Similarly, the kernel $\ker \mu_b$ of the measure μ_b consists of all bounded subsets X of the space $BC(\mathbb{R}_+, E)$ such that the functions from X are equiuniformly continuous on the half-axis $BC(\mathbb{R}_+, E)$ and all cross-sections $X(t)$ of the set X are contained in the kernel of the measure of noncompactness μ in the space E. Moreover, all functions from X tend to limits at infinity uniformly with respect to the set X.
Finally, the kernel $\ker \mu_c$ of the measure μ_c contains all bounded subsets X of $BC(\mathbb{R}_+, E)$ which are equiuniformly continuous on $BC(\mathbb{R}_+, E)$ and all cross-sections $X(t)$ of X are members of the kernel $\ker \mu$ in the Banach space E. Additionally, the thickness of the bundle formed by graphs of functions from X tends to zero at infinity.

Further, let us notice that measures of noncompactness μ_a, μ_b and μ_c defined by formulas (7), (8), (9) are not full. That means the kernels $\ker \mu_a$, $\ker \mu_b$ and $\ker \mu_c$ are proper subfamilies of the family $\mathfrak{N}_{BC(\mathbb{R}_+,E)}$.

Furthermore, the measures of noncompactness μ_a and μ_b defined by formulas (7) and (8) have the maximum property, while the measure μ_c does not have that property. Moreover, if we assume that the measure μ in the Banach space E is sublinear, then all measures μ_a, μ_b and μ_c are sublinear in the Banach space $BC(\mathbb{R}_+, E)$.

Obviously, the issue concerning the measures of noncompactness in function spaces is very broad and we are not covering the topic completely. For example, we can also consider a natural extension of our approach to the space of real functions defined and continuous on $\mathbb{R}_+$ and tempered by a given positive and continuous function $p = p(t)$ tending to zero at infinity as well as various other natural extensions.

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Summary

This chapter has the character of review. We gather both classical and new results concerning measures of noncompactness in the spaces of functions defined on bounded as well as unbounded real intervals. The classical results concern the construction of measures of noncompactness in the Banach function spaces $C([a,b],\mathbb{R})$, $BC(\mathbb{R}_+,\mathbb{R})$ and $C([a,b],E)$ whereas the new result describes such a construction in the space $BC(\mathbb{R}_+,E)$ of all bounded and continuous function defined on $\mathbb{R}_+$ with values in a Banach space E. Importantly the classical issues can be treated as guidelines for constructing a measure of noncompactness in the space $BC(\mathbb{R}_+,E)$. This construction is a quite new result and was published in 2020, see [2].

Streszczenie

Niniejszy rozdział ma charakter przeglądowy. Zbieramy zarówno klasyczne, jak i nowe wyniki dotyczące miar niezwartości w przestrzeniach funkcji określonych na ograniczonych oraz nieograniczonych przedziałach rzeczywistych. Wyniki klasyczne dotyczą konstrukcji miar niezwartości w przestrzeniach funkcyjnych Banacha $C([a,b],\mathbb{R})$, $BC(\mathbb{R}_+,\mathbb{R})$ oraz $C([a,b],E)$, natomiast nowy wynik opisuje taką konstrukcję w przestrzeni $BC(\mathbb{R}_+,E)$ wszystkich funkcji ograniczonych i ciągłych określonych na $\mathbb{R}_+$ o wartościach w przestrzeni Banacha E. Istotne jest, że zagadnienia klasyczne mogą stanowić wytyczne dla konstrukcji miary niezwartości w przestrzeni $BC(\mathbb{R}_+,E)$. Konstrukcja ta jest stosunkowo nowym wynikiem i została opublikowana w 2020 roku, patrz [2].

Chapter 5

Measures of weak noncompactness in the Banach space L^1 and the Fréchet space L^1_{loc}

Leszek Olszowy

Dedicated to Professor Józef Banaś
on the occasion of his 75th birthday

1. Introduction

Compactness is one of the most important topological concepts—it appears in the assumptions of many significant results within mathematical and functional analysis. Therefore, the ability to estimate how far the properties of a given set are from compactness seems to be very useful. One method of estimating this distance is through measures of noncompactness. The first such measure was introduced by Kuratowski [24], and later others were constructed [20–22] and their properties studied. A subsequent step in the development and maturation of the theory of measures of noncompactness was their axiomatization (see [2, 5, 8, 26]) and the inspiring influence they had on the development of fixed point theory [11, 16, 27]. This also translated into the effective development of methods for studying the solvability of nonlinear equations, mainly integral equations [10]. A further stage in

the development of this theory was the introduction of measures of weak noncompactness [3,4,17,19], as well as their axiomatization [11]. This led to a series of interesting results concerning applications (see [1,6,7,12–15,23]). However, from the point of view of applications in studying the solvability of nonlinear equations, the theory of measures of weak noncompactness still seems to be quite incomplete—useful and computationally convenient formulas for such measures are known only for a few Banach spaces, in particular for $L^1(I)$ and for the Fréchet space $L^1_{\mathrm{loc}}(\mathbb{R}_+)$. The aim of this survey chapter is to provide a concise discussion of measures of weak noncompactness in these spaces.

2. Measures of weak noncompactness in the Banach spaces L^1

First let us assume that E is an infinite-dimensional and nonreflexive Banach space with the norm $\|\cdot\|$ and the zero element θ. Denote by $\mathfrak{M}_E$ the family of all nonempty and bounded subsets of E, and by $\mathfrak{N}_E^w$ its subfamily consisting of all relatively weakly compact sets. For a subset $X \subset E$, we introduce the following notations: $\mathrm{co}(X)$ – the convex hull of the set X, $\overline{X}$ – the closure in the strong (norm) topology, $\overline{X}^w$ – the closure of the set X in the weak topology, $\overline{\mathrm{co}}(X)$ – the closure in the strong topology of the convex hull of X, $\overline{\mathrm{co}}^w(X)$ – the closure in the weak topology of the convex hull of the set X. We denote by $B_E(x,r)$ or $B(x,r)$ the ball centered at x and of radius r. We write B_r instead of $B(\theta,r)$. The restriction of a family of functions X to a set I will be denoted by $X|_I$. Finally, if A is a Lebesgue measurable subset of $\mathbb{R}$, then the symbol $|A|$ stands for the Lebesgue measure of A.

The following fact will be useful in the sequel.

Theorem 1 ([29])**.** *If a linear topological space is locally convex, then the closure of a convex set in the original topology coincides with the closure of that set in the weak topology.*

Although chronologically it was De Blasi who first introduced an example of a measure of weak noncompactness, and only after studying its properties and subsequent applications did researchers begin to propose axiomatic definitions of a measure of weak noncompactness, we shall nevertheless begin with the definition from [11].

Definition 2. A function $\psi : \mathfrak{M}_E \to \mathbb{R}_+$ is said to be a *measure of weak noncompactness* in the Banach space E if it satisfies the following conditions:

1° The family $\ker\psi := \{X \in \mathfrak{M}_E : \psi(X) = 0\}$ is nonempty and $\ker\psi \subset \mathfrak{N}_E^w$.

2° Monotonicity: $X \subset Y \Rightarrow \psi(X) \leqslant \psi(Y)$.

3° Invariance under passage to the convex hull: $\psi(\overline{\mathrm{co}}(X)) = \psi(X)$.

4° Convexity: $\psi(\lambda X + (1-\lambda)Y) \leqslant \lambda\psi(X) + (1-\lambda)\psi(Y)$ for $\lambda \in [0,1]$.

5° Generalized Cantor intersection property: If $(X_n)_{n\in\mathbb{N}} \subset \mathfrak{M}_E$ is a sequence of nonempty, weakly closed subsets of E with $X_1 \supset X_2 \supset \cdots \supset X_n \supset \ldots$ and such that $\lim_{n\to\infty} \psi(X_n) = 0$, then the set $X_\infty := \bigcap_{n=1}^{\infty} X_n$ is nonempty.

Remark 3. In view of Theorem 1, we have $\overline{\mathrm{co}}(X) = \overline{\mathrm{co}}^w(X)$ for any $X \subset E$, and thus condition 3° can be written in the equivalent form $\psi\left(\overline{\mathrm{co}}^w(X)\right) = \psi(X)$. Moreover, condition 5° implies that $X_\infty \in \ker\psi$.

Sometimes, considerations involving measures of weak noncompactness make use of additional properties.

6° Fullness: $\psi(X) = 0$ if and only if X is a relatively weakly compact set (i.e., $\ker\psi = \mathfrak{N}_E^w$).

7° Subadditivity: $\psi(X+Y) \leqslant \psi(X) + \psi(Y)$.

8° Homogeneity: $\psi(\lambda X) = |\lambda|\psi(X)$, for $\lambda \in \mathbb{R}$.

9° Max formula: $\psi(X \cup Y) = \max\{\psi(X), \psi(Y)\}$.

10° Non-singularity: $\psi(X \cup \{y\}) = \psi(X), \quad y \in E,\ X \in \mathfrak{M}_E$.

11° Sublinearity: ψ is homogenous and subadditive.

12° Regularity: ψ is full, sublinear and has the maximum property.

Proposition 4 ([5])**.** *If a function $\psi\colon \mathfrak{M}_E \to \mathbb{R}_+$ satisfies conditions* 1°, 2°, 3° *and* 10°, *then it also satisfies* 5°.

The importance of this fact lies in the observation that verifying the generalized Cantor intersection property 5° directly from the definition of a given function ψ is usually much more cumbersome than using the above result.

Remark 5. For the purposes of applying measures of weak noncompactness in the theory of nonlinear equations, conditions 1°–3° and 10° are usually sufficient, since they, by virtue of Proposition 4, imply the generalized Cantor intersection property (condition 5°), which plays a key role in the proof of many fixed point theorems formulated in terms of measures of noncompactness. For this reason, these conditions are sometimes adopted as the axioms in the definition of a measure of weak noncompactness (see [5]).

In applications, measures of noncompactness often appear in connection with various fixed point theorems. There are many such theorems — the simplest among them are Darbo-type theorems (see [16]).

Theorem 6 ([11])**.** *Assume that ψ is a measure of weak noncompactness in a Banach space E, the subset $C \subset E$ is nonempty, weakly closed, convex, bounded and the mapping $T\colon C \to C$ is weakly continuous. Moreover, suppose there exists $k \in [0,1)$ such that*

$$\psi(T(X)) \leqslant k\psi(X) \quad \text{for } X \subset C.$$

Then T has a fixed point in C.

The first significant example of a measure of weak noncompactness was introduced by De Blasi [17]

$$w(X) := \inf\{r > 0 : \exists W \in \mathfrak{N}_E^w \text{ such that } X \subset W + B_r\} \tag{1}$$

for $X \subset \mathfrak{M}_E$.

The inspiration drawn from the definition of the Hausdorff measure seems evident here. It turns out that this measure has very good properties, as stated in the theorem given below.

Theorem 7 ([17])**.** *The De Blasi measure w is regular, that is, it satisfies conditions $1^o - 12^o$ and moreover $w(B) = 1$.*

Unfortunately, it is not easy to find a convenient formula for the De Blasi measure w in a specific Banach space. Such formulas are known only for the space $L^1[0,a]$. In the paper [4], Appell and De Pascale obtained the following result.

Theorem 8. *In the Banach space $L^1[0,a]$, the De Blasi measure of weak*

noncompactness w is given by the formula

$$w(X) := \lim_{\varepsilon\to 0^+}\left\{\sup_{x\in X}\left\{\sup\left\{\int_D |x(t)|\,dt : D\subset[0,a],\ |D|<\varepsilon\right\}\right\}\right\}. \tag{2}$$

A somewhat more complex situation arises in the space $L^1(\mathbb{R}_+)$. The criterion for weak compactness in such a space is due to Dieudonné [18].

Theorem 9. *A bounded set X is relatively weakly compact in $L^1(\mathbb{R}_+)$ if and only if the following two conditions are satisfied:*

(i) for any $\varepsilon>0$ there exists $\delta>0$ such that if $|D|\leqslant\delta$ then $\int_D |x(t)|dt\leqslant\varepsilon$ for all $x\in X$,

(ii) for any $\varepsilon>0$ there is $T>0$ such that $\int_T^\infty |x(t)|dt\leqslant\varepsilon$ for any $x\in X$.

Based on this criterion, one can define for bounded subsets $X\subset L^1(\mathbb{R}_+)$ two mappings

$$c(X) := \lim_{\varepsilon\to 0^+}\left\{\sup_{x\in X}\left\{\sup\left\{\int_D |x(t)|\,dt : D\subset\mathbb{R}_+,\ |D|<\varepsilon\right\}\right\}\right\}, \tag{3}$$

$$d(X) := \lim_{T\to\infty}\left\{\sup\left\{\int_T^\infty |x(t)|dt : x\in X\right\}\right\}.$$

Next, we define the mapping $\mu\colon \mathfrak{M}_{L^1(\mathbb{R}_+)}\to\mathbb{R}_+$ by the formula

$$\mu(X) := c(X) + d(X). \tag{4}$$

In the paper [9], Banaś and Knap showed that μ is a regular weak measure of noncompactness in $L^1(\mathbb{R}_+)$ (i.e., it satisfies conditions 1°–12° from Definition 2) and that $w(X)\leqslant\mu(X)\leqslant 2\omega(X)$. Moreover, $\mu(B_r)=2r$. An exact and computationally convenient formula for the De Blasi weak measure of noncompactness is not known in the Banach space $L^1(\mathbb{R}_+)$.

3. Measures of weak noncompactness in Fréchet space $L^1_{\mathrm{loc}}(\mathbb{R}_+)$

Let us begin with general considerations. Let F be a Fréchet space over the field $\mathbb{K}$ equipped with a countable family of seminorms $(\|\cdot\|_n)_{n\in\mathbb{N}}$ and the topology on F defined by the metric

$$d(x,y) := \sup_{n\in\mathbb{N}} 2^{-n}\frac{\|x-y\|_n}{1+\|x-y\|_n}, \quad x,y\in F. \tag{5}$$

In the above metric, the entire space F is bounded; however, we will need another notion of boundedness going forward. We say that a subset $X\subset F$ is *bounded* if the numerical sets $\{\|x\|_n : x\in X\}$ are bounded for every $n=1,2,\dots$. Note that this definition of a bounded set coincides with the definition of boundedness in a locally convex topological space. We denote by $\mathfrak{M}_F$ the family of all nonempty bounded subsets of the space F, and by $\mathfrak{N}_F^w$ its subfamily consisting of all relatively weakly compact sets.

Let F' denote the topological dual of F, i.e., the space consisting of continuous linear functionals defined on F. From the definition of the metric (5) on F, the following description of such functionals follows.

Proposition 10. *A linear functional $f: F\to\mathbb{K}$ is continuous (i.e., $f\in F'$) if and only if there exist $M>0$ and $n\in\mathbb{N}$ such that*

$$\forall_{x\in F}\ |f(x)|\leqslant M\max\{\|x\|_1,\|x\|_2,\dots,\|x\|_n\}.$$

Recall further that the weak topology on F is the weakest topology making all functionals from F' continuous. Let $\mathfrak{N}_F^w$ denote the subfamily of $\mathfrak{M}_F$ consisting of all relatively weakly compact subsets of F.

In analogy to the previous ideas, we adopt the following definition [28].

Definition 11. A family of mappings $(\mu_n)_{n\in\mathbb{N}}$, where $\mu_n\colon \mathfrak{M}_F\to\mathbb{R}_+$, is called a *family of weak measures of noncompactness* in the Fréchet space F if the following conditions are satisfied:

1• The family $\ker(\mu_n) := \{X\in\mathfrak{M}_F : \mu_n(X)=0 \text{ for } n\in\mathbb{N}\}$ is nonempty and $\ker(\mu_n)\subset\mathfrak{N}_F^w$.

2• Monotonicity: $X\subset Y\Rightarrow \mu_n(X)\leqslant\mu_n(Y)$ for $n\in\mathbb{N}$.

3• Invariance under passage to the closed convex hull: $\mu_n(\overline{\text{co}}(X)) = \mu_n(X)$ for $n\in\mathbb{N}$.

4• Convexity: $\mu_n(\lambda X+(1-\lambda)Y)\leqslant\lambda\mu_n(X)+(1-\lambda)\mu_n(Y)$ for $\lambda\in[0,1]$ and $n\in\mathbb{N}$.

5• Generalized Cantor intersection property: If $(X_i)_{i\in\mathbb{N}}\subset\mathfrak{M}_F$ is a sequence of nonempty, weakly closed subsets of F with $X_1\supset X_2\supset$

$\cdots \supset X_n \supset \ldots$ and such that $\lim\limits_{i\to\infty} \mu_n(X_i) = 0$ for $n \in \mathbb{N}$, then the set $X_\infty := \bigcap_{i=1}^{\infty} X_i$ is nonempty.

Remark 12. Note that, unlike in Banach spaces E, where a single mapping $\psi : \mathfrak{M}_E \to \mathbb{R}_+$ serves as a weak measure of noncompactness, in the case of a Fréchet space F, this role is played not by one mapping but by a family of mappings $(\mu_n)_{n\in\mathbb{N}}$.

Sometimes considerations involving weak measures of noncompactness use additional properties of such measures.

$6^\bullet$ Fullness: $\mu_n(X) = 0$ for all $n \in \mathbb{N}$ if and only if X is a relatively weakly compact set (i.e. $\ker(\mu_n) = \mathfrak{N}_F^w$).

$7^\bullet$ Subadditivity: $\mu_n(X + Y) \leqslant \mu_n(X) + \mu_n(Y)$ for $n \in \mathbb{N}$.

$8^\bullet$ Homogeneity: $\mu_n(\lambda X) = |\lambda|\mu_n(X)$ for $\lambda \in \mathbb{R}, n \in \mathbb{N}$.

$9^\bullet$ Max formula: $\mu_n(X \cup Y) = \max\{\mu_n(X), \mu_n(Y)\}$ for $n \in \mathbb{N}$.

$10^\bullet$ Non-singularity: $\mu_n(X \cup \{y\}) = \mu_n(X)$ for $y \in F, X \in \mathfrak{M}_F, n \in \mathbb{N}$.

$11^\bullet$ Sublinearity: (μ_n) is homogenous and subadditive.

$12^\bullet$ Regularity: (μ_n) is full, sublinear and has the maximum property.

Proposition 13. *If the family of mappings $(\mu_n)_{n\in\mathbb{N}}$, where $\mu_n \colon \mathfrak{M}_F \to \mathbb{R}_+$, satisfies conditions $1^\bullet$, $2^\bullet$, $3^\bullet$, $10^\bullet$, then it also satisfies $5^\bullet$.*

Proof. Assume that the sequence $(X_i)_{i\in\mathbb{N}}$ satisfies the assumptions of condition $5^\bullet$ and let us arbitrarily choose elements $x_i \in X_i$ for $i = 1, 2, \ldots$. Then, for any fixed $n \in \mathbb{N}$, we have

$$\mu_n(\{x_1, x_2, ...\}) = \mu_n(\{x_i, x_{i+1}, ...\}) \leqslant \mu_n(X_i) \underset{i\to\infty}{\longrightarrow} 0,$$

that is, $\mu_n(\{x_1, x_2, ...\}) = 0$ for $n \in \mathbb{N}$, and therefore the set $\{x_1, x_2, ...\}$ is relatively weakly compact in F. Denote by

$$C := \overline{\{x_1, x_2, ...\}}^w, \quad C_i := \overline{\{x_i, x_{i+1}, ...\}}^w, \quad i = 1, 2, 3, , ,$$

Of course, these sets are nonempty, C is weakly compact in F, and thus each C_i, being a closed subset, is also weakly compact. Therefore, $(C_i)_{i\in\mathbb{N}}$ is a decreasing—and hence centered—family of nonempty compact sets, so $\bigcap_{i=1}^{\infty} C_i \neq \emptyset$. Since $C_i \subset X_i$ for $i \in \mathbb{N}$, the assertion follows $\bigcap_{i=1}^{\infty} X_i \neq \emptyset$. $\square$

Remark 14. For applications of measures of weak noncompactness in the theory of nonlinear equations, usually only conditions $1^\bullet$–$3^\bullet$ and $10^\bullet$ are required, and therefore these are sometimes taken as the axioms defining measures of weak noncompactness in Fréchet spaces.

The following concept will be useful.

Definition 15. Let U and V be locally convex topological vector spaces. A mapping $T\colon D \subset U \to V$ is said to be *ws-compact* if it is continuous and maps relatively weakly compact sets of D into relatively strongly compact ones of V.

Weak measures of noncompactness in Fréchet spaces appear in the assumptions of various fixed point theorems. Below, we present the simplest version of such a Darbo-type theorem.

Theorem 16. *Assume that C is a nonempty, closed (in the strong topology), convex, and bounded subset of the Fréchet space F, in which there is given a family of measures of weak noncompactness $(\mu_n)_{n\in\mathbb{N}}$ satisfying conditions $1^\bullet, 2^\bullet, 3^\bullet$ and $10^\bullet$. Further, assume that the mapping $T\colon C \to C$ is ws-compact and there exists a sequence $(k_n) \subset [0,1)$ such that*

$$\mu_n(T(X)) \leqslant k_n\mu_n(X), \quad X \subset C,\ n \in \mathbb{N}.$$

Then T has at least one fixed point.

Proof. Let us denote $C_1 := C, \quad C_{i+1} := \overline{\mathrm{co}}(T(C_i))$ for $i = 1, 2, \ldots$. Clearly, $C_{i+1} \subset C_i$ for $i \in \mathbb{N}$. Moreover, the sets C_i are closed in the strong topology and, by Theorem 1, they are also closed in the weak topology of the space F. Furthermore, for a fixed $n \in \mathbb{N}$ we have

$$\mu_n(C_{i+1}) = \mu_n(\overline{\mathrm{co}}(T(C_i))) = \mu_n(T(C_i)) \leqslant k_n\mu_n(C_i) \leqslant \ldots \leqslant k_n^i\mu_n(C),$$

that is, $\lim\limits_{i\to\infty} \mu_n(C_{i+1}) = 0$ for $n \in \mathbb{N}$. Proposition 13 and the properties of the sequence (C_i) established above imply that condition $5^\bullet$ can be applied, and this means that the set $C_\infty := \bigcap_{i=1}^\infty C_i$ is nonempty and $C_\infty \in \ker(\mu_n) \subset \mathfrak{N}_F^w$. Moreover C_∞ is convex, closed in the strong topology and $T\colon C_\infty \to C_\infty$. Let us further define

$$\widehat{C} := \overline{\mathrm{co}}(T(C_\infty)).$$

Since T is ws-compact, it follows that $\widehat{C}$ is also compact in the strong topology. Therefore, by the Schauder–Tychonoff theorem, there exists a fixed point for the strongly continuous mapping $T\colon \widehat{C} \to \widehat{C}$. □

We now introduce the following notation.

$$V_n(x_0, r) := \{x \in F : \|x - x_0\|_n \leqslant r\}.$$

Thus, $V_n(x_0, r)$ is the counterpart of a ball in the Fréchet space F expressed in the seminorm $\|\cdot\|_n$.

Inspired by the De Blasi measure of noncompactness in the weak topology (1) in a Banach space, one can introduce a family of weak measures of noncompactness $(w_n)_{n\in\mathbb{N}}$ in the Fréchet space F, where $w_n\colon \mathfrak{M}_F \to \mathbb{R}_+$, defined by the formula

$$w_n(X) := \inf\{r > 0 : \exists\, W \in \mathfrak{N}_F^w,\ X \subset W + V_n(\theta, r)\}.$$

It can be shown that $(w_n)_{n\in\mathbb{N}}$ is a family of weak measures of noncompactness in the Fréchet space, i.e., it satisfies conditions $1^\bullet - 12^\bullet$. In what follows, we will refer to $(w_n)_{n\in\mathbb{N}}$ as *the De Blasi family of measures of weak noncompactness* in the Fréchet space F.

Let us now denote by $L^1_{\mathrm{loc}}(\mathbb{R}_+)$ the space consisting of those functions $x : \mathbb{R}_+ \to \mathbb{R}$ that are Lebesgue measurable and such that

$$\forall_{T>0} \quad \int_0^T |x(t)|\,dt < \infty.$$

In $L^1_{\mathrm{loc}}(\mathbb{R}_+)$ a sequence of seminorms can be introduced $(\|\cdot\|_n)_{n\in\mathbb{N}}$ by the formula

$$\|x\|_n := \int_0^n |x(t)|\,dt,$$

and a metric, via (5). Clearly, $L^1_{\mathrm{loc}}(\mathbb{R}_+)$ then becomes a Fréchet space.
Let us further denote by π_n the projection operator $\pi_n : L^1_{\mathrm{loc}}(\mathbb{R}_+) \to L^1[0, n]$ given by formula $\pi_n(x) := x|_{[0,n]},\ x \in L^1_{\mathrm{loc}}(\mathbb{R}_+),\ n \in \mathbb{N}$.

It turns out that the weak topology in $L^1_{\mathrm{loc}}(\mathbb{R}_+)$ can also be introduced in an alternative way (see [25]).

Proposition 17. *The weak topology in $L^1_{\mathrm{loc}}(\mathbb{R}_+)$ is the weakest topology for which all the projection operators $\pi_n : L^1_{\mathrm{loc}}(\mathbb{R}_+) \to L^1[0, n]$ for $n \in \mathbb{N}$ are continuous, where we assume that the weak topology is taken on the space $L^1[0, n]$.*

From this, we draw the conclusion.

Corollary 18. *A subset $X \subset L^1_{\mathrm{loc}}(\mathbb{R}_+)$ is relatively compact in weak topology of $L^1_{\mathrm{loc}}(\mathbb{R}_+)$ if and only if $\pi_n(X)$ is relatively weakly compact in $L^1[0,n]$ for each $n \in \mathbb{N}$.*

We will now present a certain fact concerning the relationship between strong compactness and weak compactness in the Fréchet space $L^1_{\mathrm{loc}}(\mathbb{R}_+)$. We begin with two definitions.

Definition 19. We call a subset $X \subset L^1[0,a]$ a.e. equicontinuous on $[0,a]$, if for each $\varepsilon > 0$ there is a closed subset $D_\varepsilon \subset [0,a]$ such that $|[0,a] \setminus D_\varepsilon| \leqslant \varepsilon$ and the restriction of all functions from X to the set D_ε form an equicontinuous family of functions.

Definition 20. A subset $X \subset L^1[0,a]$ is said to be countable a.e. equicontinuous on $[0,a]$ if every sequence $(x_n) \subset X$ has a subsequence $(\bar{x}_n)$, which is a.e. equicontinuous on $[0,a]$.

The following fact describes the relationship between relative strong and weak compactness in the Fréchet space $L^1_{\mathrm{loc}}(\mathbb{R}_+)$.

Theorem 21 ([25])**.** *Assume that a subset $X \subset L^1_{\mathrm{loc}}(\mathbb{R}_+)$ is bounded. Then X is relatively compact in strong topology of the Fréchet space $L^1_{\mathrm{loc}}(\mathbb{R}_+)$ if and only if X is relatively weak compact and X is countable a.e. equicontinuous on bounded subintervals of $\mathbb{R}_+$.*

Inspired by (2), one can introduce in $L^1_{\mathrm{loc}}(\mathbb{R}_+)$ a sequence of mappings $(\mu_n)_{n\in\mathbb{N}}$, where $\mu_n \colon \mathfrak{M}_{L^1_{\mathrm{loc}}(\mathbb{R}_+)} \to \mathbb{R}_+$, defined by the formula

$$\mu_n(X) := \lim_{\varepsilon\to 0^+} \left\{ \sup_{x\in X} \left\{ \sup \left\{ \int_D |x(t)|dt : D \subset [0,n],\, |D| \leqslant \varepsilon \right\} \right\} \right\}. \tag{6}$$

This family has very good properties, since it can be shown that the following holds true.

Theorem 22 ([25])**.** *The family of mappings $(\mu_n)_{n\in\mathbb{N}}$, defined by* (6) *fulfils conditions $1^\bullet - 12^\bullet$, that is, it is a family of measures of weak noncompactness in the Fréchet space $L^1_{\mathrm{loc}}(\mathbb{R}_+)$.*

In the previous section, it was mentioned that in the Banach space $L^1(\mathbb{R}_+)$, exact formula for the De Blasi measure of weak noncompactness w

is not known; only a weak measure of noncompactness μ equivalent to the De Blasi measure is known (i.e., satisfying the estimate $w(X) \leqslant \mu(X) \leqslant 2w(X)$). The situation is different for the Fréchet space $L^1_{\mathrm{loc}}(\mathbb{R}_+)$, where convenient computational formulas are known for the family of De Blasi weak measures of noncompactness $(w_n)_{n\in\mathbb{N}}$. In fact, we have the following theorem.

Theorem 23. *In the Fréchet space $L^1_{\mathrm{loc}}(\mathbb{R}_+)$, the family of De Blasi weak measures of noncompactness $(w_n)_{n\in\mathbb{N}}$ is expressed by the formula*

$$w_n(X) = \mu_n(X), \quad n \in \mathbb{N},$$

where $(\mu_n)_{n\in\mathbb{N}}$ is given by formula (6).

Proof. Let us fix $X \in \mathfrak{M}_{L^1_{\mathrm{loc}}(\mathbb{R}_+)}$, $W \in \mathfrak{N}^w_{L^1_{\mathrm{loc}}(\mathbb{R}_+)}$, $r > 0$.
Since $V_n(\theta, r)|_{[0,n]} = B_{L^1[0,n]}(\theta, r)$ − the ball in the Banach space $L^1[0, n]$, $V_n(\theta, r)|_{[n,\infty)} = L^1_{\mathrm{loc}}[n, \infty)$ and $W|_{[0,n]} \in \mathfrak{N}^w_{L^1[0,n]}$, it follows that

$$X \subset W + V_n(\theta, r) \iff X|_{[0,n]} \subset W|_{[0,n]} + V_n(\theta, r)|_{[0,n]}$$

$$\iff X|_{[0,n]} \subset W|_{[0,n]} + B_{L^1[0,n]}(\theta, r),$$

which means $w_n(X) = w(X|_{[0,n]})$, where w is the De Blasi measure of weak noncompactness in the Banach space $L^1[0, n]$. From this and Theorem 8, we obtain

$$w_n(X) = w(X|_{[0,n]}) = \mu_n(X).$$

□

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Summary

The chapter is of a survey nature and is devoted to a brief discussion of measures of weak noncompactness in the Banach spaces $L^1[0,a]$ and $L^1(\mathbb{R}_+)$, as well as in the Fréchet space $L^1_{\rm loc}(\mathbb{R}_+)$.

Streszczenie

Rozdział ma charakter przeglądowy i jest poświęcony krótkiemu omówieniu miar słabej niezwartości w przestrzeniach Banacha $L^1[0,a]$ i $L^1(\mathbb{R}_+)$ oraz w przestrzeni Frécheta $L^1_{\rm loc}(\mathbb{R}_+)$.

Chapter 6

Recent results on matrix weighted norm inequalities

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1. Introduction

The goal of this chapter is to provide an overview of recent work on matrix weights and matrix weighted norm inequalities. In the scalar case, one weight norm inequalities have been extensively studied since the work of Muckenhoupt and others in the 1970s and there is now a rich and well developed theory. (See [43, 51, 54].) Central to their study is the Muckenhoupt A_p condition. It gives a necessary and sufficient condition for weighted strong and weak type inequalities for many of the operators of classical harmonic analysis: maximal operators, singular integral operators, square functions. They also play an important role in other fields, such as the study of degenerate elliptic PDEs. (See, for instance, [10, 45].) A parallel theory of "off-diagonal" inequalities, primarily for fractional integrals (i.e., Riesz potentials), has been developed using the $A_{p,q}$ weights of Muckenhoupt and Wheeden [16, 80].

The Muckenhoupt A_p have a rich structure. One of the deepest results in this area is the Jones factorization theorem [65], which gives a complete characterization of the class A_p in terms of the simpler class A_1. Closely

related to factorization is the Rubio de Francia extrapolation theorem [29, 97], which, in its simplest form, shows that if an operator is bounded on weighted L^2 for weights in A_2, then it is bounded on weighted L^p for all p, $1 < p < \infty$, for weights in A_p. Rubio de Francia's colleague Antonio Cordoba [50] summarized this remarkable result by saying that it showed that *there are no L^p spaces, only weighted L^2.*

A more recent problem has been to determine the sharp dependence in weighted norm inequalities on the A_p characteristic $[w]_{A_p}$, the constant that comes from the Muckenhoupt condition. This was question was first settled for the maximal operator by Buckley [8]. In a major advance, Hytönen [57] proved the sharp dependence for singular integrals is $[w]_{A_p}^{\max\{1,\frac{1}{p-1}\}}$. Central to his proof was a quantitative version of Rubio de Francia extrapolation, which allowed him to reduce the problem to proving the case $p = 2$: that is, that the sharp dependence is $[w]_{A_2}$. For this reason, the problem was referred to as the "A_2 conjecture."

Since the 1990s, there has been a great deal of interest in extending the theory of scalar weights to the setting of matrix weights: that is, $d \times d$ measurable matrix functions W that are self-adjoint, positive semidefinite, and act on vector-valued functions $\mathbf{f}$. This problem was first studied by Nazarov, Treil and Volberg [84, 101, 102, 104], who asked if the strong type weighted norm inequalities for the Hilbert transform on the real line could be extended to the matrix setting. Beyond the intrinsic interest of this problem, their original motivation came from problems in the study of multivariate random stationary processes and Toeplitz operators acting on vector-valued Hardy spaces.

They defined a class of matrix weights, again denoted by A_p, and showed that the Hilbert transform is bounded with respect to this class. This result was later extended to general singular integral operators by Christ and Goldberg [12,52]. More recently, other operators have been considered, such as square functions [61,62], and commutators [64]. Matrix $A_{p,q}$ weights were also defined and studied to prove bounds for fractional integrals [63]. Originally, the focus was on strong type inequalities, but more recently weak type inequalities have been defined and proved [25, 37].

After the proof of the A_2 conjecture for scalar weights, it was conjectured that the same result held for matrix weights. The best known result is that the constant is bounded above by $[W]_{\mathcal{A}_p}^{1+\frac{1}{p-1}-\frac{1}{p}}$, which, when $p = 2$ gives an exponent of $\frac{3}{2}$ rather than the conjectured value of 1. The case $p = 2$ was

proved by Nazarov, Petermichl, Treil and Volberg [82], and for all p in [24]. Very recently, Domelevo, Petermichl, Treil and Volberg [41] show that this conjecture is false and that when $p = 2$, $\frac{3}{2}$ is the best possible exponent. The question of sharp exponents when $p \neq 2$ is still open.

The structure of matrix A_p weights is more complicated than that of scalar weights. Bownik [6] proved that some properties of scalar weights, such as left openness (that is, if $w \in A_p$ for some $p > 1$, there exists $\epsilon > 0$ such that $w \in A_{p-\epsilon}$) no longer hold in the matrix case. In the 1990s Nazarov and Treil [84] posed two fundamental problems: extend the Jones factorization theorem and the Rubio de Francia extrapolation theorem to matrix weights. They wrote:

> *Actually, the whole theory of scalar (A_p)-weights can be transferred to the matrix case except two results...the Peter Jones factorization and the Rubio-de-Francia extrapolation theory. But today (May 1, 1996) we do not know what the analogues of these two things are in high dimensions.*

These problems were finally solved in 2022 [7]. Initially, one motivation for proving extrapolation for matrix weights was to use it to prove the matrix A_2 conjecture. However, applying extrapolation to the sharp bound for $p = 2$ yields a worse estimate than is known for other values of p. It is an interesting problem to understand why extrapolation fails in the matrix case to give the best possible dependence.

In the remainder of this chapter we will give a more detailed overview of the theory of matrix weights, emphasizing the similarities and differences with the scalar case. Though we will provide detailed references, our goal is not to provide a comprehensive history of the field; we will instead focus on some specific results which we believe are at the heart of recent research, particularly with the past five or ten years.

It is organized as follows. In Section 2. we provide an overview of scalar weighted norm inequalities, focusing on those results which are essential for understanding the subsequent development of the matrix theory. As a first step we reformulate strong type inequalities using the approach of treating the weight as a "multiplier" instead of as a "measure." This makes the transition to matrix weights easier in later sections, but we caution the reader to be mindful of this when comparing the results given here with the original literature. We introduce both the classical weak type inequality, and the so-called multiplier weak type inequalities, which were the motivation

for the definition of matrix weighted weak type inequalities. We also include a few remarks about fractional integrals and fractional maximal operators.

In Section 3. we review the scalar theory of extrapolation and factorization. We include some generalizations, including to A_p weights defined with respect to bases other than cubes, and off-diagonal extrapolation.

We then turn to the theory of matrix weights. In Section 4. we consider matrix weighted inequalities for singular integrals. We first discuss the three equivalent definitions of matrix A_p. We then give the strong type inequalities of Christ and Goldberg [12, 52] and the results proved by Nazarov, Petermichl, Treil, and Volberg [82] using the technique of convex body sparse domination. We will then consider weak type inequalities. We will show how the fine properties of matrix A_p weights can be combined with weak type inequalities to prove strong type inequalities by interpolation. We will also briefly discuss some result for fractional integrals.

In Section 5. we will explore the techniques underlying the proofs of Rubio de Francia extrapolation and Jones factorization in the matrix setting. These proofs rely on the theory of harmonic analysis on convex set-valued functions which allows us to generalize the proofs in the scalar case to the matrix case. We will then discuss some recent generalizations of extrapolation in the matrix setting. As an application of the theory of convex set-valued functions, we will sketch a new and extremely elementary proof of strong type bounds for singular integrals.

Finally, in Section 6. we will briefly consider some other results involving matrix weights. This is not intended to be a comprehensive summary. Rather, we want to highlight some recent work that we think is of interest, and give references for those wanting to explore further.

Throughout this chapter we will use the following notation. In Euclidean space the constant n will denote the dimension of $\mathbb{R}^n$, which will be the domain of our functions. The value d will denote the dimension of vector and set-valued functions. For $1 \leqslant p \leqslant \infty$, $L^p(\mathbb{R}^n)$ will denote the Lebesgue space of scalar functions, and $L^p(\mathbb{R}^n, \mathbb{R}^d)$ will denote the Lebesgue space of vector-valued functions.

By a cube we will always mean a cube in $\mathbb{R}^n$ with sides parallel to the coordinate axes. The Lebesgue measure of a cube, or of any arbitrary set E, will be denoted by $|E|$. By a weight we mean a nonnegative, locally integrable function that is positive except on a set of measure 0. We define $w(E) = \int_E w(x)\,dx$, and we let $\fint_Q w(x)\,dx = |Q|^{-1}w(Q)$.

Given $v = (v_1, \dots, v_d)^t \in \mathbb{R}^d$, the Euclidean norm of v will be denoted

by $|v|$; from context there should be no confusion with the notation for the Lebesgue measure of a set. The closed unit ball $\{v \in \mathbb{R}^d : |v| \leqslant 1\}$ will be denoted by $\mathbf{B}$. Matrices will be $d \times d$ matrices with real-valued entries. The set of all $d \times d$, symmetric, positive semidefinite matrices will be denoted by $\mathcal{S}_d$.

The letters C and c will denote constants whose value may change at each appearance which will depend implicitly on the parameters involved. If we want to show explicit dependence, we will write, for instance, $C(n, p)$. Given two quantities A and B, we will write $A \lesssim B$, or $B \gtrsim A$ if there is a constant $c > 0$ such that $A \leqslant cB$. If $A \lesssim B$ and $B \lesssim A$, we will write $A \approx B$.

2. Scalar weighted norm inequalities

To understand the motivation for the study of matrix weights, we first review some basic results in the theory of scalar weights. We will consider the weights that satisfy the Muckenhoupt A_p condition, introduced in [79]. Given $1 < p < \infty$, $w \in A_p$ if

$$[w]_{A_p} = \sup_Q \left(\fint_Q w(x)\,dx \right) \left(\fint_Q w(x)^{1-p'}\,dx \right)^{p-1} < \infty,$$

where the supremum is taken over all cubes. A weight w is in A_1 if

$$[w]_{A_1} = \sup_Q \operatorname*{ess\,sup}_{Q \ni x} w(x)^{-1} \fint_Q w(y)\,dy < \infty.$$

The quantity $[w]_{A_p}$ is referred to as the A_p characteristic of a weight.

Here, we are going to depart from the classical definition which treats the weight w as a measure, that is, the inequalities are with respect to the measure $w dx$. Instead, we will consider the weight as a multiplier: hereafter, for $1 \leqslant p \leqslant \infty$, and a nonnegative, measurable function w, we define $L^p(w)$ to be the class of measurable functions such that $wf \in L^p(\mathbb{R}^n)$ with norm $\|f\|_{L^p(w)} = \|wf\|_{L^p(\mathbb{R}^n)}$. This approach to weights was introduced by Muckenhoupt and Wheeden [80] to study fractional integrals (see below) but has also been used to extend the theory of weights to other function spaces, such as the variable Lebesgue spaces. (See, for instance, [22].)

We define the class of weights $\mathcal{A}_p$ as follows. If $1 < p < \infty$, then $w \in \mathcal{A}_p$

if

$$[w]_{\mathcal{A}_p} = \sup_Q \left(\fint_Q w(y)^p \, dy \right)^{\frac{1}{p}} \left(\fint_Q w(y)^{-p'} \, dy \right)^{\frac{1}{p'}} < \infty;$$

if $p = 1$,

$$[w]_{\mathcal{A}_1} = \sup_Q \left(\fint_Q w(y) \, dy \right) \operatorname*{ess\,sup}_{x \in Q} w(x)^{-1} < \infty;$$

and if $p = \infty$,

$$[w]_{\mathcal{A}_\infty} = \operatorname*{ess\,sup}_{x \in Q} w(x) \sup_Q \left(\fint_Q w(y)^{-1} \, dy \right) < \infty.$$

Remark 1. The two conditions A_1 and $\mathcal{A}_1$ are the same. For $1 < p < \infty$, the conditions A_p and $\mathcal{A}_p$ are equivalent under the mapping $w \mapsto w^p$: $w \in \mathcal{A}_p$ if and only if $w^p \in A_p$, and $[w]^p_{\mathcal{A}_p} = [w^p]_{A_p}$. When $p = \infty$ we have that $w \in \mathcal{A}_\infty$ if and only if $w^{-1} \in \mathcal{A}_1 = A_1$. Note that this definition of $\mathcal{A}_\infty$ is quite different from the classical A_∞ condition.

Remark 2. All of these definitions can be written in a single combined form using norms instead of integrals: for $1 \leqslant p \leqslant \infty$, $w \in \mathcal{A}_p$ if and only if

$$[w]_{\mathcal{A}_p} = \sup_Q |Q|^{-1} \|w\chi_Q\|_{L^p} \|w^{-1}\chi_Q\|_{L^{p'}} < \infty.$$

This formulation also has the advantage that it extends naturally to other Banach functions spaces: see [22, 86, 88].

The Muckenhoupt $\mathcal{A}_p$ weights arise naturally in the study of the Hardy-Littlewood maximal operator,

$$Mf(x) = \sup_Q \fint_Q |f(y)| \, dy \cdot \chi_Q(x).$$

The following theorem is due to Muckenhoupt [79]. For a more elementary proof of the strong type inequality due to Christ and Fefferman, see [11, 17].

Theorem 3. *Fix $1 \leqslant p \leqslant \infty$. Given a weight w, the following are equivalent:*

1. *$w \in \mathcal{A}_p$;*

2. *the maximal operator satisfies the weak (p,p) inequality*

$$\sup_{\lambda>0} \lambda \|\chi_{\{x:Mf(x)>\lambda\}}\|_{L^p(w)} \leqslant C\|f\|_{L^p(w)};$$

3. *if additionally $p > 1$, then*

$$\|Mf\|_{L^p(w)} \leqslant \|f\|_{L^p(w)}.$$

Remark 4. If $p < \infty$, the weak (p,p) inequality is usually written

$$\sup_{\lambda>0} \lambda^p w^p(\{x \in \mathbb{R}^n : Mf(x) > \lambda\}) \leqslant C \int_{\mathbb{R}^n} |w(x)f(x)|^p \, dx,$$

and when $1 < p < \infty$ the strong (p,p) inequality is written

$$\int_{\mathbb{R}^n} (w(x)Mf(x))^p \, dx \leqslant C \int_{\mathbb{R}^n} |w(x)f(x)|^p \, dx.$$

One advantage of the $\mathcal{A}_p$ condition is that we can naturally extend the norm inequalities to the case $p = \infty$. For the maximal operator this was done in Muckenhoupt's original work, but seems to have been mostly overlooked since then.

Muckenhoupt $\mathcal{A}_p$ weights also control weighted norm inequalities for singular integrals. A Calderón-Zygmund singular integral T is a bounded operator on $L^2(\mathbb{R}^n)$ for which there exists a kernel $K(x,y)$, defined on $\mathbb{R}^n \times \mathbb{R}^n \setminus \Delta$, where $\Delta = \{(x,x) : x \in \mathbb{R}^n\}$, such that if $f \in L^\infty_c(\mathbb{R}^n)$ and $x \notin \operatorname{supp}(f)$, then

$$Tf(x) = \int_{\mathbb{R}^n} K(x,y)f(y) \, dy.$$

The kernel satisfies the size and regularity conditions

$$|K(x,y)| \leqslant \frac{C}{|x-y|^n},$$

$$|K(x+h,y) - K(x,y)| + |K(x,y+h) - K(x,y)| \leqslant C\frac{|h|^\delta}{|x-y|^{n+\delta}},$$

where $|x-y| > 2|h|$. We say that the kernel is nondegenerate if there exist $a > 0$ and a unit vector $v \in \mathbb{R}^n$ such that if x, $y \in \mathbb{R}^n$ satisfy $x - y = tv$,

$t \in \mathbb{R}$, then

$$|K(x,y)| \geqslant \frac{a}{|x-y|^n}.$$

For example, this condition is satisfied by the Hilbert transform or any of the Riesz transforms.

In the following result, the sufficiency is due to Hunt, Muckenhoupt and Wheeden [56] for the Hilbert transform and Coifman and Fefferman [14] for general singular integrals; the necessity is due to Stein [98]. See also [43, 51, 54].

Theorem 5. *Given a Calderón-Zygmund singular integral operator T, if $1 \leqslant p < \infty$ and $w \in \mathcal{A}_p$, then the weak (p,p) inequality*

$$\sup_{\lambda>0} \lambda \|\chi_{\{x:|Tf(x)|>\lambda\}}\|_{L^p(w)} \leqslant C\|f\|_{L^p(w)}.$$

holds. Additionally, if $1 < p < \infty$, then the strong (p,p) inequality

$$\|Tf\|_{L^p(w)} \leqslant C\|f\|_{L^p(w)}$$

holds. If T is nondegenerate, then the strong and weak type inequalities imply that $w \in \mathcal{A}_p$.

The weak (p,p) inequality for singular integrals can be proved using kernel estimates and the good/bad decomposition of Calderón and Zygmund. (See, for instance, [51].) The strong type inequality was originally proved by comparing the norm of the singular integral operator to that of the maximal operator. Coifman and Fefferman [14] proved that given p, $0 < p < \infty$, and $w \in \mathcal{A}_q$ for any q, $1 \leqslant q \leqslant \infty$, there exists a constant depending on $[w]_{\mathcal{A}_q}$ such that

$$\|T^*f\|_{L^p(w)} \leqslant C\|Mf\|_{L^p(w)}. \tag{1}$$

Here, T^* is the maximal singular integral, defined by

$$T^*f(x) = \sup_{\epsilon>0} |T_\epsilon f(x)| = \sup_{\epsilon>0} \left| \int_{|x-y|>\epsilon} K(x,y) f(y)\, dy \right|,$$

which dominates the singular integral pointwise. They proved this inequality by using a good-λ inequality: they showed that there exists $\delta > 0$ such that for every γ, $\lambda > 0$ and for every cube Q,

$$w(\{x \in Q : T^*f(x) > 2\lambda, Mf(x) \leqslant \gamma\lambda\}) \leqslant C\gamma^\delta w(Q). \tag{2}$$

An alternative proof of (1) using the sharp maximal operator was given by Journé [66]; see also Alvarez and Pérez [1].

The constants in the norm inequalities in Theorems 3 and 5 depend on the $\mathcal{A}_p$ characteristic, but the original proofs were qualitative and did not give the quantitative dependence on it. Since the 1990s there has been a great deal of interest in determining the best constant in the strong (p,p) inequalities in terms of $[w]_{\mathcal{A}_p}$. This question was first considered by Buckley [8]. He proved the sharp bound for the maximal operator: for $1 < p < \infty$,

$$\|Mf\|_{L^p(w)} \leqslant C[w]_{\mathcal{A}_p}^{p'} \|f\|_{L^p(w)}.$$

He also gave a lower bound for singular integrals that was conjectured to be sharp. This conjecture became the subject of concerted effort when Astala, Iwaniec and Saksman [2] proved that sharp regularity results for solutions of the Beltrami equation hold provided that the Beurling-Ahlfors operator satisfies $\|Tf\|_{L^p(w)} \leqslant C[w]_{\mathcal{A}_p}^{p} \|f\|_{L^p(w)}$ for $p \geqslant 2$. This conjecture was extended to all Calderón-Zygmund operators: for any $p > 1$,

$$\|Tf\|_{L^p(w)} \leqslant C(n,T,p)[w]_{\mathcal{A}_p}^{\max(p,p')} \|f\|_{L^p(w)}. \tag{3}$$

By the Rubio de Francia extrapolation theorem, it suffices to prove this conjecture for $p = 2$. Results for specific singular integrals (e.g., the Beurling-Ahlfors operator, the Hilbert transform, Riesz transforms) were obtained by Petermichl and Volberg [92–94]. Partial results for the general problem were obtained by several authors (see [28,70]); the A_2 conjecture was finally proved by Hytönen [57,60].

Sharp bounds for the weak type inequalities for maximal operators and singular integral operators are also known. For the maximal operator, it follows immediately from the proof of the weak (p,p) inequality, $1 \leqslant p < \infty$, that

$$\sup_{\lambda>0} \lambda \|\chi_{\{x:Mf(x)>\lambda\}}\|_{L^p(w)} \leqslant C[w]_{\mathcal{A}_p} \|f\|_{L^p(w)};$$

this bound was first proved by Muckenhoupt [79] who also proved that it was the best possible. The sharp bound for singular integral operators is

$$\sup_{\lambda>0} \lambda \|\chi_{\{x:|Tf(x)|>\lambda\}}\|_{L^p(w)} \leqslant C[w]_{\mathcal{A}_p}^{p} \|f\|_{L^p(w)};$$

this bound was proved by Hytönen, *et al.* [58].

Hytönen's proof of the A_2 conjecture was very complicated. Soon after it was proved, however, Lerner and Nazarov [71,72,75] and Conde-Alonso and Rey [15] independently gave a new and simpler proof. As part of their proofs they introduced the technique of sparse domination, which use a generalization of the classic dyadic cubes introduced by Calderón and Zygmund. A collection of cubes

$$\mathcal{D} = \bigcup_{k\in\mathbb{Z}} \mathcal{D}_k$$

is a dyadic grid if for each $k \in \mathbb{Z}$, the cubes in $\mathcal{D}_k$ have disjoint interiors, sidelengths 2^{-k}, and their union is all of $\mathbb{R}^n$. Further, given any two cubes $P, Q \in \mathcal{D}$, we must have that $P \subset Q$, $Q \subset P$, or $P \cap Q = \emptyset$. Examples of such families include the translations of the standard dyadic grid

$$\mathcal{D}^t = \{2^{-k}[0,1)^n + m + (-1)^k t : k \in \mathbb{Z}, m \in \mathbb{Z}^n\}, \quad t \in \{0, \pm\tfrac{1}{3}\}^n.$$

These collection share the property that given any cube Q there exists t and $P \in \mathcal{D}^t$ such that $Q \subset P$ and $\ell(P) \leqslant 8\ell(Q)$. (See [16, 59].)

Let $\mathcal{D}$ be any dyadic grid. A collection $\mathcal{S} \subset \mathcal{D}$ is said to be sparse if there exist pairwise disjoint sets $\{E(Q) : Q \in \mathcal{S}\}$ and a constant $0 < \eta < 1$, such that for each Q, $E(Q) \subset Q$ and $\eta|Q| \leqslant |E(Q)|$. A sparse operator is an averaging operator of the form

$$T_{\mathcal{S}} f(x) = \sum_{Q\in\mathcal{S}} \left(\fint_Q f(y)\,dy \right) \cdot \chi_Q(x).$$

Lerner and Nazarov and Conde-Alonso and Rey showed that given a singular integral T, for every bounded function of compact support there exist a finite collection of dyadic grids $\{\mathcal{D}_k\}_{k=1}^N$ and sparse families $\{\mathcal{S}_k\}_{k=1}^N$ such that

$$|Tf(x)| \leqslant C \sum_{k=1}^N T_{\mathcal{S}_k}(|f|)(x).$$

Given this, the A_2 conjecture reduces to proving the corresponding estimates for sparse operators, which is much simpler. This is done in two steps. First, weighted L^2 bounds are proved using an argument that was essentially proved in [28]. (See also [17] where it is worked out in detail.) Then $L^p(w)$ bounds follow from the quantitative version of Rubio de Francia extrapolation. Alternatively, a direct proof that holds for all $p > 1$ was given by Moen [78].

There is another approach to weighted weak type inequalities that was first investigated in the 1970s, but has recently gotten renewed attention for its applications to matrix weights. Given a weight w, and an operator S, define S_w by

$$S_w f(x) = w(x)S(w^{-1}f)(x).$$

By a change of variables we have that

$$\|S_w f\|_{L^p} \leqslant C\|f\|_{L^p} \tag{4}$$

if and only if

$$\|Sf\|_{L^p(w)} \leqslant C\|f\|_{L^p(w)}.$$

By Chebyshev's inequality, this implies the weak (p,p) inequality

$$\sup_{\lambda>0} \lambda \|\chi_{\{x:|S_w f(x)|>\lambda\}}\|_{L^p(\mathbb{R}^n)} \leqslant C\|f\|_{L^p(\mathbb{R}^n)}. \tag{5}$$

Inequalities of this form are referred to as multiplier weak (p,p) inequalities. The following result is due to Muckenhoupt and Wheeden [81] on the real line, and was extended to all $n \geqslant 1$ in [27]. Note that the only difficult case is $p = 1$.

Theorem 6. *Given $1 \leqslant p < \infty$ and $w \in \mathcal{A}_p$, inequality* (5) *holds when S is either the Hardy-Littlewood maximal operator or a Calderón-Zygmund singular integral.*

In [37] the authors proved a quantitative version of Theorem 6. Their results are only sharp when $p = 1$, as was proved by Lerner, *et al.* [74]. Some improvements of their results when $p > 1$ were given by the same authors in [73].

Remark 7. Surprisingly, Muckenhoupt and Wheeden showed that the $\mathcal{A}_p$ condition is not necessary for multiplier weak type inequalities for either the maximal operator or the Hilbert transform, and that for $p > 1$ they found different necessary conditions for these operators. Recently, Sweeting [99] proved that their necessary condition for the maximal operator is sufficient when $p > 1$.

There is a theory of off-diagonal scalar weighted norm inequalities. These are associated to the fractional integral operator

$$I_\alpha f(x) = \int_{\mathbb{R}^n} \frac{f(y)}{|x-y|^{n-\alpha}}\, dy,$$

where $0 < \alpha < n$, and the fractional maximal operator

$$M_\alpha f(x) = \sup_Q |Q|^{\frac{\alpha}{n}} \fint |f(y)|\,dy \cdot \chi_Q(x).$$

Muckenhoupt and Wheeden [80] introduced the weight classes $\mathcal{A}_{p,q}$, where $1 \leqslant p < \frac{n}{\alpha}$ and $\frac{1}{p} - \frac{1}{q} = \frac{\alpha}{n}$, that generalize the Muckenhoupt $\mathcal{A}_p$ classes:

$$[w]_{\mathcal{A}_{p,q}} = \sup_Q < \infty. \tag{6}$$

They proved the following result.

Theorem 8. *Fix $0 < \alpha < n$ and $1 \leqslant p < \frac{n}{\alpha}$, and define q by $\frac{1}{p} - \frac{1}{q} = \frac{\alpha}{n}$. Given a weight w, if S_α is either the fractional integral operator I_α or the fractional maximal operator M_α, the following are equivalent:*

1. *$w \in \mathcal{A}_{p,q}$;*
2. *S_w satisfies the weak (p,q) inequality*

$$\sup_{\lambda>0} \lambda \|\chi_{\{x\in\mathbb{R}^n : |S_\alpha f(x)|>\lambda\}}\|_{L^q(w)} \leqslant C\|f\|_{L^p(w)};$$

3. *additionally, if $p > 1$, S_α satisfies the strong (p,q) inequality*

$$\|S_\alpha f\|_{L^q(w)} \leqslant C\|f\|_{L^p(w)}.$$

The original proof of Theorem 8 by Muckenhoupt and Wheeden was to prove norm inequalities for M_α, and then to use a good-λ inequality similar to (2) to prove norm inequalities for I_α. This approach only gives qualitative estimates for the constant. The sharp constant in terms of the $[w]_{\mathcal{A}_{p,q}}$ characteristic was proved by Lacey, *et al.* [69]. A refined version of their proofs that developed the theory of off-diagonal dyadic sparse operators, was given in [30, 31]; see also [16]. Much of this was based on the seminal work of Pérez [91].

3. Jones factorization and Rubio de Francia extrapolation

As we noted above, the proof that sparse operators satisfy weighted L^p bounds with the conjectured sharp constant follows at once from the case $p = 2$ by the Rubio de Francia extrapolation theorem. This theorem has

a long history and we refer the reader to [29] for complete details. Here we discuss a version of theorem that has several features not present in the original. First, we give a version that gives quantitative estimates for the constants. This was first proved when $p_0 < \infty$ by Dragičević, *et al.* [42]. A simpler proof is given in Duoandikoetxea [44] and also in [29].

Remark 9. This is often referred to as the "sharp constant" version of extrapolation because when applied to singular integrals it gives the sharp constants for all p when starting from $p = 2$, and so allowed proof of the A_2 conjecture. Similarly, it gives the sharp constants for the dyadic square function: see [28]. However, as we will see below for matrix weights, it may not give the best constant, even when starting from an inequality with the sharp constant.

Second, taking advantage of the uniform definition of $\mathcal{A}_p$, $1 \leqslant p \leqslant \infty$, we give a version that lets us start from inequalities when $p = \infty$. A qualitative version of this result was proved by Harboure, *et al.* [55]; they implicitly used the class of $\mathcal{A}_\infty$ weights. The quantitative version of this result was proved by Nieraeth [85] and independently in [7].

Third, we want to state an abstract version of extrapolation. Rather than working with a specific operator (which actually plays no direct role in the proof), we work with pairs of nonnegative functions $(f, g) \in \mathcal{F}$, which we will refer to as extrapolation pairs. Hereafter, we will implicitly assume that given an inequality of the form

$$\|f\|_X \leqslant C\|g\|_X, \qquad (f, g) \in \mathcal{F}, \tag{7}$$

where X is some Banach function space, we mean that this inequality holds for all pairs $(f, g) \in \mathcal{F}$ such that $0 < \|f\|_X$, $\|g\|_X < \infty$. This approach to extrapolation was introduced in [36] and systematically developed in [26,29]. Intuitively, to prove bounds for an operator T, one takes the pairs $(|Tf|, |f|)$; for a discussion of the technical details of this approach, see [17, 29].

Theorem 10. *Given p_0, $1 \leqslant p_0 \leqslant \infty$, suppose that for a family of extrapolation pairs $\mathcal{F}$ and for all $w_0 \in \mathcal{A}_{p_0}$, the inequality*

$$\|f\|_{L^{p_0}(w_0)} \leqslant N_{p_0}([w_0]_{\mathcal{A}_{p_0}})\|g\|_{L^{p_0}(w_0)}, \qquad (f, g) \in \mathcal{F},$$

holds. Then for all p, $1 < p < \infty$, and all $w \in \mathcal{A}_p$,

$$\|f\|_{L^p(w)} \leqslant N_p([w]_{\mathcal{A}_p})\|g\|_{L^p(w)}, \qquad (f, g) \in \mathcal{F},$$

where

$$N_p([w]_{\mathcal{A}_p}) = C(n,p_0,p)N_{p_0}(c(n,p_0,p)[w]_{\mathcal{A}_p}^{\max\{\frac{p}{p_0},\frac{p'}{p_0'}\}}).$$

Remark 11. One feature of this result is that it does not allow extrapolation to the endpoints: starting with $1 < p_0 < \infty$, it is not possible to get to either $p = 1$ or $p = \infty$.

We will not prove Theorem 10 here. For complete proofs (when $p < \infty$) see [29]; for a discussion of the essential ideas of the proof, see [17]. Here we want to concentrate on the key ideas underlying the proof.

The proof ultimately depends on four things. First, it requires the "duality" implicit in the definition of $\mathcal{A}_p$ weights: that for $1 \leqslant p \leqslant \infty$, $w \in \mathcal{A}_p$ if and only if $w^{-1} \in \mathcal{A}_{p'}$. Second, it requires the quantitative bound for the strong (p,p) inequality for the maximal operator discussed above: for $1 < p \leqslant \infty$, $\|Mf\|_{L^p(w)} \leqslant C[w]_{A_p}^{p'}\|f\|_{L^p(w)}$. Third, it requires the Jones factorization theorem.

Theorem 12. *Given $1 < p < \infty$ and a weight w, $w \in \mathcal{A}_p$ if and only if there exist $w_0 \in \mathcal{A}_1$ and $w_1 \in \mathcal{A}_\infty$ such that*

$$w = w_0^{\frac{1}{p}} w_1^{\frac{1}{p'}}.$$

Theorem 12 was first proved by Jones [65], but a much more elementary proof was given by Coifman, Jones, and Rubio de Francia [13]. (See also [17].) In fact, in the proof of extrapolation, we only use the easier half of the Jones factorization theorem: that if $w_0 \in A_1$ and $w_1 \in \mathcal{A}_\infty$, then $w = w_0^{\frac{1}{p}} w_1^{\frac{1}{p'}} \in \mathcal{A}_p$. This property, which we refer to as "reverse factorization," follows at once from the definition of the $\mathcal{A}_p$ classes.

Fourth, we need a means of constructing $\mathcal{A}_p$ weights, which, by applying Theorem 12, means we need to construct $\mathcal{A}_1$ and $\mathcal{A}_\infty$ weights (which are just inverses of $\mathcal{A}_1$ weights). To do so we use the Rubio de Francia iteration operator, which is also fundamental to the proof of the harder implication in the Jones factorization theorem. Fix $w \in \mathcal{A}_p$, $1 < p < \infty$. Given a nonnegative function h, define

$$\mathcal{R}h(x) = \sum_{k=0}^{\infty} \frac{M^k h(x)}{2^k \|M\|_{L^p(w)}^k},$$

where $\|M\|_{L^p(w)}$ is the operator norm of M on $L^p(w)$. It then follows from the definition and the properties of $\mathcal{A}_p$ weights that

1. $h(x) \leqslant \mathcal{R}h(x)$;
2. $\|\mathcal{R}h\|_{L^p(w)} \leqslant 2\|h\|_{L^p(w)}$;
3. $\mathcal{R}h \in A_1$ and $[\mathcal{R}h]_{A_1} \leqslant 2\|M\|_{L^p(w)} \leqslant C[w]_{A_p}^{p'}$.

Each of these estimates plays an important role in the proof.

We briefly consider three extensions of extrapolation. First, one consequence of working with abstract extrapolation pairs is that it immediately lets us prove vector-valued inequalities of the form

$$\left\| \left(\sum_{k=1}^{\infty} |wf_k|^q \right)^{\frac{1}{q}} \right\|_{L^p(\mathbb{R}^n)} \leqslant C \left\| \left(\sum_{k=1}^{\infty} |wg_k|^q \right)^{\frac{1}{q}} \right\|_{L^p(\mathbb{R}^n)} \tag{8}$$

when $1 < p, q < \infty$, $w \in \mathcal{A}_p$, and $\{(f_k, g_k)\}_{k=1}^{\infty} \subset \mathcal{F}$. It also lets us apply extrapolation to prove (classical) weak type inequalities

$$\sup_{\lambda>0} \lambda \|\chi_{\{x:|f(x)|>\lambda\}}\|_{L^p(w)} \leqslant C\|g\|_{L^p(w)}, \qquad (f,g) \in \mathcal{F}.$$

For details, see [29].

Remark 13. It is an open question whether there exists a version of Rubio de Francia extrapolation which can be applied to the multiplier weak type inequalities.

Second, we can apply extrapolation to more general classes of weights. By a basis, we mean a nonempty collection of open, bounded sets $\mathcal{B}$ in $\mathbb{R}^n$. We can then define $\mathcal{A}_p$ classes with respect to a basis: for $1 \leqslant p \leqslant \infty$, we say $w \in \mathcal{A}_{p,\mathcal{B}}$ if

$$[w]_{\mathcal{A}_{p,\mathcal{B}}} = \sup_{B\in\mathcal{B}} |B|^{-1} \|w\chi_B\|_{L^p} \|w^{-1}\chi_B\|_{L^{p'}} < \infty. \tag{9}$$

We say that a basis $\mathcal{B}$ is a Muckenhoupt basis if the maximal operator

$$M_{\mathcal{B}} f(x) = \sup_{B\in\mathcal{B}} \fint_B |f(y)|\,dy \cdot \chi_B(x) \tag{10}$$

satisfies $\|M_{\mathcal{B}}f\|_{L^p(w)} \leqslant C\|f\|_{L^p(w)}$, $1 < p \leqslant \infty$, whenever $w \in \mathcal{A}_{p,\mathcal{B}}$. There are several important examples of Muckenhoupt bases besides the basis of cubes $\mathcal{Q}$:

- the set $\mathcal{D}$ of dyadic cubes (this is implicit in the proofs of bounds for the Hardy-Littlewood maximal operator);
- the set $\mathcal{R}$ of rectangles with sides parallel to the coordinate axes (e.g., the strong maximal operator);
- more generally, the multiparameter basis $\mathcal{R}^\alpha$, $\alpha = (a_1, \dots, a_j)$, consisting of sets of the form $Q_1 \times \cdots \times Q_j$, where Q_i is a cube in $\mathbb{R}^{a_i}$ (see García-Cuerva and Rubio de Francia [51]);
- the Zygmund basis $\mathcal{Z}$ of rectangles in $\mathbb{R}^3$ with sides parallel to the coordinate axes and whose sidelengths satisfy (s, t, st) for some s, $t > 0$ (see Fefferman and Pipher [46]).

With this definition, the proofs of extrapolation (e.g., of Theorem 10) go through without any essential changes. This is done in great detail in [29]; note that the above fundamental ideas from the proof carry through immediately for Muckenhoupt bases.

Finally, we note that extrapolation can be extended to the off-diagonal setting, using the $\mathcal{A}_{p,q}$ weights (6) of Muckenhoupt and Wheeden, A qualitative version of the following result was proved (when $q_0 < \infty$) by Harboure, *et al.* [55]. A quantitative version (again when $q_0 < \infty$), which yielded the sharp constant estimates for fractional integrals, was proved by Lacey, *et al.* [69]. Their proof generalized an older proof of extrapolation that depended on a technical lemma due go García-Cuerva [49]. (See also [51].) A different, simpler proof, which does not get the best possible constant, is in [29]. (For the explicit computation of this constant, in the setting of matrix weights, see [19].) For a simpler proof of the "sharp constant" result of Lacy, *et al.*, see Duoandikoetxea [44]. For another proof, including the case $q_0 = \infty$, see the forthcoming paper [18].

Theorem 14. *Fix s, $1 \leqslant s < \infty$. Given p_0, q_0, $1 \leqslant p_0 \leqslant q_0 \leqslant \infty$, $\frac{1}{p_0} - \frac{1}{q_0} = \frac{1}{s'}$, suppose that for some family of extrapolation pairs $\mathcal{F}$ and for all $w_0 \in \mathcal{A}_{p_0,q_0}$, the inequality*

$$\|f\|_{L^{q_0}(w_0)} \leqslant N_{p_0,q_0}([w_0]_{\mathcal{A}_{p_0,q_0}})\|g\|_{L^{p_0}(w_0)}, \qquad (f,g) \in \mathcal{F},$$

holds. Then for all p, q, $1 < p \leqslant q < \infty$, $\frac{1}{p} - \frac{1}{q} = \frac{1}{s'}$, *and all* $w \in \mathcal{A}_{p,q}$,

$$\|f\|_{L^q(w)} \leqslant N_{p,q}([w]_{\mathcal{A}_{p,q}})\|g\|_{L^p(w)}, \qquad (f,g) \in \mathcal{F},$$

where

$$N_{p,q}([w]_{\mathcal{A}_{p,q}}) = C(n,p_0,p,s)N_{p_0}\big(c(n,p_0,p,s)[w]_{\mathcal{A}_{p,q}}^{\max\{\frac{q}{q_0},\frac{p'}{p_0'}\}}\big).$$

In the proof of Theorem 14, a key fact is that $w \in \mathcal{A}_{p,q}$ if and only if $w^s \in \mathcal{A}_r$. This equivalence follows immediately from the definitions.

4. Matrix weights and matrix weighted norm inequalities

We now turn to the problem of generalizing weighted norm inequalities to the setting of vector-valued functions and matrix weights. We first define some notation. Let $\mathbf{f} = (f_1, \dots, f_d)$ be a measurable $\mathbb{R}^d$-valued function. Given a singular integral T, define it acting on $\mathbf{f}$ by

$$T\mathbf{f} = (Tf_1, \dots, Tf_d).$$

It is immediate that if $1 < p < \infty$ and $\mathbf{f} \in L^p(\mathbb{R}^n, \mathbb{R}^d)$, then $\|T\mathbf{f}\|_{L^p(\mathbb{R}^n,\mathbb{R}^d)} \leqslant C\|\mathbf{f}\|_{L^p(\mathbb{R}^n,\mathbb{R}^d)}$.

To define matrix weights, recall that $\mathcal{S}_d$ is the set of $d \times d$, self-adjoint, positive semidefinite matrices. A matrix weight is a measurable function $W : \mathbb{R}^n \to \mathcal{S}_d$. We will assume that the entries of W are finite and that W is invertible almost everywhere. (This assumption is in some sense necessary: see [19].) Define a scalar weight using the operator norm of W:

$$|W(x)|_{\mathrm{op}} = \sup_{\xi\in\mathbb{R}^d,|\xi|=1} |W(x)\xi|.$$

For $1 \leqslant p \leqslant \infty$, we define the matrix weighted space $L^p(W) = L^p(W, \mathbb{R}^n, \mathbb{R}^d)$ by $\|\mathbf{f}\|_{L^p(W)} = \|W\mathbf{f}\|_{L^p(\mathbb{R}^n,\mathbb{R}^d)}$. When $p < \infty$, this becomes

$$\|\mathbf{f}\|_{L^p(W)} = \bigg(\int_{\mathbb{R}^n} |W(x)\mathbf{f}(x)|^p\,dx\bigg)^{\frac{1}{p}}.$$

Note that when $d = 1$, this reduces to the scalar space $L^p(w)$.

Remark 15. Originally, these spaces were defined for $p < \infty$ with the

matrix W replaced by $W^{\frac{1}{p}}$, so that when $d = 1$ it would reduce to the classical definition of weighted norm inequalities. Note that $W^{\frac{1}{p}}$ is well-defined since W is positive semidefinite. For a variety of reasons we believe this definition is a better one. In particular, see the discussion below on left openness.

With this notation, the problem first considered by Nazarov, Treil and Volberg can be stated as follows: given $1 < p < \infty$, prove there is a Muckenhoupt type condition on matrix weights so that the inequality

$$\|T\mathbf{f}\|_{L^p(W)} \leqslant \|\mathbf{f}\|_{L^p(W)}.$$

holds for singular integrals. Treil and Volberg [102] first solved this problem on the real line for the Hilbert transform when $p = 2$. They showed that this inequality holds if W satisfies the following analog of the A_2 condition:

$$\sup_Q \left| \left(\fint_Q W^2(x)\,dx \right)^{\frac{1}{2}} \left(\fint_Q W^{-2}(x)\,dx \right)^{\frac{1}{2}} \right|_{\mathrm{op}} < \infty. \tag{11}$$

This condition, however, cannot be extended to the case $p \neq 2$.

Ultimately, three equivalent definitions were developed for $\mathcal{A}_p$, $1 \leqslant p \leqslant \infty$, each with its own strengths and weaknesses. The first definition of matrix $\mathcal{A}_p$ was in terms of norm functions. It was conjectured by Treil [100] and used by Nazarov and Treil [84] and by Volberg [104] to prove matrix weighted norm inequalities for the Hilbert transform. Their idea was to replace matrices with norms on $\mathbb{R}^d$. Here we sketch their definition; for complete details see the above references or [7].

Recall that a function $\rho : \mathbb{R}^d \to [0, \infty)$ is a norm if, given $v, w \in \mathbb{R}^d$ and $\alpha \in \mathbb{R}$: $\rho(v) = 0$ if and only if $v = 0$; $\rho(v + w) \leqslant \rho(v) + \rho(w)$; and $\rho(\alpha v) = |\alpha|\rho(v)$. Define the dual of a norm ρ to be the norm ρ^* given by

$$\rho^*(v) = \sup_{w \in \mathbb{R}^d, \rho(w) \leqslant 1} |\langle v, w \rangle|.$$

Let $\rho : \mathbb{R}^n \times \mathbb{R}^d \to [0, \infty)$ be a measurable function such that for a.e. $x \in \mathbb{R}^n$, $\rho(x, \cdot)$ is a norm on $\mathbb{R}^d$. For instance, given a matrix weight W, we can define a norm function ρ_W by $\rho_W(x, v) = |W(x)v|$. Finally, given a cube Q, and

$1 \leqslant p < \infty$, define the average of a norm function on Q by

$$\langle \rho \rangle_{p,Q}(v) = \|\rho(\cdot, v)\|_{p,Q} = \left(\fint_Q \rho(x,v)^p \, dx \right)^{\frac{1}{p}}.$$

When $p = \infty$, let $\langle \rho \rangle_{\infty,Q}(v) = \|\rho(\cdot, v)\chi_Q\|_{L^\infty(\mathbb{R}^n)}$. For all p, $1 \leqslant p \leqslant \infty$, we have that for every $v \in \mathbb{R}^d$,

$$\langle \rho \rangle^*_{p,Q}(v) \leqslant \langle \rho^* \rangle_{p',Q}(v).$$

We define ρ to be in $\mathcal{A}_p$ if the reverse inequality holds up to a constant independent of v:

$$\langle \rho^* \rangle_{p',Q}(v) \leqslant C \langle \rho \rangle^*_{p,Q}(v)$$

When $d = 1$, if $\rho = \rho_w$ for some scalar weight w, then this reduces to the scalar $\mathcal{A}_p$ condition after rearranging terms.

Given any norm function we can associate to it a norm function induced by a matrix that is equivalent to it. Let ρ be a norm, and define K_ρ to be the closed unit ball in $\mathbb{R}^d$ with respect to this norm:

$$K_\rho = \{v \in \mathbb{R}^d : \rho(v) \leqslant 1\}. \tag{12}$$

Then K_ρ is a convex set in $\mathbb{R}^d$; associated to it is its John ellipsoid: a unique ellipsoid E of maximum volume such that $E \subset K_\rho \subset \sqrt{d}E$. We can write $E = W\mathbf{B}$, where $W \in \mathcal{S}_d$. As a consequence, $\rho(v) \approx \rho_W(v)$ for all $v \in \mathbb{R}^d$. Moreover, if ρ is a norm function, then we can choose the associated matrix function W to be measurable (see [7, Theorem 4.11]). Consequently, it suffices to restrict our attention to norms induced by matrices.

Working with matrices, Roudenko [96] gave an equivalent definition of $\mathcal{A}_p$ that more closely resembled the scalar definition: for $1 < p < \infty$, $W \in \mathcal{A}_p$ if and only if

$$[W]_{\mathcal{A}_p} = \sup_Q \fint_Q \left(\fint_Q |W^{\frac{1}{p}}(x) W^{-\frac{1}{p}}(y)|^{p'}_{\mathrm{op}} \, dy \right)^{\frac{p}{p'}} dx < \infty. \tag{13}$$

Frazier and Roudenko [47] also introduced the concept of matrix $\mathcal{A}_1$ weights: $W \in \mathcal{A}_1$ if

$$[W]_{\mathcal{A}_1} = \sup_Q \operatorname*{ess\,sup}_{x \in Q} \fint_Q |W^{-1}(x) W(y)|_{\mathrm{op}} \, dy < \infty. \tag{14}$$

The definition of $\mathcal{A}_\infty$ was given in [7]: $W \in \mathcal{A}_\infty$ if

$$[W]_{\mathcal{A}_\infty} = \sup_Q \operatorname*{ess\,sup}_{x\in Q} \fint_Q |W(x)W^{-1}(y)|_{\mathrm{op}}\, dy < \infty. \tag{15}$$

For the equivalence of this definition with the previous one, see [7]. One advantage of this definition is that it is straightforward to show that $W \in \mathcal{A}_p$, $1 \leqslant p \leqslant \infty$, if and only if the averaging operator $A_Q\mathbf{f}(x) = \fint_Q \mathbf{f}(y)\, dy \cdot \chi_Q(x)$, is bounded on $L^p(W)$. This in turn can be used to prove norm inequalities for convolution operators and norm convergence of approximate identities. (See [32].) When $d = 1$, this definition immediately reduces to the definition for scalar weights given above.

The third equivalent definition of matrix $\mathcal{A}_p$ weights is in terms of reducing operators. This approach was introduced by Volberg [104] and further developed and used by Goldberg [52]. Given a matrix $W \in \mathcal{S}_d$ and a cube Q, for $1 \leqslant p \leqslant \infty$ we define the norm $\langle \rho_W \rangle_{p,Q}$. As noted above, there exists a matrix $\mathcal{W}_Q^p$ which induces an equivalent norm. We call this matrix the reducing operator associated to W on Q. Similarly, we let $\overline{W}_Q^{p'}$ be the reducing operator associated with $\langle \rho_{W^{-1}} \rangle_{p',Q}$. Then $W \in \mathcal{A}_p$ if and only if

$$[W]_{\mathcal{A}_p}^R = \sup_Q |\mathcal{W}_Q^p \overline{W}_Q^{p'}|_{\mathrm{op}} < \infty.$$

Moreover, we have that $[W]_{\mathcal{A}_p}^R \approx [W]_{\mathcal{A}_p}$. (Again, see [7].) The reducing operator definition is very useful in technical estimates related to the maximal operator: see below. Here we note that, using the fact that if A and B are self adjoint matrices, then $|AB|_{\mathrm{op}} = |BA|_{\mathrm{op}}$, we have that $W \in \mathcal{A}_p$ if and only if $W^{-1} \in \mathcal{A}_{p'}$ and $[W^{-1}]_{\mathcal{A}_{p'}} \approx [W]_{\mathcal{A}_p}$. Using (13) above, this equivalence only follows when $p = 2$.

Norm inequalities for Calderón-Zygmund singular integrals in $\mathbb{R}^n$ were proved by Christ and Goldberg [12, 52] for $1 < p < \infty$. Reducing operators were central to their proof. Treil and Volberg had earlier noted that a key obstacle to proving matrix weighted inequalities was the lack of a maximal operator that did not lose the geometric information embedded in a vector-valued function. A key component of the proofs in [12,52] is a scalar-valued, matrix weighted maximal operator, now referred to as the Christ-Goldberg

maximal operator:

$$M_W\mathbf{f}(x) = \sup_Q \fint_Q |W(x)W^{-1}(y)\mathbf{f}(y)|\,dy \cdot \chi_Q(x).$$

The motivation for this definition comes from change of variables introduced in the scalar case when defining multiplier weak type inequalities. Note that when $d = 1$, $M_w f(x) = w(x)M(w^{-1}f)(x)$, so the boundedness of M_w on $L^p(\mathbb{R}^n)$ is equivalent to the boundedness of the Hardy-Littlewood maximal operator on $L^p(w)$. We can make a similar definition for singular integral operators; since they are linear the matrix $W(x)$ can be pulled in or out of the operator:

$$T_W\mathbf{f}(x) = W(x)^{\frac{1}{p}}T(W^{-\frac{1}{p}}\mathbf{f})(x).$$

The following theorem then holds.

Theorem 16. *Given $1 < p < \infty$ and a matrix weight $W \in \mathcal{A}_p$, then*

1. *The maximal operator M_W is bounded from $L^p(\mathbb{R}^n, \mathbb{R}^d)$ to $L^p(\mathbb{R}^n)$;*
2. *If T is a Calderón-Zygmund singular integral, then T_W is bounded on $L^p(\mathbb{R}^n, \mathbb{R}^d)$.*

Moreover, if M_W is bounded, or if T is bounded and nondegenerate, then $W \in \mathcal{A}_p$.

Remark 17 ([7])**.** When $p = \infty$, M_W is bounded on $L^\infty(\mathbb{R}^n)$ if and only if $W \in \mathcal{A}_\infty$.

The proof of Theorem 16 roughly follows the proofs of the corresponding results in the scalar case, but a number of technical obstacles arise. The first step was to prove that if $W \in \mathcal{A}_p$, $1 < p < \infty$, then M_W is bounded from $L^p(\mathbb{R}^n, \mathbb{R}^d)$ into $L^p(\mathbb{R}^n)$. This required the introduction of an auxiliary maximal operator M'_W, defined by

$$M'_W\mathbf{f}(x) = \sup_Q \fint_Q |\mathcal{W}_Q^p W^{-1}(y)\mathbf{f}(y)|\,dy \cdot \chi_Q(x).$$

The proof that the auxiliary maximal operator maps $L^p(\mathbb{R}^n, \mathbb{R}^d)$ into $L^p(\mathbb{R}^n)$ is done by first using a covering lemma argument (e.g., the Vitali covering lemma or Calderón-Zygmund cubes) to show that it satisfies a weak (p, p) inequality. In fact, by an argument which is analogous to using the reverse

Hölder inequality in the scalar case to prove that if $w \in \mathcal{A}_p$, then $w \in \mathcal{A}_{p\pm\epsilon}$, they showed that there exists $\epsilon > 0$ such that it satisfies weak $(p\pm\epsilon, p\pm\epsilon)$ inequalities. The desired strong type inequality then follows by Marcinkiewicz interpolation. Finally, while there is no direct pointwise relationship between M_W and M'_W, it follows from a stopping time argument that

$$\|M_W\mathbf{f}\|_{L^p(\mathbb{R}^n,\mathbb{R}^d)} \leqslant C\|M'_W\mathbf{f}\|_{L^p(\mathbb{R}^n,\mathbb{R}^d)}.$$

Given these bounds for the maximal operators M_W and M'_W, the fact that singular integrals are bounded is gotten by adapting the ideas of Coifman and Fefferman to prove a good-λ inequality. Define

$$T^*_W\mathbf{f}(x) = \sup_{\epsilon>0} |W^{\frac{1}{p}}(x)T_\epsilon(W^{-\frac{1}{p}}\mathbf{f})(x)|.$$

Then for every smooth function $\mathbf{f}$ with compact support, they proved that there exist constants $0 < b < 1$ and $c > 0$ such that for all $\lambda > 0$,

$$|\{x \in \mathbb{R}^n : T^*_W\mathbf{f}(x) > \lambda, \max\{M'_W\mathbf{f}(x), M_W\mathbf{f}(x)\} < c\lambda\}| \\ \leqslant \frac{1}{2}b^p|\{x \in \mathbb{R}^n : T^*_W\mathbf{f}(x) > b\lambda\}|.$$

Bounds for T^* and so for T then follow by a standard level-set argument.

As in the scalar case, this approach to proving matrix weighted norm inequalities does not yield quantitative estimates on the constant in terms of the $[W]_{\mathcal{A}_p}$ characteristic. After the proof of the scalar A_2 conjecture by Hytönen, it was conjectured that the corresponding result holds for matrix weights: for $1 < p < \infty$,

$$\|T\mathbf{f}\|_{L^p(W)} \leqslant C[W]^{\max\{p,p'\}}_{\mathcal{A}_p}\|\mathbf{f}\|_{L^p(W)}.$$

This problem was considered by Bickel, Petermichl and Wick [5], Pott and Stoica [95] and Culiuc, Di Plinio and Ou [39] when $p = 2$. Here we will consider the approach of Nazarov, Petermichl, Treil and Volberg [82]. They showed that when $p = 2$,

$$\|T\mathbf{f}\|_{L^2(W)} \leqslant C[W]^3_{\mathcal{A}_2}\|\mathbf{f}\|_{L^2(W)}.$$

Their proof is based on a deep generalization of the sparse domination estimates described above in the scalar case.

The sparse operator introduced in [82] replaces vector-valued functions by averages that are convex sets. They show that given $\mathbf{f} \in L^\infty_c(\mathbb{R}^n, \mathbb{R}^d)$, there exists a finite collection of dyadic grids $\{\mathcal{D}_k\}_{k=1}^N$ and sparse sets $\{\mathcal{S}_k\}_{k=1}^N$ such that

$$T\mathbf{f}(x) \in C\sum_{k=1}^N T_{\mathcal{S}_k}\mathbf{f}(x) = C\sum_{k=1}^N \sum_{Q\in\mathcal{S}_k} \langle\langle \mathbf{f}\rangle\rangle_Q \chi_Q(x), \tag{16}$$

where $\langle\langle f\rangle\rangle_Q$ is the convex set

$$\langle\langle \mathbf{f}\rangle\rangle_Q = \left\{ \fint_Q k(y)\mathbf{f}(y)\,dy : k \in L^\infty(Q), \|k\|_{L^\infty(Q)} \leqslant 1 \right\},$$

and the sum is the (infinite) Minkowski sum of convex sets. They referred to this estimate as a convex body sparse domination. To complete their proof, however, they did not work directly with these convex set-valued functions, but rather replaced them by vector-valued sparse operators of the form

$$\tilde{T}_{\mathcal{S}}\mathbf{f}(x) = \sum_{Q\in\mathcal{S}} \fint_Q \varphi_Q(x,y)\mathbf{f}(y)\,dy,$$

where for each Q, φ_Q is a real-valued function supported on $Q\times Q$ such that, for each x, $\|\varphi_Q(x,\cdot)\|_\infty \leqslant 1$. These they estimated using square functions.

When $1 < p < \infty$, quantitative results were proved in [24]. They showed that

$$\|T\mathbf{f}\|_{L^p(W)} \leqslant C[W]_{\mathcal{A}_p}^{p+p'-1}\|\mathbf{f}\|_{L^p(W)}.$$

They used the sparse domination result of Nazarov, *et al.* and reduced to the vector-valued sparse operators, but instead of using square function estimates, they adapted techniques from the study of scalar, two weight norm inequalities, introducing matrix analogs of the so-called $\mathcal{A}_p$ bump conditions. Consequently, they also proved two weight inequalities for singular integrals, the first such results in the matrix weight setting.

This proof, while quantitative, did not follow the approach which yielded sharp inequalities in the scalar case for sparse operators. They used Orlicz maximal function estimates which, in the scalar one weight case, were known to not produce the sharp estimate. Initially, there was hope that this proof could be refined to prove the $\mathcal{A}_2$ conjecture for matrix weights. In the scalar case the key tool is the "universal" dyadic maximal operator. Given any

weight w and a dyadic grid $\mathcal{D}$, define the dyadic operator $M_w^{\mathcal{D}}$ by

$$M_w^{\mathcal{D}} f(x) = \sup_{Q \in \mathcal{D}} \frac{1}{w(Q)} \int_Q |f(y)|\, w(y) dy \cdot \chi_Q(x). \tag{17}$$

Dyadic covering arguments yield the estimate

$$\left(\int_{\mathbb{R}^n} M_w^{\mathcal{D}} f(x)^p\, w(x) dx \right)^{\frac{1}{p}} \leqslant p' \left(\int_{\mathbb{R}^n} |f(x)|^p\, w(x) dx \right)^{\frac{1}{p}}.$$

(See [78]. While it is more natural to write this inequality with the weight acting as a measure, by replacing w by w^p we can write it with the weight acting as a multiplier.) The goal was to find an analogous operator in the matrix case and adapt the scalar proofs to use it. There is a natural candidate, based on the convex set-valued maximal operator (see below), but it was shown to not be bounded on $L^p(\mathbb{R}^n, \mathbb{R}^d)$: see [83]. Moreover, very recently the exponent 3 was shown to be the best possible when $p = 2$ by Domelevo, Petermichl, Treil, and Volberg [41]. This bound is also the best possible for sparse operators: see Treil and Volberg [103].

Remark 18. It is an open question whether the exponent $p + p' - 1$ is the best possible for all p. We are tempted to conjecture that it is, but there is no evidence either way.

For many years, only strong type inequalities were considered in the matrix case, as it did not seem possible to generalize the classical weak type inequalities to use matrix weights. But then, in [25] the authors utilized the model of multiplier weak type inequalities for scalar weights to define and prove the analogous estimates for matrix weights. The following result was proved in [25] when $p = 1$, and for $1 < p < \infty$ in [37].

Theorem 19. *Fix $1 \leqslant p < \infty$, and $W \in \mathcal{A}_p$, if T is a Calderón-Zygmund singular integral operator, then*

$$\sup_{\lambda > 0} \lambda |\{x \in \mathbb{R}^n : |T_W \mathbf{f}(x)| > \lambda\}|^{\frac{1}{p}} \leqslant C [W]_{\mathcal{A}_p}^{p+1} \|\mathbf{f}\|_{L^p(\mathbb{R}^n)}. \tag{18}$$

Moreover, the same inequality holds if T_W is replaced by the Christ-Goldberg maximal operator M_W.

Remark 20. As we noted above, the weak (p, p) inequality follows from the strong (p, p) inequality by Chebyshev's inequality. The constant above

is better in the range $1 < p < 2$ when compared to the values gotten from the best known constant in the strong type inequality. This exponent is sharp when $p = 1$ (see [74]), but it is not known what the best constant is otherwise. For some partial results, see [73].

The proof of Theorem 19 uses convex body sparse domination to reduce the problem to an estimate for a scalar weighted sparse operator. The proof then follows the same pattern as the proof of the scalar version of this result. We note that this part of the proof uses a clever restatement of the weak type norm on the lefthand side:

$$\|wT_{\mathcal{S}}(w^{-1}\,\cdot)\|_{L^p\to L^{p,\infty}} \approx \sup_{\|f\|_{L^p}=1}\ \sup_{0<|E|<\infty}\ \inf_{\substack{E'\subseteq E\\ |E|\leqslant 2|E'|}} |E|^{\frac{1}{p}-1}|\langle wT_{\mathcal{S}}(w^{-1}f),\chi_{E'}\rangle|.$$

(See [53, Exercise 1.4.14]; this approach to estimating weak type norms was previously used by Frey and Nieraeth [48].)

As an application of Theorem 19 we describe another proof of the boundedness of singular integrals in the matrix weighted setting that makes use of a recently discovered structural property of matrix $\mathcal{A}_p$ weights. As we noted above, Bownik [6] showed that for the classical definition of matrix A_p weights, left openness does not hold. More precisely, he constructed a matrix weight $W \in A_2$ that is not in A_p for any $p < 2$. However, this property does hold using the multiplier definition given above. In [34] it was shown that if $W \in \mathcal{A}_p$, $1 < p < \infty$, then there exist $0 < r < 1 < s$ such that $W \in \mathcal{A}_{rp}$ and $W \in \mathcal{A}_{sp}$. Using this fact, Penrod and Sweeting [90] showed that the strong type inequality for T_W follows immediately from Marcinkiewicz interpolation and the weak type inequality (18). Their estimate is quantitative, but gives a worse constant in terms of the $\mathcal{A}_p$ characteristic than the best known result.

Matrix weighted estimates for fractional integrals have been proved, using the matrix version of the $\mathcal{A}_{p,q}$ weights. Strong type inequalities were proved by Isralowitz and Moen [63], who also introduced and proved estimates for a fractional Christ-Goldberg maximal operator. Weak type inequalities were proved in [37]. In both cases the estimates are quantitative, but it is not known if they are sharp. Isralowitz and Moen explicitly conjecture that their result is not the best possible.

5. Convex set-valued functions, matrix weights, and extrapolation

In this section we consider the problem of extending the Rubio de Francia extrapolation theorem and the Jones factorization theorem to matrix weights. While the proofs of these two results are now relatively straightforward in the scalar case, there did not seem to be any way to generalize these proofs to the matrix setting. In particular, the fundamental obstacle was generalizing the Rubio de Francia iteration algorithm, which required iterates of the maximal operator. The Christ-Goldberg operator could not be used, since it maps vector-valued functions to scalar-valued functions.

The solution was to move beyond vector-valued functions and work with convex set-valued functions. The convex body sparse domination of Nazarov *et al.* was the first indication of the power and utility of this approach, but in [7] the authors developed it considerably. Before discussing the new techniques required for the proofs, we first state the two main theorems from [7].

Theorem 21. *Given $1 < p < \infty$ and a matrix weight W, $W \in \mathcal{A}_p$ if and only if there exist commuting matrices $W_0 \in \mathcal{A}_1$ and $W_1 \in \mathcal{A}_\infty$ such that*

$$W = W_0^{\frac{1}{p}} W_1^{\frac{1}{p'}}. \tag{19}$$

Remark 22 ([7])**.** For simplicity and ease of comparison to the scalar case, we state Theorem 21 assuming that the matrices W_0 and W_1 commute. When we factor an $\mathcal{A}_p$ weight it is always possible to get commuting matrices. For reverse factorization, we can remove this hypothesis, but to do so we must replace the product in (19) with a more complicated expression, the geometric mean of the two matrices.

To state the matrix weight version of Rubio de Francia extrapolation, we extend our convention about extrapolation pairs to pairs of vector-valued functions $(\mathbf{f}, \mathbf{g}) \in \mathcal{F}$ that satisfy inequalities of the form (7). However, we omit the assumption that they are nonnegative.

Theorem 23. *Given p_0, $1 \leqslant p_0 \leqslant \infty$, suppose that for a family of extrapolation pairs $\mathcal{F}$ and for all $W_0 \in \mathcal{A}_{p_0}$, the inequality*

$$\|\mathbf{f}\|_{L^{p_0}(W_0)} \leqslant N_{p_0}([W_0]_{\mathcal{A}_{p_0}}) \|\mathbf{g}\|_{L^{p_0}(W_0)}, \qquad (\mathbf{f}, \mathbf{g}) \in \mathcal{F},$$

holds. Then for all p, $1 < p < \infty$, and all $W \in \mathcal{A}_p$,

$$\|\mathbf{f}\|_{L^p(W)} \leqslant N_p([W]_{\mathcal{A}_p})\|\mathbf{g}\|_{L^p(w)}, \qquad (\mathbf{f},\mathbf{g}) \in \mathcal{F},$$

where the function N_p is defined exactly as in Theorem 10:

$$N_p([W]_{\mathcal{A}_p}) = C(n,p_0,p)N_{p_0}(c(n,p_0,p)[W]_{\mathcal{A}_p}^{\max\{\frac{p}{p_0},\frac{p'}{p_0'}\}}).$$

Remark 24 ([7])**.** As in the scalar case, there are some minor technical difficulties to prove weighted norm inequalities for operators using families of extrapolation pairs and Theorem 23. For a careful discussion, with specific examples.

The proofs of both Theorems 21 and 23 are long and extremely technical. Our goal here is to give a conceptual overview of the proofs and the tools that were developed for them. Underlying the proofs of Theorems 21 and 23 was the systematic philosophy of trying to replicate the proofs of factorization and extrapolation in the scalar case, especially the elementary proofs of these results in [17, 29]. As we noted above, the proof of extrapolation in the scalar case depends on four key ideas: the duality of the $\mathcal{A}_p$ condition, sharp bounds for weighted norm inequalities for the maximal operator, the Jones factorization theorem, and the Rubio de Francia iteration operator.

The duality property of the $\mathcal{A}_p$ condition, as we noted above, is a consequence of the definition of matrix $\mathcal{A}_p$ using reducing operators.

The sharp weighted bounds for the Hardy-Littlewood maximal operator were more challenging. A fundamental technical obstacle to the proof was the lack of an appropriate definition of the maximal operator that can be iterated to produce a vector-valued function. To overcome this problem, the proof passes from vector-valued functions to the larger category of convex set-valued functions, and uses a convex set-valued maximal operator. There is actually a well-developed theory of convex set-valued functions, see [3, 9]. However, it does not appear to be well-known among harmonic analysts.

We begin with some definitions about convex sets. For complete details, see [7] and the references it contains. Let $\mathcal{K}$ denote the family of all convex sets $K \subset \mathbb{R}^d$ that are closed, bounded, and symmetric: i.e., if $x \in K$, then $-x \in K$. Sometimes it is necessary also to assume K is absorbing: that is, that $0 \in \operatorname{int}(K)$. Here, however, we will not worry about this technical

hypothesis. Given a set $K \subset \mathbb{R}^d$, let $|K| = \sup\{|v| : v \in K\}$. Given a matrix W, define $WK = \{Wv : v \in K\}$; note that WK is also a convex set. Given two convex sets K and L, their Minkowski sum is the set $K + L = \{u + v : u \in K, v \in L\}$, which is again convex.

Recall (see (12)) that every norm ρ has associated to it the convex set K_ρ which is its unit ball. The converse is also true: to every convex set K these is associated to it a unique norm, ρ_K, on $\mathbb{R}^d$ such that K is its unit ball. Moreover, arguing as we did above, there exists a matrix W such that $\rho_K \approx \rho_W$. While generally the proof works with matrices $W \in \mathcal{A}_p$ or with their associated reducing operators, at a key point it is necessary to work with more general norms and the underlying convex sets.

A convex set-valued function is a map $F : \mathbb{R}^n \to \mathcal{K}$. There are several equivalent ways to define measurability of such functions F; for our purposes a useful and intuitive definition is that there exists a family $\{\mathbf{f}_k\}_{k=1}^\infty$ of measurable functions $\mathbf{f}_k : \mathbb{R}^n \to \mathbb{R}^d$, called selection functions, such that for almost every $x \in \mathbb{R}^n$,

$$F(x) = \overline{\{\mathbf{f}_k(x) : k \in \mathbb{N}\}}.$$

Given such a convex set-valued function F, we can define for each x the associated John ellipsoid function E. This ellipsoid-valued function can be chosen to be measurable. This fact appears to have been known (see, for instance, Goldberg [52]), but a proof was not in the literature; a proof was given in [7].

The integral of a convex set-valued function is defined using selection functions; this is referred to as the Aumann integral [3,4]. Given $\Omega \subset \mathbb{R}^n$ and a convex set-valued function $F : \Omega \to \mathcal{K}$, define the collection of integrable selection functions of F,

$$S^1(\Omega, F) = \{\mathbf{f} \in L^1(\Omega, \mathbb{R}^d) : \mathbf{f}(x) \in F(x)\}.$$

The Aumann integral of F is defined to be the set

$$\int_\Omega F(x)\,dx = \left\{ \int_\Omega \mathbf{f}(x)\,dx : \mathbf{f} \in S^1(\Omega, F) \right\}.$$

Since $F(x)$ is closed, bounded and convex, the Aumann integral is also a closed, convex set in $\mathbb{R}^d$. (See [7].)

Remark 25 ([82])**.** There is a close connection between the Aumann integral and the convex averages $\langle\langle \mathbf{f} \rangle\rangle_Q$ used by Nazarov, *et al.* to de-

fine convex body sparse domination. Given a vector-valued function $\mathbf{f}$, define $F_{\mathbf{f}}(x) = \overline{\text{conv}}\{\mathbf{f}(x), -\mathbf{f}(x)\}$; that is, the closed convex hull of the set $\{\mathbf{f}(x), -\mathbf{f}(x)\}$. Then $F_{\mathbf{f}}$ is a measurable, convex set-valued function, and

$$\langle\langle \mathbf{f} \rangle\rangle_Q = \fint_Q F_{\mathbf{f}}(x)\,dx.$$

Thus, the convex body sparse operator $T_{\mathcal{S}}$ is the infinite Minkowski sum of convex set-valued functions; if $\mathbf{f}$ is, for instance, bounded and has compact support, then this sum converges to a convex set-valued function.

We use the Aumann integral to define the convex set-valued maximal operator. Given $F : \mathbb{R}^n \to \mathcal{K}$, let

$$MF(x) = \overline{\text{conv}} \left(\bigcup_Q \fint_Q F(y)\,dy \cdot \chi_Q(x) \right);$$

that is, $MF(x)$ is the closed, convex hull of the union of the Aumann integral averages of F over all cubes containing x. Then MF is a measurable, convex set-valued function. (See [7].) The intuition behind this definition is that the Hardy-Littlewood maximal operator finds the largest average in magnitude, and so uses the supremum. The convex set-valued maximal operator finds the largest average in magnitude in each direction; the union of all these averages preserves information about both magnitude and direction.

Remark 26 ([7])**.** The convex set-valued maximal operator does not preserve some natural subsets of $\mathcal{K}$. For instance, if F is ellipsoid-valued (that is, $F = W\mathbf{B}$, where W is a matrix valued function) then MF need not be ellipsoid-valued.

The convex set-valued maximal operator has a number of properties that correspond to those of the Hardy-Littlewood maximal operator, replacing inequalities with set inclusion. For convex set-valued functions F and G, almost every $x \in \mathbb{R}^n$, and $\alpha \in [0, \infty)$,

1. $F(x) \subset MF(x)$;
2. $M(F + G)(x) \subset MF(x) + MG(x)$, where the sum is the Minkowski sum;
3. $M(\alpha F)(x) = \alpha MF(x)$.

The maximal operator M also satisfies L^p norm inequalities. For $1 \leqslant p < \infty$, define $L^p_{\mathcal{K}}(\mathbb{R}^n, |\cdot|)$ to be the collection of all convex set-valued functions F such that

$$\|F\|_{L^p_{\mathcal{K}}(\mathbb{R}^n,|\cdot|)} = \left(\int_{\mathbb{R}^n} |F(x)|^p \, dx \right)^{\frac{1}{p}} < \infty.$$

When $p = \infty$, define $L^\infty_{\mathcal{K}}(\mathbb{R}^n, |\cdot|)$ by $\|F\|_{L^\infty_{\mathcal{K}}(\mathbb{R}^n,|\cdot|)} = \||F|\|_{L^\infty(\mathbb{R}^n)} < \infty$. Then for $1 < p \leqslant \infty$, $\|MF\|_{L^p_{\mathcal{K}}(\mathbb{R}^n,|\cdot|)} \leqslant C\|F\|_{L^p_{\mathcal{K}}(\mathbb{R}^n,|\cdot|)}$, and when $p = 1$ a weak type inequality holds.

Remark 27 ([7])**.** Although the notation $\|\cdot\|_{L^p_{\mathcal{K}}(\mathbb{R}^n,|\cdot|)}$ is deliberately suggestive of a norm, the set $L^p_{\mathcal{K}}(\mathbb{R}^n, |\cdot|)$ is not a normed vector space, since the Minkowski sum does not have an additive inverse. It is, however, a complete metric space.

Sharp weighted norm inequalities for the convex maximal operator are governed by the matrix $\mathcal{A}_p$ weights. For $1 \leqslant p < \infty$, define $L^p_{\mathcal{K}}(\mathbb{R}^n, W) = L^p_{\mathcal{K}}(W)$ to be all convex set-valued functions F such that

$$\|F\|_{L^p_{\mathcal{K}}(\mathbb{R}^n,W)} = \left(\int_{\mathbb{R}^n} |W(x)F(x)|^p \, dx \right)^{\frac{1}{p}} < \infty.$$

When $p = \infty$, define $L^\infty_{\mathcal{K}}(\mathbb{R}^n, W)$ by $\|F\|_{L^\infty_{\mathcal{K}}(\mathbb{R}^n,W)} = \|WF\|_{L^\infty_{\mathcal{K}}(\mathbb{R}^n,|\cdot|)} < \infty$.

Theorem 28. *Given p, $1 < p \leqslant \infty$, and $W \in \mathcal{A}_p$, for every $F \in L^p_{\mathcal{K}}(W)$,*

$$\|MF\|_{L^p_{\mathcal{K}}(W)} \leqslant C(n, d, p)[W]^{p'}_{\mathcal{A}_p} \|F\|_{L^p_{\mathcal{K}}(W)}.$$

The proof of Theorem 28 uses the L^p norm inequalities for the Christ-Goldberg maximal operator M_W; the quantitative constant in terms of $[W]_{\mathcal{A}_p}$ was proved by Isralowitz and Moen [63]. It is the best possible since this is the same exponent as in the scalar case. The proof essentially bounds $\|MF\|_{L^p_{\mathcal{K}}(W)}$ by the sum of terms of the form $\|M_W \mathbf{f}\|_{L^p(\mathbb{R}^n)}$. In fact there is a converse to this inequality, showing the close connection between the Christ-Goldberg maximal operator and the convex set-valued operator: see Vuorinen [105, Lemma 3.4].

Remark 29. An important open problem is to give a direct proof of Theorem 28 that does not use the bounds for the Christ-Goldberg maximal operator. In the scalar case there is a simple proof of the strong (p, p) norm inequalities for the Hardy-Littlewood maximal operator in Theorem 3 due

to Christ and Fefferman [11] which initially seemed a good model. However, this proof uses the universal dyadic maximal operator (17) which does not appear to generalize to the setting of matrix weights. We believe there is a close connection between this problem and the problem of finding sharp estimates for sparse operators and singular integrals.

Before we can define the iteration operator, which is used to construct $\mathcal{A}_1$ weights, we need to define the analog of the $\mathcal{A}_1$ condition for convex set-valued functions. Given $F : \mathbb{R}^n \to \mathcal{K}$, we say that $F \in \mathcal{A}_1^{\mathcal{K}}$ if $MF(x) \subset CF(x)$. Denote the infimum of all such constants C by $[F]_{\mathcal{A}_1^{\mathcal{K}}}$. There is a very close relationship between convex set-valued $\mathcal{A}_1^{\mathcal{K}}$ and matrix $\mathcal{A}_1$.

Theorem 30. *Given a matrix weight W, $W \in \mathcal{A}_1$ if and only if $F = W\mathbf{B} \in \mathcal{A}_1^{\mathcal{K}}$, and $[W]_{\mathcal{A}_1} \approx [F]_{\mathcal{A}_1^{\mathcal{K}}}$.*

The proof of Theorem 30 in [7] required a careful development of the properties of norm functions and their duals on the one hand, and the properties of convex sets and their polar bodies on the other.

With this machinery we can now define the convex set-valued analog of the Rubio de Francia iteration operator. Given $F : \mathbb{R}^n \to \mathcal{K}$ and $W \in \mathcal{A}_p$, $1 < p < \infty$, define

$$\mathcal{R}F(x) = \sum_{k=0}^{\infty} \frac{M^k F(x)}{2^k \|M\|^k_{L^p_{\mathcal{K}}(W)}}.$$

This sum converges in norm and pointwise in the appropriate metric. The iteration operator $\mathcal{R}$ has the following properties which are the exact analogs of the properties of the scalar iteration operator given above:

1. $F(x) \subset \mathcal{R}F(x)$;
2. $\|\mathcal{R}F\|_{L^p_{\mathcal{K}}(W)} \leqslant 2\|F\|_{L^p_{\mathcal{K}}(W)}$;
3. $\mathcal{R}F \in \mathcal{A}_1^{\mathcal{K}}$ and $[\mathcal{R}F]_{\mathcal{A}_1^{\mathcal{K}}} \leqslant 2\|M\|_{L^p_{\mathcal{K}}(W)}$.

The final key idea we need to generalize to the matrix setting is the Jones factorization theorem, Theorem 21. And, in fact, we only need reverse factorization, which is very easy to prove in the scalar case. However, in the matrix case, the opposite is true. Given a matrix weight $W \in \mathcal{A}_p$, the construction of $W_0 \in \mathcal{A}_1$ and $W_1 \in \mathcal{A}_\infty$ follows the scalar proof in [17] closely. There is one technical obstacle: the scalar proof uses a variant of Rubio de Francia iteration operator based on the maximal operator, $M_s f(x) = M(|f|^s)^{\frac{1}{s}}$,

$s > 1$, which is a sublinear operator. To define $M_s F(x)$, requires replacing F by an appropriate ellipsoid (so that the power F^s is well-defined) and then proving that the resulting operator is sublinear.

The converse, proving reverse factorization, is very delicate and much more difficult than in the scalar case. The proof does not use the definitions of matrix $\mathcal{A}_p$ and $\mathcal{A}_1$ in terms of matrices or reducing operators. Instead, this proof requires the definition of matrix $\mathcal{A}_p$ in terms of norms as originally given by Treil and Volberg. Intuitively, the proof can be thought of as an interpolation argument between finite dimensional Banach spaces: that is, proving that the interpolation norm is in fact in $\mathcal{A}_p$.

The proof of extrapolation follows the proof of sharp constant extrapolation in the scalar case given in [29]. (For a simpler proof that produces a worse constant that can easily be adapted to the matrix setting, see also [17].) Because of the way in which the matrix $\mathcal{A}_p$ condition is defined, the proof also yields extrapolation from the endpoint $p = \infty$; this gives a quantitative version of an extrapolation result first proved in the scalar case by Harboure, Macías and Segovia [55]. (This quantitative version was proved independently by Nieraeth [85].) The proof uses reverse factorization and the Rubio de Francia iteration operator. In order to close the gap between the convex set-valued functions that this operator creates, and the ellipsoid valued functions that appear in Theorem 30 (that is, to pass between matrix $\mathcal{A}_1$ and convex set $\mathcal{A}_1$), the iteration operator is modified so that it produces ellipsoid-valued functions.

Remark 31. In the original paper [7], the authors only considered vector-valued functions which had real-valued components. This led naturally to considered convex set-valued functions in $\mathbb{R}^d$. However, in many applications it is important to consider vector-valued functions that take values in $\mathbb{C}^d$; to prove extrapolation in this setting requires developing a theory of convex set-valued functions in $\mathbb{C}^d$. Kakaroumpas and Soler i Gibert [68] have shown that all of the proofs in [7] extend to this setting; these extensions require a number of technical modifications.

Remark 32. If we apply Theorem 23 starting with the sharp constant for singular integrals when $p = 2$, we get that for $1 < p < \infty$ a bound of the form $[W]_{\mathcal{A}_p}^{3\max\{p,p'\}}$. This is always worse, when $p \neq 2$, than the bound obtained in [24]. It is unclear why extrapolation produces the sharp bounds in the scalar case but not for matrix weights.

As in the scalar theory of extrapolation, the matrix theory has been generalized in several directions. Here we describe three recent results. First, Kakaroumpas and Soler i Gibert [68] have proved that matrix extrapolation also yields vector inequalities analogous to (8). They showed that with the same hypotheses as Theorem 23, you get that

$$\left\| \left(\sum_{k=1}^{\infty} |W\mathbf{f}_k|^q \right)^{\frac{1}{q}} dx \right\|_{L^p(\mathbb{R}^n)} \leqslant C \left\| \left(\sum_{k=1}^{\infty} |W\mathbf{g}_k|^q \right)^{\frac{1}{q}} dx \right\|_{L^p(\mathbb{R}^n)},$$

when $1 < p,\, q < \infty$, $w \in \mathcal{A}_p$, and $\{(\mathbf{f}_k, \mathbf{g}_k)\}_{k=1}^{\infty} \subset \mathcal{F}$.

Remark 33. As we noted above, in the scalar case extrapolation also yields classical weak type inequalities. It is an open problem whether extrapolation can be used to prove weak type inequalities in the matrix setting.

Second, matrix extrapolation has been extended to weights and operators defined with respect to general bases. Given a basis $\mathcal{B}$, we can proceed as we did in the scalar case (see (9) and(10)) and define matrix weights $\mathcal{A}_{p,\mathcal{B}}$, $1 \leqslant p \leqslant \infty$, the Christ-Goldberg maximal operator $M_{W,\mathcal{B}}$, and the convex set-valued maximal operator $M_{\mathcal{B}}$. If we assume $\mathcal{B}$ is a matrix Muckenhoupt basis, that is, that $\|M_{W,\mathcal{B}}\mathbf{f}\|_{L^p(\mathbb{R}^n)} \leqslant C\|\mathbf{f}\|_{L^p(\mathbb{R}^n)}$ whenever $\mathcal{W} \in \mathcal{A}_p$, then the proof of Theorem 23 goes through with essentially no change. See [19].

Remark 34. In [19] the authors also investigate the properties of matrix weights. In the scalar case, while in the definition of the $\mathcal{A}_p$ classes we implicitly assumed that $0 < w(x) < \infty$ a.e. (and in fact that w and w^{-1} were in $L^p_{\mathrm{loc}}(\mathbb{R}^n)$ and $L^{p'}_{\mathrm{loc}}(\mathbb{R}^n)$), it can be shown that these are necessary conditions if the Hardy-Littlewood maximal operator satisfies a weak (p,p) inequality. (This fact is often glossed over in treatments of the Muckenhoupt condition. See [51] for details.) By a change of variables, one can redefine the Christ-Goldberg maximal operator so that it does not use the inverse W^{-1} in its definition. If one assumes that this operator satisfies a weak type inequality, then this implies that W is invertible almost everywhere. See [19] for precise definitions and details.

Third, matrix extrapolation can be extended to the off-diagonal setting. The natural conjecture is that extrapolation should hold for weights that satisfy a matrix $\mathcal{A}_{p,q}$ condition that generalizes the $\mathcal{A}_{p,q}$ weights introduced

by Muckenhoupt and Wheeden: for instance, if $1 < p \leqslant q < \infty$, $W \in \mathcal{A}_{p,q}$ if

$$[W]_{\mathcal{A}_{p,q}} = \sup_Q \left(\fint_Q \left(\fint_Q |W(x)W^{-1}(y)|_{\mathrm{op}}^{p'} \, dy \right)^{\frac{q}{p'}} \, dx \right)^{\frac{1}{q}} < \infty.$$

However, the proof required a stronger condition.

Theorem 35. *Fix s, $1 \leqslant s < \infty$, and a family of extrapolation pairs $\mathcal{F}$. Suppose that for some $p_0, q_0, 1 \leqslant p_0 \leqslant q_0 \leqslant \infty$, where $\frac{1}{p_0} - \frac{1}{q_0} = \frac{1}{s'}$ and $r_0 = \frac{q_0}{s}$, and every matrix weight W_0 with $W_0^s \in \mathcal{A}_{r_0}$,*

$$\|\mathbf{f}\|_{L^{q_0}(W_0)} \leqslant C([W^s]_{\mathcal{A}_{r_0}}) \|\mathbf{g}\|_{L^{q_0}(W_0)}, \qquad (\mathbf{f}, \mathbf{g}) \in \mathcal{F}.$$

Then for all p, q such that $1 < p \leqslant q < \infty$, $\frac{1}{p} - \frac{1}{q} = \frac{1}{p_0} - \frac{1}{q_0}$, $r = \frac{q}{s}$, and for all matrix weights W with $W^s \in \mathcal{A}_r$,

$$\|\mathbf{f}\|_{L^q(W)} \leqslant C([W^s]_{\mathcal{A}_r}) \|\mathbf{g}\|_{L^q(W)}, \qquad (\mathbf{f}, \mathbf{g}) \in \mathcal{F}.$$

The problem that arises is that, unlike in the scalar case, we no longer have that $W \in \mathcal{A}_{p,q}$ if and only if $W^s \in \mathcal{A}_r$. The latter condition is strictly stronger: see [19] for a counter-example. And it is this condition which the proof, generalizing the scalar proof, relies on. It is an open question whether Theorem 35 is true with only the weaker hypothesis that $W \in \mathcal{A}_{p,q}$.

Remark 36. The proof of Theorem 35 yields quantitative estimates on the constants in terms of $[W^s]_{\mathcal{A}_r}$; however, when reduced to the constant exponent case this proof does not give the sharp results of Lacey, *et al.* [69]. For a proof which does yield these sharp results, see the forthcoming paper [18].

The machinery developed to prove Theorems 21 and 23 provides additional tools for working with the convex body sparse domination of Nazarov, *et al.* To illustrate this, we conclude this section by sketching a very recent result in [35]. The authors give a new and relatively simple proof that convex body sparse operators, and so singular integrals, are bounded on $L^p(W)$, $1 < p < \infty$. We omit a number of (minor) technical details to emphasize the overall outline of the proof, which mimics a proof in the scalar case.

Fix p, a sparse family $\mathcal{S}$, and $\mathbf{f} \in L^p(W)$. Then by a duality argument

there exists $\mathbf{g} \in L^{p'}(W^{-1})$, $\|\mathbf{g}\|_{L^{p'}(W^{-1})} \leqslant d$, such that

$$\begin{aligned}
\|T_{\mathcal{S}}\mathbf{f}\|_{L^p_{\mathcal{K}}(W)} &\leqslant 2\int_{\mathbb{R}^n} \langle T_{\mathcal{S}}\mathbf{f}(x), \mathbf{g}(x)\rangle\,dx \\
&\leqslant C\sum_{Q\in\mathcal{S}} \left\langle \fint_Q F_{\mathbf{f}}(y)\,dy, \fint_Q \mathbf{g}(x)\,dx \right\rangle |E(Q)| \\
&= C\sum_{Q\in\mathcal{S}} \int_{E(Q)} \left\langle W(z) \fint_Q F_{\mathbf{f}}(y)\,dy, \right. \\
&\qquad\qquad \left. \fint_Q W^{-1}(z)W(y)W^{-1}(y)\mathbf{g}(y)\,dy \right\rangle dz;
\end{aligned}$$

by the definition of the convex set-valued maximal operator and the Christ-Goldberg maximal operator, and by the sharp bounds for each,

$$\begin{aligned}
&\leqslant C\sum_{Q\in\mathcal{S}} \int_{E_Q} |W(z)MF_{\mathbf{f}}(z)| M_{W^{-1}}(W^{-1}\mathbf{g})(z)\,dz \\
&\leqslant C\|MF_{\mathbf{f}}\|_{L^p_{\mathcal{K}}(W)} \|M_{W^{-1}}(W^{-1}\mathbf{g})\|_{L^{p'}(\mathbb{R}^n)} \\
&\leqslant C[W]^{p'}_{A_2} \|F_{\mathbf{f}}\|_{L^p_{\mathcal{K}}(W)} [W^{-1}]^p_{\mathcal{A}_{p'}} \|\mathbf{g}\|_{L^{p'}(W^{-1})} \\
&\leqslant C[W]^{p+p'}_{\mathcal{A}_p} \|\mathbf{f}\|_{L^p(W)}.
\end{aligned}$$

Remark 37. This proof does not give the best known estimates: when $p = 2$ it gives an exponent of 4 instead of 3. It is a very interesting question whether this approach can be refined to prove the sharp exponent.

6. Further results on matrix weights

In this section we briefly summarize some additional results involving matrix weights that we think are of interest. We note that this is a very active field and there are many other results which are not included here. Here we restrict to results related to our own work.

We begin with recent work extending matrix weights to the variable Lebesgue spaces. The variable Lebesgue spaces $L^{p(\cdot)}$ are a generalization of the classical L^p spaces, replacing the exponent p by an exponent function $p(\cdot)$. They have been studied extensively for more than 30 years; for their history and basic properties, we refer to the books [21, 40]. Scalar weighted norm inequalities in $L^{p(\cdot)}$ spaces were introduced in [22] (see also [20]),

where bounds for the maximal operator were shown to be controlled by $\mathcal{A}_{p(\cdot)}$ weights, a generalization of the Muckenhoupt condition. Extrapolation from $L^{p_0}(w_0)$ to $L^{p(\cdot)}(w)$ was proved in [38] (see also [17]); this immediately yielded bounds for singular integrals and other operators.

The problem of matrix weights in variable Lebesgue spaces has been considered in a series of recent papers by Penrod and his collaborators. In [33] they defined matrix $\mathcal{A}_{p(\cdot)}$ weights and used this to characterize averaging operators. These they used to prove that certain convolution operators are bounded and the related approximate identities converge. As an application they generalized the "$H = W$" theorem of Meyers and Serrin [77] to matrix weighted space, extending an earlier result in [32]. In [34] the authors settled an open problem in the scalar case, proving the existence of a reverse Hölder inequality for scalar $\mathcal{A}_{p(\cdot)}$ weights. As an application they proved that matrix $\mathcal{A}_{p(\cdot)}$ weights satisfy both a left and right openness property. This result was new for matrix weights even in the constant exponent case, and was used by Penrod and Sweeting [90] to prove strong type inequalities for singular integrals. Very recently, Nieraeth and Penrod [89] used left and right openness to prove that the Christ-Goldberg maximal operator is bounded on variable Lebesgue spaces, and then used a version of extrapolation to prove that singular integral operators are bounded on $L^{p(\cdot)}(W)$ when $W \in \mathcal{A}_{p(\cdot)}$. A different proof, that does not use extrapolation, will appear in [35].

Variable Lebesgue spaces are examples of Banach function spaces, and there has been some interest in extending harmonic analysis and extrapolation theory to this setting. In the scalar case, see [17,29]. Recently, Nieraeth has extended the techniques in [7] to prove a very general extrapolation theorem in matrix weighted Banach function spaces [87].

In the past 25 years there has been a great deal of work on the weighted theory for multilinear operators. The weight class $\mathcal{A}_{\vec{p}}$, $\vec{p} = (p_1, \dots, p_m)$, was introduced by Lerner, *et al.* [76]; they showed that this class was necessary and sufficient for the boundedness of a multi-sublinear maximal operator and nondegenerate multilinear singular integrals. Their results were extended to the setting of matrix weights by Kakaroumpas and Nieraeth [67]. In the scalar weighted variable Lebesgue spaces, the results of Lerner, *et al.* were extended to the bilinear setting in [23]. These results should extend to the multilinear setting, but there are numerous technical obstacles to overcome in their proof and this remains an open problem. Finally, we note that it is also an open problem to extend the results of Kakaroumpas and

Nieraeth to variable Lebesgue spaces.

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Summary

In this chapter we give an overview of recent work on matrix weights, with particular emphasis on convex body sparse domination for singular integrals, Rubio de Francia extrapolation, and Jones factorization. To provide context and motivation, we survey the comparable results in the scalar weighted case.

Streszczenie

W tym rozdziale przedstawiamy przegląd najnowszych badań nad wagami macierzowymi, ze szczególnym uwzględnieniem rzadkiej dominacji dla całek osobliwych w kontekście wypukłych ciał, ekstrapolacji w stylu Rubio de Francia oraz faktoryzacji Jonesa. Aby przedstawić kontekst i motywację, omawiamy porównywalne wyniki w przypadku wag skalarnych.

Selected Topics in Operator Theory and Topology

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